Assignment No.5

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Download all python codes from

https://github.com/Vallidevibolla/Assignment−4/blob/main/code.py

and latex-tikz codes from

https://github.com/Vallidevibolla/Assignment−4/blob/main/main.tex

Question taken from

https://github.com/gadepall/ncert/blob/main/linalg/quadratic_forms/gvv_ncert_quadratic_forms.pdf−Q.no.2.7

1 Question 2.7

Find the area of the region bounded by curve

$$\mathbf{y}^2 = \mathbf{x} \tag{1.0.1}$$

and the lines x=1,x=4 and x-axis in the first quadrant.

2 Solution

(1.0.1) can be written as

$$\mathbf{y}^2 - \mathbf{x} = 0 \tag{2.0.1}$$

(2.0.1)which has the form of

$$\mathbf{ax}^2 + 2\mathbf{bxy} + \mathbf{cy}^2 + 2\mathbf{dy} + 2\mathbf{ex} + \mathbf{f} = 0 \tag{2.0.2}$$

The matrix parameters are

$$\mathbf{V} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \mathbf{u} = \begin{pmatrix} \frac{-1}{2} \\ 0 \end{pmatrix}, \mathbf{f} = 0. \tag{2.0.3}$$

Thus, the given curve is a parabola. $\because$ $\mathbf{V}$ is diagonal and in standard form, Also,

$$\mathbf{Vp} = \mathbf{0} \tag{2.0.4}$$

$$\implies \mathbf{p} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{2.0.5}$$

with eigen parameters

$$\lambda_1 = 0, \lambda_2 = 1 \tag{2.0.6}$$

Given x=1 Using (1.0.1) we get

$$\mathbf{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \mathbf{y} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2.0.7}$$

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \mathbf{y} \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2.0.8}$$

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = 1 \tag{2.0.9}$$

Similarly, x=4 we get

$$\begin{pmatrix} 4 & 0 \end{pmatrix} \mathbf{x} = 16 \tag{2.0.10}$$

The vertex of parabola can be expressed using $\begin{pmatrix} 1 & 0 \end{pmatrix}$x=4

$$\begin{pmatrix} \mathbf{u^T} + \mathbf{np}^T \\ \mathbf{V} \end{pmatrix} \mathbf{c} = \begin{pmatrix} -f \\ \mathbf{np} - \mathbf{u} \end{pmatrix} \tag{2.0.11}$$

$$\begin{pmatrix} 1 & \frac{1}{2} \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{c} = \begin{pmatrix} -4 \\ 0 \\ \frac{-1}{2} \end{pmatrix} \tag{2.0.12}$$

$$\implies \begin{pmatrix} 1 & \frac{1}{2} \\ 0 & 0 \end{pmatrix} \mathbf{c} = \begin{pmatrix} -4 \\ \frac{-1}{2} \end{pmatrix} \tag{2.0.13}$$

$$\text{or, } \mathbf{c} = \begin{pmatrix} \frac{-13}{2} \\ 5 \end{pmatrix} \tag{2.0.14}$$

The direction vector and normal vectors are

$$\mathbf{m} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}, \mathbf{n} = \begin{pmatrix} 1 \\ -5 \end{pmatrix}. \tag{2.0.15}$$

$$\kappa = \frac{\mathbf{p}_1^T \mathbf{u}}{\mathbf{p}_1^T \mathbf{n}}, \tag{2.0.16}$$

From (2.0.16), (2.0.15) and (2.0.5),

$$\kappa = 0 \tag{2.0.17}$$

which, upon substitution in (2.0.18)

$$\begin{pmatrix} \mathbf{u} + \kappa \mathbf{n}^T \\ \mathbf{V} \end{pmatrix} \mathbf{q} = \begin{pmatrix} -f \\ \kappa \mathbf{n} - \mathbf{u} \end{pmatrix} \tag{2.0.18}$$

and simplification yields the matrix equation

$$\begin{pmatrix} 0 & \frac{-1}{2} \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{q} = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \end{pmatrix} \tag{2.0.19}$$

$$\implies \begin{pmatrix} 0 & \frac{-1}{2} \\ 0 & 1 \end{pmatrix} \mathbf{q} = \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix} \tag{2.0.20}$$

$$\text{or, } \mathbf{q} = \begin{pmatrix} \frac{-1}{4} \\ \frac{1}{2} \end{pmatrix} \tag{2.0.21}$$

Secant: The points of intersection of the line

$$L: \quad \mathbf{x} = \mathbf{q} + \mu \mathbf{m} \quad \mu \in \mathbb{R} \tag{2.0.22}$$

$$\mathbf{x}_i = \mathbf{q} + \mu_i \mathbf{m} \tag{2.0.23}$$

where

$$\mu_i = \frac{1}{\mathbf{m}^T \mathbf{V} \mathbf{m}} \left(-\mathbf{m}^T (\mathbf{V}\mathbf{q} + \mathbf{u}) \pm \sqrt{\left[\mathbf{m}^T (\mathbf{V}\mathbf{q} + \mathbf{u})\right]^2 - \left(\mathbf{q}^T \mathbf{V} \mathbf{q} + 2\mathbf{u}^T \mathbf{q} + f\right) \left(\mathbf{m}^T \mathbf{V} \mathbf{m}\right)} \right) \tag{2.0.24}$$

$\because$ $\mathbf{q}$ is the point of contact, $\mathbf{q}$ satisfies parabola equation and

$$\mathbf{q}^T \mathbf{V} \mathbf{q} + 2\mathbf{u}^T \mathbf{q} + f = 0 \tag{2.0.25}$$

Given the point of contact $\mathbf{q}$, the equation of a tangent is

$$(\mathbf{V}\mathbf{q} + \mathbf{u})^T \mathbf{x} + \mathbf{u}^T \mathbf{q} + f = 0 \tag{2.0.26}$$

From (2.0.24) we get
$\mu_1 = \frac{1}{2}, \mu_2 = \frac{-1}{2}$

The lines (2.0.9), (2.0.10) can be written in parametric form in (2.0.23) we get

$$\mathbf{x}_{i1} = \begin{pmatrix} \frac{-1}{4} \\ \frac{1}{2} \end{pmatrix} + \mu_i \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2.0.27}$$

$$\mathbf{x}_{i2} = \begin{pmatrix} \frac{-1}{4} \\ \frac{1}{2} \end{pmatrix} + \mu_i \begin{pmatrix} 0 \\ 4 \end{pmatrix} \tag{2.0.28}$$

Substituting $\mu_1 and \mu_2 values in (2.0.27), (2.0.28) we get$

$$\mathbf{K_1} = \begin{pmatrix} \frac{-1}{4} \\ 1 \end{pmatrix}, \mathbf{L_1} = \begin{pmatrix} \frac{-1}{4} \\ 0 \end{pmatrix} \tag{2.0.29}$$

$$\mathbf{K_2} = \begin{pmatrix} \frac{-1}{4} \\ \frac{5}{2} \end{pmatrix}, \mathbf{L_2} = \begin{pmatrix} \frac{-1}{4} \\ \frac{-3}{2} \end{pmatrix} \tag{2.0.30}$$

The area enclosed by parabola and line can be given as
A = Area under line - Area under curve
$\implies \boxed{\mathbf{A} = \mathbf{A_1} - \mathbf{A_3}}$
$Thus area under the lines (2.0.9), (2.0.10) i.e,$
$\mathbf{A_1} = \frac{1}{2}$

Area under the parabola(1.0.1) i.e,
$\int \mathrm{A}_3 = \sqrt{\mathbf{x}} dx$
$\boxed{\mathbf{A_3} = \frac{14}{3}}$
$Putting these values in (2) we get$
$\mathbf{A} = -4.17, -13.17 units$

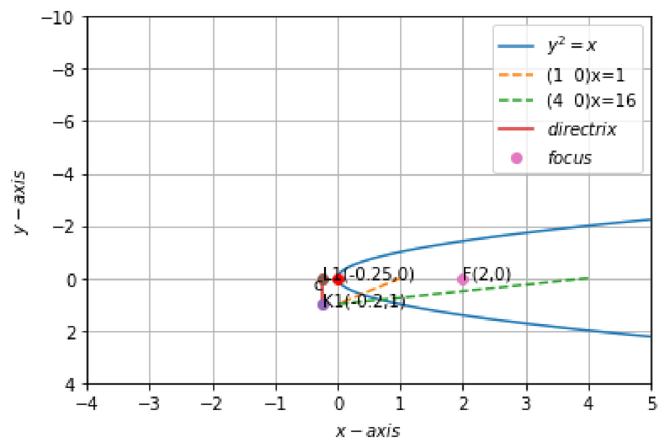

Fig. 2.1: Tangent to parabola in (1.0.1) with slope $\frac{1}{5}$.