Challenge Problem 1

K.A. Raja Babu

Download latex-tikz code from

1 CHALLENGE QUESTION 1

Show that the matrix $(t\mathbf{I} - \mathbf{nn}^T)$ in the given document is a rank 1 matrix for a parabola.

2 SOLUTION

Let

$$\mathbf{V} = (t\mathbf{I} - \mathbf{nn}^T) \qquad (2.0.1)$$

where,

$$t = \frac{\|\mathbf{n}\|^2}{e^2} \qquad (2.0.2)$$

Theorem 2.1. *For any square matrix* $\mathbf{A}$ *of order nxn having rank 1*

$$\mathbf{A}^2 = c\mathbf{A} \qquad (2.0.3)$$

where c is any scalar.

Proof. Let $\mathbf{A} = (c_1 \ c_2 \ .....c_n)$ where c_i is the i-th column of $\mathbf{A}$.

$\because \mathbf{A}$ has the rank 1,then

$$c_i \neq 0 \qquad (2.0.4)$$
$$c_i = \alpha_i c_1, i \geq 2 \qquad (2.0.5)$$

Hence,

$$\mathbf{A} = c_1(1 \ \alpha_2.....\alpha_n) = \mathbf{CR} \qquad (2.0.6)$$

Now,

$$\mathbf{A}^2 = (\mathbf{CR})(\mathbf{CR}) \qquad (2.0.7)$$
$$= \mathbf{C(RC)R} \qquad (2.0.8)$$
$$= (\mathbf{RC})\mathbf{CR} \qquad (2.0.9)$$
$$= c(\mathbf{CR}) \qquad (2.0.10)$$
$$= c\mathbf{A} \qquad (2.0.11)$$

$\square$

Now,

$$\mathbf{V}^2 = (t\mathbf{I} - \mathbf{nn}^T)(t\mathbf{I} - \mathbf{nn}^T) \qquad (2.0.12)$$
$$= (t\mathbf{I})^2 + (\mathbf{nn}^T)^2 - 2t\mathbf{Inn}^T \qquad (2.0.13)$$
$$= t^2\mathbf{I} + \|\mathbf{n}\|^2 \mathbf{nn}^T - 2t\mathbf{nn}^T \qquad (2.0.14)$$
$$= \frac{\|\mathbf{n}\|^4}{e^4}\mathbf{I} + \mathbf{nn}^T(\|\mathbf{n}\|^2 - 2t) \qquad (2.0.15)$$
$$= \frac{\|\mathbf{n}\|^4}{e^4}\mathbf{I} + \|\mathbf{n}\|^2 \mathbf{nn}^T(1 - \frac{2}{e^2}) \qquad (2.0.16)$$
$$= \frac{\|\mathbf{n}\|^4}{e^4}\mathbf{I} + \|\mathbf{n}\|^2 \mathbf{nn}^T(\frac{e^2 - 2}{e^2}) \qquad (2.0.17)$$
$$= \frac{\|\mathbf{n}\|^2}{e^2}(\frac{\|\mathbf{n}\|^2}{e^2}\mathbf{I} + \mathbf{nn}^T(e^2 - 2)) \qquad (2.0.18)$$

Now,for $e = 1$

$$\mathbf{V}^2 = \|\mathbf{n}\|^2 (\|\mathbf{n}\|^2 \mathbf{I} - \mathbf{nn}^T) \qquad (2.0.19)$$
$$\implies \mathbf{V}^2 = c\mathbf{V} \qquad (2.0.20)$$

where scalar $c = \|\mathbf{n}\|^2$.
Hence,using theorem 2.1 ,

$$rank(\mathbf{V}) = 1 \qquad (2.0.21)$$