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FURTHER NUMERICAL METHODS FOR THE FALKNER–SKAN EQUATIONS: SHOOTING AND CONTINUATION TECHNIQUES

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SUMMARY

We consider in this paper the numerical solution of the Falkner–Skan differential equation, modelling under some similarity assumptions the boundary layer equation. We look for the extremal solution of this third order differential equation. The methods we use are basically the Newton method with a shooting process, which is coupled with a continuation method: they allow us to follow the solution arcs which contain regular and turning point solutions.

KEY WORDS Falkner Skan Shooting Method Continuation Technique

1. INTRODUCTION

The Falkner–Skan equation is obtained from the dimensionless Prandtl's equations, in which is introduced a similarity assumption. It consists of a non-linear third order differential equation

$$\left.\begin{array}{l} f''' + ff'' + \beta(1-f'^2) = 0 \\ f(0) = f'(0) = 0 \\ f'(\infty) = 1 \end{array}\right\}$$

We study numerically the solutions of this equation, satisfying the initial condition:

$$f''(0) = \alpha$$

The solutions can be then characterized by their positions in the plane (α, β).

In section 2, a Newton method associated with a shooting process is used to find the regular solutions.

The introduction of the continuation equation, which consists of an arc-length constraint, allows us to treat simple turning points as regular solutions of the new problem. It is then quite easy to follow the arcs of solution containing regular and turning point solutions.

Section 4 presents numerical experiments.

1.1. The Falkner–Skan equations—theoretical results

In this section, we present the governing boundary layer equations, in the physical (x, y) plane. They correspond to a first order approximation of the incompressible Navier–Stokes equations in two dimensions. More precisely, the dimensionless Prandtl's boundary layer

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equations can be written as follows:

$$\left.\begin{aligned} &u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{\partial p}{\partial x}+\frac{\partial^2 u}{\partial x^2}, \qquad \text{in } \Omega \\ &\frac{\partial p}{\partial y}=0, \qquad \text{in } \Omega=\{(x, y);\ x>0,\ y>0\} \end{aligned}\right\} \tag{1}$$

Here p is the pressure, and (u, v) is the velocity field (referred to the external velocity $u_\infty(x)$) which satisfies the continuity equation

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{2}$$

The boundary conditions are given by natural conditions at the wall, i.e. for $y=0$, and by matching with the external flow:

$$\left.\begin{aligned} &u(x, 0)=0 \\ &v(x, 0)=v_{\text{wall}}(x) \\ &u(x, \infty)=u_\infty(x) \end{aligned}\right\} \tag{3}$$

An initial condition is given by:

$$u(0, y)=\psi(y) \tag{4}$$

In the special case of incompressible flow, a classical similarity assumption[1] applied to the set of Prandtl equations leads to

$$f'''+ff''+\beta(1-f'^2)=0 \tag{5}$$

in which the function f' of the new similarity variable $\eta\in[0, +\infty)$ constitutes a dimensionless form of the longitudinal velocity component, referred to the external velocity u_∞. The function $f(\eta)$ is proportional to the local boundary layer thickness. The parameter β plays a fundamental role. In a mathematical sense, it may take any real value, and can be considered as a 'bifurcation' parameter. Note that the solution in the special case $\beta=0$ is called the Blasius solution.

Finally, the boundary conditions associated with (5) are

$$\begin{aligned} &f(0)=f'(0)=1 \\ &f'(+\infty)=1 \end{aligned} \tag{6}$$

1.2. *Basic results*

Equations (5) and (6) were first introduced by Falkner and Skan in 1931.[2] One of the earliest studies, due to Hartree in 1937[3] gave the additional condition:

$$\forall t\geq 0, \qquad 0\leq f'(t)\leq 1$$

in order to preserve a physical meaning.

First, it is convenient to define precisely the different types of solutions of the Falkner–Skan equations which have appeared in the previous works (see the discussion in Reference 4).

1.2.1. *Definitions*

Definition 1

A *classical solution* of (5), (6) is one for which $f'(t)>0$ for $t>0$.

Definition 2

A *reverse flow solution* of (5), (6) is one for which there exists a $\tau > 0$ such that $f'(\tau) < 0$.

Definition 3

An *overshoot solution* of (5), (6) is a reverse flow solution for which there exists a τ such that $|f'(\tau)| > 1$.

1.2.2. *Solutions without overshoot.* There are two cases to consider according to the sign of β, i.e. if there is an adverse pressure gradient or not.

1.2.2.1. The case $\beta \geq 0$

Weyl[5] established the existence of a classical solution for β fixed (see Reference 6 for details). Global uniqueness holds only for $0 \leq \beta \leq 1$ in which case $f'(t) > 0$ whenever $t > 0$.[7,8] For $\beta = 0$, the Blasius solution (without pressure gradient) corresponds to the unique value of α such that $\alpha \simeq 0.49$, where $\alpha = f''(0)$.

1.2.2.2. The case $\beta < 0$

For $\beta^* < \beta < 0$, there exists an infinite number of solutions, bounded by two extremal solutions. $\beta^* = -0{\cdot}198838 \pm 10^{-6}$ is a turning (bending) point where the extremal solutions coïncide. The upper (i.e. $f''(0) \geq 0$) extremal solution is a classical one with $f'(t) \to 1$ exponentially.[9] The lower (i.e. $f''(0) \leq 0$) extremal solution was first obtained by Stewartson[10] and investigated by Hastings;[11] it is of the reverse flow type, with $f'(t) \to 1$ exponentially. All solutions between the extremal ones are characterized by an algebraic convergence of $f'(t) \to 1$.

The minimal extremal branch ends at $\beta = 0$ which is a singular limit point.[12]

1.2.2.3. The turning point β^*

The value β^* has been computed numerically by Stewartson[13] and its existence has been discussed by Iglish and Kemnitz.[7] In a physical sense, the point $(\beta^*, \alpha^* = 0)$ links reverse flow solution branches and solutions branches without separation in the (α, β) plane.

Heuristically, β^* is a turning point in the following sense: for $\beta > \beta^*$, locally there exists two extremal branches of solution, and for $\beta < \beta^*$ there is locally no extremal solution. Banks and Drazin[14] have initiated a local study near $\beta = \beta^*$. A continuation process has allowed us to follow the branch of classical solutions, especially through the turning point.

1.2.3. *Overshoot solutions.* For $\beta < \beta^*$, Stewartson has shown that all possible solutions are of the overshoot type. Numerical studies have revealed branches of overshoot solutions. The solutions of the branch labelled n, present n extrema as well as n overshoots; the number of overshoots is defined precisely as the number of zeros of the equation $f' - 1 = 0$.

Libby and Liu[15] calculated some of these branches. The last point they found for the first branch was $\beta = -1{\cdot}0060$ and $f''(0) = -1{\cdot}09$. For this solution, f' has exponential decay. So it does seem (numerically) that the first branch with overshoot ends at $\beta = -1$. The last point that Libby and Liu found for the second branch with overshoot was $\beta = -1{\cdot}9458$, $f''(0) = -1{\cdot}47$.

Following this idea, we calculated numerically some branches of extremal solutions. The numerical analysis that we did leads to a precise knowledge of seven branches, as well as their behaviour as β goes to -1.

Some theoretical insight has been given by Troy[16] who established the existence of an infinite sequence of negative β_j for which there exist solutions with j overshoots and exponential convergence of $f' \to 1$.

2. NUMERICAL METHODS WITHOUT CONTINUATION PROCESS

2.1. *Statement of the problem*

We are interested here in the numerical solution of the initial value problem

$$\left.\begin{aligned} &f'''+ff''+\beta(1-f'^2)=0\\ &f(0)=f'(0)=0\\ &f''(0)=\alpha\end{aligned}\right\} \qquad (S_\alpha)$$

and we are seeking the solutions f of (S_α) satisfying the so called extremal condition (see section 1.2.2.2):

$$\lim_{\eta\to+\infty} f'(\eta)=1 \qquad (7)$$

which amounts to specifying an admissible domain for α.

2.1.1. *Solutions of system* (S_α), *satisfying* (7). System (S_α) is equivalent to a first order differential system namely:

$$\text{Find } \{y_1, y_2, y_3\} \text{ in } (\mathscr{C}^0(\mathbb{R}^+))^3$$

such that

$$\left.\begin{aligned} &\dot{y}_1=y_2\\ &\dot{y}_2=y_3\\ &\dot{y}_3=-y_1y_3-\beta(1-y_2^2)\end{aligned}\right\} \qquad (8)$$

with the initial condition

$$y_1(0)=0, \qquad y_2(0)=0, \qquad y_3(0)=\alpha \qquad (9)$$

Among the possible solutions of (8), (9), we shall select the solutions $Y_\alpha=\{y_{1,\alpha}, y_{2,\alpha}, y_{3,\alpha}\}$, for which the extremal criterium is satisfied:

$$y_2(\infty)-1=0$$

Remark. In practice, for the numerical study, we shall replace the previous relation by

$$y_2(A)-1=0 \qquad (10)$$

with $A \gg 1$.

Define next $F:\mathbb{R}\to\mathbb{R}$ by

$$\alpha\to F(\alpha)=y_{2,\alpha}(A)-1 \qquad (11)$$

where $Y_\alpha=\{y_{1,\alpha}, y_{2,\alpha}, y_{3,\alpha}\}$ is a solution of system (8) corresponding to the initial condition (9).

Then solving (S_α), taking into account the condition (7), is equivalent to finding the roots of F in $\mathbb{R}$, namely to solving in $\mathbb{R}$ the equation

$$F(\alpha)=0 \qquad (12)$$

This last equation can be solved by a Newton's method: if the derivative of F with respect to α does not vanish at a root α^* of F, the sequence α_k defined by:

$$\alpha_{k+1}=\alpha_k-(F'(\alpha_k))^{-1}F(\alpha_k) \qquad (13)$$

converges to α^*, if α_0 belongs to a sufficiently small neighbourhood of α^*

In our case, the sequence $\{\alpha_k\}_k$ may be constructed as follows:

$$\alpha_{k+1} = \alpha_k - (y_{2,\alpha_k}(A) - 1) \Big/ \frac{\partial}{\partial \alpha_k} (y_{2,\alpha_k}(A)) \tag{14}$$

In addition, by the existence theorem of Peano, if Y is a solution of the differential system:

$$\dot{Y} = \mathscr{F}(Y) \quad \text{and} \quad Y(0) = (0, 0, \alpha)^T$$

then $\dfrac{\partial Y}{\partial \alpha}$ is a solution of the first order differential system

$$\dot{X} = \mathrm{Jac}\,(\mathscr{F})X$$

with the initial condition

$$X(0) = (0, 0, 1)^T$$

where Jac $(\mathscr{F})$ denotes the Jacobian matrix of $\mathscr{F}$.

Using this property we can write $\dfrac{\partial}{\partial \alpha_k}(y_{2,\alpha_k}(A))$ as

$$\frac{\partial}{\partial \alpha_k}(y_{2,\alpha_k}(A)) = \left(\frac{\partial}{\partial \alpha_k} y_{2,\alpha_k}\right)(A) = X_{2,\alpha_k}(A)$$

where

$$\dot{X}_{\alpha_k} = \begin{pmatrix} \dot{x}_{1,\alpha_k} \\ \dot{x}_{2,\alpha_k} \\ \dot{x}_{3,\alpha_k} \end{pmatrix} = \begin{pmatrix} x_{2,\alpha_k} \\ x_{3,\alpha_k} \\ -x_{1,\alpha_k} y_{3,\alpha_k} - y_{1,\alpha_k} x_{3,\alpha_k} + 2\beta y_{2,\alpha_k} x_{2,\alpha_k} \end{pmatrix}$$

with the initial conditions

$$\left.\begin{array}{l} x_{1,\alpha_k}(0) = 0 \\ x_{2,\alpha_k}(0) = 0 \\ x_{3,\alpha_k}(0) = 1 \end{array}\right\}$$

2.1.2. *Extremal solutions: numerical treatment.* The notion of extremal solutions was introduced for the Falkner–Skan equation by Hartree.[3] He proposed considering extremal solutions for $\beta^* < \beta < 0$ only; they can be connected then to the unique solution for $\beta \geq 0$; this will also ensure the continuity of the set of the solutions in the plane $(\beta, f''(0))$, in a neighbourhood of zero. In practice this asymptotic behaviour may be characterized by the property $f' \to 1$ exponentially.

In 1953, Stewartson[13] proposed to refine Hartree's criterion by considering the solutions defined by

$$f = \lim_{A \to +\infty} f_A$$

where f_A is a solution of the Falkner–Skan equation satisfying the initial conditions together with the final condition

$$f_A(0) = 0, \qquad f'_A(0) = 0, \qquad f'_A(A) = 1 \quad \text{for} \quad A < \infty$$

With this approach, Stewartson rediscovered the reattached flows of Hartree for $\beta^* < \beta$, and showed the existence of a branch of separated solutions for β in the same domain.

We call any solution which converges to 1 exponentially at infinity an *extremal solution.* The main interest of such solutions is that they correspond to physically stable solutions. In practice, we characterize these extremal solutions by the exponentially decay of $F(\alpha)$ with respect to A, where F is defined through (12).

An extremal solution will then minimize with respect to α the quantity

$$\Delta_\alpha = (y_{2,\alpha}(A)-1)^2 = F^2(\alpha)$$

the scale factor α corresponding to extremal solutions will maximize $\dfrac{\partial \Delta_\alpha}{\partial \alpha}$ and can be characterized by:

$$\left.\begin{aligned} &F(\alpha)=0 \\ &\frac{\partial}{\partial \alpha}(y_{3,\alpha}(A))=0 \end{aligned}\right\} \tag{15}$$

2.2. Numerical method

We shall now discuss a general method for the numerical search for extremal solutions of the Falkner–Skan equation.

The method is a shooting method, called the adjoint method. It allows computation of all the terms in the Newton iteration (14). For a given value of $f''(0)$, we integrate the Falkner–Skan system (S_α); this computation provides a value of f' at the final abscissa A; a test is then made to compare this last value with 1; if the difference $f'(A)-1$ is too large, a correction is made on $f''(0)$.

2.2.1. *The adjoint method: review.* Let us consider the general differential system:

$$\dot{y}_i = g_i(y_1, y_2, \ldots, y_n, t), \quad \text{for} \quad i=1,\ldots,n \tag{16}$$

where

$$y_i \text{ belongs to } \mathscr{C}^1(\mathbb{R})$$
$$g_i \text{ belongs to } \mathscr{C}^1(\mathbb{R}^n)$$

with the boundary conditions

$$\left.\begin{aligned} &y_i(t_0)=c_i, \quad i=1,\ldots,r \\ &y_{n+1-m}(t_f)=c_{n+1-m}, \quad m=1,2,\ldots,n-r \end{aligned}\right\} \tag{17}$$

Using a Taylor expansion, we obtain from (16) the linear variational system

$$\delta\dot{y}_i = \sum_{j=1}^{n} \frac{\partial g_i}{\partial y_i}\,\delta y_j, \qquad i=1,\ldots,n$$

which has as its adjoint system

$$\dot{x}_i = -\sum_{j=1}^{n} \frac{\partial g_j}{\partial y_i}\,x_j \tag{18}$$

Then we obtain the condition

$$\sum_{i=1}^{n} (x_i(t_f)\,\delta y_i(t_f) - x_i(t_0)\,\delta y_i(t_0)) = 0 \tag{19}$$

which connects the adjoint and variational systems.

The adjoint method is an iterative method on $y_i(t_0)$ for $i=r+1$ to n; let $y_i^{(k)}(t_0)$ be the value of y_i at the kth iteration, at the value t_0, the solution $y_i^{(k)}$ satisfies

$$\dot{y}_i^{(k)} = g_i(y_1^{(k)}, y_2^{(k)}, \ldots, y_n^{(k)}, t)$$

where $y_i^{(k)}(t_0)$ is known for $i=1$ to n.

With the help of a numerical integration method, we determine the final values $y_i^{(k)}(t_f)$; then we compute

$$\delta y_{i_m}^{(k)}(t_f) = c_{i_m} - y_{i_m}^{(k)}(t_f), \quad \text{for} \quad m=1,\ldots,n-r$$

In order to compute $\delta_{y_i}^{(k)}(t_0)$ for $i=r+1$ to $i=n$, we shall integrate $(n-r)$ times the adjoint system (18) with the condition (19); the values $\delta y_i^{(k)}(t_0)$ are the solutions of the following linear system on the interval (t_f, t_0) (backwards integration):

$$\left.\begin{aligned} &\dot{x}_i = \sum_{j=1}^{n} \frac{\partial g_i}{\partial y_j} x_j, \quad \text{for} \quad i=1,\ldots,n-r \\ &x_i^{(m)}(t_f) = \delta_{i,i_m}, \quad \delta_{i,j} \text{ is the Kroneker symbol} \end{aligned}\right\} \tag{20}$$

The condition (19) enables us to write

$$\sum_{i=r+1}^{n} x_i^{(m)}(t_0)\, \delta y_i^{(k)}(t_0) = \delta y_{i_m}^{(k)}(t_f), \qquad m=1,\ldots,n-r$$

We then obtain the following system:

$$\begin{bmatrix} \delta y_{r+1}(t_0) \\ \delta y_{r+2}(t_0) \\ \\ \delta y_n(t_0) \end{bmatrix}^{(k)} = \begin{bmatrix} x_{r+1}^{(1)}(t_0) & x_{r+2}^{(1)}(t_0) & \ldots x_n^{(1)}(t_0) \\ \vdots & & \vdots \\ \vdots \ldots\ldots & x_i^{(j)}(t_0) & \ldots\ldots \vdots \\ \vdots & & \vdots \\ x_{r+1}^{(n-r)}(t_0) & x_{r+2}^{(n-r)}(t_0) & \ldots x_n^{(n-r)}(t_0) \end{bmatrix} \begin{bmatrix} \delta y_{i_1}(t_f) \\ \delta y_{i_2}(t_f) \\ \\ \delta y_{i_{n-r}}(t_f) \end{bmatrix} \tag{21}$$

and as an initialization for the next integration, we take the value

$$y_i^{(k+1)}(t_0) = y_i^{(k)}(t_0) + \delta y_i^{(k)}(t_0), \quad \text{for} \quad i=r+1 \text{ to } i=n \tag{22}$$

(see Reference 17).

2.2.2. *Application to the Falkner–Skan equation.* We shall consider the canonical form of the system (S_α), namely

$$\left.\begin{aligned} \dot{y}_1 &= y_2 \\ \dot{y}_2 &= y_3 \\ \dot{y}_3 &= -y_1 y_3 - \beta(1-y_2^2) \end{aligned}\right\} \tag{23}$$

with the initial condition

$$\left.\begin{aligned} y_1(0) &= 0 \\ y_2(0) &= 0 \\ y_3(0) &= \alpha \end{aligned}\right\} \tag{24}$$

The corresponding variational system is

$$\left.\begin{aligned}&\begin{bmatrix}\delta\dot{y}_1\\ \delta\dot{y}_2\\ \delta\dot{y}_3\end{bmatrix}=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -y_3 & 2\beta y_2 & -y_1\end{bmatrix}\begin{bmatrix}\delta y_1\\ \delta y_2\\ \delta y_3\end{bmatrix}\\ &\begin{bmatrix}\delta y_1\\ \delta y_2\\ \delta y_3\end{bmatrix}(0)=\begin{bmatrix}0\\ 0\\ 1\end{bmatrix}\end{aligned}\right\} \tag{25}$$

and the backward adjoint system is

$$\left.\begin{aligned}&\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix}=\begin{bmatrix}0 & 0 & -y_3\\ 1 & 0 & 2\beta y_2\\ 0 & 1 & -y_1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}\\ &\text{with}\\ &\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}(A)=\begin{bmatrix}0\\ 1\\ 0\end{bmatrix}\end{aligned}\right\} \tag{26}$$

With the help of (19) and (22) we obtain the relation

$$y_3^{(k+1)}(0)=y_3^{(k)}(0)+(1-y_2^{(k)}(A))/x_3^{(k)}(0) \tag{27}$$

which defines the iterative process.

The algorithm is stopped when the extremality criterion reaches a given accuracy. The integrations of the differential systems are approximated by a fourth order Runge–Kutta method.

2.2.3. *Performances of the method.* The adjoint method allowed us to determine numerically the first seven branches of super solutions. This method is therefore effective, but it cannot be adapted to an automatic computation of the branches. Indeed, the natural progression along a constant path $\Delta\beta$ cannot be effective when the slope of the branches is large. The study of a branch has to be done point by point to adjust the path $\Delta\beta$ at each step of the computation. Numerically, it seems that all the branches with overshoots have the same singular limit points as β increases to -1.

3. CONTINUATION METHODS AND THEIR APPLICATIONS TO THE FALKNER–SKAN EQUATIONS

The previous section discussed a method for finding a solution of the Falkner–Skan equations for a fixed value of β. In this section we discuss methods for finding nearby solutions for different β to form solution branches. We describe a continuation method for solving non-linear problems and we show how this method can be adapted to the numerical computation of branches of extremal solutions of the Falkner–Skan equations. We show that this method is well suited to bifurcation problems, especially to the computation of turning points.

3.1. *Solution of non-linear problems*

The approximated problems are to find, in some finite dimensional space, the solutions of the following problem:

$$F(u) = 0 \tag{28}$$

where F is a non-linear operator defined on $\mathbb{R}^n$.

A general method, used to solve this kind of problem, is the well-known Newton method; it can be written in the case of differentiable F, as follows:

u^0 given,

u^{n+1} is computed from u^n by:

$$\left.\begin{aligned} F'(u^n)\,\delta u^n &= -F(u^n) \\ u^{n+1} &= u^n + \delta u^n \end{aligned}\right\} \tag{29}$$

This method can be easily implemented only in the special case where F' is invertible, for each u^n; moreover it requires the computation and the inversion of the matrix $F'(u^n)$ at each step of the algorithm. Nevertheless it remains efficient and easy to use, in many cases; in particular, if u^* is a simple root of (28), and, if u^0 is chosen in a suitable neighbourhood of u^*, the convergence of the sequence $\{u^n\}_n$ defined by (29) is quadratic.

Remark. We can also use a least square method, which consists of minimizing the functional defined in $\mathbb{R}^n$ by

$$J(u) = \|F(u)\|^2 \tag{30}$$

where $\|.\|$ is a convenient norm in $\mathbb{R}^n$.

Problem (28) is then transformed into an optimal control problem[18] and the minimization problem can be solved by a conjugate gradient algorithm.[19–21] This solution is efficient in general cases since it avoids the computation and the inversion of the matrix $F'(u^n)$; nevertheless it has been rejected here since the matrix $F'(u^n)$ is a 2×2 matrix invertible by hand.

3.2. Continuation methods

3.2.1. *Statement of the problem.* We shall consider a class of non-linear problems depending upon a real parameter λ:

$$G(u, \lambda) = 0 \tag{31}$$

where $G: B \times \mathbb{R} \to \mathbb{R}$, and B is a Banach space (in practice, for the solution of approximated problems, $B = \mathbb{R}^N$, and here $N = 2$).

Definition 4

A regular branch of solutions is a family of solutions of (31), depending twice continuously differentiably on a parameter s; we set

$$\Gamma_{a,b} = \{(u(s), \lambda(s)),\ s_a \leq s \leq s_b\} \tag{32}$$

Our purpose is to compute the regular branches of solutions of problem (31).

The standard approach is almost invariably to use λ, one of the naturally occurring parameters of the problem, as the parameter defining solution arcs, $u(\lambda)$. Indeed, if for $\lambda = \lambda_0$, we get an isolated solution, u_0, i.e. if the linear operator

$$G_u^0 = G_u(u_0, \lambda_0) \tag{33}$$

is an isomorphism of B onto itself and if the operator G is continuously differentiable in a neighbourhood of the solution (λ_0, u_0), the implicit function theorem shows the existence of a regular arc of solutions: $u = u(\lambda)$, for λ belonging to a suitable neighbourhood of λ_0.

Therefore, for λ given sufficiently close to λ_0, we may solve problem (31) just as problem (28).

These procedures however may fail or encounter difficulties (slow convergence for example) close to a non-isolated solution.

The basic idea to circumvent this, is due to Keller[22] and consists of using a normal parametrization

$$u = u(s)$$
$$\lambda = \lambda(s)$$

which is defined using an auxiliary equation added to the system to get the problem

$$\left.\begin{array}{r} G(u(s), \lambda(s)) = 0 \\ N(u(s), \lambda(s), s) = 0 \end{array}\right\} \tag{34}$$

where $N: B \times \mathbb{R}^2 \to \mathbb{R}$ defines the normal parameter s, on the arc of solutions.

Introduce then the new unknown $x \in \mathbb{X} = B \times \mathbb{R}$ and the operator $P: \mathbb{X} \times \mathbb{R} \to \mathbb{X}$ by

$$x(s) = (u(s), \lambda(s)) \tag{35}$$

and

$$P(x(s), s) = \begin{pmatrix} G(u(s), \lambda(s)) \\ N(u(s), \lambda(s), s) \end{pmatrix} \tag{36}$$

The new problem is to find the solution, $x(s)$ of

$$P(x(s), s) = 0 \tag{37}$$

The main interest of this new formulation is that the ordinary limit points of (31) become regular solutions of (37) (see References 22 and 23 for more details). Let us make precise the concept of a limit point: we have the following:

Definition 5

Let $\{u_0, \lambda_0\} \in B \times \mathbb{R}$ be a solution of problem (31). We say that $\{u_0, \lambda_0\}$ is a normal limit point if

$$\frac{\partial G}{\partial u}(u_0, \lambda_0)\dot{u}_0 + \frac{\partial G}{\partial \lambda}(u_0, \lambda_0)\dot{\lambda}_0 = 0 \tag{38}$$

where $\dot{u}_0$ (resp. $\dot{\lambda}_0$) denotes the derivative of u with respect to s,

$$\dim N\left(\frac{\partial G}{\partial u}(u_0, \lambda_0)\right) \equiv \operatorname{codim} R\left(\frac{\partial G}{\partial u}(u_0, \lambda_0)\right) = 1 \tag{39}$$

$$\frac{\partial G}{\partial \lambda}(u_0, \lambda_0) \notin R\left(\frac{\partial G}{\partial u}(u_0, \lambda_0)\right) \tag{40}$$

The main justification of arc length continuation follows from

Proposition 1

Any normal limit point of problem (31) is a regular solution of (37). For a proof, see Reference 23.

As a conclusion of this subsection, we have seen that any solution arcs of problem (37), composed of regular or turning points can be computed by a continuation formulation and a Newton method. For approximations of the arc length constraint, see References 24–26.

3.2. *Numerical solution of the Falkner–Skan equation*

We want to find the set of extremal solutions of the Falkner-Skan equation, which amounts to finding the branches of solutions $\{\alpha, \beta\}$, where $\{\alpha, \beta\}$ belongs to E, defined to be the set of $\{\alpha, \beta\}$, $\alpha, \beta \in \mathbb{R}$ such that there exists at least one solution of the initial value problem

$$\left.\begin{aligned} &y''' + yy'' + \beta(1 - y'^2) = 0 \\ &y(0) = y'(0) = 0 \\ &y''(0) = \alpha \end{aligned}\right\} \tag{41}$$

with the extremality condition

$$y \text{ minimizes the quantity} \qquad \Delta = |y'(A) - 1| \tag{42}$$

The methods proposed in section 2 allowed us to find solutions of (41) and (42) for α or β fixed; the iterative method acts then on one of the two parameters α or β. We meet then some difficulties in the following two cases:

(a) The solution $\{\alpha, \beta\}$ corresponds to a turning point.
(b) For a fixed value of α (resp. β), there exist solutions for at least two narrow values of β (resp. α).

It appears then necessary to implement continuation methods to overcome these difficulties.

3.2.1. *A continuation–Newton method.* Using a normal parametrization, we transform the previous problem into the following:
Find $\{\alpha, \beta\}$ in $\mathbb{R}^2$, satisfying $\alpha = \alpha(s)$, $\beta = \beta(s)$, for $s \in [s_0, s_1]$ and such that

$$\left.\begin{aligned} &y_2(A) - 1 = 0 \\ &\dot{\alpha}(s)^2 + \dot{\beta}(s)^2 = 1 \end{aligned}\right\} \tag{43}$$

where $\{y_1, y_2, y_3\}$ satisfies the canonical system

$$\left.\begin{aligned} &Y' = \begin{Bmatrix} y_1' \\ y_2' \\ y_3' \end{Bmatrix} = \begin{Bmatrix} y_2 \\ y_3 \\ -y_1 y_3 - \beta(1 - y_2^2) \end{Bmatrix} = F(Y) \text{ on } (0, A) \\ &Y(0) = \begin{Bmatrix} y_1(0) \\ y_2(0) \\ y_3(0) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ \alpha \end{Bmatrix} \end{aligned}\right\} \tag{44}$$

Here, the second equation of (43) is the arc length constraint.

The numerical solution can be done by a Newton method. The ordinary differential system (44) will be integrated by a fourth order Runge–Kutta method.

In order to compute the next point of a branch, we use two solutions on the curve: these two points will provide the path Δs, between two consecutive solutions on the branch, and an initialization point for the following solution.

Moreover, the knowledge of the two previous solutions gives as possible approximation of the arc-length constraint:

$$N(\alpha(s), \beta(s)) = \frac{\alpha(s)-\alpha_2}{\Delta s_{MM_2}} \frac{\alpha_2-\alpha_1}{\Delta s_{M_2M_1}} + \frac{\beta(s)-\beta_2}{\Delta s_{MM_2}} \frac{\beta_2-\beta_1}{\Delta s_{M_2M_1}} - 1 = 0 \tag{45}$$

where $M_1:\{\alpha_1, \beta_1\}$ and $M_2:\{\alpha_2, \beta_2\}$ denote the two previous solutions on the branch. This approximation corresponds to a first order approximation of $\dot{\alpha}(s)$ and $\dot{\beta}(s)$.

The Newton method applied to the solution of the problem

$$F(\alpha, \beta) = \begin{Bmatrix} G(\alpha, \beta) \equiv y_2(A) - 1 \\ N(\alpha, \beta) \end{Bmatrix} = 0 \tag{46}$$

may be written as follows:

$$M^{(p+1)} = M^{(p)} - (F'(M^p))^{-1} F(M^p) \tag{47}$$

where $F'(M^{(p)})$ denotes the Jacobian matrix of F, at the point $M^{(p)} = (\alpha^{(p)}(s), \beta^{(p)}(s))$.

We have

$$F' = \begin{Bmatrix} \dfrac{\partial G}{\partial \alpha} & \dfrac{\partial G}{\partial \beta} \\ \dfrac{\partial N}{\partial \alpha} & \dfrac{\partial N}{\partial \beta} \end{Bmatrix}$$

The computation of $M^{(p+1)}$ entails the inversion of the Jacobian matrix, F'; in this case, this computation step is particularly simple: the 2×2 matrix is invertible by hand.

First, let us compute the matrix, $F'(M^{(p)})$; we have

$$\frac{\partial G}{\partial \alpha} = \frac{\partial y_2(A)}{\partial \alpha} \tag{48}$$

$$\frac{\partial G}{\partial \beta} = \frac{\partial y_2(A)}{\partial \beta} \tag{49}$$

$$\frac{\partial N}{\partial \alpha} = \frac{1}{\Delta s_{MM_2}} \frac{\alpha_2-\alpha_1}{\Delta s_{M_1M_2}} \tag{50}$$

$$\frac{\partial N}{\partial \beta} = \frac{1}{\Delta s_{MM_2}} \frac{\beta_2-\beta_1}{\Delta s_{M_1M_2}} \tag{51}$$

The evaluations of $\dfrac{\partial N}{\partial \alpha}$ and $\dfrac{\partial N}{\partial \beta}$ are obvious; to compute the values of $\dfrac{\partial G}{\partial \alpha}$ and $\dfrac{\partial G}{\partial \beta}$, we differentiate system (45) with respect to α and β to get (see section 2.2.2)

$$\frac{\partial G}{\partial \alpha} = z_2(A)$$

where z_2 is the solution of the differential system

$$z' = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -y_3 & 2\beta y_2 & -y_1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \tag{52}$$

with the initial data

$$z(0)=\begin{pmatrix}0\\0\\1\end{pmatrix}$$

$\dfrac{\partial G}{\partial \beta}=w_2(A)$, where w_2 is the solution of the differential system

$$w'=\begin{bmatrix}w_1\\w_2\\w_3\end{bmatrix}=\begin{bmatrix}0 & 1 & 0\\0 & 0 & 1\\-y_3 & 2\beta y_2 & -y_1\end{bmatrix}\begin{bmatrix}w_1\\w_2\\w_3\end{bmatrix}+y_2^2-1 \tag{53}$$

with the initial data

$$w(0)=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

We then have to invert the 2×2 matrix, $F'(M^p)$ and set

$$M^{(p+1)}=M^{(p)}-(F'(M^p))^{-1}F(M^p)$$

where the quantities G, N, $\dfrac{\partial G}{\partial \alpha}$, $\dfrac{\partial G}{\partial \beta}$, $\dfrac{\partial N}{\partial \alpha}$, $\dfrac{\partial N}{\partial \beta}$ are evaluated at the point $M^{(p)}=(\alpha^{(p)}, \beta^{(p)})$.

The algorithm is stopped when both the extremality criterion and the continuation constraint reach a value less than a given precision parameter. If the continuation constraint is difficult to satisfy, the arc-length path Δs is automatically reduced.

4. NUMERICAL EXPERIMENTS

4.1. *Solutions without overshoot*

We have seen in Section 1.2.1 that the arc of solutions which does not present any overshoot points is composed of regular solutions in the plane (α, β) and of two singular points, namely the turning point (α^*, β^*) corresponding to the value $\alpha^*=0$, and a singular limit point $\alpha=\beta=0$. To compute the branch of solution going through the turning point, we have used the continuation–Newton method described in Section 3.3.1.

As initializer points, we chose two solutions (α_1, β_1) and (α_2, β_2) given by Keller[27] and corresponding to positive values of α and β. An approximate value of β^* has then been obtained:

$$\beta^*=-0{\cdot}19884$$

and the computed branch going through this point reaches the value

$$\alpha_c=-0{\cdot}062131$$
$$\beta_c=-0{\cdot}018451$$

Beyond this value, the computation was stopped by overflow; this behaviour is similar to the case where α and β are non-positive and when there are no solutions.

The solution branch is shown in Figure 1; in Figure 2 we show the evolution of the speed profile f' when α and β go to zero with negative values, towards the limit point $(0, 0)$.

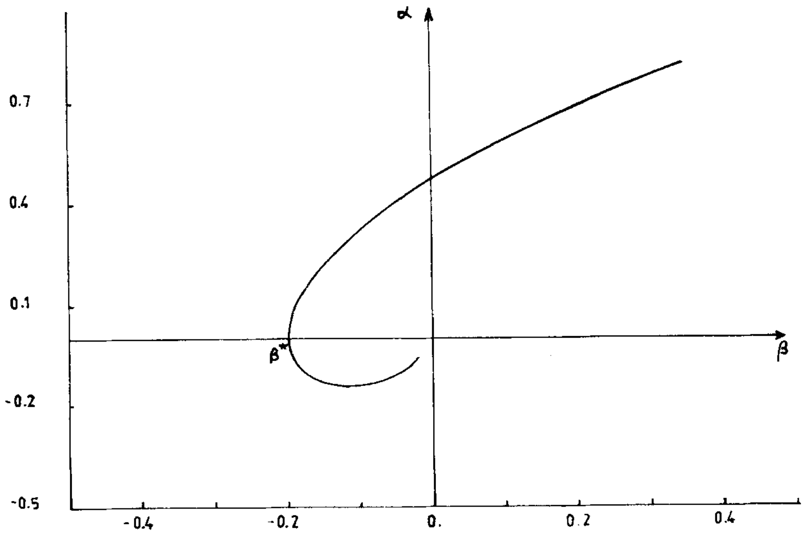

Figure 1. Solutions without overshoot

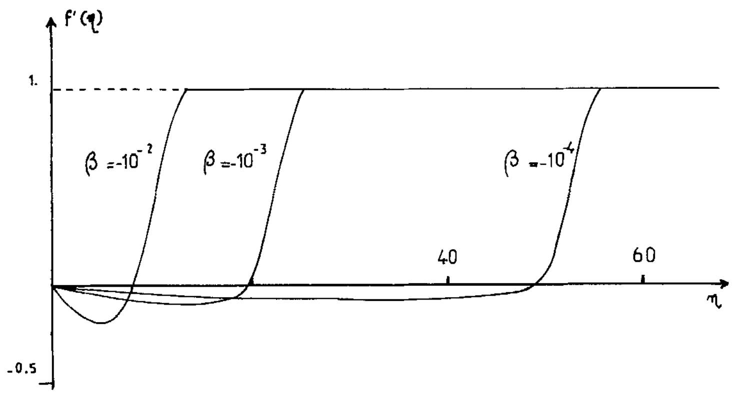

Figure 2. Evolution of the speed profile when β goes to zero

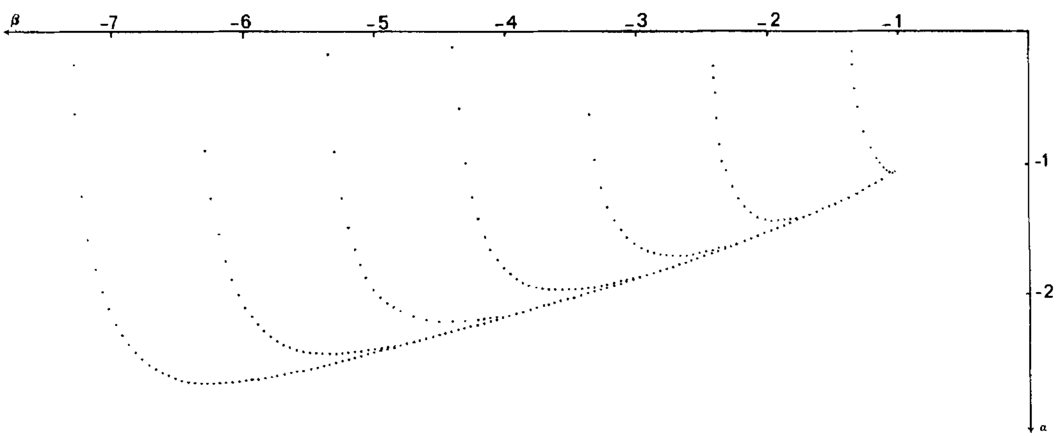

Figure 3. Seven branches of extremal solutions

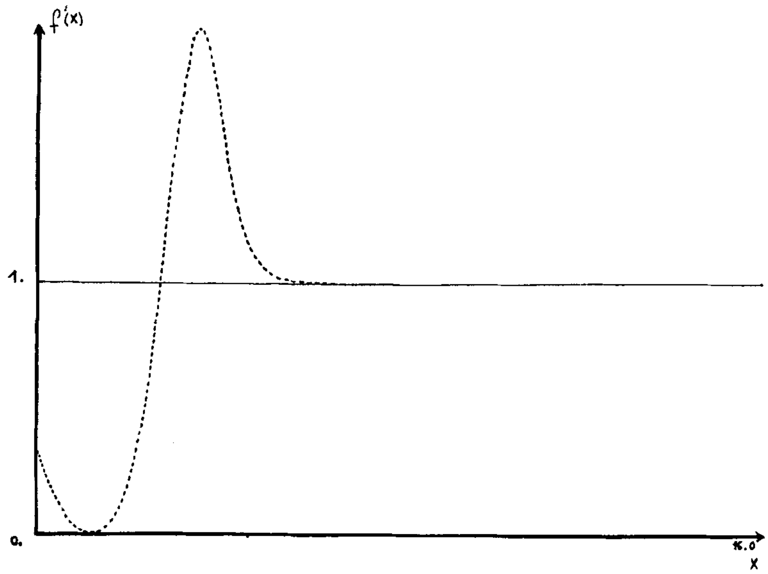

Figure 4. Shape of f' on the first branch: $\beta = -1{\cdot}1$, $\alpha = -1{\cdot}04667$

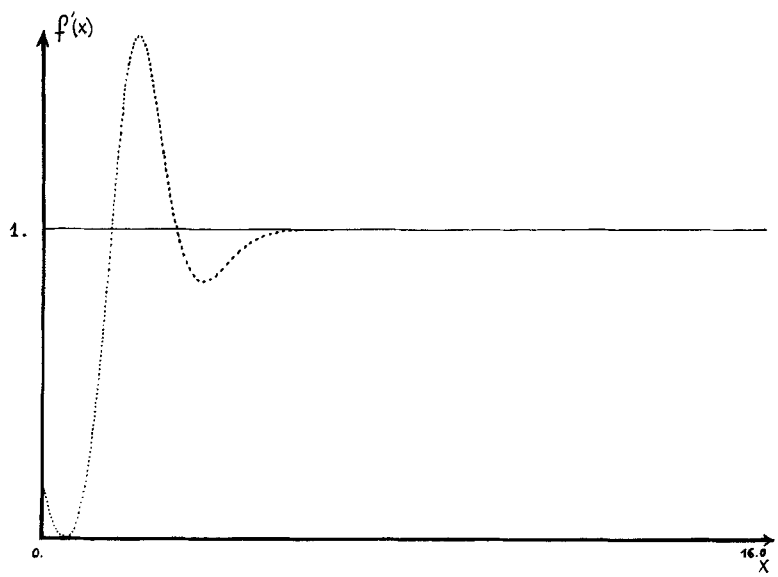

Figure 5. Shape of f' on the second branch: $\beta = -2{\cdot}3$, $\alpha = -1{\cdot}088959$

4.2. Branches of extremal solutions with overshoot

Using both methods described in Sections 2 and 3, we have obtained some interesting results concerning the first seven branches of extremal solutions. The global results are shown in Figure 3 in the (α, β) plane. Each branch can be characterized by the number of roots of the equation $f' - 1 = 0$; this can be observed in Figures 4–10, where the shape of the

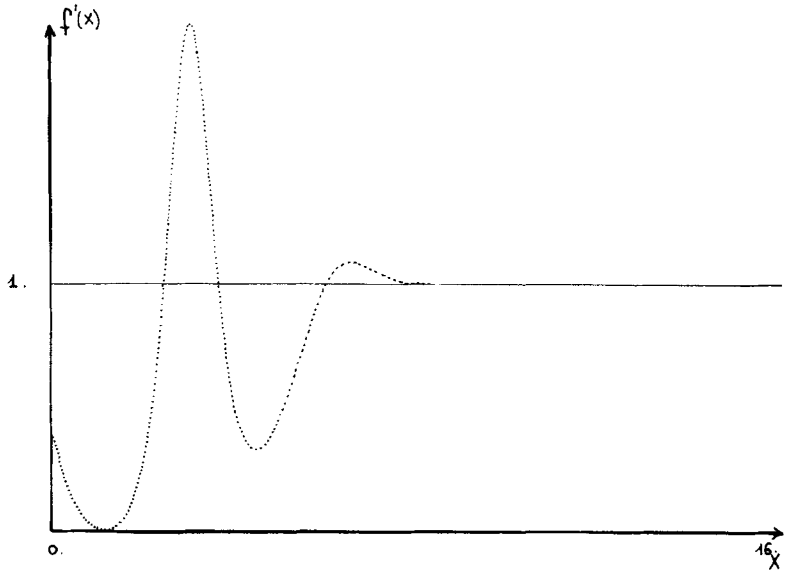

Figure 6. Shape of f' on the third branch: $\beta = -2{\cdot}3$, $\alpha = -1{\cdot}668403$

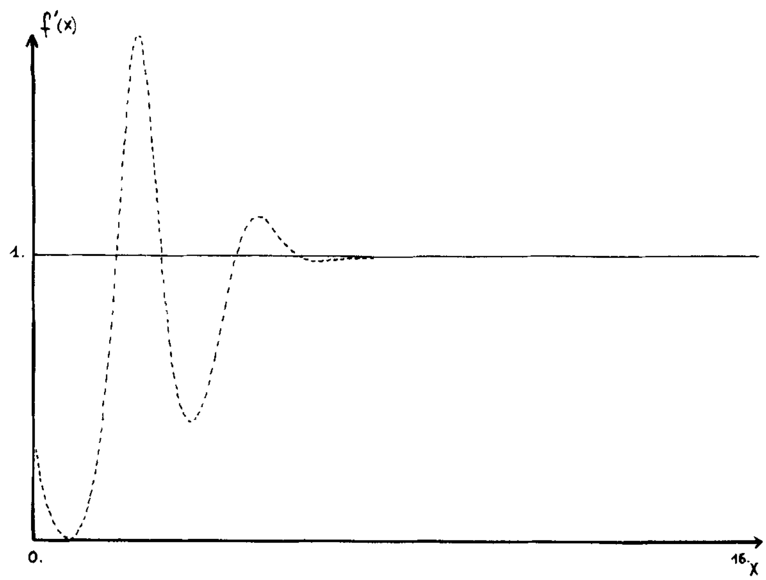

Figure 7. Shape of f' on the fourth branch: $\beta = -3{\cdot}55$, $\alpha = -2{\cdot}01556$

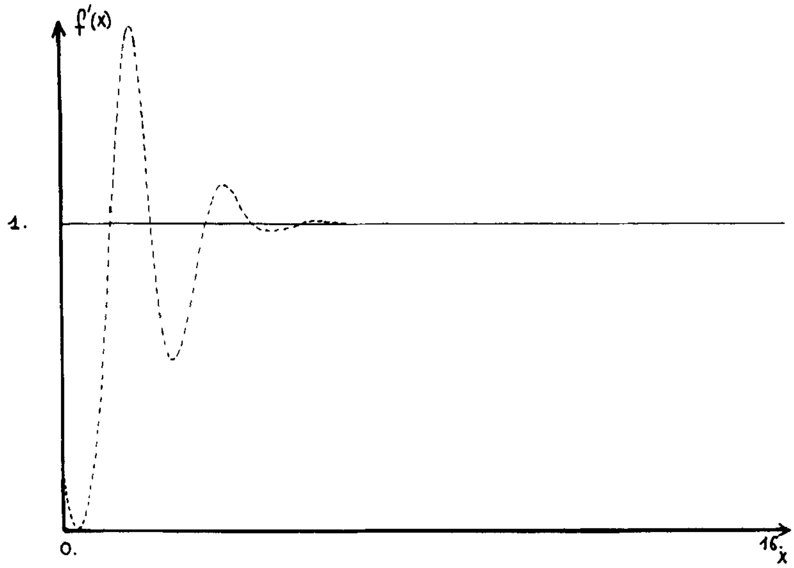

Figure 8. Shape of f_r' on the fifth branch: $\beta = -5{\cdot}099999$, $\alpha = -1{\cdot}81752$

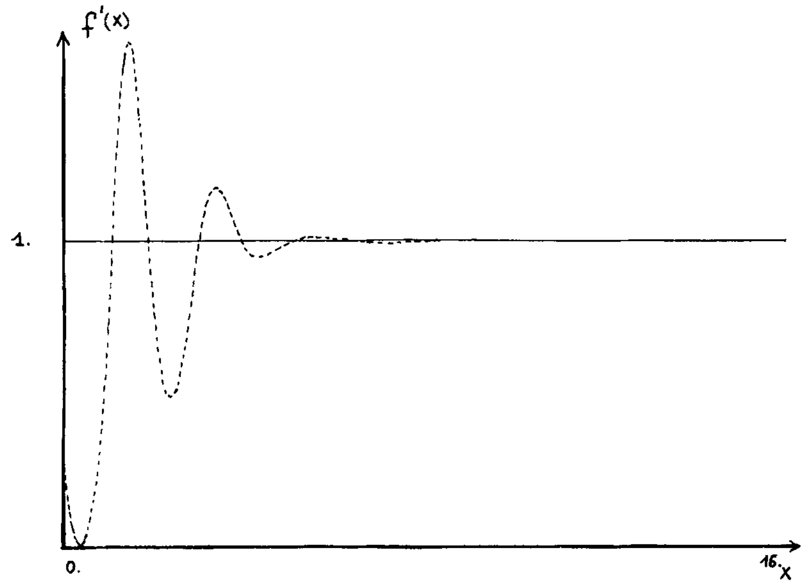

Figure 9. Shape of f' on the sixth branch: $\beta = -6$, $\alpha = -2{\cdot}12681$

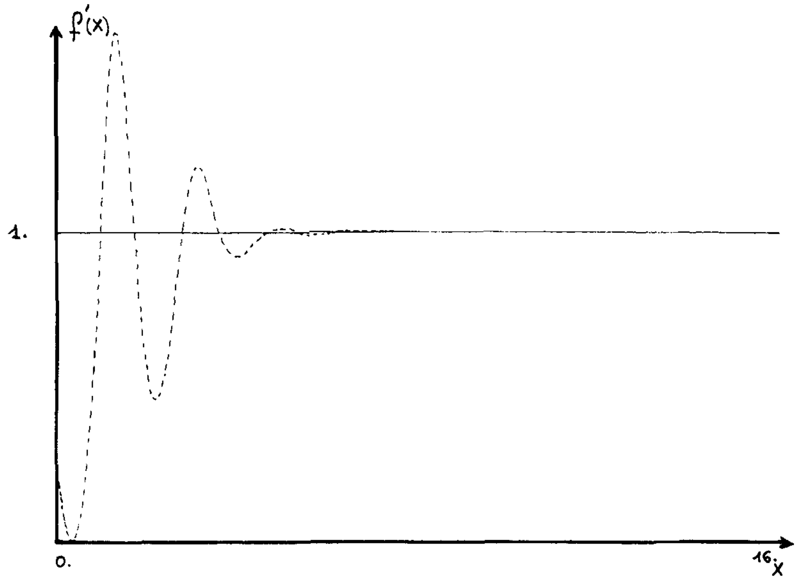

Figure 10. Shape of f' on the seventh branch: $\beta=-7$, $\alpha=-2{\cdot}24168$

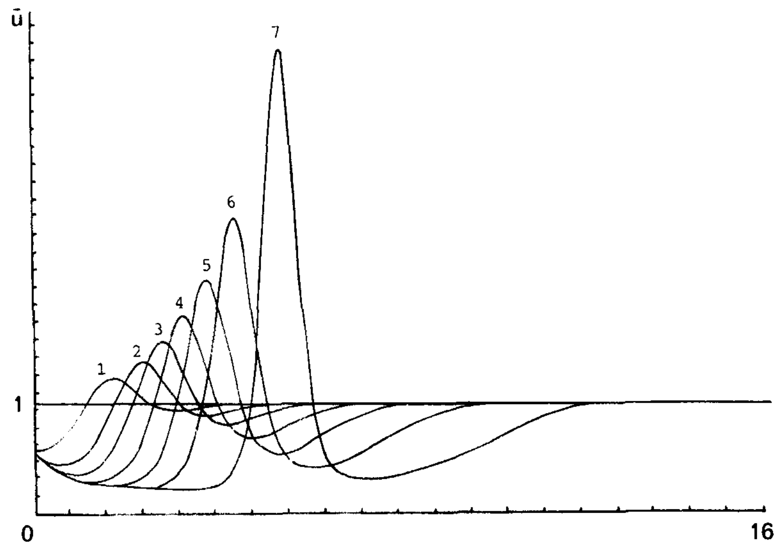

Figure 11. Evolution of overshoots on the second branch: 1. $\beta=-2{\cdot}4$, $\alpha=-0{\cdot}49018$; 2. $\beta=-2{\cdot}2$, $\alpha=-1{\cdot}30730$; 3. $\beta=-2$, $\alpha=-1{\cdot}46101$; 4. $\beta=-1{\cdot}8$, $\alpha=-1{\cdot}45781$; 5. $\beta=-1{\cdot}6$, $\alpha=-1{\cdot}39359$; 6. $\beta=-1{\cdot}4$, $\alpha=-1{\cdot}30476$; 7. $\beta=-1{\cdot}2$, $\alpha=-1{\cdot}20146$

speed f' is represented for each branch. In addition, for a given branch of extremal solutions, we can study the behaviour of f' when β goes to zero.

For example, in Figure 11, we have represented the solution f' for seven values of (α, β) on the same branch. We can observe that the amplitude of the overshoot grows as β goes to zero, and the position of the overshoot progresses to the right. It follows that the convergence of f' to the value $f'=1$, may appear for a larger value of the abscissa; this implies numerical difficulties concerning the choice of the large scale A.

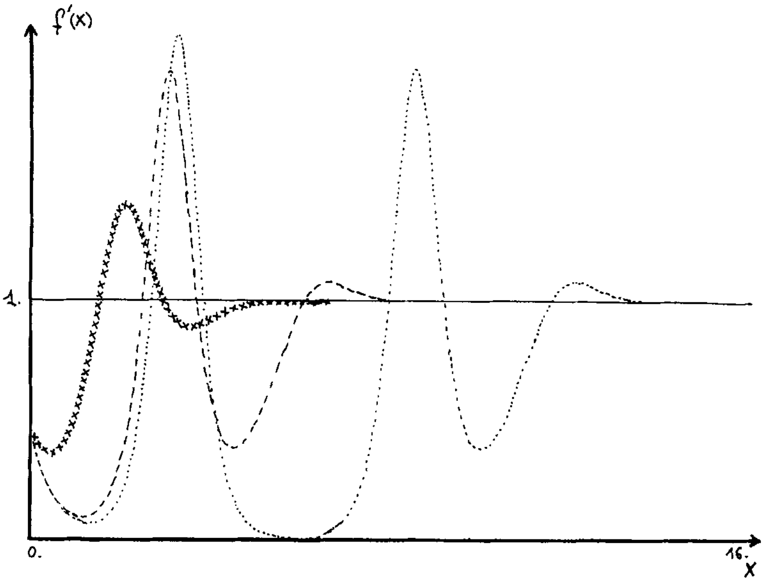

Figure 12. Three solutions for $\beta=-2{\cdot}30$: ××× $\alpha=-1{\cdot}08896$; --- $\alpha=-1{\cdot}66840$; ···· $\alpha=-1{\cdot}68599$

Concerning the asymptotic behaviour of the branches, we can see in Figure 3 that the different branches join along a 'limit branch' when β goes to zero with negative values.

It is then of most interest to look, for a fixed value of β, at the different solutions on each branch.

We can see in Figure 12 that even for two nearly equal values of α, we have obtained (see (2) and (3)) two solutions quite different. This result seems to confirm the asymptotic convergence of the branches when β goes to zero.

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