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A Numerical Solution of Falkner-Skan Equation via a Shifted Chebyshev Collocation Method

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Abstract. The Falkner-Skan equation is a nonlinear third-order boundary value problem defined on the semi-infinite interval $[0,\infty)$. This equation plays an important role to illustrate the main physical features of boundary layer phenomena. This paper presents a new collocation method for solving the Falkner-Skan equation. The proposed approach is equipped by the orthogonal Chebyshev polynomials that have perfect properties to achieve this goal. The shifted Chebyshev-Gauss points are utilized as collocation points. In addition, this method reduces solution of the problem to solution of a system of algebraic equations. Comparisons with other methods show that the proposed method is highly accurate and its convergence is very rapid.

Keywords: Falkner-Skan equation, collocation method, shifted Chebyshev-Gauss points, semi-infinite domain
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INTRODUCTION

The *Falkner-Skan equation* is

$$\frac{d^3 f}{dx^3} + f\frac{d^2 f}{dx^2} + \beta\left(1 - \left(\frac{df}{dx}\right)^2\right) = 0, \tag{1}$$

where $\beta \in \mathbb{R}$. For $\beta = 0$ the equation (1) is called *Blasius equation* that was considered for a first time by Blasius [7].

In generally, equation (1) has the following boundary conditions:

$$f(0) = 0, \tag{2}$$

$$f'(0) = 0, \tag{3}$$

$$\lim_{x\to\infty} f'(x) = 1, \tag{4}$$

here f is proportional to the stream function. The Falkner-Skan equation has provided many fruitful sources of information about the behavior of incompressible boundary layers.

The first analytical discussion for Falkner-Skan equation was given by Hartree [11], who found that in the interval $\beta \in (-0.19884, 0)$, there exists a family of unique solutions whose first order derivative $f'(x)$ tends to 1 exponentially. Then Weyl [17] for $\beta > 0$ proved that it has a solution such that $f'(x)$ increases with x and $f''(x)$ tends decreasingly to zero as $x \to \infty$. For $\beta > 0$ that is accelerating flows the velocity profiles have no points of inflection, whereas for $\beta < 0$ that is decelerated flows. For $-0.19884 < \beta \leqslant 2$ there exists the physically solution, see [15],[16].

Three schemes was presented by Cebeci *et al* [8] to solve (1). First, they used ordinary shooting to solve for the case of positive shear, then a nonlinear eigenvalue technique was introduced to solve the inverse problem and finally a parallel shooting method was employed to reduce sensitivity of the convergence. Using a coordinate transformation used by Asaithambi [6] to convert the physical problem on a semi infinite physical domain to a problem on a fixed computational domain. Then Asaithambi [2] presented a finite difference method to solve (1). Recently, Pade [14] proved a new existence and uniqueness solution of the equations governing the self-similar compressible boundary layer (1). Further, Transforming the semi-infinite domain without truncating (1) to a finite domain and without imposing the asymptotic condition was provided by Asaithambi [3] and also in [4], he presented a finite-element method to solution by using a coordinate transformation to map the semi-infinite domain of the problem to the unit interval $[0,1]$. In [5] Asaithambi presented a computational method for solving of the third-order boundary value problem characterized by (1) on a semi-infinite domain. Yang and Lan [18] obtained a new compactness results on the velocity functions and shear functions of (1). Adomian decomposition to solve (1) was used by Elgazery [9] and

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then this method was employed by Alizadeh *et al* [1]. Liu and Chang [13] developed a new numerical technique, transforming the governing equation into a nonlinear second order boundary value problem by a new transformation technique and solved it by Lie-group shooting method. The iterative method to solve (1) was used by Zhang and Chen [21] for the first time. Series was used by Yao and Chen [20] for solving the Falkner-Skan equation and Zhu *et al* [22] used the quasilinearization technique to solve (1). Using Fourier series and heat transfer was an efficient method to calculate the series of function solution by Rosales-Vera and Valencia [12]. Yang and Lan [19] worked on nonexistence of the reversed flow solution of (1). Recently, Fazio [10] used Töpfer algorithm to solve Blasius problem.

SOME FEEDBACKS TO CHEBYSHEV POLYNOMIALS

The Chebyshev polynomials of the first kind, $T_k(x)$, $k = 0, 1, \dots$, are the eigenfunctions of the singular Sturm-Liouville problem

$$\left(\sqrt{1-x^2}T_k'(x)\right)' + \frac{k^2}{\sqrt{1-x^2}}T_k(x) = 0. \tag{5}$$

They are orthogonal with respect to L_w^2 inner product on $[-1,1]$ with the weight function $w(x) = \frac{1}{\sqrt{1-x^2}}$, that means:

$$\int_{-1}^{1} w(x)T_i(x)T_j(x)dx = \frac{\pi}{2}c_{ij} \ ,$$

where

$$c_{ij} = \begin{cases} 2, & i = j = 0 \\ 1, & i = j \geq 1 \\ 0, & i \neq j. \end{cases}$$

Also they can get the recursion relation

$$T_{k+1}(x) = 2xT_k(x) - T_{k-1}(x) \ ,$$

where $T_0(x) = 1$ and $T_1(x) = x$. If they are normalized then $T_k(1) = 1$ and $T_k(x)$ can be written in terms of powers of x:

$$T_k(x) = \frac{k}{2}\sum_{i=0}^{[k/2]}(-1)^k\frac{(k-i-1)!}{i!(k-2i)!}(2x)^{k-2i} \ ,$$

where $[k/2]$ is the integer part of $k/2$.

The Chebyshev-Gauss (CG) collocation points, $z_i = \cos\frac{(2i-1)\pi}{2N+2}, \quad i = 1, 2, \dots, N+1$, are the roots of $T_{N+1}(x)$. These points have the property that

$$\int_{-1}^{1} w(z)p(z)dz = \sum_{i=1}^{N+1} w_i p(z_i),$$

where the CG quadrature weights are

$$w_i = \frac{\pi}{N+1}, \ \ i = 1, 2, \dots, N+1 \ ,$$

is exact for polynomials of degree at most $2N+1$.

The shifted Chebyshev polynomials (ShCP) on the interval $[0, L]$ are defined by:

$$\hat{T}_k(x) = T_k\left(\frac{2}{L}x - 1\right),$$

which are obtained by affine transformation from the interval $[-1,1]$. These polynomials have a set that is a complete $L_{\hat{w}}^2[0,L]$ orthogonal system with the weight function $\hat{w}(x) = \frac{1}{\sqrt{Lx-x^2}}$. Hence, any function $f \in L_{\hat{w}}^2[0,L]$ can be expanded by shifted Chebyshev polynomials as

$$f(x) = \sum_{i=0}^{\infty} f_i\hat{T}_i(x),$$

where $f_i = \frac{<f(x),\hat{T}_i(x)>_{\hat{w}}}{<\hat{T}_i(x),\hat{T}_i(x)>_{\hat{w}}}$, and $< \cdot,\cdot >_{\hat{w}}$ denotes the inner product. Further, the integral of a ShCP can be represented as a linear combination of ShCPs as follows:

$$\int_0^x \hat{T}_k(z)dz = \begin{cases} \frac{L}{2}\left(\hat{T}_0(x)+\hat{T}_1(x)\right), & k=0, \\ \frac{L}{2}\left(-\frac{1}{4}\hat{T}_0(x)+\frac{1}{4}\hat{T}_2(x)\right), & k=1, \\ \frac{L}{2}\left(\frac{(-1)^{k-1}}{k^2-1}\hat{T}_0(x)-\frac{1}{2k-2}\hat{T}_{k-1}(x)+\frac{1}{2k+2}\hat{T}_{k+1}(x)\right), & k=2,\ldots,N. \end{cases} \tag{6}$$

The shifted Chebyshev-Gauss (ShCG) collocation points, $0 < x_1 < x_2 < \cdots < x_{N+1} < L$ on the interval $[0,L]$ are obtained by shifting the CG points z_i, using the transformation

$$x_i = \frac{L}{2}(z_i+1), \quad i=1,2,\ldots,N+1. \tag{7}$$

FUNCTION APPROXIMATION

An important part in solving the Falkner-Skan equation in Eq. (1) is to approximate the value of $f''(0)$ [11]. To this end and in order to obtain stable numerical approximations, we first approximate $f''(x)$ using a series of ShCPs with unknown coefficients as follows:

$$f''(x) \simeq \sum_{i=0}^{N} a_i\hat{T}_i(x)\ , \quad 0 \leqslant x \leqslant L\ . \tag{8}$$

Now, by integrating of Eq. (8) and using Eq. (3) we have

$$f'(x) = \int_0^x f''(z)dz + f'(0) \simeq \int_0^x \sum_{i=0}^{N} a_i\hat{T}_i(z)dz = \sum_{i=0}^{N} a_i \int_0^x \hat{T}_i(z)dz. \tag{9}$$

Using the property of integral of the Chebyshev polynomials given in Eq. (6), we can simplify Eq. (9) as

$$f'(x) \simeq \sum_{i=0}^{N+1} b_i\hat{T}_i(x), \tag{10}$$

where

$$b_i = \begin{cases} \frac{L}{2}\left(a_0 - \frac{1}{4}a_1 + \sum_{j=2}^{N-1}\frac{(-1)^j}{j^2-1}a_j\right), & i=0 \\ \frac{L}{2}\left(a_0-\frac{1}{2}a_2\right), & i=1 \\ \frac{L}{2}\left(\frac{1}{2i}a_{i-1}-\frac{1}{2i}a_{i+1}\right), & i=2,3,\ldots,N-1, \\ \frac{L}{2}\left(\frac{a_{i-1}}{2i}\right), & i=N,N+1. \end{cases}$$

Similarly, the approximation of $f(x)$ is obtained as follow:

$$f(x) \simeq \sum_{i=0}^{N+2} c_i\hat{T}_i(x), \tag{11}$$

where

$$c_i = \begin{cases} \frac{L}{2}\left(b_0 - \frac{1}{4}b_1 + \sum_{j=2}^{N}\frac{(-1)^j}{j^2-1}b_j\right), & i=0 \\ \frac{L}{2}\left(b_0 - \frac{1}{2}b_2\right), & i=1 \\ \frac{L}{2}\left(\frac{1}{2i}b_{i-1} - \frac{1}{2i}b_{i+1}\right), & i=2,3,\ldots,N, \\ \frac{L}{2}\left(\frac{b_{i-1}}{2i}\right), & i=N+1,N+2. \end{cases}$$

It is observed that the coefficients b_i and c_i are obtained as linear combinations of the unknown coefficients a_i. Next, to approximate $f'''(x)$ we have by direct differentiation and using Eq. (8) that

$$f'''(x) = \frac{d}{dx}f''(x) \simeq \sum_{i=1}^{N} a_i \hat{T}'_i(x). \tag{12}$$

COLLOCATION METHOD FOR THE FALKNER-SKAN EQUATION

Our scheme for solving the Falkner-Skan equation is now described. Substituting Eqs. (8)-(12) into Eq. (1) yields

$$\sum_{i=1}^{N} a_i\hat{T}_i'(x) + \sum_{i=0}^{N+2} c_i\hat{T}_i(x)\sum_{i=0}^{N} a_i\hat{T}_i(x) + \beta\left[1 - \left(\sum_{i=0}^{N+1} b_i\hat{T}_i(x)\right)^2\right] = 0. \tag{13}$$

In addition, since our method is based on the domain truncation, we replace the condition at infinity given in Eq. (4) by the condition at the point $x = L$ which implies

$$f'(L) = 1 \Longrightarrow \sum_{i=0}^{N+1} b_i\hat{T}_i(L) = 1. \tag{14}$$

Collocating Eq. (13) at the ShCG points x_j for $j = 1,\ldots,N$, we have

$$\sum_{i=1}^{N} a_i\hat{T}_i'(x_j) + \sum_{i=0}^{N+2} c_i\hat{T}_i(x_j)\sum_{i=0}^{N} a_i\hat{T}_i(x_j) + \beta\left[1 - \left(\sum_{i=0}^{N+1} b_i\hat{T}_i(x_j)\right)^2\right] = 0. \tag{15}$$

Eq. (15) together with Eq. (14) provide a system of $N+1$ algebraic equation for the unknowns a_i, $i = 0,1,\ldots,N$, which can be solved using the Newton's iterative method. Again, it is noted that the initial conditions (2) and (3) are automatically taken into account when $f'(x)$ and $f(x)$ are approximated using Eqs. (10)-(11).

NUMERICAL RESULTS

The Falkner-Skan equation was solved using the proposed Chebyshev collocation method for different values of β, different numbers of term N considered in the series and for $L = 7$ and 8. Table (1) compares the initial slop $f''(0)$ obtained using the presented method with the values obtained in [12] by using Fourier series.

It is seen for each value of β that our method delivers the same order of accuracy with very smaller value of N.

In order to verify the accuracy and rapid convergence of our present method, in Tables (2) and (3), we have listed the values of $f''(0)$ for different β and N.

As can be seen, the convergence of the proposed method is very fast. Figure 1 shows the variation of the function $f'(x)$ with the similarity variable x for different values of the parameter β obtained with the Chebyshev collocation method for $N = 40$.

TABLE 1. Comparison of initial slopes $f''(0)$ for different values of L, N and β

	$L=7$			$L=8$		
β	[12] with $N=300$	N	$P.M.$	[12] with $N=400$	N	$P.M.$
2	1.687217	15	1.687218	1.687218	16	1.687218
0.5	0.927680	16	0.927680	0.927680	15	0.927680
−0.1	0.319270	21	0.319270	0.319270	22	0.319270
−0.15	0.216361	23	0.216362	0.216361	26	0.216361
−0.1988	0.006154	25	0.005357	0.005218	31	0.005219

TABLE 2. Values of $f''(0)$ for $L=7$ and different N and β

N	$\beta=2$	$\beta=0.5$
20	1.687218162	0.92768004
40	1.68721816920686539	0.927680039841421670
60	1.68721816920686538	0.927680039841421674

TABLE 3. Values of $f''(0)$ for $L=8$ and different N and β

N	$\beta=-0.15$	$\beta=-0.1988$
20	0.21635	0.007
40	0.216361405955	0.005218950
60	0.216361405956	0.005218955

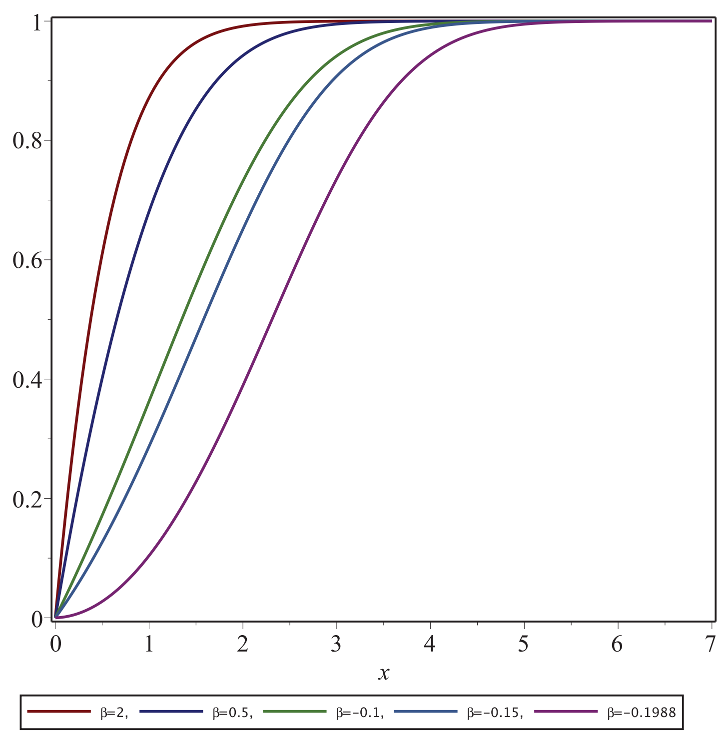

FIGURE 1. Graphs of $f'(x)$ for $N=40$ and different values of β

CONCLUSION

Using orthogonal Chebyshev polynomials a new collocation method has been proposed to solve the Falkner-Skan equation over an semi-infinite interval. This method possesses high accuracy and rapid convergence.

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