Challenge Problem 1

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Download latex-tikz code from

https://github.com/ka−raja−babu/Matrix−Theory/tree/main/ChallengeProblem1

1 Challenge Question 1

Show that the matrix $(t\mathbf{I} - \mathbf{n}\mathbf{n}^T)$ in the given document is a rank 1 matrix for a parabola.

2 Solution

Let

$$\mathbf{V} = (t\mathbf{I} - \mathbf{n}\mathbf{n}^T) \tag{2.0.1}$$

where,

$$t = \frac{\|\mathbf{n}\|^2}{e^2} \tag{2.0.2}$$

Theorem 2.1. *For any square matrix* $\mathbf{A}$ *of order nxn having rank 1*

$$\mathbf{A}^2 = c\mathbf{A} \tag{2.0.3}$$

where c is any scalar.

Proof. Let $\mathbf{A} = (c_1 \ c_2 \ .....c_n)$ where c_i is the i-th column of $\mathbf{A}$.
$\because$ $\mathbf{A}$ has the rank 1,then

$$c_1 \neq 0 \tag{2.0.4}$$
$$c_i = \alpha_i c_1, i \geq 2 \tag{2.0.5}$$

Hence,

$$\mathbf{A} = c_1(1 \ \ \alpha_2.....\alpha_n) = \mathbf{C}\mathbf{R} \tag{2.0.6}$$

Now,

$$\begin{aligned} \mathbf{A}^2 &= (\mathbf{C}\mathbf{R})(\mathbf{C}\mathbf{R}) && (2.0.7)\\ &= \mathbf{C}(\mathbf{R}\mathbf{C})\mathbf{R} && (2.0.8)\\ &= (\mathbf{R}\mathbf{C})\mathbf{C}\mathbf{R} && (2.0.9)\\ &= c(\mathbf{C}\mathbf{R}) && (2.0.10)\\ &= c\mathbf{A} && (2.0.11) \end{aligned}$$

$\square$

Now,

$$\begin{aligned} \mathbf{V}^2 &= (t\mathbf{I} - \mathbf{n}\mathbf{n}^T)(t\mathbf{I} - \mathbf{n}\mathbf{n}^T) && (2.0.12)\\ &= (t\mathbf{I})^2 + (\mathbf{n}\mathbf{n}^T)^2 - 2t\mathbf{I}\mathbf{n}\mathbf{n}^T && (2.0.13)\\ &= t^2\mathbf{I} + \|\mathbf{n}\|^2\,\mathbf{n}\mathbf{n}^T - 2t\mathbf{n}\mathbf{n}^T && (2.0.14)\\ &= \frac{\|\mathbf{n}\|^4}{e^4}\mathbf{I} + \mathbf{n}\mathbf{n}^T(\|\mathbf{n}\|^2 - 2t) && (2.0.15)\\ &= \frac{\|\mathbf{n}\|^4}{e^4}\mathbf{I} + \|\mathbf{n}\|^2\,\mathbf{n}\mathbf{n}^T(1 - \frac{2}{e^2}) && (2.0.16)\\ &= \frac{\|\mathbf{n}\|^4}{e^4}\mathbf{I} + \|\mathbf{n}\|^2\,\mathbf{n}\mathbf{n}^T(\frac{e^2-2}{e^2}) && (2.0.17)\\ &= \frac{\|\mathbf{n}\|^2}{e^2}(\frac{\|\mathbf{n}\|^2}{e^2}\mathbf{I} + \mathbf{n}\mathbf{n}^T(e^2 - 2)) && (2.0.18) \end{aligned}$$

Now,for $e = 1$

$$\begin{aligned} \mathbf{V}^2 &= \|\mathbf{n}\|^2\,(\|\mathbf{n}\|^2\,\mathbf{I} - \mathbf{n}\mathbf{n}^T) && (2.0.19)\\ \implies \mathbf{V}^2 &= c\mathbf{V} && (2.0.20) \end{aligned}$$

where scalar $c = \|\mathbf{n}\|^2$.
Hence,using theorem 2.1 ,

$$rank(\mathbf{V}) = 1 \tag{2.0.21}$$