

Assignment No.4

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Download all python codes from

<https://github.com/Vallidevibolla/Assignment-4/blob/main/code.py>

and latex-tikz codes from

<https://github.com/Vallidevibolla/Assignment-4/blob/main/main.tex>

Question taken from

https://github.com/gadepall/ncert/blob/main/linalg/quadratic_forms/gvv_ncert_quadratic_forms.pdf–Q.no.2.7

1 QUESTION 2.7

Find the area of the region bounded by curve

$$y = x^2 \quad (1.0.1)$$

and the lines $x=1, x=4$ and x -axis in the first quadrant.

2 SOLUTION

Secant: The points of intersection of the line

$$L : \mathbf{x} = \mathbf{q} + \mu \mathbf{m} \quad \mu \in \mathbb{R} \quad (2.0.1)$$

$$\mathbf{x}_i = \mathbf{q} + \mu_i \mathbf{m} \quad (2.0.2)$$

where

$$\mu_i = \frac{1}{\mathbf{m}^T \mathbf{V} \mathbf{m}} \left(-\mathbf{m}^T (\mathbf{V} \mathbf{q} + \mathbf{u}) \pm \sqrt{[\mathbf{m}^T (\mathbf{V} \mathbf{q} + \mathbf{u})]^2 - (\mathbf{q}^T \mathbf{V} \mathbf{q} + 2\mathbf{u}^T \mathbf{q} + f)(\mathbf{m}^T \mathbf{V} \mathbf{m})} \right) \quad (2.0.3)$$

$\therefore \mathbf{q}$ is the point of contact, \mathbf{q} satisfies parabola equation and

$$\mathbf{q}^T \mathbf{V} \mathbf{q} + 2\mathbf{u}^T \mathbf{q} + f = 0 \quad (2.0.4)$$

Given the point of contact \mathbf{q} , the equation of a tangent is

$$(\mathbf{V} \mathbf{q} + \mathbf{u})^T \mathbf{x} + \mathbf{u}^T \mathbf{q} + f = 0 \quad (2.0.5)$$

(1.0.1) can be written as

$$y^2 - x = 0 \quad (2.0.6)$$

(2.0.6) which has the form of

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \quad (2.0.7)$$

The matrix parameters are

$$\mathbf{V} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \mathbf{u} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \mathbf{f} = 0. \quad (2.0.8)$$

Thus, the given curve is a parabola. $\therefore \mathbf{V}$ is diagonal and in standard form, Also,

$$\mathbf{V} \mathbf{p} = \mathbf{0} \quad (2.0.9)$$

$$\Rightarrow \mathbf{p} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (2.0.10)$$

with eigen parameters

$$\lambda_1 = 0, \lambda_2 = 1 \quad (2.0.11)$$

Given $x=1$ Using (1.0.1) we get

$$\mathbf{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \mathbf{y} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (2.0.12)$$

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \mathbf{y} \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (2.0.13)$$

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = 1 \quad (2.0.14)$$

Similarly, $x=4$ we get

$$\begin{pmatrix} 4 & 0 \end{pmatrix} \mathbf{x} = 16 \quad (2.0.15)$$

The vertex of parabola can be expressed as

$$\begin{pmatrix} \mathbf{u}^T + \mathbf{n} \mathbf{p}^T \\ \mathbf{V} \end{pmatrix} \mathbf{c} = \begin{pmatrix} -f \\ \mathbf{n} \mathbf{p} - \mathbf{u} \end{pmatrix} \quad (2.0.16)$$

$$\begin{pmatrix} 1 & \frac{-1}{2} \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{c} = \begin{pmatrix} 1 \\ 0 \\ \frac{-1}{2} \end{pmatrix} \quad (2.0.17)$$

$$\Rightarrow \begin{pmatrix} 1 & \frac{-1}{2} \\ 0 & 1 \end{pmatrix} \mathbf{c} = \begin{pmatrix} 1 \\ \frac{-1}{2} \end{pmatrix} \quad (2.0.18)$$

$$\text{or, } \mathbf{c} = \begin{pmatrix} \frac{3}{2} \\ \frac{-1}{2} \end{pmatrix} \quad (2.0.19)$$

The direction vector and normal vectors are

$$\mathbf{m} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}, \mathbf{n} = \begin{pmatrix} 1 \\ -5 \end{pmatrix}. \quad (2.0.20)$$

$$\kappa = \frac{\mathbf{p}_1^T \mathbf{u}}{\mathbf{p}_1^T \mathbf{n}}, \quad (2.0.21)$$

From (2.0.21), (2.0.20) and (2.0.10),

$$\kappa = 0 \quad (2.0.22)$$

which, upon substitution in (2.0.23)

$$\begin{pmatrix} \mathbf{u} + \kappa \mathbf{n}^T \\ \mathbf{v} \end{pmatrix} \mathbf{q} = \begin{pmatrix} -f \\ \kappa \mathbf{n} - \mathbf{u} \end{pmatrix} \quad (2.0.23)$$

and simplification yields the matrix equation

$$\begin{pmatrix} 0 & \frac{-1}{2} \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{q} = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \end{pmatrix} \quad (2.0.24)$$

$$\Rightarrow \begin{pmatrix} 0 & \frac{-1}{2} \\ 0 & 1 \end{pmatrix} \mathbf{q} = \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix} \quad (2.0.25)$$

$$\text{or, } \mathbf{q} = \begin{pmatrix} \frac{-1}{4} \\ \frac{1}{2} \end{pmatrix} \quad (2.0.26)$$

From (2.0.3) we get

$$\mu_1 = \frac{1}{2}, \mu_2 = \frac{-1}{2}$$

The lines (2.0.14), (2.0.15) can be written in parametric form in (2.0.2) we get

$$\mathbf{x}_{i1} = \begin{pmatrix} \frac{-1}{4} \\ \frac{1}{2} \end{pmatrix} + \mu_i \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (2.0.27)$$

$$\mathbf{x}_{i2} = \begin{pmatrix} \frac{-1}{4} \\ \frac{1}{2} \end{pmatrix} + \mu_i \begin{pmatrix} 0 \\ 4 \end{pmatrix} \quad (2.0.28)$$

Substituting μ_1 and μ_2 values in (2.0.27), (2.0.28) we get

$$\mathbf{K}_1 = \begin{pmatrix} \frac{-1}{4} \\ 1 \end{pmatrix}, \mathbf{L}_1 = \begin{pmatrix} \frac{-1}{4} \\ 0 \end{pmatrix} \quad (2.0.29)$$

$$\mathbf{K}_2 = \begin{pmatrix} \frac{-1}{4} \\ \frac{5}{2} \end{pmatrix}, \mathbf{L}_2 = \begin{pmatrix} \frac{-1}{4} \\ \frac{-3}{2} \end{pmatrix} \quad (2.0.30)$$

The area enclosed by parabola and line can be given as

$A = \text{Area under line} - \text{Area under curve}$

$$\Rightarrow \boxed{A = A_1 - A_3}$$

Thus area under the lines (2.0.14), (2.0.15) i.e.,

$$A_1 = \frac{1}{2}$$

Area under the parabola (1.0.1) i.e.,

$$A_3 = \int \sqrt{x} dx$$

$$\boxed{A_3 = \frac{14}{3}}$$

Putting these values in (2) we get

$$A = -4.17, -13.17 \text{ units}$$

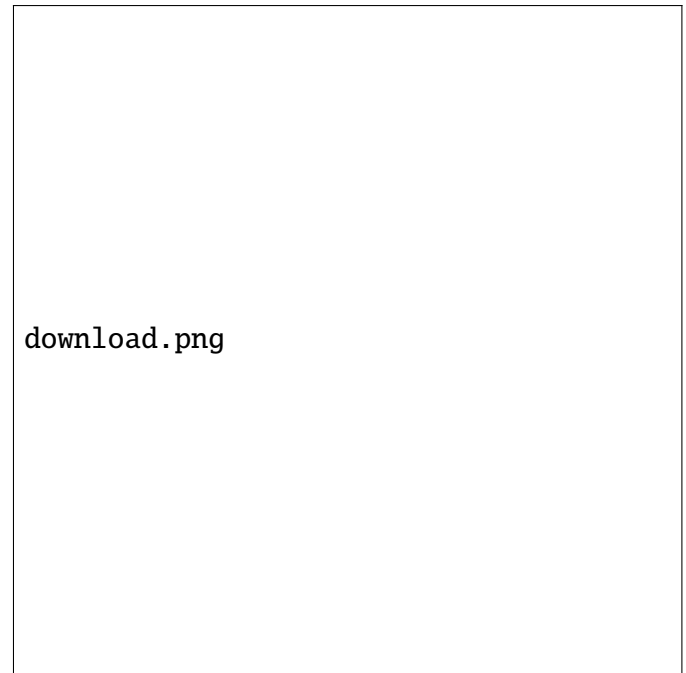


Fig. 2.1: Tangent to parabola in (1.0.1) with slope $\frac{1}{5}$.