Numerical solution for the Falkner-Skan equation using Chebyshev cardinal functions

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Acta Universitatis Apulensis No. 27/2011
ISSN: 1582-5329 pp. 229-238

NUMERICAL SOLUTION FOR THE FALKNER-SKAN EQUATION USING CHEBYSHEV CARDINAL FUNCTIONS

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Abstract. A numerical technique is presented for the solution of Falkner-Skan equation. The nonlinear ordinary differential equation is solved using Chebyshev cardinal functions. The method have been derived by first truncating the semi-infinite physical domain of the problem to a finite domain and expanding the required approximate solution as the elements of Chebyshev cardinal functions. Using the operational matrix of derivative, we reduce the problem to a set of algebraic equations. From the computational viewpoint, the solutions obtained by this method is in excellent agreement with those obtained by previous works and efficient to use.

2000 *Mathematics Subject Classification*: 65D10; 65N35; 74G15.

1. Introduction

Ordinary differential equations are important tools in solving real-world problems. A wide variety of natural phenomena are modelled by ordinary differential equations. Ordinary differential equations have been applied to many problems in physics, engineering, biology and so on. These equations have received much attention in last 20 years [12-19]

The Falkner-Skan equation arises in the study of laminar boundary layers exhibiting similarity. The solutions of the one-dimensional third-order boundary-value problem described by the well-known Falkner-Skan equation are the similarity solutions of the two dimensional incompressible laminar boundary layer equations. This is a nonlinear two-point boundary value problem for which no closed-form solutions are available.The problem is given by

$$f'''(\eta) + f(\eta)f''(\eta) + \beta(1 - f'(\eta)^2) = 0, \quad 0 \le \eta < \infty, \tag{1}$$

along with the boundary conditions

$$f(0) = 0, \quad f'(0) = 0, \quad \text{and} \quad f'(\infty) = 1, \tag{2}$$

where, in addition to the unknown function $f(\eta)$, the solution of (1.1) and (1.2) is characterized by the value of $\alpha = f''(0)$.

The mathematical treatments of this problem due to Rosenhead [1] and Weyl [2] have focused on obtaining existence and uniqueness results. Hartree [3] presented the first numerical treatment of the problem. Other numerical treatments were made available later by Smith [4], Cebeci and Keller [5], Na [6], Asaithambi [7, 8, 9] and Elgazery [10].

The outline of this paper is as follows. In Section 2, we describe Chebyshev cardinal functions and its properties and construct its operational matrix of derivative. In Section 3 the proposed method is used to approximate the solution of a model problem. As a result a set of algebraic equations are formed and a solution of the considered problem is introduced. In Section 4 the proposed method is used to approximate the solution of the Falkner-Skan equation. In Section 5, the numerical results of applying the method of this article on the model problem and Falkner-Skan equation (1) are presented. Finally a conclusion is drawn in Section 6.

2.Chebyshev Cardinal Functions

Chebyshev cardinal functions of order N in $[-1, 1]$ are defined as [11]:

$$C_j(t) = \frac{T_{N+1}(t)}{T_{N+1,t}(t_j)(t - t_j)}, \qquad j = 1, 2, ..., N + 1, \tag{3}$$

where $T_{N+1}(t)$ is second kind Chebyshev function of order $N + 1$ in $[-1, 1]$, defined by

$$T_{N+1}(t) = cos((N + 1)arccos(t)),$$

subscript t denote t-differentiation and $t_j, j = 1, 2, ..., N + 1$ are the zeros of $T_{N+1}(t)$ defined by $cos((2j - 1)/(2N + 2)), j = 1, 2, ..., N + 1$.

We change the variable $x = (t + 1)L/2$ to use these functions on $[0, L]$. Now any function $g(x)$ on $[0, L]$ can be approximated as

$$g(x) \approx \sum_{j=1}^{N+1} g(x_j)C_j(x) = G^T \Phi_N(x), \tag{4}$$

where $x_j, j = 1, 2, ..., N+1$ are the shifted points of $t_j, j = 1, 2, ..., N+1$ by transform $x = (t + 1)L/2$,

$$G = [g(x_1), g(x_2), ..., g(x_{N+1})]^T, \tag{5}$$

and

$$\Phi_N(x) = [C_1(x), C_2(x), ..., C_{N+1}(x)]^T. \tag{6}$$

2.1 The Operational Matrix of Derivative

The differentiation of vectors Φ_N in (6) can be expressed as

$$\Phi'_N = D\Phi_N, \tag{7}$$

where D is $(N+1)\times(N+1)$ operational matrix of derivative for Chebyshev cardinal functions.

The matrix D can be obtained by the following process. Let

$$\Phi'_N(x) = [C'_1(x), C'_2(x), ..., C'_{N+1}(x)]^T.$$

Using Eq.(4), any function $C'_j(x)$ can be approximated as

$$C'_j(x) = \sum_{k=1}^{N+1} C'_j(x_k)C_k(x). \tag{8}$$

Comparing Eqs.(7) and (8) we get

$$D = \begin{bmatrix} C'_1(x_1) & \dots & C'_1(x_{N+1}) \\ \vdots & & \vdots \\ C'_{N+1}(x_1) & \dots & C'_{N+1}(x_{N+1}) \end{bmatrix} \tag{9}$$

To calculate the entries $C'_j(x_k), j, k = 1, 2, ..., N+1$, we have

$$\frac{T_{N+1}(x)}{x - x_j} = \alpha \times \prod_{\substack{k=1 \\ k \neq j}}^{N+1} (x - x_k) \tag{10}$$

where $\alpha = 2^{2N+1}/L^{N+1}$ is the coefficient of x^{N+1} in the shifted Chebyshev polynomial function $T_{N+1}(x)$.

Using Eq. (10) we get

$$\frac{d}{dx}\left(\frac{T_{N+1}(x)}{x - x_j}\right) =$$

$$\alpha \times \sum_{\substack{i=1 \\ i \neq j}}^{N+1} \prod_{\substack{k=1 \\ k \neq i,j}}^{N+1} (x - x_k) = \sum_{\substack{i=1 \\ i \neq j}}^{N+1} \frac{T_{N+1}(x)}{(x - x_j)(x - x_i)}, \tag{11}$$

so we have

$$C_j'(x) = \frac{1}{T_{N+1,x}(x_j)} \times \frac{d}{dx}\left(\frac{T_{N+1}(x)}{x-x_j}\right) =$$

$$\frac{1}{T_{N+1,x}(x_j)} \sum_{\substack{i=1\\ i\neq j}}^{N+1} \frac{T_{N+1}(x)}{(x-x_j)(x-x_i)} = \sum_{\substack{i=1\\ i\neq j}}^{N+1} \frac{1}{x-x_i} C_j(x). \tag{12}$$

For $j = k$ using Eq. (12) we get

$$C_j'(x_k) = \sum_{\substack{i=1\\ i\neq j}}^{N+1} \frac{1}{x_j - x_i}, \tag{13}$$

and for $j \neq k$ using Eq.(12) we have

$$C_j'(x_k) = \frac{\alpha}{T_{N+1,x}(x_j)} \prod_{\substack{\ell=1\\ \ell\neq k,j}}^{N+1} (x_k - x_\ell). \tag{14}$$

So the entries of the matrix D can be found using Eqs. (13) and (14).

3.Description of numerical method for a model problem

Consider the boundary layer equation:

$$f'''(\eta) + f(\eta)f''(\eta) - f'(\eta)^2 - \left(M + \frac{1}{k_p}\right) f'(\eta) = 0, \quad 0 \leq \eta < \infty. \tag{15}$$

With the boundary conditions:

$$f(0) = f_w, \quad f'(0) = 1, \quad \text{and} \quad f'(\infty) = 0. \tag{16}$$

This boundary layer equation is presented for magnetohydrodynamic (MHD) flow of a viscous, incompressible and electrically conducting fluid with suction and blowing through a porous medium where M is the magnetic parameter, k_p is the permeability parameter and f_w is the mass transfer parameter, which is positive for suction and negative for injection.

Fortunately, the boundary value problem (15) together the boundary conditions (16) has an exact solution in the form:

$$f^*(\eta) = \frac{1}{z}\left(2 - 2e^{-\frac{1}{2}\eta z} + f_w z\right), \quad z = f_w + \sqrt{4 + 4\left(M + \frac{1}{k_p}\right) + f_w^2}. \tag{17}$$

To solve this problem we first truncate the semi-infinite physical domain $[0, \infty)$ of the problem to a finite domain $[0, L]$, where L is an enough large number.

Now we use Eq. (4) to approximate the function $f(\eta)$ as

$$f(\eta) = F^T \Phi_N(\eta), \tag{18}$$

where F is a $(N+1)$ unknown vector as

$$F = [f_1, f_2, \dots, f_{N+1}]^T,$$

and should be found.

By using Eq. (7) we can write

$$f'(\eta) = F^T \Phi'_N(\eta) = F^T D\Phi_N(\eta), \tag{19}$$

$$f''(\eta) = F^T D\Phi'_N(\eta) = F^T D^2\Phi_N(\eta), \tag{20}$$

and

$$f'''(\eta) = F^T D^3\Phi_N(\eta). \tag{21}$$

Using Eqs.(18)-(21) in Eq.(15) we get

$$\begin{aligned} F^T D^3\Phi_N(\eta) + \left(F^T\Phi_N(\eta)\right)\left(F^T D^2\Phi_N(\eta)\right) - \left(F^T D\Phi_N(\eta)\right)^2 \\ - \left(M + \tfrac{1}{k_p}\right) F^T D\Phi_N(\eta) = 0. \end{aligned} \tag{22}$$

By collocating Eq. (22) in $N-2$ points $\tau_j = (t_j + 1)L/2, j = 3, 4, \dots, N$, we get

$$\begin{aligned} R(\tau_j) = F^T D^3\Phi_N(\tau_j) + \left(F^T\Phi_N(\tau_j)\right)\left(F^T D^2\Phi_N(\tau_j)\right) - \left(F^T D\Phi_N(\tau_j)\right)^2 \\ - \left(M + \tfrac{1}{k_p}\right) F^T D\Phi_N(\tau_j) = 0, \\ j = 3, 4, \dots, N, \end{aligned} \tag{23}$$

where $t_j, j = 1, 2, ..., N+1$ are the zeros of $P_{N+1}(t)$.

Because of the property of cardinal functions, we have

$$\Phi_N(\tau_j) = e_j, \quad j = 3, 4, ..., N, \tag{24}$$

where e_j is the jth column of $(N+1)\times(N+1)$ identity matrix I. Using (24) in (23) we get

$$R(\tau_j) = [F^T D^3]_j + f_j[F^T D^2]_j - ([F^T D]_j)^2 - \left(M + \frac{1}{k_p}\right)[F^T D]_j = 0, \qquad j = 3, 4, \ldots, N, \tag{25}$$

where $[V]_j$ is the jth entry of row vector V.

Using Eqs. (18) and (19) in Eq. (16) we get

$$\begin{aligned} &F^T \Phi_N(0) = f_w, \\ &F^T D\Phi_N(0) = 1, \\ &F^T D\Phi_N(L) = 0. \end{aligned} \tag{26}$$

Eq. (25) together Eq. (26) gives a $(N+1)$ system of nonlinear equation, which can be solve for $f_k, k = 1, 2, \ldots, N+1$, using Newton's iterative method. So the unknown function of $f(\eta)$ can be found.

4.Applying numerical method for the FalknerSkan equation

The purpose of the present section is solving the FalknerSkan equation (1) and (2) numerically by using Chebyshev cardinal functions. Again to solve this problem we first truncate the semi-infinite physical domain $[0,\infty)$ of the problem to a finite domain $[0, L]$, where L is an enough large number. Now using Eqs.(18)-(21) in Eq. (1). we get

$$F^T D^3\Phi_N(\eta) + \left(F^T\Phi_N(\eta)\right)\left(F^T D^2\Phi_N(\eta)\right) + \beta\left(1 - \left(F^T D\Phi_N(\eta)\right)^2\right) = 0. \tag{27}$$

Collocating Eq. (27) in $N-2$ points $\tau_j, j = 3, 4, \ldots, N$, and using property of cardinal functions, we get

$$[F^T D^3]_j + f_j[F^T D^2]_j + \beta\left(1 - ([F^T D]_j)^2\right) = 0, \quad j = 3, 4, \ldots, N. \tag{28}$$

Also using Eqs.(18) and (19) in Eq.(2) we have

$$\begin{aligned} &F^T \Phi_N(0) = 0, \\ &F^T D\Phi_N(0) = 0, \\ &F^T D\Phi_N(L) = 1. \end{aligned} \tag{29}$$

Eq. (28) together Eq. (29) gives a $(N+1)$ system of nonlinear equation, which can be solve for $f_k, k = 1, 2, \ldots, N+1$, using Newton's iterative method. So the unknown function of $f(\eta)$ can be found.

3. Numerical Examples

Example 1. Consider the equation (15) with boundary conditions (16). Table 1 shows the exact and approximate solutions of $f'(\eta)$ at $\eta = 1$, using the method presented in section 3 with $N = 24$ and $L = 12$, for different values of f_w, M and K_p and compare the result with the Adomian decomposition method (ADM) presented in [10].

Table 1. Values of $f'(\eta)$ at $\eta = 1$

f_w	M	k_p	Exact solution	Presented method	ADM [10]
0.1	0.5	5	0.257999189	0.257999189	0.257999126
0.4			0.218910874	0.218910874	0.218910846
0.7			0.182683524	0.182683521	0.182683514
0.1	1.0		0.215653525	0.215653524	0.215653523
	1.5		0.183796112	0.183796110	0.183796111
	0.5	1	0.195551950	0.195551949	0.195551950
		1.5	0.218098363	0.218098363	0.218098361
		2	0.231055538	0.231055538	0.231055530

Example 2. Consider the equation (1) with boundary conditions (2). Solutions of the FalknerSkan equation for various values of β have been reported in the literature. For $\beta > 0$, the solutions obtained are called accelerating flows; the solutions corresponding to $\beta = 0$ are called constant flows; and, those corresponding to $\beta < 0$ are known as decelerating flows. Physically relevant solutions exist only for $-0.19884 < \beta \leq 2$.

In order to compare the results obtained by the present method with those of previously reported methods, the quantity $\alpha = d^2f/d\eta^2$ for $\eta = 0$ has been computed and tabulated.

Table 2 shows the approximate solutions of α, using the method presented in section 4 with $N = 24$ and $L = 6$, for different values of β, and compare the result with the previously reported methods [7–10]. Table 3 shows the CPU times to run the Maple program for different values of β.

Table 2. Comparison of computed α for $-0.1988 \leq \beta \leq 2$

β	present method	method [7,8,9]	ADM [10]
2.0000	1.687218	1.687222	1.683050
1.0000	1.232588	1.232589	1.196315
0.5000	0.927680	0.927682	0.807826
0.0000	0.469601	0.469601	0.491960
-0.1000	0.319272	0.319270	0.320189
-0.1200	0.281765	0.281762	0.278509
-0.1500	0.216376	0.216360	0.224771
-0.1800	0.128682	0.128637	0.129001
-0.1988	0.005270	0.005217	0.005227

Table 3. The CPU times to run the Maple program

β	CPU times (seconds)
2.0000	31.97
1.0000	32.09
0.5000	31.41
0.0000	24.13
-0.1000	31.56
-0.1200	31.95
-0.1500	31.36
-0.1800	33.06
-0.1988	60.02

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