Assignment No.5

Valli Devi Bolla

Download all python codes from

https://github.com/Vallidevibolla/Assignment−4/blob/main/code.py

and latex-tikz codes from

https://github.com/Vallidevibolla/Assignment−4/blob/main/main.tex

Question taken from

https://github.com/gadepall/ncert/blob/main/linalg/quadratic_forms/gvv_ncert_quadratic_forms.pdf−Q.no.2.7

1 Question 2.7

Find the area of the region bounded by curve

$$y = x^2 \tag{1.0.1}$$

and the lines x=1,x=4 and x-axis in the first quadrant.

2 solution

Secant: The points of intersection of the line

$$L: \quad \mathbf{x} = \mathbf{q} + \mu\mathbf{m} \quad \mu \in \mathbb{R} \tag{2.0.1}$$

$$\mathbf{x}_i = \mathbf{q} + \mu_i\mathbf{m} \tag{2.0.2}$$

where

$$\mu_i = \frac{1}{\mathbf{m}^T\mathbf{V}\mathbf{m}}\left(-\mathbf{m}^T\left(\mathbf{V}\mathbf{q}+\mathbf{u}\right) \pm \sqrt{\left[\mathbf{m}^T\left(\mathbf{V}\mathbf{q}+\mathbf{u}\right)\right]^2 - \left(\mathbf{q}^T\mathbf{V}\mathbf{q}+2\mathbf{u}^T\mathbf{q}+f\right)\left(\mathbf{m}^T\mathbf{V}\mathbf{m}\right)}\right) \tag{2.0.3}$$

$\because$ $\mathbf{q}$ is the point of contact, $\mathbf{q}$ satisfies parabola equation and

$$\mathbf{q}^T\mathbf{V}\mathbf{q} + 2\mathbf{u}^T\mathbf{q} + f = 0 \tag{2.0.4}$$

Given the point of contact $\mathbf{q}$, the equation of a tangent is

$$\left(\mathbf{V}\mathbf{q}+\mathbf{u}\right)^T\mathbf{x} + \mathbf{u}^T\mathbf{q} + f = 0 \tag{2.0.5}$$

(1.0.1) can be written as

$$y^2 - x = 0 \tag{2.0.6}$$

(2.0.6)which has the form of

$$ax^2 + 2bxy + cy^2 + 2dy + 2ex + f = 0 \tag{2.0.7}$$

The matrix parameters are

$$\mathbf{V} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \mathbf{u} = \begin{pmatrix} \frac{-1}{2} \\ 0 \end{pmatrix}, \mathbf{f} = 0. \tag{2.0.8}$$

Thus, the given curve is a parabola. $\because$ $\mathbf{V}$ is diagonal and in standard form, Also,

$$\mathbf{V}\mathbf{p} = \mathbf{0} \tag{2.0.9}$$

$$\implies \mathbf{p} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{2.0.10}$$

with eigen parameters

$$\lambda_1 = 0, \lambda_2 = 1 \tag{2.0.11}$$

Given x=1 Using (1.0.1) we get

$$\mathbf{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \mathbf{y}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2.0.12}$$

$$\begin{pmatrix} 1 & 0 \end{pmatrix}\mathbf{x} = \begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \mathbf{y}\begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2.0.13}$$

$$\begin{pmatrix} 1 & 0 \end{pmatrix}\mathbf{x} = 1 \tag{2.0.14}$$

Similarly, x=4 we get

$$\begin{pmatrix} 4 & 0 \end{pmatrix}\mathbf{x} = 16 \tag{2.0.15}$$

The vertex of parabola can be expressed using (2.0.14)

$$\begin{pmatrix} \mathbf{u^T} + \mathbf{n}\mathbf{p}^T \\ \mathbf{V} \end{pmatrix}\mathbf{c} = \begin{pmatrix} -f \\ \mathbf{n}\mathbf{p} - \mathbf{u} \end{pmatrix} \tag{2.0.16}$$

$$\begin{pmatrix}1 & \frac{1}{2}\\ 0 & 0\\ 0 & 0\end{pmatrix}\mathbf{c} = \begin{pmatrix}-1\\ 0\\ \frac{-1}{2}\end{pmatrix} \tag{2.0.17}$$

$$\implies \begin{pmatrix}1 & \frac{1}{2}\\ 0 & 0\end{pmatrix}\mathbf{c} = \begin{pmatrix}-1\\ \frac{-1}{2}\end{pmatrix} \tag{2.0.18}$$

$$\text{or, } \mathbf{c} = \begin{pmatrix}\frac{-7}{2}\\ 5\end{pmatrix} \tag{2.0.19}$$

The direction vector and normal vectors are

$$\mathbf{m} = \begin{pmatrix}5\\ 1\end{pmatrix}, \mathbf{n} = \begin{pmatrix}1\\ -5\end{pmatrix}. \tag{2.0.20}$$

$$\kappa = \frac{\mathbf{p}_1^T\mathbf{u}}{\mathbf{p}_1^T\mathbf{n}}, \tag{2.0.21}$$

From (2.0.21), (2.0.20) and (2.0.10),

$$\kappa = 0 \tag{2.0.22}$$

which, upon substitution in (2.0.23)

$$\begin{pmatrix}\mathbf{u} + \kappa\mathbf{n}^T\\ \mathbf{V}\end{pmatrix}\mathbf{q} = \begin{pmatrix}-f\\ \kappa\mathbf{n} - \mathbf{u}\end{pmatrix} \tag{2.0.23}$$

and simplification yields the matrix equation

$$\begin{pmatrix}0 & \frac{-1}{2}\\ 0 & 0\\ 0 & 1\end{pmatrix}\mathbf{q} = \begin{pmatrix}0\\ 0\\ \frac{1}{2}\end{pmatrix} \tag{2.0.24}$$

$$\implies \begin{pmatrix}0 & \frac{-1}{2}\\ 0 & 1\end{pmatrix}\mathbf{q} = \begin{pmatrix}0\\ \frac{1}{2}\end{pmatrix} \tag{2.0.25}$$

$$\text{or, } \mathbf{q} = \begin{pmatrix}\frac{-1}{4}\\ \frac{1}{2}\end{pmatrix} \tag{2.0.26}$$

From (2.0.3) we get
$\mu_1 = \frac{1}{2}, \mu_2 = \frac{-1}{2}$

The lines (2.0.14), (2.0.15) can be written in parametric form in (2.0.2) we get

$$\mathbf{x}_{i1} = \begin{pmatrix}\frac{-1}{4}\\ \frac{1}{2}\end{pmatrix} + \mu_i\begin{pmatrix}0\\ 1\end{pmatrix} \tag{2.0.27}$$

$$\mathbf{x}_{i2} = \begin{pmatrix}\frac{-1}{4}\\ \frac{1}{2}\end{pmatrix} + \mu_i\begin{pmatrix}0\\ 4\end{pmatrix} \tag{2.0.28}$$

Substituting $\mu_1 and \mu_2 values in (2.0.27), (2.0.28) we get$

$$\mathbf{K_1} = \begin{pmatrix}\frac{-1}{4}\\ 1\end{pmatrix}, \mathbf{L_1} = \begin{pmatrix}\frac{-1}{4}\\ 0\end{pmatrix} \tag{2.0.29}$$

$$\mathbf{K_2} = \begin{pmatrix}\frac{-1}{4}\\ \frac{5}{2}\end{pmatrix}, \mathbf{L_2} = \begin{pmatrix}\frac{-1}{4}\\ \frac{-3}{2}\end{pmatrix} \tag{2.0.30}$$

The area enclosed by parabola and line can be given as
A = Area under line - Area under curve
$\implies \boxed{\mathbf{A} = \mathbf{A_1} - \mathbf{A_3}}$
$Thus area under the lines (2.0.14), (2.0.15) i.e,$
$\mathbf{A_1} = \frac{1}{2}$

Area under the parabola(1.0.1) i.e,
$\int A_3 = \sqrt{x}dx$
$\boxed{\mathbf{A_3} = \frac{14}{3}}$
$Putting these values in (2) we get$
$\mathbf{A} = -4.17, -13.17 units$

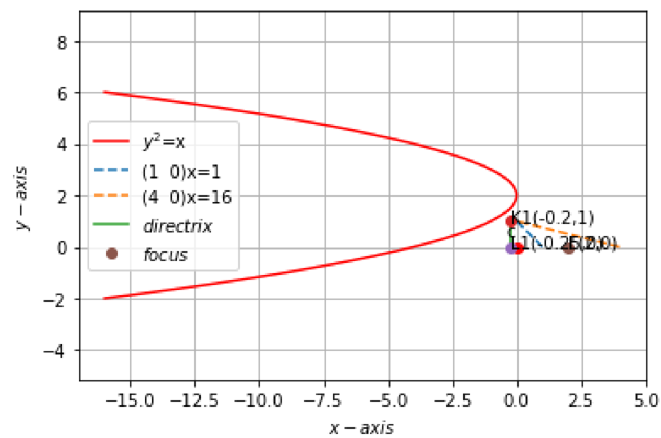

Fig. 2.1: Tangent to parabola in (1.0.1) with slope $\frac{1}{5}$.