Challenge Problem 3

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Download all python codes from

```
https://github.com/ka−raja−babu/Matrix−Theory/
    tree/main/ChallengeProblem3/Codes
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and latex-tikz codes from

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https://github.com/ka−raja−babu/Matrix−Theory/
    tree/main/ChallengeProblem3
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1 Challenge Question 3

For the quadratic equation to not have any real roots, the y coordinate should always be either positive or negative . Express this in terms of the matrix/vector parameters of the parabola .

2 Solution

The general form of a quadratic equation is

$$y = ax^2 + bx + c \tag{2.0.1}$$

$$\implies ax^2 + bx + c - y = 0 \tag{2.0.2}$$

which can be written in vector form as

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{2.0.3}$$

where

$$\mathbf{V} = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \tag{2.0.4}$$

$$\mathbf{u} = \begin{pmatrix} \frac{b}{2} \\ \frac{-1}{2} \end{pmatrix} \tag{2.0.5}$$

$$f = c \tag{2.0.6}$$

Using eigenvalue decomposition,

$$\mathbf{D} = \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix} \tag{2.0.7}$$

$$\mathbf{P} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{2.0.8}$$

Also,

$$\eta = \mathbf{u}^T\mathbf{p_1} \tag{2.0.9}$$

$$= \frac{-1}{2} \tag{2.0.10}$$

Now,for y coordinate to be always positive,two conditions need to be satisfied:

1) y-coordinate of vertex **c** of parabola needs to be always positive.
2) Coefficient a of x^2 needs to be always positive.

∴ For condition 1:

$$\begin{pmatrix} \mathbf{u}^T + \eta\mathbf{p_1}^T \\ \mathbf{V} \end{pmatrix}\mathbf{c} = \begin{pmatrix} -f \\ \eta\mathbf{p_1} - \mathbf{u} \end{pmatrix} \tag{2.0.11}$$

$$\mathbf{p_1}^T\mathbf{c} > 0 \tag{2.0.12}$$

For condition 2:

$$\mathbf{p_2}^T\mathbf{V}\mathbf{p_2} > 0 \tag{2.0.13}$$

And,for y coordinate to be always negative,two conditions need to be satisfied:

1) y-coordinate of vertex **c** of parabola needs to be always negative.
2) Coefficient a of x^2 needs to be always negative.

∴ For condition 1:

$$\begin{pmatrix} \mathbf{u}^T + \eta\mathbf{p_1}^T \\ -\mathbf{V} \end{pmatrix}\mathbf{c} = \begin{pmatrix} -f \\ \eta\mathbf{p_1} - \mathbf{u} \end{pmatrix} \tag{2.0.14}$$

$$\mathbf{p_1}^T\mathbf{c} < 0 \tag{2.0.15}$$

For condition 2:

$$\mathbf{p_2}^T\mathbf{V}\mathbf{p_2} < 0 \tag{2.0.16}$$

Hence,condition for non-real roots in terms of matrix/vector parameters of the parabola is

$$\boxed{\begin{pmatrix} \mathbf{p_1}^T\mathbf{c} \\ \mathbf{p_2}^T\mathbf{V}\mathbf{p_2} \end{pmatrix} \succ 0} \tag{2.0.17}$$

or

$$\boxed{\begin{pmatrix} \mathbf{p_1}^T\mathbf{c} \\ \mathbf{p_2}^T\mathbf{V}\mathbf{p_2} \end{pmatrix} \prec 0} \tag{2.0.18}$$

3 Examples

1)

$$y = 21x^2 - 28x + 10 \tag{3.0.1}$$

Here,

$$\mathbf{V} = \begin{pmatrix} 21 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{u} = -\begin{pmatrix} 14 \\ \frac{1}{2} \end{pmatrix}, f = 10 \tag{3.0.2}$$

Using eigenvalue decomposition,

$$\mathbf{D} = \begin{pmatrix} 0 & 0 \\ 0 & 21 \end{pmatrix}, \mathbf{P} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{3.0.3}$$

$\therefore$Vertex $\mathbf{c}$ is given by

$$\begin{pmatrix} -14 & -1 \\ 21 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{c} = \begin{pmatrix} -10 \\ 14 \\ 0 \end{pmatrix} \tag{3.0.4}$$

$$\implies \begin{pmatrix} -14 & -1 \\ 21 & 0 \end{pmatrix} \mathbf{c} = \begin{pmatrix} -10 \\ 14 \end{pmatrix} \tag{3.0.5}$$

$$\implies \mathbf{c} = \begin{pmatrix} \frac{2}{3} \\ \frac{2}{3} \end{pmatrix} \tag{3.0.6}$$

Now,

$$\mathbf{p_1}^T \mathbf{c} = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{2}{3} \\ \frac{2}{3} \end{pmatrix} \tag{3.0.7}$$

$$= \frac{2}{3} \tag{3.0.8}$$

and,

$$\mathbf{p_2}^T \mathbf{V} \mathbf{p_2} = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 21 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{3.0.9}$$

$$= 21 \tag{3.0.10}$$

$\because$

$$\begin{pmatrix} \mathbf{p_1}^T \mathbf{c} \\ \mathbf{p_2}^T \mathbf{V} \mathbf{p_2} \end{pmatrix} = \begin{pmatrix} \frac{2}{3} \\ 21 \end{pmatrix} \succ 0 \tag{3.0.11}$$

Hence,the given equation does not have any real roots.

2)

$$y = 6x^2 - x - 2 \tag{3.0.12}$$

Here,

$$\mathbf{V} = \begin{pmatrix} 6 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{u} = -\begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}, f = -2 \tag{3.0.13}$$

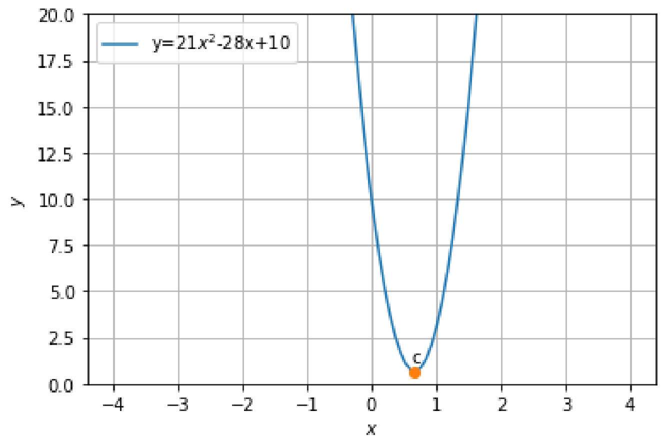

Fig. 3.1: $y = 21x^2 - 28x + 10$

Using eigenvalue decomposition,

$$\mathbf{D} = \begin{pmatrix} 0 & 0 \\ 0 & 6 \end{pmatrix}, \mathbf{P} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{3.0.14}$$

$\therefore$Vertex $\mathbf{c}$ is given by

$$\begin{pmatrix} \frac{-1}{2} & -1 \\ 6 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{c} = \begin{pmatrix} 2 \\ \frac{1}{2} \\ 0 \end{pmatrix} \tag{3.0.15}$$

$$\implies \begin{pmatrix} \frac{-1}{2} & -1 \\ 6 & 0 \end{pmatrix} \mathbf{c} = \begin{pmatrix} 2 \\ \frac{1}{2} \end{pmatrix} \tag{3.0.16}$$

$$\implies \mathbf{c} = \begin{pmatrix} \frac{1}{12} \\ \frac{-49}{24} \end{pmatrix} \tag{3.0.17}$$

Now,

$$\mathbf{p_1}^T \mathbf{c} = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{12} \\ \frac{-49}{24} \end{pmatrix} \tag{3.0.18}$$

$$= \frac{-49}{24} \tag{3.0.19}$$

and,

$$\mathbf{p_2}^T \mathbf{V} \mathbf{p_2} = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 6 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{3.0.20}$$

$$= 6 \tag{3.0.21}$$

$\because$

$$\begin{pmatrix} \mathbf{p_1}^T \mathbf{c} \\ \mathbf{p_2}^T \mathbf{V} \mathbf{p_2} \end{pmatrix} = \begin{pmatrix} \frac{-49}{24} \\ 6 \end{pmatrix} \not\succ 0 \tag{3.0.22}$$

Hence,the given equation has real roots.

3)

$$y = -x^2 - 4x - 5 \tag{3.0.23}$$

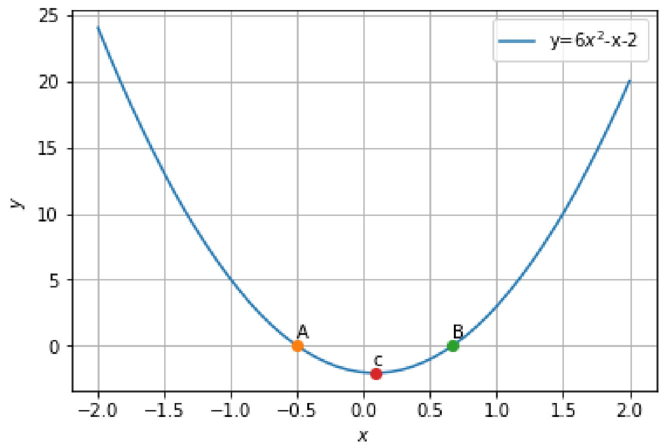

Fig. 3.2: $y = 6x^2 - x - 2$

Here,

$$\mathbf{V} = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{u} = -\begin{pmatrix} 2 \\ \frac{1}{2} \end{pmatrix}, f = -5 \tag{3.0.24}$$

Using eigenvalue decomposition,

$$\mathbf{D} = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}, \mathbf{P} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{3.0.25}$$

$\therefore$Vertex $\mathbf{c}$ is given by

$$\begin{pmatrix} -2 & -1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{c} = \begin{pmatrix} 5 \\ 2 \\ 0 \end{pmatrix} \tag{3.0.26}$$

$$\implies \begin{pmatrix} -2 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{c} = \begin{pmatrix} 5 \\ 2 \end{pmatrix} \tag{3.0.27}$$

$$\implies \mathbf{c} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \tag{3.0.28}$$

Now,

$$\mathbf{p_1}^T \mathbf{c} = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix} \tag{3.0.29}$$

$$= -1 \tag{3.0.30}$$

and,

$$\mathbf{p_2}^T \mathbf{V} \mathbf{p_2} = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{3.0.31}$$

$$= -1 \tag{3.0.32}$$

$\because$

$$\begin{pmatrix} \mathbf{p_1}^T \mathbf{c} \\ \mathbf{p_2}^T \mathbf{V} \mathbf{p_2} \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix} < 0 \tag{3.0.33}$$

Hence,the given equation does not have real roots.

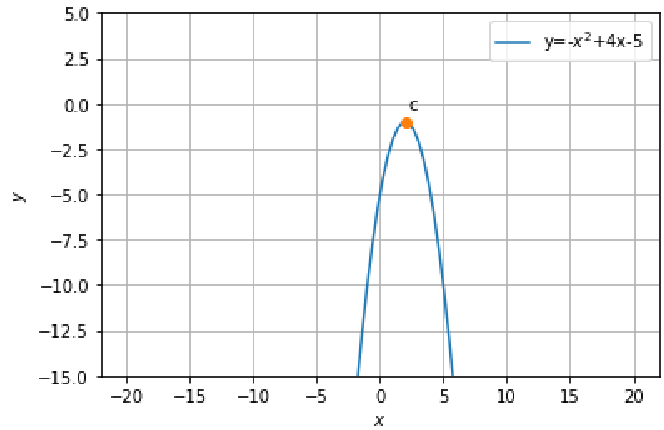

Fig. 3.3: $y = -x^2 + 4x - 5$

4)

$$y = -x^2 - 4x + 9 \tag{3.0.34}$$

Here,

$$\mathbf{V} = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{u} = -\begin{pmatrix} 2 \\ \frac{1}{2} \end{pmatrix}, f = 9 \tag{3.0.35}$$

Using eigenvalue decomposition,

$$\mathbf{D} = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}, \mathbf{P} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{3.0.36}$$

$\therefore$Vertex $\mathbf{c}$ is given by

$$\begin{pmatrix} -2 & -1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{c} = \begin{pmatrix} -9 \\ 2 \\ 0 \end{pmatrix} \tag{3.0.37}$$

$$\implies \begin{pmatrix} -2 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{c} = \begin{pmatrix} -9 \\ 2 \end{pmatrix} \tag{3.0.38}$$

$$\implies \mathbf{c} = \begin{pmatrix} 2 \\ 13 \end{pmatrix} \tag{3.0.39}$$

Now,

$$\mathbf{p_1}^T \mathbf{c} = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 13 \end{pmatrix} \tag{3.0.40}$$

$$= 13 \tag{3.0.41}$$

and,

$$\mathbf{p_2}^T \mathbf{V} \mathbf{p_2} = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{3.0.42}$$

$$= -1 \tag{3.0.43}$$

$\because$

$$\begin{pmatrix} \mathbf{p_1}^T\mathbf{c} \\ \mathbf{p_2}^T\mathbf{V}\mathbf{p_2} \end{pmatrix} = \begin{pmatrix} 13 \\ -1 \end{pmatrix} \nless 0 \tag{3.0.44}$$

Hence,the given equation has real roots.

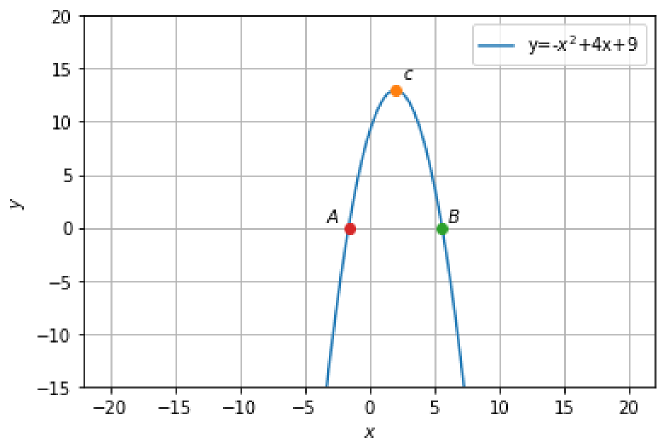

Fig. 3.4: $y = -x^2 + 4x + 9$