Assignment 6

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Download all python codes from

```
https://github.com/ka−raja−babu/Matrix−Theory/
    tree/main/Assignment6/Codes
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and latex-tikz codes from

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https://github.com/ka−raja−babu/Matrix−Theory/
    tree/main/Assignment6
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1 Question No. 2.29

Find the coordinates of the focus, axis, the equation of the directrix and latus rectum of the parabola $y^2 = 8x$.

2 Solution

All parameters of parabola $y^2 = 8x$ can be summarised in table 2.1 .

Note : Given general formula is valid only when parabola is in standard form i.e. $|\mathbf{V}| = 0$ and $\lambda_1 = 0$.

Given parabola is

$$y^2 = 8x \tag{2.0.1}$$

$$\implies y^2 - 8x = 0 \tag{2.0.2}$$

Vector form of given parabola is

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} + 2 \begin{pmatrix} -4 & 0 \end{pmatrix} \mathbf{x} + 0 = 0 \tag{2.0.3}$$

$\therefore$

$$\mathbf{V} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \mathbf{u} = \begin{pmatrix} -4 \\ 0 \end{pmatrix}, f = 0 \tag{2.0.4}$$

$\because |\mathbf{V}| = 0$ and $\lambda_1 = 0$ i.e. it is in standard form $\therefore$

$$\mathbf{P} = \mathbf{I} \implies \mathbf{p_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{2.0.5}$$

$$\eta = \mathbf{u}^T \mathbf{p_1} = -4 \tag{2.0.6}$$

Parameter	Symbol	Value	General Formula
Vertex	$\mathbf{c}$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} \mathbf{u}^T + \eta \mathbf{p_1}^T \\ \mathbf{V} \end{pmatrix} \mathbf{c} = \begin{pmatrix} -f \\ \eta \mathbf{p_1} - \mathbf{u} \end{pmatrix}$
Focal Length	β	2	$\frac{1}{4} \left| \frac{2\eta}{\lambda_2} \right|$
Focus	$\mathbf{F}$	$\begin{pmatrix} 2 \\ 0 \end{pmatrix}$	$\mathbf{F} = \mathbf{c} + \mathbf{a}^T$
Axis		$\begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} = 0$	$\mathbf{x} = \mathbf{F} + k\mathbf{m}$
Axis direction vector	$\mathbf{v}$	$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$	$\mathbf{v}^T \mathbf{x} = 0$
Directrix normal vector	$\mathbf{v_1}$	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$	$\mathbf{v_1} \mathbf{v}^T = 0$
Directrix		$\begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = -2$	$\mathbf{v_1}^T \mathbf{x} = -\beta$
Latus rectum normal vector	$\mathbf{v_2}$	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$	$\mathbf{v_2} \mathbf{v}^T = 0$
Latus rectum		$\begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = 2$	$\mathbf{v_2}^T \mathbf{x} = \beta$
End points of latus rectum	$\mathbf{M}, \mathbf{N}$	$\begin{pmatrix} 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ -4 \end{pmatrix}$	$\begin{pmatrix} \beta \\ \pm y(\beta) \end{pmatrix}$
Length of latus rectum	l	8	$\|\mathbf{M} - \mathbf{N}\|$

TABLE 2.1: Parameters of parabola $y^2 = 8x$

The vertex $\mathbf{c}$ is given by

$$\begin{pmatrix} -8 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{c} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{2.0.7}$$

$$\implies \mathbf{c} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.0.8}$$

The focal length β is given by

$$\beta = \frac{1}{4}\left|\frac{2\eta}{\lambda_2}\right| = \frac{1}{4}\left|\frac{-8}{1}\right| = 2 \tag{2.0.9}$$

The focus $\mathbf{F}$ is given by

$$\mathbf{a} = \frac{-2\eta\begin{pmatrix}1 & 0\end{pmatrix}}{4} = \begin{pmatrix}2 & 0\end{pmatrix} \tag{2.0.10}$$

$$\mathbf{F} = \mathbf{c} + \mathbf{a}^T = \begin{pmatrix}2\\0\end{pmatrix} \tag{2.0.11}$$

$\because$ Axis of parabola passes through both vertex and focus .
$\therefore$ Axis of parabola is given by

$$\mathbf{m} = \mathbf{F} - \mathbf{c} = \begin{pmatrix}2\\0\end{pmatrix} \tag{2.0.12}$$

$$\mathbf{x} = \mathbf{F} + k\mathbf{m} = \begin{pmatrix}2+2k\\0\end{pmatrix} \tag{2.0.13}$$

$$\implies \begin{pmatrix}0 & 1\end{pmatrix}\mathbf{x} = 0 \tag{2.0.14}$$

$$\implies \mathbf{v} = \begin{pmatrix}0\\1\end{pmatrix} \tag{2.0.15}$$

$\because$ Vertex of parabola is at equal distance from focus and directrix and is perpendicular to axis.
$\therefore$ Directrix of parabola is given by

$$\mathbf{v_1}\mathbf{v}^T = 0 \tag{2.0.16}$$

$$\implies \mathbf{v_1} = \begin{pmatrix}1\\0\end{pmatrix} \tag{2.0.17}$$

So ,

$$\mathbf{v_1}^T\mathbf{x} = -\beta \tag{2.0.18}$$

$$\implies \begin{pmatrix}1 & 0\end{pmatrix}\mathbf{x} = -2 \tag{2.0.19}$$

$\because$ Latus rectum of parabola passes through focus and is perpendicular to axis.
$\therefore$ Latus rectum of parabola is given by

$$\mathbf{v_2}\mathbf{v}^T = 0 \tag{2.0.20}$$

$$\implies \mathbf{v_2} = \begin{pmatrix}1\\0\end{pmatrix} \tag{2.0.21}$$

So ,

$$\mathbf{v_2}^T\mathbf{x} = \beta \tag{2.0.22}$$

$$\implies \begin{pmatrix}1 & 0\end{pmatrix}\mathbf{x} = 2 \tag{2.0.23}$$

End points of latus rectum are

$$\mathbf{M} = \begin{pmatrix}2\\4\end{pmatrix} \tag{2.0.24}$$

$$\mathbf{N} = \begin{pmatrix}2\\-4\end{pmatrix} \tag{2.0.25}$$

So,the length of latus rectum l is

$$l = \|\mathbf{M} - \mathbf{N}\| = 8 \tag{2.0.26}$$

Plot of given parabola

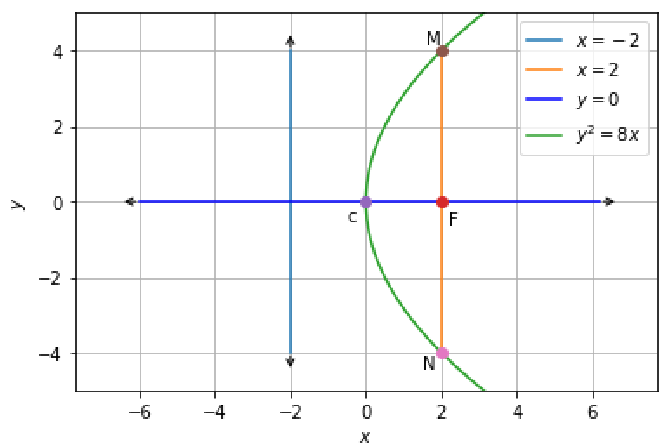

Fig. 2.1: Parabola $y^2 = 8x$

3 Appendix

General equation of a conic is given by

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \tag{3.0.1}$$

and can be expressed as

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{3.0.2}$$

where

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix}a & b\\ b & c\end{pmatrix} \tag{3.0.3}$$

$$\mathbf{u} = \begin{pmatrix}d & e\end{pmatrix} \tag{3.0.4}$$

Using Affine Transformation,

$$\mathbf{x} = \mathbf{P}\mathbf{y} + \mathbf{c} \tag{3.0.5}$$

Using Eigenvalue Decomposition,

$$\mathbf{P}^T\mathbf{V}\mathbf{P} = \mathbf{D} \tag{3.0.6}$$

such that

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \tag{3.0.7}$$

$$\mathbf{P} = \begin{pmatrix} \mathbf{p_1} & \mathbf{p_2} \end{pmatrix}, \quad \mathbf{P}^T = \mathbf{P}^{-1} \tag{3.0.8}$$

Substituting (3.0.5) in (3.0.2),we get

$$(\mathbf{Py} + \mathbf{c})^T \mathbf{V}(\mathbf{Py} + \mathbf{c}) + 2\mathbf{u}^T(\mathbf{Py} + \mathbf{c}) + f = 0 \tag{3.0.9}$$

which can be expressed as

$$\mathbf{y}^T\mathbf{P}^T\mathbf{VPy} + 2(\mathbf{Vc} + \mathbf{u})^T\mathbf{Py} + \mathbf{c}^T\mathbf{Vc} + 2\mathbf{u}^T\mathbf{c} + f = 0 \tag{3.0.10}$$

From (3.0.10) and (3.0.6),

$$\mathbf{y}^T\mathbf{P}^T\mathbf{VPy} + 2(\mathbf{Vc} + \mathbf{u})^T\mathbf{Py} + \mathbf{c}^T\mathbf{Vc} + 2\mathbf{u}^T\mathbf{c} + f = 0 \tag{3.0.11}$$

When $|\mathbf{V}| = 0, \lambda_1 = 0$ and

$$\mathbf{Vp}_1 = 0, \mathbf{Vp}_2 = \lambda_2\mathbf{p}_2 \tag{3.0.12}$$

Then,substituting (3.0.8) in (3.0.10),

$$\mathbf{y}^T\mathbf{Dy} + 2\left(\mathbf{c}^T\mathbf{V} + \mathbf{u}^T\right)\begin{pmatrix} \mathbf{p}_1 & \mathbf{p}_2 \end{pmatrix}\mathbf{y} + \mathbf{c}^T\left(\mathbf{Vc} + \mathbf{u}\right) + \mathbf{u}^T\mathbf{c} + f = 0 \tag{3.0.13}$$

$$\implies \mathbf{y}^T\mathbf{Dy} + 2\begin{pmatrix} \left(\mathbf{c}^T\mathbf{V} + \mathbf{u}^T\right)\mathbf{p}_1 & \left(\mathbf{c}^T\mathbf{V} + \mathbf{u}^T\right)\mathbf{p}_2 \end{pmatrix}\mathbf{y} + \mathbf{c}^T\left(\mathbf{Vc} + \mathbf{u}\right) + \mathbf{u}^T\mathbf{c} + f = 0 \tag{3.0.14}$$

$$\implies \mathbf{y}^T\mathbf{Dy} + 2\begin{pmatrix} \mathbf{u}^T\mathbf{p}_1 & \left(\lambda_2\mathbf{c}^T + \mathbf{u}^T\right)\mathbf{p}_2 \end{pmatrix}\mathbf{y} + \mathbf{c}^T\left(\mathbf{Vc} + \mathbf{u}\right) + \mathbf{u}^T\mathbf{c} + f = 0 \tag{3.0.15}$$

$$\implies \lambda_2 y_2^2 + 2\left(\mathbf{u}^T\mathbf{p}_1\right)y_1 + 2y_2\left(\lambda_2\mathbf{c} + \mathbf{u}\right)^T\mathbf{p}_2 + \mathbf{c}^T\left(\mathbf{Vc} + \mathbf{u}\right) + \mathbf{u}^T\mathbf{c} + f = 0 \tag{3.0.16}$$

which is the equation of a parabola.
From (3.0.16), by comparing the coefficients of y_2^2 and y_1, the focal length of the parabola is obtained as

$$\beta = \frac{1}{4}\left|\frac{2\mathbf{u}^T\mathbf{p}_1}{\lambda_2}\right| \tag{3.0.17}$$

Now,by choosing

$$\eta = \mathbf{u}^T\mathbf{p}_1 \tag{3.0.18}$$

such that

$$\mathbf{P}^T\left(\mathbf{Vc} + \mathbf{u}\right) = \eta\begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{3.0.19}$$

$$\mathbf{c}^T\left(\mathbf{Vc} + \mathbf{u}\right) + \mathbf{u}^T\mathbf{c} + f = 0 \tag{3.0.20}$$

Multiplying (3.0.19) by $\mathbf{P}$ yields

$$(\mathbf{Vc} + \mathbf{u}) = \eta\mathbf{p}_1, \tag{3.0.21}$$

which, upon substituting in (3.0.20) results in

$$\eta\mathbf{c}^T\mathbf{p}_1 + \mathbf{u}^T\mathbf{c} + f = 0 \tag{3.0.22}$$

(3.0.21) and (3.0.22) can be clubbed together to obtain

$$\begin{pmatrix} \mathbf{u}^T + \eta\mathbf{p_1}^T \\ \mathbf{V} \end{pmatrix}\mathbf{c} = \begin{pmatrix} -f \\ \eta\mathbf{p_1} - \mathbf{u} \end{pmatrix} \tag{3.0.23}$$

So,when $|V|$=0 ,(3.0.2) can be expressed as

$$\mathbf{y}^T\mathbf{Dy} = -2\eta\begin{pmatrix} 1 & 0 \end{pmatrix}\mathbf{y} \tag{3.0.24}$$

From (3.0.24),focus $\mathbf{F}$ can be calculated as

$$\mathbf{a} = \frac{-2\eta\begin{pmatrix} 1 & 0 \end{pmatrix}}{4} \tag{3.0.25}$$

$$\mathbf{F} = \mathbf{c} + \mathbf{a}^T \tag{3.0.26}$$

By using $\mathbf{c}$ from (3.0.23) and $\mathbf{F}$ from (3.0.26),axis can be calculated as

$$\mathbf{m} = \mathbf{F} - \mathbf{c} \quad (\mathbf{m} \text{ is direction vector}) \tag{3.0.27}$$

$$\mathbf{x} = \mathbf{F} + k\mathbf{m} \quad (k \in \mathbb{R}) \tag{3.0.28}$$

$$\implies \mathbf{vx} = 0 \quad (\mathbf{v} \text{ is direction vector of } \mathbf{x}) \tag{3.0.29}$$

Directrix is always perpendicular to axis such that

$$\mathbf{v_1}\mathbf{v}^T = 0 \quad (\mathbf{v_1} \text{ is normal vector}) \tag{3.0.30}$$

Using (3.0.17),(3.0.29) and (3.0.30),directrix can be expressed as

$$\mathbf{v_1}^T\mathbf{x} = -\beta \tag{3.0.31}$$

Also,latus rectum is always perpendicular to axis such that

$$\mathbf{v_2}\mathbf{v}^T = 0 \quad (\mathbf{v_2} \text{ is normal vector}) \tag{3.0.32}$$

Using (3.0.17),(3.0.29) and (3.0.32),latus rectum can be expressed as

$$\mathbf{v_2}^T\mathbf{x} = \beta \tag{3.0.33}$$

Using (3.0.1) and (3.0.17),intersection points of latus rectum can be expressed as

$$\mathbf{M} = \begin{pmatrix} \beta \\ y(\beta) \end{pmatrix} \tag{3.0.34}$$

$$\mathbf{N} = \begin{pmatrix} \beta \\ -y(\beta) \end{pmatrix} \tag{3.0.35}$$

Now,using (3.0.34) and (3.0.35),length of latus rectum can be expressed as

$$l = \|\mathbf{M} - \mathbf{N}\| \tag{3.0.36}$$