SPECTRAL METHODS ON THE SEMI-INFINITE LINE

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Abstract

Scientific computing has an important role in applied mathematics. Many problems that occur in physics and engineering can be modeled by linear or nonlinear differential equations. The main topic of this thesis is the solution of Blasius and Lane-Emden type equations which are nonlinear ordinary differential equations on a semi-infinite interval. The Blasius equation is a third-order nonlinear ordinary differential equation. The Lane-Emden type equations have been considered by many mathematicians. An orthogonal Laguerre basis is proposed to provide an effective and simple way to improve the solution by spectral methods. Through comparisons among the exact solutions of Horedt [54] and the series solutions of Wazwaz [84], Liao [61], and Ramos [76], and the current work, it is shown that the present work provides an effective approach for Lane-Emden type equations; also it is confirmed by the numerical results that this approach has exponentially convergence rate. In the Blasius equation, the second derivative at zero is an important point of the function, so we have computed It and compared the result with other well-known methods and show that the present solution is accurate.

Keywords: Nonlinear differential equations, Spectral methods, Semi-infinite intervals, Rational Legendre functions, Lane-Emden equation, Scaled Laguerre functions, Collocation method.

To my parents, my sister and my brother

"Human beings are members of a whole,
In creation of one essence and soul.
If one member is afflicted with pain,
Other members uneasy will remain.
If you have no sympathy for human pain,
The name of human you cannot retain."

— *Shahnameh*, HAKIM ABUL-QASIM FERDOWSI TUSI, *between c. 977 and 1010 AD*

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Preface

In this thesis Lagrangian, collocation and Tau methods are proposed for solving the singular Lane-Emden and Blasius equations which are nonlinear ordinary differential equations on a semi-infinite interval. Collocation, Galerkin, and Tau methods are applied for solving the Lane-Emden problem, and according to the results, the solution of the Tau method is the most accurate. The derivative and product-matrices of the modified generalized Laguerre functions are presented. These matrices, in conjunction with the Tau method, are then utilized to reduce the solution of the Lane-Emden equation to that of a system of algebraic equations. We also present a comparison of this work with some well-known results and show that the present solution is highly accurate.

The first and second chapters serve mainly to introduce different numerical methods for solving differential equations. Weighted Residual methods are introduced in the first chapter. The second chapter considers different spectral methods. The third chapter is devoted to the introduction of basis functions and polynomials and, especially Laguerre functions. Additional results on spectral methods are also presented in this chapter. The fourth chapter of the thesis discusses in detail the different spectral methods that are used to solve problems in semi-infinite domains. In the fifth chapter we introduce and derive the Lane-Emden equation. In this chapter previous methods for solving this problem are documented. The sixth chapter considers the Blasius differential equation. In this chapter we use collocation method to solve the Blasius equation. Finally, we implement the Lagrangian, collocation and Tau methods for solving the Lane-embden equation in the following chapters.

Chapter 1

Discretizations of differential equations

It is not possible to find the exact solution of the majority of differential equations. Even if we can find an exact analytical solution, it may not be possible to compute the solution. However, we can use numerical methods to approximate the solution.
Throughout this thesis, we will consider a real function $u(x)$ on (a, b). This function can be expanded with respect to a set of basis functions φ_i:

$$u = \sum_{j=0}^{\infty} a_i \varphi_i.$$

If we use an orthogonal basis the coefficients are:

$$a_i = \frac{< u, \varphi_i >}{< \varphi_i, \varphi_i >}$$

Let us assume that we have a set of basis functions $\{\varphi_1(x), \varphi_2(x), \ldots\}$. We say the basis functions are orthogonal if:

$$\int \varphi_i(x)\varphi_j(x) = \begin{cases} 0, & i \neq j \\ \gamma_j, & i = j; \end{cases}$$

if $\gamma_j = 1$ for all j, then the basis is called orthonormal.
This set of basis functions is defined in an infinite dimensional Hilbert space namely, $L^2(a, b)$ with the inner product

$$< f, g >= \int_a^b f(x)g(x)w(x)dx. \tag{1.1}$$

Here $w(x)$ is a nonnegative weight function. We can classify numerical methods according to the basis functions of such an expansion:

- Finite Difference: overlapping local polynomials of low order (i.e., truncated Taylor series).
- Finite Element: local polynomials of fixed degree.
- Spectral: global smooth functions (e.g., truncated Fourier series).

Over the last three decades spectral methods have been used for solving different differential equations describing a multitude of physical systems. One of the benefits of spectral methods is high accuracy in approximating smooth functions. The error tends to zero faster than any fixed power of N, and the order of approximation is restricted only by the global smoothness of the approximated function. Such behavior is known as spectral accuracy.

1.1 Weighted residual method

The weighted residual method (WRM) is a general method for solving differential equations. This approach is credited to Crandall [27]. This particular method was improved and implemented by Finlayson [36, 35].

We want to solve the differential equation (written in operator form)

$$L(u) = 0, \tag{1.2}$$

with the initial and boundary condition formulated as $I(u) = 0$ and $S(u) = 0$, respectively. The solution $u(x)$ is approximated as a linear combination of some basis functions $\varphi_j(x)$. That is,

$$u(x) \cong u_N(x) = \sum_{j=0}^{N} a_j \varphi_j(x), \tag{1.3}$$

a_j are called expansion coefficients and $u_N(x)$ is the approximated solution. In (WRM) we insert the approximate solution (1.3) into the differential equation (1.2) to obtain the residual function as below:

$$Res(x; a_0, \ldots, a_N) = L(u_N), \tag{1.4}$$

for initial and boundary conditions we get:

$$Res_I(x; a_0, \ldots, a_N) = I(u_N), \quad Res_b(x; a_0, \ldots, a_N) = S(u_N) \tag{1.5}$$

The purpose of WRM's is to find the expansion coefficients a_j so that the residual functions Res,Res_I and Res_b become minimized. In WRM's methods there is a set of functions $(\chi_0(x), \chi_1(x), \ldots, \chi_N(x))$ which is chosen to be orthogonal to the residual function. These functions are called test functions.

$$\int_X Res(x)\chi_i(x)dx = 0, \quad i = 1, 2, ..., N.$$

and

$$\int_X Res_I(x)\chi_i(x)dx = 0, \quad i = 1, 2, ..., N.$$

$$\int_X Res_b(x)\chi_i(x)dx = 0, \quad i = 1, 2, ..., N.$$

The result is a set of equations for the unknown constants a_i.

Chapter 2

Spectral methods

Choosing different test functions will result in different WRM methods. Some approaches are listed below:

1. Sub-domain method
2. Least Squares method
3. Collocation method
4. Galerkin method
5. Tau method

Collocation, Galerkin and Tau methods are called spectral methods.

2.0.1 Sub-domain method

In this approach the domain is divided into N sub-domains called D_i. Then test functions are defined as :

$$\Psi_j(x) = \begin{cases} 1, & x \in D_j \\ 0, & Else \end{cases} \tag{2.1}$$

Choosing those test functions, the equations for finding the coefficients a_j are:

$$\begin{aligned}
&\langle Res(x), \Psi_j(x)\rangle = \int_D Res(x)\Psi_j(x)dx, \\
&= \Sigma \int_{D_j} Res(x)\Psi_j(x)dx, \\
&= \int_{D_j} Res(x)\Psi_j(x)dx, \quad k = 0, 1, \ldots, N
\end{aligned} \tag{2.2}$$

choosing bigger N values will result in smaller sub-domains. This approach is similar to the finite volume method which is a well-known technique in fluid dynamics. The sub-domain method was introduced by Biezeno [12]. Papers [11, 73, 13] used this method to solve differential equations.

2.0.2 Least squares method

In this method rather than minimizing the residual function, the coefficients a_j will be chosen such that the following functional is minimized.

$$S = \int_D Res(x)Res(x)dx = \langle Res(x), Res(x)\rangle.$$

In order to minimize this function, we will set the derivatives of S with respect to the coefficients a_j to zero:

$$\frac{\partial S}{\partial a_j} = 2\int_D Res\frac{\partial Res}{\partial a_j}dx = 0.$$

Therefore the test functions are

$$\Psi_j = \frac{\partial Res}{\partial a_j}.$$

This method involves significant computational effort.

2.1 Spectral methods

Spectral methods are a subcategory of the weighted residual method. In spectral methods a function $u(x)$ is either an infinite series of basis functions, or it is approximated by a finite series of basis function. The choice of basis functions in spectral methods distinguishes them from other numerical approaches such as finite element and finite difference. Spectral methods use basis functions which are smooth and nonzero over the whole domain; however, in the finite element method, basis functions are nonzero only in subdomains. That is to say, spectral methods are global in nature, finite elements and finite differences are local.
Spectral methods can be divided into two sub-categories:

1. Collocation
2. Non-Interpolating

2.1.1 Collocation method

In collocation methods a set of points are chosen which are called interpolation or collocation points. There are natural methods for picking collocation points. Gaussian quadrature points can be used as collocation points; moreover, roots of basis functions can be used as collocation points. In collocation methods the coefficients of a_j interpolation series can be found by setting the residual function zero in the collocation points. In this method, the test or weight functions are delta dirac functions.

$$\Psi_j(x) = \delta(x - x_j),$$

where x_j are collocation points. Since

$$\langle u, \delta(x - x_j)\rangle = u(x_j),$$

we have:

$$Res(x_k; a_0, \ldots, a_N) = 0, \quad k = 0, 1, \ldots, N, \tag{2.3}$$

this means that the differential equations should be satisfied exactly at the $N+1$ collocation points. Using more collocation points will make the value of the residual function closer to zero in more points. In theory, because of spectral accuracy of these methods, the residual function approaches zero in the whole domain. It means if N approaches ∞ then $I_N u(x)$ approaches to the function $u(x)$. This method was first proposed by Slater [52] and Barta [6] for solving differential equations, and the collocation method was first used by Frazer [39] et al in 1937 for solving differential equations. Lanczos [59] used Chebyshev points as collocation points.

The non-Interpolating methods include Tau and Galerkin methods. In these methods there are no specific meshpoints or collocation points. Coefficients ,a_j, of interpolation series of a function $u(x)$ are found using the inner product of $u(x)$ and basis functions.
Choosing orthogonal basis functions will make the computation faster. The Galerkin method is credited to Boris Galerkin; however, it was studied in more detail by Collatz[80], Finlayson [35] and papers [46, 31, 30, 29, 32, 37, 45].

In the non-Interpolating methods, after inserting the approximation function $u_N(x)$ into residual function, the following equation is generated:

$$Res(x; a_0, \ldots, a_n) = \sum_{j=0}^{\infty} r_j(a_0, \ldots, a_N)\varphi_j(x).$$

If φ_j are orthogonal functions then the coefficients r_j can be determined by :

$$r_j = \frac{\langle Res, \varphi_j \rangle}{\langle \varphi_j, \varphi_j \rangle}.$$

As more coefficients become zero the residual function becomes smaller and smaller. The coefficients a_j can be found by setting the first $N+1$ coefficients r_j zero.

$$r_j = 0, \quad j = 0, 1, \ldots, N.$$

This is equivalent to:

$$\langle Res, \varphi_j \rangle = 0, \quad j = 0, 1, \ldots, N. \tag{2.4}$$

In the Galerkin method basis functions are smooth functions, and they have derivatives of all orders and satisfy all boundary conditions. The Tau method is virtually the same as the Galerkin method ; however, its basis function do not have to satisfy the boundary conditions. The Tau method is often used for solving non-periodic boundary condition problems.

This method minimizes the residual function like the Galerkin method; however, it differs in that the boundary condition is also a constraint. For example, consider a differential equation with the condition $u(-1) = u(1) = 0$. Applying the Tau method leads to the following equations:

$$\langle Res, \varphi_j \rangle = 0, \quad j = 0, 1, \ldots, N,$$

and

$$\sum_{j=0}^{N} a_j \varphi_i(\pm 1) = 0.$$

In the Galerkin method we must choose basis functions to satisfy

$$\Phi_j(\pm 1) = 0. \quad j = 0, 1, \ldots, N.$$

Assume $\phi(x)$ is a vector of functions:

$$\phi(x) = [\phi_0(x), \phi_1(x), \ldots, \phi_{N-1}(x)]^T, \tag{2.5}$$

where $\phi_0(x), \phi_1(x), \ldots, \phi_{N-1}(x)$ are a finite set of basis functions. The derivative of the vector $\phi(x)$ can be expressed by

$$\frac{d\phi(x)}{dx} = \mathbf{D}\phi(x), \tag{2.6}$$

where $\mathbf{D}$ is an NXN matrix which is called the differentiation matrix. Parand, et al. [67] derived the differentiation matrix of rational Chebyshev functions and proved that, the general form of the matrix $\mathbf{D}$ is a lower-Hessenberg matrix. The rational Chebyshev functions, denoted by $R_n(x)$, are defined by [16]

$$R_n(x) = T_n(y) = \cos(nt), \tag{2.7}$$

where L is a constant parameter, $T_n(y)$ is the well-known Chebyshev polynomial and

$$y = \frac{x-L}{x+L}; \quad y \in [-1, 1], \qquad t = 2\cot^{-1}\left(\sqrt{\frac{x}{L}}\right); \quad t \in [0, \pi]. \tag{2.8}$$

These functions are orthogonal with respect to the weight function $w_R(x) = \sqrt{L}/[\sqrt{x}(x+L)]$. The product of two vectors $\phi(x)$ (which is defined in 2.5) and it's transpose can be defined as

$$\phi(x)\phi(x)^T \mathbf{A} \simeq \widetilde{\mathbf{A}}^T \phi(x) \tag{2.9}$$

Where $\widetilde{\mathbf{A}}$ is the product matrix for the vector $\mathbf{A}$. In the Tau method, derivative and product matrices are defined. These matrices can be found using recursive relations of orthogonal basis functions. The Tau method transfers the differential equation into a system of algebraic equations. In this thesis, the recursive relations of Lauguere functions are used to make the matrices. While there are inherent differences between the Galerkin and tau methods, when applied in this context, they give comparable results numerically.

2.2 Basis functions and polynomials

2.3 Numerical integration

Numerical integration and Lagrange interpolation are related to each other as one method for integration is approximating the function $u(x)$ with a polynomial and then integrating the polynomial. Given that the interpolation polynomial can be integrated exactly, the only source of error is the interpolation error.
Assuming $N+1$ different points $x_0, x_1, \ldots, x_N$, the Lagrange basis polynomials are defined as:

$$\ell_j(x) = \Pi_{0\leq m\leq N, m\neq j} \frac{x - x_m}{x_j - x_m}. \tag{2.10}$$

Assume we approximate the function $u(x)$ with Lagrangian polynomials in interval $[a, b]$.

$$u(x) \simeq \sum_{j=0}^{N} \ell_j(x) u(x_j). \tag{2.11}$$

We use this formula for approximation of the integral of $u(x)$ so we get:

$$\int_a^b u(x)\rho(x)dx \approx \sum_{j=0}^{N} u(x_j) \int_a^b \ell_j(x)\rho(x)dx = \sum_{j=0}^{N} \lambda_j u(x_j)$$

where

$$\lambda_j = \int_a^b \ell_j(x)\rho(x)dx,$$

and λ_j are called Lagrangian weights.

If $u(x)$ is a polynomial with degree smaller than $N+1$ then the integration is exact since relation (2.11) is exact. Assume $\Re_N$ is the space of all polynomials with degree $\leq N$. Take $\{P_k\}_{k=0,1,\ldots}$ to be a set of mutually orthogonal polynomials in the interval I=$[0, \infty)$ with the weight function $\rho(x)$.

$$\int_0^\infty P_r P_s \rho(x)dx = 0, \quad r \neq s,$$

where the degree of P_k is exactly k. This system is complete in the space $L^2_\rho(I)$. $L^2_\rho(a,b)$ is the space of all functions whose norm with a weight is finite:

$$L^2_\rho(I) = \{v : \|v\|_\rho < \infty\}, \tag{2.12}$$

where the norm is defined as:

$$\|v\|_\rho = \left(\int_0^\infty \mid (v(\xi)) \mid^2 \rho(\xi) d\xi\right)^{1/2}.$$

A function $u \in L^2_\rho(I)$ can be expressed like this:

$$u = \sum_{j=0}^{\infty} a_i p_i,$$

where;

$$a_i = \frac{1}{\|p_i\|^2_\rho} \int_0^\infty u(\xi) p_i(\xi) \rho(\xi) d\xi.$$

For each $N > 0$ we define the operation $S_N : L^2_\rho(I) \to P_N$ as:

$$S_N u = \sum_{j=0}^{N} a_i p_i.$$

According to the orthogonality relation:

$$\langle S_N u, v\rangle = \langle u, v\rangle, \quad \forall v \in \Re_N,$$

when N approaches ∞ then completeness of $\Re_N$ will result in :

$$\| \; u - S_N(u) \; \|_\rho \to 0.$$

Completeness means that the error of approximation using the norm $\|v\|_\rho$ in (2.12) is negligible by using N basis function where N is sufficiently large.

2.4 Choosing basis functions

One of the most important questions in spectral method is choosing suitable basis functions. Basis functions should have the following properties:

- Easy to compute
- Fast Convergence rate
- Completeness

In periodic problems, choosing suitable basis functions is easy. The trigonometric series, i.e., the Fourier series is an obvious option [38, 18]; however, in non-periodical problems finding suitable basis functions is challenging [18]. Basis functions in spectral methods are usually orthogonal functions. Using orthogonal functions make the computation faster. The greatest benefit of orthogonality is the ability to find the coefficients of expansion series easily:

$$\langle f(x), \varphi_j \rangle = \sum_{j=0}^{N} a_j (\varphi_j, \varphi_i) \Longrightarrow a_j = \frac{\langle f(x), \varphi_j \rangle}{\langle \varphi_j, \varphi_j \rangle}.$$

In this thesis we use some known orthogonal polynomials and functions. It has been proven that [38, 18] Chebyshev and Legendre polynomials are suitable orthogonal polynomials for approximation in the interval $[-1, 1]$. In semi-infinite domains Legendre and Laguerre polynomials are the best options; in infinite intervals Sinc functions, Hermite polynomials and rational Chebyshev functions have the least approximation errors [25, 17, 48, 47].

Below will consider some properties of Gaussian Integration.
If the integration points include both end points of the integration interval, we use Gauss-Lobatto formula. If one end point is included in the integration points we use the Gauss-Radau formula.

Theorem 1 (Gaussian Integration) *There are positive numbers λ_0, λ_1, … and λ_N such that $\forall f(\xi) \in \mathcal{P}_{2N+1}$*

$$\int_a^b f(\xi)\rho(\xi)\mathrm{d}\xi = \sum_{j=0}^{N} \lambda_j f(\xi_j), \tag{2.13}$$

where the ξ_j are the roots of the polynomial $p_{N+1}(\xi)$ [22].

Theorem 2 (Gauss Radau Integration) *There are positive numbers* λ_0, λ_1, ... *and* λ_N *such that* $\forall f(\xi) \in \mathcal{P}_{2N}$

$$\int_a^b f(\xi)\rho(\xi)\mathrm{d}\xi = \sum_{j=0}^{N} \lambda_j f(\xi_j), \tag{2.14}$$

where ξ_j *are the roots of the polynomial* $p_{N+1}(\xi) - \frac{p_{N+1}(c)}{p_N(c)} p_N(\xi)$ *and* c *is a boundary point,* c*=*a *or* c*=*b *[22].*

Theorem 3 (Gauss Lobatto Integration) *There are positive numbers* λ_0, λ_1, ... ,λ_N *such that* $\forall f(\xi) \in \mathcal{P}_{2N-1}$

$$\int_a^b f(\xi)\rho(\xi)\mathrm{d}\xi = \sum_{j=0}^{N} \lambda_j f(\xi_j), \tag{2.15}$$

where ξ_j *are the roots of the polynomial* $q(\xi) = p_{N+1}(\xi) + c p_N(\xi) + d p_{N-1}(\xi)$ *where* c *and* d *are choosen such that* $q(a) = q(b) = 0$ *[22].*

Chapter 3

Laguerre functions

This section is devoted to the introduction of the basic notions and working tools concerning orthogonal generalized Laguerre polynomials. More specifically, we present some properties of scaled Laguerre functions concerning projection.

The derivation of background information follows an exposition in paper [81]; a paper which the writer contributed to.

The Laguerre approximation has been widely used for the numerical solution of differential equations on semi-infinite intervals. Let

$$\mathfrak{R}_N = span\{1, x, \dots, x^{2N-2}\}, \mathfrak{P}_N = span\{1, x, \dots, x^{N-1}\}, \tag{3.1}$$

$L_n^\alpha(x)$ (generalized Laguerre polynomial) is the nth eigenfunction of the Sturm-Liouville problem [8, 26, 51]:

$$\begin{aligned} &x\frac{d^2}{\mathrm{d}x^2}L_n^\alpha(x) + (\alpha + 1 - x)\frac{\mathrm{d}}{\mathrm{d}x}L_n^\alpha(x) + nL_n^\alpha(x) = 0, \\ &\qquad\qquad x \in I = [0, \infty) \qquad n = 0, 1, 2, ... \end{aligned} \tag{3.2}$$

The generalized Laguerre polynomials are defined with the following recurrence formula:

$$\begin{aligned} &L_0^\alpha(x) = 1, \quad L_1^\alpha(x) = 1 + \alpha - x, \\ &nL_n^\alpha(x) = (2n - 1 + \alpha - x)L_{n-1}^\alpha(x) \\ &-(n + \alpha - 1)L_{n-2}^\alpha(x), \qquad n \geq 2, \qquad \alpha > -1, \end{aligned} \tag{3.3}$$

The generalized Laguerre polynomials satisfy the following relation:

$$\partial_x L_n^\alpha(x) = -\sum_{k=0}^{n-1} L_k^\alpha(x). \tag{3.4}$$

These are orthogonal polynomials for the weight function $w_\alpha = x^\alpha e^{-x}$.

$$\int_0^{+\infty} L_n^\alpha(x) L_m^\alpha(x) w_\alpha(x) \mathrm{d}x = \left(\frac{\Gamma(n+1+\alpha)}{n!}\right) \delta_{nm}.$$

Theorem 4 (Laguerre Gauss Integration) *If the ξ_j are the roots of polynomial $L_{N+1}(x)$, one method for evaluating the roots of $L_{N+1}(x)$ is as below [50, 79]:*
First the values y_k, $\quad k = 1, ..., N$, are the solutions of:

$$y_k - \sin y_k = 2\pi \frac{N-k+3/4}{2N+\alpha+1}, \qquad k = 1, \ldots, N, \tag{3.5}$$

then the values z_k are defined as below:

$$z_k = [\cos(\frac{1}{2} y_k)]^2, \qquad k = 1, \ldots, N, \tag{3.6}$$

then the Gaussian points ξ_j can be computed as

$$\xi_j = 2(2N+\alpha+1) z_k + \frac{1}{6(2N+\alpha+1)[\frac{5}{4(1-z_k)^2} - \frac{1}{1-z_k} - 1 + 3\alpha^2]}, \tag{3.7}$$

and the weights are

$$\lambda_j \quad = \quad \frac{\Gamma(\alpha+N)}{N!} \left(L_N^\alpha(x_j) \frac{\mathrm{d}}{\mathrm{d}x} L_{N-1}^\alpha(x_j)\right)^{-1}, \quad j = 0, 2, \ldots, N.$$

Theorem 5 (Laguerre Gauss-Radau Integration) *Let $N \geq 1$ be an integer and we denote with $x_{j,N}^\alpha$, $j = 1, \ldots, N-1$ the zeroes of $\frac{d}{dx} L_N^\alpha$ and the point $x_{0,N}^\alpha = 0$ (the only boundary point of the interval $[0, +\infty)$); it can be shown that $x_{j,N}^\alpha \geq 0$, $j = 0, \ldots, N-1$ [8] and the corresponding weights are:*

$$w_{0,N}^\alpha = \frac{(\alpha+1)\Gamma^2(\alpha+1)(N-1)!}{\Gamma(N+\alpha+1)}$$
$$w_{j,N}^\alpha = \frac{\Gamma(\alpha+N)}{N!} \left(L_N^\alpha(x_{j,N}^\alpha) \frac{d}{dx} L_{N-1}^\alpha(x_{j,N}^\alpha)\right)^{-1}, \quad j = 1, 2, ..., N-1.$$

The following quadrature formula with error term is known:

$$\begin{aligned} \int_0^{+\infty} f(x) w_\alpha(x) \mathrm{d}x \quad &= \quad \sum_{j=0}^{N-1} f(x_{j,N}^\alpha) w_{j,N}^\alpha \\ &+ \left(\frac{\Gamma(N+\alpha+1)}{(N)!(2N)!}\right) f^{2N-1}(\xi), \quad 0 < \xi < \infty, \end{aligned} \tag{3.8}$$

in particular, the second term on the right hand side vanishes when f is a polynomial of degree at most $2N-2$. For convenience, we shall write $x^{\alpha}_{j,N} = x_j$ and $w^{\alpha}_{j,N} = w_j$.

We define Modified generalized Laguerre functions (which we denote (MGL) functions) ϕ_j as follows:

$$\phi_j(x) = \exp(-x/(2L))L_j^{\alpha}(x/L), \quad L > 0. \tag{3.9}$$

This system is an orthogonal basis [42, 83] with weight function $w(x) = \frac{x}{L}$ and

$$< \phi_n, \phi_m >_{w_L} = \left(\frac{\Gamma(n+2)}{L^2 n!}\right)\delta_{nm},$$

where δ_{nm} is the Kronecker delta symbol.

Figure 11 shows that when n gets larger, it is difficult to obtain the graphical representations of Laguerre polynomials. An initial gentle behavior explodes to severe oscillations, for increasing values of x [40]. This will lead to unstable algorithms when we apply collocation methods[40]. Funaro [40] introduced a scaling function and appropriate numerical procedures in order to limit these unpleasant phenomena.

We define scaled Laguerre functions $\{\ell_n\}$ as follows:

$$\ell_0(x) = 1, \quad \ell_n(x) = S_n(x/k)L_n^1(x/L), \quad n = 1, 2, ... \tag{3.10}$$

where $L > 0$ is a constant and $L_n^1(x)$ are the generalized Laguerre polynomials for $\alpha = 1$ and $S_n(x)$ is defined as follows:

$$S_0(x) = 1, \quad S_n(x) = \left(\binom{n+1}{n}\prod_{t=1}^{n}(1 + x/(4t))\right)^{-1}, \quad n = 1, 2, ... \tag{3.11}$$

we denote scaled Laguerre functions with (Slf) .
Figures 11,12 show that using (MGL) and (Slf) will limit severe oscillations.

Boyd [18, 20] offered guidelines for optimizing the map parameter L where $L > 0$. Numerical results are not very sensitive to L because $dError/dL = 0$ at the minimum itself, so the error varies very slowly with L around the minimum. A little trial and error is usually sufficient to find a value that is nearly optimum. In general, there is no way to avoid a small amount of trial and error in choosing L when solving problems on an unbounded domain.

Experience and the asymptotic approximations of [20] can help, but some experimentation is always necessary as Boyd explains in his book [18].

Thus we obtain the following formula:

$$\begin{aligned}
&\ell_0(x) = 1, \quad \ell_1(x) = \frac{4(2k-x)}{2(x+4k)}, \\
&\ell_n(x) = \frac{4n}{(n+1)(4n+x/k)}[(2n-x/k)\ell_{n-1}(x) \\
&-\frac{4(n-1)^2}{4n+x/k-4}\ell_{n-2}(x)], \qquad n \geq 2, \qquad \alpha > -1,
\end{aligned} \tag{3.12}$$

this system is an orthogonal basis[40].

In [40], some of the relations of scaled Laguerre functions are shown. Laguerre-Gauss-Radau points and generalized Laguerre-Gauss-type interpolation were introduced in [57] . Funaro [40] defines a new set of Gauss type weights:

$$\begin{aligned}
&\widetilde{w}_{0,N} = \frac{w_0}{S_N(0)^{-2}} = \frac{2\Gamma(N+2)}{N(N!)} \\
&\widetilde{w}_{j,N} = \frac{w_j}{S_N(x_j)^{-2}} = \\
&\frac{\Gamma(N+2)}{4N^2(N!)}\frac{4N+x_j}{\ell_N(x_j)}\left(\frac{\mathrm{d}}{\mathrm{d}x}\ell_{N-1}(x_j) + \ell_{N-1}(x_j)\sum_{m=1}^{N-1}\frac{1}{4m+x_j}\right)^{-1}, \\
&j = 1, 2, ..., N-1,
\end{aligned} \tag{3.13}$$

where we have, $N \geq 2$ [40]

$$\frac{\mathrm{d}}{\mathrm{d}x}[S_{N-1}(x)]^{-1} = [S_{N-1}(x)]^{-1}\sum_{m=1}^{N-1}\frac{1}{4m+x_j}, \qquad x \in I = [0, \infty)$$

And we have:

$$\int_0^{+\infty} p^2(x)xe^{-x}\mathrm{d}x = \sum_{j=0}^{N-1}\widetilde{p}^2(x_j)\widetilde{w}_{j,N}, \qquad \forall p \in \mathfrak{P}_N,$$

where $\widetilde{p} = pS_N$

3.1 Gaussian integration in semi-infinite domains

Take $R_{k=0,1,\ldots}$ to be a set of mutually orthogonal functions (rational Chebyshev or Laguerre functions) in the interval $[0,\infty)$ with the weight function $w(x)$:

$$\int_0^{+\infty} R_r(x)R_s(x)w(x)dx = 0, \quad r \neq s,$$

$R_k(x)$ is derived from the corresponding orthogonal polynomials (Laguerre or Chebyshev polynomials) which has a weight function $\rho(x)$. The following relation holds between weight functions.

$$w(x)dx = \rho(\xi)d\xi.$$

This system is complete in Hilbert space $L^2_w(0,\infty)$ with norm

$$\|v\|_w = \left(\int_0^{\infty} \mid v(x) \mid^2 w(x)dx\right)^{1/2}.$$

Consider $\Re_N$ to be the space generated by the functions $R_k(x)$:

$$\Re_N = \text{span}\left\{R_0, R_1, \ldots, R_N\right\}. \tag{3.14}$$

A function $u \in L^2_w(0,\infty)$ can be expanded

$$u = \sum_{i=0}^{\infty} a_i R_i, \tag{3.15}$$

where a_i can be found using the following relation:

$$a_i = \frac{1}{\|R_i\|_w^2} \int_a^{\infty} u(x)R_i(x)w(x)dx. \tag{3.16}$$

For each $N > 0$, we define the operation $P_N : L^2_w(0,\infty) \to \Re_N$

$$P_N u = \sum_{j=0}^{N} a_j R_j.$$

According to the orthogonality relation:

$$\langle P_N u, v\rangle = \langle u, v\rangle, \quad \forall v \in \Re_N,$$

completeness of R_i will result in

$$\| u - P_N(u) \|_w \to 0.$$

$\forall u \in L^2_w(0,\infty)$ when N approaches ∞.

Theorem 6 (Gaussian integration in semi-finite domain) *There are positive numbers ω_0, ω_1, ... ,ω_N such that $\forall f(x) \in \mathcal{R}_{2N+1}$*

$$\int_0^\infty f(x)w(x)\mathrm{d}x = \sum_{j=0}^{N} \omega_j f(x_j), \tag{3.17}$$

x_j are the roots of $R_N(\xi)$ [21].

Theorem 7 (Gauss Radau integration in semi-finite domain) *There are positive numbers ω_0, ω_1, ... ,ω_N such that $\forall f(\xi) \in \mathcal{R}_{2N}$*

$$\int_0^\infty f(\xi)w(x)\mathrm{d}x = \sum_{j=0}^{N} \omega_j f(\xi_j), \tag{3.18}$$

ξ_j are the roots of the function $R_{N+1}(\xi) - \frac{R_{N+1}(0)}{R_N(0)} R_N(\xi)$ [22].

Theorem 8 (Gauss Lobatto integration in semi-finite domain) *There are positive numbers ω_0, ω_1, ... ,ω_N such that $\forall f(\xi) \in \mathcal{R}_{2N-1}$*

$$\int_0^\infty f(\xi)\rho(\xi)\mathrm{d}\xi = \sum_{j=0}^{N-1} \lambda_j f(\xi_j) + \lambda_N \lim_{\xi\to\infty} f(\xi), \tag{3.19}$$

where ξ_j are the roots of $q(\xi) = R_{N+1}(\xi) + cR_N(\xi) + dR_{N-1}(\xi)$ and c and d are such that $q(0) = \lim_{\xi\to\infty} q(\xi) = 0$ [22].

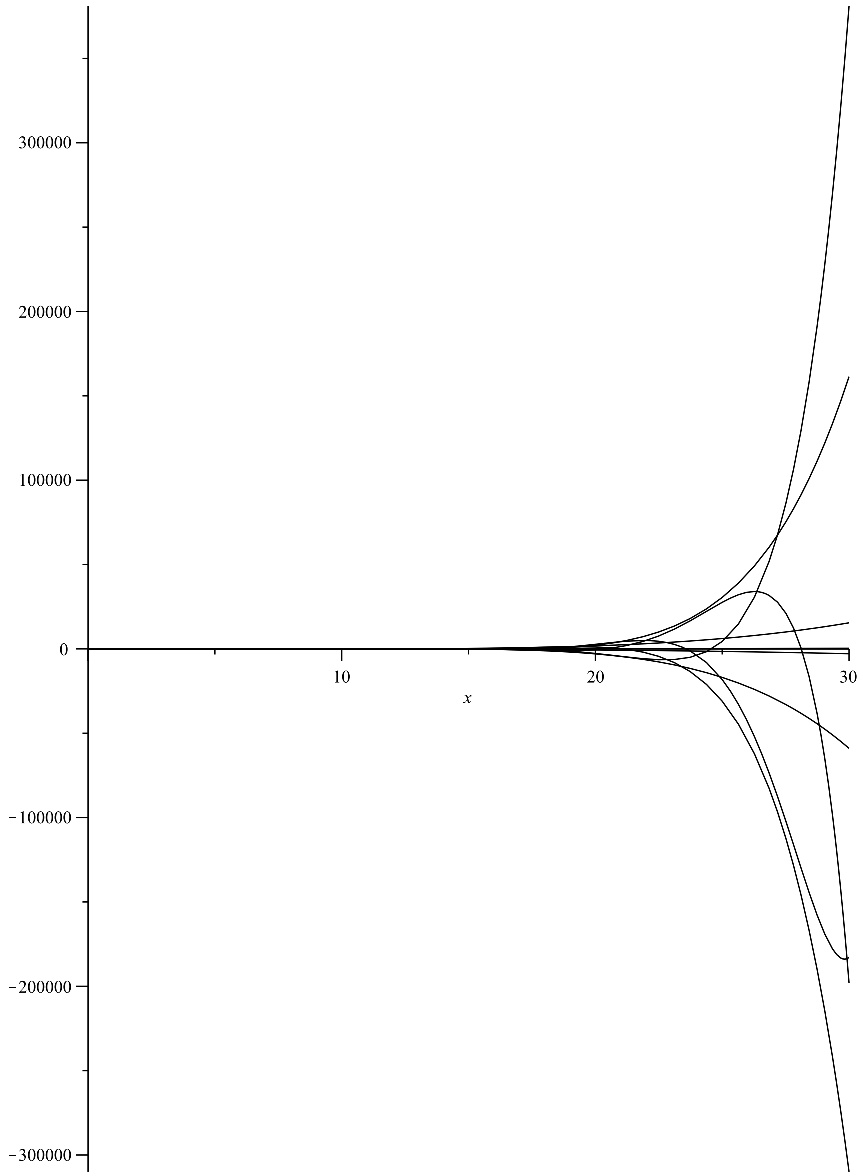

Figure 3.1: Graph of the generalized Laguerre polynomials $L_1^1(x)$, $L_1^1(x)$, $\ldots$,$L_1^{10}(x)$ for $x \in [0, 30]$

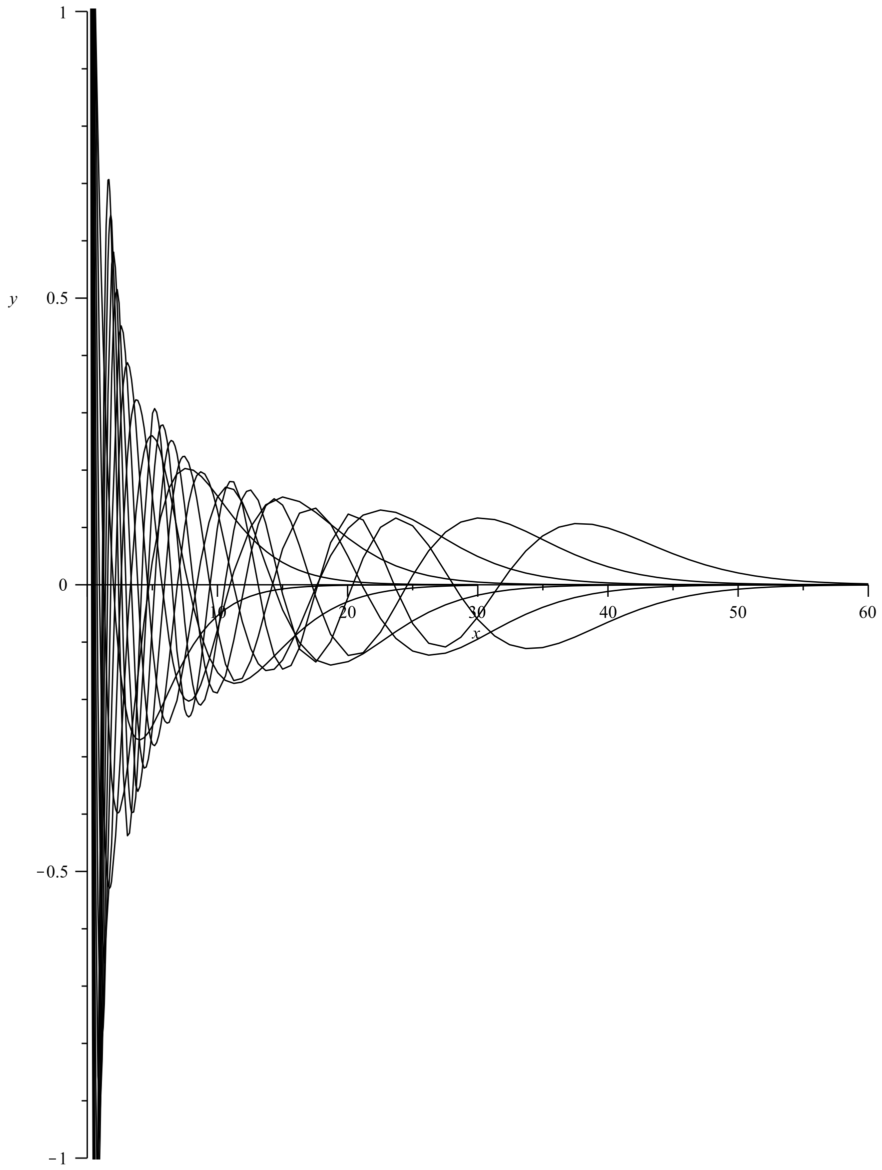

Figure 3.2: Graph of the generalized Laguerre functions $\phi_1(x)$, $\phi_2(x)$, $\ldots$,$\phi_10(x)$ for $x \in [0, 60]$

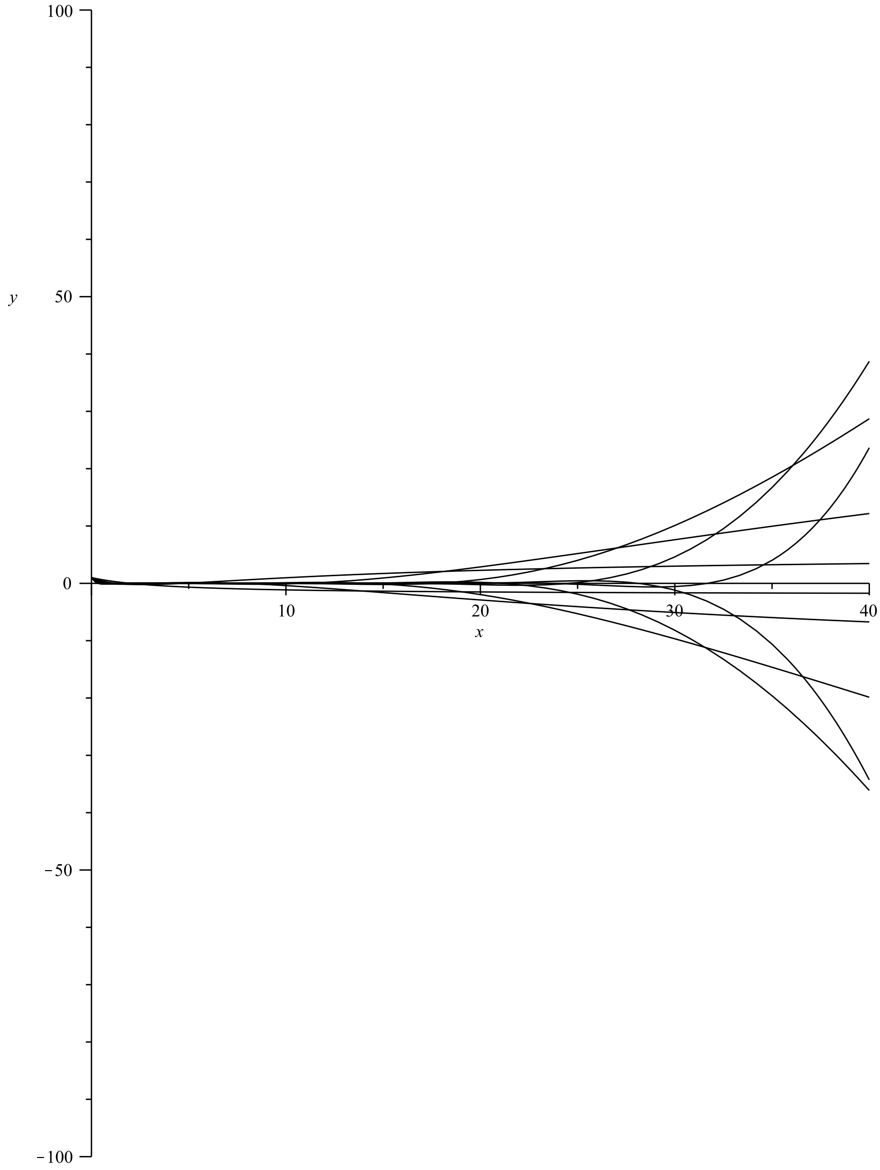

Figure 3.3: Graph of the generalized scaled functions $S_1(x)$, $S_2(x)$, $\ldots$,$S_10(x)$ for $x \in [0, 40]$

Chapter 4

Solving problems in semi-infinite domains

Spectral methods have been successfully applied in the approximation of differential boundary value problems defined in unbounded domains. There are different solution techniques. Among these, one approach consists of using the collocation method based on the nodes of Gauss formulas related to unbounded intervals [49]. This involves computations with orthogonal polynomials, such as Laguerre polynomials. However, there is a shortage of numerical methods for the solution of partial differential equations over semi-infinite and infinite domains. We can apply different spectral methods that are used to solve problems in semi-infinite domains. The first approach is using Laguerre polynomials [49, 78, 79, 63]. Guo [49] suggested a Laguerre-Galerkin method for the Burgers equation and Benjamin-Bona-Mahony (BBM) equation on a semi-infinite interval.

In [78] Shen proposed spectral methods using Laguerre functions and analyzed them for model elliptic equations on regular unbounded domains. It was shown that spectral-Galerkin approximations based on Laguerre functions are stable and convergent with spectral accuracy in the Sobolev spaces.

Maday, et al. [63] proposed a collocation method for solving partial differential equations. They introduced a general presentation of the method and a description of the derivation discretization matrices and then determined the optimum estimations in the adapted Hilbert norms.

Siyyam [79] applied two numerical methods to solve the differential equations using the

Laguerre Tau method. He generated linear systems and solved them.

The second approach is reformulating the original problem in the semi-infinite domain as a singular problem in a bounded domain by variable transformation and then using the Jacobi polynomials to approximate the resulting singular problem [50]. We can use mapping functions to reformulate the original problem. Mapping functions are $x = \varphi(\xi)$ where $\varphi : [-1, 1] \to [0, \infty)$. Using this mapping function we can study the behavior of the $u(x)$ by the function $v(\xi) = u(\varphi(\xi))$. It means that substitution of $\varphi(\xi)$ into x transfers the problem from semi-infinite interval to bounded domain. When the mapping function $\varphi(\xi)$ is a smooth function in the interval $[-1, 1]$, the accuracy of the method is of infinite order. Some of these mapping functions are

The Algebraic mapping:

$$x = L\frac{1+\xi}{1-\xi}. \tag{4.1}$$

The Logarithmic mapping:

$$x = \frac{L}{2}ln\frac{1+\xi}{1-\xi}. \tag{4.2}$$

The Algebraic mapping is more suitable for the functions which decrease very slowly at infinity like the function $\frac{1}{x}$. The logarithmic and exponential mapping are suitable for functions which have fast decrease at infinity [18].

The third approach is replacing the semi-infinite domain with the $[0, L]$ interval by choosing L sufficiently large. This method is named domain truncation [18].

The fourth approach of spectral methods is based on rational orthogonal functions. Boyd [18] defined a new spectral basis, named rational Chebyshev functions on the semi-infinite interval, by mapping to the Chebyshev polynomials.

The algebraic mapping converts the Chebyshev polynomials [18] in ξ into rational Chebyshev functions in x; rational Chebyshev functions are defined as follows [18]:

$$R_n(x) = T_n(\frac{\xi - L}{\xi + L}), \tag{4.3}$$

where L is a constant parameter, $T_n(y)$ is the well-known Chebyshev polynomial [18].
The expansion at infinity has two parameters n and L. $L > 0$ is the scaling parameter. On

a semi-infinite domain, there is always a parameter that must be determined experimentally.

Guo, et al. [49] introduced a new set of rational Legendre functions which is orthogonal in $L^2(0, +\infty)$. They applied a spectral scheme using the rational Legendre functions for solving the Korteweg-de Vries equation on the half line. Boyd, et al. [16] applied collocation methods on a semi-infinite interval and compared rational Chebyshev, Laguerre and mapped Fourier sine.

The authors of [69, 67, 68] applied a spectral method to solve nonlinear ordinary differential equations on semi-infinite intervals. Their approach was based on a rational Tau method. They obtained the operational matrices of derivative and product of rational Chebyshev and Legendre functions and then they applied these matrices together with the Tau method to reduce the solution of these problems to the solution of a system of algebraic equations.

In this thesis, we apply the different spectral methods to a singular form of Lane-Emden type initial value problems directly. The result shows the higher accuracy of the Tau method compared to the other methods.

Chapter 5

Collocation Method for the Blasius Equation

5.1 Introduction

5.1.1 Blasius equation

A great deal of interest has been focused on the steady flow of viscous incompressible fluids. Keulegen [58] investigated the case of two parallel streams, where the upper stream was moving and the lower one was at rest. An approximate solution has been obtained for this model. Lock [62] studied two cases, where the lower stream was at rest as well as when it was in motion. In [74], Potter extended the study to two fluids of different viscosities and densities, where both fluids were moving co-current with different velocities. The velocity distribution in the boundary layers was well addressed [74]. The existence of a solution for this model was successfully addressed and established in [2] by using the technique of Weyl [86].

A broad class of analytical solutions methods and numerical solutions methods were used to handle this problem.

The Blasius equation is known as the mother of all boundary-layer equations in fluid mechanics. Many different, but related, equations have been derived for a multitude of fluid-mechanical situations, for instance, the Falkner- Skan equation [9]. Mathematical model described by ordinary differential equation:

$$y''' + \frac{1}{2}yy'' = 0, \tag{5.1}$$

with boundary conditions:

$$y(0) = y'(0) = 0, \quad y'(\infty) = 1. \tag{5.2}$$

5.2 Solution of Blasius Equation

We apply MGL functions (3.2) collocation method to the Blasius Equation introduced in (5.1) subject to the conditions (5.2). At first we try to find the coefficients of the interpolant of $y(x)$:

$$I_N y(x) = x + \sum_{j=0}^{N+1} a_j \phi_j(x). \tag{5.3}$$

To find the unknown coefficients a_j's, we substitute the truncated series into the (5.1) and boundary conditions in (5.2).

$$\left.\frac{\mathrm{d}^3 I_N y(x)}{\mathrm{d}x}\right|_{x=x_j} + \frac{1}{2}\frac{\mathrm{d}^2 I_N y(x)}{\mathrm{d}x} I_N y(x)\Big|_{x=x_j} = 0, \quad j = 0, \ldots, N-1, \tag{5.4}$$

$$I_N y(x)\Big|_{x=0} = 0, \qquad \left.\frac{\mathrm{d}I_N y(x)}{\mathrm{d}x}\right|_{x=0} = 0, \tag{5.5}$$

where the x_j are the roots of the MGL functions (3.2). Note that the approximation function satisfies the third boundary condition so we omit this condition here. The initial values required to start Newton's iterative method have been chosen as $\frac{x}{L}$.

The second derivative at zero, $\alpha = y''(0)$, plays an important role in the function. A highly accurate numerical solution of the Blasius equation has been provided by Howarth [56], who obtained $\alpha = y''(0) = 0.332057$. The problem is solved with the BVP4C function in MATLAB and the code provided in the book [15]. In the code the value of 10 is used instead of infinity and the value of α is $\alpha = y''(0) = 0.332100$ with relative error 0.012%. Abbasbandy [1] used Adomian's decomposition method to obtain $\alpha = y''(0) = 0.333729$ with 0.383% accuracy, also Tajvidi et al. [82] calculated $\alpha = y''(0) = 0.333329$ with 0.009% accuracy. Howarth [56] also obtained $y'(1) = 0.32979$ and $y(1) = 0.16557$, He [53] obtained these with relative errors 16.68% and 6.32% respectively. The approximations of the $y''(0)$, $y^{'}(1)$, $y(1)$ obtained by this method show that our results are highly accurate. In Table 1, the resulting values of $y(1)$, $y^{'}(1)$, $\alpha = y^{''}(0)$ together with L and accuracies (0.0006%), (0.08%), (0.006%) using the present method with $N = 5, 7, 9$ are presented.

Tables $2, 3, 4$ show the numerical values of $y, y'$and y'' with those of Howarth [56], respectively. The results show the higher accuracy of the MGL function collocation method compared to the other methods. Moreover, if we use the value of 200 instead of infinity in BVP4C function, it takes more than 2 hours to solve the problem; however, applying the MGL function collocation method and using the same PC it takes less than 4 minutes to solve this problem.

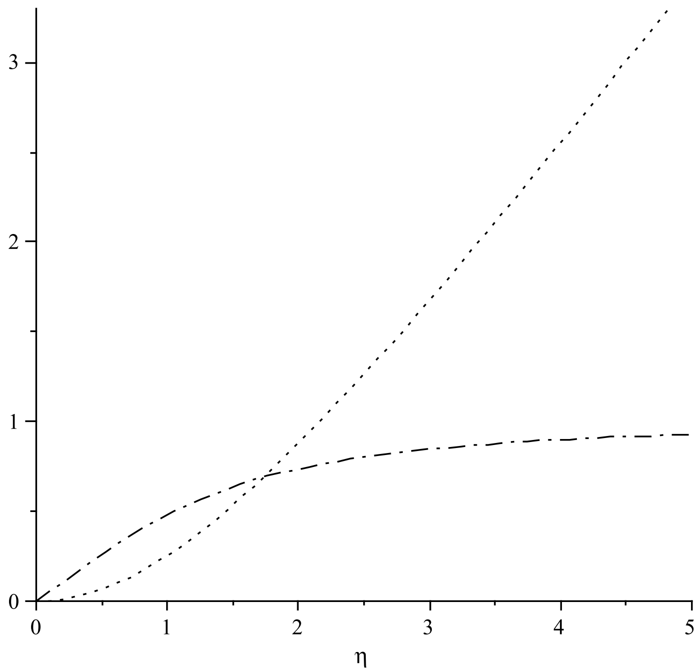

Figure 5.1: Approximations of $y(x)$ (SOLID) , $y'(x)$ (dotted line) for Blasius equation obtained by present method.

Table 5.1: The resulting values of $\alpha = y(1)$,$y^{'}(1)$,$y^{''}(0)$ together with L and relative errors(%) using the present method

N	L	$y(1)$	$error(\%)^{y}$	$y^{'}(1)$	$error(\%)^{y'}$	$y^{''}(0)$	$error(\%)^{y''}$
5	0.78430	0.16548	0.050	0.32851	0.30	0.332093	0.31209
7	0.78209	0.16556	0.006	0.32944	0.10	0.332070	0.32090
9	0.78121	0.16556	0.006	0.32951	0.08	0.332059	0.0006

Table 5.2: Approximation of $y(x)$ for present method, BVP4C function and solutions of Howarth [56]

x	Present method	BVP4C solutions	solutions of Howarth [56]
0.00	0.000000	0.000000	0.000000
1.00	0.16556	0.165627	0.16557
2.00	0.65085	0.650000	0.65003
3.00	1.39617	1.396099	1.39682
4.00	2.30552	2.305243	2.30576
5.00	3.28385	3.283334	3.28329
6.00	4.27968	4.279634	4.27964
7.00	5.27921	5.279323	5.27926
8.00	6.27919	6.279234	6.27923
9.006	7.27921	7.288065	7.27923

Table 5.3: Approximation of $y'(x)$ for present method, BVP4C function and solutions of Howarth [56]

x	Present method	BVP4C solutions	solutions of Howarth [56]
0.00	0.00000	0.00000	0.00000
1.00	0.32951	0.32980	0.32979
2.00	0.62937	0.62983	0.62977
3.00	0.84605	0.84600	0.84605
4.00	0.95571	0.95559	0.95552
5.00	0.99182	0.99154	0.99155
6.00	0.99838	0.99900	0.99898
7.00	0.99979	0.99998	0.99992
8.00	0.99999	1.00000	1.00000
9.006	1.00000	1.00000	1.00000

Table 5.4: Approximation of $y''(x)$ for present method, BVP4C function and solutions of Howarth [56]

x	Present method	BVP4C solutions	solutions of Howarth [56]
0.00	0.332059	0.332100	0.33206
1.00	0.323613	0.323097	0.32301
2.00	0.266836	0.266898	0.26675
3.00	0.161709	0.161409	0.16136
4.00	0.064702	0.064209	0.06424
5.00	0.015758	0.015989	0.01591
6.00	0.002503	0.002409	0.00240
7.00	0.000189	0.000239	0.00022
8.00	0.000054	0.000000	0.00001
9.006	0.000000	0.000000	0.00000

Chapter 6

Lane-Emden type equations

This derivation of background follows an exposition in paper [81], a paper which the writer contributed to.

A Lane-Emden type equation has the following form:

$$y''(x) + \frac{\alpha}{x}y(x) + f(x)g(y) = h(x), \quad \alpha x \geq 0, \tag{6.1}$$

where initial conditions are:

$$\begin{aligned} &a. \quad y(0) = A, \\ &b. \quad y'(0) = B, \end{aligned} \tag{6.2}$$

where α, A and B are real constants and $f(x)$, $g(y)$ and $h(x)$ are some given functions. For special forms of $g(y)$, the well-known Lane-Emden equations were used to model several phenomena in mathematical physics and astrophysics such as the theory of stellar structure, the thermal behavior of a spherical cloud of gas, isothermal gas spheres, theory of thermionic currents, radiatively cooling, self-gravitating gas clouds, in the mean-field treatment of a phase transition in critical absorption or in the modeling of clusters of galaxies. [23, 10, 4, 14]. In this section we consider the standard Lane-Emden equation.

6.1 The Lane-Emden equation

In the study of stellar structure [23] an important mathematical model described by the second-order ordinary differential equation

$$xy'' + 2y' + xg(y) = 0, \qquad x > 0, \tag{6.3}$$

with the initial conditions

$$y(0) = 1, \qquad y'(0) = 0, \tag{6.4}$$

arises, where $g(y)$ is some given function of y [23]. Among the most popular forms of $g(y)$ is

$$g(y) = y^m, \tag{6.5}$$

This equation is the standard Lane-Emden equation. It was first proposed by Lane [5] and studied in more detail by Emden [34].

The physically interesting range of m is $0 \leq m \leq 5$. Numerical and perturbation approaches to solve equation (6.3) with $g(y) = y^m$ have been considered by various authors. It has been claimed in the literature that only for $m = 0, 1$ and 5 the solutions of the Lane-Emden equation(also called the polytropic differential equations) could be given in closed form.

In fact, for $m = 5$, only a 1-parameter family of solutions is presented. The so called generalized Lane-Emden equations of the first kind have been looked at in Goenner and Havas [44] and Goenner [43].

6.1.1 The path of deriving the Lane-Emden equation.

This equation is one of the basic equations in the theory of stellar structure and has been the focus of many studies[10, 4, 14]. We simply begin with the Poisson equation and the condition for hydrostatic equilibrium:

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{GM(r)}{r^2}, \tag{6.6}$$

$$\frac{\mathrm{d}M(r)}{\mathrm{d}r} = 4\pi\rho r^2, \tag{6.7}$$

where G is the gravitational constant, P is the pressure, $M(r)$ is the mass of a star at a certain radius r, and ρ is the density, at a distance r from the center of a spherical star. Combination of these equations yields the following equation, which as should be noted, is an equivalent form of the Poisson equation.

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{r^2}{\rho}\frac{\mathrm{d}P}{\mathrm{d}r}\right) = -4\pi G\rho. \tag{6.8}$$

From these equations one can obtain the Lane-Emden equation through the simple supposition that the density is simply related to the pressure, while remaining independent of the temperature. We already know that in the case of a degenerate electron gas that the pressure and density are $\rho \sim P^{\frac{3}{5}}$ [23], assuming that such a relation exists for other states of the star we are led to consider a relation of the following form:

$$P = K\rho^{1+\frac{1}{m}}, \tag{6.9}$$

where K and m are constants.

At this point it is important to note that m is the polytropic index which is related to the ratio of specific heats of the gas comprising the star. Based upon these assumptions we can insert this relation into our first equation for the hydrostatic equilibrium condition and from this rewrite equation to:

$$\left[\frac{K(m+1)}{4\pi G}\lambda^{\frac{1}{m}-1}\right]\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}y}{\mathrm{d}r}\right) = -y^m, \tag{6.10}$$

where the additional alternation to the density of this expression has λ representing the central density of the star and y that of a related dimensionless quantity that are both related to ρ through the following relation

$$\rho = \lambda y^m. \tag{6.11}$$

Additionally, if we place this result into the Poisson equation, we obtain a differential equation for the mass, with a dependance upon the polytropic index m. Though the differential equation is seemingly difficult to solve, this problem can be partially alleviated by the introduction of an additional dimensionless variable x, given by the following [23]:

$$r = ax, \tag{6.12}$$

$$a = \left[\frac{K(m+1)}{4\pi G}\lambda^{\frac{1}{m}-1}\right]^{\frac{1}{2}}. \tag{6.13}$$

Inserting these relations into our previous relations we obtain the famous form of the Lane-Emden equation, given below:

$$\frac{1}{x^2}\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2\frac{\mathrm{d}y}{\mathrm{d}x}\right) = -y^m. \tag{6.14}$$

Taking these simple relations we will have the Lane-Emden equation with $g(y) = y^m$,

$$y'' + \frac{2}{x}y' + y^m = 0, \qquad x > 0. \tag{6.15}$$

Numerical and perturbation approaches to solve equation Eq. (6.5) with $g(y) = y^m$ have been considered by various authors [10, 4, 14].

6.2 White-dwarf equation

The gravitational potential of the degenerate white-dwarf stars can be modeled by the so-called white-dwarf equation which is Eq. (6.3) when $g(y)$ is given by

$$g(y) = (y^2 - C)^{3/2}. \tag{6.16}$$

The initial conditions are

$$y(0) = 1, \qquad y'(0) = 0. \tag{6.17}$$

If $C = 0$, Eq. (6.17) reduces to the Lane-Emden equation of index $m = 3$.

6.3 Previous methods for solving this problem

Many analytic methods have been used to solve Lane-Emden equations, the main difficulty arises in the singularity of the equation at $x = 0$. Currently, most techniques in use for handling the Lane-Emden-type problems are based on either series solutions or perturbation techniques.
Bender et al. [10] proposed a perturbative technique for solving nonlinear differential equation such as Lane-Emden.

Shawagfeh [77] applied a nonperturbative approximate analytic solution for the Lane-Emden equation using the Adomian decomposition method. Adomian proposed a new method to solve some functional equations [41, 3]. This method and its modifications [55] have been used to solve singular and non-singular ordinary differential equations. Also, ADM is a method for solving a wide range of problems whose mathematical models yield equations involving algebraic, differential, integral and integro-differential equations [41, 3]. This method is based on the search for a solution in the form of a series and on decomposing the nonlinear operator into a series in which the terms are calculated recursively using Adomian polynomials [3]. ADM is applied to various differential equations. For example, this approach is used to obtain numerical solution for the Falkner-Skan equation [33]. In [85] ADM is applied to the third-order dispersive partial differential equation.

Wazwaz [84] employed the Adomian decomposition method with an alternate framework designed to overcome the difficulty of the singular point. Liao [61] provided an analytic algorithm for Lane-Emden type equations. This algorithm logically contains the well-known Adomian decomposition method. Parand and Razzaghi [69] presented a numerical technique to solve higher order ordinary differential equations such as Lane-Emden. Their approach was based on a rational Legendre Tau method. Bataineh et al. [7] obtained analytic solutions of singular initial value problems (IVPs) of the Emden-Fowler type by the homotopy analysis method (HAM).

In this thesis, we implement the Lagrangian and collocation method for solving the Lane-Emden type equation; we also apply the Tau method for solving (6.3) where $g(y)$ is (6.5), the solution of which is then approximated by modified generalized Laguerre function with unknown coefficients. The operational matrices of derivative and product of modified generalized Laguerre functions are given. These matrices together with the Tau method are then utilized to evaluate the unknown coefficients and find approximate solutions for $y(x)$. The Tau method was invented by Lanczos [60] in 1938.

Chapter 7

Lagrangian method using Laguerre polynomials

7.1 The Lagrangian Method for Solving Lane-Emden Type Equation Arising in Astrophysics on Semi-infinite Domains

This derivation of background follows an exposition in paper [70]; a paper which the writer contributed to.

The Lagrangian basis of generalized Laguerre polynomials (3.2) (which we denoted GLP) of order p at the Gauss-Radau-Laguerre quadrature points in $\mathbb{R}^+$ is [40]:

$$\ell_j(x) = \begin{cases} \frac{xL_N^\alpha(x)}{\eta_j \frac{d}{dx} L_N^\alpha(\eta_j)} \frac{1}{x-\eta_j}, & j = 1, ..., N, \\ \frac{L_N^\alpha(x)}{L_N^\alpha(0)}, & j = 0, \end{cases} \tag{7.1}$$

where $\eta_j, j = 1, 2.., N$ are the N GLP-Radau points.

The derivative operator of GLP is:

$$d_{ij} = \ell_j^{'}(\eta_i),$$

moreover, for any polynomial p of degree at most N, one gets:

$$p^{'}(\eta_i) = \sum_{j=0}^{N} d_{ij} p(\eta_j).$$

Funaro [40] obtained the derivative matrix of GLP(D_N):

$$d_{ij}=\begin{cases}\dfrac{\eta_i \frac{d}{dx}L_N^\alpha(\eta_i)}{\eta_j \frac{d}{dx}L_N^\alpha(\eta_j)}\dfrac{1}{\eta_i-\eta_j} & i,j=1,...,N, i\neq j,\\ \dfrac{1-\alpha+\eta_i}{2\eta_i} & i=j=1,...,N,\\ \dfrac{\frac{d}{dx}L_N^\alpha(\eta_i)}{L_N^\alpha(0)} & i=1,...,N, j=0,\\ -\dfrac{L_N^\alpha(0)}{\eta_j^2\frac{d}{dx}L_N^\alpha(\eta_j)} & j=1,...,N, i=0,\\ -\dfrac{N}{\alpha+1} & i=j=0,\end{cases} \tag{7.2}$$

the second derivative operator is obtained either by squaring D_N or by evaluating $\ell_j''(\eta_i)$:

$$\ell_{ij}''(\eta_i)=\begin{cases}\dfrac{\frac{d}{dx}L_N^\alpha(\eta_i)((1-\alpha+\eta_i)(\eta_i-\eta_j)-2\eta_i)}{\eta_j(\eta_i-\eta_j)^2\frac{d}{dx}L_N^\alpha(\eta_j)} & i,j=1,...,N, i\neq j,\\ \dfrac{(\eta_i-\alpha)^2}{3\eta_i^2}-\dfrac{N-1}{3\eta_i} & i=j=1,...,N,\\ -\dfrac{(\alpha+1-\eta_i)\frac{d}{dx}L_N^\alpha(\eta_i)}{\eta_i L_N^\alpha(0)} & i=1,...,N, j=0,\\ -\dfrac{2(N+\alpha+1)L_N^\alpha(0)}{\eta_i^3(\alpha+1)\frac{d}{dx}L_N^\alpha(\eta_j)} & i=1,...,N, j=0,\\ \dfrac{N(N-1)}{(\alpha+1)(\alpha+2)} & i=j=0.\end{cases} \tag{7.3}$$

Laguerre polynomials are not suitable for computations [40], and also Lagrangian interpolation of Laguerre polynomials is not suitable for solving some differential equations, such as Lane-Emden equations because of their boundary conditions. So we use Lagrangian interpolation of MGL functions (3.9). At first we must find the Lagrangian basis and derivative operators of MGL functions.

Let $\Gamma_N^\alpha(x)=e^{-x/2}L_N^\alpha(x)$ and by substitution of the x with η_i we have,

$$\frac{d}{dx}\Gamma_N^\alpha(\eta_i)=e^{-\eta_i/2}\frac{d}{dx}L_N^\alpha(\eta_i). \tag{7.4}$$

Lemma 1 *The Lagrangian basis of* $\Gamma_N^\alpha(x)=e^{-x/2}L_N^\alpha(x)$ *is,*

$$\widehat{\ell}_i(x)=\ell_i(x)\frac{e^{-x/2}}{e^{-\eta_i/2}}, \tag{7.5}$$

where $\ell_j(x)$ are the Lagrangian basis of Laguerre polynomials.

Proof: Suppose $\gamma\frac{x\Gamma_N^\alpha(x)}{x-\eta_j}$ is the Lagrangian basis of $\Gamma_N^\alpha(x)$; using relation (7.4) we can find a constant γ,

$$\begin{aligned}
&\widehat{\ell}_i(\eta_i) = 0,\\
&\lim_{x\to\eta_j}\gamma\frac{x\Gamma_N^\alpha(x)}{x-\eta_j} = \gamma\lim_{x\to\eta_j}(\ell_N^\alpha(x) + x\frac{d}{dx}\Gamma_N^\alpha(x)) = 1\\
&\Rightarrow \gamma = \frac{1}{\eta_j e^{-\eta_i/2}\frac{d}{dx}L_N^\alpha(\eta_i)},
\end{aligned}$$

so

$$\widehat{\ell}_i(x) = \frac{1}{\eta_j e^{-\eta_i/2}\frac{d}{dx}L_N^\alpha(\eta_i)}\frac{xL_N^\alpha(x)}{x-\eta_j}, \tag{7.6}$$

and with comparison of (7.1) and (7.6) lemma 1 is proved.

By Eq. (7.5) the derivative operator of $\Gamma_N^\alpha(x) = e^{-x/2}L_N^\alpha(x)$ (we denote by $\widehat{D}_N$) is:

$$\widehat{d}_{ij} = d_{ij}\frac{e^{-\eta_j/2}}{e^{-\eta_i/2}} - 1/2\delta_{ij}, \qquad i,j = 1,...,N, \tag{7.7}$$

where matrix d_{ij} is defined in Eq. (7.2). As pointed out before, the second derivative operator is obtained either by squaring $\widehat{D}_N$ or by evaluating $\widehat{\ell}_j''(\eta_i)$. For evaluating $\widehat{\ell}_j''(\eta_i)$ we can use the following relation:

$$\widehat{\ell''}_i(\eta_j) = \frac{1}{4}\delta_{ij} - d_{ij}\frac{e^{-\eta_j/2}}{e^{-\eta_i/2}} + \frac{e^{-\eta_j/2}}{e^{-\eta_i/2}}\ell_{ij}''(\eta_j), \qquad i,j = 1,...,N, \tag{7.8}$$

and $\ell_i''(\eta_j)$ is defined in Eq. (7.3). It is obvious that MGL function is $\Gamma(x/L)$, so Lagrangian basis of MGL function is $\widehat{\ell}_i(x/L)$, and derivative operators can be obtained easily.

7.1.1 Function Approximation Using the Lagrangian Method

Setting $b_j = y(L\eta_j)$ we define the interpolant of $y(x)$ by

$$I_N y(x) = \sum_{j=0}^{N} b_j\widehat{\ell}_j(x/L), \tag{7.9}$$

7.1.2 Numerical results

To apply Lagrangian basis of MGL functions to the Lane-Emden equation introduced in Eq. (6.3) and Eq. (6.5) with boundary conditions Eq. (6.4), we try at first to find the coefficients of the interpolant of $y(x)$ in Eq. (7.9). To find the unknown coefficients b_j, we substitute the truncated series into Eq. (6.3) with $g(y)$ introduced in Eq. (6.5) and boundary conditions in Eq. (6.4). So we have:

$$\eta_i \sum_{j=0}^{N} b_j \frac{1}{L^2} \widehat{d_{ij}^{(2)}} + 2 \sum_{j=0}^{N} b_j \frac{1}{L} \widehat{d}_{ij} + L\eta_i b_i^m = 0, \quad i = 1, ..., N-1 \tag{7.10}$$

$$\begin{aligned} &b_0 = 1, \\ &\sum_{j=0}^{N} b_j \widehat{d_{0j}} = 0. \end{aligned} \tag{7.11}$$

We have $N-1$ equations in Eq. (7.10), that generate a set of $N+1$ nonlinear equations with the boundary equations in Eq. (7.11).
The b_j's are the expansion coefficients associated with the family $\{\widehat{\ell}_j(x/L)\}$. A semilogarithmic plot of $|b_j|$ versus j is also useful to determine a good choice of L when the exact solution is unknown. One can run the code for several different L and then plot the coefficients from each run on the same graph. The best L is the choice that gives the most rapid decrease of the coefficients [18] because it will increase the convergence rate of the Lagrangian method.

Table 7.1 shows the approximations of $y(x)$ for standard Lane-Emden with $m = 3$ obtained by the method proposed in this thesis for $N = 15$ and $k = 1$, and those obtained by Horedt [54].

Tables 7.2,7.3,7.4 show the comparison of the first zero of y, between Padé approximation used by [10], method in [65] and the present method for $m = 2, 3, 4$, respectively. These tables show that the current method has an exponential convergence rate. Additionally, high convergence rate and good accuracy are obtained by the proposed method using a relatively low numbers of data points.

Figures 7.1 shows the resulting graph of Lane-Emden for $N = 20$, $m = 2, 3, 4$.

Table 7.1: Approximation of $y(x)$ for Lagrangian method, solutions of Horedt [54] for $m = 3$

x	Present method	solutions of Horedt [54]	Absolute error
0.000	1.000000	1.000000	$0.00e-06$
0.100	0.998323	0.998336	$1.30e-06$
0.500	0.959821	0.959839	$1.20e-06$
1.000	0.855057	0.855058	$1.00e-06$
5.000	0.110820	0.110820	$0.00e-06$
6.000	0.043718	0.043738	$2.00e-06$
6.800	0.004165	0.004168	$3.00e-06$
6.896	0.000035	0.000036	$1.00e-06$

Table 7.2: Comparison the first zero of y, between Padé approximation used by Bender[10], method in [65] and the Lagrangian method for $m = 2$.

N	Present method	method in [65]	Bender	Exact value
15	4.353112	4.352875	4.3603	4.35287460
19	4.352873	–		

Table 7.3: Comparison the first zero of y, between Padé approximation used by [10] and the Lagrangian method for $m = 3$.

N	Present method	Bender	Exact value
15	6.895841	7.0521	6.89684862
19	6.896845		

Table 7.4: Comparison the first zero of y, between Padé approximation used by [10], method in [65] and the Lagrangian method for $m = 4$.

N	Present method	method in [65]	Bender	Exact value
15	14.971531	14.971546	17.967	14.9715463
19	14.971546	14.971546		

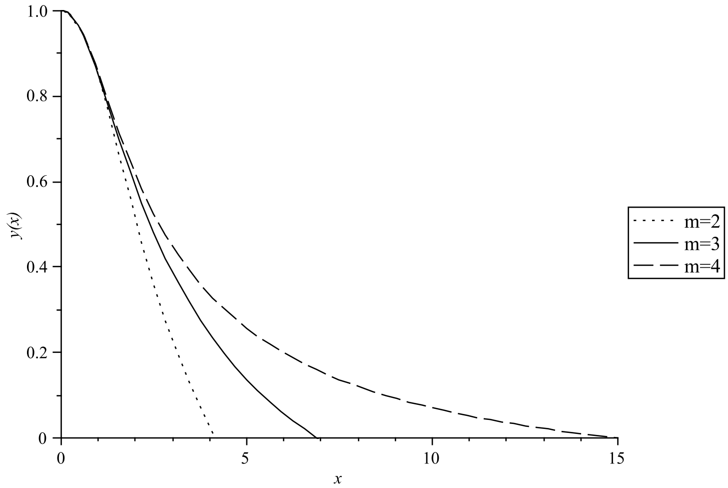

Figure 7.1: Lane-Emden equation graph obtained by Lagrangian method.

Chapter 8

Collocation Method for Solving Lane-Emden Type Equation

8.1 Scaled Laguerre Collocation Method for a Lane-Emden Equation

To apply the scaled Laguerre collocation method to the standard Lane-Emden equation introduced in Eq. (6.3) and Eq. (6.5) with boundary conditions Eq. (6.4). To apply the collocation method we try at first to find the coefficients of the interpolant of $y(x)$:

$$I_N y(x) = \sum_{j=0}^{N} a_j \ell_j(x). \tag{8.1}$$

To find the unknown coefficients a_j, we substitute the truncated series into the Eq. (6.3) with $g(y)$ introduced in Eq. (6.5) and boundary conditions in Eq. (6.4). So we have

$$x \sum_{j=0}^{N} a_j \ell_j''(x) + 2 \sum_{j=0}^{N} a_j \ell_j'(x) + x \left(\sum_{j=0}^{N} a_j \ell_j(x) \right)^m = 0, \tag{8.2}$$

$$\sum_{j=0}^{N} a_j \ell_j(0) = 1, \quad \sum_{j=0}^{N} a_j \ell_j'(0) = 0. \tag{8.3}$$

By replacing x in Eq. (8.2) with the $N-1$ collocation points which are roots of the function $\frac{\mathrm{d}}{\mathrm{d}x} L_N^{\alpha}$, we have $N-1$ equations that generate a set of $N+1$ nonlinear equations with boundary conditions in Eq. (8.3).

For solving these nonlinear equations we can use the function fsolve in Matlab. The internal Matlab solving command of fsolve approximates the solution however requires that the nonlinear equations be defined in vector form. Using the paper [19] the discretization of the functions $\cos([\pi/2]x)$ is used as the starting guess for solving this equation. As the command fsolve approximates the solution only to about 7 decimal places, Newton method is also implemented for this problem in order to increase the number of decimal places.

Because the Lame-Emden equations is nonlinear, another method that can be applied to solve the equations is the Newton-Kantorovich iteration method [64]. The Newton-Kantorovich iteration (also known as quasi-linearization) treats nonlinear terms as perturbation about linear ones in order to approximates the solution of nonlinear differential equation. Unlike perturbation theories however, this method is not rooted in the existence of some type of small parameter. We employed the NewtonKantorovich iteration to reduce the problem to a sequence of linear differential equations. The differential equation was solved by truncating the MGL function series to N terms and then applying the collocation method at N collocation points. The paper [19] is used to find the Jacobian matrix, residual vector and initial guess for solving the linearized equation.

Tables 8.1 and 8.2 represent the approximations of $y(x)$ for standard Lane-Emden with $m = 3, 4$ obtained by the method proposed in this thesis for $N = 7$ and $L = 0.679$, and those obtained by Horedt [54]. These tables show that the current method has an exponential convergence rate. Additionally, high convergence rates and good accuracy are obtained by the proposed method using relatively low numbers of data points.

Table 8.3 represents coefficients of the Laguerre functions of the standard Lane-Emden equations for $m = 2, 3$ and 4 respectively, this table also confirm that this approach has an exponential convergence rate.

Table 8.1: Comparison of $y(x)$ values of standard Lane-Emden equation, for the present method and exact values given by Horedt [54], for m=3

x	Present method	Exact value	Absolute error
0.0	1.00000000	1.0000000	$0.00e+00$
0.1	0.99864783	0.9983358	$1.40e-06$
0.5	0.95987485	0.9598391	$2.99e-06$
1.0	0.85508465	0.8550576	$1.99e-06$
5.0	0.11089375	0.1108198	$3.89e-07$
6.0	0.04379475	0.0437380	$1.12e-06$
6.8	0.00419265	0.0041678	$1.05e-05$
6.9	0.00003603	0.0000360	$9.79e-08$

Table 8.2: Comparison of $y(x)$ values of standard Lane-Emden equation, for the present method and exact values given by Horedt [54], for m=4

x	Present method	Exact value	Error
0.0	1.0000000	1.0000000	$0.00e+00$
0.1	0.9988364	0.9983367	$2.51e-04$
0.2	0.9938264	0.9933862	$2.48e-04$
0.5	0.9600275	0.9603109	$2.05e-04$
1.0	0.8610563	0.8608138	$1.93e-04$
5.0	0.2358837	0.2359227	$8.59e-05$
10.0	0.0596105	0.0596727	$6.22e-05$
14.0	0.0083756	0.0083305	$2.47e-05$
14.9	0.0005784	0.0005764	$4.59e-07$

Table 8.3: Coefficients of the Laguerre functions of the standard Lane-Emden equations for $m = 2, 3$ and 4 respectively

i	a_i			i	a_i		
	$m=2$	$m=3$	$m=4$		$m=2$	$m=3$	$m=4$
0	$-5.28412352348e-01$	$-4.409930495e-01$	$-3.851122457e-01$	13	$-$	$-2.4548768173e-02$	$-$
1	$-2.0672342425e-01$	$-1.5728958674e-01$	$1.1585045625e-01$	14	$-$	$-1.7916420281e-02$	$-$
2	$-2.1013492467e-01$	$-1.7607958676e-01$	$-1.65764836e-01$	15	$-$	$-1.0200227258e-02$	$-$
3	$-1.2898713573e-01$	$-1.1378968348e-01$	$9.28541832865e-03$	16	$-$	$-6.5268030714e-03$	$-$
4	$-1.3634595733e-01$	$-1.2995495868e-01$	$-7.1551456445e-02$	17	$-$	$-2.7962896018e-03$	$-$
5	$-8.7619794765e-02$	$-9.6296843568e-02$	$-9.8809867837e-03$	18	$-$	$-1.5765572392e-03$	$-$
6	$-7.2750404957e-02$	$-9.8373545696e-02$	$-4.8372345689e-02$	19	$-$	$-3.7895054857e-04$	$-$
7	$-3.9156804957e-02$	$-7.9430045686e-02$	$-9.4556005968e-03$	20	$-$	$-2.4542997154e-04$	$-$
8	$-2.6813994854e-02$	$-7.8340445688e-02$	$-2.6810038674e-02$				
9	$-9.5249904867e-03$	$-6.2915945668e-02$	$-6.6016456456e-03$				
10	$-4.1991048674e-03$	$-5.7157720774e-02$	$-1.1910412312e-02$				
11	$-$	$-4.3579589433e-02$	$-1.1402951223e-03$				
12	$-$	$-3.6177724390e-02$	$-3.1134305650e-03$				

8.2 Solving the white-dwarf equation

For solving the white-dwarf equation which is introduced in Eq. (6.3) and Eq. (6.16) with boundary conditions Eq. (6.17), we substitute the truncated series in Eq. (8.1) into the Eq. (6.3) with $g(y)$ introduced in Eq. (6.16) and boundary conditions in Eq. (6.17) to find the unknown coefficients a_j's. So we have

$$x\sum_{j=0}^{N} a_j \ell_j''(x) + 2\sum_{j=0}^{N} a_j \ell_j'(x) + x((\sum_{j=0}^{N} a_j \ell_j(x))^2 - C)^{3/2} = 0, \tag{8.4}$$

$$\sum_{j=0}^{N} a_j \ell_j(0) = 1, \quad \sum_{j=0}^{N} a_j \ell_j'(0) = 0. \tag{8.5}$$

By replacing x in Eq. (8.4) with the $N-1$ collocation points which are roots of the function $\frac{\mathrm{d}}{\mathrm{d}x} L_N^1$, we have $N-1$ equations that generates a set of $N+1$ nonlinear equations with boundary equations in (8.5).

Figures 8.1 shows the resulting graph of white-dwarf.

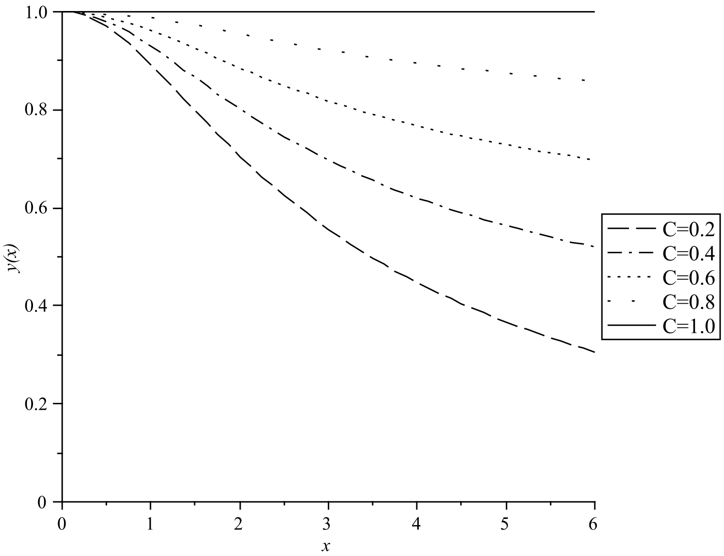

Figure 8.1: white-dwarf equation graph obtained by collocation method.

8.3 Lane-Emden type equations

In this section we apply the MGL collocation method to solve some well-known Lane-Emden type equations.

To apply the collocation method, we have

$$x\sum_{j=0}^{N} a_j\phi_j''(x) + \alpha\sum_{j=0}^{N} a_j\phi_j'(x) + f(x)g((\sum_{j=0}^{N} a_j\phi_j(x))) = 0, \tag{8.6}$$

To find the unknown coefficients a_j's, we substitute the truncated series into boundary conditions in Eq. (6.2). So, we have a set of $N+1$ nonlinear equations which can be solved by Newton method and the starting guess $\cos([\pi/2]x)$ for unknown coefficients a_is. We use Laguerre-Gauss points in the mentioned nonlinear equations.

Figure 8.2 represents standard Lane-Emden equation for $m = 1.5$, 2, 2.5, 3 and 4. Figure 8.3 shows a logarithmic graph of absolute coefficients $|a_i|$ of Laguerre function of standard Lane-Emden for $m = 3$. This figure confirms that this approach has an exponential convergence rate.

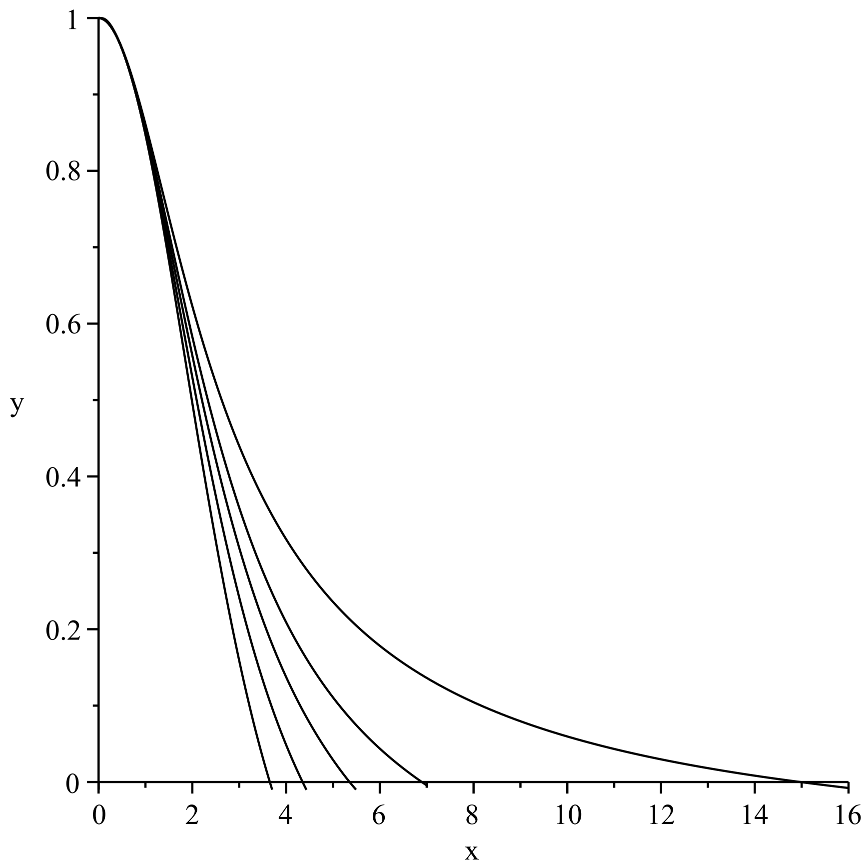

Figure 8.2: Graph of standard Lane-Emden equation for $m = 1.5$, 2, 2.5, 3 and 4

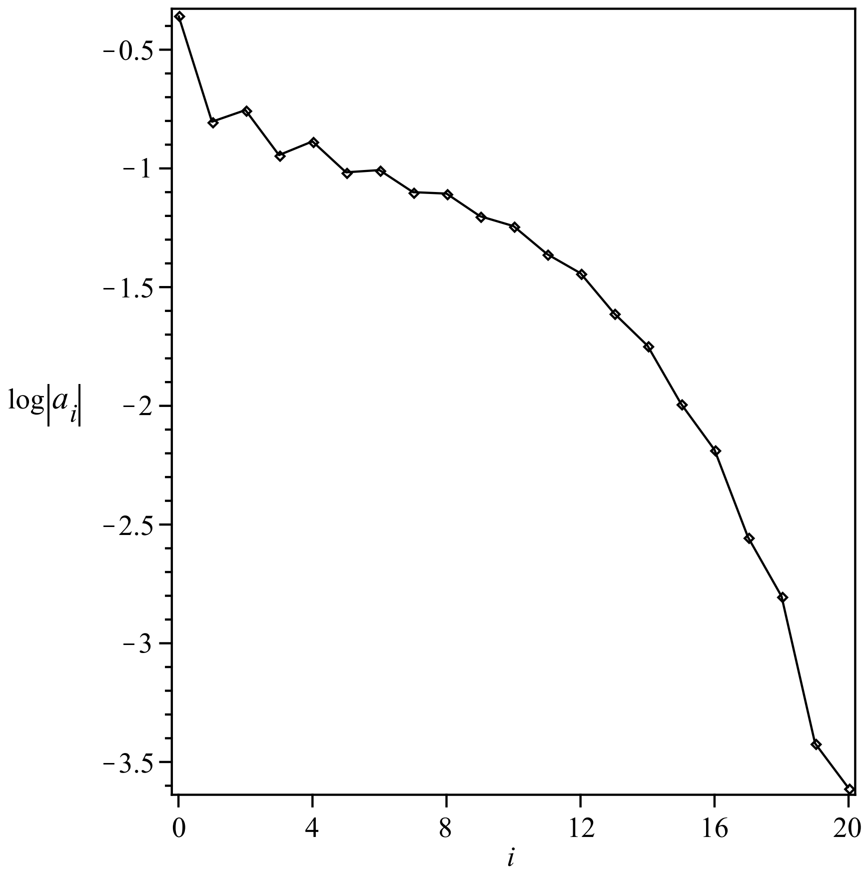

Figure 8.3: Logarithmic graph of absolute coefficients $|a_i|$ of Laguerre function of standard Lane-Emden for $m = 3$

8.3.1 The homogeneous Lane-Emden type equations

Example 1. ***(The isothermal gas spheres equation)***

For $f(x) = 1$, $g(y) = e^y$, $A = 0$ and $B = 0$ Eq. (6.1) is the isothermal gas sphere equation.

$$y''(x) + \frac{2}{x}y'(x) + e^{y(x)} = 0, \quad x \geq 0, \tag{8.7}$$

with the following boundary conditions:

$$y(0) = 0,$$
$$y'(0) = 0.$$

This model can be used to find different parameters such as pressure, mass of stars at a given radius in the isothermal gas spheres. In the isothermal case temperature remains constant because system is in contact with an outside thermal reservoir. For a thorough discussion of the formulation of Eq. (8.7), see [28].
A series solution obtained by Wazwaz [84], Liao [61], Singh et al. [66] and Ramos [76] by using methods like Adomian decomposition method (ADM), homotopy analysis method (HAM), modified homotopy analysis method (MHAM) and series expansion respectively:

$$y(x) \simeq -\frac{1}{6}x^2 + \frac{1}{5.4!}x^4 - \frac{8}{21.6!}x^6 + \frac{122}{81.8!}x^8 - \frac{61.67}{495.10!}x^{10}. \tag{8.8}$$

This type equation has been also solved by [4, 7] with series solutions and ADM methods respectively. We apply the MGL collocation method to solve this equation.
Substitution of MGL functions in Eq. (8.7) will result in:

$$x\sum_{j=0}^{N} a_j\phi_j''(x) + 2\sum_{j=0}^{N} a_j\phi_j'(x) + e^{\sum_{j=0}^{N} a_j\phi_j(x)} = 0, \tag{8.9}$$

and boundary condition:

$$\sum_{j=0}^{N} a_j\phi_j(0) = 0, \quad \sum_{j=0}^{N} a_j\phi_j'(0) = 0. \tag{8.10}$$

Table 8.4 shows the comparison of $y(x)$ obtained by the method proposed in this thesis with ($N = 23$, $L = 2$), and those obtained by Wazwaz [84]. This table shows that the current method has an exponential convergence rate. Additionally, high convergence rate and accurate results are obtained by the proposed method using relatively low numbers of data points.

Table 8.4: Comparison of $y(x)$, between present method and series solution given by Wazwaz [84] for isothermal gas sphere equation

x	Present method	Wazwaz	Absolute error
0.0	0.0000000000	0.0000000000	$0.00e+00$
0.1	-0.0016637935	-0.0016658339	$2.04e-06$
0.2	-0.0066746502	-0.0066533671	$2.12e-05$
0.5	-0.0411836501	-0.0411539568	$2.96e-05$
1.0	-0.1588281737	-0.1588273537	$8.20e-07$
1.5	-0.3380175682	-0.3380131103	$4.45e-06$
2.0	-0.5598283654	-0.5599626601	$1.34e-04$
2.5	-0.8063493756	-0.8100196713	$3.67e-03$

The resulting graph of the isothermal gas spheres equation in comparison to the presented method and those obtained by Wazwaz [84] are shown in Figure 8.4.

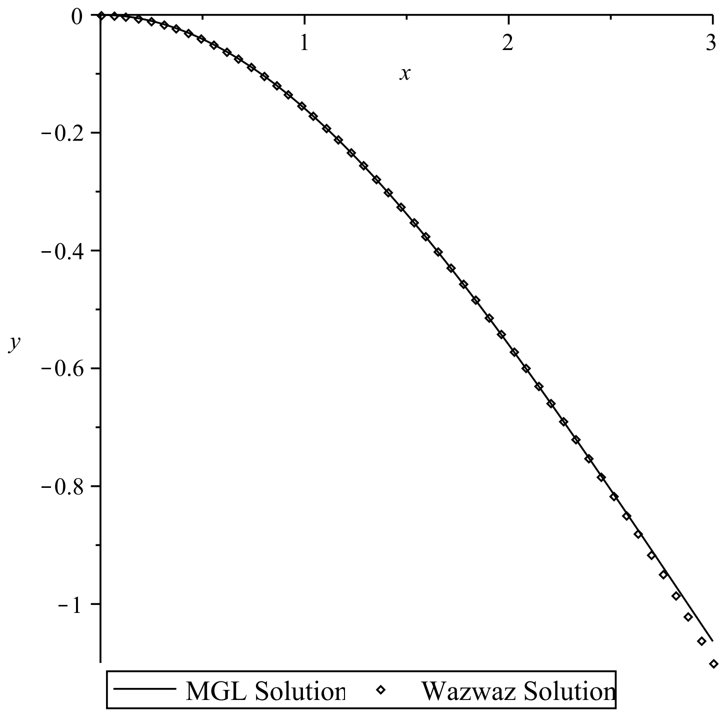

Figure 8.4: Graph of isothermal gas sphere equation in comparison with Wazwaz solution [84]

The logarithmic graph of absolute coefficients of Laguerre functions of standard isothermal gas spheres is shown in Figure 8.5. This graph shows that the method has an exponential convergence rate.

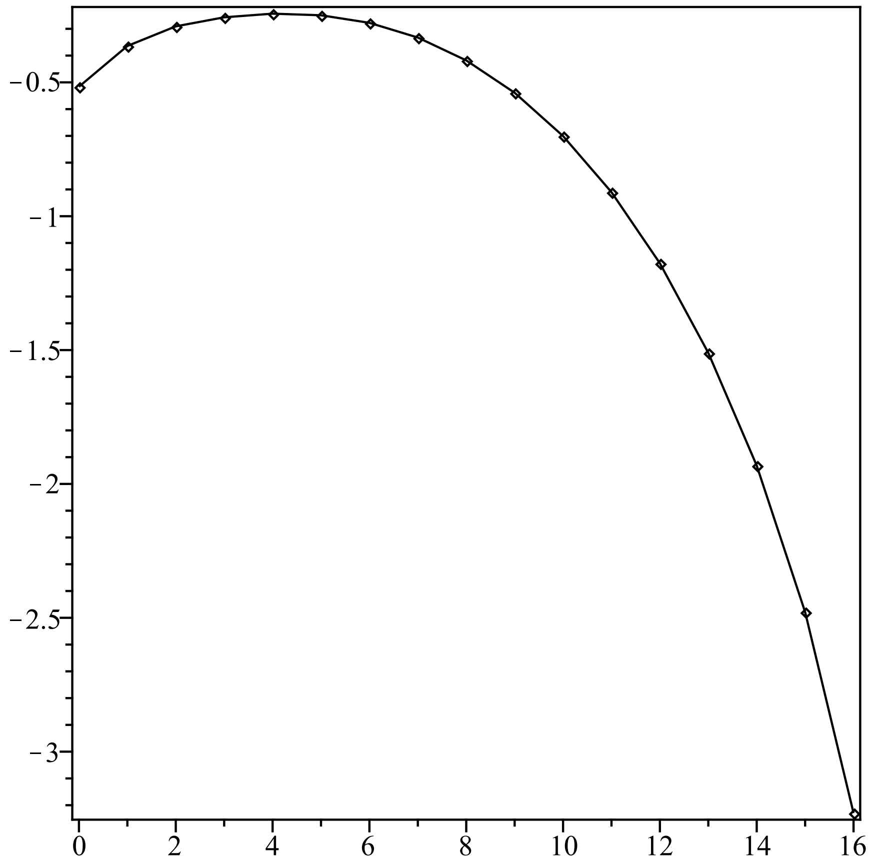

Figure 8.5: Logarithmic graph of absolute coefficients $|a_i|$ of Laguerre function of isothermal gas sphere equation

Example 2.

For $f(x) = 1$, $g(y) = \sinh(y)$, $A = 1$ and $B = 0$, the Eq. (6.1) will be one of the Lane-Emden type equations that is:

$$y''(x) + \frac{2}{x}y'(x) + \sinh(y) = 0, \quad x \geq 0, \tag{8.11}$$

subject to the boundary conditions

$$y(0) = 1,$$
$$y'(0) = 0,$$

A series solution obtained by Wazwaz [84] by using ADM:

$$\begin{aligned} y(x) \quad &\simeq 1 - \frac{(e^2-1)x^2}{12e} + \frac{1}{480}\frac{(e^4-1)x^4}{e^2} - \frac{1}{30240}\frac{(2\,e^6+3\,e^2-3\,e^4-2)x^6}{e^3} \\ &+ \frac{1}{26127360}\frac{(61\,e^8-104\,e^6+104\,e^2-61)x^8}{e^4} \end{aligned}$$

Applying the MGL collocation method to this differential equation (Eq. 8.11) results in following equation.

$$x\sum_{j=0}^{N} a_j\phi_j''(x) + 2\sum_{j=0}^{N} a_j\phi_j'(x) + \sinh(\sum_{j=0}^{N} a_j\phi_j(x)) = 0, \tag{8.12}$$

and boundary conditions:

$$\sum_{j=0}^{N} a_j\phi_j(0) = 1, \quad \sum_{j=0}^{N} a_j\phi_j'(0) = 0. \tag{8.13}$$

Table 8.5 shows the comparison of $y(x)$ obtained by the method proposed in this thesis with ($n = 13$, $L = 1.6$), and those obtained by Wazwaz [84].

Table 8.5: Comparison of $y(x)$, between present method and series solution given by Wazwaz [84] for example number 3

x	Present method	Wazwaz	Error
0.0	1.0000000000	1.0000000000	$0.00e+00$
0.1	0.9981183653	0.9980428414	$7.55e-05$
0.2	0.9922758837	0.9921894348	$8.64e-05$
0.5	0.9520372659	0.9519611019	$7.61e-05$
1.0	0.8183082346	0.8182516669	$5.65e-05$
1.5	0.6254884756	0.6258916077	$4.03e-04$
2.0	0.4066483246	0.4136691039	$7.02e-03$

The resulting graph of the Eq. 8.11 in comparison to the presented method and those obtained by Wazwaz [84] are shown in Figure 8.6.

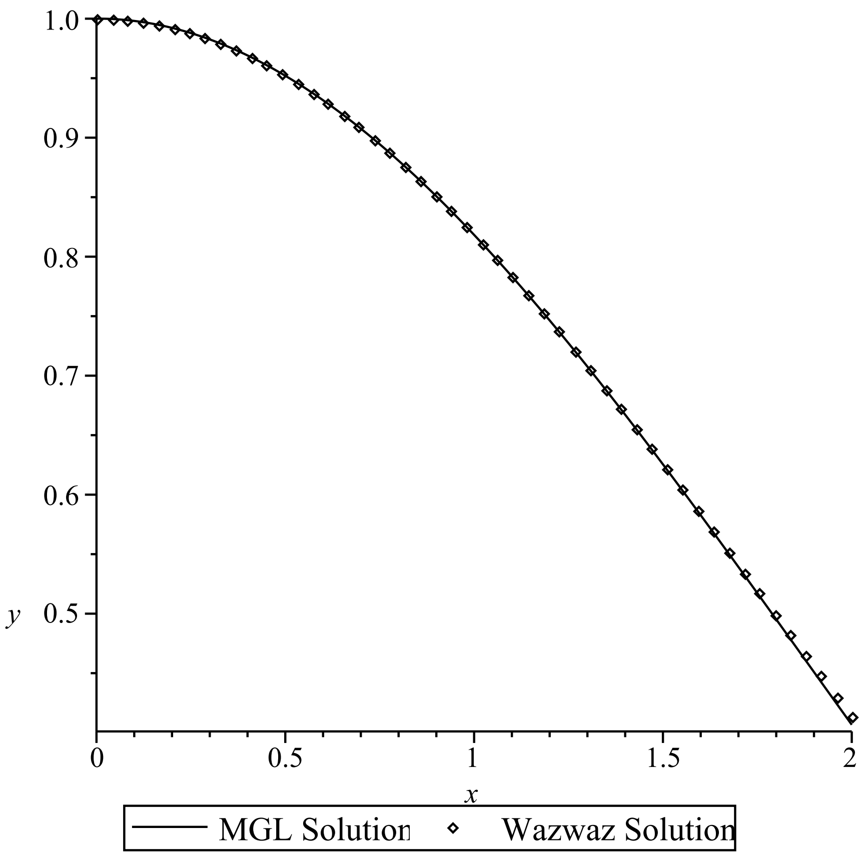

Figure 8.6: Graph of equation example 3 in comparing the presented method and Wazwaz solution [84]

Example 3.

For $f(x) = 1$, $g(y) = \sin(y)$, $A = 1$ and $B = 0$, Eq. (6.1) will be one of the Lane-Emden type equations that is:

$$y''(x) + \frac{2}{x}y'(x) + \sin(y) = 0, \quad x \geq 0, \tag{8.14}$$

subject to the boundary conditions

$$\begin{aligned} y(0) &= 1, \\ y'(0) &= 0, \end{aligned}$$

Wazwaz [84] used ADM method to find the following series solution:

$$\begin{aligned} y(x) \quad &\simeq 1 - \tfrac{1}{6}k_1x^2 + \tfrac{1}{120}k_1k_2x^4 + k_1(\tfrac{1}{3024}k_1^2 - \tfrac{1}{5040}k_2^2)x^6 \\ &\qquad +k_1k_2(-\tfrac{113}{3265920}k_1^2 + \tfrac{1}{362880}k_2^2)x^8 \\ &+k_1(\tfrac{1781}{898128000}k_1^2k_2^2 - \tfrac{1}{399168000}k_2^4 - \tfrac{19}{23950080}k_1^4)x^{10}, \end{aligned}$$

where $k_1 = \sin(1)$ and $k_2 = \cos(1)$.

We apply the MGL collocation method to solve this type equation (Eq. 8.14). Therefore, we construct the following system of equations:

$$x\sum_{j=0}^{N} a_j\phi_j''(x) + 2\sum_{j=0}^{N} a_j\phi_j'(x) + \sin\,(\sum_{j=0}^{N} a_j\phi_j(x)) = 0, \tag{8.15}$$

$$\sum_{j=0}^{N} a_j\phi_j(0) = 1, \quad \sum_{j=0}^{N} a_j\phi_j'(0) = 0. \tag{8.16}$$

By replacing x in Eq. (8.15) with the $N-1$ collocation points which are roots of the function $\frac{\mathrm{d}}{\mathrm{d}x}L_N^1$, we have $N-1$ equations which generate a set of $N+1$ nonlinear equations with a set of boundary equation in (8.16).

Table 8.6 shows the comparison of $y(x)$ obtained by the method demonstrated in this thesis with $(N = 17, L = 1.76)$, and those obtained by Wazwaz [84].

Table 8.6: Comparison of $y(x)$, between present method and series solution given by Wazwaz [84] for example number 4

x	Present method	Wazwaz	Error
0.0	1.0000000000	1.0000000000	$0.00e+00$
0.1	0.9986094857	0.9985979358	$1.15e-05$
0.2	0.9944085662	0.9943962733	$1.00e-05$
0.5	0.9651894865	0.9651777886	$1.16e-05$
1.0	0.8636893856	0.8636811027	$8.28e-06$
1.5	0.7050594856	0.7050419247	$1.75e-05$
2.0	0.5064604957	0.5063720330	$8.84e-05$

The resulting graph of the solution of Eq. (8.14) with the present method in comparison to that obtained by Wazwaz [84] are shown in Figure 8.7.

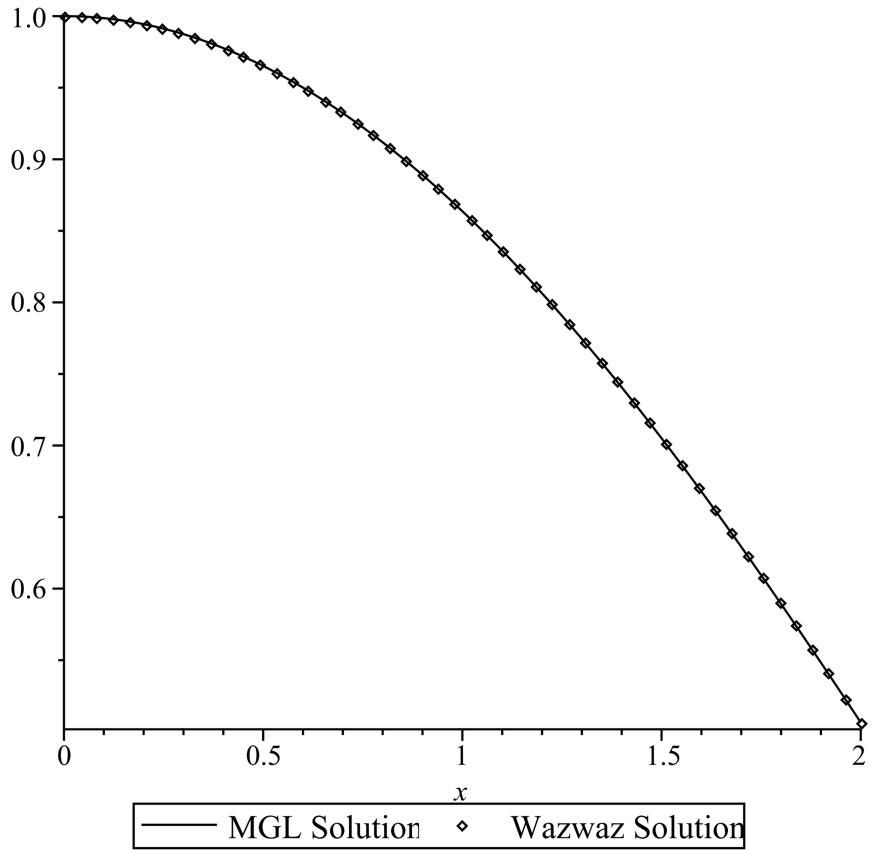

Figure 8.7: Graph of equation example 4 in comparing the presented method and Wazwaz solution [84]

Example 4.

For $f(x) = 1$, $g(y) = 4(2e^y + e^{y/2})$, $A = 0$ and $B = 0$, Eq. (6.1) will be one of the Lane-Emden type equations that is:

$$y''(x) + \frac{2}{x}y'(x) + 4(2e^y + e^{y/2}) = 0, \quad x \geq 0, \tag{8.17}$$

subject to the boundary conditions

$$y(0) = 0,$$
$$y'(0) = 0.$$

The analytical solution of the equation is

$$y(x) = -2ln(1 + x^2) \tag{8.18}$$

This equation has been solved in [24, 87] with homotopy perturbation method (HPM) and variational iteration method (VIM) respectively.

We apply the MGL collocation method to solve this type equation (Eq. 8.17).

$$x\sum_{j=0}^{N} a_j\phi_j''(x) + 2\sum_{j=0}^{N} a_j\phi_j'(x) + 4(2e^{(\sum_{j=0}^{N} a_j\phi_j(x))} + e^{(\sum_{j=0}^{N} a_j\phi_j(x))/2}) = 0, \tag{8.19}$$

and boundary conditions are:

$$\sum_{j=0}^{N} a_j\phi_j(0) = 0, \quad \sum_{j=0}^{N} a_j\phi_j'(0) = 0. \tag{8.20}$$

(8.20).

Table 8.7 shows the comparison of $y(x)$ obtained by the method proposed in this thesis with $(N = 23, L = 2)$, and analytic solution Eq. (8.18).

Table 8.7: Comparison of $y(x)$, between present method and exact solution for example number 5

x	Present method	Exact value	Error
0.00	0.0000000000	0.0000000000	$0.00e+00$
0.01	-0.0001970387	-0.0001999900	$2.95e-06$
0.10	-0.0198960394	-0.0199006617	$3.94e-06$
0.50	-0.4462840293	-0.4462871026	$3.02e-06$
1.00	-1.3862939575	-1.3862943611	$9.31e-07$
2.00	-3.2188760293	-3.2188758249	$5.00e-07$
3.00	-4.6051700394	-4.6051701860	$8.10e-07$
4.00	-5.6664278465	-5.6664266881	$7.69e-07$
5.00	-6.5161930238	-6.5161930760	$6.64e-07$
6.00	-7.2218368377	-7.2218358253	$5.48e-07$
7.00	-7.8240468274	-7.8240460109	$1.70e-07$
8.00	-8.3487738534	-8.3487745398	$1.09e-06$
9.00	-8.813450854	-8.8134384945	$1.21e-05$
10.00	-9.2302035487	-9.2302410337	$3.83e-05$

The resulting is graph of the Eq. (8.17) with the present method in comparison to the analytic solution are shown in Figure 8.8.

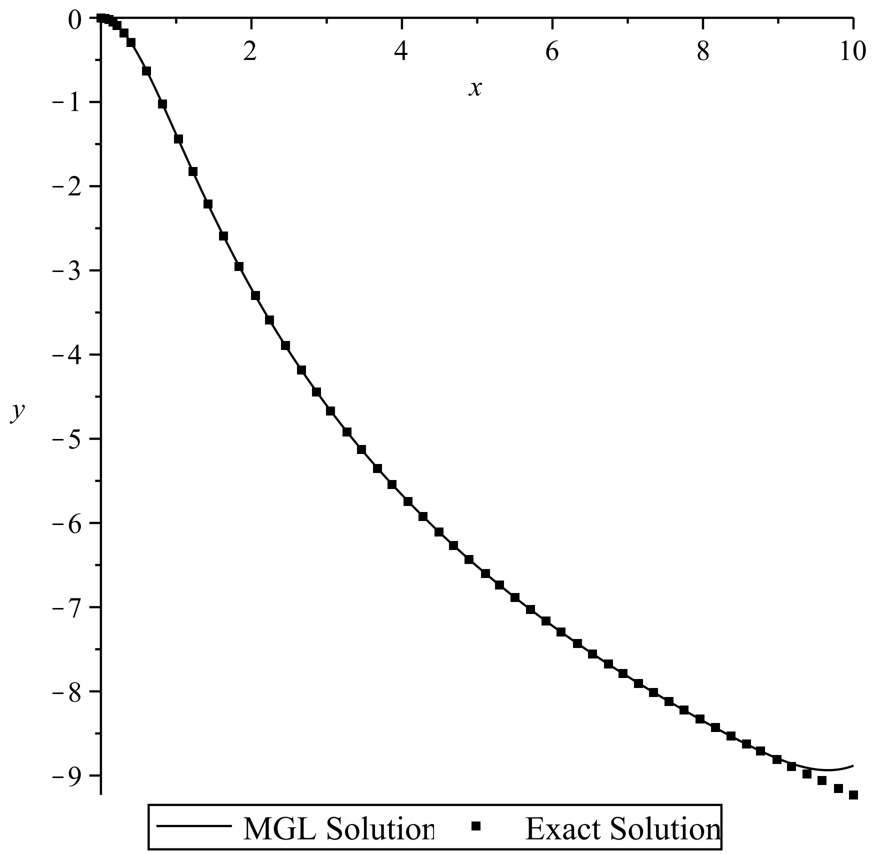

Figure 8.8: Graph of equation example 5 in comparing the presented method and analytic solution

Chapter 9

Tau method for Solving Lane-Emden type Equation

9.1 A solution to the Lane-Emden equation in the theory of stellar structure utilizing the Tau method

9.1.1 Function Approximation

This derivation of background follows an exposition in paper [81]; a paper which the writer contributed to. A function $f(x)$ defined over the interval $I = [0, \infty)$ can be expanded as

$$f(x) = \sum_{i=0}^{\infty} a_i \phi_i(x), \tag{9.1}$$

where

$$a_i = \frac{< f, \phi_i >_w}{< \phi_i, \phi_i >_w}, \tag{9.2}$$

If the infinite series in Eq. (9.1) is truncated after N terms, then it can be written as

$$f(x) \simeq \sum_{i=0}^{N-1} a_i \phi_i(x) = A^T \phi(x), \tag{9.3}$$

with

$$A = [a_0, a_1, a_2, \ldots, a_{N-1}]^T, \tag{9.4}$$

$$\phi(x) = [\phi_0(x), \phi_1(x), \ldots, \phi_{N-1}(x)]^T, \tag{9.5}$$

9.1.2 The derivative operational matrix

The derivative of the vector $\phi(x)$ defined in Eq. (3.9) can be expressed as

$$\phi^{'}(x) = D\phi(x), \tag{9.6}$$

where D is the $N \times N$ operational matrix for derivative. By derivative of (MGL) functions we have following relation:

$$\frac{d}{dx}\phi_n = -\frac{1}{2L}\exp(-x/(2L))L_n^1(x/L) + \exp(-x/(2L))\frac{d}{dx}L_n^1(x/L). \tag{9.7}$$

Using Eqs. (3.4), (9.6) and (9.7) matrix D can be expressed as a lower triangular matrix with $\frac{-1}{2L}$ entries on the main diagonal and entries below the main diagonal are $\frac{-1}{L}$.
For $N = 5$ the matrix D is like below:

$$D = \frac{-1}{L}\begin{pmatrix} 1/2 & 0 & 0 & 0 & 0 \\ 1 & 1/2 & 0 & 0 & 0 \\ 1 & 1 & 1/2 & 0 & 0 \\ 1 & 1 & 1 & 1/2 & 0 \\ 1 & 1 & 1 & 1 & 1/2 \end{pmatrix} \tag{9.8}$$

9.1.3 The product operational matrix

The product of two (MGL) functions vectors defined in Eq. (3.9) can be expressed as

$$\phi(x)\phi(x)^T A \simeq \widetilde{A}^T\phi(x), \tag{9.9}$$

where $\widetilde{A}$ is an $N \times N$ product operational matrix for the vector A. Using Eq. (9.9) and the orthogonal property, the elements $\widetilde{A}_{i,j}, (i = 0, \dots, N-1, j = 0, \dots, N-1)$ of the matrix $\widetilde{A}$ can be calculated from [82]:

$$\widetilde{A}_{i,j} = \left(L^2\frac{n!}{\Gamma(n+2)}\right)\sum_{k=0}^{N-1} a_k g_{ijk}, \tag{9.10}$$

where g_{ijk} is given by

$$g_{ijk} = \int_0^{\infty} \phi_i(x)\phi_j(x)\phi_k(x)w(x)dx. \tag{9.11}$$

9.1.4 Tau method for solving Lane-Emden equation

In this section, we use the MGL functions tau method to solve the Lane-Emden equation. For using MGL functions, at first we multiply both sides of Eq. (6.3) by $e^{-x/2}$ where $g(y)$ is in Eq. (6.5) so:

$$e^{-x/2}xy'' + 2e^{-x/2}y' + xe^{-x/2}y^m = 0, \qquad x > 0, \tag{9.12}$$

we now express all terms in Eq. (9.12) by MGL functions as:

$$e^{-x/2} = \sum_{i=0}^{N-1} E_i\phi_i(x) = E^T\phi(x). \tag{9.13}$$

$$e^{-x/2}x = \sum_{i=0}^{N-1} B_i\phi_0(x) = B^T\phi(x), \tag{9.14}$$

From Eq. (9.3) we can deduce the following relations:

$$y^2(x) = A^T\phi(x)\phi(x)^TA = A^T\widetilde{A}\phi(x), \tag{9.15}$$

$$y^3(x) = A^T\widetilde{A}\phi(x)\phi(x)^TA = A^T\widetilde{A}^2\phi(x), \tag{9.16}$$

and

$$y^m(x) = A^T\widetilde{A}^{m-2}\phi(x)\phi(x)^TA = A^T\widetilde{A}^{m-1}\phi(x), \tag{9.17}$$

$$B = [1, \frac{-1}{2}, 0, \dots, 0]^T$$
$$E = [1, 0, 0, 0, \dots, 0]^T$$

Using Eqs. (9.14)-(9.17) we get

$$e^{-x/2}xy^{''}(x) = A^TD^{(2)}\phi(x)\phi(x)^TB = A^TD^{(2)}\widetilde{B}\phi(x). \tag{9.18}$$

$$e^{-x/2}y^{'}(x) = A^TD\phi(x)\phi(x)^TE = A^TD\widetilde{E}\phi(x). \tag{9.19}$$

$$xe^{-x/2}y^m(x) = A^T\widetilde{A}^{m-1}\widetilde{B}\phi(x), \tag{9.20}$$

where D^2 is the second power of the matrix D given in Eq. (9.8) and using Eq. (9.1). Using Eqs. (9.18)-(9.20) the residual $Res(x)$ for Eq. (9.12) can be written as

$$Res(x) = [A^TD^2\widetilde{B} + 2A^TD\widetilde{E} + A^T\widetilde{A}^{m-1}\widetilde{B}]\phi(x). \tag{9.21}$$

As in a typical Tau method, we generate $N-1$ algebraic equations by applying

$$< Res(x), \phi_i(x) >= 0, \quad i = 0, \ldots, N-2. \tag{9.22}$$

And from Eqs. (6.4) we get

$$y(0) = A^T \phi(0) = 1, \quad y^{'}(0) = A^T D\phi(0) = 0, \tag{9.23}$$

Eq. (9.22) generates a set of algebraic equations and with two Eqs. (9.23) generates a set of $N+1$ nonlinear algebraic equations, which can be solved by the Newton method and starting point $\cos([\pi/2]x)$ for the unknown coefficients a_i, in Eq. (9.1).

The problem is solved with ODE45 function in MATLAB and the source code is shown in the appendix. Table 9.1 shows the approximations of $y(x)$ for standard Lane-Emden with $m = 3$ obtained by Tau method proposed in this thesis for $N = 7$ and $L = 0.679$, and those obtained by Horedt [54] and ODE45 function. This Table shows that the method has high accuracy.

Tables 9.2,9.3 and 9.4 show the comparison of the first zero of y, between Padé approximation used by [10], [75], ODE45 function and the present method for $m = 2, 3, 4$, and $L = 0.676, 0.677, 0.679$ respectively.

Table 9.1: Approximation of $y(x)$ for the Tau method, ODE45 function and solutions of Horedt [54] for $m = 3$

x	Present method	ODE45 function	solutions of Horedt [54]
0.000	1.000000	1.000000	1.000000
0.100	0.998313	0.998325	0.998336
0.500	0.959835	0.959833	0.959839
1.000	0.855084	0.855016	0.855058
5.000	0.110820	0.110937	0.110820
6.000	0.043708	0.043890	0.043738
6.800	0.004155	0.004334	0.004168
6.896	0.000026	0.000026	0.000036

Table 9.2: Comparison of the first zero of y, between Padé approximation used by [10], [75], ODE45 function and the Tau method for $m = 2$.

N	Present method	Bender [10]	Ramos [75]	ODE45 function	Exact value
11	4.33112	4.3603	4.35086	4.3481	4.35287460
14	4.32874				

Table 9.3: Comparison of the first zero of y, between Padé approximation used by [10], [75], ODE45 function and the Tau method for $m = 3$.

N	Present method	Bender [10]	Ramos [75]	ODE45 function	Exact value
11	6.89851	7.0521	6.89312	6.85331	6.89684862
14	6.89848				

Table 9.4: Comparison of the first zero of y, between Padé approximation used by [10], [75], ODE45 function and the Tau method for $m = 4$.

N	Present method	Bender [10]	Ramos [75]	ODE45 function	Exact value
11	14.97113	17.967	14.96518	14.91456	14.9715463
14	14.97154				

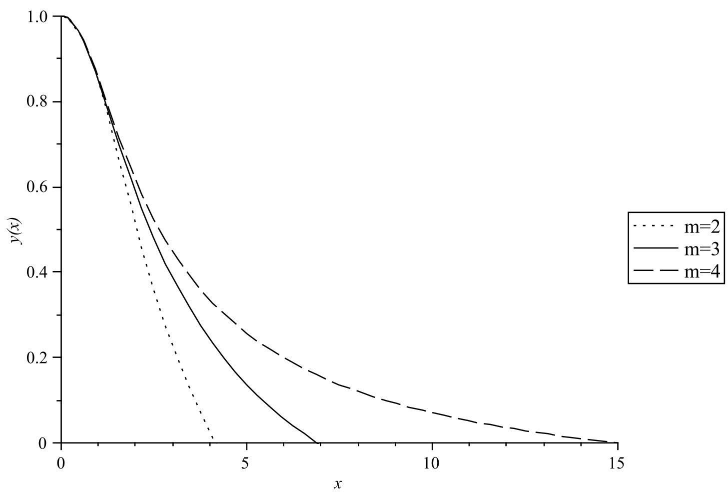

Figure 9.1: Lane-Emden equation graph obtained by the Tau method.

Figure 9.1 shows the resulting graph of Lane-Emden for $N = 14$, $m = 2, 3, 4$ and $L = 0.679$.

Chapter 10

Conclusions and Future Work

The fundamental goal of this thesis is to construct an approximation to the solution of the nonlinear Lane-Emden equation which is singular at $x = 0$ in a semi-infinite interval. A set of orthogonal polynomials and functions are proposed to provide an effective but simple way to improve the convergence of the solution using different spectral methods. Through the comparisons among the solutions of Horedt, the approximate solutions of Bender and the current work, more accurate solutions for Lane-Emden equations are provided.
A number of open problems can be solved using the methods in this thesis in semi-infinite domain; moreover, quasi-linearisation can provide a linear form of the differential equations and utilizing more collocation points will be possible. Quasi-linearisation, map parameter, and scaled (or modified) basis functions, can be utilized for obtaining more accurate results for solving some nonlinear differential equation like Blasius equation. The Blasius equation is a third order nonlinear ordinary differential equation, which arises in the problem of the two-dimensional laminar viscous flow over a semi-infinite flat plane.

The branch of physics that deals with dynamics of electrically conducting fluids is called Magnetohydrodynamics (MHD). Dynamo theory is the branch of astrophysics that involves the study of generation and maintenance of magnetic fields. The physical mechanism of regenerating the earth's magnetic field, by virtue of endemic fluid motion, is referred to as geodynamo. The Earth's magnetic field is known as geomagnetic. The source of geomagnetic originates from electrically conducting liquid iron which include the outer core layer of the earth [72]. According to some research, the changes in the geomagnetic are a result of the departures of steady state outer core fluid motions, which are due to perturbations of geodynamo and changes in boundary conditions of the outer core fluid motions. The only external case that can lead to a change in the boundary conditions of fluid motions is a change in the Earth's moment of inertia. One of the phenomena that could lead to a change in the Earth's moment of inertia, would be a change in the global sea level. According to the conservation of momentum, changes in the momentum of the earth will result in variations in the rotation rate of the mantle. These variations will lead to an azimuthal velocity shear at the top of the fluid core. One of the challenging question in magnetohydrodynamics is whether significant changes in the Earth's moment of inertia, originally at the surface and spins-up the mantle layer of the earth, could result in a magnetic reversal of the earth's field [72].

The change of an astrophysical-scale magnetic field B embedded in an electrically conducting fluid with velocity v and electrical conductivity σ is described by the following differential equation that usually is called the hydromagnetics induction equation [71]:

$$\frac{\partial B}{\partial t} = \nabla \times (v \times B) + \nabla \times (\eta \nabla \times B), \tag{10.1}$$

where $\eta = \frac{1}{\sigma \mu_0}$
is the magnetic diffusivity. If $\eta = constant$, then the induction relationship can be described as:

$$\frac{\partial B}{\partial t} = \nabla \times (v \times B) + \eta \nabla^2 B. \tag{10.2}$$

And if the $\sigma \Rightarrow \infty$ then there is no sink or diffusion term.

$$\frac{\partial B}{\partial t} = \nabla \times (v \times B). \tag{10.3}$$

Both velocity and magnetic fields are solenoidal so

$$B = \nabla \times [A(r, \Theta) e_\Phi] + B(r, \Theta) e_\Phi. \tag{10.4}$$

$$A = \nabla \times [\Psi(r, \Theta) e_\Phi] + v(r, \Theta) e_\Phi. \tag{10.5}$$

Given assumptions like

- Asymmetric convective motions
- Incompressibility $\nabla . v = 0$

- $\eta = constant$

Then Eq. (10.2) is expanded in component form as

$$\frac{\partial A^{(i)}}{\partial t} = \frac{\eta_i}{\eta_o}\nabla^2_\phi A^{(i)}. \tag{10.6}$$

$$\frac{\partial B^{(i)}}{\partial t} = \frac{\eta_i}{\eta_o}\nabla^2_\phi B^{(i)}. \tag{10.7}$$

$$\frac{\partial A^{(o)}}{\partial t} = \nabla^2_\phi A^{(o)} + \alpha B^{(o)} + \aleph_2(\psi, A^{(o)}). \tag{10.8}$$

$$\frac{\partial B^{(o)}}{\partial t} = \nabla^2_\phi B^{(o)} + \aleph_1(v, A^{(o)}) - \aleph_1(B^{(o)}, \psi) + \tag{10.9}$$

$$\alpha e_\Phi.\nabla \times [\xi \nabla \times (A^{(o)} e_\Phi)], \tag{10.10}$$

where

$$\nabla_2^\Phi \equiv (\nabla^2 - \frac{1}{r^2 sin^2\Theta}). \tag{10.11}$$

The Φ-component of the spherical asymmetric vector diffusion operator and field quantities are expressed as:

$$\aleph_1(X_1, X_2) = e_\Phi.\nabla \times [\chi_1 e_\Phi \times \nabla \times (\chi_2 e_\Phi)]. \tag{10.12}$$

$$\aleph_2(X_1, X_2) = e_\Phi.[\nabla \times (\chi_1 e_\Phi) \times \nabla \times (\chi_2 e_\Phi)], \tag{10.13}$$

where the superscripts (i) and (o) are inner and outer core quantities, respectively. There are two boundary conditions at the core-mantle interfaces. The boundary conditions are because of continuity and the conservation of magnetic fields.

$$B^{(o)}(R_o, \theta, t) = B^{(m)}(R_o, \theta, t), \quad \eta_{(o)}\frac{1}{r}\frac{\partial r B^{(o)}}{\partial r}|_{r=R_o^-} = \eta_{(m)}\frac{1}{r}\frac{\partial r B^{(m)}}{\partial r}|_{r=R_o^+}. \tag{10.14}$$

Moreover, we can add the following boundary conditions:

$$B^{(o)}(0, \theta) = B^{(m)}(R_m, \theta) = 0 \tag{10.15}$$

The goal of future work will be to solve the MHD differential equation using the basis function and the spectral methods used in this thesis while working under the assumption that a simple azimuthal core fluid is modeled as $v = r\Omega(r)sin(\theta)$. The results will be used to investigate whether significant changes in the Earth's moment of inertia could result in a magnetic reversal of the earth's field.

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