

Assignment 6

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Download all python codes from

<https://github.com/ka-raja-babu/Matrix-Theory/tree/main/Assignment6/Codes>

and latex-tikz codes from

<https://github.com/ka-raja-babu/Matrix-Theory/tree/main/Assignment6>

1 QUESTION No. 2.29

Find the coordinates of the focus, axis, the equation of the directrix and latus rectum of the parabola $y^2 = 8x$.

2 SOLUTION

All parameters of parabola $y^2 = 8x$ can be summarised in table 2.1.

Note : Given general formula is valid only when parabola is in standard form i.e. $|\mathbf{V}| = 0$ and $\lambda_1 = 0$.

¹ Procedure to find axis :

- 1) Calculate vertex \mathbf{c} and focus \mathbf{F} .
- 2) Find equation of line joining \mathbf{c} and \mathbf{F} using $\mathbf{x} = \mathbf{F} + k\mathbf{m}$ where $\mathbf{m} = \mathbf{F} - \mathbf{c}$ and $k \in \mathbb{R}$.

² Procedure to find directrix :

- 1) Calculate its direction vector \mathbf{v}_1 by using $\mathbf{v}^T \mathbf{v}_1 = 0$.
- 2) Equate $\mathbf{v}_1^T \mathbf{x}$ to $-\beta$ to get final equation.

³ Procedure to find latus rectum :

- 1) Calculate its direction vector \mathbf{v}_2 by using $\mathbf{v}^T \mathbf{v}_2 = 0$.
- 2) Equate $\mathbf{v}_2^T \mathbf{x}$ to β to get final equation.

Parameter	Symbol	Value	General Formula
Vertex	\mathbf{c}	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} \mathbf{u}^T + \eta \mathbf{p}_1^T \\ \mathbf{V} \end{pmatrix} \mathbf{c} = \begin{pmatrix} -f \\ \eta \mathbf{p}_1 - \mathbf{u} \end{pmatrix}$
Focal Length	β	2	$\frac{1}{4} \left \frac{2\eta}{\lambda_2} \right $
Focus	\mathbf{F}	$\begin{pmatrix} 2 \\ 0 \end{pmatrix}$	$\mathbf{F} = \mathbf{c} + \mathbf{a}^T$
Axis ¹		$\begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} = 0$	$\mathbf{x} = \mathbf{F} + k\mathbf{m}$
Axis direction vector	\mathbf{v}	$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$	$\mathbf{v}^T \mathbf{x} = 0$
Directrix direction vector	\mathbf{v}_1	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$	$\mathbf{v}^T \mathbf{v}_1 = 0$
Directrix ²		$\begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = -2$	$\mathbf{v}_1^T \mathbf{x} = -\beta$
Latus rectum direction vector	\mathbf{v}_2	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$	$\mathbf{v}^T \mathbf{v}_2 = 0$
Latus rectum ³		$\begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = 2$	$\mathbf{v}_2^T \mathbf{x} = \beta$
End points of latus rectum	\mathbf{M}, \mathbf{N}	$\begin{pmatrix} 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ -4 \end{pmatrix}$	$\begin{pmatrix} \beta \\ \pm y(\beta) \end{pmatrix}$
Length of latus rectum	l	8	$\ \mathbf{M} - \mathbf{N}\ $

TABLE 2.1: Parameters of parabola $y^2 = 8x$

Given parabola is

$$y^2 = 8x \quad (2.0.1)$$

$$\Rightarrow y^2 - 8x = 0 \quad (2.0.2)$$

Vector form of given parabola is

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} + 2 \begin{pmatrix} -4 & 0 \end{pmatrix} \mathbf{x} + 0 = 0 \quad (2.0.3)$$

\therefore

$$\mathbf{V} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \mathbf{u} = \begin{pmatrix} -4 \\ 0 \end{pmatrix}, f = 0 \quad (2.0.4)$$

$\therefore |\mathbf{V}| = 0$ and $\lambda_1 = 0$ i.e. it is in standard form

\therefore

$$\mathbf{P} = \mathbf{I} \Rightarrow \mathbf{p}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (2.0.5)$$

$$\eta = \mathbf{u}^T \mathbf{p}_1 = -4 \quad (2.0.6)$$

The vertex \mathbf{c} is given by

$$\begin{pmatrix} -8 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{c} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad (2.0.7)$$

$$\Rightarrow \mathbf{c} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad (2.0.8)$$

The focal length β is given by

$$\beta = \frac{1}{4} \left| \frac{2\eta}{\lambda_2} \right| = \frac{1}{4} \left| \frac{-8}{1} \right| = 2 \quad (2.0.9)$$

The focus \mathbf{F} is given by

$$\mathbf{a} = \frac{-2\eta \begin{pmatrix} 1 & 0 \end{pmatrix}}{4} = \begin{pmatrix} 2 & 0 \end{pmatrix} \quad (2.0.10)$$

$$\mathbf{F} = \mathbf{c} + \mathbf{a}^T = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \quad (2.0.11)$$

\therefore Axis of parabola passes through both vertex and focus .

\therefore Axis of parabola is given by

$$\mathbf{m} = \mathbf{F} - \mathbf{c} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \quad (2.0.12)$$

$$\mathbf{x} = \mathbf{F} + k\mathbf{m} = \begin{pmatrix} 2 + 2k \\ 0 \end{pmatrix} \quad (2.0.13)$$

$$\Rightarrow \begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} = 0 \quad (2.0.14)$$

$$\Rightarrow \mathbf{v} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (2.0.15)$$

\therefore Vertex of parabola is at equal distance from focus and directrix and is perpendicular to axis.

\therefore Directrix of parabola is given by

$$\mathbf{v}^T \mathbf{v}_1 = 0 \quad (2.0.16)$$

$$\Rightarrow \mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (2.0.17)$$

So ,

$$\mathbf{v}_1^T \mathbf{x} = -\beta \quad (2.0.18)$$

$$\Rightarrow \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = -2 \quad (2.0.19)$$

\therefore Latus rectum of parabola passes through focus and is perpendicular to axis.

\therefore Latus rectum of parabola is given by

$$\mathbf{v}^T \mathbf{v}_2 = 0 \quad (2.0.20)$$

$$\Rightarrow \mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (2.0.21)$$

So ,

$$\mathbf{v}_2^T \mathbf{x} = \beta \quad (2.0.22)$$

$$\Rightarrow \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = 2 \quad (2.0.23)$$

End points of latus rectum are

$$\mathbf{M} = \begin{pmatrix} 2 \\ 4 \end{pmatrix} \quad (2.0.24)$$

$$\mathbf{N} = \begin{pmatrix} 2 \\ -4 \end{pmatrix} \quad (2.0.25)$$

So, the length of latus rectum l is

$$l = \|\mathbf{M} - \mathbf{N}\| = 8 \quad (2.0.26)$$

Plot of given parabola

3 APPENDIX

General equation of a conic is given by

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \quad (3.0.1)$$

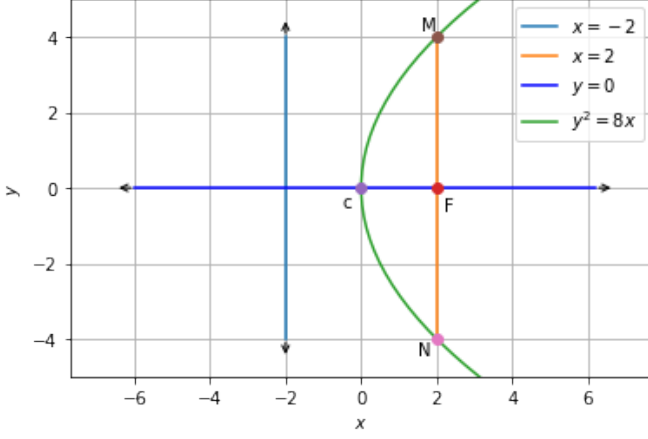


Fig. 2.1: Parabola $y^2 = 8x$

and can be expressed as

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (3.0.2)$$

where

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \quad (3.0.3)$$

$$\mathbf{u} = \begin{pmatrix} d & e \end{pmatrix} \quad (3.0.4)$$

Using Affine Transformation,

$$\mathbf{x} = \mathbf{P} \mathbf{y} + \mathbf{c} \quad (3.0.5)$$

Using Eigenvalue Decomposition,

$$\mathbf{P}^T \mathbf{V} \mathbf{P} = \mathbf{D} \quad (3.0.6)$$

such that

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \quad (3.0.7)$$

$$\mathbf{P} = (\mathbf{p}_1 \quad \mathbf{p}_2), \quad \mathbf{P}^T = \mathbf{P}^{-1} \quad (3.0.8)$$

Substituting (3.0.5) in (3.0.2), we get

$$(\mathbf{P} \mathbf{y} + \mathbf{c})^T \mathbf{V} (\mathbf{P} \mathbf{y} + \mathbf{c}) + 2\mathbf{u}^T (\mathbf{P} \mathbf{y} + \mathbf{c}) + f = 0 \quad (3.0.9)$$

which can be expressed as

$$\mathbf{y}^T \mathbf{P}^T \mathbf{V} \mathbf{P} \mathbf{y} + 2(\mathbf{V} \mathbf{c} + \mathbf{u})^T \mathbf{P} \mathbf{y} + \mathbf{c}^T \mathbf{V} \mathbf{c} + 2\mathbf{u}^T \mathbf{c} + f = 0 \quad (3.0.10)$$

From (3.0.10) and (3.0.6),

$$\mathbf{y}^T \mathbf{P}^T \mathbf{V} \mathbf{P} \mathbf{y} + 2(\mathbf{V} \mathbf{c} + \mathbf{u})^T \mathbf{P} \mathbf{y} + \mathbf{c}^T \mathbf{V} \mathbf{c} + 2\mathbf{u}^T \mathbf{c} + f = 0 \quad (3.0.11)$$

When $|\mathbf{V}| = 0, \lambda_1 = 0$ and

$$\mathbf{V} \mathbf{p}_1 = 0, \mathbf{V} \mathbf{p}_2 = \lambda_2 \mathbf{p}_2 \quad (3.0.12)$$

Then, substituting (3.0.8) in (3.0.10),

$$\mathbf{y}^T \mathbf{D} \mathbf{y} + 2(\mathbf{c}^T \mathbf{V} + \mathbf{u}^T)(\mathbf{p}_1 \quad \mathbf{p}_2) \mathbf{y} + \mathbf{c}^T (\mathbf{V} \mathbf{c} + \mathbf{u}) + \mathbf{u}^T \mathbf{c} + f = 0 \quad (3.0.13)$$

$$\Rightarrow \mathbf{y}^T \mathbf{D} \mathbf{y} + 2((\mathbf{c}^T \mathbf{V} + \mathbf{u}^T) \mathbf{p}_1 \quad (\mathbf{c}^T \mathbf{V} + \mathbf{u}^T) \mathbf{p}_2) \mathbf{y} + \mathbf{c}^T (\mathbf{V} \mathbf{c} + \mathbf{u}) + \mathbf{u}^T \mathbf{c} + f = 0 \quad (3.0.14)$$

$$\Rightarrow \mathbf{y}^T \mathbf{D} \mathbf{y} + 2(\mathbf{u}^T \mathbf{p}_1 \quad (\lambda_2 \mathbf{c}^T + \mathbf{u}^T) \mathbf{p}_2) \mathbf{y} + \mathbf{c}^T (\mathbf{V} \mathbf{c} + \mathbf{u}) + \mathbf{u}^T \mathbf{c} + f = 0 \quad (3.0.15)$$

$$\Rightarrow \lambda_2 y_2^2 + 2(\mathbf{u}^T \mathbf{p}_1) y_1 + 2y_2 (\lambda_2 \mathbf{c} + \mathbf{u})^T \mathbf{p}_2 + \mathbf{c}^T (\mathbf{V} \mathbf{c} + \mathbf{u}) + \mathbf{u}^T \mathbf{c} + f = 0 \quad (3.0.16)$$

which is the equation of a parabola.

From (3.0.16), by comparing the coefficients of y_2^2 and y_1 , the focal length of the parabola is obtained as

$$\beta = \frac{1}{4} \left| \frac{2\mathbf{u}^T \mathbf{p}_1}{\lambda_2} \right| \quad (3.0.17)$$

Now, by choosing

$$\eta = \mathbf{u}^T \mathbf{p}_1 \quad (3.0.18)$$

such that

$$\mathbf{P}^T (\mathbf{V} \mathbf{c} + \mathbf{u}) = \eta \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (3.0.19)$$

$$\mathbf{c}^T (\mathbf{V} \mathbf{c} + \mathbf{u}) + \mathbf{u}^T \mathbf{c} + f = 0 \quad (3.0.20)$$

Multiplying (3.0.19) by \mathbf{P} yields

$$(\mathbf{V} \mathbf{c} + \mathbf{u}) = \eta \mathbf{p}_1, \quad (3.0.21)$$

which, upon substituting in (3.0.20) results in

$$\eta \mathbf{c}^T \mathbf{p}_1 + \mathbf{u}^T \mathbf{c} + f = 0 \quad (3.0.22)$$

(3.0.21) and (3.0.22) can be clubbed together to obtain

$$\begin{pmatrix} \mathbf{u}^T + \eta \mathbf{p}_1^T \\ \mathbf{V} \end{pmatrix} \mathbf{c} = \begin{pmatrix} -f \\ \eta \mathbf{p}_1 - \mathbf{u} \end{pmatrix} \quad (3.0.23)$$

So, when $|\mathbf{V}|=0$, (3.0.2) can be expressed as

$$\mathbf{y}^T \mathbf{D} \mathbf{y} = -2\eta \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{y} \quad (3.0.24)$$

From (3.0.24), focus \mathbf{F} can be calculated as

$$\mathbf{a} = \frac{-2\eta \begin{pmatrix} 1 & 0 \end{pmatrix}}{4} \quad (3.0.25)$$

$$\mathbf{F} = \mathbf{c} + \mathbf{a}^T \quad (3.0.26)$$

By using \mathbf{c} from (3.0.23) and \mathbf{F} from (3.0.26), axis can be calculated as

$$\mathbf{m} = \mathbf{F} - \mathbf{c} \quad (\mathbf{m} \text{ is direction vector}) \quad (3.0.27)$$

$$\mathbf{x} = \mathbf{F} + k\mathbf{m} \quad (k \in \mathbb{R}) \quad (3.0.28)$$

$$\implies \mathbf{v}^T \mathbf{x} = 0 \quad (\mathbf{v} \text{ is direction vector of } \mathbf{x}) \quad (3.0.29)$$

Directrix is always perpendicular to axis such that

$$\mathbf{v}^T \mathbf{v}_1 = 0 \quad (\mathbf{v}_1 \text{ is normal vector}) \quad (3.0.30)$$

Using (3.0.17), (3.0.29) and (3.0.30), directrix can be expressed as

$$\mathbf{v}_1^T \mathbf{x} = -\beta \quad (3.0.31)$$

Also, latus rectum is always perpendicular to axis such that

$$\mathbf{v}^T \mathbf{v}_2 = 0 \quad (\mathbf{v}_2 \text{ is normal vector}) \quad (3.0.32)$$

Using (3.0.17), (3.0.29) and (3.0.32), latus rectum can be expressed as

$$\mathbf{v}_2^T \mathbf{x} = \beta \quad (3.0.33)$$

Using (3.0.1) and (3.0.17), intersection points of latus rectum can be expressed as

$$\mathbf{M} = \begin{pmatrix} \beta \\ y(\beta) \end{pmatrix} \quad (3.0.34)$$

$$\mathbf{N} = \begin{pmatrix} \beta \\ -y(\beta) \end{pmatrix} \quad (3.0.35)$$

Now, using (3.0.34) and (3.0.35), length of latus rectum can be expressed as

$$l = \|\mathbf{M} - \mathbf{N}\| \quad (3.0.36)$$