

Challenge Problem 3

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Download all python codes from

<https://github.com/ka-raja-babu/Matrix-Theory/tree/main/ChallengeProblem3/Codes>

and latex-tikz codes from

<https://github.com/ka-raja-babu/Matrix-Theory/tree/main/ChallengeProblem3>

1 CHALLENGE QUESTION 3

For the quadratic equation to not have any real roots, the y coordinate should always be positive. Express this in terms of the matrix/vector parameters of the parabola.

2 SOLUTION

The general form of a quadratic equation is

$$y = ax^2 + bx + c \quad (2.0.1)$$

$$\implies ax^2 + bx + c - y = 0 \quad (2.0.2)$$

which can be written in vector form as

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (2.0.3) \quad 1)$$

where

$$\mathbf{V} = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \quad (2.0.4)$$

$$\mathbf{u} = \begin{pmatrix} \frac{b}{2} \\ -\frac{1}{2} \end{pmatrix} \quad (2.0.5)$$

$$f = c \quad (2.0.6)$$

Using eigenvalue decomposition,

$$\mathbf{D} = \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix} \quad (2.0.7)$$

$$\mathbf{P} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (2.0.8)$$

Also,

$$\eta = \mathbf{u}^T \mathbf{p}_1 \quad (2.0.9)$$

$$= \frac{-1}{2} \quad (2.0.10)$$

Now, for y coordinate to be always positive, two conditions need to be satisfied:

- 1) y-coordinate of vertex \mathbf{c} of parabola needs to be always positive.
- 2) Coefficient a of x^2 needs to be always positive.

\therefore For condition 1:

$$\begin{pmatrix} \mathbf{u}^T + \eta \mathbf{p}_1^T \\ \mathbf{V} \end{pmatrix} \mathbf{c} = \begin{pmatrix} -f \\ \eta \mathbf{p}_1 - \mathbf{u} \end{pmatrix} \quad (2.0.11)$$

$$\mathbf{p}_1^T \mathbf{c} > 0 \quad (2.0.12)$$

For condition 2:

$$\mathbf{p}_2^T \mathbf{V} \mathbf{p}_2 > 0 \quad (2.0.13)$$

Hence, condition for non-real roots in terms of matrix/vector parameters of the parabola is

$$\begin{pmatrix} \mathbf{p}_1^T \mathbf{c} \\ \mathbf{p}_2^T \mathbf{V} \mathbf{p}_2 \end{pmatrix} > 0 \quad (2.0.14)$$

3 EXAMPLES

$$y = 21x^2 - 28x + 10 \quad (3.0.1)$$

Here,

$$\mathbf{V} = \begin{pmatrix} 21 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{u} = -\begin{pmatrix} 14 \\ \frac{1}{2} \end{pmatrix}, f = 10 \quad (3.0.2)$$

Using eigenvalue decomposition,

$$\mathbf{D} = \begin{pmatrix} 0 & 0 \\ 0 & 21 \end{pmatrix}, \mathbf{P} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (3.0.3)$$

\therefore Vertex \mathbf{c} is given by

$$\begin{pmatrix} -14 & -1 \\ 21 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{c} = \begin{pmatrix} -10 \\ 14 \\ 0 \end{pmatrix} \quad (3.0.4)$$

$$\implies \begin{pmatrix} -14 & -1 \\ 21 & 0 \end{pmatrix} \mathbf{c} = \begin{pmatrix} -10 \\ 14 \end{pmatrix} \quad (3.0.5)$$

$$\implies \mathbf{c} = \begin{pmatrix} \frac{2}{3} \\ \frac{2}{3} \end{pmatrix} \quad (3.0.6)$$

Now,

$$\mathbf{p}_1^T \mathbf{c} = (0 \ 1) \begin{pmatrix} 2 \\ 3 \end{pmatrix} \quad (3.0.7)$$

$$= \frac{2}{3} \quad (3.0.8)$$

and,

$$\mathbf{p}_2^T \mathbf{V} \mathbf{p}_2 = (1 \ 0) \begin{pmatrix} 21 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (3.0.9)$$

$$= 21 \quad (3.0.10)$$

\therefore

$$\begin{pmatrix} \mathbf{p}_1^T \mathbf{c} \\ \mathbf{p}_2^T \mathbf{V} \mathbf{p}_2 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} \\ 21 \end{pmatrix} > 0 \quad (3.0.11)$$

Hence, the given equation does not have any real roots.

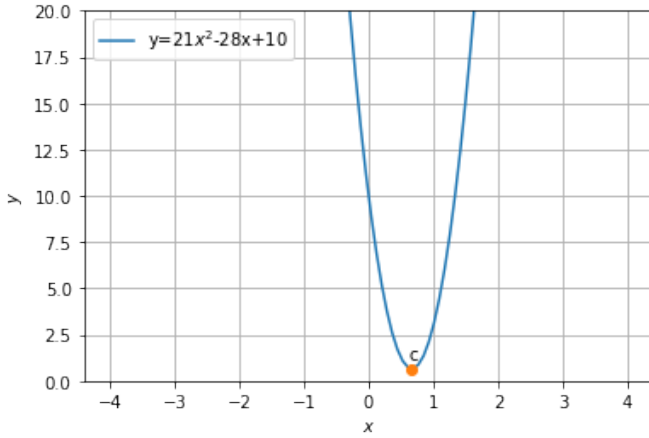


Fig. 3.1: $y = 21x^2 - 28x + 10$

2)

$$y = 6x^2 - x - 2 \quad (3.0.12)$$

Here,

$$\mathbf{V} = \begin{pmatrix} 6 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{u} = -\begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}, f = -2 \quad (3.0.13)$$

Using eigenvalue decomposition,

$$\mathbf{D} = \begin{pmatrix} 0 & 0 \\ 0 & 6 \end{pmatrix}, \mathbf{P} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (3.0.14)$$

\therefore Vertex \mathbf{c} is given by

$$\begin{pmatrix} -\frac{1}{2} & -1 \\ 6 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{c} = \begin{pmatrix} 2 \\ \frac{1}{2} \\ 0 \end{pmatrix} \quad (3.0.15)$$

$$\Rightarrow \begin{pmatrix} -\frac{1}{2} & -1 \\ 6 & 0 \end{pmatrix} \mathbf{c} = \begin{pmatrix} 2 \\ \frac{1}{2} \end{pmatrix} \quad (3.0.16)$$

$$\Rightarrow \mathbf{c} = \begin{pmatrix} \frac{1}{12} \\ -\frac{49}{24} \end{pmatrix} \quad (3.0.17)$$

Now,

$$\mathbf{p}_1^T \mathbf{c} = (0 \ 1) \begin{pmatrix} \frac{1}{12} \\ -\frac{49}{24} \end{pmatrix} \quad (3.0.18)$$

$$= \frac{-49}{24} \quad (3.0.19)$$

and,

$$\mathbf{p}_2^T \mathbf{V} \mathbf{p}_2 = (1 \ 0) \begin{pmatrix} 6 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (3.0.20)$$

$$= 6 \quad (3.0.21)$$

\therefore

$$\begin{pmatrix} \mathbf{p}_1^T \mathbf{c} \\ \mathbf{p}_2^T \mathbf{V} \mathbf{p}_2 \end{pmatrix} = \begin{pmatrix} -\frac{49}{24} \\ 6 \end{pmatrix} \neq 0 \quad (3.0.22)$$

Hence, the given equation has real roots.

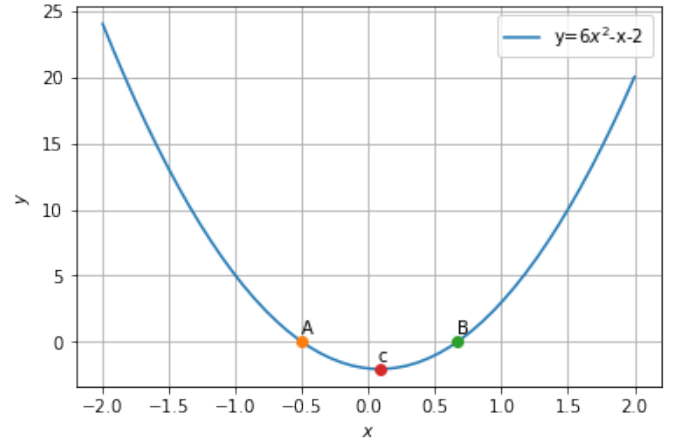


Fig. 3.2: $y = 6x^2 - x - 2$