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VIRTUAL MOBILITY IN DECISION SCIENCES

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Education and Culture DG

Lifelong Learning Programme

CHAPTER 3
ELECTRE MODELS
(A brief summary)

[VR_COI002]

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Abstract

This open access file summarizes the theory for the VRTUOSI course Multi-Criteria Decision Aiding. Examples and some extensions presented on the course are omitted here.

ELECTRE Methods are based on pairwise comparisons of actions (outranking relations) that avoid complete compensation among the criteria. This file presents basic ideas along with the first method in this family: ELECTRE I. Later, it presents more sophisticated outranking relations and the ELECTRE TRI sorting method, probably the ELECTRE method most used in practice nowadays.

Outranking relations

Value function models compute a global value for each action, allowing to rank them. Outranking methods follow a different principle: they compare actions pair by pair, as in a tournament. Let us start by presenting this idea as implemented by ELECTRE methods.

Performances table

	$g_1(.)$	$g_2(.)$	...	$g_n(.)$
a_1	$g_1(a_1)$	$g_2(a_1)$	...	$g_n(a_1)$
a_2	$g_1(a_2)$	$g_2(a_2)$	...	$g_n(a_2)$
...	...	...	...	...
a_m	$g_1(a_m)$	$g_2(a_m)$	...	$g_n(a_m)$

- We depart again from a given table of performances.
- $g_j(a_i)$ denotes the performance of action a_i ($i=1,...,m$) on the j-th criterion ($j=1,...,n$), expressed on an ordinal or cardinal scale.
- In Chapter 1 you learned how to build such a table.

Basic ideas of ELECTRE

- Given a set of actions, let us consider a pair (a_x, a_y)
- We will say a_x <u>outranks</u> a_y (denoted $a_x\ S\ a_y$) if there are good arguments to sustain that a_x <u>is at least as good as</u> a_y (in other words, a_x <u>is not worse than</u> a_y)
  - If $g_j(a_x)$ is not worse than $g_j(a_y)$, then we say that the jth criterion is concordant with $a_x\ S\ a_y$
  - If $g_j(a_x)$ is worse than $g_j(a_y)$, then we say that the jth criterion is discordant with $a_x\ S\ a_y$
  - Note that a criterion can be both concordant and with $a_x\ S\ a_y$ and also with $a_y\ S\ a_x$, namely when $g_j(a_x)=g_j(a_y)$
- We wish to know whether $a_x\ S\ a_y$ and also whether $a_y\ S\ a_x$, for all pairs of interest

Basic ideas of ELECTRE

	$g_1(.)$	$g_2(.)$	$g_3(.)$	$g_4(.)$	$g_5(.)$
a_x	75	80	60	55	80
a_y	55	80	60	50	50

[scale ranging from zero (worst) to 100 (best]

- a_x is preferred to a_y, since it dominates the latter:

$$a_x \, S \, a_y, \;\; \sim(a_y \, S \, a_x)$$

~ denotes negation

Basic ideas of ELECTRE

	$g_1(.)$	$g_2(.)$	$g_3(.)$	$g_4(.)$	$g_5(.)$
a_x	55	80	60	55	80
a_y	55	80	60	55	80

- a_x is indifferent to a_y (since performances are equal)

$$a_x \, S \, a_y, \; a_y \, S \, a_x$$

Basic ideas of ELECTRE

	$g_1(.)$	$g_2(.)$	$g_3(.)$	$g_4(.)$	$g_5(.)$
a_x	75	80	60	70	80
a_y	65	70	50	60	90

- Subjective: does a_x *outrank* a_y ?
  - In favor: criteria 1,2,3,4 (concordant coalition)
  - Against: criterion 5 (discordant coalition)

Basic ideas of ELECTRE

	$g_1(.)$	$g_2(.)$	$g_3(.)$	$g_4(.)$	$g_5(.)$
a_x	75	80	60	70	80
a_y	65	70	50	60	90

- if:
  - Weight of (1,2,3,4) is “big enough” and opposition from criterion (5) is “small” $\Rightarrow a_x \, S \, a_y$
  - Weight of (5) is not “big enough” $\Rightarrow \sim(a_y \, S \, a_x)$
- Then $a_x \, S \, a_y \wedge \sim(a_y \, S \, a_x) \Rightarrow a_x \, \boldsymbol{P} \, a_y$

  (preference relation $\boldsymbol{P}$: a_x *preferred to* a_y)

Basic ideas of ELECTRE

	$g_1(.)$	$g_2(.)$	$g_3(.)$	$g_4(.)$	$g_5(.)$
a_x	75	80	60	70	80
a_y	65	70	60	75	90

- if:
  - Weight of (1,2,3) is “big enough” and opposition from criteria (4,5) is “small” $\Rightarrow a_x \, S \, a_y$
  - Weight of (3,4,5) is “big enough” and opposition from criteria (1,2) is “small” $\Rightarrow a_y \, S \, a_x$
- Then $a_x \, S \, a_y \wedge a_y \, S \, a_x \Rightarrow a_x \, \boldsymbol{I} \, a_y$

  (indifference relation ***I***: a_x *indifferent to* a_y)

Basic ideas of ELECTRE

	$g_1(.)$	$g_2(.)$	$g_3(.)$	$g_4(.)$	$g_5(.)$
a_x	75	80	60	70	0
a_y	65	70	50	60	90

- if:
  - Weight of (1,2,3,4) is “big enough” but opposition from criterion (5) is very strong (veto) $\Rightarrow \sim(a_x \, S \, a_y)$
  - Weight of (5) is not “big enough” $\Rightarrow \sim(a_y \, S \, a_x)$
- then $\sim(a_x \, S \, a_y) \wedge \sim(a_y \, S \, a_x) \Rightarrow a_x \, \boldsymbol{R} \, a_y$ (incomparability relation ***R***: a_x *incomparable to* a_y)

Basic ideas of ELECTRE

In summary we have the following basic relations of preference

$a_x\ S\ a_y$?	$a_y\ S\ a_x$?	Situation
Yes	No	$a_x\ P\ a_y$
No	Yes	$a_y\ P\ a_x$
Yes	Yes	$a_x\ I\ a_y$
No	No	$a_x\ R\ a_y$

- Notes
  - A global value is not computed for each action
  - Accepts incomparability and no-compensation

The family of ELECTRE Methods

- ELECTRE I (1967): choice, crisp S relation
- ELECTRE IS (1984): choice, valued S relation
- ELECTRE II (1971): ranking, crisp S relation
- ELECTRE III (1978): ranking, valued S relation
- ELECTRE IV (1982): ranking, valued S relation, no weights on criteria
- ELECTRE TRI (1992): sorting, valued S relation

  Notes:

  - Crisp S means a Yes/No relation (either outranks or not)
  - Valued S means that a credibility degree for thr outranking is computed in the interval [0,1]

ELECTRE I for selection problems

ELECTRE I was the first outranking method (1968). It starts by building a crisp outranking relation and then exploits it with the objective of identifying the best action (or a small subset of actions containing the best one)

ELECTRE I

- Developed in the 1960's in France by consultants from the SEMA group including Bernard Roy (the main proponent of ELECTRE methods)
- Specific to the problem of selecting the best (one) action
  - 1st step: To construct a crisp outranking relation

    For each (a_x,a_y), does $a_x\ S\ a_y$? And does $a_y\ S\ a_x$?
  - 2nd step: To exploit the outranking relation (finding a kernel)

    Purpose: to select a minimal set of candidates to become the most preferred action

ELECTRE I: Constructing S

- Given *(a,b)*, ***a S b*** if the following conditions are both true:

Concordance: $$c(a,b) = \sum_{j:\Delta_j(a,b)\geq 0} k_j \geq c$$

(non)**Discordance:** $\forall j: -\Delta_j(a,b) \leq v_j$

notation:

$\Delta_j(a,b)$ is the advantage of a over b on criterion g_j

k_j is the importance coefficient (weight) of g_j

(let us assume $k_1,\ldots,k_n \geq 0$ and $k_1+\ldots+k_n=1$)

$c(a,b)$ is the concordance index,

c is the concordance threshold

v_j is the veto threshold of g_j

ELECTRE I: Constructing S

- Given *(a,b)*, *a S b* if the following conditions are both true:

Concordance: $$c(a,b) = \sum_{j:\Delta_j(a,b)\geq 0} k_j \geq c$$

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k_j is the importance coefficient (weight) of g_j

(let us assume $k_1,\ldots,k_n \geq 0$ and $k_1+\ldots+k_n=1$)

$c(a,b)$ is the concordance index,

c is the concordance threshold

v_j is the veto threshold of g_j

The concordance index is equal to the sum of the weights of the criteria that agree with *aSb* (the criteria in which *a* is as good as *b*, or better).
The concordance condition holds if this sum (the total concordant weight) attains the required majority threshold c.

ELECTRE I: Constructing S

- Given *(a,b), a S b* if the following conditions are both true:

Concordance: $$c(a,b) = \sum_{j:\Delta_j(a,b)\geq 0} k_j \geq c$$

(non)**Discordance:** $\forall j: -\Delta_j(a,b) \leq v_j$

The discordance condition for *aSb* holds if there is no discordant criterion in which *b* is better than *a* by a difference greater than the criterion's veto threhold.

notation:

$\Delta_j(a,b)$ is the advantage of *a* over *b* on criterion g_j

k_j is the importance coefficient (weight) of g_j

(let us assume $k_1,\ldots,k_n \geq 0$ and $k_1+\ldots+k_n=1$)

$c(a,b)$ is the concordance index,

c is the concordance threshold

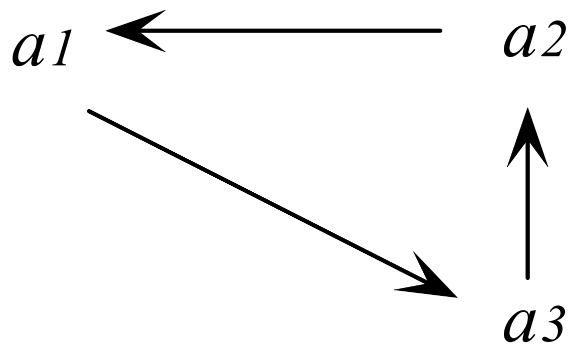
v_j is the veto threshold of g_j

ELECTRE I:
Exploiting S – Finding the kernel

- Purpose: to select a minimal set of candidates to become the most preferred action.
- Definition of kernel of a graph:
  - External stability: $\forall a_y \notin N, \exists a_x \in N: a_x \; S \; a_y$ (justification for excluding actions outside of the kernel)
  - Internal stability : $\forall a_y \in N, \exists a_x \in N: a_x \; S \; a_y$ (absence of justification to exclude any action from the kernel).

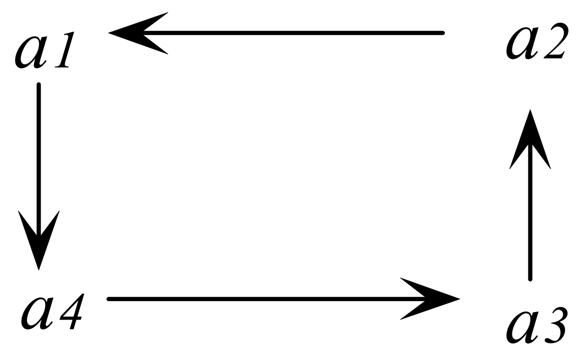
ELECTRE I: Exploiting S – Finding the kernel

- A kernel may not always exist, or it may not be unique.
- For instance for the S relation depicted on the right there is no kernel
  - *A kernel cannot consist of a single action (e.g., a_1) because that would leave one action outside the kernel (in this case a_2) that is not outranked by the action in the kernel, thus violating external stability*
  - *But a kernel cannot consist of more than on action (e.g., a_1 and a_2) because one outranks the other thus violating internal stability*



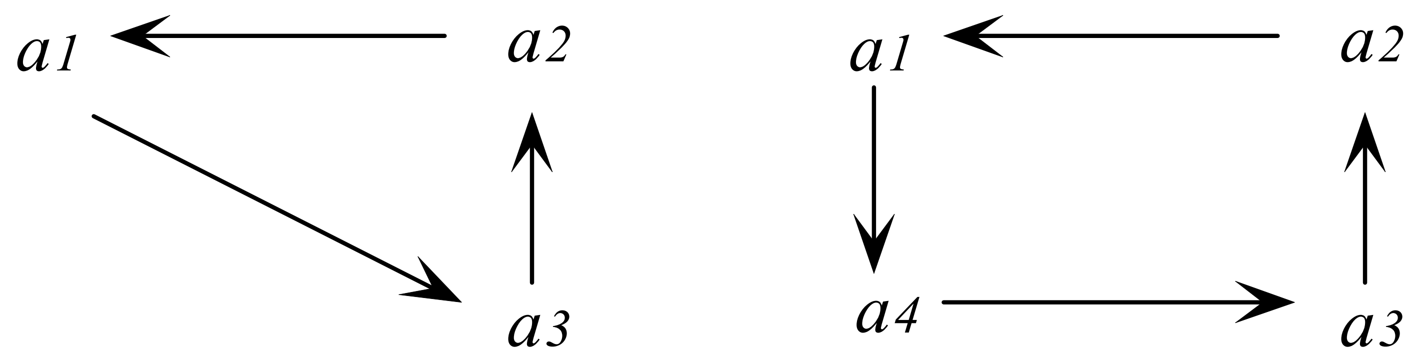
ELECTRE I:
Exploiting S – Finding the kernel

- As another example, for the S relation depicted on the right there more than one kernel
  - *A kernel can consist of a_1 and a_3 because there is no outranking between them (the arrows are not transitive) and the alternatives left out of the kernel are outranked by actions in the kernel (a_1 justifies exclusion of a_4, while a_3 justifies exclusion of a_2)*
  - *But if we choose a kernel that consists of a_2 and a_4 then it verifies internal and external stability equally well: no outrankings between a_2 and a_4, a_2 justifies exclusion of a_1, while a_4 justifies exclusion of a_3)*



ELECTRE I:
Exploiting S – Finding the kernel

- In the previous two examples the outranking graph contains a cycle:



- The existence of a unique kernel is guaranteed if the graph is acyclic.
- A solution: to consider actions in a circuit as being indifferent, treating them as a single class inheriting incoming and outgoung arcs.

ELECTRE I:
Exploiting S – Finding the kernel

Algorithm to find a kernel K

Intial step: reduce all cycles to indifference classes

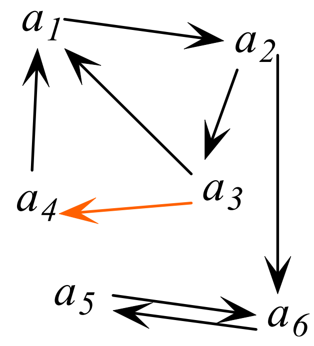
- Place in K all actions that are not outranked
- Remove all actions outranked by actions placed in K
- If there are actions outside of K, reiterate from 1, otherwise, stop.

ELECTRE I: Remarks

- Advantages (vs. additive MAVT model)
  - No strong axioms and conditions to verify
  - Works with any type of scales, including qualitative scales
  - Importance coefficients k_j truly reflect the "criteria" weights ("voting power" analogy) independently of the scales.
  - Alerts for incomparabilities (actions that are too different)

ELECTRE I: Remarks

- Disadvantages (vs. additive MAVT model)
  - Specific to the problem of selecting the best (one) action *(it does not allow to rank the actions)*
  - Exploitation difficulties *(multiple actions in the kernel)*
  - Lack of independence with regard to third actions *(for instance, if a_3 did not outrank a_4 then class $[a_5, a_6]$ would be in the kernel)*



  - Sudden transition from S to not S as data changes

Valued outranking relations

ELECTRE I works with crisp outranking relations: given a pair (a,b), the statement aSb is established to be true or false.

We will now learn a model that considers that an outranking can be partially true, computing a credibility degree for it. This model is used by ELECTRE III and ELECTRE TRI.

Credibility degree for *a S b*

Let Δ_j denote the advantage of an action *a* over another action *b* according to a criterion $g_j(.)$:

$$\Delta_j = \begin{cases} g_j(a) - g_j(b), & \text{if criterion } g_j \text{ is to be maximized} \\ g_j(a) - g_j(b), & \text{if criterion } g_j \text{ is to be minimized} \end{cases}$$

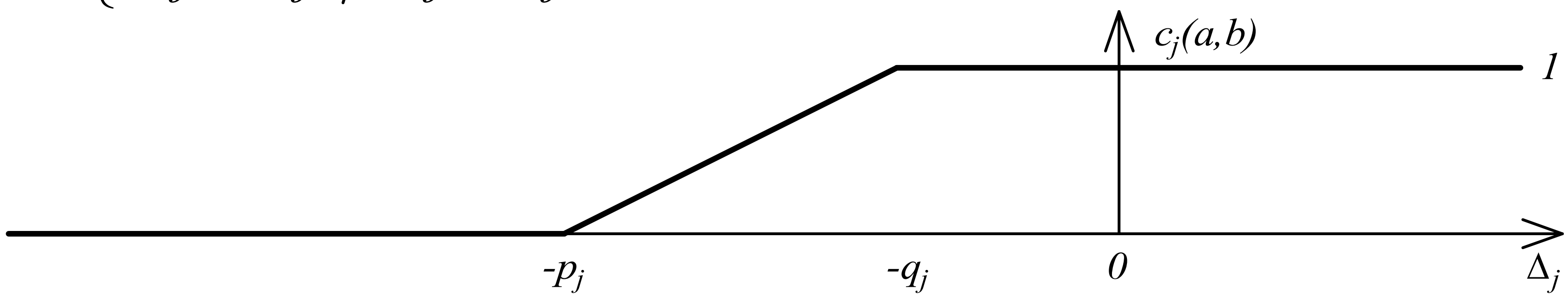
- For ELECTRE I, if $\Delta_j \geq 0$ then criterion g_j is fully concordant with *aSb*. On the other hand, if $\Delta_j < 0$, even if the difference is almost zero, then $g_j(.)$ is not concordant with *aSb*
- For ELECTRE I, if $-\Delta_j \geq v_j$ then criterion g_j opposes a veto to *aSb*, even if the threshold is surpassed by a negligible amount
- We will now avoid these sudden transitions

Credibility degree for *a S b*

Concordance

- Definition of a single-criterion concordance index
  how far does the criterion agree with aSb?

$$c_j(a,b) = \begin{cases} 1, & \text{if } \Delta_j \geq -q_j \\ 0, & \text{if } \Delta_j < -p_j \\ (p_j + \Delta_j)/(p_j - q_j), & \text{otherwise} \end{cases}$$



with

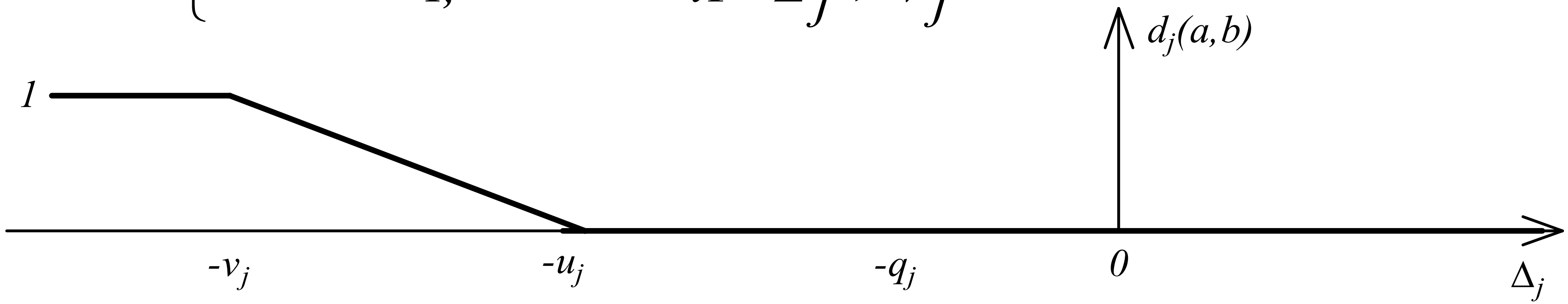
q_j = indifference threshold (disadvantage tolerated)

p_j = preference threshold (disadvantage that corresponds to absence of concordance)

Credibility degree for *a S b*
Discordance

- Definition of a single-criterion discordance index
  how much does the criterion oppose a veto to aSb?

$$d_j(a,b) = \begin{cases} 0, & \text{if } -\Delta_j \le u_j \\ \left(-\Delta_j - u_j\right)/\left(v_j - u_j\right), & \text{if } u_j < -\Delta_j \le v_j \\ 1, & \text{if } -\Delta_j > v_j \end{cases}$$



with

u_j = non-discordance threshold (disadvantage for which a partial veto begins). Originally $u_j = p_j$

v_j = veto threshold (disadvantage originating a total veto).

Credibility degree for *a S b*

Aggregation of concordance and discordance

- The global concordance index given weights k_j (still assuming the sum of the weights is 1) is a weighted sum

$$c(a,b) = \sum_{j=1}^{n} c_j(a,b).k_j$$

- The global discordance index is the maximum discordance

$$d^{max}(a,b) = \max_{j \in \{1,...,n\}} d_j(a,b)$$

Credibility degree for *a S b*
Aggregation of concordance and discordance

- Credibility index for *a S b*

$$\sigma(a,b) = \min\{c(a,b),\, 1 - d^{max}(a,b)\}$$

Alternatively, original variant (Yu, 1992), with $p_j = u_j$

$$\sigma(a,b) = c(a,b) \prod_{\substack{j \in \{1,\ldots,n\}: \\ d_j(a,b) > c(a,b)}} \frac{1 - d_j(a,b)}{1 - c(a,b)}$$

Yu, W., ELECTRE TRI. Aspects méthodologiques et guide d'utilisation, Document du LAMSADE, No. 74, Université Paris-Dauphine. 1992

Another variant (Mousseau and Dias, 2004 and IRIS software)

$$\sigma(a,b) = c(a,b) \prod_{j \in \{1,\ldots,n\}} \left(1 - d_j(a,b)\right)$$

Mousseau V., L. Dias, Valued Outranking Relations in ELECTRE Providing Manageable Disaggregation Procedures, European Journal of Operational Research, Vol. 156, No. 2, pp. 467-482, 2004

ELECTRE TRI for sorting problems

ELECTRE TRI is a method to sort actions into predefined categories, based on valued outranking relations. It has been successfully applied in finance, agriculture, priority setting, territorial zoning, and other areas.

ELECTRE TRI

- Method exclusively for ordinal sorting
- Evaluates actions in absolute terms, not a contest
  The actions are not compared against each other. They are compared to predefined standards
- Places each action into a category chosen from an <u>ordered</u> set
  - C^1: worst category
  - ...
  - C^k: best category

ELECTRE TRI – examples of categories

Action to sort
(a company)

Market position
Managers' curriculum
Expenses / Sales
Total debt / Total assets
Earnings / Capital
Etc.

ELECTRE TRI

<u>Risk category</u>

5) Very low (best)
4) Low
3) Medium
2) High
1) Very High

- Category definitions (subjective)
- Relative importance of each criterion (subjective)
- Veto possibility (subjective)

ELECTRE TRI – examples of categories

Action to sort
(job candidate)

Academic record
Previous experience
Language skills
Etc.

ELECTRE TRI

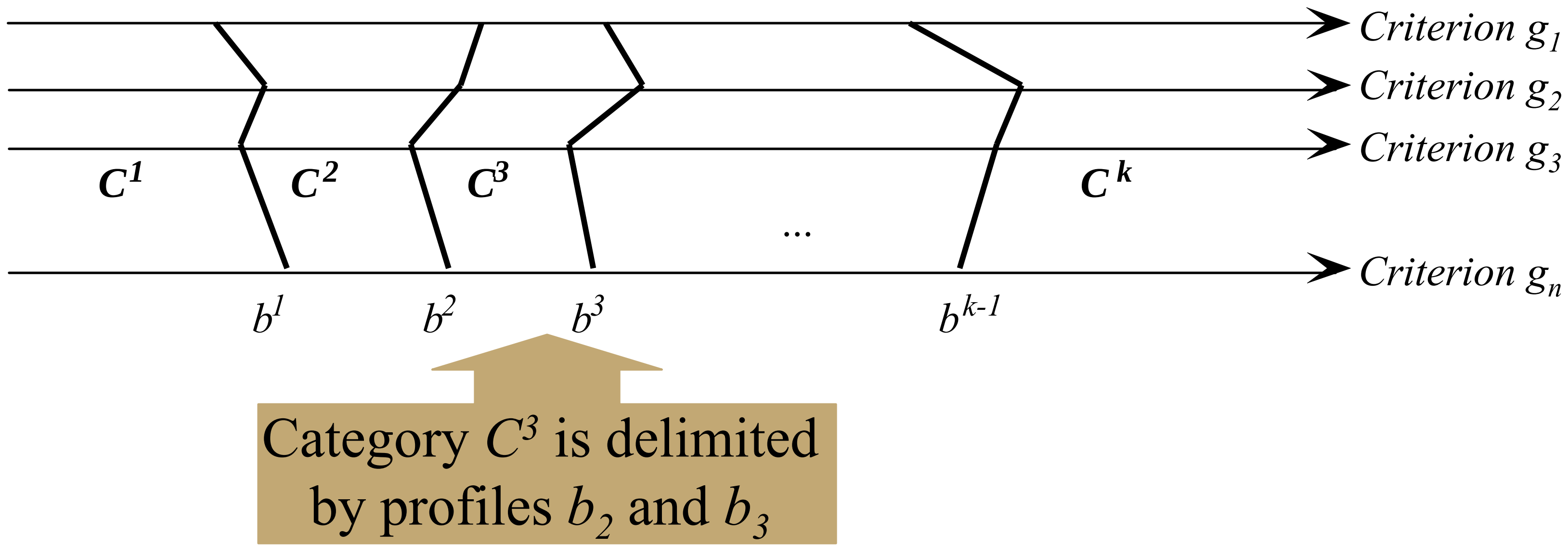
<u>Decision category</u>

3) Invite for interview
2) Keep in database
1) Ignore

- Category definitions (subjective)
- Relative importance of each criterion (subjective)
- Veto possibility (subjective)

ELECTRE TRI
Defining categories

- Categories 1, 2, …, k are delimited by profiles (reference actions) b^1, …, b^{k-1} such that
  - b^h dominates b^{h-1} ($h = 2, \dots, k-1$);
  - Each action to be sorted can be indifferent to at most one profile.



ELECTRE TRI
Pessimistic assignment rule

- Each action is compared to the profiles $b^1, \ldots, b^{k-1}$
  - a_i is assigned to category C^1 if and only if it does not outrank b^{h-1}

$$a_i \rightarrow C^1 \Leftrightarrow \sim a_i \text{ S } b^1$$

  - a_i is assigned to category C^k if and only if outranks b^{k-1}

$$a_i \rightarrow C^k \Leftrightarrow a_i \text{ S } b^{k-1}$$

  - otherwise a_i is assigned to a category C^h such that a_i outranks the category's lower bound, and it does not outrank the category's upper bound

$$a_i \rightarrow C^h \Leftrightarrow a_i \text{ S } b^{h-1} \wedge \sim a_i \text{ S } b^h$$

ELECTRE TRI
Pessimistic assignment rule

- A possible algorithm
  - $h \leftarrow 1$
  - WHILE a_i S b^h and $h<k$ DO

    $h \leftarrow h+1$
  - END WHILE
  - a_i is assigned to category C^h

ELECTRE TRI
Optimistic assignment rule

- Each action is compared to the profiles $b^1, \ldots, b^{k-1}$
  - $a_i \rightarrow C^1$ if and only if b^1 is preferred to a_i

$$a_i \rightarrow C^1 \Leftrightarrow \sim a_i \text{ S } b^1 \wedge b^1 \text{ S } a_i$$

  - $a_i \rightarrow C^k$ if and only if b^{k-1} is not preferred to a_i

$$a_i \rightarrow C^k \Leftrightarrow a_i \text{ S } b^{k-1} \vee \sim b^{k-1} \text{ S } a_i$$

  - otherwise a_i is assigned to a category C^h such that b^h is preferred to a_i but b^{h-1} is not preferred to a_i

$$a_i \rightarrow C^h \Leftrightarrow (\sim a_i \text{ S } b^h \wedge b^h \text{ S } a_i) \wedge (a_i \text{ S } b^{h-1} \vee \sim b^{h-1} \text{ S } a_i)$$

ELECTRE TRI
Optimistic assignment rule

- A possible algorithm
  - $h \leftarrow 1$
  - WHILE ~b^h S a_i and $h<k$ DO

    $h \leftarrow h+1$
  - END WHILE
  - a_i is assigned to category C^h

ELECTRE TRI

Assignment rules

- The category resulting from the optimistic assignment (let us call it C^{opt}) is at least as good as the category resulting from the pessimistic assignment (lets call it C^{pes})
- If C^{opt} does not coincide with C^{pes}, this means that the action being sorted is incomparable with the intermediate profiles:

$$a_i \, R \, b^{pes}, \ldots, a_i \, R \, b^{opt\text{-}1}$$

Recall that $a_i \, R \, b$ *means that* $a_i \, S \, b$ *but* $\sim b \, S \, a_i$

ELECTRE TRI

Parameters required

- It is necessary to define the category profiles $b^1, \ldots, b^{k-1}$
- The valued outranking relation use introduced in the last section will be used: for each pair (a_i,b), with a_i being an action to sort and b^h being a category profile we can compute the credibility index for $a_i\, S\, b^h$, denoted $\sigma(a_i,b^h)$
- This computation is based on the parameters q_j (indifference threshold), p_j (preference threshold), u_j (non-discordance threshold), v_j (veto threshold), and k_j (importance coefficient; weight), for j=1,…,n
- An additional parameter λ (cutting level) is required to define the required credibility:

$$a_i\, S\, b^h \iff \sigma(a_i,b^h) \geq \lambda$$

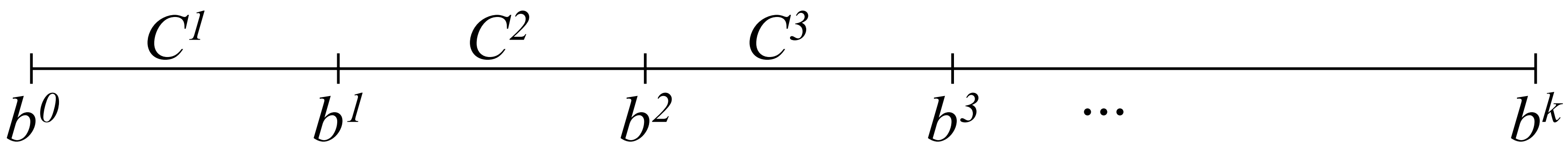
The MAVT alternative

- Given a multiattribute utility / value model

$$v(a_i) = f\left(v_1(a_i), v_2(a_i), \ldots, v_n(a_i)\right)$$

  it is possible to:
  - Define category bounds $b^0 < b^1 < \ldots < b^k$
  - Implement a simple sorting rule:

$$a_i \to C^h \iff v(a_i) \geq b^{h-1} \wedge v(a_i) < b^h$$



ELECTRE TRI: Remarks

- Advantages (vs. additive MAVT model)
  - No strong axioms and conditions to verify
  - Works with any type of scales, including qualitative
  - Importance coefficients k_j truly reflect the “criteria” weights (“voting power” analogy) independently of the scales.
  - Allows putting a penalty on a very weak performances on some criterion

ELECTRE TRI: Remarks

- Disadvantages (vs. additive MAVT model)
  - Large number of parameters
  - Relatively complex computations

Robustness analysis for ELECTRE TRI

As in MAVT (recall chapter 2) robustness analysis assesses which conclusions are robust given some constraints on the parameter values. Here we consider constraints on weights.

Robustness analysis

- Robustness analysis consists in defining a set of constraints that the parameter values should satisfy and then study which conclusions are robust, that is, which conclusions are valid for all vectors of parameters that satisfy the constraints.
- For instance, we may ask the client to rank the criteria weights without asking them to provide a precise figures…
- …and then make computations to learn robust intervals for the assignment
- A software such as IRIS can be used (developed at the University of Coimbra in cooperation with Univ. Paris Dauphine).
- It is also available as a plugin to the Decision Desktop platform

Further reading

The main references for this chapter are:

- M.G. Rogers, M. Bruen & L-Y. Maystre, *ELECTRE and Decision Support: Methods and Applications in Engineering and Infrastructure Investment*. Kluwer Academic Publishers, Boston, 1999.
  - Chapter 3 (pp. 45-86)
- Valerie Belton & Theodore Stewart, *Multiple Criteria Decision Analysis: An Integrated Approach*. Kluwer Academic Publishers, Boston, 2002.
  - Chapter 8 (pp. 233-260)