Problem Set #1 - Answer Sheet

0. $\binom{12}{4} = 495$

There is a typo, it should be $\binom{12}{4}$ rather than $\binom{11}{4}$.

1. 4^{11}

Each of the 11 positions can independently be any of the 4 nucleotides. By the step rule of counting, the total number of sequences is: $4 \times 4 \times \cdots \times 4 = 4^{11}$.

2.

a. $26!$

There are 26 unique letters, and we are arranging all of them. The total number of permutations is simply: $26!$

b. $25! \times 2!$

Treat the pair (Q, U) as a single “super letter.” This gives 25 elements to arrange:

- $25!$ ways to order them.
- Inside the pair, Q and U can be $2! = 2$ orders.

So the total number is: $25! \times 2!$

c. $22! \times 5!$

Consider the 5 vowels as a single group:

- 22! ways to arrange the 21 consonants plus this vowel group.
- The vowels within the group can be arranged in 5! ways.

Total number of permutations: $22! \times 5!$

d. $\binom{21+1}{5} \times 22! \times 5!$

First, arrange the 21 consonants in a line. This creates 22 “gaps”:

- 1 gap before the first consonant,
- 20 gaps between consonants,
- 1 gap after the last consonant.

We must place the 5 vowels in 5 of these 22 gaps so that no two vowels are adjacent.

- There are $\binom{22}{5}$} ways to choose the gaps.
- The consonants can be arranged in 21! ways.
- The vowels can be arranged in 5! ways.

So the total number is: $\binom{21+1}{5} \times 22! \times 5!$

3.

a. $\frac{104!}{(2!)^{52}}$

If all 104 cards were unique, there would be $104!$ arrangements. But since there are 2 identical copies of each card type (52 types), we must divide by 2! for each pair, giving: $\frac{104!}{(2!)^{52}}$

b. $52 * 52 = 2704$

Each of the 2 cards can be one the the 13 values and 4 suits, which gives 52 possible outcomes for each card. Therefore there're $52 * 52 = 2704$ dixtinct pairs.

4.

a. $\binom{10+4-1}{4-1} = 286$

First, we must account for the minimum investments: $1 + 2 + 3 + 4 = \$10M$.

This leaves: $20 - 10 = \$10M$ to allocate freely among the 4 companies.

This is equivalent to the number of non-negative integer solutions to:
$x_1 + x_2 + x_3 + x_4 = 10$.

By the "stars and bars" theorem (divider method), the number of solutions is:
$\binom{10+4-1}{4-1} = \binom{13}{3}$.

b. $\binom{14}{4} + \binom{13}{4} + \binom{12}{4} + \binom{11}{4} = 2541$

Similar to a, but there're 4 cases: we have to invest company 1, 2, 3 / company 1, 2, 4 / company 1, 3, 4 / company 2, 3, 4, which corresponds to (n, r) equals to $(14{,}4)$, $(13{,}4)$, $(12{,}4)$, $(11{,}4)$.

5. $\dfrac{99!}{(3!)^{33} \times 33!} = 2.25075 \times 10^{93}$

Imagine arranging all 99 students in a line: there are 99! permutations.

- Since the order within each group of 3 students does not matter, for each group we must divide by 3!.
- Since the order of the 33 groups does not matter, we also divide by 33!.

Thus, the total number of partitions is: $\dfrac{99!}{(3!)^{33} \times 33!}$.

6. $\sum_{s=0}^{k} \binom{s+n-1}{n-1}$

We are counting all vectors whose entries sum to at most k. For each possible total sum $s \in \{0,1,\ldots,k\}$, the number of vectors where the entries sum to exactly s is: $\binom{s+n-1}{n-1}$, by the divider method.

Summing over all s from 0 to k gives: $\sum_{s=0}^{k} \binom{s+n-1}{n-1}$.

7.

a. $\binom{m+n-2}{n-1}$

To reach the destination (⭐), the robot need to take $(m-1)+(n-1) = m+n-2$ steps in total. Which of these steps goes down, and which of these steps goes right determines the path. There're $\binom{m+n-2}{n-1}$ combinations of stepping down and stepping right, and therefore there're $\binom{m+n-2}{n-1}$ paths.

b. $\binom{m+n-3}{n-1}$

Similar to question a, however, since the first step is determined, we can only determine the following $(m-2)+(n-1)=m+n-3$ steps. Therefore there're $\binom{m+n-3}{n-1}$ paths.

c. $2 \times (m-2) \times (n-2)$

Three times of changinf direction correxpons to two cases:

1. right steps -> down steps -> right steps -> down steps
2. down steps -> right steps -> down steps -> right steps

For case 1, we need to divide all $(m-1)$ right steps to 2 groups, each group contains at least one right step. Similarly, we need to divide all $(n-1)$ down steps to 2 groups, and each group contains at least one down step. Therefore, there're $\binom{m-1-1}{1} \times \binom{n-1-1}{1}$ permutations for all right and down steps.

Case 2 has the same number of permutations as case 1.

By adding the number of permutations of case 1 and case 2, we can get the total number of eligible permutations.

8.

a. $\binom{12}{4} = 495$

We are distributing 8 identical hits among 5 computers.
This is the number of non-negative integer solutions to: $x_1 + x_2 + x_3 + x_4 + x_5 = 8$.
Using the divider ("stars and bars") method: $\binom{8+5-1}{5-1} = \binom{12}{4}$.

b. $\dfrac{12!}{5! \times 3! \times 4!} = 27720$

We can use the variation of the divider method. Recall the divider method that $N = \dfrac{(n+r-1)!}{n! \times (r-1)!}$, the $n! \times (r-1)!$ serves as the denominator because there're n and $(r-1)$ identical items. In this question, 5 and 3 of the 8 requests are considered identical, therefore the denominator should be $n_1! \times n_2! \times (r-1)!$, where $n_1 = 5$ and $n_2 = 3$, and the number of distribution should be $\dfrac{(n_1 + n_2 + (r-1))!}{n_1! \times n_2! \times (r-1)!}$.

9.

a. $\dfrac{1}{n}$

There're 2 ways to think of it.

The first way is intuitive, the probability of failure of first try is $\dfrac{n-1}{n}$, the probability of failure of first try is $\dfrac{n-2}{n-1}$, and so on. Therefore, the probability of tehfirst success on the first try is $\dfrac{1}{n}$.

The second way uses the probability of equally likely outcomes. The number of oucomes is n because the success can happen on any of the n tries. The number of outcomes that the first success happens on the 5th try is 1, therefore $P = \dfrac{1}{n}$.

b. $(\dfrac{n-1}{n})^4 \times \dfrac{1}{n}$

The probability of failure is $\dfrac{n-1}{n}$, and the probability of success is $\dfrac{1}{n}$. Because each experiment is independent, therefore we can use the probability of and.

10.

a. $\dfrac{4 \times \binom{13}{5}}{\binom{52}{5}} = 0.00198$

There're $\binom{13}{5}$ ways to choose 5 cards in one suit (contains 13 cards), since there're 4 suits, the number of possible flush is $4 \times \binom{13}{5}$. Because there're $\binom{52}{5}$ number of ways to choose 5 cards from the 52 cards, then $P = \dfrac{4 \times \binom{13}{5}}{\binom{52}{5}}$.

b. $\dfrac{\binom{4}{1} \times \binom{13}{2} \times \binom{3}{1} \times \binom{13}{2} \times \binom{2}{1} \times \binom{13}{1}}{\binom{52}{5}} = 0.04754$

We first choose 1 suit from the 4 suits and choose 2 cards from the 13 cards of that suit. Similarly, we choose 1 suit from the lasting 3 suits and choose 2 cards from the 13 cards it…

Because there're $\binom{52}{5}$ number of ways to choose 5 cards from the 52 cards, then

$$P = \frac{\binom{4}{1} \times \binom{13}{2} \times \binom{3}{1} \times \binom{13}{2} \times \binom{2}{1} \times \binom{13}{1}}{\binom{52}{5}}$$

c. $\dfrac{\binom{4}{1} \times \binom{13}{3} \times \binom{3}{1} \times \binom{2}{1} \times \binom{13}{1}}{\binom{52}{5}} = 0.02113$

Similar to b

11. $\dfrac{\binom{12}{4} \binom{8}{3} \dfrac{20!}{(2!)^4 (4!)^3}}{12^{20}}$

Step-by-step reasoning:

1. Choose which 4 users get 2 emails each: $\binom{12}{4}$.

2. Choose which 3 users get 4 emails each: $\binom{8}{3}$.

3. Partition the 20 emails into the 4 groups of 2 emails and 3 groups of 4 emails: $\dfrac{20!}{(2!)^4 (4!)^3}$.

4. Denominator:
Each email independently goes to any of 12 users: 12^{20}.

So the probability is: $$\frac{\binom{12}{4}\binom{8}{3}\frac{20!}{(2!)^4\,(4!)^3}}{12^{20}}.$$

12. See the code file