SOME COURSE NOTES ON AEROSERVOELASTICITY
DIPARTIMENTO INGEGNERIA AEROSPAZIALE, POLITECNICO DI MILANO, ITALY

RUDIMENTS OF FLIGHT FLUTTER TESTING PROCEDURES AND ANALYSES

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Scope and content

The aim of this notes is to provide a simple, yet somewhat exhaustive picture, of a few significant aspects regarding flight flutter testing. It should be noticed that within the course of Aeroservoelasticity they serve the scope of their title as well as a reuse and remastering of ideas and concepts already seen.
Paraphrasing Leonardo's: "He who loves practice without theory is like the sailor who boards ship without a rudder and compass and never knows where he may cast.", we will try to highlight the importance of flight flutter testing, with the implied need of obtaining a matching between analyses and experiments, by saying: "He who loves theory without practice is like the sailor who boards ship without verifying if rudder and compass are working well, even if he knows he may shipwreck."
The presentation will span: test procedures, test instrumentation, possible excitations, experimental data analysis and model identification techniques. The identification part will be treated both in time and frequency domains. The notes are organized as follows: the first 3 chapters describe some of the main aspects of flight flutter testing, with attention to instrumentation in chapter 2, exciters and the choice of their excitation commands in chapter 3. Chapter 4 frames the way to interprete what will be identified. Then some identification tools are presented in chapters 5 and 6. Chapter 5 deals with the very basic concepts of identification in the time domain, its properties and estimation methods, which are eventually applied to flutter prediction along with the determination of the uncertainties of the obtainable V-g plots. An outline of the multivariable case is also provided. Chapter 6 deals with frequency domain identification along with its application to flutter testing.

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Chapter 1

An Introduction to Flight Flutter Testing

1.1 Introduction

It is a well known fact that aeroelasticity plays an essential role in aircraft design. The adoption of thin wings, supercritical wings, optimized lighter and thus more deformable structures, jumbo aircraft, all moving stabilizers, T configurations and, generally speaking, any unconventional design, has increased the probability of flutter instabilities. Flutter is determined by the negative interaction among aerodynamic, elastic, inertia, controls and servosystems, which can produce structural oscillations, at times unstable, often leading to possible vehicle damages or even to its complete loss.
High performance aircrafts should be more prone to flutter instabilities. In spite of this, some "amateur" aircraft have shown themselves susceptible to this problem too. In conclusion, without accurate investigations, it is impossible to exclude the presence of flutter and of some other aeroelastic problems.
The study of aeroelasticity is performed using both experimental and theoretical abilities, which should be strictly connected. The experimental activity is necessary also to validate, and eventually update, the theoretical model, which in turn will permit to analyze all of the possible configurations of weight, load and balance which are impossible to cover experimentally.
Aeroelastic analyses follow the scheme reported in figure 1.1, [27]. Before starting flight testing, a servo-elasto-mechanical validation of the structural model is often, if not always, carried out through a ground vibration test. Such a practice might be preceded by some static only verifications with the double aim of verifying:

- the structural stiffness, indirectly by measuring influence coefficients.
- the effects of possible structural non-linearities due to, e.g., local instabilities, plays, static friction.

Ground vibrations and in flight flutter testing should also provide definite clues about some characteristics, typically related to control surfaces and their possible servoing, aerodynamic forces in transonic and/or significantly separated flows, whose modeling is often difficult.

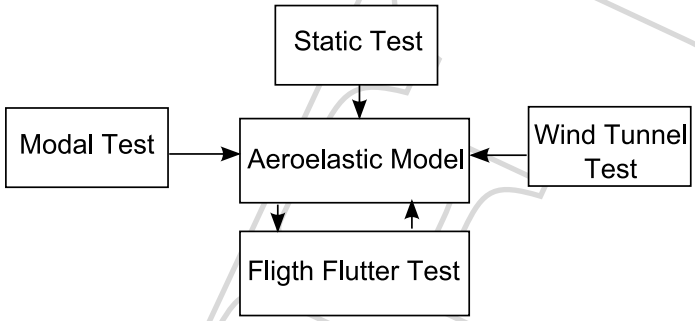


Figure 1.1: `Aeroelastic Analysis`

Flight flutter tests can be divided into two categories:

- development and surveys carried out on aircraft models and prototypes, to investigate and learn about their aeroelastic behavior, possibly unconventional.
- qualifications carried out on aircraft which will go to production, the certifying tests being required by regulatory authorities to release an airworthiness certification. Such a test must demonstrate:
  - good damping within the all flight envelope;
  - a suitable flutter safety coefficient, typically freedom from flutter up to $1.15 - 1.2V_D$.

It should be clear that requirements on the flutter safety coefficient ($1.15 - 1.2V_D$.) cannot be demonstrated in flight, because V_D represent the maximum speed reachable by an aircraft. So, it is necessary to make an extrapolation on the base of the behavior shown within the admissible flight envelope.
Regulations ask also to verify the correlation between tests and numerical simulations. If such a correlation is not adequate, e.g. the damping obtained from numerical analyses is higher than that of tests, it is often asked to provide a valid explanation of the possible causes leading to differing results.
As it will be made clear later on, the model is continually updated on the base of newly acquired experimental data, so to safely extrapolate its aeroelastic behavior to successive test points. In fact an outcome of each flutter test campaign can be seen as a newly gained asset producing improved capabilities toward better future tests.
A summary of relevant civil and military regulations is presented in the appendix. Through their

reading it is possible to understand the request imposed on response calculations and what should be verified experimentally [1].
It should noted that in the military, where a certification in the true sense does not exist nowadays, customers' requests, possibly differing from some of the standard requirements, must also be taken into consideration.

1.2 Flight Flutter Test Characteristic Aspects

The flight experimentation regarding aeroelastic phenomena is particularly critical from the safety point of view. Flight flutter tests, in fact, are in the category of the so called "*hazardous flight tests*".
Every type of flight activity requiring a special preparation, modifications of the aircraft included, in order to guarantee the safety of the crew and of the aircraft itself, is known under that name.[2]
In this category there are also:

- stall, post stall and spin tests;
- weapons certification;
- minimum control velocity, approach to stall;
- flights at a high angle of attack;

The primary purpose of flight flutter tests is to check the absence of aeroelastic instabilities within the flight envelope, i.e. the whole altitude, dynamic pressure (and consequently Mach number), at which the aircraft use is permitted, see figure 1.2. Flight flutter tests lead to the so called flight envelope opening and, because of its potential danger, the related activities are performed in steps, starting from a restricted flight envelope within which the inhexistence of troublesome aeroelastic phenomena is hopefully assured by previously carried out analyses and/or experiences.
Only after each new part of the envelope has been verified from the aeroelastic point of view, performances, the related flight qualities and all of the systems functionalities can be verified, extensively and in full safety.

There are different ways to provide indications about the aeroelastic stability of an aircraft. Here, we follow the experimental reconstruction of the V-g diagram, see figure 1.3. Clearly it is not the only possible approach, but is the one here chosen because of its simplicity and, from this course teaching point of view, because it affords the possibility of reciting already discussed topics. Conceptually, the experimental reconstruction of the V-g diagram, consists in the experimental tracking of the identified eigenvalues/poles of the system.

[1] Incidentally: In the past FAR-JAR-23 established that for low end aircraft with a conventional architecture, the absence of flutter, as well as any other detriment to aeroelastic functionalities of an aircraft, could be demonstrated through tests, calculations or both. When T tail configuration appeared in this category also, regulations began asking to always carry out flight tests and numerical-experimental correlations.

[2] The installation of the anti-spin parachute for high incidence tests is the most typical. In the past, there were some cases of civil aircraft in which an ejection system for the pilot was installed.

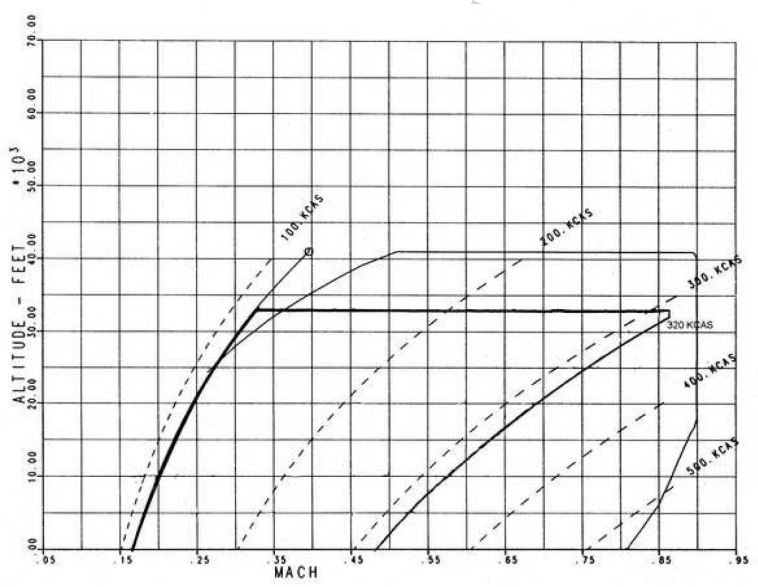


Figure 1.2: Flight Envelope

Therefore, at each further step in the flight envelope opening, an evaluation of the damping and frequency of each aeroelastic mode of interest is carried out.

An important aspect of flight flutter testing is extrapolation. After analyzing and verifying a test

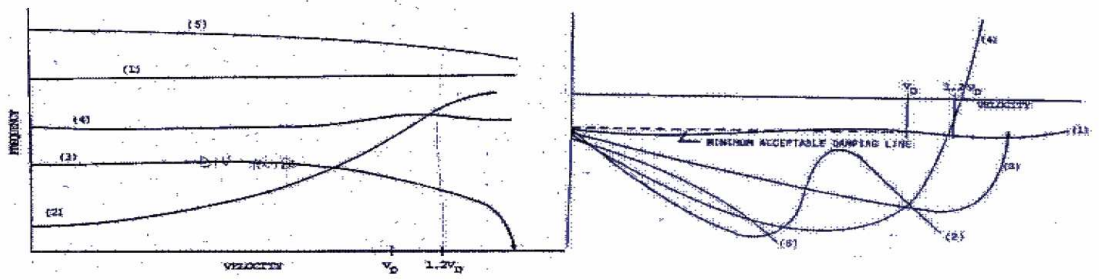


Figure 1.3: V-g diagram:damping and frequency versus flight velocity

point, it is necessary to move to the following one, in the safest possible manner. On the base of the already processed experimental data, the problem becomes then one of understanding how to move from a test point to the next, without risking an instability and safely trying to reduce testing times and costs. In fact flutter tests must be concluded in all safety, but at the same time as quickly as possible: the longer the aeroelastic testing campaign, the later the aircraft will be sent into production. The standard procedure asks for:

- execution of the test for an established test point;
- processing the related experimental data;
- correlation between experimental results and numerical analyses. If any lack of congruence should occur it is necessary to understand its cause; therefore computations are often

repeated trying to find which ”button” of the numerical model must be ”pushed” to obtain a good correlation;

- after the correlation at a certain part of the flight envelope is achieved, forward projections[3] both for the numerical analyses and for the test are made. If the comparison related to a determined test point between the extrapolated projection of the test and the result provided by the computation on the correlated numerical model is satisfactory, it is then possible to proceed with the opening of the envelope to the next test points.

As said certification tests must demonstrate freedom from flutter as far as $1.15 - 1.2V_D$. Being such a speed not reachable an extrapolation of the results can be used to plausibly determine the requested safety against a possible flutter onset.
Because of the large use of extrapolation flight flutter tests are often seen as ”an exploration of the unknown”. We have to proceed in steps on the base of the previous information only, without a safe betting on what is ahead.
From this point of view, the most critical situation which can arise is a flutter caused by a sudden decrease of a so far highly damped aeroelastic mode, called explosive flutter. These flutters are characterized by a trend of significant growth of the damping at the increasing of the flight speed, with a flutter suddenly manifesting itself at a next velocity increase. The slope with which the damping curve crosses the velocity axis is often linked to the flutter violence: the higher the slope, the more violent the flutter is.
It is generally assumed that flutter occurs in an advanced part of the flight envelope. Thus its preceding parts are usually safely covered in a relatively faster way, with the opening of the advanced flight envelope proceeding afterward in a more cautious mode on the base of the analyses and of the data of the previous flight tests.
In general, the experimental approach is to schedule just a single or more test points for a determined flight. Carrying out many test points for flight is clearly an asset which require extensive and reliable capabilities to extrapolate the damping trends in real time, without risks and as quickly as possible to avoid keeping an aircraft idly flying while waiting for a go on. Flutter flight tests consist in exciting the system and measuring either its response and excitation (better) or, depending on the chosen excitation, just the resulting responses. Then damping and frequencies are obtained by applying suitable identification techniques to the measured data. To some extent it is possible to view in flight flutter tests as being similar to ground vibration tests.

1.3 Test Procedure

A typical flutter test consists in stabilizing the aircraft at an assigned speed and altitude. In the case of tests carried out at speeds higher than the maximum reachable in horizontal flight, one should choose whether to maintain a constant Mach number (constant condition for the compressibility phenomena) or indicated speed (constant dynamic pressure), while loosing some altitude. In figure 1.4 a typical planning of test points for a high performance aircraft can be seen.

[3] In fact, the extrapolation projection regards only the test, while ”forward projections” to the next envelope point obtained by the simulations are simply the results given by the analysis on a correlated model.

Once the aircraft has been stabilized, the chosen excitation system is activated for the needed

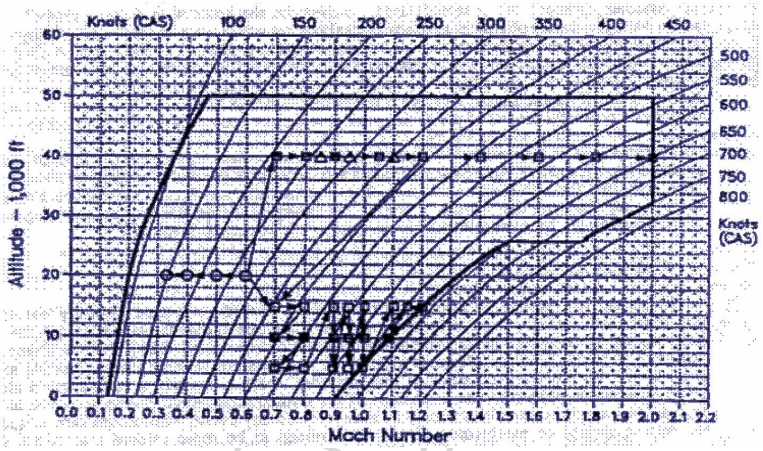


Figure 1.4: Planning of test points

time. Then a data analysis is carried out and if good results are obtained, so that an extrapolation can be inferred with an acceptable confidence, the aircraft is taken to the next test point.
This point can be at the same speed, but with a different excitation, altitude or load factor. As we will see, in fact, according to the used excitation type (pyrotechnical exciters, for example) it might be better to repeat the test. When changing the altitude, it is possible to maintain the Mach number unchanged, increasing the dynamic pressure. Usually increasing the load factor increases the amplitude, while frequencies remain unchanged.
To guarantee some consistency of the obtainable results, at least a test should be repeated under the same conditions. As we will see, the model on which the experimental evaluation is based, is assumed as linear or linearized at the trimmed test reference condition so, for example, a test repetition with varying amplitude excitations can either validate such an assumption or negate it, eventually revealing possible non-linearities that could make it implausible.
On the other hand, flight tests are expensive, so the repetition of a test point is often performed only in the case of a real suspected uncertainty and/or when close to critical conditions.
Even if a V-g reconstruction is a viable procedure, the related identification is usually complemented with other clues based, for example, on the coalescence of near modes or peak-hold.
The coalescence of two close modes is controlled by making a comparison on the evolution of some frequency spectra with increasing dynamic pressure, whereas at increasing values we will notice resonant frequencies related to 2 modes, typically a bending and a torsional one, coming closer and closer, see figure 1.5 The peak-hold method consists in analyzing a spectrum variation while increasing the dynamic pressure, see figure 1.6. Typically at lower dynamic pressures we can notice a high damped mode with a wide base or with a light response. By increasing the speed, the response presents a more pronounced peak and a less wide base. Drawing the amplitude inverse according to the critical dynamic pressure, fig. 1.7, near the critical dynamic pressure the amplitude inverse goes to zero. By extrapolating measures to zero, an estimation of the instability dynamic pressure is obtained.
In the post flight analysis more complex methods are applied to the acquired data, up to fulfilling the scope of an experimental evaluation of the aeroelastic model. Thanks to the increased available computational power, the cost, in terms of time, of the data processing phase is much

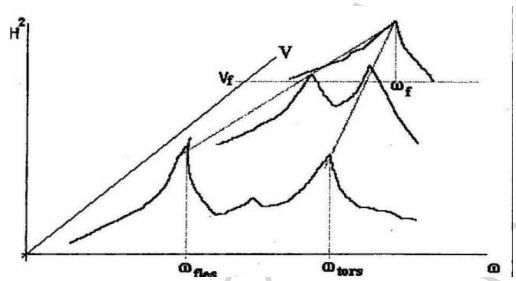


Figure 1.5: Coalescence of flexional and torsional modes increasing the dynamic pressure

more reduced than it could have been some years ago. As already said, in addition to the off-line processing, there is nowadays also the tendency to perform real time analyses, which consists in exciting, processing and identifying during the testing phase, so that it is possible to obtain damping and frequency trends immediately .

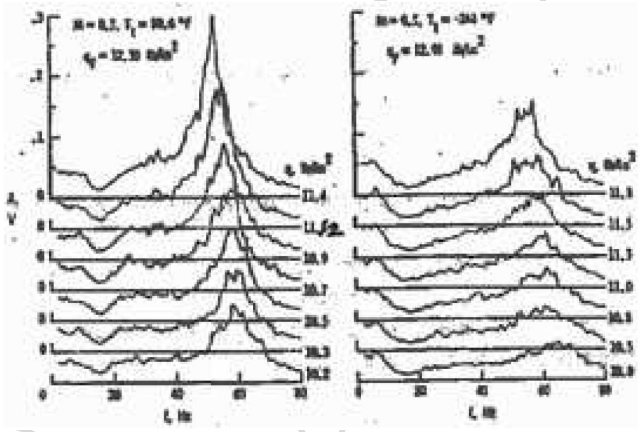


Figure 1.6: Response Amplitude in function of dynamic pressure

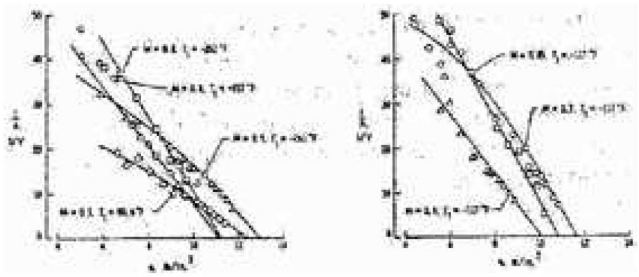


Figure 1.7: Extrapolation of the peaks amplitude's inverse

Chapter 2

Flight flutter test instrumentation

This chapter deals with the design of the test set up in terms of exciters/sensors positions and sensors types, along with a possible recording of test data. Aspects regarding the choice of what types of exciters to adopt and the related excitation commands, will be treated in the next chapter.

2.1 Position of exciters and sensors

The first problem which must be solved in the planning of a test is the choice of the excitation command and the number, type, and position of exciters and sensors. It should be taken into account that a test might be designed to measure the aeroelastic modes of the aircraft also. Nonetheless here we will consider only the experimental reconstruction of the V-g diagram.
In the planning of a test, such a choice is significantly linked to the type of data processing and how it should be carried out. All the systems used to apply an excitation or to measure a response have qualities, defects and limitations to be taken into account.

The problem of positioning exciters and sensors can in principle be easily formulated using two concepts of control system theory in the time domain: the observability and controllability properties of a dynamic Linear Time Invariant (LTI) system.
Regarding the exciters position it is important to maximize the controllability of the systems, therefore the exciters are placed so to guarantee that all modes in the band of interest are well excited. This choice can nonetheless affect the implied equivalent linearization to be used for the flutter onset identification. So, even if it is not always a simple matter, a compromise between these two requirements has to be found.
Instead, the positioning of sensors brings in the concept of system observability: the number and position of transducers must be adequate to guarantee a good measure of the response quality for all modes participating in any response of interest.
The problem can be analyzed from a mathematical point of view by recalling the concepts of controllability and observability, exploiting them to find the ”optimal” sensors and exciters positions.

The state representation of a linear dynamic system is given by:[1]

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases} \tag{2.1}$$

The algebraic controllability and observability conditions for an LTI system, [22], set that the system is controllable and observable if and only if the controllability test matrix Q and the algebraic observability test matrix R are full rank, where:

$$Q = \begin{bmatrix} B & AB & \cdots & A^{(n-1)}B \end{bmatrix} \tag{2.2}$$

$$R = \begin{bmatrix} C^T & A^TC^T & \cdots & A^{(n-1)^T}C^T \end{bmatrix} \tag{2.3}$$

Another controllability and observability rank test [15] is Popov-Belevitch-Hautus (PBH):

- the pair (A,B) is controllable iff:

$$row\,rank\,[\lambda I - A, B] = n \quad \forall\,\lambda \tag{2.4}$$

- the pair (A,C) is observable iff:

$$column\,rank\,\left[\lambda I - A^T, C^T\right] \forall\,\lambda \tag{2.5}$$

The rank concept is on-off, so it is better to use something giving a continuous measure of controllability and observability. In [22] the conditioning number is suggested. Whenever on-off controllability and observability criteria are used, a way to measure controllability and observability is that of comparing the conditioning number of the controllability and observability matrices, as a function of a differing positions of exciters and sensor [47].
Other measurable ways to ascertain controllability and observability are: the verification of the coefficients of modal input-output matrices and Gramians. The first can be explained by using the modal transformation $x = Xq$, so that in modal coordinates the system becomes:

$$\begin{cases} \dot{q} = \tilde{A}q + \tilde{B}u \\ y = \tilde{C}q + Du \end{cases} \quad \begin{cases} \tilde{A} = X^{-1}AX \\ \tilde{B} = X^{-1}B \\ \tilde{C} = CX \end{cases} \tag{2.6}$$

It should then be simple to understand that the conditions of controllability and observability are here translated into choosing excitations points so that $X^{-1}B$ does not have any null row for the modes in the frequency band of interest, and in choosing measurement points so that CX has no null column. Moreover, the lower the modal amplitude is, for either a generic generalized force or measure, the less controllable or observable the system is.
It is important to underline that X contains all the eigenvectors of A. Then in the case of multiple eigenvalues, this test must be satisfied for all of their possible multiple eigenvectors, which are

[1] In the case of an accelerometer the D matrix will be present only if the transfer function of an accelerometer is considered ideal, i.e. with no phase delay and infinite bandwidth. Otherwise only C will appear in the output relation.

infinite since a linear combination of them is still an eigenvector. A final controllability and observability check is based on:

$$G_c := \int_0^\infty e^{At} B B^T e^{A^T t} dt$$
$$G_o := \int_0^\infty e^{A^T t} C^T C e^{At} dt \tag{2.7}$$

the so called controllability, G_c, and observability, G_o, Gramians which, due to the infinite horizon of equations 2.7, are defined for asymptotically stable systems only. Such a constraint is clearly satisfied in our case, since we are working within the flight envelope exactly with the scope of ascertaining its safe aeroservoelastic stability extension with the stated aim of determining where instabilities will occur, while avoiding them absolutely. The system is thus controllable and observable if these matrices are not singular[2]. In the case of an LTI system is possible to prove that the condition on the Gramians are implicitly satisfied if the conditions on the rank of the controllability and observability matrices are as well (iff condition).
Differently from algebraic criteria, the use of Gramians, in the case the system is not fully controllable or observable, could provide more informations about the system properties. Thanks to their spectral decomposition, it is possible to obtain a quantitative measure of controllability and observability. So looking at which part of the eigenspace of the Gramian is significantly contributing to the Gramian itself, we can infer which components are more or less controllable/observable. It is important to keep in mind that if any re-scaling of the physical quantities of the model has to be taken into account, the two Gramians will reflect such a scaling, since they will then refer to different input/output quantities. Summarizing, the positioning of sensors and exciters through the optimization of the controllability and observability properties of the system is clearly iterative and should based on the use of some numerical optimization techniques. Therefore the real unknowns of the problem will be the position of sensors and exciters maximizing controllability and observability. In such a regard it should be noted that the exploitation of the previously mentioned concepts requires a modern, i.e. state space, aeroservoelastic formulation. Moreover precise models will lead to quite a sizeable number of states so that, due to the bad conditioning caused by the repeated matrix powers in them, it will be difficult the use a performance index based on controllability/observability matrices, e.g. maximization of their lowest singular value, as a base for the optimization procedure, leaving the adoption of performance indices based on Gramians as the most viable choice.
In practice, we define a function based on a suitable norm, e.g. using the eigenvalues of balanced Gramians [14] or rank measures, then optimize such a function with respect to positions, so determining the matrices B e C associated to an appropriate state representation of our system.
However, from a practical point of view, the choice of the excitation and measure points ends in following some simple rules:

[2]Remember that the asymptotic stability of an LTI system is required to compute the Gramians. From a computational point of view they are obtained by solving the two Lyapunov equations:

$$AG_c + G_c A^T + BB^T = 0$$
$$A^T G_o + G_o A + C^T C = 0$$

- the exciters are often far less than sensors, they are usually placed at the points exhibiting the highest modal responses for key modes, because applying the excitation there well assures an adequate controllability. These points are very often the tips of the aerodynamic surfaces.

- the sensors, typically accelerometers and strain gages, are installed where the amplitude of the measured response is higher. Being it so, accelerometers are placed far from the modal nodes and at, or near, the points in which modes exhibit a high amplitude. On the other hand, strain gages must be placed at points of high stresses. The problem of sensors placement is often less critical and easier, because of their number being far larger than that of actuators.

In choosing sensors and exciters it is important to take into consideration also the intrusiveness of the related devices to be installed on the aircraft. In practice, this problem concerns more the exciters than the sensors, which have a negligible mass and size and are mostly placed inside the structure.
For the exciters instead there is the need of a thorough verification that they do not modify the aeroelastic behavior of the aircraft. As it will be shown in the next chapter, such devices can have a conspicuous mass and can also modify the aerodynamics of the aircraft. The flutter behavior of the aircraft with or without the excitation device should substantially remain unchanged. If some intrusion cannot be avoided it is then important to verify that it can be appropriately taken into account in flutter analyses reproducing it so to well correlate with tests results. That makes it is possible to provide reliable corrected flutter analyses for the maiden parent aircraft to the certificating authority.
As said, the number of sensors is, typically, greater than that of the exciters. Often flight flutter tests are conducted using only a single excitation command. That does not imply the use of a single exciting device, but that all the exciters will be driven by a unique signal. Typically the excitation command is given to the exciters so as to excite symmetric and antisymmetric modes. However, the aspects regarding the number of excitation commands as system input, will be detailed later on, when talking about the identification.
An advantage linked to the great number of sensors presently used in flutter tests is that it greatly reduces the risk of missing any sensible information about an aeroelastic mode at a given frequency. In fact, even if it might happen that a sensor close to a modal node, looses its observability contribution to that mode, making its measure much more noisy also, there will be other sensors capable of well observing that mode. Furthermore, having a great number of sensors means an opportunity of getting the shape of the aeroelastic modes also.
Notice that, thanks to the low cost of sensors, nowadays so many sensors are installed on the aircraft, that any problem of observability has practically disappeared.

2.2 Instrumentation for flight flutter test

A further aim of this paragraph is to give a short introductive picture on the instrumentation used in flight flutter tests, in relation to possible sensors data recording and telemetry.
With the nowadays flooding of cheap, massive and easily available consumer data recording tools, e.g. high capacity disks, solid state disks and mobile telecoms, including satellite relaying,

the problem of recording/communicating data from an aircraft to ground is going to become no more such. In fact the use of standard consumer appliances in what once was the reign of costly and specialised recording and transmission tool, methods and techniques, is one of the facets of the so called Custom-Off-The-Shelf (COTS) exploitation in the aerospace field, and other fields as well. Nonetheless we do not hasten to recall two common recording/transmission techniques that are still in use in flight flutter testing:

- analog: FM which stands for Frequency Modulation;
- digital: PCM which stands for Pulse Code Modulation;

FM: FM recording is analogical so, for an assigned carrier signal, it contains all of the measurable frequencies. The working principle is the same as that of radio waves, where information is represented by the frequency variations with respect to a nominal frequency, called carrier frequency. Each measured parameter has its own recording channel;

PCM: differently from FM recording there is a map of parameters which is acquired in a sequential way. If a parameter is present only at a time within the map, its sampling frequency is the inverse of the time needed to read all of the parameters. The way to increase the sampling frequency of a chosen parameter is to put it many times in the map. The limitations of this recording technique are clear since the number of positions present in the map, the transfer and the recording time can assume only a finite value. If a PCM is used, the data are already recorded in digital form, in this case it is important to remember that resolution depends on the range and number of bits used to represent the number[3].

In both cases, any data recording uses magnetic tapes, with a preference for PCM recording, since when an FM recording is data should be digitalized anyhow, as the only way of processing them is nowadays invariably using a computer.
Sometimes data are not only recorded on board but they are also transmitted to the ground via telemetry. This choice is adopted substantially for reasons due to the impossibility of boarding the recording system, or because of the need of performing a real time analyses on the flight data whenever an adequate computational power is not available on board, a fading constraint more and more. In this case the decoding is performed in real time to permit a preliminary data analysis. Let us remember that, to prevent the loss of data, in case of on board recording failures or in the, worst and sad, case of a loss of the whole aircraft and crew, telemetered data are preferebly recorded also at the ground base.
It is interesting to notice that the telemetric transmission is in FM at the highest frequency, about GHz, so the aircraft must be ”seen” by the telemetry antenna. For this reason the quality of telemetric data can be lowered because of the possible loss of radio connections or simply because of the limits of the transmission rate.

[3]For example, consider the static pressure measured by the anemometer, the measure range is included between 100 and 1100 mbar.If the internal representation is 12 bit, the resolution available is $\Delta P = ; \frac{1100-100}{2^{12}} = \frac{1000}{4096} = 0.24\,mb$

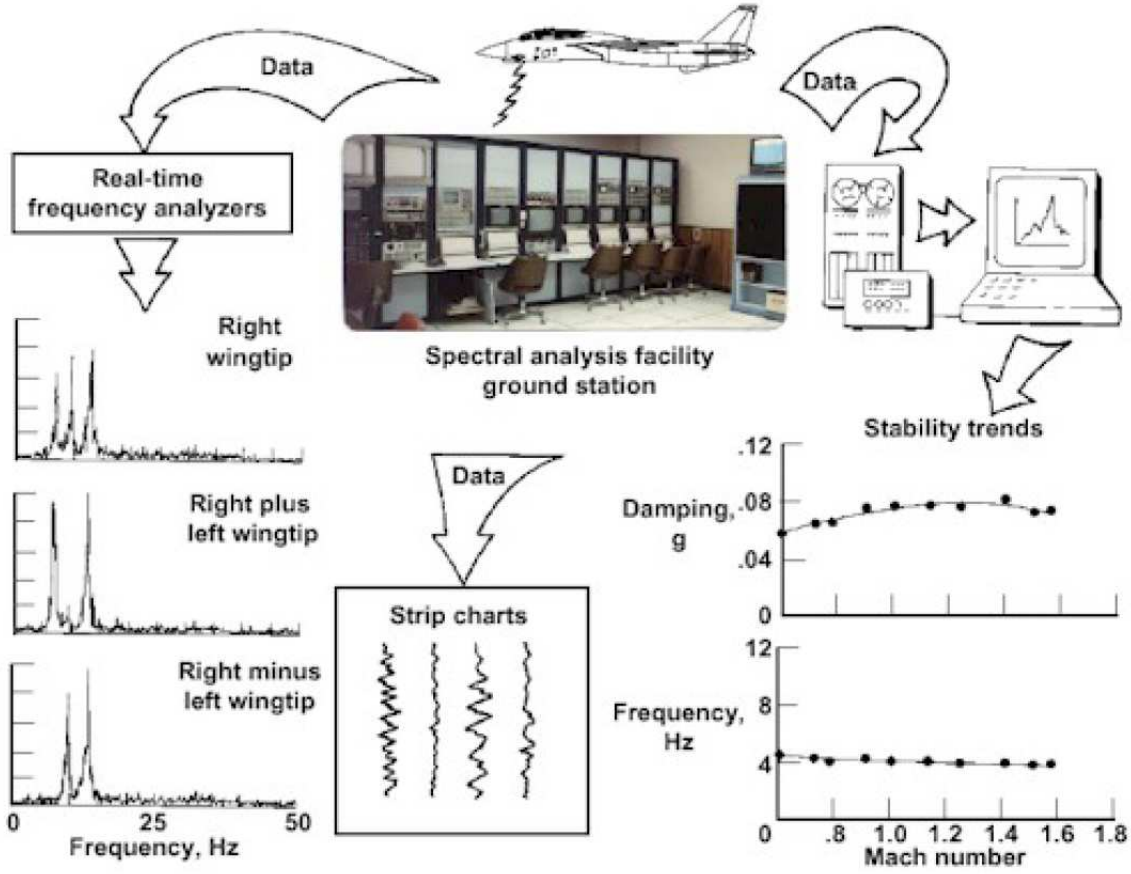


Figure 2.1: `Test Configuration`

Chapter 3

Excitation

The excitation of the aeroelastic system is a fundamental part of flight flutter testing. It will be very difficult to identify any type of aeroelastic instability without a suitable excitation. The term suitable stands for the capability of an excitation to introduce into the structure a quantity of energy adequate to excite all of the significant modes, with an amplitude level making it possible to correctly identify stability. For an example, we remind to what happened to Transavia PL12/T-400 (a small aircraft for agricultural use,1986), [37]. In the initial phase of its flight tests, the excitation system was limited to the atmospheric turbulence and impulses on the flight controls. During these tests there was no flutter instability. During a following flight, not dedicated to any flutter investigation, in adverse weather conditions, the aircraft showed violent oscillations of the rudder and tail. This experience teaches us that an unsuitable excitation level could lead to:

- the impossibility of recognizing an approaching instability, (flutter onset);
- wrong evaluation of the damping;
- an excessive scatter for the identified parameters.

As we said in the previous chapter, we will consider only the case of having a single excitation command, which can involve the presence of more than one exciters, but with all of the exciters controlled by the same signal. Nonetheless, the single excitation assumption deserves an important comment as, even if the driving signal is unique to all the exciters and they are supposedly of the same making, the measured forces and moments transmitted by each exciters, will be different eventually. This is because exciters will not be exactly the same and differences in the installation of each exciters and added noise will occur. Choosing to retain one, or more than one input, applied to the system, means also choosing if considering or not the transfer function between the excitation signal and the effective excitation to the structure. A choice that might end in having implications not only for the test phase, since instrumenting the exciters mounts for measuring the forces and moments transmitted to the structure can be required, but also in the following identification phase, as it will be shown in the next chapters.
As we have underlined in chapter 2, exciters are placed mostly at the tips of the aerodynamic surfaces because at those positions they generally will better excite all of the modes of interest.
In a flight test, differently from a ground vibration test, there are always unmodeled and uncontrollable disturbances, e.g. atmospheric turbulence.

with respect to them, it is interesting to observe that if the aircraft has a significant geometric symmetry, as it happens in most cases, it is possible to actuate the exciters symmetrically and antisymmetric, so managing the excitation of either symmetric or antisymmetric modes only. In practice, this is obtained by feeding the actuators with a single command, either in phase or counter phase to left and right exciters. This kind of excitation can produce an uncontrollable model error if the aircraft is not perfectly symmetric. As said, these inconsistencies can be partly solved by measuring the actual excitation at each application point and using a multi input treatment of the related experimental data.

3.1 Type of excitation

Once the exciters positions have been established, so to provide an adequate controllability, a suitable level of excitation has then to be applied, covering the all frequency range of interest.
In particular, it would be useful to have an excitation spectrum constant enough within the frequency range of interest. Clearly, lowly damped modes will be more excited, because a lower excitation energy is requested when we are close to a resonant frequency. Conceptually we should think about introducing a uniform energy into the system.
A first important subdivision of the excitation techniques is between:

- measurable excitations;
- unmeasurable excitations;

The excitation mode will influence also the following identification of the model parameters. If we want to perform the identification in the frequency domain, it is better to obtain the frequency response function from measures, and so conceptually we need to measure the excitation. If this is not possible, we must take into account that some identification techniques will be applicable, some other not.
Moreover, when it is possible to measure the excitation, we must ask ourselves whether it is convenient or not to do so. This decision must be taken by looking at the successive identification techniques to be used.
Another point, if a measurable excitation is chosen, is what to measure: whether to measure the control signal given to an exciter or its very output. Clearly the difference between one and the other is simply the transfer function of the excitation device. If we measure the control signal, we must consider the transfer function of the exciter unless it has a very large bandwidth. Typically it is common to measure the excitation force transmitted to the structure, e.g. by instrumenting the exciter mount.
In conclusion, it is very important to keep in mind that in taking these decisions, there is not a completely wrong choice or a completely right one. The final decision is the best compromise between many needs and the processing abilities of whom performs the test.
In conclusion, regarding the excitation choice, usually different types of excitation methods are used during a flight flutter test.
Typically the flight flutter test planning is made of many phases: first of all the low part of the flight envelope, in which flutter is not expected, is freed rapidly enough with simpler types of excitations then, as flutter is approached, the most suitable excitation is established.

3.1.1 Impulsive excitation

An impulsive excitation can be obtained using any exciter, by simply applying, as fast as possible, an impulsive command to the device, irrespective of it being measurable or not.
The main idea is very simple: if the applied impulse can be retained an ideal one, the system response will be similar to the impulsive response of the system. Being it so by frequency transforming the acquired system response we will obtain a good approximation of the system transfer function directly. Moreover, if the applied impulse is close to an ideal one, we can also neglect its measure.
Notice that if the system is well damped, the use of an impulsive excitation might result in being somewhat unsuitable. In fact its transient response will decay rapidly and so the contribution of any persistent noise to any measured response risks of being highly significant in term of signal to noise ratio. Such a problem is worsened for well damped systems where the hiding within the noise will appear rather soon, so aggravating the loss of resolution associated to well damped impulse responses. The problem is now to understand how we can apply an impulse, which, at least in the frequency range of interest, can be retained as an ideal one. We have at our disposal the choice of two parameters:

- duration
- intensity

The duration of the impulse can be estimated, very roughly, by the inverse of the maximum frequency which we are interested in. The impulse must be suitable to excite well all the frequencies in the desired frequency range, and so its spectrum must be "as flat as possible" at least within the assigned range. To understand what we intend with "as flat as possible" we can consider the case of a rectangular impulse:

$$f(t) = \begin{cases} \frac{Q}{a} & -a < t < a \\ 0 & everywhere \end{cases} \tag{3.1}$$

whose Fourier transform is given by:

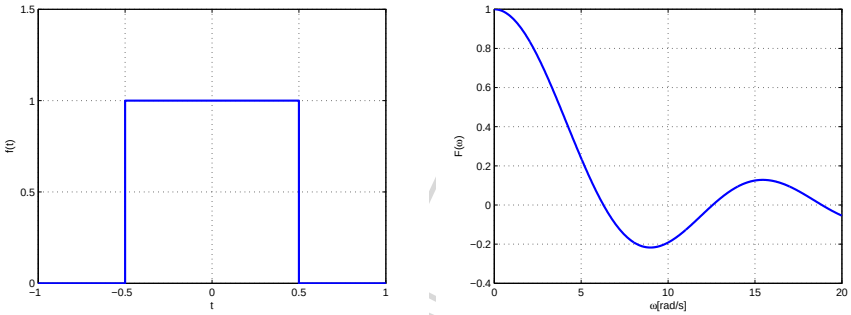
$$F(\omega) = \int_{-\infty}^{+\infty} f(t) e^{-j\omega t} dt = Q \int_{-a}^{a} e^{-j\omega t} dt = -Q \frac{e^{j\omega t}}{j\omega}\Big|_{-a}^{a} = \tag{3.2}$$

$$= 2Q \frac{sin(\omega a)}{\omega} = 2Qa \frac{sin(\omega a)}{\omega a} \quad \omega \neq 0 \tag{3.3}$$

Defining $A = 2Qa$ the area of the rectangular impulse, calling $x = \omega a$, we can rewrite 3.3 as:

$$F = A \frac{sin(x)}{x} \tag{3.4}$$

In choosing a we must guarantee that the frequency range of interest is included in the range of those frequencies for which the amplitude spectrum values normalized to the area of the rectangular impulse F/A will appertain to the range $[1 - 0.8]$.
Notice that for an appropriately assigned impulse duration it is not important if the excitation law is rectangular or not, as what will count should be just an equivalent pulse area.
Regarding the intensity we should keep in mind, as it will well explained in paragraph 3.2, that what is important is the affording of an excitation apt to introducing into the system a suitable energy level.



(a) `Rectangular Pulse` (b) `Fourier Transform of the Rectangular Pulse`

Figure 3.1: `Rectangular Pulse in time and frequency domain`

A simple procedure to estimate the duration a

According to what we have just remarked, the correct duration must be at least such that $\overline{B} = F/A = 0.8$. So recalling 3.4, the problem is to solve the non-linear equation:

$$\overline{B} = \frac{sin(a\omega)}{a\omega} \tag{3.5}$$

with respect to a, once we have fixed ω and the desired $\overline{B}$. Nevertheless we can simplify such a calculation by observing that we are interested to a low frequency range and the duration of the impulse is small, so we can substitute to the function $sin(a\omega)/a\omega$ its Mac-Laurin series:

$$\frac{sin(a\omega)}{a\omega} \simeq 1 - \frac{1}{6}a^2\omega^2 + \frac{1}{120}a^4\omega^4 + \cdots \tag{3.6}$$

Keeping the series to the second order and substituting it into 3.5 we obtain:

$$\overline{B} = \frac{sin(a\omega)}{a\omega} \rightarrow \overline{B} \simeq 1 - \frac{1}{6}a^2\omega^2 \rightarrow a \simeq \frac{\sqrt{6(1-\overline{B})}}{\omega} \tag{3.7}$$

leading to:

$$\omega = 2\pi f \rightarrow a \simeq \frac{\sqrt{6(1-\overline{B})}}{2\pi f} \tag{3.8}$$

Equation 3.7 and 3.8 express the relation between the impulse duration and the frequency content we want to excite.
Assuming 0.8 as a suitable value for the ratio $\overline{B} = F/A$, equation 3.7 and 3.8 can rewritten as:

$$a = \frac{1.09}{\omega} \tag{3.9}$$

$$a = \frac{0.17}{f} \tag{3.10}$$

Hence, if we are interested to well excite up to 10 Hz we should apply an equivalent impulse whose duration is 0.02 s, instead of rough estimation of 0.1 s, obtained by taking the inverse of the maximum frequency of interest.

3.1.2 Frequency sweep

Theoretically, the ideal way of performing a flight flutter test for each frequency in the band of interest, is to perform a series of flight, using each time a different amplitude, verify if we have a correct linearized and then proceed to another frequency in the same way. Clearly this strategy cannot be applied for economic and time reasons.
Alternatively, the frequency sweep technique is used in practice. This kind of excitation should be implemented in a way allowing to measure the excitation forces. The main idea is to use excitation forces that have the characteristic of exciting all of the frequencies of interest, but differently from an impulse, all the corresponding spectrum is not acted upon simultaneously, but swept over so to excite each frequency for an adequate period of time.
Their general form is given by:

$$u(t) = A sin(f(t)) \tag{3.11}$$

where A is the amplitude and $f(t)$ is the time law which assign the frequency variation, its instantaneous value being: $\omega = \frac{df}{dt}$, and can be linear, logarithmic or exponential,. Generally, equation 3.11 is written as:

$$u(t) = A sin(\omega(t)\, t) \tag{3.12}$$

where, given the frequency range of interest, $[\omega_{min} - \omega_{max}]$, and a T_s sweep time, $\omega(t)$ will be:

- linear sweep:

$$\omega(t) = \omega_{min} + \frac{\omega_{max} - \omega_{min}}{T_s} t \quad \rightarrow \quad \omega(t) = at + b \tag{3.13}$$

$$\begin{cases} a = \frac{\omega_{max} - \omega_{min}}{T_s} \\ b = \omega_{min} \end{cases} \tag{3.14}$$

- exponential sweep, [46]:

$$\omega(t) = \frac{\omega_1 T}{\ln\left(\frac{\omega_1}{\omega_2}\right)} \left(e^{\frac{t}{T} \ln\left(\frac{\omega_1}{\omega_2}\right)} - 1 \right) \rightarrow \omega(t) = A\left(e^{Bt} - 1\right) \tag{3.15}$$

$$\begin{cases} A = \frac{\omega_1 T}{\ln\left(\frac{\omega_1}{\omega_2}\right)} \\ B = \frac{1}{T} \ln\left(\frac{\omega_1}{\omega_2}\right) \end{cases} \tag{3.16}$$

- logarithmic sweep:

$$\omega(t) = \frac{\omega_1 T_s}{\left(\frac{\omega_1}{\omega_2} - 1\right)} \ln(1 + Bt) \rightarrow A ln(1 + Bt) \tag{3.17}$$

$$\begin{cases} A = \frac{\omega_1 T_s}{\left(\frac{\omega_1}{\omega_2} - 1\right)} \\ B = \frac{1}{T_s} \ln\left(\frac{\omega_1}{\omega_2} - 1\right) \end{cases} \tag{3.18}$$

The sweep time, i.e. the total duration of an excitation, determines the frequency resolution and the number of acquired samples. So the longer the sweep time is, the higher the frequency

resolution is. If the sweep time is too short, the risk is to lose the contribution of some mode in the response. If a mode is excited its transient is influenced by its own damping, and the excitation time must considered also under this aspect. The lower the damping is, the slower the sweep time must be. The ideal condition of the excitation should be to use as slow a sweep as the time required to reach an almost steady harmonic response. In this situation at each frequency we will see almost that frequency only. Often, this strategy leads to an excessive test time. So a compromise between reducing test time and having a good resolution has to be taken, recovering the transfer function by processing an often fast continuous transient response.
It is possible to prescribe an "optimum" sweep rate for a given system, taking into consideration its prevailing damping levels. By way of example we cite that the ISO Standard prescribes the following maximum sweep rate through a resonance, [20]:

$$S_{max} < 216\,\omega_r^2\,\zeta_r^2 \quad Hz/min \quad linear \quad sweep \tag{3.19}$$

$$S_{max} < 310\,\omega_r^2\,\zeta_r^2 \quad octaves/min \quad logarithmic \quad sweep \tag{3.20}$$

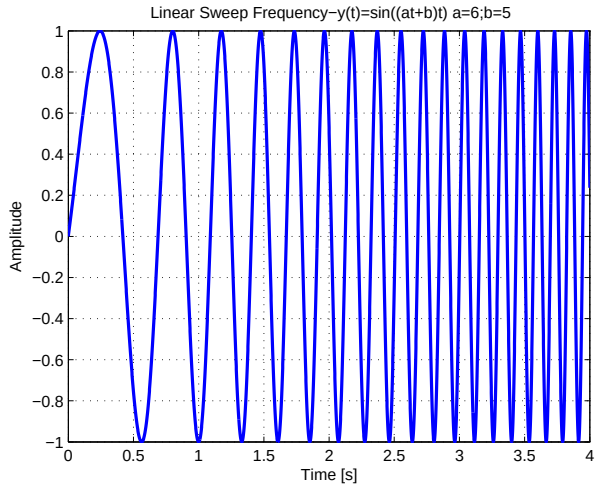
The Power Spectral Density (PSD) of a linear sweep shows two peaks, at the lower and higher nominal sweep frequencies respectively. Between the two extremes, the PSD is almost constant, so the linear sweep well excites all the modes within the considered range.
The logarithmic sweep persists with more energy at lower frequencies and less at the higher ones.
During the same flight stabilization more than one excitation is often applied. In fact for increasing the quality of experimental data, we have T_s, which determine the frequency resolution, while the number of repetitions can improve the signal to noise ratio though averages.
It is known that improvements obtained through a mean of a number of repeated measures, is given by $\sqrt{N}$, making it evident that a high cleaning of noisy signals requires many repetitions. Moreover, in view of a following identification in the frequency domain, these excitations might be repeated periodically.

Chirp signal excitation

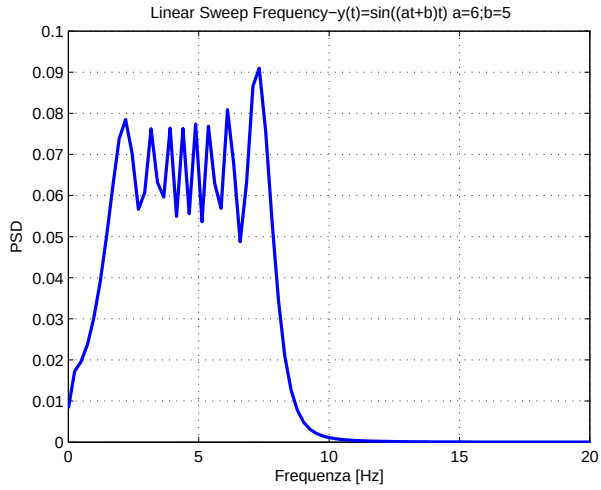
The chirp signal consist of a short duration signal sweep which has the form shown in 3.4. The chirp excitation is a transient excitation. The frequency content of a chirp can be precisely chosen by through its period and starting and finishing frequencies, possibly tailored so to optimize its processing.
As for the normal sweeps mentioned before it is common to repeat the transient event more than once, averaging the results to get the final outcome.

Random

A random signal, by its definition, is unpredictable and the values it assumes do not follow any deterministic law, hence the signal is not repeatable. In our case such a signal is implemented using deterministic samples, well satisfying the associated amplitude and frequency casual properties.
A random signal can be characterized only using statistical indicators, e.g. its power spectral density in the frequency domain or the probability density in the time domain.
Once more, the use of a random signal requires it has an approximately constant PSD in the



(a) `Time History`



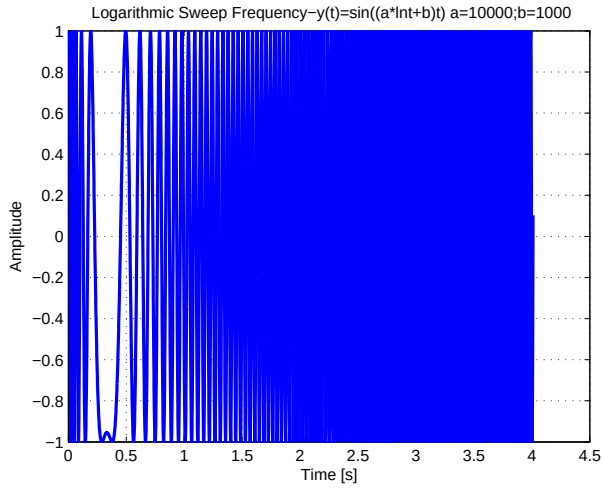
(b) `Power Spectral Density`

Figure 3.2: `Linear Frequency Sweep`

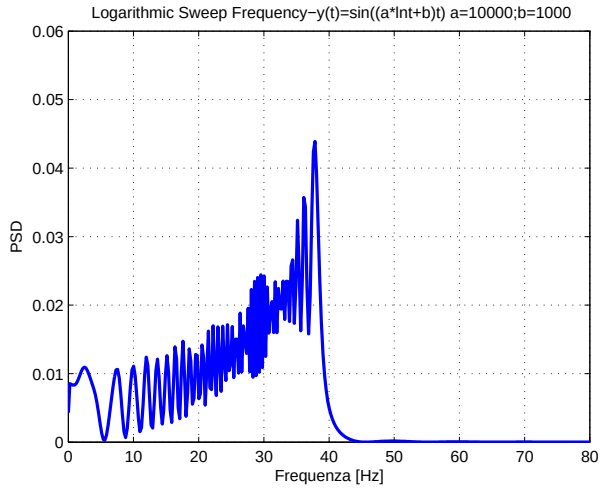
frequency range of interest. An advantage of this kind of excitation is the relative simplicity with which it is possible to define its frequency content through a shaping filter [45].

3.1.3 Excitation using turbulence

Atmospheric turbulence is an always present disturbance that can be converted to being an excitation source. The main problem in using turbulence is the impossibility of both measuring and controlling its level and so the excitation it provides.
Turbulence has a low pass characteristic, hence it allows to well excite all those frequencies that are roughly included within its bandwidth. Since it is not measurable, the turbulence, provided it affords an adequate excitation level, should be considered as a suitable excitation source only if it had a significantly constant spectrum for the frequency range of interest. The relevance of



(a) Time History

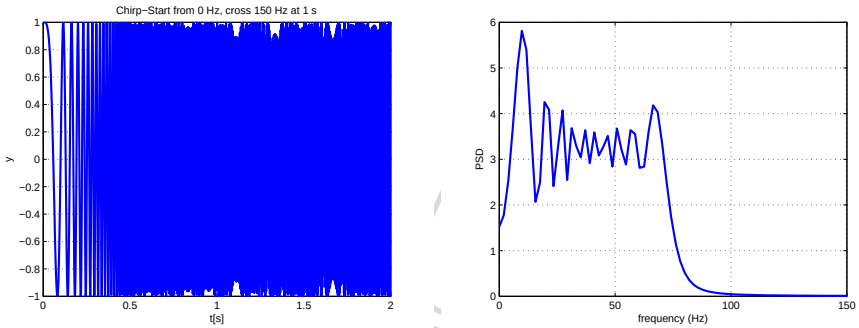


(b) Power Spectral Density

Figure 3.3: Logarithmic Frequency Sweep

such a point can be inferred intuitively by noting that the frequency response peaks for all those frequencies outside the constant spectrum part will appear lower than they are for real.
Since we cannot measure the associated level we cannot know if the related peaks trends, at changing flight conditions, are softened because of the damping or because of an unsuitable excitation. Clearly, a bad estimation of the response peaks, will bring a bad estimation of the model parameter, as it will be underlined in the next chapters dealing with the V-g identification in time and frequency domains. In essence turbulence can be used as a suitable excitation only for low frequencies.
For a better comprehension of the problem we consider an example regarding a 3 degrees of freedoms mechanical system[1]. Figure 3.5 reports a scheme of the considered system. It is made of 3 masses, connected togethers with springs and dampers. In the table below, the dynamic

[1]The system is taken from L.Meirovitch, Analytical Methods in vibration, chapter 9, Damped system.



(a) Time History (b) Power Spectral Density

Figure 3.4: Chirp Excitation

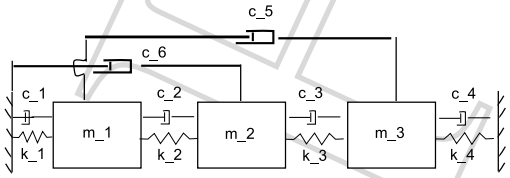


Figure 3.5: Mechanical Test System

characteristics in terms of frequencies and damping are presented.

	Frequency-rad/s	**Damping**
Mode 1	8.12	3.47e-002
Mode 2	13.1	2.54e-002
Mode 3	17.8	2.13e-002

Suppose to excite the system using a random force applied to the third mass and to identify the transfer function of the displacement of the first mass, starting from acquired data. A low-pass filter, with a cut-off frequency of 15 rad/s, is assumed to shape the random force, its transfer function modulus is represented in figure 3.6.

In figure 3.7, a comparison between both the modulus of the related transfer function and the

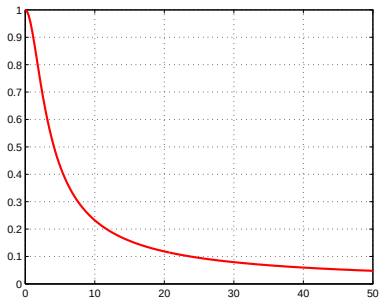


Figure 3.6: Filter transfer function

ideally identified square root of the response power spectral density is presented. We notice that the peaks, most of all those at higher frequencies, are softened by the simulated acquisition of the result obtainable without measuring the random disturbance. Hence, they appear lower than how they are for real.

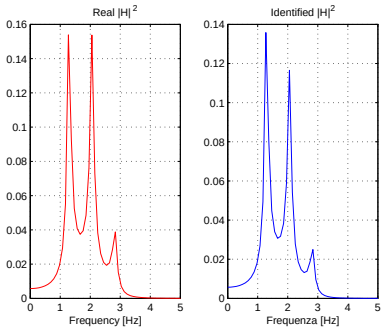
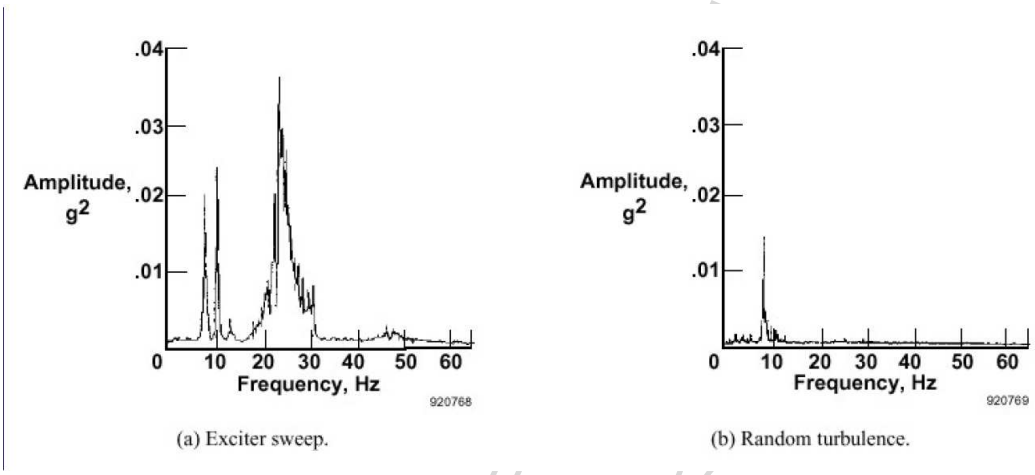


Figure 3.7: Comparison between the modulus of the transfer function and the pseudo identified one

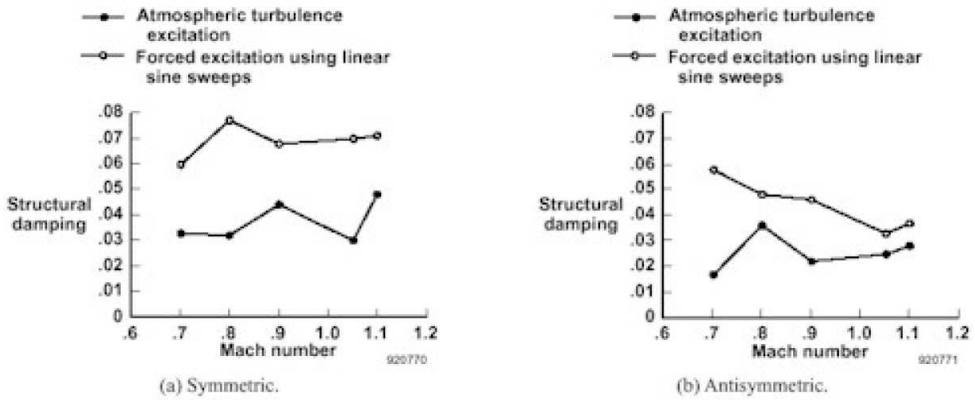
3.2 The importance of an appropriate excitation level

The real nature of the system is non-linear, so our aim is to obtain an equivalent linearized model through a suitable excitation.The requirement of having a correct equivalent linearization guarantees that the identified coefficients do not vary significantly if the excitation amplitude varies within acceptable bounds, and so that frequencies and damping do not vary with the variation of the excitation amplitude.
The equivalent linearization involves the use of an excitation whose level should not be either too high, the risk is that of damaging the structure, trivial and rare, or unduly amplifying nonlinearities like plays and contacts, or too low, otherwise the system response cannot overcome noises and disturbance in general and the subsequent identification of the characteristics parameter will not be accurate; the main risk is that of a mistake in identifying damping. Still the excitation level should be significant enough; it is important to keep in mind that, because of an uncontrollable environment, the quality of measure in flight tests is always lower than that of ground tests. In figure 3.8, taken from [13], the comparison between two Frequency Response Functions (FRF) pertaining to the same frequency band is shown. They are obtained with two different excitation types, random turbulence and frequency sweep. The response levels are clearly different and therefore the quality of the identified FRF. Trying to identify the aeroelastic damping through a pseudo FRF related to turbulence can lead to serious mistakes, for the reasons pointed out in paragraph 3.1.3. In figure 3.9 we see the comparison of the damping identification for symmetric and antisymmetric bending wing modes identified by analyzing the FRF due to frequency sweep and random turbulence excitations: the damping trends are very different, [13].
The excitation level, regardless of the excitation method, must be such to guarantee the equivalent linearization and the persistent excitation of the system. We have talked about the equivalent linearization and its importance at the beginning of the paragraph, while regarding the persistence of an excitation we simply state that it is an important necessary condition for granting a



(a) Exciter sweep. (b) Random turbulence.

Figure 3.8: Acceleration response due to sweep excitation and random turbulence



(a) Symmetric. (b) Antisymmetric.

Figure 3.9: Structural damping identification

well conditioned identifiability of the system.
In fact the loss of identifiability has a practical implication in the solution of the numerical problems associated to the application of an identification technique. As it will be made clear later on, all the identification techniques lead to the solution of a linear system, whose coefficient matrix is well conditioned only if a persistent excitation is applied.

3.3 Exciters

The general characteristics which an excitation system must satisfy are:

- to guarantee a significant level of excitation;
- to cover the frequency range of interest;
- to have a contained weight, so to affect the least the modal characteristics of the structure, being as little intrusive as possible in general;
- to negligibly change the aerodynamics, being as little intrusive as possible once more;
- the power consumption characteristics must be sufficiently contained to be satisfied by that available on board as it is.

The most common excitation devices are:

- flight controls;
- pyrotechnical exciters;
- inertia exciters;
- electrodynamic exciters;
- aerodynamic exciters (vanes, spoilers, spinning cylinders);
- atmospheric turbulence.

3.3.1 Flight controls

Flight controls historically have been the first system used in flight flutter tests. A problem regarding their use could be their somewhat limited frequency band and the possibility to suitably measure the related input to the structure, without changing too much what they are. Clearly the evaluation of their suitability is also connected to the their mode of use and actuation.
On small aircraft, whose command chain is purely mechanic, the actuator is the pilot himself. In this case, the effective band is very small, only few Hz, 1-2 Hz. If the actuators are servoed, the effective band can be significantly larger, even if it is likely confined to a few Hz, 3-5 Hz, again. At present, thanks to FBW flight controls and to much improved servo actuators, for some types of aircraft, big transport jumbos with critical low frequency modes [2], or military mounting new high performance actuators, it is possible to use the flight control system in a special test mode for flutter testing. For example a military aircraft equipped with advanced actuators can have the possibility of achieving an usable band of up to 20 Hz, by just changing the flight control operational mode in software. A flight control system can be used to generate practically all of the mentioned command histories. The main advantage in the use of this excitation device is linked to its low intrusiveness both from the aerodynamic and the structural point of view. In fact, it is not required to install any particular device, at most the excitation control box which is installed in the cabin.
A disadvantage can be the need of installing special actuators, as we have explained previously, because the standard ones do not permit to reach the requested frequency range.

[2] The Airbus A380 has more than 10 vibration modes below 5 Hz

So flight controls are an often used excitation source, most of all with modern aircraft equipped with FBW flight system controls. The input they provides to the following identification phases is generally their deflection.

3.3.2 Pyrotechnical exciters

Generally known as ”bonkers”, ”thrusters”, ”ballistic exciters” or ”impulse generators”. Historically this device was one of the first used to excite the structure, circa 1940. It consists in small rockets fed with solid propellent, which burn for a short time, so producing a transient response, with varying possible thrust levels. Generally speaking they are simple and light and so do not modify the system if used singly. The problem comes from the need of installing many of them at once, in which case they can cause noticeable changes to modal characteristics and aerodynamics. Their cons are:

- they can be fired only once;
- the frequency band they are able to excite depends on the duration of the pyrotechnical charge. To appropriately excite a frequency band of interest it is often necessary to install more than a single piece, with different combustion laws.
- Typically they are fired one at a time, because it is difficult to phase the simultaneous firing of more than one.
- A measure of the force they apply to the structure is usually not available, it would be of little use anyhow, as it will be seen in the identification chapters.

Catalogues of these devices report their force level and burning time, possibly along with equivalent impulse.

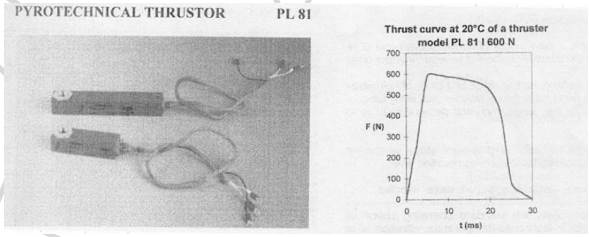


Figure 3.10: Bonkers

3.3.3 Mechanical inertial exciters

There fundamentally two types of inertia exciters, i.e. those based on rotating masses and those based on oscillating masses.
The devices based on rotating masses are made of rotors counter rotating at the same angular

velocity, with an eccentric mass mounted on each of them. The excitation is produced by the centrifugal force acting on the each eccentric mass:

$$F = m\omega^2 r \tag{3.21}$$

where ω is the angular speed and r the offset of the mass with respect to the motor axis.
Since the two masses have the same eccentricity and counter rotate at the same speed they generate a force along a desired fixed direction, whose intensity and frequency content can be varied by controlling offset and speed. It is thus simple to provide any viable swept excitation of interest.

To generate a significant excitation force it might be required to use devices whose size can

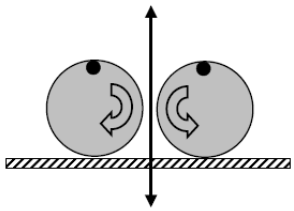


Figure 3.11: Schematic Working of a rotating inertial Exciter

affect the aerodynamics and/or alter the modal characteristics, because of the use of a sizeable mass. So mechanical inertia exciters could become somewhat intrusive. These kinds of devices are suitable for those aircraft, typically the military ones, carrying under wing loads, like war loads or fuel tips and tanks, in which case it is possible to substitute part of their mass with an exciter, or even the whole part with an excitation pod.
In addition to rotating mass inertia exciters, there are also oscillating exciters, like that shown in the figure 3.12.
As an example we report the features, in terms of exciter mass and force intensity, of two exciters, one used on the Convair F102-A (Delta Dagger) and the other used on B1-B.
On the F102-A a it had a weight of 4 kg and provided a force ranging between 100 to 1500 N, on the B1-B was a 20 kg mass was used and the maximum force obtainable was of about 2500 N. By measuring the rotation/oscillation speed and the offset, the applied force is indirectly measured by using the mentioned formula. Velocity and offset servoing are usually precise enough that the related command can be directly taken as their measures.

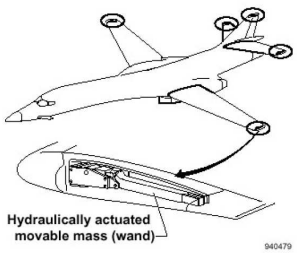


Figure 3.12: Inertial Exciters

3.3.4 Inertia electrodynamic exciter

These devices are also known with the name of electrodynamic shaker, and are based on voice coil motors.
The name ”voice coil” derives from the origins of the earliest devices, which were scaled up from the drive mechanisms of bassos loud speakers. These exciters are made of a movable coil, of lenght l, in a permanent magnetic field, B, so that by controlling the current, i, flowing into it it is possible to generate a desired force according to:

$$F = B\,l\,i \tag{3.22}$$

A simplified scheme of a voice coil motor is presented in figure 3.13. This exciters is very

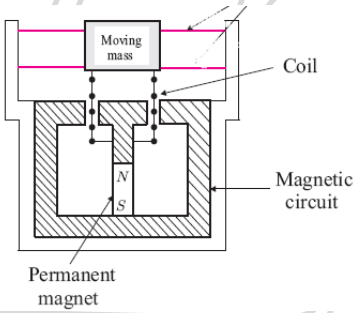


Figure 3.13: `Working scheme of an Electrodynamic exciter`

versatile, but has a limited stroke and, since it needs a reaction mass to generate a force at the mount, it cannot generate a constant force, meaning it has difficulties at the lowest frequencies also. Generally the bulkier the exciter is, the bigger the excitation level. Hence, if an high excitation level is required, the needed exciter can become very intrusive. In short, even if it is more versatile than counter rotating masses, it is of their same kind, so it shares most of their pros and cons too. The applied force is simply measured as the inertia force obtained through an accelerometer mounted onto the, known, oscillating mass.

3.3.5 Aerodynamic exciters: vanes

Oscillating vanes are aerodynamic appendices, mostly mounted at the wing/tail tips. They are fixed to a shaft oscillating around a mean angle, hydraulically-mechanically-electrically powered. The oscillation produces a controllable varying aerodynamic force on the aircraft. The level of the excitation force depends on the vanes size, dynamic pressure and rotation angle. Its main advantages are:

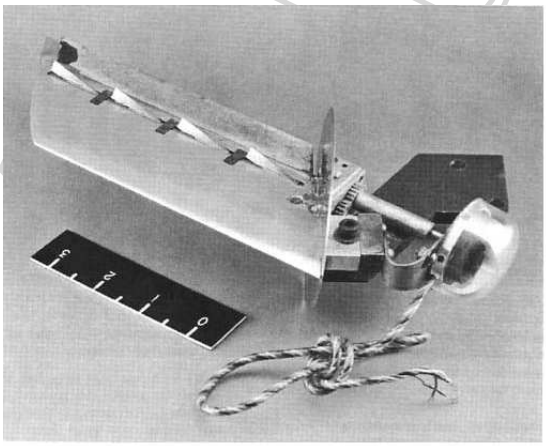
- a good excitation at low frequencies;
- the amplitude at high frequencies is limited only by the actuating system;
- the excitation frequency and amplitude, at a given velocity, can be easily controlled;
- the time history is repeatable;

- a measure of the excitation force can be obtained either by instrumenting its mount point appropriately, better, or using its deflection as input.

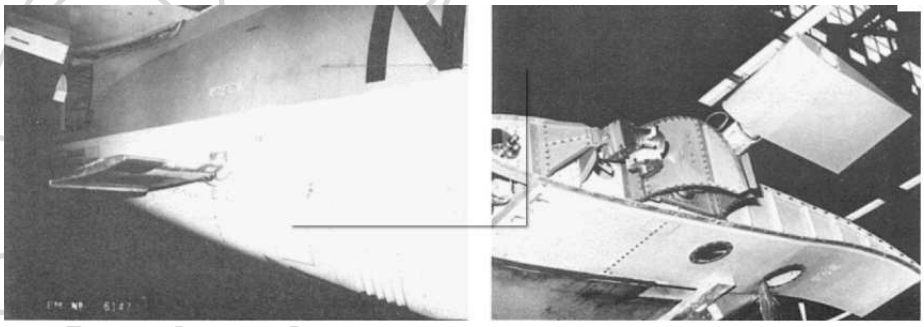
Clearly the control signal must have a suitable frequency content to excite all modes of interest. On the other hand, their main disadvantages are:

- the maximum force intensity available depends on the square of velocity;
- the total mass of the system can be relevant;
- its placement at the tip of lifting surfaces generates an aerodynamic disturbance, i.e. aerodynamics vanes could be a somewhat intrusive excitation device.
- a significant power can be required to provide the wanted exciting motions.

Their placement at the tips of an aerodynamic surface, where a significant roll up tip vortex exist, makes their design and actuation not so simple.



(a)



(b)

Figure 3.14: Aerodinamic Vanes

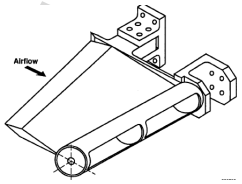
3.3.6 Aerodynamic exciters: rotating cylinder

Such an excitation system is made up of three main components:

- a cockpit control panel, which should be installed so to allows the pilot to control the excitation to be applied, according to the mission testing plan;
- an electronic box containing the needed hardware, i.e. signals conditioning and the close loop motion controller.
- a fixed exciter vane mounted at a surface tip. Such a vane consists of two parts: a leading diamond-shaped symmetrical airfoil section and a rotating slotted cylinder, driven by an electrical motor, at the trailing edge.

The system incorporates several operating modes: constant frequency, linear or logarithmic sine sweeps and a quick-stop feature for free decay measurements. By choosing between a high/low force amplitude mode it is also possible to act on excitation level. The installed device can be intrusive from the modal point of view because of the mass distribution changes it causes. It is intrusive from the aerodynamic point of view also, somewhat less than oscillating vane though, because of the way it generate the exciting flow. The excitation force, generated by this device,

(a) Rotating Cylinder-Real Exciter



(b) Rotating Cylinder-Schematic Draw

(c) Rotating Cylinder-Installation

Figure 3.15: Rotating Cylinder

is at a frequency which is twice that of the rotating cylinder. In fact, the two slots cuts deflect the flow upward and downward alternatively, so generating a periodic lift. To better understand the

working mode of the device, one should refer to the figure below, fig. 3.16, a slight modification of what found in [13]. First of all, we can choose a reference point on the cylinder, X, which is on the opening of the slot. During a rotation X will occupies several positions, denoted with A,B,C,D and E, to which correspond different values of the excitation force. Hence when X is:

- at point A, the flow trough the slots is straight on, no lift force is generated;
- at point B, the flow is completely deflected downward and the lift force reaches its maximum positive value;
- at point C, the cylinder has turned 90° with respect to the starting point, the flow straightly passes trough the slots so no lift force is generated again;
- at point D, the flow is completely deflected upward and the lift force reaches its maximum negative value;
- at point E, the cylinder has turned 180° with the respect the starting point, the flow is straight again and the lift is null as well again.

It can be easily noticed that an half rotation of the cylinder, corresponds to a single full forcing period, hence a complete rotation leads to: $f_{force} = 2\, f_{cyl}$. The amplitude of the excitation force depends, at a given dynamic pressure, from the amount of the slot opening, which regulates the stream flow trough the slots themselves. To control the slot opening there are two strategies, which can be combined. The first is to act on the rotational direction of the cylinder drive motor: in fact, reversing it causes half of the span wise slot opening to be blocked by an inner cylinder within the inboard slot. Closing the inboard slot allows to attenuate the excitation force by half in flight. When the device is used at high dynamic pressures, it is necessary to install a plug, spanning 25 percent of the outboard slot opening. That because wind tunnel tests have demonstrated that at high pressure the vane can generate a higher force than the desired one. Hence, in conclusion, when the exciter is used in the high force mode, only the plug in is inserted and the slotted cylinder is 75% open. In low force mode instead, the inner cylinder is at the inboard slot and the plug is inserted also, so that the slotted cylinder is 25% open.
The exciter system is capable of excitation frequencies up to 50 Hz, and the force level can by measured by instrumenting the mount of the device with strain gage bridges measuring the applied shear forces and moments.
Generally, the characteristics of this device are similar to those of the oscillating aerodynamic vanes, but it is lighter, less intrusive aerodynamically, of simple construction and since the power required to rotate the slotted cylinders is minimal, the use of a low wattage motor is possible. So, since it can use a 28 V source, it can be readily use a readily available aircraft power supply.

3.3.7 Atmospheric Turbulence

It does not need any particular device to be mounted on board while exciting all the symmetric-antisymmetric modes at the same time.
Turbulence is the random variation of free stream air velocity either in intensity or in direction, it is generated by the variation of atmospheric conditions. It has the following disadvantages:

- it is unmeasurable;

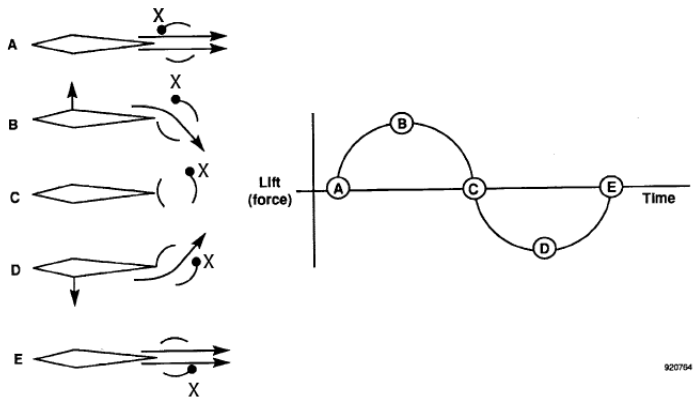


Figure 3.16: Rotating Cylinder Working scheme

- its intensity is neither controllable nor repeatable and the excitation it provides can be inadequate for a reliable system identification;
- the previous point is worsened by the relatively low signal to noise ratio it affords, so data acquisition has to last for a somewhat long time to improve its statistic properties;
- it excites low frequency modes only;

For an idea of the frequency content of turbulence one could refer to the turbulence models frequently used to evaluate aircraft performances, i.e. the Dryden and Von Karman models. The power spectra for these models are defined analytically in MIL-F-8785C. Each model consists of three velocity spectra, corresponding to the three axes associated with a body-fixed coordinate system. The frequency variable is spatially referred to a ”frozen” turbulence field, a concept that has been known to produce realistic results for normal aircraft speeds. The advantage of the spatially referenced spectra is that the turbulence characteristics are independent from the aircraft speed. The reader is referred to such a reference for the related formulas and figures.

Chapter 4

Identification: introduction

Identification is a numerical processing phase applied to acquired test data. Its aim is to reconstruct the behavior of a system at hand through its excitation with suitable input, measuring its response and, if viable, the input itself.
So we can classify our identification techniques accordingly to having or not available measured input and the related processing can be carried out both in the time and frequency domain.

4.1 Disturbances and their modeling

We will assume that our experimental activity is based on the assumption that the LTI system to be tested is either linear or linearized around a reference steady test condition. Moreover we should appropriately care of some important operating factors, in particular:

- Non-linearities of the aeroelastic system, possible modeling errors and the fact that in flight there is no possibility of controlling external environmental conditions;
- The excitation force acting on the system is given by the summation of the desired excitation, as generated by the mounted exciter, at times disturbed by implementation specific disturbances;
- there are uncertainties connected to measures, usually called measurement noise, which can be linked to the measure system, its intrusiveness and the way we treat the acquired measures in between acquisition and processing.
- last but not least, atmospheric turbulence is always present; it enters the system and appears at its output after being filtered by the system itself.

The scope of this paragraph is to show how disturbances can be framed and interpreted for paving the way to standard identification schemes. At a given flight condition the set of equation describing our aeroelastic aircraft in the Laplace domain is:

$$\begin{cases} \left(s^2[M] + s[C] + [K] - q[H_{am}(s)]\right)\{q\} = [B](\{u\} + \{n_u\}) + \frac{q}{V_\infty}[H_{ag}(s)]\ \{v_g\} \\ \{y\} = [C(s)]\ q + \{n_y\} \end{cases} \tag{4.1}$$

where $\{n_u\}$ and $\{n_y\}$ are, respectively, the control and measurements noises and $\frac{q}{V_\infty}[H_{ag}]\{v_g\}$ the turbulence contribution.
In term of transfer matrix, the aeroelastic system can be represented by:

$$[H(s)] = [C][Z(s)]^{-1}[B] \qquad [Z(s)] = \left(s^2[M] + s[C] + [K] - q[H_{am}]\right) \tag{4.2}$$

Disturbances are then assumed to be generated by unknown asymptotically stable, LTI, shaping filters excited by ergodic white noises. For possible tests design purposes the most used turbulence models are von Karman and Dryden, while measurements noises can be estimated from an appropriate dynamic model of sensors.
Referring to figure 4.1: we have:

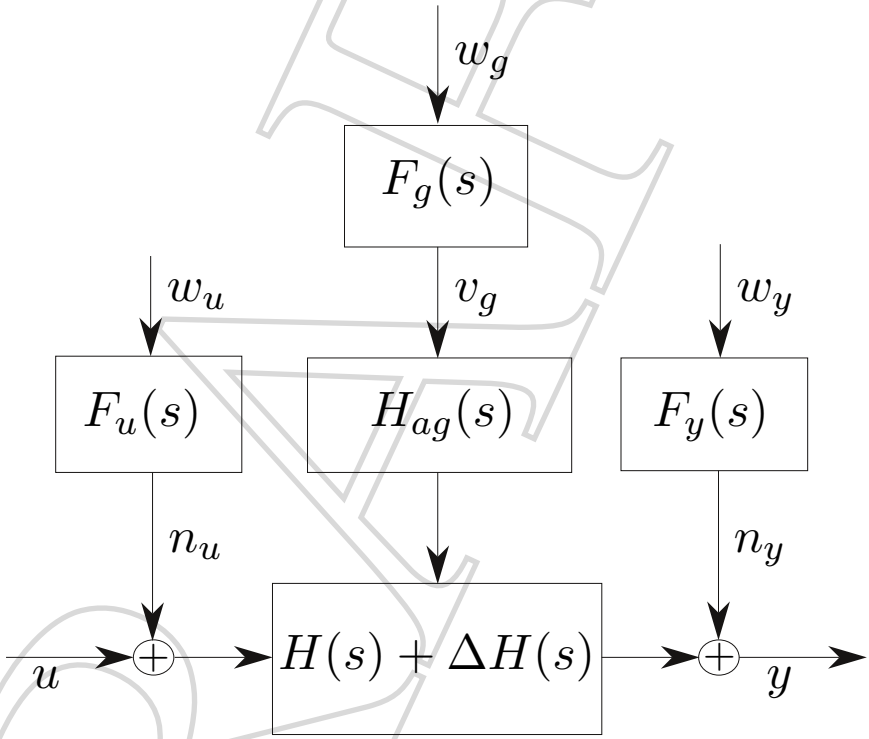


Figure 4.1: `System and disturbances`

- $[H(s)]$ is the transfer matrix of the aeroelastic aircraft;
- $[F_u]$, $[F_g]$ and $[F_y]$ are the shape filters of: control noise, turbulence and measurement noise.

The presence of modeling errors is easily taken into account, in an unstructured manner, by adding a perturbation matrix, $[\Delta H]$, to the nominal transfer matrix, so that $[\Delta H]$ represents the global effect of all the modeling errors on the nominal plant. These errors can be due to different uncertainties and imprecise approximations for the mass, damping, stiffness or generalized aerodynamic forces. The unstructured operator, hence, does not allow to quantify the relative influences of each modeling error source on the nominal aircraft model, but defines just an omni

comprehensive bound on each coefficient of its nominal transfer matrix.
Introducing an additive unstructured uncertainty $[\Delta H]$ we can write the relation between the measured input/output, u and y:

$$\{y\} = ([H(s)] + [\Delta H(s)])\,(\{u\} + ([F_u]\{w_u\}) + \frac{q}{V_\infty}[C][Z(s)]^{-1}[H_{ag}][F_g])\{w_g\} - [F_y]\{w_y\} \quad (4.3)$$

which can be shortened as:

$$\{y\} = [H(s)]\{u\} + \{n\} \quad (4.4)$$

with:

$$\{n\} = [\Delta H(s)]\{u\} + ([H(s)] + [\Delta H(s)])[F_u]\{w_u\} + \quad (4.5)$$

$$+ \frac{q}{V_\infty}[C][Z(s)]^{-1}[H_{ag}][F_g]\{w_g\} - [F_y]\{w_y\} \quad (4.6)$$

$[H]$ can be written as a Left Matrix Fraction Description (LMFD), $[A]^{-1}[B]$. In a similar way $\{n\}$ could be seen as the output of unknown shaping filters driven by a generic white noise $\{w\}$, so that: $\{n\} = [A_n]^{-1}[B_n]\{w\}$, an LMFD again, hence:

$$\{y\} = [A(s)]^{-1}[B(s)]]\{u\} + [A_n(s)]^{-1}[B_n(s)]\{w\} \quad (4.7)$$

It is important to recall that the above relation is assumed to be written in term of the actual measurements. Looking at the system and disturbances scheme above, that should be pretty clear in relation to y, while it is a bit undefined either if we have actually measured an undisturbed u, applying a noise corrupted version of it to the system, as the figure suggests, or picked up the very noise affected u applied to the system. The real situation could be an in-between, so we are happy with a scheme which might not care about such a detail but does care that the measured input is different from what is applied to the system. That said we can write also:

$$[A(s)]\{y\} = [B(s)]\{u\} + [A(s)][A_n]^{-1}[B_n(s)]\{w\} \quad (4.8)$$

or

$$[A(s)]\{y\} = [B(s)]\{u\} + [C(s)]\{w\} \quad (4.9)$$

with:

$$[C(s)] = [A(s)][A_n(s)]^{-1}[B_n] \quad (4.10)$$

so that:

$$\{n\} = [C(s)]\{w\} \quad (4.11)$$

eventually ending with the possible extreme simplification of assuming n to be a white noise, so that:

$$[A(s)]\{y\} = [B(s)]\{u\} + \{w\} \quad (4.12)$$

The shaping of $\{n\}$ through $[A_n]^{-1}[B_n]\{w\}$ corresponds to an Infinite Impulse Response filter (IIR) while $\{n\} = [C]\{w\}$ can be viewed as a Finite Impulse Response (FIR) filter, both to be asintotically stable. The latter notion is related to a possibile discretized approximation of the convolution:

$$\{n(t)\} = \int_{-\infty}^{+\infty}[c(\tau)]\{w(t-\tau)\}d\tau \quad (4.13)$$

through the truncation: $n(z) = \sum_{i=0}^{N} c_i z^{i-n}$, of its discretized infinite sum.
Considering a SISO case their corresponding representations become:

$$y = \frac{B(s)}{A(s)}u + \frac{B_n(s)}{A_n(s)}w \tag{4.14}$$

$$A(s)y = B(s)u + C(s)w \tag{4.15}$$

$$A(s)y = B(s)u + w \tag{4.16}$$

The choice of one representation respect to the other, up to the immediately above extreme simplification assuming any noise and system model disturbance as white, will affect also the methods and domain (time or frequency) used for their identification.
In these notes we will use the above continuous schemes in the frequency domain preferring their corresponding difference models in the time domain. Then the related representation term of the z transform will be:

$$y = \frac{B(z)}{A(z)}u + \frac{B_n(z)}{A_n(z)}w \tag{4.17}$$

$$A(z)y = B(z)u + C(z)w \tag{4.18}$$

$$A(z)y = B(z)u + w \tag{4.19}$$

As it is well known they will result in difference equation in the time domain, ready for the use of sampled responses and excitations invariably available nowadays. As it will be shown, in the next chapter, dealing with the time domain identification, the representation through an IIR filter will be linked to the so called Generalized Least Squares (GLS) method, while the one using a FIR filter will result in a so called Extended Least Squared (ELS) scheme.
The main advantage of a time domain identification is that it is linear, differently from the frequency domain one which is non linear, as it will be pointed out in the next chapters. However, the frequency domain identification is often used because the knowledge of the Frequency Response Function (FRF) is still a useful tool to complete our understanding of the aeroelastic behavior of an aircraft.

Another possible framing The previous frequency presentation of the system along with its noises and model related disturbances will serve mainly for framing the schemes we will present for its identification in the time domain. In fact there could be different ways to structure the various terms making up the system abstraction adopted above. Thus, in view of introducing a scheme that will better suite the following identification in the frequency domain, we notice that system and gust/turbulence disturbances are indistinguishable and can be considered as one by redefining, with a somewhat procedural programming like notational abuse, $\{n\}$ as:

$$\{n_y\} = \{n_y\} - [\Delta H(s)]\left(\{u\} + \{n_u\}\right) - \frac{q}{V_\infty}[C][Z(s)]^{-1}[H_{ag}][F_g])\{w_g\} \tag{4.20}$$

therefore writing:

$$\{y\} + \{n_y\} = [H(s)]\left(\{u\} + \{n_u\}\right) \tag{4.21}$$

We anticipate that in the following transfer matrix/function identification we will significantly forget the structure of the above newly defined $\{n_y\}$, for example by assuming uncorrelated cross power spectral densities between $\{y\}$, $\{n_y\}$, $\{\mathrm{u}\}$ and $\{n_u\}$. That is clearly not the case

when system uncertainties and turbulence will be significant. Therefore we dare saying that a time identification scheme could be a better choice in relation to the processing of flutter tests data to recover the system eigenvalues/poles.

4.2 Single or multiple input?

As we have underlined in the previous chapter, in spite of many exciters could be installed, they are often driven only with a single excitation command. That allows to consider that only one input is applied to the system, hence if we consider only one output at a time, we have to identify a Single Input Single Output (SISO) system.
However, if we measure local forces and moments transmitted by each exciter to the structure, we will find that different values are applied at each excitation point, in spite of all the exciters being nominally the same and driven by the same command. The differences in local forces and moments could be due to differences at exciters mountings and to the realization disturbances. Hence, even if only one excitation command is used, the system turns out to be inherently multi-input.
It will be seen that there is no much difference between processing a SISO or a Multi Input Single Output (MISO) model. So the problem of appropriately treating single or multiple input is mostly related to the our ability of getting excitations measures at all mounting points or not. So, whether we have single or multiple input, more complex schemes will be required instead if we decide to process more output simultaneously at a time, i.e. in the so called Single Input Multi Output (SIMO) and Multi Input Multi Output (MIMO) cases. In the next chapters we will focus mainly on identification techniques for the SISO/MISO cases, a short outline about SIMO/MIMOs being given for the identification in time domain only.

Chapter 5

Identification in the time domain

From a conceptual point of view, this chapter can be divided into two parts:

- the first part presents the fundamental concepts of time domain identification techniques. At first a short outline of time domain models is given, then the following item will be explored;
  - the Least Squares method (LS), which is first presented in an intuitive way, followed by an analysis of its statistical properties;
  - the Extended Least Squares Method (ELS);
  - Multivariable Identification Techniques (MIT), as a short outline only.
- Then the second part of the chapter applies the concepts previously presented to flight flutter testing, by presenting their use, both for the case of measurable and unmeasurable input first, then the main aspects of using a time domain identification technique to obtain the V-g diagram.

5.1 Fundamentals concepts

Our aim is that of finding a model capable of well identifying the flutter behaviour. The notation used will call Auto Regressive (AR) the model part, A, related to the system output y, eXogenous (X) the part, B, of the input variable u and Moving Average (MA) what pertains to w, C. [1].
As outlined in the previous chapter several models are available for the difference equation representing the aeroelastic process, [28] and [29]. In particular, we will dub Eqs. 4.19 and 4.18 as:

- Auto Regressive with eXogenous input (ARX), Eq. 4.19;
- Auto Regressive (AR), Eq. 4.19 without B;

[1] Some authors, instead, use w for the exogenous part and u for the moving average one

- Auto Regressive with Moving Average and eXogenous input (ARMAX), Eq. 4.18;
- Auto Regressive with Moving Average input (ARMA) Eq. 4.18 without B;

ARX model

An ARX model is defined by the following synthetic form:

$$A(z)y = B(z)u + w \tag{5.1}$$
$$A(z) = 1 - a_1 z^{-1} - a_2 z^{-2} - \cdots a_{na} z^{-na} \tag{5.2}$$
$$B(z) = b_1 z^{-1} + b_1 z^{-2} + \cdots b_{nb} z^{-nb} \tag{5.3}$$

expanded into an explicit difference equation:

$$y_k = a_1 y_{k-1} + a_2 y_{k-2} + \cdots a_{na} y_{k-na} + \tag{5.4}$$
$$+ b_1 u_{k-1} b_2 u_{k-2} + \cdots b_{nb} y_{k-nb} + w_k \tag{5.5}$$

An ARX scheme should be suitable for the identification of a system excited by a measurable excitation combined with external/modelling disturbances approximated through a generic unknown white noise.

AR model

In synthetic notation AR models are described by the following difference equation:

$$A(z)y = w \tag{5.6}$$

whose expanded form is:

$$y_k = a_1 y_{k-1} + a_2 y_{k-2} + \cdots a_{na} y_{k-na} + w_k \tag{5.7}$$

An AR scheme should be suitable for the identification of a system excited by unmeasured impulse input combined with generic unknown disturbances approximated through a white noise. If no impulse is applied than the system is excited just by the related white noise.

ARMAX model

The synthetic expression of an ARMAX models is:

$$A(z)y = B(z)u + C(z)w \tag{5.8}$$
$$C(z) = 1 + c_1 z^{-1} + c_1 z^{-2} + \cdots c_{nc} z^{-nc} \tag{5.9}$$

to be exapanded into the following explicit difference equations:

$$y_k = a_1 y_{k-1} + a_2 y_{k-2} + \cdots a_{na} y_{k-na} + \tag{5.10}$$
$$+ b_1 u_{k-1} b_2 u_{k-2} + \cdots + b_{nb} u_{k-nb} + \tag{5.11}$$
$$+ w_k + c_1 w_{k-1} + \cdots + c_{nc} w_{k-nc} \tag{5.12}$$

An ARMAX scheme should be suitable for the identification of a system excited through a measurable excitation combined with generic unknown disturbances, as shaped by $C(z)$.

ARMA model

Finally e synthetic ARMA model is:

$$A(z)y = C(z)w \tag{5.13}$$

its difference equation being:

$$y_k = a_1 y_{k-1} + a_2 y_{k-2} + \cdots a_{na} y_{k-na} + w_k + c_1 w_{k-1} + \cdots + c_{nc} w_{k-nc} \tag{5.14}$$

An ARMA scheme should be suitable for the identification of a system excited by unmeasured impulse input combined with generic unknown disturbances, shaped accordingly to $C(z)$. If no impulse is applied than the system is excited just by the unmeasured disturbance related to $C(z)$.

Each of the parts of the above finite differences schemes is characterized by a parameter n which specifies the number of previous time instant to consider for determining its order. Therefore we indicate with na, nb and nc the order, respectively, of the autoregressive, exogenous and moving average parts. The values of these parameters specify the complexity of the assumed model.
In our presentation we will deal mainly with the scalar case (SISO-MISO). Nonetheless, all of the models to be presented can be extended to the multivariable case, (MISO-MIMO) where y, u and w will be column vectors of p elements. The coefficients a_i, b_j, c_k,with $i = 1, 2 \cdots na$, $j = 1, 2 \cdots nb$, $k = 1, 2 \cdots nc$ will become matrices whose size is, respectively, $p\ x\ p$ for the autoregressive and moving average part, $p \times n_b$ for the exogenous part and $p \times n_c$ for the disturbance part [19].

5.2 The Least Squares Method (LSM)

5.2.1 Introduction

Supposing to be in the SISO or, after combining all of the input in a single one, MISO case, at the end of a test we will have N sample of the time history of the input, u, and of the output, y. Assuming by way of example an ARX scheme, we can write the output at the generic time instant k as a linear combination of a certain number of the passed input and output values, respectively n_a and n_b, hence:

$$y_k = a_1 y_{k-1} + a_2 y_{k-2} + \cdots + a_{na} y_{k-na} + b_1 u_{k-1} + b_2 u_{k-2} + \cdots + b_{nb} u_{k-nb} \tag{5.15}$$

where the n_a coefficients a_i and the n_b coefficients b_i are unknowns. So assuming that for a good identification we will need some kind of averaging, so to smooth out disturbance effects, we know that far more samples than unknown model parameters will likely be available. Therefore it will not be possible to satisfy our model equation at each sample and we can anticipate the existence of an algebraic residual $\{\varepsilon\}$ at once. To evaluate the unknown coefficients, we can write equation 5.15 for each of N-q available measures, where $q = max(n_a, n_b)$, obtaining a linear system. In fact to write equation 5.15, the values of the input and output up to, respectively, n_a and n_b previous time instants are needed, generally $n_a \neq n_b$. Hence, equation 5.15 should be written beginning after q time instants have elapsed, i.e. starting from the qth measure of the

input and output.
The so resulting linear system will be overdetermined, since it is reasonable to retain that the sum of n_a and n_b will be less than the number of the sampled measures N. Then we write our system as:

$$\begin{Bmatrix} y_q \\ y_{q+1} \\ y_{q+2} \\ \\ \vdots \\ y_N \end{Bmatrix} = \begin{bmatrix} y_{q-1} & y_{q-2} & \cdots & y_{q-na} & u_{q-1} & u_{q-2} & \cdots & u_{q-nb} \\ y_q & y_{q-1} & \cdots & y_{q-na+1} & u_q & u_{q-1} & \cdots & u_{q-nb+1} \\ y_{q+1} & y_q & \cdots & y_{q-na+2} & u_{q+1} & u_q & \cdots & u_{q-nb+2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ y_{q-1+N} & y_{q-2+N} & \cdots & y_{q-na+N} & u_{q-1+N} & u_{q-2+N} & \cdots & u_{q-nb+N} \end{bmatrix} \begin{Bmatrix} a_1 \\ \vdots \\ a_{na} \\ b_1 \\ \vdots \\ b_{nb} \end{Bmatrix} + \begin{Bmatrix} \varepsilon_q \\ \varepsilon_{q+1} \\ \varepsilon_{q+2} \\ \vdots \\ \varepsilon_N \end{Bmatrix} \tag{5.16}$$

followed by its compacted matrix form:

$$\{y\} = [A]\{\phi\} + \{\varepsilon\} \tag{5.17}$$

where A is a rectangular $(N-q)\ x\ (n_a + n_b)$ matrix, and $\{\phi\}$, is the $(n_a + n_b)$ is the unknown vector of the model parameters.
To evaluate the coefficients of our overdetermined model, we elect to use the celebrated Least Squares (LS) technique;

$$min_{(a,b)}\ J = \frac{1}{2}\{\varepsilon\}^T\{\varepsilon\} \tag{5.18}$$

where $\{\varepsilon\}$ is the vector of the $(N-q)\times 1$ algebraic residuals. Being $\{\varepsilon\} = \{y\} - [A]\{\phi\}$, the functional J can be explicitly written as a function of the unknown vector $\{\phi\}$:

$$\begin{aligned} J = \frac{1}{2}\{\varepsilon\}^T\{\varepsilon\} = \frac{1}{2}(\{y\} - [A]\{\phi\})^T(\{y\} - [A]\{\phi\}) = \\ = \frac{1}{2}(\{y\}^T\{y\} - 2\{\phi\}^T[A]^T\{y\} + \{\phi\}^T[A]^T[A]\{\phi\}) \end{aligned} \tag{5.19}$$

whose variation is:

$$\delta J = \delta\{\phi\}^T\left(-[A]^T\{y\} + [A]^T[A]\{\phi\}\right) \tag{5.20}$$

so that, being all of $\delta\{\phi\}$ independent, we have:

$$\{\phi\} = ([A]^T[A])^{-1}[A]^T\{y\} \tag{5.21}$$

Notice that we have provided a simple and purely algebraic interpretation of the vector $\{\varepsilon\}$ as the residual of an overdetermined system. As we are going to see shortly it is also possible to provide a physical explanation for it.

5.2.2 An outline on the practical solution of the Least Squares problem

As we have shown the solution of an LS problem requires the inversion, better say factorization, of $[A]^T[A]$. Such a matrix is, unfortunately, frequently ill conditioned [40]. Notice that for a square matrix the conditioning number of the $[A]^T[A]$ is worse than that of [A], because its eigenvalues will be the square of those of [A] and so are more spreaded with respect to each

other [2]. A way to preserve the conditioning number of $[A]^T[A]$ [40] is to use the QR factorization of $[A]$, as its will afford a conditioning close to that of $[A]$ itself. We introduce the QR factorization reporting what found in [31]:

1 *QR factorization*

If [A] is a $R^{m\times n}$ matrix with $m \geq n$ it admits a QR factorization, such that:

$$[A] = [Q][R] \tag{5.22}$$

where [Q] is a $R^{m\times m}$ orthogonal matrix, i.e. $[Q]^T[Q] = [I]$,while [R] is a trapezoidal $R^{m\times n}$ with null rows starting from the n+1 th row.

Using the QR factorization of [A] for an LS problem we have:

$$\begin{aligned} [A]^T[A]\{\phi\} = [A]^T\{b\} \rightarrow [R]^T[Q]^T[Q][R]\{\phi\} = [R]^T[Q]^T\{b\} \\ [R]^T[R]\{\phi\} = [R]^T[Q]^T\{b\} \end{aligned} \tag{5.23}$$

so that the very final solution ends in being a forward-backward substitution "a la Cholesky". Various methods are available to compute the QR factorization of [A], an exhaustive review being presented in [18].
A similarly stable approach is the LDL^T factorization to $[A]^T[A]$. When [A] is a symmetric matrix, it is a particular case of the LDM^T factorization. Reporting what found in [18] we have:

2 *LDM^T – LDL^T factorization*

If all the leading principal submatrices of [A] are non singular, then there exist a unique lower triangular matrix[3] [L] and [M] and a unique diagonal matrix [D], such that $[A] = [L][D][M]^T$.

If [A] is symmetric [M]=[L], hence:

$$[A] = [L][D][L]^T \tag{5.24}$$

Once this factorization has been computed, the solution of the original linear system, $[A]\{x\} = \{b\}$, can be reduced to the sequential solution of two triangular and a diagonal system:

$$[L]\{y\} = \{b\} \quad \rightarrow \quad [D]\{z\} = \{y\} \quad \rightarrow \quad [L^T]\{x\} = \{z\} \tag{5.25}$$

whose total computational cost is, roughly, of $n^3/6$ flops. The factorization matrices [L] and [D] could be computed by the following recursive relations:

$$D_j = A_{jj} - \sum_{k=1}^{j-1} L_{jk}^2 D_k \tag{5.26}$$

$$L_{ij} = \frac{1}{D_j}\left(A_{ij} - \sum_{k=1}^{j-1} L_{ik}L_{jk}D_k\right) \qquad \text{for } i > j. \tag{5.27}$$

[2] Remember that the conditioning number of a matrix [A] is given by: $K = \frac{\lambda_{max}}{\lambda_{min}}$ where λ_{min} and λ_{max} are the moduli of the minimum and maximum eigenvalues [31].

[3] A unitary triangular matrix is a matrix whose diagonal element are unitary

It is worth remarking that while the QR and LDL^T factorizations have roughly the same numerical stability, a QR factorization will be more expensive from the computational cost point of view. For a deeper discussion on the most used factorizations in estimation problems, see [38].

5.2.3 Some statistical properties of the Least Squares method

Having found $\{\phi\}$ on the base of a mere algebraic interpretation of the solution of an overdetermined linear system of equations, we want now to view it as an estimator of an ARX model, with a particular care for the expected value and variance of the identified coefficients $\{\phi\}$. So we reuse its compact form to represent an ARX model as:

$$\{y\} = [A]\{\phi\} + \{w\} \tag{5.28}$$

Recalling that, in framing our identification scheme, all of our measures noise and model disturbances have gone into either n or w, we can consider $\{y\}$, u and, consequently, $[A]$ as being deterministic quantities, we now assume each component of $\{\phi\}$ and $\{w\}$ as being an unknown stochastic variable, with $\{w\}$ having a null mean and a variance:

$$[\sigma^2_{ww}] = E\left[\{w\}\{w\}^T\right] = \lambda^2[I] \tag{5.29}$$

with a yet unknown λ^2 to be appropriately determined in the following. The structure of the above variance implies:

- uncorrelated noises due to the diagonal structure of the variance matrix;
- each component w_i is taken from the same population; a reasonable assumption if one considers the relatively short duration of a test. [4].

While the acquired data can be associated to a specific value of $\{\phi\}$ we have no immediate clue about it. So, as said above, we will treat it too as a set of stochastic variables, trying to determine its mean and variance.

Expected value estimation

Recalling that $\{w\}$ has a null mean and $E(\{y\}) = \{y\}$, by taking the expected value of both sides of 5.28 we can write:

$$\{y\} = [A]E(\{\phi\}) + E(\{w\}) = [A]\{\bar{\phi}\} \tag{5.30}$$

What above is nothing but the overdetermined system we solved before using LS, so we can rewrite $\{\bar{\phi}\}$ as:

$$\{\bar{\phi}\} = ([A]^T[A])^{-1}[A]^T\{y\} \tag{5.31}$$

Then, using 5.28, we can determine the relation between $\{\bar{\phi}\}$ and the true unknown $\{\phi\}$ through the following steps:

$$\begin{aligned}\{\bar{\phi}\} &= ([A]^T[A])^{-1}[A]^T\{y\} = ([A]^T[A])^{-1}[A]^T([A]\{\phi\} + \{w\}) = \\ &= \{\phi\} + ([A]^T[A])^{-1}[A]^T\{w\}\end{aligned} \tag{5.32}$$

[4] It is important to remember that at each time we have got a sample of a population and all of the $\{w\}$ are different, their mean value representing a sample mean among all the possible realization.

so that the deviation from the mean value is:

$$\{\bar{\phi}\} - \{\phi\} = \{\Delta\phi\} = ([A]^T[A])^{-1}[A]^T\{w\} \tag{5.33}$$

Taking the expected value of the previous formula, after recalling, once more, that $E(\{w\})) = 0$, we see that $E(\{\Delta\phi\}) = 0$ so $\{\bar{\phi}\} = \{\phi\}$. Therefore the LS estimation of an ARX scheme will not be polarized for the assumed noise scheme.

Variance estimation

Having determined $\{\bar{\phi}\}$ we proceed to determining its variance.
We should recall that λ^2 in also still unknown so we will have to estimate it too. We have then to:

- show how to evaluate the variance of the components of the vector $\{\phi\}$, once the variance of the white noise $\{w\}$ is available;
- provide a viable way to estimate the variance of λ^2.

Variance estimation of the components of $\{\phi\}$ In view of what presented before, after recalling the definition: $[\sigma^2_{\Phi\Phi}] = E(\{\Delta\phi\}\{\Delta\phi\}^T)$ and along with the previously found relation: $\{\Delta\phi\} = ([A]^T[A])^{-1}[A]^T\{w\}$, we can write:

$$[\sigma^2_{\Phi\Phi}] = E[\{\Delta\phi\}\{\Delta\phi\}^T] = ([A]^T[A])^{-1}[A]^T E[\{w\}\{w\}^T][A]([A]^T[A])^{-1} \tag{5.34}$$

Recalling that $E(\{w\}\{w\}^T) = \lambda^2[I]$ and substituting it into the previous equation we obtain:

$$[\sigma^2_{\Phi\Phi}] = \lambda^2([A]^T[A])^{-1}[A]^T[A]([A]^T[A])^{-1} \quad \Rightarrow \quad [\sigma^2_{\Phi\Phi}] = \lambda^2([A]^T[A])^{-1} \tag{5.35}$$

If the intensity of $\{w\}$, i.e. λ^2, was known, from 5.35 we could obtain $[\sigma^2_{\Phi\Phi}]$.

A posteriori estimation of the white noise $\{w\}$ variance We turn back to our old algebraic residual, $\{\varepsilon\}$, related to the LS solution of the overdetermined system but, as shown before, now we read it in term of $\{\bar{\phi}\}$, writing:

$$\{\varepsilon\} = \{y\} - [A]\{\bar{\phi}\} = [A]\{\phi\} + \{w\} - [A]\{\bar{\phi}\} = \{w\} - [A]\{\Delta\phi\} \tag{5.36}$$

$$= \{w\} - [A]([A]^T[A])^{-1}[A]^T\{w\} = ([I] - [A]([A]^T[A])^{-1}[A]^T)\{w\} \tag{5.37}$$

so that, calling:

$$[D] = [I_N] - [A]([A]^T[A])^{-1}[A]^T \tag{5.38}$$

we can write:

$$\{\varepsilon\} = [D]\{w\} \tag{5.39}$$

It is important to remark that $[D]$ is symmetric and idempotent, i.e. one for which $[D]^i = [D]$, for whatever i.
Eq. 5.39 is highy interesting, as it establishes a direct connection between $\{\varepsilon\}$ and $\{w\}$, which we exploit to connect the related variances estimate:

$$E(\{\varepsilon\}\{\varepsilon\}^T) = [\sigma^2_{\varepsilon\varepsilon}] = [D]E(\{w\}\{w\}^T)[D]^T = [D][\sigma^2_{ww}][D]^T = \lambda^2[D] \tag{5.40}$$

Then we estimate λ^2 from the trace of the above equation as:

$$\lambda^2 = \frac{tr([\sigma^2_{\varepsilon\varepsilon}])}{tr([D])} \tag{5.41}$$

It should then be easy to verify that $tr([D]) = N - q - na - nb = N - \hat{q}$, $\hat{q} = q + na + nb$, so that we are left choosing how to estimate $tr([\sigma_{\varepsilon\varepsilon}])$ from the limited sample $\{\varepsilon\}$ we have. For that we write:

$$tr([\sigma_{\varepsilon\varepsilon}]) = tr(E(\{\varepsilon\}\{\varepsilon\}^T)) = E(tr(\{\varepsilon\}\{\varepsilon\}^T)) = E(\{\varepsilon\}^T\{\varepsilon\}) = \frac{\{\varepsilon\}^T\{\varepsilon\}}{N} = \frac{s^2_{\varepsilon\varepsilon}}{N} \tag{5.42}$$

[5] so that:

$$\lambda^2 = \frac{s^2_{\varepsilon\varepsilon}}{N(N-\hat{q})} \tag{5.43}$$

using the simpler: $\lambda^2 \simeq \frac{s^2_{\varepsilon\varepsilon}}{N^2}$, whenever $N \gg \hat{q}$.
Therefore a practical approach to the LS identification of an ARX model can be summarized as follows:

- chose the model order and build the coefficient matrix [A], using the available input/output data;
- calculate $\{\bar{\phi}\} = ([A]^T[A])^{-1}[A]^T\{y\}$;
- calculate $\{\varepsilon\}$, $s_{\varepsilon\varepsilon} = \{\varepsilon\}^T\{\varepsilon\}$ and $\lambda^2 = \frac{s^2_{\varepsilon\varepsilon}}{N(N-\hat{q})}$, or more simply $\lambda^2 \simeq \frac{s^2_{\varepsilon\varepsilon}}{N^2}$, whenever $N \gg \hat{q}$.

5.3 Extended Least Squares method

We have just seen that LS applied to an ARX scheme provides an unbiased estimation of the model parameters, and hence of the poles of the system, if $\{w\}$ is a white noise.
We want now to consider the case in which the noise is not white but colored, which we have seen to be the most appropriate scheme for our uncertain and noisy models. In the previous chapter, we presented two representations for the colored disturbances shaping filter, IIR and FIR, suggesting that they, even if equivalent from the modeling point of view, could have affected the identification method. In this paragraph we are going to consider coloring through a moving average FIR shaping.
To such an end we adopt an ARMAX formulation, writing the related difference equation at any time k as:

$$\begin{aligned} y_k = &-a_1 y_{k-1} - a_2 y_{k-2} - \cdots - a_{na} y_{k-na} + b_1 u_{k-1} + b_2 u_{k-2} + \ldots b_{nb} u_{k-nb} + w_k + \\ &+ c_1 w_{k-1} + c_2 w_{k-2} + \cdots + c_{nc} w_{k-nc} \end{aligned} \tag{5.44}$$

whose number of unknown parameters is now increased, with respect to an ARX scheme, by the nc coefficients $\{c\}$ of the shaping filter. Therefore we have a total of $n = na + nb + nc$ unknowns,

[5] Remind the well known formula for the standard deviation of a sampled mean: $\sigma = \varepsilon/\sqrt{N-1}$

n being the order of the ARMAX model.
By simply reorganizing the unknowns vector, [35], and defining an extended parameters vector, we can frame the structure of an ARMAX estimation in the same way as a compacted ARX scheme. In fact if we introduce the extended parameters vector:

$$\{\phi\} = \{a_1 \ \dots \ a_{na} \ b_1 \ \dots \ b_{nb} \ c_1 \ \dots \ c_{nc}\}^T \tag{5.45}$$

and the row matrix:

$$[A_k] = \begin{bmatrix} -y_{k-1} & \dots & -y_{k-na} & u_{k-1} & \dots & u_{k-na} & w_{k-1} & \dots & w_{nc} \end{bmatrix} \tag{5.46}$$

the above kth equation can be synthesized as:

$$y_k = [A_k]\{\phi\} + w_k \tag{5.47}$$

which can be seen has the *kth* equation of the following compacted form:

$$\{y\} = [A]\{\phi\} + \{w\} \tag{5.48}$$

To obtain the extended model parameters $\{\phi\}$, we can write equation 5.44 for each of the N-q available measures, where, in this case, q=max(na,nb,nc) and, analogously the previous case, the compacted set of equations 5.48 is written starting from the qth time instant.
Unfortunately, measurements of the disturbance $\{w\}$ are not available, but, assuming $\{\phi\}$ has been given, it can only be estimated by writing:

$$\{w\} = \{y\} - [A]\{\phi\} \tag{5.49}$$

Since both the model parameters and the noise sequence are unknown, they need each other, hence there is a clear need of an iterative process. Indicating with the apex j the generic *jth* iteration a possible iterated solution will look as follows:

1. At the starting point, we retain $\{w\}$ as white and $\{c\} = 0$. Hence eq. 5.44 is equal to eq. 4.19 and can be solved as an ARX scheme:

$$[A]^0\{\phi\}^0 = \{y\} \tag{5.50}$$

   so obtaining a first estimation of the autoregressive and exogenous parts of the model:

$$\{\phi\}^0 = \left\{\hat{a}_1^0 \ \hat{a}_2^0 \ \dots \ \hat{a}_{na}^0 \quad \hat{b}_1^0 \ \hat{b}_2^0 \ \dots \ \hat{b}_{nb}^0 \quad 0_1 \ 0_2 \ \dots \ 0_{nc}\right\}^T \tag{5.51}$$

2. Estimate the noise sequence using equation:

$$\{w\}^j = \{y\} - [A]^{j-1}\{\phi\}^{j-1} \tag{5.52}$$

   where $k = q \dots N$ and $i = 1 \dots nc$.

3. Thanks to the noise sequence $\{w\}^j$ we can obtain a new estimation of the model parameters through an LS solution of the overdetermined linear system:

$$[A]^j\{\phi\}^j = \{y\} \tag{5.53}$$

where $\{\phi\}^j$ is given by:

$$\{\phi\}^j = \left\{ a_1^j \ \dots \ a_{na}^j \ b_1^j \ \dots \ b_{nb}^j \ c_1^j \ \dots \ c_{nc}^j \right\}^T \tag{5.54}$$

while $[A_j]$ is given by:

$$\begin{bmatrix} y_{q-1} & \cdots & y_{q-na} & u_{q-1} & \cdots & u_{q-nb} & w_{q-1}^{j-1} & \cdots & w_{q-nc}^{j-1} \\ y_q & \cdots & y_{q-na+1} & u_q & \cdots & u_{q-nb+1} & w_q^{j-1} & \cdots & w_{q-nc+1}^{j-1} \\ y_{q+1} & \cdots & y_{q-na+2} & u_{q+1} & \cdots & u_{q-nb+2} & w_{q+1}^{j-1} & \cdots & w_{q-nc+2}^{j-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ y_{N-1} & \cdots & y_{N-na} & u_{N-1} & \cdots & u_{N-nb} & w_{N-1}^{j-1} & \cdots & w_{N-nc}^{j-1} \end{bmatrix} \tag{5.55}$$

During the iterations $[A]^j$ changes just its partition associated to $\{w\}$ so, recalling $[A]^0$, available since the very first iteration and related to the coefficients $\{a\}$ and $\{b\}$ and $[W]^j$ the rectangular matrix $(N-q) \times nc$ which contains the disturbances sequence acting on all the N-q measures used for the parameter estimation, we can write:

$$[A]^j = \begin{bmatrix} [A]^0 & [W]^j \end{bmatrix} \tag{5.56}$$

where:

$$[W]^j = \begin{bmatrix} w_{q-1}^j & w_{q-2}^j & \cdots & w_{q-i}^j & \cdots & w_{q-nc}^j \\ w_q^j & w_{q-1}^j & \cdots & w_{q+1-i}^j & \cdots & w_{q+1-nc}^j \\ w_{q+1}^j & w_q^j & \cdots & w_{q+2-i}^j & \cdots & w_{q+2-nc}^j \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ w_{k-1}^j & w_{k-2}^j & \cdots & w_{k-i}^j & \cdots & w_{k-nc}^j \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ w_{N-1}^j & w_{N-2}^j & \cdots & w_{N-i}^j & \cdots & w_{N-nc}^j \end{bmatrix} \tag{5.57}$$

What above implies that during the iterated process we have either to compute only $[A]^{0T}[W]^j$ and $[W]^{jT}[W]^j$ within:

$$[A]^{jT}[A]^j = \begin{bmatrix} [A]^{0T}[A]^0 & [A]^{0T}[W]^j \\ ([A]^{0T}[W]^j)^T & [W]^{jT}[W]^j \end{bmatrix} \tag{5.58}$$

or proceed to the QR/LDL factorizations of the related parts.

4. Check convergence, looking if a suitable norm of the difference between the estimated parameters at two successive iterations is less than an assigned tolerance:

$$|| \{\phi^j\} - \{\phi^{j-1}\} || < \varepsilon \tag{5.59}$$

if the convergence criterion is satisfied, exit from the iterative process, go to step 2 otherwise.

Notice that during the iterative process it is important to verify that the estimated parameter of the FIR shaping filter are associated to a stable impulse response, otherwise convergence problems could arise, as well explained in [2].

5.4 Identification of multivariable models

In chapter 2 and 3 we remarked that even if more exciters are installed, often they are typically driven by a unique command. Afterward we casted some doubts on the fact that even a single command resulted in a truly single input. We did observe that for many practical reasons local exciters forces and moments could end in being different. Therefore the system becomes intrinsically multi input. What above leads to three possible situations: MISO, SIMO, MIMO. From the practical processing point of view, as we observed in chapter 4 already, a MISO scheme is equivalent to a SISO, In fact all of the measured input can be stacked so to become a single one and we can work as for a SISO scheme.
So, from an algorithmic point view, the only difference is whether we want to process a single or many output at a time. For our main scope, the recovery of eigenvalues/poles of the system, we note that a very precise and undisturbed model should result in autoregressive parts providing the same values, whatever the model used. In practice it will never be that way. So if we work on a single output, SISO-MISO, or on different clusters of them, SIMO-MIMO, we will end in obtaining different results, some having slight some large discrepancies. So an "a posteriori judging art" is required to interpret what is good and what is not. Processing all of the output in a single batch identification may appear as a way to avoid multiple results. Apart from possibly being not a viable option there is no guarantee that it is the best one either. Nonetheless clustering a few output simultaneously within a single identification can help in providing an improved consistency and better disturbance rejection. In short the capability of managing SIMO-MIMO schemes can be a tool providing some help for a better processing of flutter test data.
Therefore we are going to provide a brief extension of the concepts presented for the SISO(MISO) case to SIMO-MIMO. To such an end let us see how it could be possible to reformulate the Least Squares and Extended Least Squares methods for SIMO-MIMO schemes.
We assume to have:

- ma measures y^i with $i = 1 \dots ma$, which can be represented by the output vector $\{\mathrm{y}\}$:

$$\{y\}^T = \{y^1 \quad y^2 \quad y^3 \quad \dots \quad y^{ma}\} \tag{5.60}$$

- mb inputs u^j with $j = 1 \dots mb$, which can be represented by the input vector $\{\mathrm{u}\}$:

$$\{u\}^T = \{u^1 \quad u^2 \quad u^3 \quad \dots \quad u^{mi}\} \tag{5.61}$$

and an additive noise which can be white or colored. As we have underlined many times, using a suitable shaping filter, each colored noise component can be generated from a white noise. It is then useful to define the vector of unmodeled-unmeasured additive white noise $\{w\}$, which clearly will have the same number of components as the output vector:

$$\{w\}^T = \{w^1 \quad w^2 \quad w^3 \quad \dots \quad w^{ma}\} \tag{5.62}$$

We can write a relationship between the ma output of the system, the mb input and the additive measure noise, choosing a FIR representation for its shaping filter, obtaining [2]:

$$\{y_k\} = \sum_{l=1}^{na} [A_l]\{y_{k-l}\} + \sum_{p=1}^{nb} [B_p]\{u_{k-p}\} + \{w_k\} + \sum_{r=1}^{nc} [C_r]\{w_{k-r}\} \tag{5.63}$$

where the generic matrices $[A_l]$, $[B_p]$ and $[C_r]$ are given by:

$$[A_l] = \begin{bmatrix} a^l_{11} & a^l_{12} & a^l_{13} & \dots & a^l_{1i} & \dots & a^l_{1ma} \\ a^l_{21} & a^l_{22} & a^l_{23} & \dots & a^l_{2i} & \dots & a^l_{2ma} \\ a^l_{31} & a^l_{32} & a^l_{33} & \dots & a^l_{3i} & \dots & a^l_{3ma} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ a^l_{i1} & a^l_{i2} & a^l_{i3} & \dots & a^l_{ii} & \dots & a^l_{ima} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a^l_{ma1} & a^l_{ma2} & a^l_{ma3} & \dots & a^l_{mai} & \dots & a^l_{mama} \end{bmatrix}_{ma\times ma} \tag{5.64}$$

$$[B_p] = \begin{bmatrix} b^l_{11} & b^l_{12} & b^l_{13} & \dots & b^l_{1i} & \dots & b^l_{1mb} \\ b^l_{21} & b^l_{22} & b^l_{23} & \dots & b^l_{2i} & \dots & b^l_{2mb} \\ b^l_{31} & b^l_{32} & b^l_{33} & \dots & b^l_{3i} & \dots & b^l_{3mb} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ b^l_{i1} & b^l_{i2} & b^l_{i3} & \dots & b^l_{ii} & \dots & b^l_{imb} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ b^l_{mb1} & b^l_{mb2} & b^l_{mb3} & \dots & b^l_{mbi} & \dots & b^l_{mbmb} \end{bmatrix}_{ma\times mb} \tag{5.65}$$

$$[C_r] = \begin{bmatrix} c^l_{11} & c^l_{12} & c^l_{13} & \dots & c^l_{1i} & \dots & c^l_{1ma} \\ c^l_{21} & c^l_{22} & c^l_{23} & \dots & c^l_{2i} & \dots & c^l_{2ma} \\ c^l_{31} & c^l_{32} & c^l_{33} & \dots & c^l_{3i} & \dots & c^l_{3ma} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ c^l_{i1} & c^l_{i2} & c^l_{i3} & \dots & c^l_{ii} & \dots & c^l_{ima} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ c^l_{ma1} & c^l_{ma2} & c^l_{ma3} & \dots & c^l_{mai} & \dots & c^l_{mama} \end{bmatrix}_{ma\times ma} \tag{5.66}$$

Notice that all the $[A_l]$,$[B_p]$ and $[C_r]$ matrices are unknown. In the MIMO case, therefore,the unknowns are the coefficient of all these matrices, in particular, observe that:

- the generic coefficient a^l_{ij} represents the contribution to the ith measure y^i at time k, due to the jth measure y^j at time k-l, with $i, j = 1 \dots ma$ and $l = 1 \dots na$, where na is the order of the autoregressive part of the model;
- the generic coefficient b^p_{ij} represents the contribution to the ith measure y^i at time k, due to the jth measure u^j at time k-p, with $j = 1 \dots mb$ and $p = 1 \dots nb$, where mb is the number of the inputs forcing the system, while nb is the order of the exogenous part of the model;
- the generic coefficient c^r_{ij} represents the contribution to the measure noise of the ith measure y^i at time k, due to the white noise on the jth measure y^j at time k-r, w^i_{k-r} ,with $j = 1 \dots ma$ and $r = 1 \dots nc$, where nc is the order of the moving average part of the model.

The total number of unknowns in the general case of a colored noise, is equal to t, where $t = na \times (ma \times ma) + nb \times (ma \times mb) + nc \times (ma \times ma)$. After introducing suitable matrices and vectors, equation 5.63 can be rewritten in the standard form:

$$\{y_k\} = [\vartheta]\{\varphi_k\} + \{w_k\} \tag{5.67}$$

where:

$$[\vartheta] = [[A_1]\,[A_2]\,\ldots\,[A_{na}]\,[B_1]\,[B_2]\,\ldots\,[B_{nb}]\,[C_1]\,[C_2]\,\ldots\,[C_{nc}]]_{ma\times t} \tag{5.68}$$

and:

$$\{\varphi_k\} = \left\{\{y_{k-1}\}\;\;\{y_{k-2}\}\;\;\ldots\;\;\{y_{k-na}\}\;\;\{u_{k-1}\}\;\;\{u_{k-2}\}\;\;\ldots\;\;\{u_{k-nb}\}\;\;\{w_{k-1}\}\right\}^T_{t\times 1} \tag{5.69}$$

Supposing of having at our disposal N time instants, we can collocate equation 5.67 starting from the qth, q=max(na,nb,nc), time instant obtaining an overdetermined linear system of N-q equations, whose unknowns are the coefficients of the matrices themselves. Notice that, as we have underlined many times in this chapter, the first q time instant are rejected for a more consistent starting of the estimation method.
As we are used to have the unknowns at the right side of the coefficient i matrix, before collocating equation 5.67 we transpose it, obtaining:

$$\{y_k\}^T = \{\varphi_k\}^T[\vartheta]^T + \{w_k\}^T \tag{5.70}$$

Hence collocating for $k = q\ldots N$ we obtain:

$$\{y_q\}^T = \{\varphi_q\}^T[\vartheta]^T + \{w_q\}^T \tag{5.71}$$
$$\{y_{q+1}\}^T = \{\varphi_{q+1}\}^T[\vartheta]^T + \{w_{q+1}\}^T \tag{5.72}$$
$$\{y_{q+2}\}^T = \{\varphi_{q+2}\}^T[\vartheta]^T + \{w_{q+2}\}^T \tag{5.73}$$
$$\vdots \tag{5.74}$$
$$\{y_N\}^T = \{\varphi_N\}^T[\vartheta]^T + \{w_N\}^T \tag{5.75}$$
$$\tag{5.76}$$

Organizing all the vectors $1\times ma\ \{y_k\}^T$ and all the vectors $1\times t\ \{\varphi\}^T$ in, respectively, an $N\times ma$ matrix $[\mathcal{B}]$ and an $N\times t$ matrix $[\mathcal{A}]$:

$$[\mathcal{B}] = \begin{bmatrix} \{y_q\}^T \\ \{y_{q+1}\}^T \\ \{y_{q+2}\}^T \\ \vdots \\ \{y_N\}^T \end{bmatrix}_{N\times ma} \qquad [\mathcal{A}] = \begin{bmatrix} \{\phi_q\}^T \\ \{\phi_{q+1}\}^T \\ \{\phi_{q+2}\}^T \\ \vdots \\ \{\phi_N\}^T \end{bmatrix}_{N\times t} \tag{5.77}$$

we will obtain an overdetermined linear system whose unknowns are the elements of the $t\times$ matrix $[\vartheta]$, which can be solved in the usual LS sense:

$$[\mathcal{A}][\vartheta] = [\mathcal{B}] \rightarrow [\vartheta] = ([\mathcal{A}]^T[\mathcal{A}])^{-1}[\mathcal{A}]^T[\mathcal{B}] \tag{5.78}$$

If the additive measure noise is white, no shaping filter is required, hence the number of the total unknowns is reduced, since we have to compute uniquely the coefficients of the l and p matrices associated respectively to the autoregressive and the exogenous part of the model. In fact equation 5.63 can be rewritten as:

$$\{y_k\} = \sum_{l=1}^{na}[A_l]\{y_{k-l}\} + \sum_{p}^{nb}[B_p]\{u_{k-p}\} + \{w_k\} \tag{5.79}$$

Instead, if the additive measure noise is white a moving average filter is required and the very same ELS iteration, previously presented for the SISO (MISO) scheme, can be used, with the care of maintaining matrices $[C]_i$ of the noise shaping filters stable.

5.5 How to choose the model order

In the previous chapter, we have presented the fundamental concepts of the estimation of the model parameters, without saying anything about the choice of the model order. This paragraph will provide a few related simple helping guidelines.
The model order can be synthesized through the total number of the parameters used. First of all, to verify if the chosen model order is correct, we can perform a whiteness test on the residual of the estimated model. The basic assumption of this criterion is that the model is adequate if its residuals are white. One of the most used whiteness test is Anderson's, [28], which is based on the sample covariance function, defined as:

$$\gamma(\tau) = \frac{1}{N}\sum_{t=1}^{N-\tau} \varepsilon(t)\varepsilon(t+\tau)\ \tau > 0 \tag{5.80}$$

divided by the an estimation of the variance of the signal $\gamma(0)$, to evaluate a normalized covariance function $\rho(\tau)$:

$$\rho(\tau) = \frac{\gamma(\tau)}{\gamma(0)} \tag{5.81}$$

It can demonstrated, [29], that if ε is white the function $\rho(\tau)$ has, for N large enough, a Gaussian distribution with a null expected value and variance 1/N.
Then Anderson test consists in setting a confidence level, indicated with α, $\alpha \in [0,1]$. Then we:

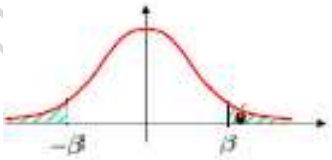


Figure 5.1: `Anderson Test`

- find a value β such that the tails of the Gaussian distribution outside the range $[-\beta,\beta]$ cover an area equal to α;
- consider a number of M evaluations of $\rho(\tau)$:

$$\begin{bmatrix}\rho(0) & \rho(1) & \dots & \rho(M)\end{bmatrix}$$

- count the number n of the evaluation of $\rho(\tau)$ which are in the range $[-\beta/\sqrt{(N)},\beta/\sqrt{(N)}]$. If:

$$\frac{n}{M} < \alpha \tag{5.82}$$

  $\varepsilon()$ can be considered as a white noise and the model order chosen is correct.

The order of an assigned model should be also optimized. For that we define J, a quadratic index aiming at qualifying how the identified model is representative of the experimental data:

$$J(\phi_N) = \frac{1}{N}|\varepsilon|^2 \tag{5.83}$$

We will often find that J decreases by increasing the model order, up to a point where it stabilizes then increases again eventually. The related outcome can be useful and is used indeed. Nonetheless, drawing the model order out J is a somewhat subjective evaluation of its adequacy. In fact, choosing too high an order might result in a over parametrization of the model itself, leading to having some coefficients, those corresponding to the superfluous part of the parametrization, characterized by an high degree of uncertainty. That is due to the fact that there is a conceptual error in the use of $J(\phi_N)$ as an indicator for evaluating the correctness of the identified model, because we are trying to validate the model using the same data used to identify it.
A way to solve the problem is, for example, to introduce an objective evaluation measuring the accuracy of the identified model on data different from those used in the identification phase, [28]. Aiming at a short outline of the most used, more rational, criteria for the estimation of a model order, we will present the FPE, AIC and MDL criteria. They are derived from different considerations, but for all of them the optimal order is that corresponding to the minimum if a suitable defined function. For a more detailed discussion of what is here summarized see [28].
The Final Prediction Error (FPE) criterion evaluates the correctness of the identified model with respect to possible data realizations. It is given by:

$$FPE = \frac{N+n}{N-n} J(\phi)^n \tag{5.84}$$

The Aikake Information Criterion (AIC), instead, results from the minimization of a suitable

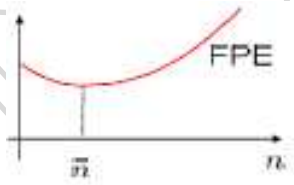


Figure 5.2: `FPE Criterion`

distance between the real data probability and the one that will be produced by a given model. The AIC criterion tends to favor low order models if there are few data available. It is given by:

$$AIC = 2\frac{n}{N} + ln(J(\phi_N)^n) \tag{5.85}$$

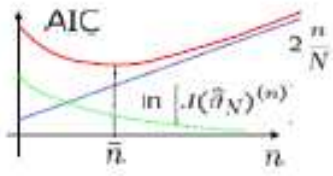


Figure 5.3: `AIC Criterion`

At last, the Minimum Description Length MDL criterion takes its origin form the information

theory: the correct complexity of a model is that for which the complete description of data through the model requires the lowest bit number with respect to any other model. It is given by:

$$MDL = (\ln N)\,\frac{n}{N} + \ln[J(\phi)^n] \tag{5.86}$$

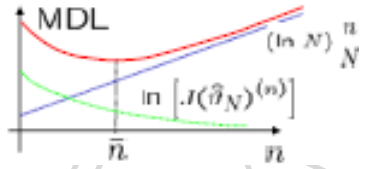


Figure 5.4: `MDL Criterion`

In general, it is not assured that the presented criteria have a unique minimum. Moreover, the different criteria will often provide different results.
So determining the model order is not simple undertaking and cannot leave the ability and experience of the experimenter aside. Typically, at first it is better to take into consideration different models, with different grade of complexity, setting the order of each of their parts a priori and then carry out the identification. The second step is to apply the whiteness test to each identified model, if it is not satisfied the model it is possible that the chosen order and/or model is not correct and we could try MDL, AIK, FPE as a help in setting them.

5.6 Using a time domain identification for flight flutter testing

The aim of this paragraph is to provide some guidelines about the choice of a suitable model fitting a given flutter test design and data. The reading of reference [1] is suggested for a simple, yet meaningful, application of the topics treated in this paragraph, taking notice that the their presented results based on the Maximum Likelihood (ML) criterion will substantially be the same obtainable by using the ELS scheme of these notes.

Identification with measured input If the excitation is measured our model must contain both the autoregressive and exogenous part. We can then opt for an ARX model, if the disturbance noise can be assumed to be white, an ARMAX being better otherwise. Accordingly an LS and ELS (GLS) schemes are required.

Identification with unmeasured excitation The unmeasurable excitation can be of stochastic nature, like turbulence, or impulsive, so the related models will have no exogenous part. Referring to turbulence, the time history of the output can be identified either through an AR or an ARMA model. As we have underlined at paragraph 3.1.3 of chapter 3, turbulence should be used for low frequency models. Nonetheless, to some extent, using and ARMA, scheme in place of the simpler AR one, can improve the result related to poles somewhat outside its low frequency passband, see [1].

Referring to the case of a well chosen impulsive excitation we remark at first that one could even think to measure it. Unfortunately, being it impulsive, we will end in acquiring just too few samples, collocated within the identification warm up phase only. It is thus better to process just the free response following the impulse, so ending in using a model with unmeasurable input anyhow. To better understand what said above we can make a practical example. Suppose that we want to excite up to 30 Hz. Eq. 3.8 states that we have to use a 0.0057 s impulse. A suitable sampling rate for such a case could be 500 Hz, so that if we measure the input we end in acquiring only three samples of it, which are likely to be significantly less than the number of exogenous coefficients to be used. An AR model could then be used but the inevitable disturbances make an ARMA scheme far better. To further support such a suggestion it should be noticed that the higher the system damping is the worse will be the quality of the identification, as the impulse response ends in being swiftly hidden within persistent errors and noises, caused for example by the always applied turbulence.

It is then seen that turbulence and impulse excitations should be processed invariably by using an ARMA identification scheme, once more [1] provides significant clues to support such a conclusion. The reader of this notes consulting the previously mentioned references should take into account that the their reported results associated to the use of the Maximum Likelihood (ML) method should be practically the same as those obtainable using an ELS (GLS) scheme.

5.6.1 How to obtain the V-g Diagram

We want to show how the presented method can be used to reconstruct a V-g plot experimentally. For that we have to take into consideration:

- the calculation of the roots, frequencies and damping factors;
- the calculation of their uncertainty to help in safely opening the flight envelope.

Calculation of the roots Whatever our well chosen model type is the determination of the aeroelastic eigenvalues/poles requires just its AR part, i.e. the eigenvalues/poles $A(z)([A(z)])$. So the identified system will be stable if the roots/eigenvlaues of $A(z)([A(z)])$ have $|z| < 1$, i.e. z is inside the unit circle. To plot a V-g diagram it is nonetheless preferable to study the stability of the roots of A using their corresponding values in the continuous complex frequency plane. Calling Δt the sampling we recall that:

$$z = e^{s\,\Delta t} = \rho\, e^{j(\theta + 2k\pi)} \tag{5.87}$$

and solving it with respect to $s = \sigma + j\omega$ we have:

$$s = \frac{1}{\Delta t} log\, z = \frac{log\,\rho}{\Delta t} + i\frac{1}{\Delta t}(\theta + 2k\pi) \tag{5.88}$$

so that, safely taking the principal of the logarithm, i.e. $k = 0$, since we are sure to have avoided any aliasing, we have:

$$\sigma = \frac{log\,\rho}{\Delta t} \qquad \omega = \frac{\theta}{\Delta t} \tag{5.89}$$

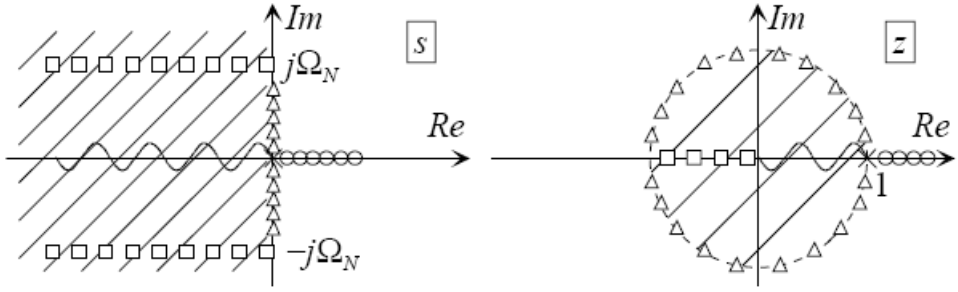


Figure 5.5: `Sampling Transformation`

Estimation of the roots variance For sake of simplicity, i.e. to avoid evaluating the sensitivities of the eigenvalues associated to the polynomial matrix of SIMO-MIMO schemes, we will take into account only SISO-MISO polynomial, associated to their regressive part *A*. On the base of such a choice it will simple to define a confidence interval for the roots of A, [1].
So let us assume z_0 is a generic root of A, i.e. one that satisfies the equation:

$$z_0^n + a_1 z_0^{n-1} + a_2 z_0^{n-2} + \cdots a_n = 0 \tag{5.90}$$

The sensitivity of $log\, z_0$ with respect to a real parameter p is given by:

$$\frac{\partial (log z_0)}{\partial p} = \frac{1}{z_0} [\lambda] \frac{\partial \{a\}}{\partial p} \tag{5.91}$$

where $[\lambda]$ is the row vector:

$$[\lambda] = -\frac{[z_0^{n-1}\, z_0^{n-2} \cdots 1]}{(n z_0^{n-1} + a_1 (n-1) z_0^{n-2} + \cdots a_{n-1})} \tag{5.92}$$

Since z_0 is a complex number, then 5.91 can be rewritten in terms of the real and the imaginary parts of $s = \sigma + j\omega$:

$$\frac{\partial \sigma_0}{\partial p} = \frac{1}{\Delta t} Real(\frac{1}{z_0}[\lambda]) \frac{\partial \{a\}}{\partial p} \qquad \frac{\partial \omega_0}{\partial p} = \frac{1}{\Delta t} Imag(\frac{1}{z_0}[\lambda]) \frac{\partial \{a\}}{\partial p} \tag{5.93}$$

Indicating with $[\sigma^2_{\sigma\sigma}]$ and $[\sigma^2_{\omega\omega}]$ the variance matrices respectively of σ e ω, i.e. the variance matrices of the roots, with $[\sigma^2_{aa}]$ being the partition of $[\sigma^2_{\Phi\Phi}]$ related to the autoregressive part $\{a\}$ shown in par. 5.2.3, we can write them as:

$$[\sigma^2_{\sigma\sigma}] = [\Sigma]\,[\sigma^2_{aa}]\,[\Sigma]^T \qquad [\sigma^2_{\omega\omega}] = [\Omega]\,[\sigma^2_{aa}]\,[\Omega]^T \tag{5.94}$$

with:

$$[\Sigma] = \frac{1}{\Delta t}\left[\frac{\partial \sigma_0}{\partial a_1} \quad \frac{\partial \sigma_0}{\partial a_2} \quad \cdots \quad \frac{\partial \sigma_0}{\partial a_n}\right]^T \qquad [\Omega] = \frac{1}{\Delta t}\left[\frac{\partial \omega_0}{\partial a_1} \quad \frac{\partial \omega_0}{\partial a_2} \quad \cdots \quad \frac{\partial \omega_0}{\partial a_n}\right]^T \tag{5.95}$$

One should remark the similarity of the above formulae with those of errors propagation seen in a previous course on measurements in engineering. Figure 5.6 shows a rough, exaggerated, sketch

of the appearance of a fluttering root of an experimental V-g diagram, the ellipsoid depicting uncertainties computed as suggested above.

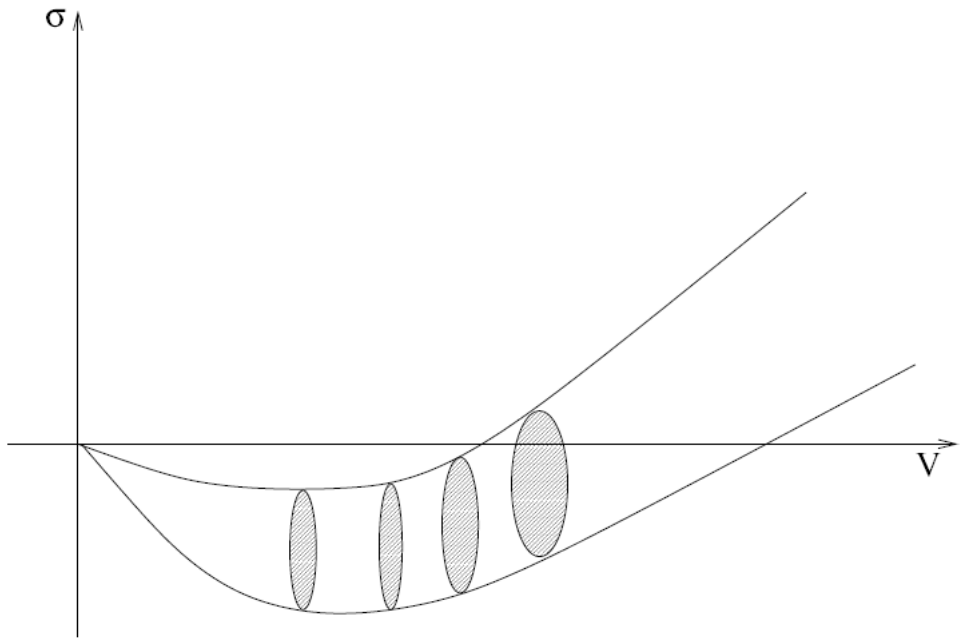


Figure 5.6: `Experimental V-g diagram`

5.6.2 Doing it all recursively

We end our short tour within time domain identification by hinting at the basics of an interesting alternative way of solving LS problems.
Calling $[a_k]$ the kth row of the matrix $[A]$, at the ith measurement the normal coefficient matrix of the least squares problem, $[N]_{i-1}$, can be computed as:

$$[N]_i = [A(t)]_i^T\,[A(t)]_i = \Sigma_i([a_k]^T[a_k]) \tag{5.96}$$

therefore at the $(i+1)th$ time we will have:

$$[N]_{i+1} = [N]_i \;+\; [a_{i+1}]^T[a_{i+1}] \tag{5.97}$$

where $[a_{i+1}]$ is obtained by shifting $[a_i]$ to the left while appending the $(i+1)th$ measure of $\{u\}$ to its right. Then, assuming we have already available $[N]_i^{-1}$, we can calculate $[N]_{i+1}^{-1}$ using the Sherman-Morrison formula [6], i.e.:

$$[N]_{i+1}^{-1} = [N]_i^{-1} \;-\; \frac{[N]_i^{-1}\,[a_{i+1}]^T[a_{i+1}]\,[N]_i^{-1}}{1 \;+\; [a_{i+1}]\,[N]_i^{-1}\,[a_{i+1}]^T} \tag{5.98}$$

[6] See [51], or simply Google Wikipedia for "Sherman Morrison formula". It can be interpreted also as an unsteady Kalman filter [23].

With it we can update the inverse of a symmetric matrix of order n, such as $[N]$, with order of $2n^2$ multiply add operations, possibly carrying out the solution of an LS problem keeping apace with data acquisition in strict real time. The same can be said for the discrete roots calculation, thus allowing to follow the stabilization of our poles as data are acquired. In practice it is likely that, while updating the inverse and the right hand side at each sampling instant, we will carry out the solution and the calculation of the discrete poles only after having repeated the updates a few times, the choice of the most convenient tactics being in the hands of the engineers following a test.
In such a case we would likely gather a batch of m rows of $[A]$, from the $(i+1)th$ to the $(i+m)th$, so that we can use the symmetric Woodbury identity (formula) [7] to carry out the corresponding batch update of $[N]$ in a single step, i.e.:

$$[N]_{i+m}^{-1} = [N]_i^{-1} - [N]_i^{-1} [A_{im}]^T ([I] + [A_{im}] [N]_i^{-1} [A_{im}]^T)^{-1} [A_{im}][N]_i^{-1} \tag{5.99}$$

where:

$$[A_{im}] = \begin{bmatrix} [a_{i+1}] \\ [a_{i+2}] \\ . \\ [a_{i+m}] \end{bmatrix} \tag{5.100}$$

The recursive updates can begin either after having accumulated enough samples to make it possible a single first batch evaluation of $[N]$ or by initializing the inverse of $[N]$ with a scalar matrix, $\alpha\ [I]$, recursively updating it from the very beginning while waiting to carry out any solution till a true inverse is available, i.e. after having updated it for a number of times $\geq n$. Such an approach might suffer from numerical instabilities, but stabler recursive updates are available for the numerically better QR and LDL^T approaches.
We will exploit and expand such a technique in another part of our course, when it will be used for indirect in flight adaptive damping enhancement and flutter stabilization.

[7] See references mentioned in the previous footnote. It is a generalization of the Sherman Morrison formula, also known as Sherman Morrison Woodbury formula.

Chapter 6

Identification in the Frequency Domain

In this chapter we will treat identification methods in the frequency domain, dividing them into two main schemes:

- single degree of freedom methods, in which modal characteristics are identified by isolating each mode within a set of well separated modes.
- multi degree of freedom methods, in which all the modes of interest are identified at the same time.

The identification in the frequency domain is based on the evaluation of Transfer Functions/Matrices (TF/TM). Therefore the aim of this chapter is to describe the fundamentals of a transfer matrix identification from input/output measures, finally presenting a few identification methods. We will analyze:

- single degree of freedom methods:
  - Peak picking;
  - Kennedy-Panku circle approximation;
- multi degrees of freedom methods:
  - Rational function approximation;
  - Residuals and pole factors approximation;

6.1 Determination of Frequency Response (FR)

First of all it is necessary to determine the FR from the measured data acquired during a test. Toward such and end let us recall the frequency domain input/output scheme we previously set up:

$$\{y\} + \{n_y\} = [H(\omega)](\{u\} + \{n_u\}) \tag{6.1}$$

Anticipating a bit what will be said further on we assume that we can carry out many tests, transforming the related input/output measures from the time to the frequency domain so to end with having available the above equation at so many frequency values to make it possible a least squares determination of $[H(\omega)]$. It is likely useful to say that actual tests repetitions are required only for time limited, i.e. transient, time histories. In fact in the case of stationary random signals, mostly ergodic in our case, the different samples to be used will simply be the pieces of a single record, long enough to make it possible to extract chunks whose duration is adequate to represent the randomness contained in the signal and to satisfy resolution requirements, without impeding a substantial overdetermination for the previously suggested least squares estimates. In such a view it should be recalled also that the effective noise cleaning will come in proportion to the inverse square root of the number, N, of available time histories [1]. If a significant smoothing is required that imposes quite a severe constraint on testing time and cost, inevitably forcing some compromises on a test planning and data processing, as it often is for many, almost all, engineering activities. We have now to recall that least squares fits of a function are based on the assumption that there is an independent measure having negligible noise, with all of the uncertainties being associated to the dependent measure. Using a very simple example we can say that if the simple linear algebraic relation $y = a\,u$ is to be identified on the base of a set of measures, y_i and u_i, it should be used as it is if u is the least affected by noise, reverting to $u = y/a$ otherwise. We will later hint to what should be done in the case that both input and output are substantially noisy. Calling Σ_N the summation over all the N available tests/chunks, what just said leads us to the following least squares solution [2]:

$$\Sigma_N(\{y\}\{u\}^*) = [H(\omega)]\Sigma_N(\{u\}\{u\}^*) \tag{6.2}$$

for the case in which $\{n_u\}$ can be retained negligible, while the following should be used when it will be so for $\{n_y\}$:

$$\Sigma_N(\{y\}\{y\}^*) = [H(\omega)]\Sigma_N(\{u\}\{y\}^*) \tag{6.3}$$

Clearly, the latter equation implies that the number of input must be equal to that of the output. For their implementation only the transforms of the measured data $\{y\}$ and $\{u\}$ will be available and we will define the above implied coefficient matrices as:

$$[S_{yy}] = \Sigma_N(\{y\}\{y\}^*) \qquad [S_{uu}] = \Sigma_N(\{u\}\{u\}^*) \tag{6.4}$$

$$[S_{yu}] = \Sigma_N(\{y\}\{u\}^*) \qquad [S_{uy}] = \Sigma_N(\{u\}\{y\}^*) \tag{6.5}$$

[1] Review the basics of the course on the statistical treatment of measures.

[2] Reminding the calculations of the stationary conditions for quadratic performances used in the previous course on structural dynamics and controls, $\frac{d\,(AB^*)}{d\,B} = A$ and the following notation: $(.)^* = Conj(.)^T$, we can sketch the related proofs. For the case of $\{u\}$ being the cleanest, $\{n_u\} \simeq 0$, signal we have: $min\ \Sigma_N((\{y\} + \{n_y\} - [H(\omega)]\{u\})^*(\{y\} + \{n_y\} - [H(\omega)]\{u\})) = min\ \Sigma_N\ Tr((\{y\} + \{n_y\} - [H(\omega)]\{u\}(\{y\}^* + \{n_y\}^* - \{u\}^*[H(\omega)]^*))$, whose stationary condition is: $\Sigma_N(\{y\}\{u\}^*) - [H(\omega)]\Sigma_N(\{u\}\{u\}^* - \{n_y\}\{u\}^*) = 0$, wildly assuming $\Sigma_N(\{n_y\}\{u\}^*) \simeq 0$, we obtain the equation reported in the text. The case of $\{y\}$ being less noisier, which implies that $\{n_y\}$ should be negligible and $\Sigma_N(\{n_u\}\{y\}^*) \simeq 0$, is quite similar but, under the constraint of an equal number of input and output, minimises $[H(\omega)]^{-1}$ using: $min\ \Sigma_N(([H(\omega)]^{-1}\{y\} - \{u\} - \{n_u\})^*([H(\omega)]^{-1}\{y\} - \{u\} - \{n_u\}))$. Then following the same previous steps it leads to $[H(\omega)]^{-1}\Sigma_N(\{y\}\{y\}^*) - \Sigma_N(\{u\}\{y\}^*) = 0$, from which one finds what reported in the text once more.

We recall that theoretical auto and cross power spectra, $[\Phi(\omega)]$, are the transform of the related auto and cross covariances, $[K(\tau)]$, which for finite length records are defined as:

$$[K_{yy}(\tau)] = \int_{-\infty}^{\infty} \{\Delta y(t)\}\{\Delta y(t+\tau)\}^T dt \qquad [K_{uu}(\tau)] = \int_{-\infty}^{\infty} \{\Delta u(t)\}\{\Delta u(t+\tau)\}^T dt \tag{6.6}$$

$$[K_{yu}(\tau)] = \int_{-\infty}^{\infty} \{\Delta y(t)\}\{\Delta u(t+\tau))\}^T dt \qquad [K_{uy}(\tau)] = \int_{-\infty}^{\infty} \{\Delta u(t)\}\{\Delta y(t+\tau)\}^T dt \tag{6.7}$$

so that [3]:

$$[\Phi_{yy}(\omega)] = \{\Delta y(-\omega)\}\{\Delta y(\omega)\}^T \qquad [\Phi_{uu}(\omega)] = \{\Delta u(-\omega)\}\{\Delta u(\omega)\}^T \tag{6.8}$$

$$[\Phi_{yu}(\omega)] = \{\Delta y(-\omega)\}\{\Delta u(\omega)\}^T \qquad [\Phi_{uy}(\omega)] = \{\Delta u(-\omega)\}\{\Delta y(\omega)\}^T \tag{6.9}$$

The use of Δ, difference between a history and its mean value along time, in the definition of auto and cross covariances reminds us that they will coincide with their auto and cross correlations only if their along time records mean is null, so that we have to subtract such a mean before taking the Fourier transform of each of their record. Therefore, we have the following relations [4]:

$$[\Phi_{yy}(\omega)] \simeq \frac{1}{N}[S_{yy}]^T \qquad [\Phi_{uu}(\omega)] \simeq \frac{1}{N}[S_{uu}]^T \tag{6.10}$$

$$[\Phi_{yu}(\omega)] \simeq \frac{1}{N}[S_{uy}]^T \qquad [\Phi_{uy}(\omega)] \simeq \frac{1}{N}[S_{yu}]^T \tag{6.11}$$

so that, $1/N$ averaging and transposed conjugation apart, the matrices $[S]$ are estimate of the related power spectral densities. We can thus write:

$$[S_{yu}(\omega)] = [H_1(\omega)][S_{uu}(\omega)] \qquad [S_{yy}(\omega)] = [H_2(\omega)][S_{uy}(\omega)] \tag{6.12}$$

from which we have:

$$[H_1(\omega)] = [S_{yu}(\omega)][S_{uu}(\omega)]^{-1} \qquad [H_2(\omega)] = [S_{yy}(\omega)[S_{uy}(\omega)]^{-1} \tag{6.13}$$

where we have used the subscript 1 and 2 to distinguish the two different identifications, whereas it should be clearly $[H_1] = [H_2] = [H]$ for the ideal case of no noises/disturbances and $N \to \infty$. Recalling that: $\{y\} = [H](\{u\} + \{n_u\}) - \{n_y\}$, we could have written also:

$$\Sigma_N(\{y\}\{y\}^*) = [H(\omega)]\Sigma_N(\{u\}\{y\}^*) = [H(\omega)]\Sigma_N(\{u\}\{u\}^*)[H(\omega)]^* + \tag{6.14}$$

$$+ [H(\omega)]\Sigma_N(\{u\}\{n_u\}^*)[H(\omega)]^* - [H(\omega)]\Sigma_N(\{u\}\{n_y\}^*) \tag{6.15}$$

[3] For limited duration, transient, signals, taking $[\Phi_{yu}]$ as an example we have: $[\Phi_{yu}(\omega)] = \int_{-\infty}^{\infty}[K_{yu}(\tau)]e^{-j\omega\tau}d\tau = \int_{-\infty}^{\infty}\{\Delta y(t)\}\{\Delta u(t+\tau))\}^T e^{-j\omega\tau}dtd\tau = \int_{-\infty}^{\infty}\{\Delta y(t)\}e^{-j(-\omega)t}\{\Delta u(v))\}^T e^{-j\omega v}dtdv = \{\Delta y(-\omega)\}\{\Delta u(\omega)\}^T$. To be noticed that $\{\Delta y(-\omega)\}$ is the conjugate of $\{\Delta y(\omega)\}$. In the case of ergodic signals we instead have: $[\Phi_{yu}(\omega)] = \frac{lim}{T\to\infty}\frac{1}{2T}\int_{-T}^{T}[K_{yu}(\tau)]e^{-j\omega\tau}d\tau = \frac{lim}{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\{\Delta y(t)\}\{\Delta u(t+\tau))\}^T e^{-j\omega\tau}dtd\tau = \frac{lim}{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\{\Delta y(t)\}e^{-j(-\omega)t}\{\Delta u(v))\}^T e^{-j\omega v}dtdv = \frac{lim}{T\to\infty}\frac{1}{2T}\{\Delta y(-\omega,T)\}\{\Delta u(\omega,T)\}^T$. The related limited cannot be carried out analytically on experimental data. Therefore it is evaluated numerically on a record long enough. That means that we end in estimating the power spectra using an ensemble average, in place of an along record one, in the same way as we had transient data.

[4] Since $\{y(-\omega)\} = Conj(\{y(\omega)\})$, for any couple of vectors $\{y\}$ and $\{u\}$ we have $\{y(-\omega)\}\{u(\omega)\}^T = (((\{u(\omega)\}\{y(\omega)\}^*)^T$

which, assuming: $\Sigma_N(\{u\}\{n_y\}^*) \simeq 0$ and $\Sigma_N(\{u\}\{n_u\}^*) \simeq 0$, using $[H_0]$ for $[H]$ to remark yet another way to estimate it, we obtain:

$$[S_{yy}(\omega)] = [H_0(\omega)][S_{uu}(\omega)][H_0(\omega)]^* \tag{6.16}$$

The above formulae are thus those corresponding to the theoretical ones:

$$[\Phi_{uy}(\omega)] = [\Phi_{uu}(\omega)][H(\omega)]^T \qquad [\Phi_{yy}(\omega)] = [\Phi_{yu}(\omega)][H(\omega)]^T \tag{6.17}$$

$$[\Phi_{yy}(\omega)] = [H(-\omega)][\Phi_{uu}(\omega)][H(\omega)]^T \tag{6.18}$$

which are the Fourier transform of:

$$[K_{uy}(\tau)] = \int_{-\infty}^{+\infty} [K_{uu}(\tau - v)][h(v)]^T dv \qquad [K_{yy}(\tau)] = \int_{-\infty}^{+\infty} [K_{yu}(\tau - v)][h(v)]^T dv \tag{6.19}$$

$$[K_{yy}(\tau)] = \int_{-\infty}^{\infty} [h(v)][K_{uu}(\tau + v - p)][h(p)]^T dv dp \tag{6.20}$$

In practice we have noises on both input and output so the best estimate will be an appropriate combination of H_1 and H_2. To find it we should resort to total least squares [5], but we do not expand that further and make us happy with a simple mean defined as:

$$[H_3] = \frac{[H_1] + [H_2]}{2} \tag{6.21}$$

In the SISO case the previous relations become:

$$H_1(\omega) = \frac{S_{py}(\omega)}{S_{uu}(\omega)} \qquad H_2(\omega) = \frac{S_{yy}(\omega)}{S_{uy}(\omega)} \qquad H_3(\omega) = \frac{H_1 + H_2}{2} \tag{6.22}$$

$$S_{yy}(\omega) = |H_0(\omega)|^2 \Phi_{uu}(\omega) \tag{6.23}$$

with the one immediately above providing an estimation of the transfer function modulus:

$$H_0 = \sqrt{\frac{S_{yy}}{S_{uu}}} \tag{6.24}$$

H_1 and H_2 are complex functions of ω from which it is possible to obtain the corresponding phase/gain relationship.

[5] If one is interested in pursuing the subject further a look at [51] and [52] can be of help. The specialization of total least squares to the identification of FRMs, often called $[H_v]$, can be found in [20]. A higly readable and concise presentation, with a simple code sample and further references, can be found at: http://en.wikipedia.org/wiki/Total_least_squares.

Getting a full transfer function from its modulus This estimation of the FRF is suitable when the excitation is not measurable and only the response is available.
In such a case, we cannot apply the previously presented estimators of the FRF, because we have just the power spectral density of the ouput, Φ_{yy}, without any information about the input. However, for a SISO system, if the input PSD Φ_{uu} can be retained significantly constant, at least over a frequency range of interest, we might neglect the constant value of the input spectrum and retain that $\Phi_{yy} \simeq |H(\omega)|^2$. Hence starting from the spectrum of the output we are able to obtain an estimate of the absolute value of the FRF as: $H = \sqrt{\Phi_{yy}}$.
However, the knowledge of the absolute value of the FRF is not sufficient to the complete estimation of FRF of the system, we need also to know its phase. If the system can be assumed to be minimum phase, the full FRF can be evaluated directly from its absolute value, thanks to Bode's formula:

$$\phi = \angle H(j\overline{\omega}) = \frac{\overline{\omega}}{10\pi \ln} \int_0^\infty \frac{|H(j\omega)|_{db} + |H(j\overline{\omega})|_{db}}{\omega^2 + \overline{\omega}^2} d\omega \tag{6.25}$$

The minimum phase assumption is not verified always, but, in our case, the poles of the system are in the left half plane, because we tested without having a flutter onset, while it is possible to have some zeros in the right half plane. Hence, reconstructing the phase of the system using Bode's formula might lead to a false result, producing a wrong estimation of system zeros. However, from our point of view, we are interested in computing the system frequencies and dampings, hence, the only information we need to know with a good precision is about the poles and we can thus accept any phase reconstruction which does not alter them.

Coherence function It is possible to establish a direct correlation index between the input and the output by defining a coherence function γ^2_{uy}, whose value ranges between 0 and 1. For a MIMO system, the coherence functions is a matrix given by [20]:

$$[\gamma^2_{uy}] = [\Phi_{yy}]^{-1}[\Phi_{yu}][\Phi_{uu}]^{-1}[\Phi_{uy}] \tag{6.26}$$

to be estimated by taking into account (6.10) and (6.11). The scalar case can be rewritten as:

$$\gamma^2_{uy}(\omega) = \frac{|\Phi_{uy}(\omega)|^2}{\Phi_{uu}(\omega)\Phi_{yy}(\omega)} \simeq \frac{|S_{uy}(\omega)|^2}{S_{xx}(\omega)S_{yy}(\omega)} = \frac{H_1(\omega)}{H_2(\omega)} \tag{6.27}$$

In an ideal case, in which there is no noise at the input/output and the system is perfectly linear, $\gamma^2_{xy} = 1$, in the MIMO case it will be so for any coefficient of the coherence matrix. Saying that there is a high coherence means that the measured output are directly connected to any given input. Many causes can lead to a low coherence between the input and the output. Usually the more significant are related to:

- exogenous noise in the measures and the system;
- resolution errors in spectral estimations;
- non linearities between input and output;

In this case, H_1 and H_2 represent only an upper and a lower bound, between which the real value of H is contained:

$$|H_1(\omega)| \leq |H(\omega)| \leq |H_2(\omega)| \tag{6.28}$$

It should be also clear that any actual coherence calculation will be based on the use of the S estimates.

Final considerations In the case where on the system there is no noise $H_1 = H_2 = H$, where H is the ideal frequency response. In the real case, in which noise is present, H_1 and H_2 are two estimators that give a certain band within which the frequency response function is included, giving an inferior and a superior limit to the maximum variation of the frequency response function of the system. The real frequency response function of the system is contained between every spectral line between the values of H_1 and H_2. The mean value for each spectral line between H_1 and H_2 is H_3. H_3 is not often used because if we draw H_1, H_2 and H, we notice that H is never in the middle. The lower the noise, the more, for each spectral line, the band formed by H_1 and H_2 tends to converge towards a unique spectral line.
Looking at the distance between H_1 and H_2 gives a rough indication of the measure qualities: the stronger the noise effect at the input or at the output the higher the spread is.

Often in literature there are statements supporting that the estimation of H_1 is better near resonances, whereas that of H_2 near anti-resonances.
These statements should be clear, according to the results which we have just presented: H_1 represents the optimal estimation of the frequency response of the system in presence of noise only at the output.
Exciting a system far from one of its resonance frequency, needs to use a high excitation level to have a well measurable response level, because we are trying to transfer energy in the system without exploiting the amplification factor characteristic typical of the resonance. So if the input level is high enough we can assume that the noise effect on the measure of the input itself will be negligible. Differently, on the response measures the noise effect could be relevant most if the response level itself was too low. This case is similar to the ideal one, in which noise effects are only at the output.
Near a resonance we have the opposite situation: output are high but the input level is very low and so the input measure is dominated by noise. This case, instead, is similar to the idealized one, in which the noise effect is only at the input: H_2,as previously shown, will be the optimal estimation of the frequency response of the system.
These considerations should explain the different behavior between H_1 and H_2, near and far system resonance frequencies respectively.

6.1.1 Rudiments on the practical estimation of FRM/FRFs

The aim of this paragraph is to provide a short outline of the techniques used to compute the FRF estimators starting from data acquired in the test phase.
Suppose that data have been sampled with an anti-aliasing filter, if the pass band of sensors has not done it yet adequately, according to the Shannon Theorem. Nowadays input/output measures are digitally sampled at discrete times and a Discrete Fourier Transform (DFT) is invariably used, thus implicitly imposing a periodicity in both time and frequency.
Clearly records have finite length, the DFT allows to compute the spectrum of a signal starting from a finite length record if and only if the Shannon Theorem is respected and the circular repetition of the record correctly restores the original signal. In particular, this last condition can be satisfied if and only if the record contains an integer multiple of any periodic signal component. If this condition is violated, the circular repetition of the record does not restore the original signal but introduces some discontinuities in correspondence of each repetition, an

artifact which does not exist in the reality, being it due to the use of finite length records. These errors which are also known as truncation errors because if we could use infinite length records they will not occur.
What said justifies the often use periodic repetition of excitations, we mentioned in the general description if forcing functions. Whenever it cannot be avoided the truncation errors give origin to undesired phenomena, both in the time and frequency domain, which should be cared of as much as possible. In particular, the energy of the signal is redistributed over a wider frequency range generating spurious additional spectral lines, which do not exist for the original signal, due to the presence of the discontinuities caused by the fractional periodic record repetitions [45]. The consequent redistribution of the energy of the signal on a wider frequency interval results in a smearing of the signal spectrum, which increases the uncertainty in measuring the frequency content of the signal. Because the information has been smeared, it is not possible in practice to find exact maxima of the main lobes, and hence an estimation of the main contributions to the frequency content of the analyzed signal cannot be exact. This phenomena is known as leakage. It can be demonstrated that the effect of the finite record length corresponds to convolving the original transform of the signal with the transform of $sin(\omega/2)/(\omega/2)$ function [48]. In fact, considering a finite length, T, record of the generic signal $x(t)$ is equal to multiply the signal for the so called boxcar, $b(t)$, function[6]. Hence:

$$x_T(t) = b(t)x(t)$$

thanks to the convolution theorem, the Fourier transform of u_p is given by:

$$\mathcal{F}[x_b(t)] = X_p(\omega) = \int_{-\infty}^{+\infty} U(\eta)X(\omega-\eta)d\eta = \int_{-\infty}^{\infty} U(\eta)\frac{sin(\pi(f-\eta))}{\pi(f-\eta)} =$$
$$= \int_{-\infty}^{\infty} X(\omega-\eta)\frac{sin(\pi(f-\eta))}{\pi(f-\eta)}d\eta$$

A solution for the leakage problem, is suggested from observing that the leakage is originated because of jumps at the end of the record, so decaying as $1/\omega$ and linked to Gibbs phenomenon. Hence the solution is that of multiplying the acquired signal for a continuous weighting window function, whose duration is T and identically zero outside T, before applying the DFT transform. The most used weighting window functions:

- Rectangular, boxcar, function;
- Hamming function;
- Hanning function;
- Flap top;
- Exponential (mostly for transient records);

[6]The boxcar function is defined as: $b(t) = sca(t) - sca(t-T)$, where T is observation time, which is connected to the record length.

For a discussion of their properties and utilization, see [45] and [48]. Notice that limited duration, transient, signals are not affected by leakage because, by definition and if they are acquired correctly, any related asymptotic system response starts and ends inside the acquired record. For this reason, transient signal are often called self windowing.

To improve the estimation of the spectrum of stationary, mainly ergodic for us, random signals, minimizing the effect of the exogenous noise, a good strategy is to apply a linear averaging on the signal.
The main idea is that, if we have a record length long enough, we can divide the time history in more segments and obtain its spectrum simply by averaging the spectra computed on each segment. This operation is the same as extracting the mean value of a signal, in fact, to average time segments is equivalent to average their Fourier transform, since both operations are linear.
It is clear that if the considered segments were uncorrelated the ones with the others, this procedure will converge to the mean value, i.e. in a null mean process, to zero. However, since the segments are not statistically independent, hence they are correlated, performing the linear averaging will result in a better estimation of the signal spectrum, since the effect due to the exogenous noise will be smoothed by the mean operation. Remember that this filtering technique has an efficiency of $\sqrt{N}$ where N is the number of the sample used to evaluate the mean.
Clearly, every time a time history is truncated in more segments, this will result in a loss of frequency resolution, and the longer the segment is, the smaller this loss will be.
However, even if the resolution is too wide, the linear average allows to recover the bias error due to smearing, because, even if we can not increase the resolution adding more point, thanks to the linear averaging we can reduce the smearing errors effect in correspondence of all those points in which the spectrum is evaluated, reducing the variability.

6.2 Single degree of freedom methods

6.2.1 Peak picking (Half Power Bandwidth)

This method is perhaps the simplest method and is used during flight flutter tests as a quick look preliminary analysis.
The method is based on the hypothesis that when there is a good separation between any couple of modes of interest, each of them can be treated as standing alone. Its application for not excessively damped modes leads to the following scheme:

- The single resonance peaks are located on the amplitude diagram of the FR; the frequency at the response peak is taken as being the natural frequency ω_r.[7]
- Calling $|\tilde{\alpha}|$ the maximum response amplitude, we identify the bounding frequencies, ω_a and ω_b, defining a range $\Delta\omega = \omega_b - \omega_a$ in correspondence to which the amplitude is $\frac{|\tilde{\alpha}|}{\sqrt{2}}$; ω_a and ω_b are called "middle power points";

[7] In fact the peak corresponds to the damped frequency, which is given by $\omega_r = \omega_n \sqrt{(1-\zeta^2)}$, where ω_n is the natural frequency. If damping is low, as it is often our case, the difference between the two frequencies is very small.

- The dissipation, η_r, and damping, ζ_r factors are then given by:

$$\eta_r = \frac{\omega_b^2 - \omega_a^2}{\omega_r^2} \simeq \frac{\Delta\omega}{\omega_r} \qquad \zeta_r = \frac{\eta_r}{2} = \frac{\Delta\omega}{2\omega_r} \tag{6.29}$$

If there is a good separation the global response close to the frequency band considered is due only to the considered mode, its residual, i.e. its modal constant:

$$\frac{X_p}{F_q} = \frac{A_{pqr}}{-\sigma_r + j(\omega - \omega_r)} \tag{6.30}$$

can then be estimated as:

$$|\tilde{\alpha}| = \frac{A_r}{\omega_r^2 \eta_r} \quad \Rightarrow \quad A_r = |\tilde{\alpha}|\,\omega_r^2 \eta_r \tag{6.31}$$

The main limitations of the peak picking method can be summarized as follows:

- The estimations of damping and modal constant depend mainly upon the care with which the maximum amplitude is evaluated, so an adequate frequency resolution is required.
- The lowly damped single mode assumption might be too rough.

6.2.2 Kennedy-Panku circular approximation

The original approach was proposed by Kennedy and Panku as a way to approximate structural damping, also known as hysteretic damping. Under appropriate assumptions it can be extended to viscous damping and complex modes. The assumptions required for such an extension are:

- modes are only lightly coupled when close to a resonance and the contribution coming in from both the high and low frequency sides can be approximated with a single complex bias constant g = R + jI;
- the system is slightly damped; the frequency response of the structure within the frequency range in which the r-mode is dominant can be obtained using the following modification of Eq. 6.30:

$$\frac{X_p}{F_q} = \frac{U_{pqr} + jV_{pqr}}{-\sigma_r + j(\omega - \omega_d r)} + (R + jI) \tag{6.32}$$

  where X_p is the response at point p, F_q being the excitation at point q; and (R + jI) includes the contribution of the conjugate value also.

If the complex constant is neglected and the mode is normalize for a unit amplitude, we can write:

$$\Re\left(\frac{X_p}{F_q}\right) = -\frac{(\omega - \omega_r)}{(\omega - \omega_r)^2 + \sigma_r^2} \tag{6.33}$$

$$\Im\left(\frac{X_p}{F_q}\right) = -\frac{\sigma_r}{(\omega - \omega_r)^2 + \sigma_r^2} \tag{6.34}$$

$$\tag{6.35}$$

from it we have:

$$\left[\Re\left(\frac{X_p}{F_q}\right)\right]^2 + \left[Im\left(\frac{X_p}{F_q}\right) - \frac{1}{2\sigma_r}\right]^2 = \left[\frac{1}{2\sigma_r}\right]^2 \tag{6.36}$$

which is clearly the equation of a circle.
In short, the contribution of a generic mode to the response of the system can be represented by a circle in the complex frequency plane. Then, provided we know the complex constant and the modal coefficient, its center and diameter are given by:

$$x_c = \left(R - \frac{U_{pqr}}{2\sigma_r}\right) \tag{6.37}$$

$$y_c = \left(I - \frac{V_{pqr}}{2\sigma_r}\right) \tag{6.38}$$

$$d = \frac{\sqrt{U_{pqr}^2 + V_{pqr}^2}}{\sigma_r} \tag{6.39}$$

The modal displacement either expands or reduces the diameter and rotates the circle in the in the complex frequency plane. The method can be verified by looking at how the frequency response shape at a resonant frequency is close to a circular one. The more the response shape is close to a circle, the more the result is precise.
It can be demonstrated that the resonance frequency is where the phase angle variation in function of frequency is maximum, $\frac{\partial\phi}{\partial\omega} = 0$. The damping ratio, ζ, can be obtained from:

$$\zeta = \left|\frac{\omega_a - \omega_b}{2\omega_r}\right| \tag{6.40}$$

where ω_a and ω_b are the frequencies at $\pm 90^\circ$ with respect to the damped resonance frequency.
The phase angle of the complex modal coefficient, ϕ_{pqr}, is given by the angle between a straight line from the resonance point, ω_r to the center of the circle and the imaginary axis.

$$\phi_{pqr} = \arctan\left(\frac{U_{pqr}}{V_{pqr}}\right) = \frac{\pi}{2} + \arg(A_{pqr}) \tag{6.41}$$

The circle fit technique allows a better separation of modes, it is efficient and fast and can be used to obtain a complex mode. Nonetheless, to achieve the best result, it is usually based on a LS approximation iterated over an increased number of points at each frequency of interest. It is then possible to somewhat verify if there is any significant hidden modal coupling, as if there is none the fit will not change, to a reasonable extent, while increasing the number of points. It is remarked that the Kennedy-Panku method is naturally embedded and improved it the pole/residuals factorization to be presented shortly ahead.

6.3 Multi degree of freedom methods

The biggest limitation of single degree identification methods is the need of a significant separation between modes, multi-degrees methods allow to overcome it.
To represent a FR, we will consider two parametrized schemes, whose coefficients must be determined through an appropriate identification procedure: the pole/residuals factorization and rational function approximation.

6.3.1 Approximation through a pole/residuals factorization

The main idea of this method is to express the transfer function in the form of partial fractions [20]:

$$H(s) = \sum_{i=1}^{n}\left[\frac{A_i}{s+p_i} + \frac{A_i^*}{s+p_i^*}\right] \rightarrow H(\omega) = \sum_{i=1}^{n}\left[\frac{A_i}{j\omega+p_i} + \frac{A_i^*}{j\omega+p_i^*}\right] \tag{6.42}$$

Residual parts can be added to represent possible modes outside a frequency range of interest. After selecting a frequency band of interest, given by $[\omega_a, \omega_b]$, the residuals will take into account the approximate contribution of the poles below ω_a and of those beyond ω_b. residualization of all the dynamics at a frequency lower than ω_a, while, the stiffness residual represents the residualization of all the dynamics at a frequency higher than ω_b. In fact, if we consider a generic pole $\frac{1}{s+a}$ and want to residualize it for the high frequency response, i.e. for $\frac{a}{s} \ll 1$, expanding in series we obtain a polynomial in $\frac{a}{s}$. Instead, if we want to residualize for the low frequency response, i.e. for $\frac{s}{a} \ll 1$, the related series expansion will be a polynomial in $\frac{s}{a}$.
Thus our factorized expansion will be composed of three sums, aimed at the best approximation within $[\omega_a, \omega_b]$:

$$H(\omega) = \frac{a_1}{j\omega} + \frac{a_2}{(j\omega)^2} + \cdots + \sum_{i=1}^{n}\left[\frac{A_i}{j\omega+p_i} + \frac{A_i^*}{j\omega+p_i^*}\right] + k + b_1(j\omega) + b_2(j\omega)^2 + \cdots \tag{6.43}$$

and remembering that A_i and p_i are complex numbers, calling A_i^* and p_i^* their complex conjugates and limiting the residuals to the second order, we can rewrite the previous equation as follows:

$$H(\omega) = \frac{a_1}{j\omega} + \frac{a_2}{(j\omega)^2} + \sum_{i=1}^{n}\left[\frac{U_i + jV_i}{j(\omega+\omega_i)+\sigma_i} + \frac{U_i - jV_i}{j(\omega-\omega_i)+\sigma_i}\right] + k + b_1(j\omega) + b_2(j\omega)^2 \tag{6.44}$$

We then obtain a total set of 4n+5 unknowns parameters. In fact we have 4 parameters σ_i, ω_i, U_i e V_i for each of the n terms of the summation, and the five parameters a_1, a_2, k, b_1, b_2, hence:

$$\begin{aligned} &H(\omega, \{\gamma\}) = H(\omega, \gamma_1, \gamma_2 \cdots \gamma_{4n+5}) \\ &\{\gamma\} = \{a_1, a_2, \sigma_1, \omega_1, U_1, V_1, \sigma_2, \omega_2, U_2, V_2, \cdots, \sigma_n, \omega_n, U_n, V_n, k, b_1, b_2\}^T \end{aligned} \tag{6.45}$$

To determinate the unknown parameters, we must impose 4n+5 conditions at least. Fortunately, $H(\omega)$ is known for a number frequencies, say m, far larger than $(4n+5)$ so that, indicating with $\bar{H}(\omega_k)$ the estimated FR at the kth frequency ω_k, $\omega_a \leq \omega_k \leq \omega_b$, we can collocate our model so to obtain an overdetermined non linear system of equations whose *kth* residual will be:

$$\begin{aligned} E_k(\{\gamma\}) &= \bar{H}(\omega_k) - H(\omega_k, \{\gamma\}) = \\ &= \bar{H}(\omega_k) - \frac{a_1}{j\omega_k} - \frac{a_2}{(j\omega_k)^2} - \sum_{i=1}^{n}\left[\frac{U_i + jV_i}{j(\omega_k+\omega_i)+\sigma_i} + \frac{U_i - jV_i}{j(\omega_k-\omega_i)+\sigma_i}\right] \\ &- k - b_1(j\omega_k) - b_2(j\omega_k)^2 \end{aligned} \tag{6.46}$$

Writing the residual for each of the m frequencies, we will obtain We can then try to solve it in the LS sense by minimizing: [8]:

$$F = \{E\}^*\{E\} = \sum_{k=1}^{m} E_k(\{\gamma\})^* E_k(\{\gamma\}) \tag{6.47}$$

[8] Remember that the operator * applied to a vector means the transpose of the conjugate.

Rewriting what above in term of the real and the imaginary part of the residual $\{\mathrm{E}\}$ we have:

$$\begin{aligned} F &= \sum_{k=1}^{m} E_k(\{\gamma\})^* E_k(\{\gamma\}) = \sum_{k=1}^{m} (Re(E_k(\{\gamma\})) - jIm(E_k(\{\gamma\}))) (Re(E_k(\{\gamma\})) + jIm(E_k(\{\gamma\}))) = \\ &= \sum_{k=1}^{m} \left(Re(E_k\{\gamma\})^2 + Im(E_k\{\gamma\})^2 - Im(E_k\{\gamma\}) Re(E_k\{\gamma\}) + Re(E_k\{\gamma\}) Im(E_k\{\gamma\}) \right) = \\ &\sum_{k=1}^{m} \left(Re(E_k\{\gamma\})^2 + Im(E_k\{\gamma\})^2 \right) = \{Re(E)\}^T \{Re(E)\} + \{Im(E)\}^T \{Im(E)\} \end{aligned} \tag{6.48}$$

where $\{Re(E)\}$ and $\{Im(E)\}$ are two $m \times 1$ real vectors containing, respectively, the real and the imaginary part of the residual vector $\{\mathrm{E}\}$.
The related stationarity condition with respect the unknown parameters is:

$$\min_{\{\gamma\}} F \rightarrow \frac{\partial F}{\partial \{\gamma\}} = 0 \rightarrow \frac{\partial}{\partial \{\gamma\}} \left(\{Re(E)\}^T \{Re(E)\} + \{Im(E)\}^T \{Im(E)\} \right) = 0 \tag{6.49}$$

and leads to a non linear system of equations.
The advantage of this parametrization of a FRF is the possibility of appropriately identifying a model part related to a frequency range of interest only. In such a view the Kennedy-Panku method of the previous paragraph, can be seen as a particular case of our factorization scheme, whereas we use only a single term summation and a simplified out of range residual.

6.3.2 Approximation with rational functions

This method assigns the following model to a FRF: [20] and [25]. Hence:

$$H(s) = \frac{N(s)}{D(s)} \tag{6.50}$$

$$N(s) = a_0 + a_1 s + \ldots + a_\alpha s^\alpha \tag{6.51}$$

$$D(s) = 1 + b_1 s + \ldots + b_\beta s^\beta \tag{6.52}$$

$N(s)$ and $D(s)$ being two polynomials of order α and β respectively, $D(s)$ being normalized by imposing $b_0 = 1$. The roots of $D(s)$ are the poles/eigenvalues of our aeroelastic system. [9]
The total number of unknowns are the polynomial coefficients a_i and b_j, for a total of $m = \alpha + \beta + 1$ unknowns.
The solution procedure is much similar to the one adopted for the factorized scheme, i.e. the residuals at each known FR point, $\bar{H}(\omega_i)$, at each frequency ω_i, within is written as:

$$E_i = \bar{H}(\omega_i) - H(\omega_i) = \bar{H}(\omega_i) - \frac{N(\omega_i)}{D(\omega_i)} \tag{6.53}$$

giving rise again to an ovedetermined system of non linear equations, which are solved along the same line previously seen for the factorized approach.

[9] Notice that rational functions need a normalization, otherwise they are undetermined. Conceptually we could chose any coefficient; usually $b_0 = 1$, to grant that the denominator will never be singular.

6.4 Gauss-Newton (GN) Method

The calculation of the coefficients of a rational or factorized residuals approximation leads to a non linear overdetermined system of equations. A way to solve it is to use the so called Gauss-Newton method.
In presenting its fundamental ideas, we will consider only a very simple scalar case. Suppose that f(x) is a non-linear function, the solution through the Least Squares technique leads to impose the minimum condition:

$$min_x \frac{1}{2} f(x)^2 \quad \Rightarrow \quad f(x)\, f'(x) \,=\, 0 \tag{6.54}$$

Now we can apply a Newton-Raphson (NR) technique, leading to finding the roots of the linearized expression, around a trial solution $\bar{x}$, $\tilde{f}(\bar{x}) + \frac{\partial \tilde{f}(\bar{x})}{\partial x} = 0$, where $\tilde{f}(\bar{x})$ is the non-linear equation whose roots we want to find. In our case the application of the NR method to $f(x)\, f'(x) = 0$ implies the need of calculating second order derivatives:

$$f(\bar{x})\, f'(\bar{x}) \,+\, (\, f'(\bar{x})\, f'(\bar{x}) \,+\, f(\bar{x})\, f''(\bar{x}))\, \Delta x \,=\, 0 \tag{6.55}$$

$$\bar{x} \,=\, \bar{x} \,+\, \Delta x \tag{6.56}$$

We should note that if $f(\bar{x})$ the term $f(\bar{x})\, f''(\bar{x})$ might be negligible. Since in our case $f(\bar{x})$ represents the residual, if the model used is apt to well reproducing our responses, the residual should be small and the term $f(\bar{x})\, f''(\bar{x})$ can be neglected, without jeopardizing the convergence toward the sought solution.
This method is a kind of modified NR method and is known as the Gauss-Newton (GN) method. Hence a GN approach can be seen as the specialization of a modified Newton-Raphson (NR) method in which the non linear equations associated to the search of a close to zero stationary value of a function are solved through an approximated Jacobian matrix obtained by ignoring the mentioned contribution.
It is then important to remark that the GN method can be obtained also through the search of the minimum of an approximate quadraticization, i.e.:

$$min_{\Delta x} \frac{1}{2} (\, f(\bar{x}) \,+\, f'(\bar{x})\, \Delta x)^2 \quad \Rightarrow \quad (\, f(\bar{x}) \,+\, f'(\bar{x})\, \Delta x)\, f'(\bar{x}) \,=\, 0 \tag{6.57}$$

Once the problem has been linearized, we can solve it with linear LS. The trivial presentation here made can be used also for multivariable LS, in which case the Hessian matrix will take the place of our simpler second derivative. Summarizing, the application of the GN method leads to:

- Linearizing the residual;
- Solving the over-determined linear system in the LS sense to obtain the variation of the unknown coefficients of the model;
- Iterating till the true non linear LS residual reaches a minimum and/or the solution remains unchanged.

When the assumption $f(\bar{x})\, f''(\bar{x})$ is well satisfied the GN convergence rate is close to that of a full NR scheme. Problems can arise at its starting point, when the very first trial might be far from the true solution. In such a case using a few searches based on a mere gradient of the

LS function, i.e. $f(\bar{x})\, f'(\bar{x})$ can be better, resuming a GN scheme as the assumption $f(\bar{x})\, f''(\bar{x})$ becomes satisfied more and more. One such a scheme of solution is Levenberg-Marquardt (LM) technique, [36].
LM transitions from a gradient to a GN based iteration by gradually reducing a relatively high quantity added to the diagonal of the Jacobian matrix $\frac{\partial f_i}{\partial x_k}$, as the residual reduces its norm. In fact, adding high terms to the diagonal leads to a preponderance of the diagonal as opposed to the other terms of the matrix, the diagonal representing the residual gradient. In short using an LM technique means following the residual gradient direction till GN can be used reliably, thus speeding up convergence.

6.4.1 The Gauss-Newton method applied to factorized TFs

Linearizing Eq. 6.42, supposing the variations with respect to any given value, we obtain:

$$\bar{H}(\omega) \;=\; H(\omega,\{\gamma_v\}) \;+\; \sum_{i=1}^{4n+5} \frac{\partial H}{\partial \gamma_i}(\omega,\{\gamma_v\})\,\Delta\gamma_i \tag{6.58}$$

where:

- $\{\gamma_v\}$ is the available approximated vector at any iteration;
- $\{\Delta\gamma_i\}$ its variation;
- $\frac{\partial H}{\partial \gamma_i}$ are the derivatives of H which are obtained by differentiating Eq. 6.42 with respect to $\{\gamma_v\}$:

$$\frac{\partial H}{\partial a_1} = \frac{1}{j\omega} \quad \frac{\partial H}{\partial a_2} = \frac{1}{(j\omega)^2} \quad \frac{\partial H}{\partial k} = 1 \quad \frac{\partial H}{\partial b_1} = j\omega \quad \frac{\partial H}{\partial b_2} = (j\omega)^2 \tag{6.59}$$

$$\frac{\partial H}{\partial \sigma_i} = \left[-\frac{U_i + jV_i}{(\sigma_i + j(\omega+\omega_i))^2} - \frac{U_i - jV_i}{(\sigma_i + j(\omega-\omega_i))^2} \right] \tag{6.60}$$

$$\frac{\partial H}{\partial j\omega_i} = -j\frac{\partial H}{\partial \omega_i} = -j\left[-j\frac{U_i + jV_i}{(\sigma_i + j(\omega+\omega_i))^2} + j\frac{U_i - jV_i}{(\sigma_i + j(\omega-\omega_i))^2} \right] = \tag{6.61}$$

$$= \left[-\frac{U_i + jV_i}{(\sigma_i + j(\omega+\omega_i))^2} + \frac{U_i - jV_i}{(\sigma_i + j(\omega-\omega_i))^2} \right]$$

$$\frac{\partial H}{\partial U_i} = \left[\frac{1}{\sigma_i + j(\omega+\omega_i)} + \frac{1}{\sigma_i - j(\omega-\omega_i)} \right] \tag{6.62}$$

$$\frac{\partial H}{\partial V_i} = \left[\frac{j}{\sigma_i + j(\omega+\omega_i)} - \frac{j}{\sigma_i - j(\omega-\omega_i)} \right] \tag{6.63}$$

Hence we can write the linearized residual at the kth frequency as simply as:

$$E_{lin_k} \;=\; \bar{H}(\omega_k) \;-\; H(\omega_k,\{\gamma_v\}) \;-\; \sum_{i=1}^{4n+5} \frac{\partial H}{\partial \gamma_i}(\omega_k,\gamma_v)\,\Delta\gamma_i \tag{6.64}$$

Remembering that $\bar{H}(\omega)$ is known at a set of m discrete frequencies, we can rewrite the previous equation in a vectorial form:

$$\{E_{lin}\} = \{\bar{H}\} - \{\tilde{H}_v\} - [\Lambda]\{\Delta\gamma\} = \{\Delta H\} - [\Lambda]\{\Delta\gamma\} \tag{6.65}$$

where $\{\varepsilon_{lin}\}$ is the linearized residual vector, $m \times 1$, $\{\bar{H}\}$ is the vector of the measured FR, $\{\tilde{H}_v\}$ is an $m \times 1$ vector of the factorized function approximation evaluated at each frequency ω_k using the approximation of the unknown parameter given by the previous iteration. $[\Lambda]$ is an $m \times 4n+5$ matrix collecting the gradients of the rational function approximation with respect to each unknown parameters, evaluated at each frequency ω_k. The generic coefficient Λ_{ki} is given by:

$$\Lambda_{ki} = \frac{\partial H}{\partial \gamma_i}(\omega_k, \{\gamma_v\}) \tag{6.66}$$

We can then determine $\{\Delta\gamma\}$ through linear LS.

$$F = \{E_{lin}\}^* \{E_{lin}\} \tag{6.67}$$

which, as suggested by Eq. 6.48, can be rewritten as:

$$F = \{Re(E_{lin})\}^T \{Re(E_{lin})\} + \{Im(E_{lin})\}^T \{Im(E_{lin})\} \tag{6.68}$$

where $\{Re(E_{lin})\}$ and $\{Im(E_{lin})\}$ are, respectively, the real and the imaginary part of the linearized residual vector which, thanks to the expression of the linearized residual, is:

$$\begin{cases} \{Re(E_{lin})\} = \{Re(\Delta H)\} - [Re(\Lambda)]\{\Delta\gamma\} \\ \{Im(E_{lin})\} = \{Im(\Delta H)\} - [Im(\Lambda)]\{\Delta\gamma\} \end{cases} \tag{6.69}$$

Imposing the stationarity condition, as suggested by Eq. 6.49, we obtain:

$$\frac{\partial}{\partial \Delta\gamma}\left(\{Re(E_{lin})\}^T \{Re(E_{lin})\} + \{Im(E_{lin})\}^T \{Im(E_{lin})\}\right) = 0 \tag{6.70}$$

where thanks to Eq. 6.69, we can write:

$$\begin{aligned} \{Re(E_{lin})\}^T \{Re(E_{lin})\} = {} & \{Re(\Delta H)\}^T \{Re(\Delta H)\} + 2\{\Delta\gamma\}^T [Re(\Lambda)]^T [Re(\Lambda)]\{\Delta\gamma\} + \\ & - 2\{\Delta\gamma\}^T [Re(\Lambda)]^T \{Re(\Delta H)\} \end{aligned} \tag{6.71}$$

$$\begin{aligned} \{Im(E_{lin})\}^T \{Im(E_{lin})\} = {} & \{Im(\Delta H)\}^T \{Im(\Delta H)\} + 2\{\Delta\gamma\}^T [Im(\Lambda)]^T [Im(\Lambda)]\{\Delta\gamma\} + \\ & - 2\{\Delta\gamma\}^T [Im(\Lambda)]^T \{Im(\Delta H)\} \end{aligned} \tag{6.72}$$

Therefore, after substituting in Eq. 6.70 and neglecting constant terms, we obtain:

$$\begin{aligned} & \frac{\partial}{\partial \Delta\gamma}\left[\{\Delta\gamma\}^T \left(Re[\Lambda]^T Re[\Lambda] + Im[\Lambda]^T Im[\Lambda]\right)\{\Delta\gamma\} + \right. \\ & \left. - \{\Delta\gamma\}^T \left([Re(\Lambda)]^T \{Re(\Delta H)\} + [Im(\Lambda)]^T \{Im(\Delta H)\}\right)\right] = 0 \end{aligned} \tag{6.73}$$

Calling:

$$[A] = Re[\Lambda]^T Re[\Lambda] + Im[\Lambda]^T Im[\Lambda] \tag{6.74}$$

$$\{b\} = [Re(\Lambda)]^T \{Re(\Delta H)\} + [Im(\Lambda)]^T \{Im(\Delta H)\} \tag{6.75}$$

we can rewrite it as:

$$\frac{\partial}{\partial \Delta\gamma}\left(\{\Delta\gamma\}^T[A]\{\Delta\gamma\} - \{\Delta\gamma\}^T\{b\}\right) = 0 \tag{6.76}$$

which leads to solving the linear system:

$$[A]\{\Delta\gamma\} = \{b\} \tag{6.77}$$

from which we have:

$$\{\Delta\gamma\} = [A]^{-1}\{b\} \tag{6.78}$$

$$\{\gamma_v\} = \{\gamma_v\} + \{\Delta\gamma\} \tag{6.79}$$

The iterative solution is stopped when either an appropriate norm of the variations is small enough or because the minimum of E^2 is reached. Generally, to enhance convergence, the iterative procedure is implemented so that some poles, appearing to diverge, can be kept fixed while the iteration continues over the rest of them. Typically the diverging modes are those whose modal coefficients are very small. Clearly a precise and quick convergence is achieved only if a good initial guess is available. Such an initial guess is often determined by combining the previously seen single mode approximations.

6.4.2 Gauss-Newton method applied to a rational TFs

The linearization of a rational function approximation for a FR is:

$$\begin{aligned} H(\omega,\{\Delta a\},\{\Delta b\}) &= H(\omega,\{a\}_v,\{b\}_v) + \\ &+ \sum_{i=1}^{\alpha+1}\frac{\partial H}{\partial a_i}(\omega,\{a\}_v,\{b\}_v)\Delta a_i + \sum_{j=1}^{\beta}\frac{\partial H}{\partial b_j}(\omega,\{a\}_v,\{b\}_v)\Delta b_j \end{aligned} \tag{6.80}$$

where:

$$\frac{\partial H}{\partial a_i}(\omega,\{a\}_v,\{b\}_v) = \frac{\partial H}{\partial N}\frac{\partial N}{\partial a_i} = \frac{1}{D(\omega,\{b\}_v)}(j\omega)^{i-1} \tag{6.81}$$

$$\frac{\partial H}{\partial b_j}(\omega,\{a\}_v,\{b\}_v) = \frac{\partial H}{\partial D}\frac{\partial D}{\partial b_j} = -\frac{N(\{a\}_v)}{D^2(\omega,\{b\}_v)}(j\omega)^{j} \tag{6.82}$$

Eq. 6.80 can be rewritten in the same form as Eq. 6.58, by simply introducing and defining a suitable vector $\{\gamma\}$ containing the parameters a_i and b_j. For that we write:

$$\{\gamma\} = \{\{a\} \quad \{b\}\}^T \tag{6.83}$$

$$H(\omega,\{\Delta\gamma\}) = H(\omega,\{\gamma\}_v) + \sum_{i=1}^{m}\frac{\partial H}{\partial \gamma_i}(\omega,\{\gamma\}_v)\Delta\{\gamma_i\} \tag{6.84}$$

Hence, the linearized residual at the kth frequency ω_k is given by:

$$E_{lin_k} = \bar{H}(\omega_k) - H(\omega_k,\{\gamma\}_v) - \sum_{i=1}^{m}\frac{\partial H}{\partial \gamma_i}(\omega_k,\{\gamma\}_v)\Delta\{\gamma_i\} \tag{6.85}$$

which can rewritten in the vectorial form:

$$\{E\}_{lin} = \{\bar{H}\} - \{H\}_v - [\Lambda]\{\Delta\gamma\} = \{\Delta H\} - [\Lambda]\{\Delta\gamma\} \tag{6.86}$$

Applying the same solution method, described in the previous paragraph for a factorized scheme, we will obtain:

$$\{\Delta\gamma\} = [A]^{-1}\{b\} \tag{6.87}$$
$$\{\gamma_v\} = \{\gamma_v\} + \{\Delta\gamma\} \tag{6.88}$$

to which we can apply the same comments in relation to how to start the and stop the iterated solution.

Estimating the parameters variance and order The residuals $\{E\}$ and the matrix $[A]$ of equations like 6.78 and 6.87 serve the same scope as $\{\varepsilon\}$ and $([A]^T[A])^{-1}$ seen in the identification in the time domain. After calling $q = n_a + n_b$ we can then paraphrase the determination of the variances of $\{\gamma\}$, $[\sigma_{\gamma\gamma}]$, as:

- chose the model order and build the coefficient matrix [A], using the available input/output data;
- calculate $\{\gamma\} = [A]^{-1}\{b\}$;
- calculate $\{E\}$, $s_{EE} = \{E\}^T\{E\}$ and $\lambda^2 = \frac{s^2_{EE}}{N(N-q)}$, or more simply $\lambda^2 \simeq \frac{s^2_{\varepsilon\varepsilon}}{N^2}$, whenever $N \gg q$
- $[\sigma_{\gamma\gamma}] = \lambda^2[I]$;

N being the number of frequencies at which we have evaluated a TF. The same can be said for the calculation of the sensitivity of the root of $D(s)$, obtainable from $[\sigma_{\gamma\gamma}]$, which is an even easier task since there is no need to translate them from the z to the s plane. The order of the model is usually chose so to minimize s_{EE}, whereas such a performance index tend to increase as the model order increases, up a point where little or no improvement is obtained.

Other solution methods for a rational function approximation

Sometimes, two other solution techniques can be used as in place of GN. They are often less performing and somewhat inadequate for a consistent estimation of the unknown parameters variance.
The first one tries to conduct the overdetermined non linear system to a trivially linear one to which apply a standard linear LS. Recalling Eq. 6.50 we see that it would be possible to write:

$$D(j\omega_i)\bar{H}(j\omega_i) - N(j\omega_i) = 0 \tag{6.89}$$

evaluating this equation for each available frequency we obtain an overdetermined linear system of N equations in m unknowns, whose ith equation is given by:

$$\begin{aligned} &D(j\omega_i)\overline{H}_i - N(j\omega_i) = 0 \\ &\left(1 + b_1(j\omega_i) + b_2(j\omega_i)^2 + \ldots + b_\beta(j\omega_i)^\beta\right)\overline{H}_i = \quad a_0 + a_1(j\omega_i) + a_2(j\omega_i)^2 + \\ &\qquad\qquad \ldots + a_\alpha(j\omega_i)^\alpha \end{aligned} \tag{6.90}$$

rewritten as:

$$a_0 + a_1(j\omega_i) + a_2(j\omega_i)^2 + \ldots + a_\alpha(j\omega_i)^\alpha - b_1(j\omega_i)\overline{H}_i - b_2(j\omega_i)^2\overline{H}_i - \ldots + \\ - b_\beta(j\omega_i)^\beta\overline{H}_i = \overline{H}_i \tag{6.91}$$

to evidence its unknowns and the ith component of the right side. As it should be simple to see, such an equation is complex with real unknowns parameters. Instead of following what we did with GN we here split it explicitly into two equations one, with even powers of $(j\omega)$, for its real part:

$$a_0 - \omega_i^2 a_2 - \omega_i^4 a_4 + \ldots + Re(\bar{H}_i)\omega_i^2 b_2 + Re(\bar{H}_i)\omega_i^4 b_4 + \ldots = Re(\bar{H}_i) \tag{6.92}$$

and one, with odd power of $(j\omega)$, for its imaginary part:

$$\omega_i a_1 + \omega_i^3 a_3 + \ldots - Im(\bar{H}_i)\omega_i b_1 - Im(\bar{H}_i)\omega_i^3 b_3 + \ldots = Im(\bar{H}_i) \tag{6.93}$$

Then we end into a final system given by:

$$[A]\{\gamma\} = \{b\} \tag{6.94}$$

where [A] is a $2N \times m$ real matrix:

$$[A] = \begin{bmatrix} Re(\tilde{a}_1) & Im(\tilde{a}_1) & Re(\tilde{a}_2) & Im(\tilde{a}_2) & Re(\tilde{a}_3) & Im(\tilde{a}_3) & \ldots & Re(\tilde{a}_N) & Im(\tilde{a}_N) \end{bmatrix}^T \tag{6.95}$$

where $\tilde{\{a_i\}}$ represent the ith Eq. 6.91, collocated at the ith frequency and $\{b\}$ is the $2N \times 1$ real right hand side vector:

$$\{b\} = \begin{bmatrix} Re(\{\bar{H}_1\}) & Im(\{\bar{H}_1\}) & Re(\{\bar{H}_2\}) & Im(\{\bar{H}_2\}) & \ldots & Re(\{\bar{H}_N\}) & Im(\{\bar{H}_N\}) \end{bmatrix}^T \tag{6.96}$$

so that standard LS will provide:

$$\{\gamma\} = ([A]^T[A])^{-1}[A]^T\{b\} \tag{6.97}$$

Such an approach, albeit being very simple, misses the weight of the denominator at each frequency, eventually optimizing a LS performance whose residuals differ wildly from those of its original parent formulation, so producing unsuitable results in many cases. To remedy to such a problem the second technique leaves unchanged the problem formulation but proceeds by applying the following iteration to Eq. 6.50:

$$\frac{D_k(j\omega)H(j\omega) - N_k(j\omega)}{D_{k-1}(j\omega)} = 0 \tag{6.98}$$

k being the iteration index, possibly starting the related iteration process by using the previously presented scheme. Provided it converges to a solution, by using the denominator of a previous iteration, $D_{k-1}(j\omega)$, the above equation will converge to a residual that is the same as that of the corresponding GN approach. From a practical point of view, at each iteration we have to solve a linear overdetermined LS problem. Hence, writing the *ith* equation of the system, at the *kth* iteration, we have:

$$\frac{D_k(j\omega_i)\overline{H}_i - N_k(j\omega_i)}{D_{k-1}(\omega_i)} = 0 \tag{6.99}$$

which, after putting into evidence its unknowns and right side, becomes:

$$\begin{aligned}&\tfrac{1}{\delta_i}a_0^k + a_1^k\tfrac{(j\omega_i)}{\delta_i} + a_2^k\tfrac{(j\omega_i)^2}{\delta_i} + \ldots + a_\alpha^k\tfrac{(j\omega_i)^\alpha}{\delta_i} - b_1^k\tfrac{(j\omega_i)\overline{H}_i}{\delta_i} - b_2^k\tfrac{(j\omega_i)^2\overline{H}_i}{\delta_i} - \ldots + \\ &- b_\beta^k\tfrac{(j\omega_i)^\beta\overline{H}_i}{\delta_i} = \tfrac{1}{\delta_i}\overline{H}_i\end{aligned} \tag{6.100}$$

$$\delta_i = \left(1 + b_1^{k-1}(j\omega_i) + b_2^{k-1}(j\omega_i)^2 + \ldots + b_\beta^{k-1}(j\omega_i)^\beta\right) \tag{6.101}$$

so that any of its iterations can proceed much like as the simpler first approximation previously presented.

6.5 Frequency domain identification applied to aeroelasticity

6.5.1 Introduction

The aim of this paragraph is to show how the previously explained concepts can be applied to a flight flutter test. As we have remarked in the previous chapter, the test phase can be designed for either measuring or not the excitation input. Clearly, such a design choice has also implications on the following identification phase. In fact in treating the identification in the time domain, we have already seen that measuring or not the input affects the choice of the process model, and therefore of the identification technique used. We want now to see the related implications in the frequency domain, showing their connection with an available estimated FRF. Finally one should notice that, despite having treated only SISO identifications in the frequency domain, it remains valid the comment, already made for the time domain, that MISO systems can use the vary same techniques. In fact it would be just a matter of maintaining the same poles, or denominator, adding as many residual terms, or numerators, as are the number of input.

V-g diagram with measured input

If the input is measured, we can: calculate the auto-interspectra, estimate $H(\omega)$, identify its analytic approximation using any of the previously presented techniques. Then the system poles needed for a V-g reconstruction, along with their uncertainty bounds, can be determined either directly from the factorized form or computed from the denominator of a rational function approximation. The related partition of the Jacobian matrices can be used for estimating their variances, as they are for the factorized form or from the denominator roots for the rational form. In the latter case the variances of the roots of the denominator can eventually be calculated from the variance of the denominator coefficients by following the same scheme presented for the time domain identification.

V-g diagram without measured input

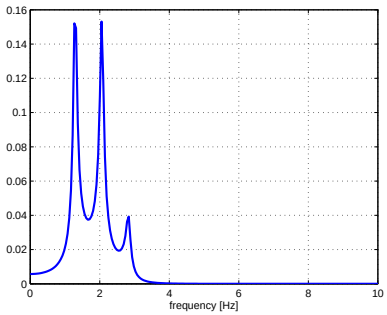
In flight flutter tests, the identification techniques for unmeasured input must be applied when impulsive excitations or turbulence are used. In both cases, the problem to solve is how to derive an equivalent pseudo Transfer Function (TF) without having available any input.

In the case of an impulsive excitation, once the impulse as been appropriately designed for the frequency content we want to excite, e.g. as it has been shown in paragraph 3.1.1 of chapter 3, the situation in not dissimilar from what we saw in the time domain. So we have to remark, once more, the need of taking a lot of cautiousness in choosing an impulsive excitation because if the system is well damped we have a response that might end in providing an inadequate resolution, even with a low noise level. Things are worsened substantially with actual disturbance levels, as its true response is well hidden within noise quite soon. Moreover trying to improve the signal to noise of an impulse response by truncating the processed window time might end in worsening the resolution even more. That noticed we can easily obtain an estimation of the FRF, directly its TF in fact, by simply taking the Fourier Transform of measured responses. To improve the related estimation one has often to fight the competing requirements of averaging more firings, toward a better estimation of a TF per flight condition, against speeding up tests. An often taken compromise between decreased resolution and a better signal to noise ratio is to use an exponential window, i.e. multiply the recorded impulse response by $e^{\alpha t}$, with α being a parameter to be chosen so to decrease the tail noise in the response without loosing to much resolution. Such a windowing has the advantage of limiting a few problems, we saw, associated to the Fourier Transform of truncated records. The bias it causes for the identified damping can, in principle, be eliminated by subtracting its value from the real part of the poles obtained after having carried out the identification of the related TF.

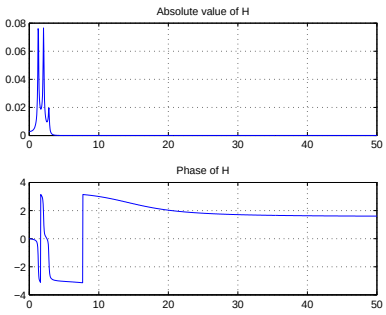
When the unmeasurable input is random, e.g. turbulence, a viable solution is possible only if it can be assumed as being assumed to have a significantly constant PSD over the frequency range of interest. In such a case a possible identification approach can then be that of evaluating a pseudo TF, as explained in the paragraph on the FRF estimators.
Therefore, we have to recall that turbulence, beside being neither controllable nor measurable, so that we have no information on its actual intensity and frequency content during a flight test at hand, will provide us just with the response spectra Φ_{yy}. They in turn will meaningfully resemble a true $|H(\omega)|^2$ only if we can retain Φ_{gg} significantly constant over the frequency range of interest. In fact, in chapter 3, talking about turbulence as an excitation source, paragraph 3.1.3, we have pointed out that turbulence is a suitable excitation source only for those frequencies in correspondence of which its spectrum can be considered constant, otherwise the peaks of the system frequency response will appear lower than they are for real, therefore hinting at a higher damping.

Pseudo impulsive response method applied to a turbulence response In paragraph 6.1, we have shown how it is possible to recover the minimum phase transfer function of a system from the sole knowledge of its modulus. An alternative and somewhat simpler approach will be illustrated in the following. The method is based on the observation that the autocovariance of the output of a system, K_{yy}, obtained by antitransforming an available squared modulus of its transfer functions, albeit being symmetric in time, shows, on both of its sides, an acausal trend whose decaying behavior is quite similar to that of its corresponding impulsive response. In practice using just its right part we obtain something that is quite similar to the system casual impulse response. That is why we will call it a pseudo-impulse response. One of the things distinguishing such a pseudo-impulse response from its true counterpart are its initial conditions, which may be far from being related to the real ones. Nonetheless it maintains the decay characteristics of a true impulse response, i.e. what we are interested in. Thus by appropriately transforming it back

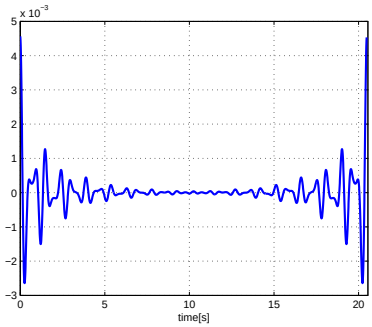
into the frequency domain we have a pseudo transfer function, to which we can apply any of the identification techniques in the frequency domain which has been shown in this notes.
As already said we care little of missing the true initial conditions of the true physical response and the only thing we have to care of is to obtain its correct transform in the frequency domain. For that we have to note that the autocovariance is symmetric and has its maximum value at zero. Knowing that its antitransform will converge at 1/2 of a jump discontinuity we have to force such a values before carrying out its transformation to the frequency domain, to get its related pseudo-transfer function. To achieve a better comprehension of the pseudoimpulsive response method we will present a simple example, considering the dynamics of a system which we have presented in figure 3.5, in relation to the use of the turbulence as an excitation force. The figures displayed below are related to the position output of the mass chart 1 of paragraph 3 in paragraph 3.1.3. Clearly, the measurements of positions is not a realistic output in the case of flight flutter test. Nonetheless such an example is adequate to illustrate the whole procedure. In figure 6.5.1, we present both the "analytic " form of $|H^2|$, and H, between the position of the mass chart 1 and the force applied on the mass chart 3, as well as the related covariance. The latter has been obtained by applying an inverse Fourier transform to $|H|^2$. When processing real experimental data we will clearly use the estimated, noisy, power spectral density.



(a) Analytic $|H|^2$



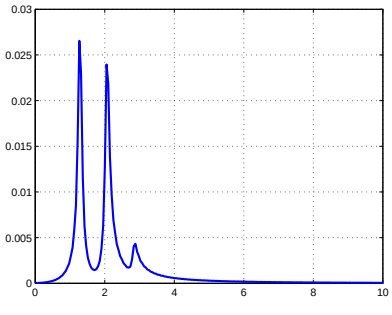
(b) Analytic TF



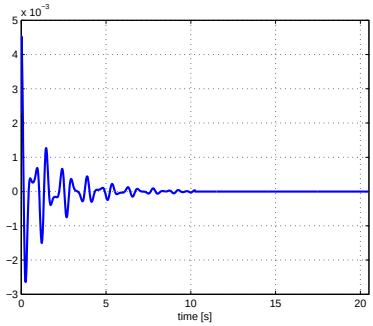
(c) Output Covariance

Figure 6.1: True system model

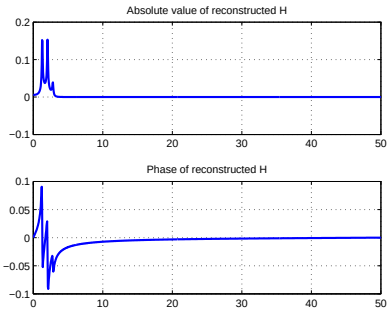
Then, the autocovariance is truncated. The corresponding pseudo transfer function H can then be easily obtained by applying the Fourier transform to it, after setting its value at zero as 1/2 of the one available. In figure 6.5.1 we present the truncated covariance from which we have obtained the transfer function shown. Observing the trend of the reconstructed $|H|^2$, we notice that the



(a) Reconstructed $|H|^2$



(b) Truncated output covar.



(c) Reconstructed TF

Figure 6.2: Identified system model

peaks are in the same position of the analytic ones, but their amplitude is lower. This because the reconstructed H has been obtained by transforming the truncated version of the autocovariance, which contains lower power with respect the original signal. However, the peaks maintain the same scale as the analytic model, hence, applying the previously presented identification techniques to so reconstructed pseudo TF, we will be able to identify the system poles along with the related variances, i.e. what we are interested at.

Appendix A

Airworthiness Code

A.1 MIL-A-8870C(AS)-General Requirement

Construction, materials and design shall be such that there will be: No flutter, buzz, divergence, aeroservoelastic, aerothermoelastic instabilities, including sustained limit amplitude instabilities, of the airplane weapon system consistent with the requirements of 3.1.1 $\cdots$.

3.1.1 Aeroelastic Stability Analysis, wind tunnel and laboratory tests, and airplane ground and flight tests (up to design limit speed) shall demonstrate that flutter, divergence and other related aeroelastic or aeroservoelastic instability boundaries occur outside the 1.15 times design limit speed envelope. The airplane shall meet the following stability design requirements for both normal and emergency conditions:

1. Margin: Fifteen percent equivalent airspeed, Ve, margin on the applicable design limit speed envelope, both at constant altitude and constant Mach number.

2. Damping: The damping coefficient, g, for any critical flutter mode or for any significant dynamic response mode shall be at least three percent (0.03) for all altitudes or flight speeds up to design limit speed.

A.2 EASA 23.629: Flutter

1. It must be shown by the methods of (2) and either (3) or (4) of this paragraph, that the aeroplane is free from flutter, control reversal and divergence for any condition of operation within the limit V-n envelope and at all speeds up to the speed specified for the selected method. In addition:

   (a) Adequate tolerances must be established for quantities which affect flutter; including speed, damping, mass balance and control system stiffness; and

(b) The natural frequencies of main structural components must be determined by vibration tests or other approved methods.

2. Flight flutter tests must be made to show that the aeroplane is free from flutter, control reversal and divergence and to show by these tests that:

   (a) Proper and adequate attempts to induce flutter have been made within the speed range up to V_D;

   (b) The vibratory response of the structure during the test indicates freedom from flutter;

   (c) A proper margin of damping exists at V_D; and

   (d) There is no large and rapid reduction in damping as V_D is approached.

3. Any rational analysis used to predict freedom from flutter, control reversal and divergence must cover all speeds up to 1.2 V_D.

4. Compliance with the rigidity and mass balance criteria, pages 4-12, in Airframe and Equipment Engineering Report No. 45, as corrected, ”Simplified Flutter Prevention Criteria”, published by the Federal Aviation Administration, may be accomplished to show that the aeroplane is free from flutter, control reversal, or divergence if:

   (a) V_D-M_D for the aeroplane is less than 260 Knots (EAS) and less than Mach 0.5;

   (b) The wing and aileron flutter prevention criteria, as represented by the wing torsional stiffness and aileron balance criteria, are limited to use to aeroplanes without large mass concentrations (such as engines, floats, or fuel tanks in outer wing panels) along the wing span; and

   (c) The aeroplane:

      i. Does not have a T-tail or other unconventional tail configurations;

      ii. Does not have unusual mass distributions or other unconventional design features that affect the applicability of the criteria; and

      iii. Has fixed-fin and fixed-stabiliser surfaces.

5. For turbo-propeller powered aeroplanes, the dynamic evaluation must include:

   (a) Whirl mode degree of freedom which takes into account the stability of the plane of rotation of the propeller and significant elastic, inertial and aerodynamic forces; and

   (b) Propeller, engine, engine mount and aeroplane structure stiffness and damping variations appropriate to the particular configuration.

6. Freedom from flutter, control reversal and divergence up to VD/MD must be shown as follows:

   (a) For aeroplanes that meet the criteria of sub-paragraphs (4)(a) to (4)(c) of this paragraph, after the failure, malfunction, or disconnection of any single element in any tab control system.

(b) For aeroplanes other than those described in sub-paragraph (6)(a) of this paragraph, after the failure, malfunction, or disconnection of any single element in the primary flight control system, any tab control system, or any flutter damper.

7. For aeroplanes showing compliance with the fail-safe criteria of JAR 23.571 and 23.572, the aeroplane must be shown by analysis to be free from flutter up to V_D-M_D after fatigue failure, or obvious partial failure of a principal structural element.

8. For aeroplanes showing compliance with the damage-tolerance criteria of JAR 23.573, the aeroplane must be shown by analysis to be free from flutter up to V_D-M_D with the extent of damage for which residual strength is demonstrated.

9. For modifications to the type design which could affect the flutter characteristics compliance with sub-paragraph (1) of this paragraph must be shown, except that analysis alone, which is based on previously approved data, may be used to show freedom from flutter, control reversal and divergence for all speeds up to the speed specified for the selected method.

A.3 EASA 25.629: Flutter Deformation and Failsafe Criteria

1. General. Compliance with this paragraph must be shown by calculations, resonance tests, or other tests found necessary by the Authority. Full scale flight flutter tests at speeds up to V_{DF}-M_{DF} for the critical aeroplane flutter modes must be conducted when:

   (a) M_D is equal to or greater than 0.8 M;

   (b) The adequacy of flutter analysis and tunnel tests have not been established by previous experience with aircraft having similar design features; or

   (c) The conditions specified in sub-paragraph (1)(a) or (b) of this paragraph exist, and modifications to the type design have a significant effect on the critical flutter modes.

2. Flutter and Divergence prevention. The dynamic evaluation of the aeroplane must include an investigation of the significant elastic, inertia, and aerodynamic forces associated with the rotations and displacements of the plane of the propeller. In addition, the following apply:

   (a) The aeroplane must be designed to be free from flutter and divergence (unstable structural distortion due to aerodynamic loading) for all combinations of altitude and speed encompassed by the V_D-M_D versus altitude envelope enlarged at all points by an increase of 20% in equivalent airspeed at both constant Mach number and constant altitude, except that [the envelope may be limited to a maximum Mach number of 1.0 when M_D is less than 1.0 at] all design altitudes and the following is established:

      i. A proper margin of damping exists at all speeds up to M_D; and

      ii. There is no large and rapid reduction in damping as M_D is approached.

   (b) If concentrated balance weights are used on control retraces, their effectiveness and strength, including supporting structure, must be substantiated.

3. Loss of control due to structural deformation. The aeroplane must be designed to be free from control reversal and from undue loss of longitudinal, lateral, and directional stability and control, as a result of structural deformation (including that of the control surface covering) at speeds up to the speed prescribed in sub-paragraph (b) of this paragraph for flutter prevention.

4. Fail-safe criteria. The following fail-safe criteria must be met:

   (a) It must be shown, by analysis or tests, that the aeroplane is free from such flutter or divergence that would preclude safe flight, at any speed up to V_D, after each of the following:

      i. Each of the failures, malfunctions, or adverse conditions listed in sub-paragraph (4)(d) of this paragraph.

      ii. Any other combination of [failures, malfunctions, or adverse conditions not shown to be extremely] improbable.

   (b) If a failure, malfunction, or adverse condition described in sub-paragraph (4)(d) of this paragraph is simulated during a flight test in showing compliance with this paragraph, the maximum speed investigated need not exceed V_{FC} if it is shown, by correlation of the flight test data with other test data or analysis, that hazardous flutter or divergence will not occur at any speed up to V_D.

   (c) The structural failures described in sub-paragraphs (4)(d)(i) and (ii) of this paragraph need not be considered in showing compliance with this paragraph if engineering data substantiate that the probability of their occurrence is negligible by showing that the structural element is designed with:

      i. Conservative static strength margins for each ground and flight loading conditions specified in this JAR-25; or

      ii. Sufficient fatigue strength for the loading spectrum expected in operation.

   (d) The failures, malfunctions, or adverse conditions use to show compliance with this paragraph are as follows:

      i. Failure of any single element of the structure supporting any engine, independently mounted propeller shaft, large auxiliary power unit, or large externally mounted aerodynamic body (such as an external fuel tank).

      ii. Any single failure of the engine structure, on turbo-propeller aeroplanes, that would reduce the yaw or pitch rigidity of the propeller rotational axis.

      iii. Absence of propeller aerodynamic forces resulting from the feathering of any single propeller, and, for aeroplanes with four or more engines, the feathering of the critical combination of two propellers. In addition, any single feathered propeller must be paired with the failures, specified in (4)(d)(i) of this sub-paragraph, involving failure of any single element of the structure supporting any engine or independently mounted propeller shaft, and the failures specified in (4)(d)(ii) of this sub-paragraph.

      iv. Any single propeller rotating at the highest likely overspeed. Failure of each principal element selected for compliance with JAR 25.571 (b). Safety following a failure may be substantiated by showing that losses in rigidity or changes in

frequency, mode shape, or damping are within the parameter variations shown to be satisfactory in the flutter and divergence investigations.

v. Any single failure or malfunction, or combinations thereof, in the flight control system considered under JAR 25.671, 25.672 and 25.1309, and any single failure in any flutter damper system. Investigation of forced structural vibration than flutter, resulting from failures, malfunctions, or adverse conditions in the automatic flight control system may be limited to airspeed up to V_C.

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