Voronoi cells in Diestel-Leader Graph

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Motivation and Background

Let $p > 1$. The (infinite) Diestel-Leader graph $DL(p,p)$ is a Cayley graph for lamplighter group $\mathbb{Z}_p \wr \mathbb{Z}$ [Woe04]. Note that $DL(p,q)$ is a Cayley graph only when $p = q$. Diestel-Leader graphs $DL(p,q)$ are horocyclic products of two homogeneous trees T_p and T_q where each vertex of T_p (resp. T_q) has one parent and p (resp. q) children.

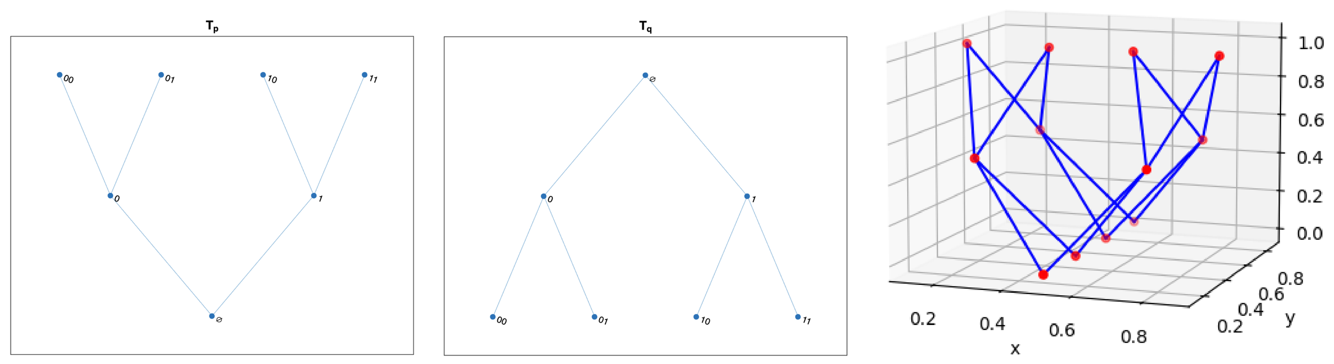

Fig. 1: Left: Finite trees T_2, T_2 of height 3, oriented in opposite directions
Right: The horocyclic product of these trees, $DL_3(2,2)$

We define a finite Diestel-Leader graph $DL_n(p,q)$ to be the graph with vertices

$$V(DL_n(p,q)) = \{(x,y) \in T_p \times T_q : h(x) = -h(y)\},$$

where n is the height of T_p and T_q and h is the height of x and y in the trees T_p and T_q, respectively. Two vertices (x_1, y_1) and (x_2, y_2) are adjacent if x_1 is adjacent to x_2 in T_p and y_1 is adjacent to y_2 in T_q. The distance between (x_1, y_1) and (x_2, y_2) is given by the formula

$$d((x_1,x_2),(y_1,y_2)) = d(x_1,y_1) + d(x_2,y_2) - |h(x_1) - h(x_2)| \quad (1)$$

We think of Diestel-Leader graphs as metric spaces using the graph metric, where each edge has length one and the distance between two vertices is the length of the shortest path between them. Finite Diestel-Leader graphs $DL_i(p,q)$ are approximations of their infinite counterparts as they fit into an ascending sequence of inclusions whose union is $DL(p,q)$:

$$DL_2(p,q) \subset DL_3(p,q) \subset \ldots DL_i(p,q) \subset \ldots DL(p,q).$$

Suppose we have a set of m points called nuclei distributed across a plane. The *Voronoi diagram* associated with these nuclei partitions the plane into precisely m cells, each encompassing the region closest to its nucleus [Bel23]. Voronoi diagrams exist in daily life: for example, cross-sections of horizontally-cut garlic, ancient porcelains from China, and giraffes' coats are Voronoi diagrams.

Fig. 2: Voronoi cells in nature [Geo][Wik][Gra]

The shapes of Voronoi cells depend on the metric of the space. [Bel23].

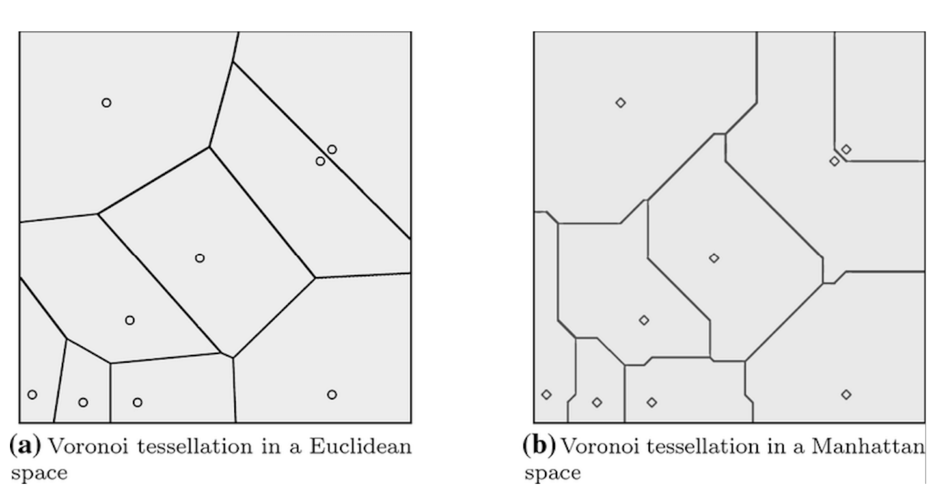

Fig. 3: Voronoi diagrams on a plane with the Euclidean metric (left) and Manhattan metric (right) [Str22]

Define the Problem

Our goal is to study the geometry of Diestel-Leader graphs by computing Voronoi diagrams on finite approximations of Diestel-Leader graphs.

Model

In this project, we take finite approximations of the Diestel-Leader graph $DL(2,2)$. We generate Voronoi diagrams for finite Diestel-Leader graphs by randomly assigning nuclei and growing balls around each nucleus at the same rate. We encountered problems with points that are equidistant from multiple nuclei. We considered two methods to resolve this issue, the first of which assigns all points equidistant from two or more nuclei to a new nucleus. The following figure shows a Voronoi diagram with two nuclei on $DL_4(2,2)$ where points equidistant from the two nuclei are assigned to a new nucleus.

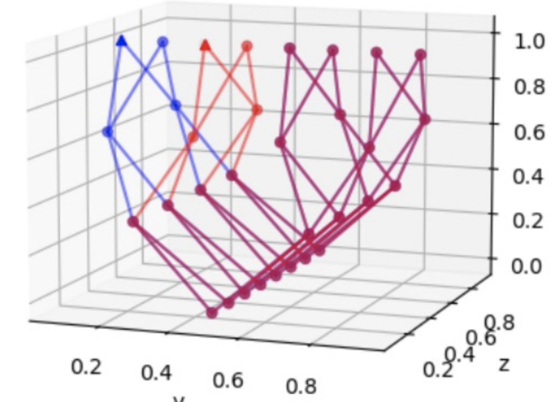

Fig. 4: Voronoi diagram with two nuclei on $DL_4(2,2)$. The cells of the two original nuclei are colored red and blue, and points equidistant from the two nuclei are assigned to a new nucleus and colored purple.

The problem with this method is that often, most of the graph will be covered by cells around new nuclei. To eliminate this problem, we decided to assign each nucleus a "priority." When a point is equidistant from multiple nuclei, it gets assigned to the Voronoi cell around the nucleus of the highest priority. Prioritizing nuclei allows us to construct a Voronoi diagram on Diestel-Leader graphs without adding new nuclei.

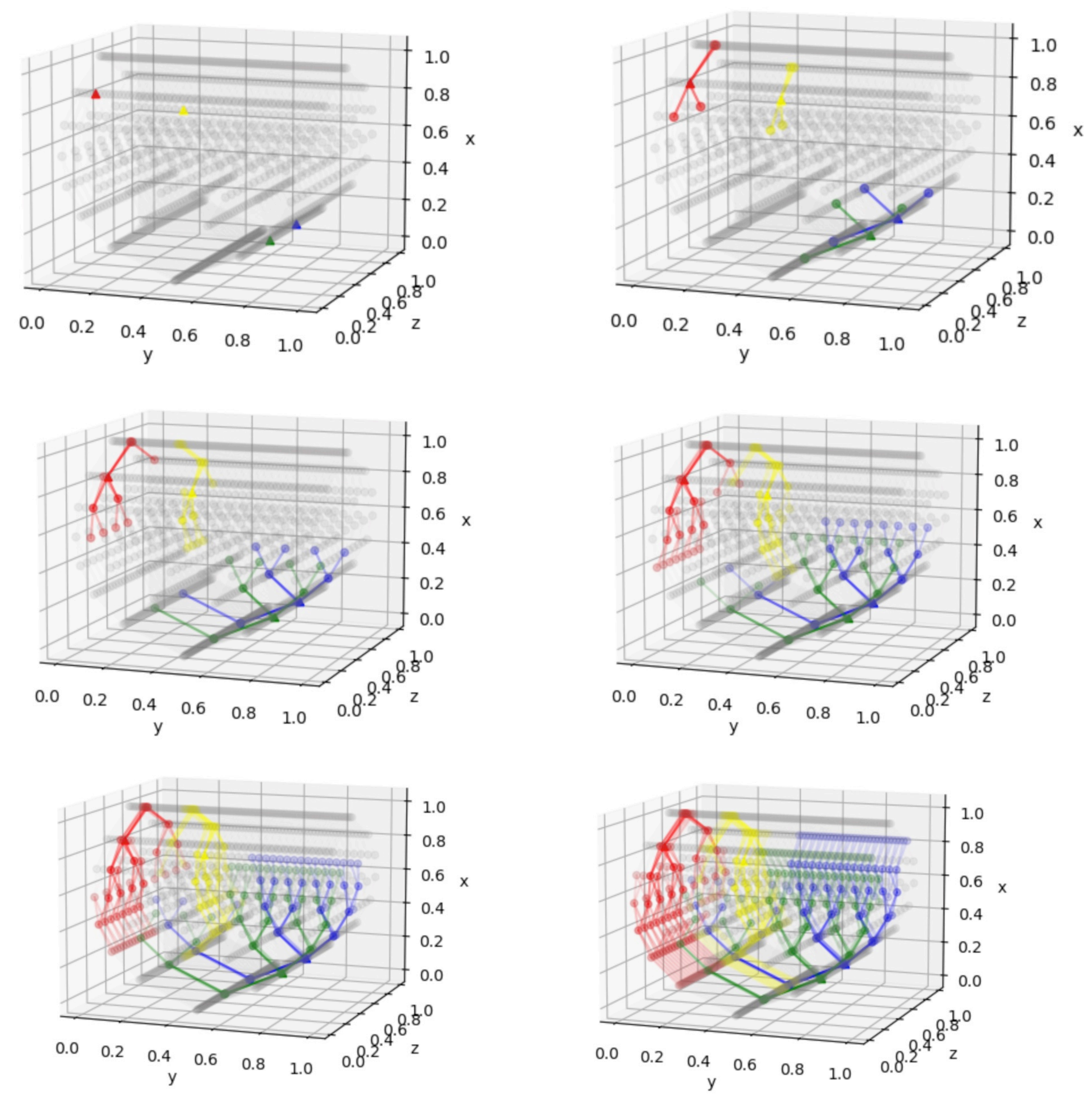

Fig. 5: Growing balls around four nuclei in $DL_8(2,2)$. Each picture corresponds to balls around the nuclei of different radius

Results

We found the distance function between two vertices in Diestel-Leader graph: see Equation (1). We also plotted Voronoi diagrams on Diestel-Leader graphs of various heights with various numbers and locations of nuclei. Below is a Voronoi tessellation of $DL_8(2,2)$ where each vertex had a 5% chance of being a nucleus of a Voronoi cell. Note that this is the final Voronoi tessellation associated with Figure 5.

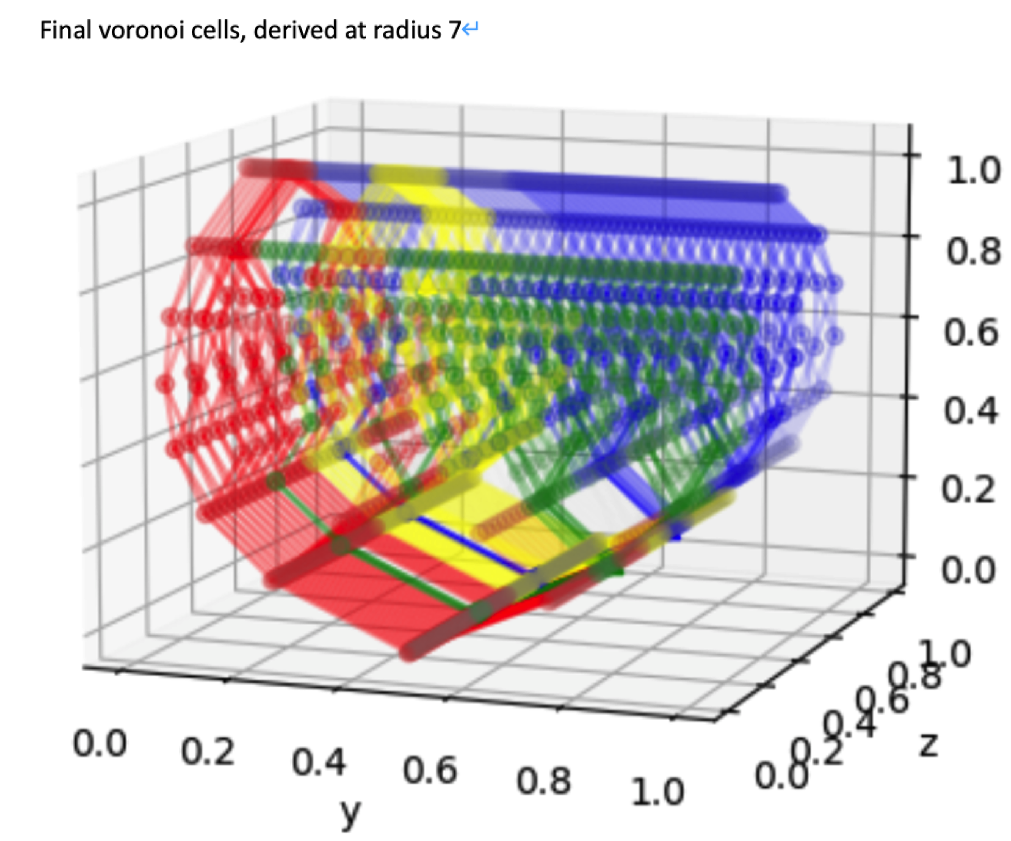

Fig. 6: Voronoi tessellation of $DL_8(2,2)$ with four nuclei

Future Ideas

In our future work, we will compute several more Voronoi tessellations of finite Diestel-Leader graphs. Our aim is to increase the heights of our finite Diestel-Leader graphs and decrease the probability that a vertex is a nucleus. Ultimately, we hope to use these Voronoi diagrams to form a conjecture about what Voronoi diagrams look like for infinite Diestel-Leader graphs.

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