Reciprocity of the Wigner derivative for spherical tetrahedra

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Today

Abstract

The Wigner derivative is the partial derivative of dihedral angle with respect to opposite edge length in a tetrahedron, all other edge lengths remaining fixed. We compute the inverse Wigner derivative for spherical tetrahedra, namely the partial derivative of edge length with respect to opposite dihedral angle, all other dihedral angles remaining fixed. We show that the inverse Wigner derivative is actually equal to the Wigner derivative. These computations are motivated by the asymptotics of the classical and quantum 6j symbols for SU(2).

1 Introduction

In his seminal book on group theory and quantum mechanics from 1959 ([8], see also [1]), Wigner studied the classical 6j symbols for SU(2) which encode the associator data for the tensor category of representations of SU(2). He related the 6j symbol $\begin{Bmatrix} J & j_2 & j' \\ j_1 & j_3 & j \end{Bmatrix}$ to a Euclidean tetrahedron with side lengths given by j_1, j_2, j_3, J, j, j' and gave a heuristic argument that the square of this 6j symbol should (on average, for large spins) be proportional to the partial derivative $\frac{\partial \theta}{\partial j'}$ of the dihedral angle θ at edge j with respect to the length of the opposite edge j', all other lengths being held fixed (see Figure 1a) $\boldsymbol{Insert figures (a), (b) here}$ If the lengths j_1, j_2, j_3, J, j are held constant, then P can still traverse the indicated circle, changing j'. The probability of a given tetrahedron occuring is proportional to $\frac{\partial \theta}{\partial j'}(j, \cdots, j')$ where θ is the dihedral angle at the edge with length j. Taken from [8]. (b) The Wigner derivative for a spherical tetrahedra is $\frac{\partial \theta}{\partial l'}$, the partial derivative of dihedral angle with respect to opposite edge length, all oher lengths held fixed.

In 1968 the physicists Ponzano and Regge conjectured a more refined formula for the asymptotics of the classical 6j symbols, which included an oscillatory term. This formula was first proved rigorously by Roberts in 1999, using geometric quantization techniques [6], and since then a number of other proofs have been given [$\boldsymbol{Still need to put references here}$].

In 2003 Taylor and Woodward gave a corresponding formula for the geometry of spherical tetrahedra [**still need some references here**]. In their outline of a possible geometric proof of their formula (this approach was later made rigorous by Marché and Paul [4]), the partial derivative of dihedral angle with respect to opposite edge length (this time for a spherical tetrahedron) again played a crucial role. Following Taylor, we call this the Wigner derivative (see Figure 1b).

Given a spherical tetrahedron with vertices v_0, v_1, v_2, v_3 and edge lengths l_{ij}, let G be the length Gram matrix, $G_{ij} = \cos(l_{ij})$. Taylor and Woodward' s formula for the Wigner derivative is as follows [1], of which we also we give an independent proof.

Theorem 1.1. *The Wigner derivative for a spherical tetrahedron is*

$$\frac{\partial \theta}{\partial l'}(l_{ij}) = \frac{\sin l \sin l'}{\sqrt{\det G}} \tag{1}$$

where θ is the interior dihedral angle at the edge with length l and $l'$is the length of the opposite edge (see Fig. 1b).

Unlike a Euclidean tetrahedron, a spherical tetrahedron is determined up to isometry by its six edge lengths as well as by its six dihedral angles. So there is a 1-1 correspondence (see (10) in Section 2 for an explicit formula)

$$(l_{01}, l_{02}, l_{03}, l_{12}, l_{13}, l_{23}) \longleftrightarrow (\theta_{01}, \theta_{02}, \theta_{03}, \theta_{12}, \theta_{13}, \theta_{23})$$

and it makes sense to ask about the inverse Jacobian matrix $\frac{\partial l_{ij}}{\partial \theta_{kl}}$ and in particular the inverse Wigner derivative $\frac{\partial l'}{\partial \theta}$ in Figure 1b. Indeed, in our work we were led to consider this inverse Jacobian as it shows up in the stationary phase approximation for a conjectural integral formula for the quantum 6j symbols. The main result of this paper is as follows.

Theorem 1.2. *The inverse Wigner derivative for a spherical tetrahedron is*

$$\frac{\partial l'}{\partial \theta}(\theta_{ij}) = \frac{\sin l \sin l'}{\sqrt{\det G}} \tag{2}$$

Comparing with formula (1) for the Wigner derivative, we obtain the following corollary.

Corollary 1.3 (Reciprocity of the Wigner derivative)**.** *For spherical tetrahedra, the Wigner derivative and the inverse Wigner derivative are equal:*

$$\frac{\partial \theta}{\partial l'}(l_{ij}) = \frac{\partial l'}{\partial \theta}(\theta_{ij}) \tag{3}$$

Remark 1.4. *In the proof of [[7]-published Prop 2.4.1.(n)] and in [[7], Prop 2.2.0.5] Taylor and Woodward show that*

$$\frac{\partial l'}{\partial \theta} = \frac{\sqrt{\det G}}{\sin l \sin l'} \tag{4}$$

which is the reciprocal of our formula in 1.2 and thus seems to contradict it. What is going on? The answer is that they are different partial derivatives as different sets of variables are being held constant. In our formula (2), θ is changing while deeping the five remaining dihedral angles constant, while in Taylor and Woodward' s formula (3), θ is changing while all lengths excluding l' are being held constant. It is interesting that these two different partial derivatives are reciprocals of each other. To the best of our knowledge, formula (2) and its corollary (3) are new (see [5] for related work).

[1] Note that the actual statement of Proposition 2.4.1.(n) in [[7]-Published] contains a typo. The left hand side should be $\frac{\partial \theta_{ab}}{\partial l_{cd}}$ not $(\frac{\partial \theta_{ab}}{\partial l_{cd}})^{-1}$

Overview of paper *In Section 2 we show, as a warm-up result, that reciprocity of the Wigner derivative holds for spherical triangles.*

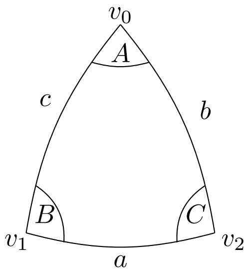
2 Reciprocity of the Wigner derivative for spherical triangles

In this section we review some elementary spherical trigonometry, and prove reciprocity of the Wigner derivative for spherical triangles. This serves as a warm-up example before tackling spherical tetrahedra.

Consider a spherical triangle $\Delta \subseteq S^2$ as in Figure 2 with vertices $v_0, v_1, v_2 \in S^2$,

$$\Delta := \{t_0 v_0 + t_1 v_1 + t_2 v_2 : t_0, t_1, t_2 \geq 0\} \cap S^2.$$

Let a, b, c be the lengths of its edges and A (resp. B, resp. C) be the angle opposite to the edge of length a (resp. b, resp. c).



The sine rule says that

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C} \tag{5}$$

The cosine law expresses the interior angles in terms of the side lengths,

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c} \tag{6}$$

while the dual cosine law expresses the side lengths in terms of the interior angles:

$$\cos a = \frac{\cos A + \cos B \cos C}{\sin B \sin C} \tag{7}$$

For the sine and cosine laws, see [Wikipedia cosine law]. From (6) and (7) we obtain

$$\frac{\partial A}{\partial a} = \frac{\sin a}{\sin A \sin b \sin c}, \frac{\partial a}{\partial A} = \frac{\sin A}{\sin a \sin B \sin C} \tag{8}$$

To write these formulas in the form of (1) and (2), we introduce the length Gram matrix $G_{ij} = \cos l(v_i, v_j)$

$$G = \begin{pmatrix} 1 & \cos c & \cos b \\ \cos c & 1 & \cos a \\ \cos b & \cos a & 1 \end{pmatrix}$$

The following fact is well known [Wikipedia], but we include a proof in order to be self-contained.

Lemma 2.1. $\sqrt{\det G} = \sin A \sin b \sin c$

Proof. By row operations we obtain

$$\begin{aligned}
\det G &= \begin{pmatrix} 1 & \cos c & \cos b \\ 0 & 1-\cos^2 c & \cos a - \cos b \cos c \\ 0 & \cos a - \cos b \cos c & 1-\cos^2 b \end{pmatrix} \\
&= \sin^2 b \sin^2 c - (\cos a - \cos b \cos c)^2 \\
&= \sin^2 b \sin^2 c - \sin^2 b \sin^2 c \ (\frac{\cos a - \cos b \cos c}{\sin b \sin c})^2 \\
&= \sin^2 b \sin^2 c \ (1-\cos^2 A) \\
&= \sin^2 b \sin^2 c \ \ \sin^2 A
\end{aligned}$$

where we have used the cosine law in the second last step. □

This allows us to prove the formula for the Wigner derivative and its inverse for spherical triangles, and show that they are equal.

Theorem 2.2 (Wigner reciprocity for spherical triangles)**.** *For spherical triangles, we have*

$$\frac{\partial A}{\partial a}(a,b,c) = \frac{\sin a}{\sqrt{\det G}} = \frac{\partial a}{\partial A}(A,B,C)$$

Proof. The first equation follows directly from (8a) and Lemma 2.1. The second equation follows from:

$$\begin{aligned}
\frac{\frac{\partial A}{\partial a}}{\frac{\partial a}{\partial A}} &= \frac{\sin^2 a \sin B \sin C}{\sin^2 A \sin b \sin c} \qquad \text{(by 8)} \\
&= 1 \qquad \text{(by the sine rule)}
\end{aligned}$$

□

InsertFigure2here

3 Reciprocity of the Wigner derivative for spherical tetrahedra

Now consider a spherical tetrahedron $\Delta \subseteq S^3$ with vertices $v_0, v_1, v_2, v_3 \in S^3$ as in Figure 2(a),

$$\Delta := \{t_0 v_0 + t_1 v_1 + t_2 v_2 + t_3 v_3 : t_0, t_1, t_2, t_3 \geq 0\} \cap S^3.$$

The edge lengths $a, \cdots, f$ and interior dihedral angles $A, \cdots, F$ are shown, as well as the inner angles at v_0, v_2 and v_3. The link $Lk(v)$ of a vertex v is the spherical triangle with edge lengths given by the inner angles at v. In $Lk(v_0)$, let Γ be the interior angles opposite the edge with length γ. The following fact is used in [3]; here we give an explicit proof.

Lemma 3.1. $\Gamma = E$.

Proof. By acting with an appropriate element of $SO(4)$, we can rotate Δ so that its vertices are in standard position:

$$v_0 = (1,0,0,0), v_i = (\cos\theta_i, \sin\theta_i n_i) \quad i = 1, \cdots, 3$$

here $n_i \in S^2$ are the vertices of $Lk(v_0)$ as in Figure 2b:

$$n_1 = (0,0,1), n_2 = (\sin\alpha, 0, \cos\alpha), n_3 = (\sin\beta\cos\gamma, \sin\beta\sin\gamma, \cos\beta)$$

By definition,

$$\cos E = -w_2.w_3$$

where $w_2, w_3 \in \mathbb{R}^4$ are the outward unit normals to the faces $v_0v_1v_3$ and $v_0v_1v_2$ of Δ respectively. Clearly we have

$$w_2 = (0, a_2), w_3 = (0, a_3)$$

where $a_2, a_3 \in \mathbb{R}^3$ are the outward unit normals to the edges n_1n_3 and n_1n_2 of $Lk(v_0)$ respectively (see Figure 2b). But by definition

$$\cos\Gamma = -a_2.a_3$$

which shows that $\Gamma = E$. □

Lemma 3.2. *In the spherical tetrahedron Δ, the Wigner derivative and inverse Wigner derivative are:*

$$\frac{\partial E}{\partial f}(a,b,c,d,e,f) = \frac{\sin f}{\sin E \sin\alpha \sin\beta \sin a \sin b}$$

$$\frac{\partial f}{\partial E}(A,B,C,D,E,F) = \frac{\sin E}{\sin f \sin\kappa \sin\sigma \sin A \sin B}$$

Proof. For the Wigner derivative,

$$E = E(\alpha,\beta,\gamma) \quad \text{(using cosine rule for } Lk(v_0)\text{; see Fig. 1)} \tag{9}$$

$$E = E(\alpha(a,c,e), \beta(b,d,e), \gamma(a,b,f)) \quad \text{(using cosine rule for triangles } v_0v_1v_2, v_0v_1v_3 \text{ and } v_0v_2v_3\text{)} \tag{10}$$

and so by the chain rule,

$$\begin{aligned} \frac{\partial E}{\partial f} &= \frac{\partial E}{\partial\gamma}\frac{\partial\gamma}{\partial f} \\ &= \left(\frac{\sin\gamma}{\sin E \sin\alpha \sin\beta}\right)\left(\frac{\sin f}{\sin\gamma \sin a \sin b}\right) \\ &= \frac{\sin f}{\sin E \sin\alpha \sin\beta \sin a \sin b}. \end{aligned}$$

For the inverse Wigner derivative,

$$\begin{aligned} f &= f(\kappa,\sigma,\gamma) \quad \text{(by dual cosine rule for triangle } v_0v_2v_3\text{)} \\ &= f(\kappa(A,C,F), \sigma(B,D,F), \gamma(A,B,E)) \quad \text{(By dual cosine rule for } Lk(v_2), Lk(v_3), Lk(v_0)\text{; see Fig. 1)} \end{aligned}$$

and so by the chain rule,

$$\begin{aligned}\frac{\partial f}{\partial E} &= \frac{\partial f}{\partial \gamma}\frac{\partial \gamma}{\partial E} \\ &= \left(\frac{\sin\gamma}{\sin f \sin\kappa \sin\sigma}\right)\left(\frac{\sin E}{\sin\gamma \sin A \sin B}\right) \\ &= \frac{\sin E}{\sin f \sin\kappa \sin\sigma \sin A \sin B}\end{aligned}$$

□

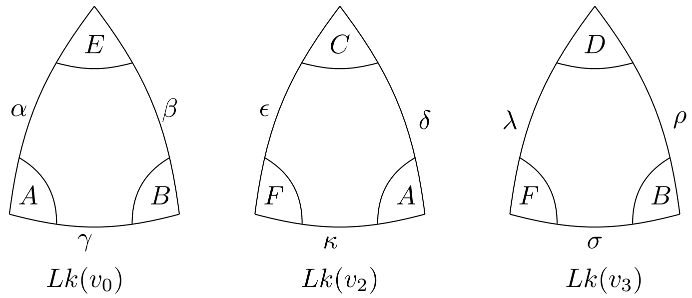


Figure 1: The links of v_0, v_2 and v_3

To write these derivatives in the form (1) and (2), we will need the 4×4 Gram matrix $G_{ij} = \cos(l(v_i, v_j))$.

Lemma 3.3. *(see [2])* $\sqrt{\det G} = \sin a \sin b \sin e \sin\alpha \sin\beta \sin E.$

Proof.

$$\begin{aligned}\det G &= \det\begin{pmatrix} 1 & \cos e & \cos a & \cos b \\ \cos e & 1 & \cos c & \cos d \\ \cos a & \cos c & 1 & \cos f \\ \cos b & \cos d & \cos f & 1\end{pmatrix} \\ &= \det\begin{pmatrix} 1 & \cos e & \cos a & \cos b \\ 0 & 1-\cos^2 e & \cos c - \cos a\cos e & \cos d - \cos b \cos e \\ 0 & \cos c - \cos a \cos e & 1 - \cos^2 a & \cos f - \cos a \cos b \\ 0 & \cos d - \cos b \cos e & \cos f - \cos a \cos b & 1 - \cos^2 b\end{pmatrix} \\ &= \sin^2 a \sin^2 b \sin^2 e \det\begin{pmatrix} 1 & \cos\alpha & \cos\beta \\ \cos\alpha & 1 & \cos\gamma \\ \cos\beta & \cos\gamma & 1\end{pmatrix} \\ &= \sin^2 a \sin^2 b \sin^2 e \det G'\end{aligned}$$

where G' is the Gram matrix of $Lk(v_0)$. Now use Lemma 2.1. □

Note that Lemmas 3.2 and 3.3 combine to give a different proof of Taylor and Woodward' s formula for the Wigner derivative.

Theorem 3.4. *(see [7]) The Wigner derivative for a spherical tetrahedron (see Figure 2a) is*

$$\frac{\partial E}{\partial f}(a,b,c,d,e,f) = \frac{\sin e \sin f}{\sqrt{\det G}}$$

We can now prove our main results.

Theorem 3.5. *The inverse Wigner derivative for a spherical tetrahedron (see Figure 2a) is*

$$\frac{\partial f}{\partial E}(A,B,C,D,E,F) = \frac{\sin e \sin f}{\sqrt{\det G}}$$

Corollary 3.6 (Reciprocity of the Wigner derivative)**.** *For spherical tetrahedra, the Wigner derivative is equal to the inverse Wigner derivative.*

Proof. Both results follow from:

$$\begin{aligned}
\frac{\frac{\partial E}{\partial f}}{\frac{\partial f}{\partial E}} &= \frac{\sin^2 f \sin A \sin B \sin \kappa \sin \sigma}{\sin^2 E \sin \beta \sin \alpha \sin b \sin a} \\
&= \frac{\sin^2 f}{\sin^2 E}\frac{\sin^2 E}{\sin^2 \gamma}\frac{\sin^2 \gamma}{\sin^2 f} \quad \text{(using sine law for } Lk(v_0) \text{ and triangle } v_0v_2v_3) \\
&= 1
\end{aligned}$$

□

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