The Geometry of Diestel-Leader Graphs and Voronoi Tessellations inside Diestel-Leader Graphs

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Abstract

In this paper, we explore the geometry of Diestel-Leader graphs $DL_n(p,q)$ by analyzing Voronoi diagrams in their finite approximations. We introduce an algorithm for generating Diestel-Leader graphs, and simulate the growth of Voronoi Cells in these graphs under various configurations. In addition, we address the problem of tessellation generation in Diestel-Leader graphs on equidistant points from multiple nuclei by assigning each nuclei with different priority. We demonstrate our findings through graphical representations, illustrating the tessellations in Diestel-Leader graphs of $DL_8(2,2)$.

1 Introduction

Given a group G and a generating set $S \subseteq G$ with $e \notin S$ and $S = S^{-1}$, the Cayley graph $\mathrm{Cay}(G,S)$ is defined as a graph where the vertices correspond to the elements of G and for each $g \in G$ and $s \in S$, there is an edge between vertices g and gs. Let (X_1, d_1) and (X_2, d_2) be metric spaces. A mapping $\phi : X_1 \to X_2$ is a *quasi-isometry* if there exist constants $A > 0$ and $B \geq 0$ such that for all $x_1, y_1 \in X_1$ and $x_2 \in X_2$:

(i) $d_2(\phi(x_1), \phi(x_2)) \leq Ad_1(x_1, y_1) + B$ and $d_1(x_1, y_1) \leq \frac{1}{A} d_2(\phi(x_1), \phi(x_2)) + B$

(ii) $d_2(x_2, \phi(X_1)) \leq B$

Mathematician Wolfgang Woess asked a question in his work in 1990 [Woe14]

> *"Is there a (connected, locally finite, infinite) vertex-transitive graph that is not quasi-isometric to some Cayley graph?"*

In response to this question, mathematicians Diestel and Leader introduced the concept of Diestel-Leader graph without a proof in 2001 [DL01]. Diestel-Leader graphs $DL(p,q)$ are horocyclic products of two homogeneous trees T_p and T_q where each vertex of T_p (respectively, T_q) has one parent and p (respectively, q) children [Woe14]. Displayed in Figure 1 are the two homogeneous trees, T_p and T_q, derived from the $DL_3(2,2)$ graph. Initially beginning from an empty set, the nodes branching to the left are labeled as 0, while those branching to the right are labeled as 1.

Suppose we have a set of m points called nuclei distributed across a plane. The Voronoi diagram associated with these nuclei partitions of the plane into precisely m

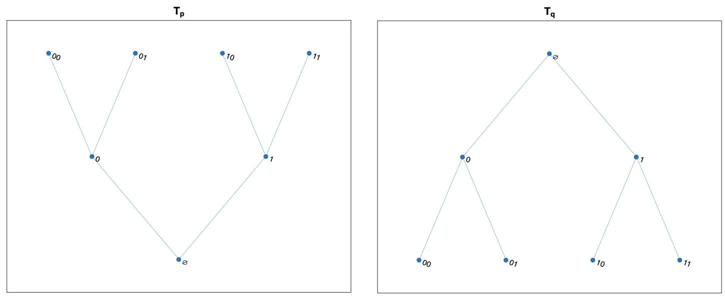

Figure 1: Two homogeneous trees T_p and T_q

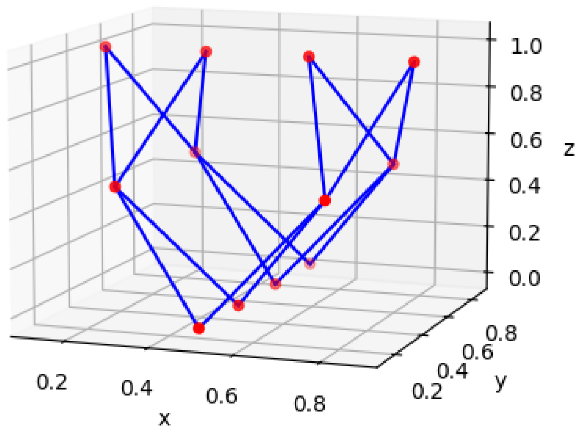

Figure 2: The horocyclic product of those two homogeneous trees will generate $DL_3(2, 2)$

cells, each encompassing the region closest to its nucleus [Bel23]. Voronoi diagrams on plane offer insights into the distribution of space by identifying the smallest and largest cell areas within a given metric space. This characteristic has practical implications for numerous applications in computational geometry, biology, and the physical sciences. For instance, a leading supply chain company may use Voronoi diagrams to decide where to build a major distribution center to house merchandise that is far enough from its other centers, yet close enough to offer services and adequately compete with major proximal competitors[Dob05].

Voronoi partitions also have significant applications across various scientific fields and daily life. For example, Voronoi diagrams were utilized by German mathematician Dirichlet [Dir50] in his work on quadratic forms, these partitions were based on points of a lattice in the Euclidean plane [89]. In daily life, for example, the cross-sections of horizontally-cut garlic, ancient porcelains from China, and giraffes' coats are Voronoi diagrams. Our goal is to study the geometry of Diestel-Leader graphs by computing Voronoi diagrams on finite approximations of Diestel-Leader graphs.

In Section 3 we discuss our preliminaries and definition of Diestel-Leader graph and Voronoi diagram. In Section 4 we explain the method and algorithmn we used for our experiments in stimulating Voronoi tessellation growth in Diestel-Leader graph. In Section 5 we display our result of stimulation on $DL_8(2, 2)$ with assigned priority, and discuss the major results of the research. In Section 6, we discuss our limitations in this research and explore our future expectations on this work.

2 Acknowledgements

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3 Background

The lamplighter group over $\mathbb{Z}$ is defined as the wreath product $\mathbb{Z}_q \wr \mathbb{Z}$, where $\mathbb{Z}$ denotes the set of integers and $\mathbb{Z}_q$ represents the cyclic group of order q. Let $p > 1$. The Cayley graph corresponding to this group, with respect to a canonical generating set, is identified as the Diestel-Leader graph $DL(q, q)$ [Woe04]. Let $w \in \partial T_q$. The *Busemann function* [Esc23] with respect to w is the map $h_w \colon T \to \mathbb{Z}$ defined by

$$h_w(x) = d(x, x * o) - d(o, x * o).$$

Definition 1. Consider two *Busemann* pairs (X_1, h_1) and (X_2, h_2) over a common linearly ordered set L, which could be $\mathbb{R}$ or $\mathbb{Z}$ depending on the nature of the metric space X. The *horocyclic product* of X_1 and X_2, denoted as $X_1 \times_h X_2$, is defined as the set of pairs $(x_1, x_2) \in X_1 \times X_2$ for which the condition $h(x_1) + h(x_2) = 0$ holds [Woe14].

Definition 2. A *finite Diestel-Leader graph* $DL_n(p, q)$ to be the graph with vertices

$$V(DL_n(p, q)) = \{(x, y) \in T_p \times T_q : h(x) = -h(y)\} \tag{1}$$

where n is the height of T_p and T_q and h is the height function of x and y in the trees T_p and T_q, respectively. Two vertices (x_1, y_1) and (x_2, y_2) are adjacent if x_1 is adjacent to x_2 in T_p and y_1 is adjacent to y_2 in T_q.

Definition 3. A geodesic ray is an infinite sequence $(v_n)_{n\in\mathbb{N}}$ of vertices of T such that $d(v_i, v_j) = |i - j|$, for all $i, j \in \mathbb{N}$ [Esc23]. Let each edge has length of one. We define the distance between two nodes in Diestel-Leader graph as the geodesic ray between two nodes.

The distance between (x_1, y_1) and (x_2, y_2) is given by the formula

$$d((x_1, y_1), (x_2, y_2)) = d(x_1, y_1) + d(x_2, y_2) - |h(x_1) - h(x_2)| \tag{2}$$

We think of Diestel-Leader graphs as metric spaces using the graph metric, where each edge has length one and the distance between two vertices is the length of the shortest path between them. Finite Diestel-Leader graphs $DL_i(p, q)$ are approximations of their infinite counterparts as they fit into an ascending sequence of inclusions whose union is $DL(p, q)$:

$$DL_2(p, q) \subset DL_3(p, q) \subset \ldots \subset DL_i(p, q) \subset \ldots \subset DL(p, q).$$

Definition 4. For a finite set S of objects p_i in a space M, the computation of the *Voronoi diagram* involves partitioning M into regions, $R(p_i, S)$. Each region $R(p_i, S)$ encompasses all points in M that are closer to p_i than to any other object p_j in S [89].

Definition 5. Generating nuclei randomly by deciding whether a given vertex was a nucleus with probability $0 < \mathbb{P} < 1$, We define *priority* as the sequence for a nucleus in Diestel-Leader graph to "occupy" the area when there is a node that is equidistant from more than one nucleus.

4 Method

In this project, we take finite approximations of the Diestel-Leader graph $DL_n(2,2)$. We generate Voronoi diagrams for finite Diestel-Leader graphs by randomly assigning each vertex a probability $0 < \mathbb{P} < 1$ of being a nucleus for a Voronoi cell and growing balls around each nucleus at the same rate.

We first developed an algorithm to generate $DL_n(2,2)$ graph by computing its coordinates and edges. The algorithm generates a set of three-dimensional coordinates based on input parameters n, p, and q, and then transforms these coordinates using a mathematical formula. It also constructs an edge library to represent connections between the transformed coordinates. The process involves iterating through different combinations of i, j, and k values within specified ranges to generate initial coordinates. These coordinates are then transformed by scaling and translating them based on the input parameters. Additionally, the algorithm identifies potential child coordinates for each initial coordinate and records these relationships in the edge library. Finally, the transformed coordinates and edge library are returned as the output of the algorithm.

We try to simulate the growth of the Voronoi tessellation inside Diestel-Leader graphs by the following algorithm. We choose our nucleus with the probability of $0 < \mathbb{P} < 1$ at first. Then as the radius r grows, we mark the nodes that are contained by the Voronoi tessellation of that nucleus. Finally, the Voronoi tessellation with different nucleus will stop growing when there forms boundaries and every node in Diestel-Leader graph has been grouped.

Algorithm 1: Voronoi Cells Algorithm

```
Data: A specific Diestel-Leader graph
Result: Voronoi Cells of radius r
Parameter Parameter: t: total number of nodes
Parameter: n: number of nuclei
Parameter: r: radius of Voronoi Cells
for i ← 0 to n − 1 do
    for j ← 0 to t − 1 do
        using distance formula Equation (2) calculate the distance d of each
         node[j] to nuclei[i]
        if d <= r then
            mark node[j];
        end
    end
end
```

We encountered problems with points that are equidistant from multiple nuclei. We considered two methods to resolve this issue, the first of which assigns all points equidistant from two or more nuclei to a new nucleus. The following figure shows a Voronoi diagram with two nuclei on $DL_4(2,2)$ where points equidistant from the two nuclei are assigned to a new nucleus. The result is shown in Figure 3.

The problem with the first method is that often, most of the graph will be covered by cells around new nuclei. To eliminate this problem, we decided to assign each nucleus a *priority* based on the ordering. When a point is equidistant from multiple nuclei, it gets assigned to the Voronoi cell around the nucleus of the highest priority. Prioritizing nuclei allows us to construct a Voronoi diagram (also referred to as tesselation) on Diestel-Leader

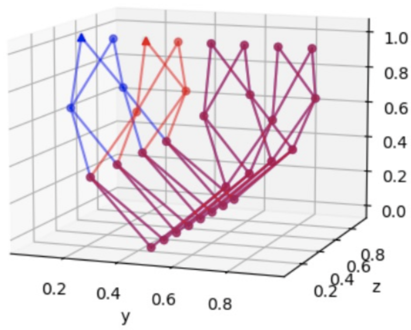

Figure 3: Voronoi diagram with two nuclei on $DL_4(2,2)$. The cells of the two original nuclei are colored red and blue, and points equidistant from the two nuclei are assigned to a new nucleus and colored purple.

graphs without adding new nuclei.

As illustrated in Figure 4, our initial step involves generating a $DL_8(2,2)$ graph. For clarity in presentation, only the nodes from the first step are displayed. We then designate nuclei by randomly selecting 4 nodes within $DL_8(2,2)$. Subsequently, the tessellation process commences, wherein nodes and edges associated with a nucleus are marked with distinct colors. This tessellation is incrementally expanded by a length of 1 in each step, and this process is repeated until all nodes in the graph are colored.

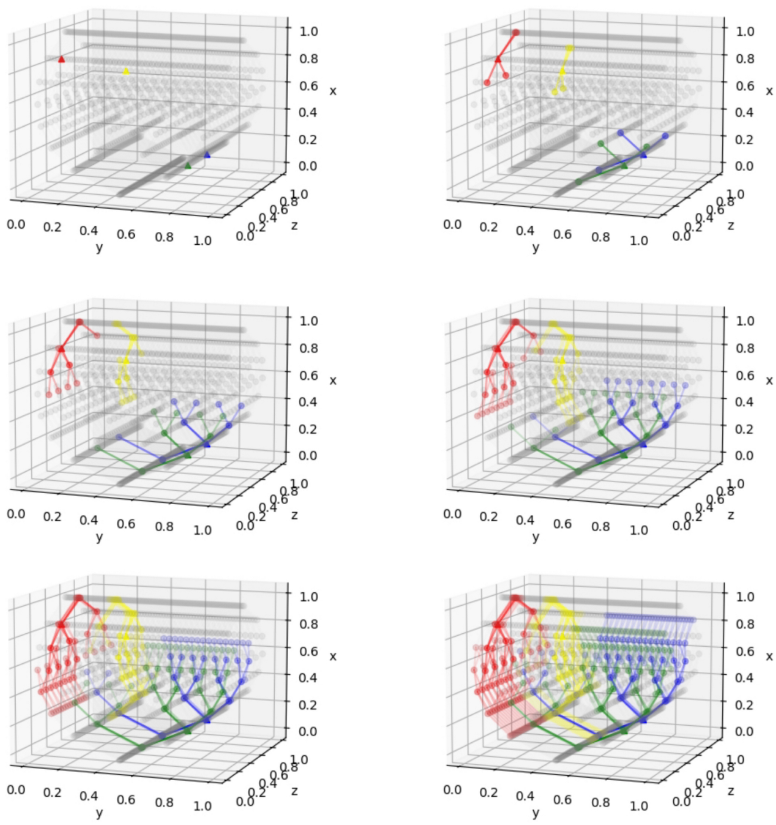

Figure 4: Growing balls around four nucleus in $DL_8(2,2)$. Each picture corresponds to balls around the nucleus of different radius. Sequence: top to bottom, left to right.

5 Conclusion

We discover the distance function between two vertices in Diestel-Leader graph, see Equation (2), at the beginning of the research. We also plotted Voronoi diagrams on Diestel-Leader graphs of various heights with various numbers and locations of nuclei. Below is a Voronoi tessellation of $DL_8(2,2)$ where each vertex had a 5% chance of being a nucleus of a Voronoi cell. Note that this is the final Voronoi tessellation associated with Figure 6. This methodical approach allows us to observe the growth pattern and the ultimate formation of the tessellation across the Diestel-Leader graph.

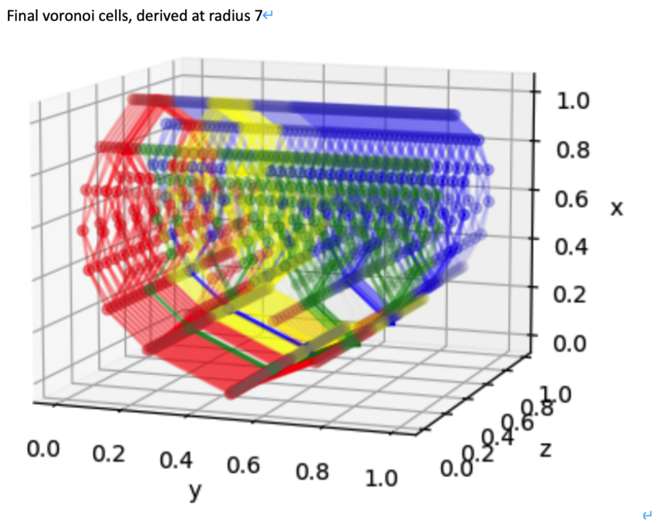

Figure 5: Voronoi tessellation of $DL_8(2,2)$ with four nucleus and priority

6 Discussion

In this research, we investigate the finite setting of Diestel-Leader graph, which traditionally be infinite. The limitation of our algorithms is that it needs further optimization to run infinite setting. Our current model uses double for-loop to go through all nucleus and all nodes to determine which vertex is included by the different Voronoi tessellations, which makes the time complexity and computation complexity growing in the order of n^2. Meanwhile the total number of nodes n grows exponentially as we increase the height of Diestel-Leader graph. As a result, our current model is not useful for large, or infinite Diestel-Leader Graph. In our future work, we will compute several more Voronoi tessellations of finite Diestel-Leader graphs. Our aim is to increase the heights of our finite Diestel-Leader graphs and decrease the probability that a vertex is a nucleus. Ultimately, we hope to use these Voronoi diagrams to form a conjecture about what Voronoi diagrams look like for infinite Diestel-Leader graphs with small value of p.

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