MEF 3D – Fortaleza de Alexa

Ecuación

$$\frac{d}{dx}\left(\frac{d}{dx}\left(EI\,\frac{d}{dx}\left(\frac{d}{dx}\vec{w}\right)\right)\right)=\vec{f}$$

$EI = 46$

$$\vec{f}=\begin{bmatrix}22\\29\\46\end{bmatrix}$$

Sustituire la letra w por H para evitar confusiones mas adelante con las funciones de peso, en el modelo quedaria de la siguiente forma

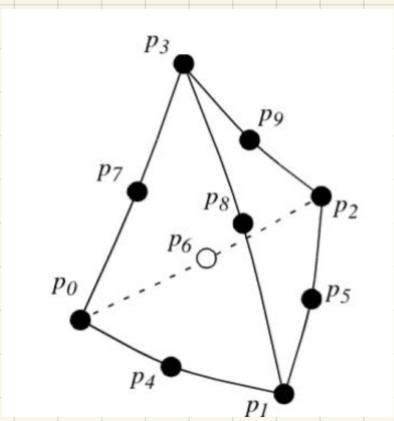
$$\frac{d}{dx}\left(\frac{d}{dx}\left(EI\,\frac{d}{dx}\left(\frac{d}{dx}\vec{H}\right)\right)\right)=\vec{f}$$

$$\vec{H}=\begin{bmatrix}H_x\\H_y\\H_z\end{bmatrix}\rightarrow\vec{H}=\begin{bmatrix}\alpha\\\beta\\\gamma\end{bmatrix}$$

Step 2: Interpolation

Se utilizara Funciones de forma cuadraticas para un elemento tetrahedro

Por tanto se necesitan 10 funciones de forma para la interpolacion.



Las funciones a utilizar

$$N_1=(1-\epsilon-\eta-\phi)(2(1-\epsilon-\eta-\phi)-1)$$
$$N_2=\epsilon(2\epsilon-1)$$
$$N_3=\eta(2\eta-1)$$
$$N_4=\phi(2\phi-1)$$
$$N_5=4\epsilon\eta$$
$$N_6=4\eta\phi$$
$$N_7=4\epsilon\phi$$
$$N_8=4\epsilon(1-\epsilon-\eta-\phi)$$
$$N_9=4\eta(1-\epsilon-\eta-\phi)$$
$$N_{10}=4\phi(1-\epsilon-\eta-\phi)$$

Por tanto

$$\alpha \approx N_1\alpha_1 + N_2\alpha_2 + N_3\alpha_3 + N_4\alpha_4 + N_5\alpha_5 + N_6\alpha_6 + N_7\alpha_7 + N_8\alpha_8 + N_9\alpha_9 + N_{10}\alpha_{10}$$

$$\beta \approx N_1\beta_1 + N_2\beta_2 + N_3\beta_3 + N_4\beta_4 + N_5\beta_5 + N_6\beta_6 + N_7\beta_7 + N_8\beta_8 + N_9\beta_9 + N_{10}\beta_{10}$$

$$\gamma \approx N_1\gamma_1 + N_2\gamma_2 + N_3\gamma_3 + N_4\gamma_4 + N_5\gamma_5 + N_6\gamma_6 + N_7\gamma_7 + N_8\gamma_8 + N_9\gamma_9 + N_{10}\gamma_{10}$$

$$\alpha \approx \begin{bmatrix} N_1 & N_2 & N_3 & N_4 & N_5 & N_6 & N_7 & N_8 & N_9 & N_{10} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \\ \alpha_7 \\ \alpha_8 \\ \alpha_9 \\ \alpha_{10} \end{bmatrix} \Rightarrow \alpha \approx N\alpha$$

$$\beta \approx \begin{bmatrix} N_1 & N_2 & N_3 & N_4 & N_5 & N_6 & N_7 & N_8 & N_9 & N_{10} \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \\ \beta_5 \\ \beta_6 \\ \beta_7 \\ \beta_8 \\ \beta_9 \\ \beta_{10} \end{bmatrix} \Rightarrow \beta \approx N\beta$$

$$\gamma \approx \begin{bmatrix} N_1 & N_2 & N_3 & N_4 & N_5 & N_6 & N_7 & N_8 & N_9 & N_{10} \end{bmatrix} \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4 \\ \gamma_5 \\ \gamma_6 \\ \gamma_7 \\ \gamma_8 \\ \gamma_9 \\ \gamma_{10} \end{bmatrix} \Rightarrow \gamma \approx N\gamma$$

$$\vec{H} \approx \begin{bmatrix} N\alpha \\ N\beta \\ N\gamma \end{bmatrix} \approx \begin{bmatrix} N & 0 & 0 \\ 0 & N & 0 \\ 0 & 0 & N \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} \rightarrow \vec{H} \approx N^* H$$

Step 3: Model Aproximation

$$\frac{d}{dx}\left(\frac{d}{dx}\left(EI\,\frac{d}{dx}\left(\frac{d}{dx}\vec{H}\right)\right)\right)=\vec{f}$$

$$\frac{d}{dx}\left(\frac{d}{dx}\left(EI\,\frac{d}{dx}\left(\frac{d}{dx}NH\right)\right)\right)=\vec{f}$$

$$R=\frac{d}{dx}\left(\frac{d}{dx}\left(EI\,\frac{d}{dx}\left(\frac{d}{dx}NH\right)\right)\right)-\vec{f}$$

Step 4: W.R.M

$$\iiint w\,R\,dx\,dy\,dz=0$$

$$\int_V w\,R\,dV=0$$

$$\int_V w\left[\frac{d}{dx}\left(\frac{d}{dx}\left(EI\,\frac{d}{dx}\left(\frac{d}{dx}NH\right)\right)\right)-\vec{f}\right]dV=0$$

Tenemos 10 incognitas por tanto necesitamos 10 ecuaciones

$$\int_V w_1\left[\frac{d}{dx}\left(\frac{d}{dx}\left(EI\,\frac{d}{dx}\left(\frac{d}{dx}NH\right)\right)\right)-\vec{f}\right]dV=0$$

$$\int_V w_2\left[\frac{d}{dx}\left(\frac{d}{dx}\left(EI\,\frac{d}{dx}\left(\frac{d}{dx}NH\right)\right)\right)-\vec{f}\right]dV=0$$

$$\int_V w_3\left[\frac{d}{dx}\left(\frac{d}{dx}\left(EI\,\frac{d}{dx}\left(\frac{d}{dx}NH\right)\right)\right)-\vec{f}\right]dV=0$$

$$\int_V w_4\left[\frac{d}{dx}\left(\frac{d}{dx}\left(EI\,\frac{d}{dx}\left(\frac{d}{dx}\mathbf{NH}\right)\right)\right)-\vec{f}\right]dV=0$$

$$\int_V w_5\left[\frac{d}{dx}\left(\frac{d}{dx}\left(EI\,\frac{d}{dx}\left(\frac{d}{dx}\mathbf{NH}\right)\right)\right)-\vec{f}\right]dV=0$$

$$\int_V w_6\left[\frac{d}{dx}\left(\frac{d}{dx}\left(EI\,\frac{d}{dx}\left(\frac{d}{dx}\mathbf{NH}\right)\right)\right)-\vec{f}\right]dV=0$$

$$\int_V w_7\left[\frac{d}{dx}\left(\frac{d}{dx}\left(EI\,\frac{d}{dx}\left(\frac{d}{dx}\mathbf{NH}\right)\right)\right)-\vec{f}\right]dV=0$$

$$\int_V w_8\left[\frac{d}{dx}\left(\frac{d}{dx}\left(EI\,\frac{d}{dx}\left(\frac{d}{dx}\mathbf{NH}\right)\right)\right)-\vec{f}\right]dV=0$$

$$\int_V w_9\left[\frac{d}{dx}\left(\frac{d}{dx}\left(EI\,\frac{d}{dx}\left(\frac{d}{dx}\mathbf{NH}\right)\right)\right)-\vec{f}\right]dV=0$$

$$\int_V w_{10}\left[\frac{d}{dx}\left(\frac{d}{dx}\left(EI\,\frac{d}{dx}\left(\frac{d}{dx}\mathbf{NH}\right)\right)\right)-\vec{f}\right]dV=0$$

Lo que resulta en

$$\int_V \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \\ w_6 \\ w_7 \\ w_8 \\ w_9 \\ w_{10} \end{bmatrix}\left[\frac{d}{dx}\left(\frac{d}{dx}\left(EI\,\frac{d}{dx}\left(\frac{d}{dx}\mathbf{NH}\right)\right)\right)-\vec{f}\right]dV=0$$

$$\longrightarrow \int_V \mathbf{W}\left[\frac{d}{dx}\left(\frac{d}{dx}\left(EI\,\frac{d}{dx}\left(\frac{d}{dx}\mathbf{NH}\right)\right)\right)-\vec{f}\right]dV=0$$

Step 5: Galerkin

$$\mathbf{W} = \mathbf{N}^T$$

$$\int_V \mathbf{N}^T \left[\frac{d}{dx}\left(\frac{d}{dx}\left(EI \frac{d}{dx}\left(\frac{d}{dx}\mathbf{N}\mathbf{H}\right)\right)\right) - \vec{f}\right] dV = 0$$

$$\int_V \mathbf{N}^T \left[\frac{d}{dx}\left(\frac{d}{dx}\left(EI \frac{d}{dx}\left(\frac{d}{dx}\mathbf{N}\mathbf{H}\right)\right)\right)\right] dV - \int_V \mathbf{N}^T \vec{f}\, dV = 0$$

Strong form:

$$\left(\int_V \mathbf{N}^T \left[\frac{d}{dx}\left(\frac{d}{dx}\left(EI \frac{d}{dx}\left(\frac{d}{dx}\mathbf{N}\right)\right)\right)\right] dV\right)\mathbf{H} = \int_V \mathbf{N}^T \vec{f}\, dV = 0$$

Step 6: Integración by Parts

$$\int_V \mathbf{N}^T \left[\frac{d}{dx}\left(\frac{d}{dx}\left(EI \frac{d}{dx}\left(\frac{d}{dx}\mathbf{N}\right)\right)\right)\right] dV$$

$$\int U\, dV = UV - \int V\, dU$$

$$U = \mathbf{N}^T \qquad dV = \frac{d}{dx}\left(\frac{d}{dx}\left(EI \frac{d}{dx}\left(\frac{d}{dx}\mathbf{N}\right)\right)\right)$$

$$dU = \frac{d}{dx}\mathbf{N}^T \qquad V = \frac{d}{dx}\left(EI \frac{d}{dx}\left(\frac{d}{dx}\mathbf{N}\right)\right)$$

$$= N^T\left(\frac{d}{dx}\left(EI\,\frac{d}{dx}\left(\frac{d}{dx}N\right)\right)\right)\Big|_V - \int \frac{d}{dx}N^T\left(\frac{d}{dx}\left(EI\,\frac{d}{dx}\left(\frac{d}{dx}N\right)\right)\right)dV$$

Aplicamos nuevamente integración por partes

$$-\int \frac{d}{dx}N^T\left(\frac{d}{dx}\left(EI\,\frac{d}{dx}\left(\frac{d}{dx}N\right)\right)\right)dV$$

$$U = \frac{d}{dx}N^T \qquad dV = \frac{d}{dx}\left(EI\,\frac{d}{dx}\left(\frac{d}{dx}N\right)\right)$$

$$dU = \frac{d}{dx}\left(\frac{d}{dx}N^T\right) \qquad V = EI\,\frac{d}{dx}\left(\frac{d}{dx}N\right)$$

$$= -\left(\frac{d}{dx}N^T\left(EI\frac{d}{dx}\left(\frac{d}{dx}N\right)\right)\right)\Big|_V - \left(\int\left(\frac{d}{dx}\left(\frac{d}{dx}N^T\right)\right)\left(EI\frac{d}{dx}\left(\frac{d}{dx}N\right)\right)dV\right)$$

$$= -\frac{d}{dx}N^T\left(EI\frac{d}{dx}\left(\frac{d}{dx}N\right)\right)\Big|_V + \int\left(\frac{d}{dx}\left(\frac{d}{dx}N^T\right)\right)\left(EI\frac{d}{dx}\left(\frac{d}{dx}N\right)\right)dV$$

Sustituyendo

$$\int_V N^T\left[\frac{d}{dx}\left(\frac{d}{dx}\left(EI\,\frac{d}{dx}\left(\frac{d}{dx}N\right)\right)\right)\right]dV$$

$$N^T\left(\frac{d}{dx}\left(EI\,\frac{d}{dx}\left(\frac{d}{dx}N\right)\right)\right)\Big|_V - \frac{d}{dx}N^T\left(EI\frac{d}{dx}\left(\frac{d}{dx}N\right)\right)\Big|_V + \int\left(\frac{d}{dx}\left(\frac{d}{dx}N^T\right)\right)\left(EI\frac{d}{dx}\left(\frac{d}{dx}N\right)\right)dV$$

$$\int\left(\frac{d}{dx}\left(\frac{d}{dx}\mathbf{N}^T\right)\right)\left(EI\frac{d}{dx}\left(\frac{d}{dx}\mathbf{N}\right)\right)dV - \underbrace{\left[\mathbf{N}^T\left(\frac{d}{dx}\left(EI\,\frac{d}{dx}\left(\frac{d}{dx}\mathbf{N}\right)\right)\right) - \frac{d}{dx}\mathbf{N}^T\left(EI\frac{d}{dx}\left(\frac{d}{dx}\mathbf{N}\right)\right)\right]\Bigg|_V}_{\text{Condiciones de contorno se ignoran por el momento}}$$

Por tanto

$$\left(\int\left(\frac{d}{dx}\left(\frac{d}{dx}\mathbf{N}^T\right)\right)\left(EI\frac{d}{dx}\left(\frac{d}{dx}\mathbf{N}\right)\right)dV\right)\mathbf{H} = \int_V \mathbf{N}^T\vec{f}\,dV$$

$$\mathbf{K} = \int_V\left(\frac{d}{dx}\left(\frac{d}{dx}\mathbf{N}^T\right)\right)\left(EI\frac{d}{dx}\left(\frac{d}{dx}\mathbf{N}\right)\right)dV$$

$$\mathbf{b} = \int_V \mathbf{N}^T\vec{f}\,dV$$

Final Local Sistem

$$\mathbf{K}\mathbf{H} = \mathbf{b}$$

$$\mathbf{K}^{-1}\cdot\mathbf{K}\cdot\mathbf{H} = \mathbf{K}^{-1}\cdot\mathbf{b}$$

$$\mathbf{H} = \mathbf{K}^{-1}\mathbf{b}$$