Assignment 6

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Download all python codes from

```
https://github.com/ka-raja-babu/Matrix-Theory/
    tree/main/Assignment6/Codes
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and latex-tikz codes from

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https://github.com/ka-raja-babu/Matrix-Theory/
    tree/main/Assignment6
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1 Question No. 2.29

Find the coordinates of the focus, axis, the equation of the directrix and latus rectum of the parabola $y^2 = 8x$.

2 Appendix

All parameters of parabola $y^2 = 8x$ can be summarised in table 2.1 .

Note : Given general formula is valid only when parabola is in standard form i.e. $|\mathbf{V}| = 0$ and $\lambda_1 = 0$.

[1] Procedure to find axis :

1) Calculate vertex $\mathbf{c}$ and focus $\mathbf{F}$.
2) Find equation of line joining $\mathbf{c}$ and $\mathbf{F}$ using $\mathbf{x} = \mathbf{F} + k\mathbf{m}$ where $\mathbf{m} = \mathbf{F} - \mathbf{c}$ and $k \in \mathbb{R}$.

[2] Procedure to find directrix :

1) Calculate its direction vector $\mathbf{v_1}$ by using $\mathbf{v}^T\mathbf{v_1} = 0$.
2) Equate $\mathbf{v_1}^T\mathbf{x}$ to -β to get final equation.

[3] Procedure to find latus rectum :

1) Calculate its direction vector $\mathbf{v_2}$ by using $\mathbf{v}^T\mathbf{v_2} = 0$.
2) Equate $\mathbf{v_2}^T\mathbf{x}$ to β to get final equation.

Parameter	Symbol	Value	General Formula
Vertex	$\mathbf{c}$	$\begin{pmatrix}0\\0\end{pmatrix}$	$\begin{pmatrix}\mathbf{u}^T + \eta\mathbf{p_1}^T\\ \mathbf{V}\end{pmatrix}\mathbf{c} = \begin{pmatrix}-f\\ \eta\mathbf{p_1} - \mathbf{u}\end{pmatrix}$
Focal Length	β	2	$\frac{1}{4}\left\|\frac{2\eta}{\lambda_2}\right\|$
Focus	$\mathbf{F}$	$\begin{pmatrix}2\\0\end{pmatrix}$	$\mathbf{F} = \mathbf{c} + \mathbf{a}^T$
Axis[1]		$\begin{pmatrix}0 & 1\end{pmatrix}\mathbf{x} = 0$	$\mathbf{x} = \mathbf{F} + k\mathbf{m}$
Axis direction vector	$\mathbf{v}$	$\begin{pmatrix}0\\1\end{pmatrix}$	$\mathbf{v}^T\mathbf{x} = 0$
Directrix direction vector	$\mathbf{v_1}$	$\begin{pmatrix}1\\0\end{pmatrix}$	$\mathbf{v}^T\mathbf{v_1} = 0$
Directrix[2]		$\begin{pmatrix}1 & 0\end{pmatrix}\mathbf{x} = -2$	$\mathbf{v_1}^T\mathbf{x} = -\beta$
Latus rectum direction vector	$\mathbf{v_2}$	$\begin{pmatrix}1\\0\end{pmatrix}$	$\mathbf{v}^T\mathbf{v_2} = 0$
Latus rectum[3]		$\begin{pmatrix}1 & 0\end{pmatrix}\mathbf{x} = 2$	$\mathbf{v_2}^T\mathbf{x} = \beta$
End points of latus rectum	$\mathbf{M}, \mathbf{N}$	$\begin{pmatrix}2\\4\end{pmatrix}, \begin{pmatrix}2\\-4\end{pmatrix}$	$\begin{pmatrix}\beta\\ \pm y(\beta)\end{pmatrix}$
Length of latus rectum	l	8	$\|\|\mathbf{M} - \mathbf{N}\|\|$

TABLE 2.1: Parameters of parabola $y^2 = 8x$

Theorem 2.1. *General equation of a conic is given by*

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \tag{2.0.1}$$

and can be expressed as

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{2.0.2}$$

where

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \tag{2.0.3}$$

$$\mathbf{u} = \begin{pmatrix} d & e \end{pmatrix} \tag{2.0.4}$$

Theorem 2.2. (2.0.2) *can be expressed as*

$$\mathbf{y}^T\mathbf{D}\mathbf{y} + 2(\mathbf{V}\mathbf{c} + \mathbf{u})^T\mathbf{P}\mathbf{y} + \mathbf{c}^T\mathbf{V}\mathbf{c} + 2\mathbf{u}^T\mathbf{c} + f = 0 \tag{2.0.5}$$

Proof. Using Affine Transformation,

$$\mathbf{x} = \mathbf{P}\mathbf{y} + \mathbf{c} \tag{2.0.6}$$

Using Eigenvalue Decomposition,

$$\mathbf{P}^T\mathbf{V}\mathbf{P} = \mathbf{D} \tag{2.0.7}$$

such that

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \tag{2.0.8}$$

$$\mathbf{P} = \begin{pmatrix} \mathbf{p_1} & \mathbf{p_2} \end{pmatrix}, \quad \mathbf{P}^T = \mathbf{P}^{-1} \tag{2.0.9}$$

Substituting (2.0.6) in (2.0.2),we get

$$(\mathbf{P}\mathbf{y} + \mathbf{c})^T\mathbf{V}(\mathbf{P}\mathbf{y} + \mathbf{c}) + 2\mathbf{u}^T(\mathbf{P}\mathbf{y} + \mathbf{c}) + f = 0 \tag{2.0.10}$$

which can be expressed as

$$\mathbf{y}^T\mathbf{P}^T\mathbf{V}\mathbf{P}\mathbf{y} + 2(\mathbf{V}\mathbf{c} + \mathbf{u})^T\mathbf{P}\mathbf{y} + \mathbf{c}^T\mathbf{V}\mathbf{c} + 2\mathbf{u}^T\mathbf{c} + f = 0 \tag{2.0.11}$$

From (2.0.11) and (2.0.7),

$$\mathbf{y}^T\mathbf{D}\mathbf{y} + 2(\mathbf{V}\mathbf{c} + \mathbf{u})^T\mathbf{P}\mathbf{y} + \mathbf{c}^T\mathbf{V}\mathbf{c} + 2\mathbf{u}^T\mathbf{c} + f = 0 \tag{2.0.12}$$

□

Theorem 2.3. *When* $|\mathbf{V}| = 0, \lambda_1 = 0$ *and*

$$\mathbf{V}\mathbf{p}_1 = 0, \mathbf{V}\mathbf{p}_2 = \lambda_2\mathbf{p}_2 \tag{2.0.13}$$

Then, (2.0.5) *can be expressed as*

$$\mathbf{y}^T\mathbf{D}\mathbf{y} = -2\eta\begin{pmatrix} 1 & 0 \end{pmatrix}\mathbf{y} \tag{2.0.14}$$

Proof. Substituting (2.0.9) in (2.0.11),

$$\mathbf{y}^T\mathbf{D}\mathbf{y} + 2\left(\mathbf{c}^T\mathbf{V} + \mathbf{u}^T\right)\begin{pmatrix} \mathbf{p}_1 & \mathbf{p}_2 \end{pmatrix}\mathbf{y} + \mathbf{c}^T\left(\mathbf{V}\mathbf{c} + \mathbf{u}\right) + \mathbf{u}^T\mathbf{c} + f = 0 \tag{2.0.15}$$

$$\implies \mathbf{y}^T\mathbf{D}\mathbf{y} + 2\begin{pmatrix} \left(\mathbf{c}^T\mathbf{V} + \mathbf{u}^T\right)\mathbf{p}_1 & \left(\mathbf{c}^T\mathbf{V} + \mathbf{u}^T\right)\mathbf{p}_2 \end{pmatrix}\mathbf{y} + \mathbf{c}^T\left(\mathbf{V}\mathbf{c} + \mathbf{u}\right) + \mathbf{u}^T\mathbf{c} + f = 0 \tag{2.0.16}$$

$$\implies \mathbf{y}^T\mathbf{D}\mathbf{y} + 2\begin{pmatrix} \mathbf{u}^T\mathbf{p}_1 & \left(\lambda_2\mathbf{c}^T + \mathbf{u}^T\right)\mathbf{p}_2 \end{pmatrix}\mathbf{y} + \mathbf{c}^T\left(\mathbf{V}\mathbf{c} + \mathbf{u}\right) + \mathbf{u}^T\mathbf{c} + f = 0 \tag{2.0.17}$$

$$\implies \lambda_2 y_2^2 + 2\left(\mathbf{u}^T\mathbf{p}_1\right)y_1 + 2y_2\left(\lambda_2\mathbf{c} + \mathbf{u}\right)^T\mathbf{p}_2 + \mathbf{c}^T\left(\mathbf{V}\mathbf{c} + \mathbf{u}\right) + \mathbf{u}^T\mathbf{c} + f = 0 \tag{2.0.18}$$

which is the equation of a parabola.

Thus,(2.0.5) can be expressed as (2.0.14) by choosing

$$\eta = \mathbf{u}^T\mathbf{p}_1 \tag{2.0.19}$$

such that

$$\mathbf{P}^T\left(\mathbf{V}\mathbf{c} + \mathbf{u}\right) = \eta\begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{2.0.20}$$

$$\mathbf{c}^T\left(\mathbf{V}\mathbf{c} + \mathbf{u}\right) + \mathbf{u}^T\mathbf{c} + f = 0 \tag{2.0.21}$$

$$\therefore \mathbf{y}^T\mathbf{D}\mathbf{y} = -2\eta\begin{pmatrix} 1 & 0 \end{pmatrix}\mathbf{y} \tag{2.0.22}$$

□

Lemma 2.1. *Focal length of a parabola is given by*

$$\beta = \frac{1}{4}\left|\frac{2\mathbf{u}^T\mathbf{p}_1}{\lambda_2}\right| \tag{2.0.23}$$

Proof. From (2.0.18), by comparing the coefficients of y_2^2 and y_1, the focal length of the parabola is obtained as

$$\beta = \frac{1}{4}\left|\frac{2\mathbf{u}^T\mathbf{p}_1}{\lambda_2}\right| \tag{2.0.24}$$

□

Lemma 2.2. *Vertex of a parabola when it is in* *standard form is given by*

$$\begin{pmatrix} \mathbf{u}^T + \eta \mathbf{p_1}^T \\ \mathbf{V} \end{pmatrix} \mathbf{c} = \begin{pmatrix} -f \\ \eta \mathbf{p_1} - \mathbf{u} \end{pmatrix} \tag{2.0.25}$$

Proof. Multiplying (2.0.20) by $\mathbf{P}$ yields

$$(\mathbf{Vc} + \mathbf{u}) = \eta \mathbf{p}_1, \tag{2.0.26}$$

which, upon substituting in (2.0.21) results in

$$\eta \mathbf{c}^T \mathbf{p}_1 + \mathbf{u}^T \mathbf{c} + f = 0 \tag{2.0.27}$$

(2.0.26) and (2.0.27) can be clubbed together to obtain

$$\begin{pmatrix} \mathbf{u}^T + \eta \mathbf{p_1}^T \\ \mathbf{V} \end{pmatrix} \mathbf{c} = \begin{pmatrix} -f \\ \eta \mathbf{p_1} - \mathbf{u} \end{pmatrix} \tag{2.0.28}$$

□

Lemma 2.3. *Focus of a parabola when it is in standard form is given by*

$$\mathbf{a} = \frac{-2\eta \begin{pmatrix} 1 & 0 \end{pmatrix}}{4} \tag{2.0.29}$$

$$\mathbf{F} = \mathbf{c} + \mathbf{a}^T \tag{2.0.30}$$

Proof. From (2.0.22),focus $\mathbf{F}$ can be calculated as

$$\mathbf{a} = \frac{-2\eta \begin{pmatrix} 1 & 0 \end{pmatrix}}{4} \tag{2.0.31}$$

$$\mathbf{F} = \mathbf{c} + \mathbf{a}^T \tag{2.0.32}$$

□

Definition 1. *Axis of a parabola always passes through both vertex and focus*

Lemma 2.4. *Axis of a parabola when it is standard form is given by*

$$\mathbf{x} = \mathbf{F} + k\mathbf{m} \quad (k \in \mathbb{R}) \tag{2.0.33}$$

Proof. Using definition 1, axis can be calculated by finding equation of line joining vertex and focus.
Using $\mathbf{c}$ from (2.0.28) and $\mathbf{F}$ from (2.0.32),
axis can be calculated as

$$\mathbf{m} = \mathbf{F} - \mathbf{c} \quad (\mathbf{m} \text{ is direction vector}) \tag{2.0.34}$$

$$\mathbf{x} = \mathbf{F} + k\mathbf{m} \quad (k \in \mathbb{R}) \tag{2.0.35}$$

$$\implies \mathbf{v}^T \mathbf{x} = 0 \quad (\mathbf{v} \text{ is direction vector of } \mathbf{x}) \tag{2.0.36}$$

□

Definition 2. *Vertex of parabola is at equal distance from focus and directrix and directrix is perpendicular to axis.*

Lemma 2.5. *Directrix of a parabola when it is in standard form is given by*

$$\mathbf{v_1}^T \mathbf{x} = -\beta \tag{2.0.37}$$

Proof. Using definition 2,directrix is always perpendicular to axis such that

$$\mathbf{v}^T \mathbf{v_1} = 0 \quad (\mathbf{v_1} \text{ is direction vector of directrix}) \tag{2.0.38}$$

Using (2.0.24) and (2.0.38),directrix can be expressed as

$$\mathbf{v_1}^T \mathbf{x} = -\beta \tag{2.0.39}$$

□

Definition 3. *Latus rectum of parabola passes through focus and is perpendicular to axis.*

Lemma 2.6. *Latus rectum of a parabola when it is in standard form is given by*

$$\mathbf{v_2}^T \mathbf{x} = \beta \tag{2.0.40}$$

Proof. Using definition 3,latus rectum is always perpendicular to axis such that

$$\mathbf{v}^T \mathbf{v_2} = 0 \quad (\mathbf{v_2} \text{ is direction vector of latus rectum}) \tag{2.0.41}$$

Using (2.0.24) and (2.0.41),latus rectum can be expressed as

$$\mathbf{v_2}^T \mathbf{x} = \beta \tag{2.0.42}$$

□

Lemma 2.7. *End points of latus rectum of a parabola when it is in standard form is given by*

$$\mathbf{M} = \begin{pmatrix} \beta \\ y(\beta) \end{pmatrix} \tag{2.0.43}$$

$$\mathbf{N} = \begin{pmatrix} \beta \\ -y(\beta) \end{pmatrix} \tag{2.0.44}$$

Proof. Using (2.0.1) and (2.0.24),end points of latus rectum can be expressed as

$$\mathbf{M} = \begin{pmatrix} \beta \\ y(\beta) \end{pmatrix} \tag{2.0.45}$$

$$\mathbf{N} = \begin{pmatrix} \beta \\ -y(\beta) \end{pmatrix} \tag{2.0.46}$$

$\square$

Lemma 2.8. *Length of latus rectum is given by*

$$l = \|\mathbf{M} - \mathbf{N}\| \tag{2.0.47}$$

Proof. Using (2.0.45) and (2.0.46),length of latus rectum can be expressed as

$$l = \|\mathbf{M} - \mathbf{N}\| \tag{2.0.48}$$

$\square$

3 SOLUTION

Given parabola is

$$y^2 = 8x \tag{3.0.1}$$

$$\implies y^2 - 8x = 0 \tag{3.0.2}$$

Vector form of given parabola is

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} + 2 \begin{pmatrix} -4 & 0 \end{pmatrix} \mathbf{x} + 0 = 0 \tag{3.0.3}$$

$\therefore$

$$\mathbf{V} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \mathbf{u} = \begin{pmatrix} -4 \\ 0 \end{pmatrix}, f = 0 \tag{3.0.4}$$

$\because$ $|\mathbf{V}| = 0$ and $\lambda_1 = 0$ i.e. it is in standard form $\therefore$

$$\mathbf{P} = \mathbf{I} \implies \mathbf{p_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{3.0.5}$$

$$\eta = \mathbf{u}^T \mathbf{p_1} = -4 \tag{3.0.6}$$

The vertex $\mathbf{c}$ is given by

$$\begin{pmatrix} -8 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{c} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{3.0.7}$$

$$\implies \mathbf{c} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{3.0.8}$$

The focal length β is given by

$$\beta = \frac{1}{4} \left| \frac{2\eta}{\lambda_2} \right| = \frac{1}{4} \left| \frac{-8}{1} \right| = 2 \tag{3.0.9}$$

The focus $\mathbf{F}$ is given by

$$\mathbf{a} = \frac{-2\eta \begin{pmatrix} 1 & 0 \end{pmatrix}}{4} = \begin{pmatrix} 2 & 0 \end{pmatrix} \tag{3.0.10}$$

$$\mathbf{F} = \mathbf{c} + \mathbf{a}^T = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \tag{3.0.11}$$

$\because$ Axis of parabola passes through both vertex and focus .
$\therefore$ Axis of parabola is given by

$$\mathbf{m} = \mathbf{F} - \mathbf{c} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \tag{3.0.12}$$

$$\mathbf{x} = \mathbf{F} + k\mathbf{m} = \begin{pmatrix} 2 + 2k \\ 0 \end{pmatrix} \tag{3.0.13}$$

$$\implies \begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} = 0 \tag{3.0.14}$$

$$\implies \mathbf{v} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{3.0.15}$$

$\because$ Vertex of parabola is at equal distance from focus and directrix and directrix is perpendicular to axis.
$\therefore$ Directrix of parabola is given by

$$\mathbf{v}^T \mathbf{v_1} = 0 \tag{3.0.16}$$

$$\implies \mathbf{v_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{3.0.17}$$

So ,

$$\mathbf{v_1}^T \mathbf{x} = -\beta \tag{3.0.18}$$

$$\implies \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = -2 \tag{3.0.19}$$

$\because$ Latus rectum of parabola passes through focus and is perpendicular to axis.
$\therefore$ Latus rectum of parabola is given by

$$\mathbf{v}^T \mathbf{v_2} = 0 \tag{3.0.20}$$

$$\implies \mathbf{v_2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{3.0.21}$$

So ,

$$\mathbf{v_2}^T \mathbf{x} = \beta \tag{3.0.22}$$

$$\implies \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = 2 \tag{3.0.23}$$

End points of latus rectum are

$$\mathbf{M} = \begin{pmatrix} 2 \\ 4 \end{pmatrix} \tag{3.0.24}$$

$$\mathbf{N} = \begin{pmatrix} 2 \\ -4 \end{pmatrix} \tag{3.0.25}$$

So,the length of latus rectum l is

$$l = \|\mathbf{M} - \mathbf{N}\| = 8 \tag{3.0.26}$$

Plot of given parabola

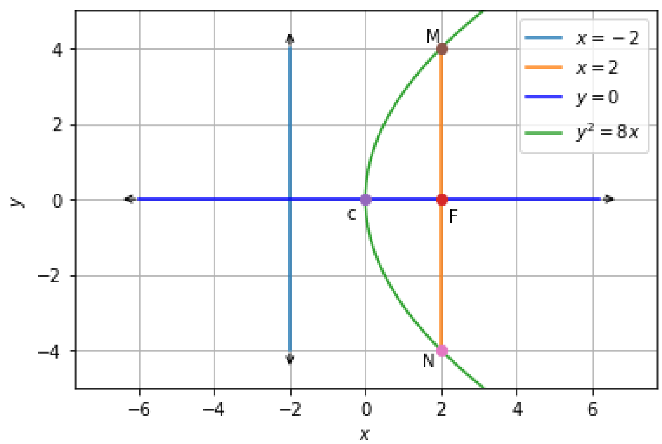

Fig. 3.1: Parabola $y^2 = 8x$