Challenge Problem 1

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Download latex-tikz code from

https://github.com/ka−raja−babu/Matrix−Theory/tree/main/ChallengeProblem1

1 Challenge Question 1

Show that the matrix $(t\mathbf{I} - \mathbf{n}\mathbf{n}^T)$ in the given document is a rank 1 matrix for a parabola.

2 Solution

Let

$$\mathbf{V} = (t\mathbf{I} - \mathbf{n}\mathbf{n}^T) \tag{2.0.1}$$

where,

$$t = \frac{\|\mathbf{n}\|^2}{e^2} \tag{2.0.2}$$

Theorem 2.1. *For any square matrix $\mathbf{A}$ of order nxn having rank 1*

$$\mathbf{A}^2 = c\mathbf{A} \tag{2.0.3}$$

where c is any scalar.

Proof. According to Cayley Hamilton Theorem, characteristic polynomial $p(\mathbf{A})$ for a nxn matrix $\mathbf{A}$ is given by

$$p(\mathbf{A}) = \mathbf{A}^n + c_{n-1}\mathbf{A}^{n-1} + ... + c_1\mathbf{A} + (-1)^n |\mathbf{A}| \mathbf{I}_n = 0 \tag{2.0.4}$$

For n=2,

$$p(\mathbf{A}) = \mathbf{A}^2 + c_1\mathbf{A} + (-1)^2 |\mathbf{A}| \mathbf{I}_2 = 0 \tag{2.0.5}$$

Rank = 1 implies $|\mathbf{A}| = 0$
$\therefore$

$$\mathbf{A}^2 + c_1\mathbf{A} = 0 \tag{2.0.6}$$

$$\implies \mathbf{A}^2 = c\mathbf{A} \tag{2.0.7}$$

where $c = -c_1$ ☐

Now,

$$\mathbf{V}^2 = (t\mathbf{I} - \mathbf{n}\mathbf{n}^T)(t\mathbf{I} - \mathbf{n}\mathbf{n}^T) \tag{2.0.8}$$

$$= (t\mathbf{I})^2 + (\mathbf{n}\mathbf{n}^T)^2 - 2t\mathbf{I}\mathbf{n}\mathbf{n}^T \tag{2.0.9}$$

$$= t^2\mathbf{I} + \|\mathbf{n}\|^2 \mathbf{n}\mathbf{n}^T - 2t\mathbf{n}\mathbf{n}^T \tag{2.0.10}$$

$$= \frac{\|\mathbf{n}\|^4}{e^4}\mathbf{I} + \mathbf{n}\mathbf{n}^T(\|\mathbf{n}\|^2 - 2t) \tag{2.0.11}$$

$$= \frac{\|\mathbf{n}\|^4}{e^4}\mathbf{I} + \|\mathbf{n}\|^2 \mathbf{n}\mathbf{n}^T(1 - \frac{2}{e^2}) \tag{2.0.12}$$

$$= \frac{\|\mathbf{n}\|^4}{e^4}\mathbf{I} + \|\mathbf{n}\|^2 \mathbf{n}\mathbf{n}^T(\frac{e^2 - 2}{e^2}) \tag{2.0.13}$$

$$= \frac{\|\mathbf{n}\|^2}{e^2}(\frac{\|\mathbf{n}\|^2}{e^2}\mathbf{I} + \mathbf{n}\mathbf{n}^T(e^2 - 2)) \tag{2.0.14}$$

Now,for $e = 1$

$$\mathbf{V}^2 = \|\mathbf{n}\|^2 (\|\mathbf{n}\|^2 \mathbf{I} - \mathbf{n}\mathbf{n}^T) \tag{2.0.15}$$

$$\implies \mathbf{V}^2 = c\mathbf{V} \tag{2.0.16}$$

where scalar $c = \|\mathbf{n}\|^2$.
Hence,using theorem 2.1 ,

$$rank(\mathbf{V}) = 1 \tag{2.0.17}$$