Module 1 – Lesson 07 – Minimization of Circuits and K-maps

Karnaugh Maps – (aka K-maps)

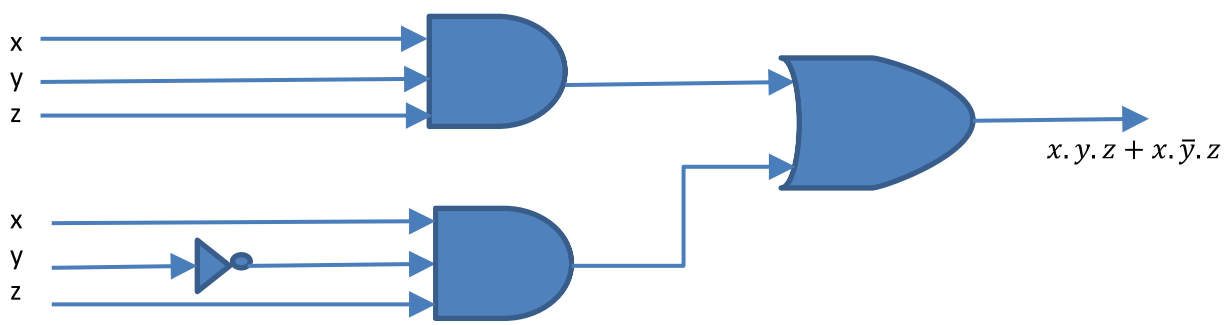
Minimization of Circuits

Combinatorial circuit efficiency is dependent on the number and the arrangement of its gates. When designing combinatorial circuits, we begin by constructing a table specifying the output given each possible combination of input values. A sum-of-products expansion of a circuit can always be used to find a set of logic gates that can be used to implement the circuit. However, the sum-of-products expansion may not be the most efficient solution; it may contain more terms than necessary (redundant minterms).

Example 1:

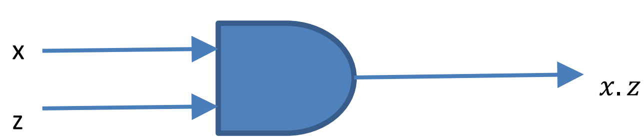
Suppose a circuit is represented by the following sum of product equation:

$$x \cdot y \cdot z + x \cdot \bar{y} \cdot z$$



If we were to use our knowledge of Boolean algebra, we could simplify this equation to be equal to:

$$x \cdot z$$



The above example shows that combining terms in the sum-of-products expansion of a circuit can lead to a simpler expression for the circuit – an example of ***minimization of the Boolean function***. It is important to simplify circuits to use as few gates as possible as it increases reliability and reduces cost to produce a chip. By minimizing a circuit, you also make it possible to fit more circuits on the same chip.

The method we will examine to simplify sum-of-products expansion is Karnaugh maps (K-maps).
Karnaugh Maps (K-maps)

To reduce the number of terms in a Boolean expression representing a circuit, we must find terms to combine. A ***Karnaugh Map*** is a graphical method for finding terms to combine in a Boolean function for a small number of inputs or variables. The goal is to use as few blocks as possible to cover all the 1s in the K-Map.

Simplify two-variable expansions of Boolean expressions using K-maps

In two-variable Boolean functions, there are four (2^2) possible minterms in the sum-of-products expansion. The K-map consists of a square split into four cells, where a 1 is placed in the cell representing a minterm that is present in the expansion.

Cells are ***adjacent*** if the minterms they represent differ in exactly one literal. For example, $\bar{x}y$ is adjacent to xy and $\bar{x}\bar{y}$.

Example 2: Find K-maps for the following:

$xy + \bar{x}y$

	y	$\bar{y}$
x		
$\bar{x}$		

$x\bar{y} + \bar{x}y$

	y	$\bar{y}$
x		
$\bar{x}$		

$x\bar{y} + \bar{x}y + \bar{x}\bar{y}$

	y	$\bar{y}$
x		
$\bar{x}$		

We can identify minterms that can be combined based on the K-map. If two adjacent cells in the K-map both have 1s, the minterms represented by these cells can be combined into a product involving just one of the variables.

$xy + \bar{x}y$

	y	$\bar{y}$
x		
$\bar{x}$		

$x\bar{y} + \bar{x}y$

	y	$\bar{y}$
x		
$\bar{x}$		

$x\bar{y} + \bar{x}y + \bar{x}\bar{y}$

	y	$\bar{y}$
x		
$\bar{x}$		

Exercise 1: Find K-maps for the following and simplify the sum-of-products expansion: $x\bar{y} + xy + \bar{x}\bar{y}$

Simplify three-variable expansions of Boolean expressions using K-maps

In three-variable Boolean functions, there are eight (2^3) possible minterms in the sum-of-products expansion. The K-map consists of a rectangle divided into eight cells, where a 1 is placed in the cell representing a minterm that is present in the expansion. Remember that two cells are adjacent if the minterms they represent differ in ***exactly*** one literal.

To simplify a sum-of-products expansion in three variables, use the K-map to identify blocks of minterms that can be combined.

Rules for Grouping together adjacent cells containing 1's

- Groups must contain 1, 2, 4, 8 cells.
- Groups must contain only 1 (and X if don't care is allowed).
- Groups may be horizontal or vertical, but not diagonal.
- Groups should be as large as possible.
- Each cell containing a 1 must be in at least one group.
- Groups may overlap.
- Groups may wrap around the table. The leftmost cell in a row may be grouped with the rightmost cell and the top cell in a column may be grouped with the bottom cell.
- There should be as few groups as possible.

Example 3: Blocks of cells that can be combined:

	yz	$y\bar{z}$	$\bar{y}\bar{z}$	$\bar{y}z$
x	xyz	$xy\bar{z}$	$x\bar{y}\bar{z}$	$x\bar{y}z$
$\bar{x}$	$\bar{x}yz$	$\bar{x}y\bar{z}$	$\bar{x}\bar{y}\bar{z}$	$\bar{x}\bar{y}z$

$$\bar{y}\bar{z} = x\bar{y}\bar{z} + \bar{x}\bar{y}\bar{z}$$

	yz	$y\bar{z}$	$\bar{y}\bar{z}$	$\bar{y}z$
x	xyz	$xy\bar{z}$	$x\bar{y}\bar{z}$	$x\bar{y}z$
$\bar{x}$	$\bar{x}yz$	$\bar{x}y\bar{z}$	$\bar{x}\bar{y}\bar{z}$	$\bar{x}\bar{y}z$

$$\bar{x}z = \bar{x}yz + \bar{x}\bar{y}z$$

	yz	$y\bar{z}$	$\bar{y}\bar{z}$	$\bar{y}z$
x	xyz	$xy\bar{z}$	$x\bar{y}\bar{z}$	$x\bar{y}z$
$\bar{x}$	$\bar{x}yz$	$\bar{x}y\bar{z}$	$\bar{x}\bar{y}\bar{z}$	$\bar{x}\bar{y}z$

$$\bar{z} = xy\bar{z} + x\bar{y}\bar{z} + \bar{x}y\bar{z} + \bar{x}\bar{y}\bar{z}$$

	yz	$y\bar{z}$	$\bar{y}\bar{z}$	$\bar{y}z$
x	xyz	$xy\bar{z}$	$x\bar{y}\bar{z}$	$x\bar{y}z$
$\bar{x}$	$\bar{x}yz$	$\bar{x}y\bar{z}$	$\bar{x}\bar{y}\bar{z}$	$\bar{x}\bar{y}z$

$$\bar{x} = \bar{x}yz + \bar{x}y\bar{z} + \bar{x}\bar{y}\bar{z} + \bar{x}\bar{y}z$$

	yz	$y\bar{z}$	$\bar{y}\bar{z}$	$\bar{y}z$
x	xyz	$xy\bar{z}$	$x\bar{y}\bar{z}$	$x\bar{y}z$
$\bar{x}$	$\bar{x}yz$	$\bar{x}y\bar{z}$	$\bar{x}\bar{y}\bar{z}$	$\bar{x}\bar{y}z$

$$1 = xyz + xy\bar{z} + x\bar{y}\bar{z} + x\bar{y}z + \bar{x}yz + \bar{x}y\bar{z} + \bar{x}\bar{y}\bar{z} + \bar{x}\bar{y}z$$

<u>Example 4</u>: Use K-maps to minimize these sum-of-products expansions:

a) $xy\bar{z} + x\bar{y}\bar{z} + \bar{x}yz + \bar{x}\bar{y}\bar{z}$

	yz	$y\bar{z}$	$\bar{y}\bar{z}$	$\bar{y}z$
x				
$\bar{x}$				

b) $x\bar{y}z + x\bar{y}\bar{z} + \bar{x}yz + \bar{x}\bar{y}z + \bar{x}\bar{y}\bar{z}$

	yz	$y\bar{z}$	$\bar{y}\bar{z}$	$\bar{y}z$
x				
$\bar{x}$				

Exercise 3: Use a K-map to minimize the sum-of-products expansion:

$$xyz + xy\overline{z} + x\overline{y}z + x\overline{y}\,\overline{z} + \overline{x}yz + \overline{x}\,\overline{y}z + \overline{x}\,\overline{y}\,\overline{z}$$

	yz	$y\overline{z}$	$\overline{y}\,\overline{z}$	$\overline{y}z$
x				
$\overline{x}$				

More Exercises

1. Draw the K-map to simplify the following expressions and draw the resulting simplified circuit.

a) $xyz + \overline{x}yz$

b) $xy\overline{z} + x\overline{y}\,\overline{z} + \overline{x}y\overline{z} + \overline{x}\,\overline{y}\,\overline{z}$

c) $xyz + xy\overline{z} + \overline{x}yz + \overline{x}y\overline{z}$

d) $xyz + x\overline{y}z + x\overline{y}\,\overline{z} + \overline{x}yz + \overline{x}y\overline{z} + \overline{x}\,\overline{y}\,\overline{z}$

Combinational Logic Circuit Design

You have learnt how to obtain the Boolean expressions from both logic circuits and truth tables. You've also learned how to draw logic circuits given a Boolean expression. Furthermore, you've leant how to create a sum of product and then simplify the Boolean expression using both Boolean algebra and K-maps. Next you will learn how to design combinatorial circuits for real world problems.

Combinational logic circuits design comprises the following steps

1. From the design specification, obtain the truth table
2. From the truth table, derive the Sum of Product Boolean Expression.
3. Use K-maps to minimize the Boolean expression. The simpler the Boolean expression, the less logic gates will be used.
4. Use logic gates to draw the simplified Boolean Expression.

Example

A bank wants to install an alarm system with movement sensors. The bank has 3 sensors labeled A,B, and C.

- A : Arm Alarm
- B : Motion Sensor
- C : Door Open Sensor

To prevent false alarms produced by a single sensor activation, the alarm will be triggered only when at least two sensors activate simultaneously. Design the circuit that will meet the following specifications.