

# Module 1 - Lesson 04: Boolean Logic & Algebra

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## Boolean Logic

- You start off with the idea that a single simple statement  $P$  is either true or false; it can't be anything in between (this is called the law of the excluded middle).
- Then you can form other statements, which are true or false, by combining these initial simple statements together using the fundamental operators And, OR and NOT to create compound statements, aka, Boolean expressions.

## Logical Operators

The following tables summarize the behaviour of each of the logical operators.

A	B	A AND B
0	0	0
0	1	0
1	0	0
1	1	1

A	B	A OR B
0	0	0
0	1	1
1	0	1
1	1	1

A	NOT A
0	1
1	0

A	B	A XOR B
0	0	0
0	1	1
1	0	1
1	1	0

Suppose simple statement  $A$  stands for “this apple is red” and simple statement  $B$  stands for “this apple is sweet”. The compound statement  $A \text{ AND } B$  will stand for “this apple is red and sweet”. This statement will only be TRUE if both simple statements were true separately. If we replace the AND with an OR to say,  $A \text{ OR } B$ , then the compound statement is TRUE if either one of the simple statements or both are true separately.

When combining two simple statements using an exclusive OR (XOR) operator, the output will only be TRUE if and only if one of the two simple statements is TRUE at once. In other words  $A \text{ XOR } B$  will result in TRUE if the simple statements have a different truth value.

## Implication Operator: $A \rightarrow B$

$A \rightarrow B$ : If  $A$  then  $B$

$A \rightarrow B$  is false only when  $A$  is true and  $B$  is false. In all other situations it is true.

Bi-conditional Operator:  $A \leftrightarrow B$

**$A \leftrightarrow B$ :  $A$  if and only if  $B$**

$A \leftrightarrow B$  is true when  $A$  and  $B$  are the same and false when  $A$  and  $B$  are different

### Order of Operations for logical operators

Similarly to the order of operation we learnt in grade school (BEDMAS), there is a specific order of operation for Boolean algebra. When evaluating a Boolean expression, we must consider the following order:

- Brackets
- NOT
- AND
- OR/XOR (from left to right)
- Implication
- Bi-conditional

### Symbols used to represent the logical operators

- NOT is denoted as a bar on top of the variable or a  $\sim$  in front of the variable
- AND is denoted as ' $*$ ' or ' $\bullet$ '.
- OR is denoted as ' $+$ '
- XOR is denoted as ' $\oplus$ '

### Examples

$$\begin{aligned}(A * B * C) + (A * \sim(B * C)), \text{ where } A = 1, B = 0, C = 1 \\(1 * 0 * 1) + (1 * \sim(0 * 1)) \\(0) + (1 * \sim(0)) \\0 + (1 * 1) \\0 + 1 \\1\end{aligned}$$

$$\begin{aligned}\sim(P) * Q \leftrightarrow (P + \sim Q), \text{ where } P = 1, Q = 1 \\ \sim(1) * 1 \leftrightarrow (1 + \sim 1) \\ 0 * 1 \leftrightarrow (1 + 0) \\ 0 \leftrightarrow 1 \\ 0\end{aligned}$$

## Exercises

Find the truth value of each of the expressions below, using the truth values shown.

a)  $\sim A + (B * C) * (\sim(A + C))$ , where  $A = 0, B = 1, C = 0$

b)  $A \oplus (B * C)$ , where  $A = 1, B = 0, C = 1$

c)  $\sim(P * Q) + (P + Q)$ , where  $P = 1, Q = 0$

d)  $\sim(P * Q) \oplus (P + Q)$ , where  $P = 1, Q = 0$

e)  $D * \sim(E * F) \rightarrow \sim(D * E) * F$ , where  $D = 0, E = 0, F = 1$

## Logic Definitions

- **Logical equivalence** – Two compound statements are logically equivalent if they produce the same Boolean logical output.
- **Tautology** – A compound statement that produces **true** output for all possible truth values of its compound statement variables.
- **Contradiction** – A compound statement that produces **false** output for all possible truth values of its compound statement variables.

## Boolean algebra

- Just as you would use algebraic laws/identities to expand and simplify algebraic expressions, you can use similar Boolean algebraic laws/identities to simplify Boolean expressions.
- Boolean expressions are typically a representation of electronic circuits or programmer's combination of logical operations. Boolean algebra is used to simplify these circuits and/or logic.
- All Boolean expression result in either a 'True' or 'False' value.

# Boolean Identities (Logically Equivalent Statements)

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Figure 1:

**T1: Commutative laws**

(a)  $A + B = B + A$

(b)  $A * B = B * A$

**T2: Associative laws**

(a)  $(A + B) + C = A + (B + C)$

(b)  $(A * B) * C = A * (B * C)$

**T3: Distributive laws**

(a)  $A * (B + C) = A * B + A * C$

(b)  $A + (B * C) = (A + B) * (A + C)$

**T4: Idempotent laws**

(a)  $A + A = A$

(b)  $A * A = A$

**T5: Absorption/Redundancy laws**

(a)  $A + (A * B) = A$

(b)  $A * (A + B) = A$

**T6: Identity laws**

(a)  $A + 0 = 0 + A = A$

(b)  $A * 1 = 1 * A = A$

**T7: Boundedness laws**

(a)  $A + 1 = 1 + A = 1$

(b)  $A * 0 = 0 * A = 0$

**T8: Complement laws**

(a)  $A + \sim A = \sim A + A = 1$

(b)  $A * \sim A = \sim A * A = 0$

**T9: Demorgan's Theorem**

(a)  $\sim(A + B) = \sim A * \sim B$

(b)  $\sim(A * B) = \sim A + \sim B$

**T10: Law of Double Negation**

$\sim(\sim A) = A$

**Example 1:**

Show that the left-hand-side (LHS) of the equation is logically equivalent to the right-hand-side (RHS) of the equation. Use the laws and identities in Figure 1 and label all your steps with the corresponding law.

$$A + (\sim A * B) = A + B$$

LHS:  $A + (\sim A * B)$

$(A + \sim A) * (A + B)$

$(1) * (A + B)$

$(A + B)$

T3b – Distributive Laws

T8a – Complement Law

T6b – Identity Law

RHS:  $(A + B)$

**Therefore: LHS = RHS**

### Example 2:

Show that the left-hand-side (LHS) of the equation is logically equivalent to the right-hand-side (RHS) of the equation. Use the laws and identities in Figure 1 and label all your steps with the corresponding law.

$$\sim(A + B) + \sim A * B = \sim A$$

LHS:	$(\sim A * \sim B) + (\sim A * B)$	T9a – Demorgan’s Law
	$\sim A * (\sim B + B)$	T3a – Distributive Law
	$\sim A * 1$	T8a – Complement Law
	$\sim A$	T6b – Identity Law

RHS:  $\sim A$

Therefore: LHS = RHS

### Exercises

Using the Identities and Laws of Boolean Algebra, simplify the following expressions. List the specific law used for every simplification.

- $P + (P + \bar{Q})$
- $A + B * \sim A$
- $A + (B + \sim A)$
- $\sim(A + \sim B) * A$
- $\sim(\sim B * A) + A$
- $A * B + A * (B + C)$
- $\sim((A * B) + (A * \sim B))$
- $A * \sim B * C + A * B * C + A * C$