Challenge Problem 1

K.A. Raja Babu

Download all python codes from

```
https://github.com/ka-raja-babu/Matrix-Theory/
    tree/main/ChallengeProblem1/Codes
```

and latex-tikz codes from

```
https://github.com/ka-raja-babu/Matrix-Theory/
    tree/main/ChallengeProblem1
```

1 Challenge Question 1

Show that the matrix $(t\mathbf{I} - \mathbf{n}\mathbf{n}^T)$ in the given document is a rank 1 matrix for a parabola.

2 Solution

Given :

$$\mathbf{V} = (t\mathbf{I} - \mathbf{n}\mathbf{n}^T) \tag{2.0.1}$$

is a symmetric matrix.
$\therefore$

$$\mathbf{V} = \mathbf{V}^{\mathbf{T}} = \mathbf{V}^{-1} \tag{2.0.2}$$

Lemma 2.1. *Let* $\mathbf{A}$ *be a mxn matrix.Then,*

$$\rho(\mathbf{A}) = \rho(\mathbf{A}^{\mathbf{T}}\mathbf{A}) \tag{2.0.3}$$

Proof. Let $\mathbf{x} \in N(\mathbf{A})$ where $N(\mathbf{A})$ is the null space of $\mathbf{A}$.
So,

$$\mathbf{A}\mathbf{x} = 0 \tag{2.0.4}$$
$$\implies \mathbf{A}^T\mathbf{A}\mathbf{x} = 0 \tag{2.0.5}$$
$$\implies \mathbf{x} \in N(\mathbf{A}^T\mathbf{A}) \tag{2.0.6}$$

Hence,

$$N(\mathbf{A}) \subseteq N(\mathbf{A}^T\mathbf{A}) \tag{2.0.7}$$

Again,let $\mathbf{x} \in N(\mathbf{A}^T\mathbf{A})$
So,

$$\mathbf{A}^T\mathbf{A}\mathbf{x} = 0 \tag{2.0.8}$$
$$\implies \mathbf{x}^T\mathbf{A}^T\mathbf{A}\mathbf{x} = 0 \tag{2.0.9}$$
$$\implies (\mathbf{A}\mathbf{x})^T(\mathbf{A}\mathbf{x}) = 0 \tag{2.0.10}$$
$$\implies \mathbf{A}\mathbf{x} = 0 \tag{2.0.11}$$
$$\implies \mathbf{x} \in N(\mathbf{A}) \tag{2.0.12}$$

Hence,

$$N(\mathbf{A}^T\mathbf{A}) \subseteq N(\mathbf{A}) \tag{2.0.13}$$

$\therefore$

$$N(\mathbf{A}^T\mathbf{A}) = N(\mathbf{A}) \tag{2.0.14}$$
$$\implies \dim(N(\mathbf{A}^T\mathbf{A})) = \dim(N(\mathbf{A})) \tag{2.0.15}$$
$$\implies \rho(\mathbf{A}^T\mathbf{A}) = \rho(\mathbf{A}) \tag{2.0.16}$$

$\square$

Now,using (2.0.3) ,

$$\rho(\mathbf{V}) = \rho(\mathbf{V}^T\mathbf{V}) \tag{2.0.17}$$
$$\implies \rho(\mathbf{V}) = \rho(\mathbf{V}^{-1}\mathbf{V}) \tag{2.0.18}$$
$$\implies \rho(\mathbf{V}) = \rho(\mathbf{I}) \tag{2.0.19}$$
$$\implies \rho(\mathbf{V}) = 2 \tag{2.0.20}$$

In case of parabola where $\mathbf{V}$ is singular,

$$\rho(\mathbf{V}) < 2 \tag{2.0.21}$$
$$\implies \rho(\mathbf{V}) = 1 \tag{2.0.22}$$

In case of ellipse,hyperbola and circle where $\mathbf{V}$ is non-singular,

$$\rho(\mathbf{V}) = 2 \tag{2.0.23}$$