Assignment 6

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Download all python codes from

```
https://github.com/ka−raja−babu/Matrix−Theory/
    tree/main/Assignment6/Codes
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and latex-tikz codes from

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https://github.com/ka−raja−babu/Matrix−Theory/
    tree/main/Assignment6
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1 Appendix

Parameter	Symbol	General Formula	Value
Vertex	$\mathbf{c}$	$\begin{pmatrix} \mathbf{u}^T + \eta \mathbf{p_1}^T \\ \mathbf{V} \end{pmatrix} \mathbf{c} = \begin{pmatrix} -f \\ \eta \mathbf{p_1} - \mathbf{u} \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$
Focal Length	β	$\frac{1}{2} \left\lvert \frac{\eta}{\lambda_2} \right\rvert$	2
Focus	$\mathbf{F}$	$\mathbf{F} = \mathbf{c} + \frac{-2\eta\begin{pmatrix} 1 & 0 \end{pmatrix}^T}{4}$	$\begin{pmatrix} 2 \\ 0 \end{pmatrix}$
Axis		$(\mathbf{Vc}+\mathbf{u})^T \mathbf{x} = 0$	$\begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} = 0$
Directrix		$(\mathbf{Vc}+\mathbf{u})^T(\mathbf{x}+\beta) + \mathbf{u}^T\mathbf{c} + \mathbf{f} = 0$	$\begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = -2$
Latus rectum		$(\mathbf{Vc}+\mathbf{u})^T(\mathbf{x}-\beta) + \mathbf{u}^T\mathbf{c} + \mathbf{f} = 0$	$\begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = 2$
End points of latus rectum	κ	$\mathbf{u}^T\kappa = -\frac{(\kappa^T \mathbf{V}\kappa + f)}{2}$	$\begin{pmatrix} 2 \\ \pm 4 \end{pmatrix}$
Length of latus rectum	l	$\left\lVert \beta(\mathbf{Vc}+\mathbf{u})^T \right\rVert$	8

TABLE 1.1: Parameters of parabola $y^2 = 8x$

All parameters of parabola $y^2 = 8x$ can be summarised in table 1.1 .
Note : Given general formula is valid only when parabola is in standard form i.e. $|\mathbf{V}| = 0$ and $\lambda_1 = 0$.

Lemma 1.1. *General equation of a conic is given by*

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \tag{1.0.1}$$

and can be expressed as

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{1.0.2}$$

where

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \tag{1.0.3}$$

$$\mathbf{u} = \begin{pmatrix} d & e \end{pmatrix} \tag{1.0.4}$$

Lemma 1.2. (1.0.2) *can be expressed as*

$$\mathbf{y}^T\mathbf{D}\mathbf{y} + 2(\mathbf{Vc}+\mathbf{u})^T\mathbf{P}\mathbf{y} + \mathbf{c}^T\mathbf{V}\mathbf{c} + 2\mathbf{u}^T\mathbf{c} + f = 0 \tag{1.0.5}$$

where

$$\mathbf{x} = \mathbf{P}\mathbf{y} + \mathbf{c} \tag{1.0.6}$$

$$\mathbf{P}^T\mathbf{V}\mathbf{P} = \mathbf{D} \tag{1.0.7}$$

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \tag{1.0.8}$$

$$\mathbf{P} = \begin{pmatrix} \mathbf{p_1} & \mathbf{p_2} \end{pmatrix} \tag{1.0.9}$$

Lemma 1.3. (1.0.5) *can be expressed as*

$$\mathbf{y}^T\mathbf{D}\mathbf{y} = \mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f \quad (|\mathbf{V}| \neq 0) \tag{1.0.10}$$

$$\mathbf{y}^T\mathbf{D}\mathbf{y} = -2\eta\begin{pmatrix} 1 & 0 \end{pmatrix}\mathbf{y} \quad (|\mathbf{V}| = 0) \tag{1.0.11}$$

where

$$\eta = \mathbf{u}^T\mathbf{p_1} \tag{1.0.12}$$

Lemma 1.4. *Focal length of a parabola is given by*

$$\beta = \frac{1}{2}\left\lvert \frac{\mathbf{u}^T\mathbf{p_1}}{\lambda_2} \right\rvert \tag{1.0.13}$$

Lemma 1.5. *Vertex of a parabola when it is in standard form is given by*

$$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} \quad (|\mathbf{V}| \neq 0) \tag{1.0.14}$$

$$\begin{pmatrix} \mathbf{u}^T + \eta \mathbf{p_1}^T \\ \mathbf{V} \end{pmatrix} \mathbf{c} = \begin{pmatrix} -f \\ \eta \mathbf{p_1} - \mathbf{u} \end{pmatrix} \quad (|\mathbf{V}| = 0) \tag{1.0.15}$$

Lemma 1.6. *Focus of a parabola when it is in standard form is given by*

$$\mathbf{F} = \begin{pmatrix} -f \\ \eta \mathbf{p_1} - \mathbf{u} \end{pmatrix} \begin{pmatrix} \mathbf{u}^T + \eta \mathbf{p_1}^T \\ \mathbf{V} \end{pmatrix}^{-1} + \frac{-2\eta \begin{pmatrix} 1 & 0 \end{pmatrix}^T}{4} \tag{1.0.16}$$

Proof. From (1.0.11) and (1.0.15),focus $\mathbf{F}$ is given by

$$\mathbf{F} = \mathbf{c} + \frac{-2\eta \begin{pmatrix} 1 & 0 \end{pmatrix}^T}{4} \tag{1.0.17}$$

$$\implies \mathbf{F} = \begin{pmatrix} -f \\ \eta \mathbf{p_1} - \mathbf{u} \end{pmatrix} \begin{pmatrix} \mathbf{u}^T + \eta \mathbf{p_1}^T \\ \mathbf{V} \end{pmatrix}^{-1} + \frac{-2\eta \begin{pmatrix} 1 & 0 \end{pmatrix}^T}{4} \tag{1.0.18}$$

□

Lemma 1.7. *Normal vector at any point* $\mathbf{q}$ *of a conic section is obtained as*

$$\mathbf{n} = \mathbf{V}\mathbf{q} + \mathbf{u} \tag{1.0.19}$$

Lemma 1.8. *Axis of a conic is given by*

$$(\mathbf{V}\mathbf{c} + \mathbf{u})^T \mathbf{x} = 0 \tag{1.0.20}$$

Proof. Using (1.7),Normal vector at vertex is given by

$$(\mathbf{V}\mathbf{c} + \mathbf{u})^T \tag{1.0.21}$$

So,axis is given as

$$(\mathbf{V}\mathbf{c} + \mathbf{u})^T \mathbf{x} = 0 \tag{1.0.22}$$

□

Lemma 1.9. *Given the point of contact* $\mathbf{q}$*,equation of tangent is given by*

$$(\mathbf{V}\mathbf{q} + \mathbf{u})^T \mathbf{x} + \mathbf{u}^T \mathbf{q} + f = 0 \tag{1.0.23}$$

Lemma 1.10. *Directrix of a conic is given by*

$$(\mathbf{V}\mathbf{c} + \mathbf{u})^T (\mathbf{x} + \beta) + \mathbf{u}^T \mathbf{c} + \mathbf{f} = 0 \tag{1.0.24}$$

Proof. Using (1.0.23),directrix is given by

$$(\mathbf{V}\mathbf{c} + \mathbf{u})^T (\mathbf{x} + \beta) + \mathbf{u}^T \mathbf{c} + \mathbf{f} = 0 \tag{1.0.25}$$

□

Lemma 1.11. *Latus rectum of a conic is given by*

$$(\mathbf{V}\mathbf{c} + \mathbf{u})^T (\mathbf{x} - \beta) + \mathbf{u}^T \mathbf{c} + \mathbf{f} = 0 \tag{1.0.26}$$

Proof. Using (1.0.23),latus rectum is given by

$$(\mathbf{V}\mathbf{c} + \mathbf{u})^T (\mathbf{x} - \beta) + \mathbf{u}^T \mathbf{c} + \mathbf{f} = 0 \tag{1.0.27}$$

□

Lemma 1.12. *End points of latus rectum of a conic is given by*

$$\mathbf{u}^T \kappa = -\frac{(\kappa^T \mathbf{V} \kappa + f)}{2} \tag{1.0.28}$$

where

$$\kappa = \begin{pmatrix} \beta \\ \mathbf{y} \end{pmatrix} \tag{1.0.29}$$

Proof. Substituting $x = \kappa$ in (1.0.1),end points of latus rectum are

$$\mathbf{u}^T \kappa = -\frac{(\kappa^T \mathbf{V} \kappa + f)}{2} \tag{1.0.30}$$

□

Lemma 1.13. *Length of latus rectum is given by*

$$l = \left\| \beta (\mathbf{V}\mathbf{c} + \mathbf{u})^T \right\| \tag{1.0.31}$$

Proof. Using (1.0.26) ,length of latus rectum can be expressed as

$$l = \left\| \beta (\mathbf{V}\mathbf{c} + \mathbf{u})^T \right\| \tag{1.0.32}$$

□

2 Question No. 2.29

Find the coordinates of the focus, axis, the equation of the directrix and latus rectum of the parabola $y^2 = 8x$.

3 solution

Given parabola is

$$y^2 = 8x \tag{3.0.1}$$

$$\implies y^2 - 8x = 0 \tag{3.0.2}$$

Vector form of given parabola is

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} + 2\begin{pmatrix} -4 & 0 \end{pmatrix} \mathbf{x} + 0 = 0 \tag{3.0.3}$$

$\therefore$

$$\mathbf{V} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \mathbf{u} = \begin{pmatrix} -4 \\ 0 \end{pmatrix}, f = 0 \tag{3.0.4}$$

$\because |\mathbf{V}| = 0$ and $\lambda_1 = 0$ i.e. it is in standard form
$\therefore$

$$\mathbf{P} = \mathbf{I} \implies \mathbf{p_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{3.0.5}$$

$$\eta = \mathbf{u}^T \mathbf{p_1} = -4 \tag{3.0.6}$$

The vertex $\mathbf{c}$ is given by

$$\begin{pmatrix} -8 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{c} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{3.0.7}$$

$$\implies \mathbf{c} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{3.0.8}$$

The focal length β is given by

$$\beta = \frac{1}{4}\left|\frac{2\eta}{\lambda_2}\right| = \frac{1}{4}\left|\frac{-8}{1}\right| = 2 \tag{3.0.9}$$

The focus $\mathbf{F}$ is given by

$$\mathbf{F} = \mathbf{c} + \frac{-2\eta\begin{pmatrix} 1 & 0 \end{pmatrix}^T}{4} \tag{3.0.10}$$

$$\implies \mathbf{F} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \end{pmatrix} \tag{3.0.11}$$

$$\implies \mathbf{F} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \tag{3.0.12}$$

Axis of parabola is given by

$$k(\mathbf{Vc} + \mathbf{u})^T \mathbf{x} = 0 \quad (k \in \mathbb{R}) \tag{3.0.13}$$

$$\implies k\begin{pmatrix} -4 & 0 \end{pmatrix} \mathbf{x} = 0 \tag{3.0.14}$$

$$\implies \begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} = 0 \tag{3.0.15}$$

Directrix of parabola is given by

$$(\mathbf{Vc} + \mathbf{u})^T (\mathbf{x} + \beta) + \mathbf{u}^T \mathbf{c} + \mathbf{f} = 0 \tag{3.0.16}$$

$$\implies \begin{pmatrix} -4 & 0 \end{pmatrix} (\mathbf{x} + \mathbf{2}) = 0 \tag{3.0.17}$$

$$\implies \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = -2 \tag{3.0.18}$$

Latus rectum of parabola is given by

$$(\mathbf{Vc} + \mathbf{u})^T (\mathbf{x} - \beta) + \mathbf{u}^T \mathbf{c} + \mathbf{f} = 0 \tag{3.0.19}$$

$$\implies \begin{pmatrix} -4 & 0 \end{pmatrix} (\mathbf{x} - \mathbf{2}) = 0 \tag{3.0.20}$$

$$\implies \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = 2 \tag{3.0.21}$$

End points of latus rectum are

$$\mathbf{u}^T \kappa = -\frac{(\kappa^T \mathbf{V} \kappa + f)}{2} \tag{3.0.22}$$

$$\implies \begin{pmatrix} -4 & 0 \end{pmatrix} \kappa = -\frac{\kappa^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \kappa + 0}{2} \tag{3.0.23}$$

$$\implies \kappa = \begin{pmatrix} 2 \\ \pm 4 \end{pmatrix} \tag{3.0.24}$$

Length of latus rectum l is

$$l = \left\|\beta(\mathbf{Vc} + \mathbf{u})^T\right\| \tag{3.0.25}$$

$$\implies l = \left\|2\begin{pmatrix} -4 & 0 \end{pmatrix}\right\| \tag{3.0.26}$$

$$\implies l = 8 \tag{3.0.27}$$

Plot of given parabola

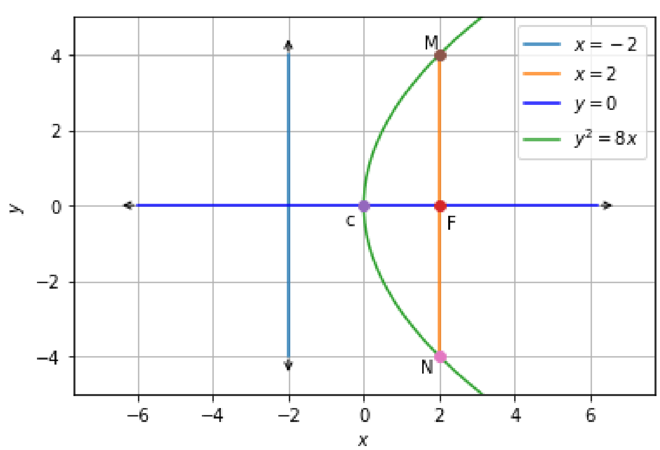

Fig. 3.1: Parabola $y^2 = 8x$

4 Generalisation

4.1 Circle

4.1.1 Property:

$$\mathbf{V} = \mathbf{D} = \mathbf{P} = \mathbf{I} \tag{4.1.1}$$

4.1.2 Standard Form: From (1.0.2),

$$\mathbf{x}^T\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{4.1.2}$$

4.1.3 Centre: From (1.0.14),

$$\mathbf{c} = -\mathbf{u} \tag{4.1.3}$$

4.1.4 Radius: From (1.0.10),

$$\mathbf{r} = \sqrt{\mathbf{u}^T\mathbf{u} - f} \tag{4.1.4}$$

4.2 Ellipse

4.2.1 Property:

$$|\mathbf{V}| > 0 \tag{4.2.1}$$

$$\lambda_1 > 0, \lambda_2 < 0 \tag{4.2.2}$$

4.2.2 Standard Form: From (1.0.10),

$$\frac{\mathbf{y}^T\mathbf{D}\mathbf{y}}{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f} = 1 \tag{4.2.3}$$

4.2.3 Centre: From (1.0.14),

$$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} \tag{4.2.4}$$

4.2.4 Axes: From (1.0.10),

$$\begin{cases} \sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u}-f}{\lambda_1}} \\ \sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u}-f}{\lambda_2}} \end{cases} \tag{4.2.5}$$

4.3 Hyperbola

4.3.1 Property:

$$|\mathbf{V}| < 0 \tag{4.3.1}$$

$$\lambda_1 > 0, \lambda_2 < 0 \tag{4.3.2}$$

4.3.2 Standard Form: From (1.0.10),

$$\frac{\mathbf{y}^T\mathbf{D}\mathbf{y}}{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f} = 1 \tag{4.3.3}$$

4.3.3 Centre: From (1.0.14),

$$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} \tag{4.3.4}$$

4.3.4 Axes: From (1.0.10),

$$\begin{cases} \sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u}-f}{\lambda_1}} \\ \sqrt{\frac{f-\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u}}{\lambda_2}} \end{cases} \tag{4.3.5}$$

4.4 Parabola

4.4.1 Property:

$$|\mathbf{V}| = 0 \tag{4.4.1}$$

$$\lambda_1 = 0 \tag{4.4.2}$$

4.4.2 Standard Form: From (1.0.11),

$$\mathbf{y}^T\mathbf{D}\mathbf{y} = -2\eta\begin{pmatrix}1 & 0\end{pmatrix}\mathbf{y} \tag{4.4.3}$$

4.4.3 Centre: From (1.0.15),

$$\begin{pmatrix}\mathbf{u}^T + \eta\mathbf{p_1}^T \\ \mathbf{V}\end{pmatrix}\mathbf{c} = \begin{pmatrix}-f \\ \eta\mathbf{p_1} - \mathbf{u}\end{pmatrix} \tag{4.4.4}$$

4.4.4 Focal length: From (1.0.13),

$$\beta = \frac{1}{2}\left|\frac{\mathbf{u}^T\mathbf{p}_1}{\lambda_2}\right| \tag{4.4.5}$$

4.4.5 Focus: From (1.0.16),

$$\mathbf{F} = \begin{pmatrix}-f \\ \eta\mathbf{p_1} - \mathbf{u}\end{pmatrix}\begin{pmatrix}\mathbf{u}^T + \eta\mathbf{p_1}^T \\ \mathbf{V}\end{pmatrix}^{-1} + \frac{-2\eta\begin{pmatrix}1 & 0\end{pmatrix}^T}{4} \tag{4.4.6}$$

4.4.6 Axis: From (1.0.20),

$$k(\mathbf{V}\mathbf{c} + \mathbf{u})^T\mathbf{x} = 0 \tag{4.4.7}$$

4.4.7 Directrix: From (1.0.24),

$$(\mathbf{V}\mathbf{c} + \mathbf{u})^T(\mathbf{x} + \beta) + \mathbf{u}^T\mathbf{c} + \mathbf{f} = 0 \tag{4.4.8}$$

4.4.8 Latus Rectum: From (1.0.26),

$$(\mathbf{V}\mathbf{c} + \mathbf{u})^T(\mathbf{x} - \beta) + \mathbf{u}^T\mathbf{c} + \mathbf{f} = 0 \tag{4.4.9}$$

4.4.9 End points of latus rectum: From (1.0.28),

$$\mathbf{u}^T\kappa = -\frac{(\kappa^T\mathbf{V}\kappa + f)}{2} \tag{4.4.10}$$

4.4.10 Length of latus rectum: From (1.0.31),

$$l = \left\|\beta(\mathbf{V}\mathbf{c} + \mathbf{u})^T\right\| \tag{4.4.11}$$

Conic	Property	Standard Form	Standard Parameters
Circle	$\mathbf{V} = \mathbf{D} = \mathbf{P} = \mathbf{I}$	$\mathbf{x}^T\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0$	1)Centre : $\mathbf{c} = -\mathbf{u}$ 2)Radius : $\mathbf{r} = \sqrt{\mathbf{u}^T\mathbf{u} - f}$
Ellipse	$\|\mathbf{V}\| > 0$ $\lambda_1 > 0, \lambda_2 < 0$	$\frac{\mathbf{y}^T\mathbf{D}\mathbf{y}}{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u}-f} = 1$	1)Centre : $\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u}$ 2)Axes : $\begin{cases} \sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u}-f}{\lambda_1}} \\ \sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u}-f}{\lambda_2}} \end{cases}$
Hyperbola	$\|\mathbf{V}\| < 0$ $\lambda_1 > 0, \lambda_2 < 0$	$\frac{\mathbf{y}^T\mathbf{D}\mathbf{y}}{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u}-f} = 1$	1)Centre : $\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u}$ 2)Axes : $\begin{cases} \sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u}-f}{\lambda_1}} \\ \sqrt{\frac{f-\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u}}{\lambda_2}} \end{cases}$
Parabola	$\|\mathbf{V}\| = 0$ $\lambda_1 = 0$	$\mathbf{y}^T\mathbf{D}\mathbf{y} = -2\eta\begin{pmatrix}1 & 0\end{pmatrix}\mathbf{y}$	1)Centre: $\begin{pmatrix}\mathbf{u}^T + \eta\mathbf{p_1}^T \\ \mathbf{V}\end{pmatrix}\mathbf{c} = \begin{pmatrix}-f \\ \eta\mathbf{p_1} - \mathbf{u}\end{pmatrix}$ 2)Focal Length: $\beta = \frac{1}{2}\left\|\frac{\mathbf{u}^T\mathbf{p}_1}{\lambda_2}\right\|$ 3)Focus: $\mathbf{F} = \begin{pmatrix}-f \\ \eta\mathbf{p_1} - \mathbf{u}\end{pmatrix}\begin{pmatrix}\mathbf{u}^T + \eta\mathbf{p_1}^T \\ \mathbf{V}\end{pmatrix}^{-1} + \frac{-2\eta\begin{pmatrix}1 & 0\end{pmatrix}^T}{4}$ 4)Axis: $k(\mathbf{V}\mathbf{c} + \mathbf{u})^T\mathbf{x} = 0$ 5)Directrix: $(\mathbf{V}\mathbf{c} + \mathbf{u})^T(\mathbf{x} + \beta) + \mathbf{u}^T\mathbf{c} + \mathbf{f} = 0$ 6)Latus Rectum: $(\mathbf{V}\mathbf{c} + \mathbf{u})^T(\mathbf{x} - \beta) + \mathbf{u}^T\mathbf{c} + \mathbf{f} = 0$ 7)End points of latus rectum : $\mathbf{u}^T\kappa = -\frac{(\kappa^T\mathbf{V}\kappa + f)}{2}$ 8)Length of latus rectum: $l = \left\|\beta(\mathbf{V}\mathbf{c} + \mathbf{u})^T\right\|$

TABLE 4.1: Generalisation of conic