Challenge Problem 1

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Download all python codes from

```
https://github.com/ka−raja−babu/Matrix−Theory/
    tree/main/ChallengeProblem1/Codes
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and latex-tikz codes from

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https://github.com/ka−raja−babu/Matrix−Theory/
    tree/main/ChallengeProblem1
```

1 Challenge Question 1

Show that the matrix $(t\mathbf{I} - \mathbf{n}\mathbf{n}^T)$ in the given document is a rank 1 matrix for a parabola.

2 Solution

Given :

1) Directrix : $\mathbf{n}^T\mathbf{x} = c$
2) Eccentricity : e
3) t=$\frac{\|n\|^2}{e^2}$

Definition 1. *Rank of a singular matrix of nxn is always less than n.*

Definition 2. *Rank of a matrix is zero only if it is a null matrix.*

Proof. Let $\mathbf{n} = \begin{pmatrix} a \\ b \end{pmatrix}$.
$\therefore$

$$\|\mathbf{n}\|^2 = a^2 + b^2 \tag{2.0.1}$$

$$\mathbf{n}\mathbf{n}^T = \begin{pmatrix} a^2 & ab \\ ab & b^2 \end{pmatrix} \tag{2.0.2}$$

Now,

$$t = \frac{a^2 + b^2}{e^2} \tag{2.0.3}$$

$$\implies t\mathbf{I} = \begin{pmatrix} \frac{a^2+b^2}{e^2} & 0 \\ 0 & \frac{a^2+b^2}{e^2} \end{pmatrix} \tag{2.0.4}$$

$\therefore$

$$(t\mathbf{I} - \mathbf{n}\mathbf{n}^T) = \begin{pmatrix} \frac{(1-e^2)a^2+b^2}{e^2} & -ab \\ -ab & \frac{a^2+(1-e^2)b^2}{e^2} \end{pmatrix} \tag{2.0.5}$$

For a parabola,$e = 1$.
$\therefore$

$$(t\mathbf{I} - \mathbf{n}\mathbf{n}^T) = \begin{pmatrix} b^2 & -ab \\ -ab & a^2 \end{pmatrix} \tag{2.0.6}$$

$$\implies \left|t\mathbf{I} - \mathbf{n}\mathbf{n}^T\right| = 0 \tag{2.0.7}$$

Using def. 1,

$$\text{Rank of } (t\mathbf{I} - \mathbf{n}\mathbf{n}^T) < 2 \tag{2.0.8}$$

$\because$ $(t\mathbf{I} - \mathbf{n}\mathbf{n}^T)$ is not a null matrix.
$\therefore$ Using def. 2,

$$0 < \text{Rank of } (t\mathbf{I} - \mathbf{n}\mathbf{n}^T) < 2 \tag{2.0.9}$$

Hence,using (2.0.8) and (2.0.9),

$$\text{Rank of } (t\mathbf{I} - \mathbf{n}\mathbf{n}^T) = 1 \tag{2.0.10}$$

□