Challenge Problem 3

K.A. Raja Babu

Download latex-tikz codes from

https://github.com/ka−raja−babu/Matrix−Theory/tree/main/ChallengeProblem3

1 Challenge Question 3

For the quadratic equation to not have any real roots, the y coordinate should always be positive . Express this in terms of the matrix/vector parameters of the parabola .

2 Solution

The general form of a quadratic equation is

$$y = ax^2 + bx + c \tag{2.0.1}$$

$$\implies ax^2 + bx + c - y = 0 \tag{2.0.2}$$

which can be written in vector form as

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{2.0.3}$$

where

$$\mathbf{V} = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \tag{2.0.4}$$

$$\mathbf{u} = \begin{pmatrix} \frac{b}{2} \\ \frac{-1}{2} \end{pmatrix} \tag{2.0.5}$$

$$f = c \tag{2.0.6}$$

Using eigenvalue decomposition,

$$\mathbf{D} = \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix} \tag{2.0.7}$$

$$\mathbf{P} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{2.0.8}$$

and

$$\eta = \mathbf{u}^T\mathbf{p_1} \tag{2.0.9}$$

$$= \frac{-1}{2} \tag{2.0.10}$$

Now,for y coordinate to be always positive,two conditions need to be satisfied:

1) y-coordinate of vertex **c** of parabola needs to be always positive.
2) Coefficient a of x^2 needs to be always positive.

$\therefore$ For condition 1:

$$\mathbf{c} = \begin{pmatrix} \mathbf{u}^T + \eta\mathbf{p_1}^T \\ \mathbf{V} \end{pmatrix}^{-1} \begin{pmatrix} -f \\ \eta\mathbf{p_1} - \mathbf{u} \end{pmatrix} \tag{2.0.11}$$

$$\mathbf{p_1}^T\mathbf{c} > 0 \tag{2.0.12}$$

For condition 2:

$$\mathbf{p_2}^T\mathbf{V}\mathbf{p_2} > 0 \tag{2.0.13}$$

Hence,condition for non-real roots in terms of matrix/vector parameters of the parabola is

$$\begin{pmatrix} \mathbf{p_1}^T\mathbf{c} \\ \mathbf{p_2}^T\mathbf{V}\mathbf{p_2} \end{pmatrix} \succ 0 \tag{2.0.14}$$