

# Assignment 6

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Download all python codes from

<https://github.com/ka-raja-babu/Matrix-Theory/tree/main/Assignment6/Codes>

and latex-tikz codes from

<https://github.com/ka-raja-babu/Matrix-Theory/tree/main/Assignment6>

All parameters of parabola  $y^2 = 8x$  can be summarised in table 1.1 .

Note : Given general formula is valid only when parabola is in standard form i.e.  $|\mathbf{V}| = 0$  and  $\lambda_1 = 0$  .

**Lemma 1.1.** *General equation of a conic is given by*

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \quad (1.0.1)$$

and can be expressed as

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (1.0.2)$$

where

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \quad (1.0.3)$$

$$\mathbf{u} = \begin{pmatrix} d & e \end{pmatrix} \quad (1.0.4)$$

**Lemma 1.2.** (1.0.2) can be expressed as

$$\mathbf{y}^T \mathbf{D} \mathbf{y} + 2(\mathbf{V}\mathbf{c} + \mathbf{u})^T \mathbf{P} \mathbf{y} + \mathbf{c}^T \mathbf{V} \mathbf{c} + 2\mathbf{u}^T \mathbf{c} + f = 0 \quad (1.0.5)$$

where

$$\mathbf{x} = \mathbf{P} \mathbf{y} + \mathbf{c} \quad (1.0.6)$$

$$\mathbf{P}^T \mathbf{V} \mathbf{P} = \mathbf{D} \quad (1.0.7)$$

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \quad (1.0.8)$$

$$\mathbf{P} = \begin{pmatrix} \mathbf{p}_1 & \mathbf{p}_2 \end{pmatrix} \quad (1.0.9)$$

**Lemma 1.3.** (1.0.5) can be expressed as

$$\mathbf{y}^T \mathbf{D} \mathbf{y} = \mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f \quad (|\mathbf{V}| \neq 0) \quad (1.0.10)$$

$$\mathbf{y}^T \mathbf{D} \mathbf{y} = -2\eta \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{y} \quad (|\mathbf{V}| = 0) \quad (1.0.11)$$

where

$$\eta = \mathbf{u}^T \mathbf{p}_1 \quad (1.0.12)$$

**Lemma 1.4.** *Focal length of a parabola is given by*

$$\beta = \frac{1}{2} \left| \frac{\mathbf{u}^T \mathbf{p}_1}{\lambda_2} \right| \quad (1.0.13)$$

## 1 APPENDIX

Parameter	Symbol	General Formula	Value
Vertex	$\mathbf{c}$	$\begin{pmatrix} \mathbf{u}^T + \eta \mathbf{p}_1^T \\ \mathbf{V} \end{pmatrix} \mathbf{c} = \begin{pmatrix} -f \\ \eta \mathbf{p}_1 - \mathbf{u} \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$
Focal Length	$\beta$	$\frac{1}{2} \left  \frac{\eta}{\lambda_2} \right $	2
Focus	$\mathbf{F}$	$\mathbf{F} = \mathbf{c} + \frac{-2\eta \begin{pmatrix} 1 & 0 \end{pmatrix}^T}{4}$	$\begin{pmatrix} 2 \\ 0 \end{pmatrix}$
Axis		$(\mathbf{V}\mathbf{c} + \mathbf{u})^T \mathbf{x} = 0$	$\begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} = 0$
Directrix		$(\mathbf{V}\mathbf{c} + \mathbf{u})^T (\mathbf{x} + \beta) + \mathbf{u}^T \mathbf{c} + f = 0$	$\begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = -2$
Latus rectum		$(\mathbf{V}\mathbf{c} + \mathbf{u})^T (\mathbf{x} - \beta) + \mathbf{u}^T \mathbf{c} + f = 0$	$\begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = 2$
End points of latus rectum	$\kappa$	$\mathbf{u}^T \kappa = -\frac{(\kappa^T \mathbf{V} \kappa + f)}{2}$	$\begin{pmatrix} 2 \\ \pm 4 \end{pmatrix}$
Length of latus rectum	$l$	$\ \beta(\mathbf{V}\mathbf{c} + \mathbf{u})^T\ $	8

TABLE 1.1: Parameters of parabola  $y^2 = 8x$

**Lemma 1.5.** Vertex of a parabola when it is in standard form is given by

$$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} \quad (|\mathbf{V}| \neq 0) \quad (1.0.14)$$

$$\begin{pmatrix} \mathbf{u}^T + \eta \mathbf{p}_1^T \\ \mathbf{V} \end{pmatrix} \mathbf{c} = \begin{pmatrix} -f \\ \eta \mathbf{p}_1 - \mathbf{u} \end{pmatrix} \quad (|\mathbf{V}| = 0) \quad (1.0.15)$$

**Lemma 1.6.** Focus of a parabola when it is in standard form is given by

$$\mathbf{F} = \begin{pmatrix} -f \\ \eta \mathbf{p}_1 - \mathbf{u} \end{pmatrix} \begin{pmatrix} \mathbf{u}^T + \eta \mathbf{p}_1^T \\ \mathbf{V} \end{pmatrix}^{-1} + \frac{-2\eta \begin{pmatrix} 1 & 0 \end{pmatrix}^T}{4} \quad (1.0.16)$$

*Proof.* From (1.0.11) and (1.0.15), focus  $\mathbf{F}$  is given by

$$\mathbf{F} = \mathbf{c} + \frac{-2\eta \begin{pmatrix} 1 & 0 \end{pmatrix}^T}{4} \quad (1.0.17)$$

$$\Rightarrow \mathbf{F} = \begin{pmatrix} -f \\ \eta \mathbf{p}_1 - \mathbf{u} \end{pmatrix} \begin{pmatrix} \mathbf{u}^T + \eta \mathbf{p}_1^T \\ \mathbf{V} \end{pmatrix}^{-1} + \frac{-2\eta \begin{pmatrix} 1 & 0 \end{pmatrix}^T}{4} \quad (1.0.18)$$

**Lemma 1.7.** Normal vector at any point  $\mathbf{q}$  of a conic section is obtained as

$$\mathbf{n} = \mathbf{V}\mathbf{q} + \mathbf{u} \quad (1.0.19)$$

**Lemma 1.8.** Axis of a parabola when it is in standard form is given by

$$(\mathbf{V}\mathbf{c} + \mathbf{u})^T \mathbf{x} = 0 \quad (1.0.20)$$

*Proof.* Using (1.7), Normal vector at vertex is given by

$$(\mathbf{V}\mathbf{c} + \mathbf{u})^T \quad (1.0.21)$$

So, axis is given as

$$(\mathbf{V}\mathbf{c} + \mathbf{u})^T \mathbf{x} = 0 \quad (1.0.22)$$

□

**Lemma 1.9.** Given the point of contact  $\mathbf{q}$ , equation of tangent is given by

$$(\mathbf{V}\mathbf{q} + \mathbf{u})^T \mathbf{x} + \mathbf{u}^T \mathbf{q} + f = 0 \quad (1.0.23)$$

**Lemma 1.10.** Directrix of a parabola when it is in standard form is given by

$$(\mathbf{V}\mathbf{c} + \mathbf{u})^T (\mathbf{x} + \beta) + \mathbf{u}^T \mathbf{c} + \mathbf{f} = 0 \quad (1.0.24)$$

*Proof.* Using (1.0.23), directrix is given by

$$(\mathbf{V}\mathbf{c} + \mathbf{u})^T (\mathbf{x} + \beta) + \mathbf{u}^T \mathbf{c} + \mathbf{f} = 0 \quad (1.0.25)$$

□

**Lemma 1.11.** Latus rectum of a parabola when it is in standard form is given by

$$(\mathbf{V}\mathbf{c} + \mathbf{u})^T (\mathbf{x} - \beta) + \mathbf{u}^T \mathbf{c} + \mathbf{f} = 0 \quad (1.0.26)$$

*Proof.* Using (1.0.23), latus rectum is given by

$$(\mathbf{V}\mathbf{c} + \mathbf{u})^T (\mathbf{x} - \beta) + \mathbf{u}^T \mathbf{c} + \mathbf{f} = 0 \quad (1.0.27)$$

□

**Lemma 1.12.** End points of latus rectum of a parabola when it is in standard form is given by

$$\mathbf{u}^T \kappa = -\frac{(\kappa^T \mathbf{V} \kappa + f)}{2} \quad (1.0.28)$$

where

$$\kappa = \begin{pmatrix} \beta \\ \mathbf{y} \end{pmatrix} \quad (1.0.29)$$

*Proof.* Substituting  $x = \kappa$  in (1.0.1), end points of latus rectum are

$$\mathbf{u}^T \kappa = -\frac{(\kappa^T \mathbf{V} \kappa + f)}{2} \quad (1.0.30)$$

□

**Lemma 1.13.** Length of latus rectum is given by

$$l = \|\beta(\mathbf{V}\mathbf{c} + \mathbf{u})^T\| \quad (1.0.31)$$

*Proof.* Using (1.0.26), length of latus rectum can be expressed as

$$l = \|\beta(\mathbf{V}\mathbf{c} + \mathbf{u})^T\| \quad (1.0.32)$$

□

## 2 QUESTION No. 2.29

Find the coordinates of the focus, axis, the equation of the directrix and latus rectum of the parabola  $y^2 = 8x$ .

## 3 SOLUTION

Given parabola is

$$y^2 = 8x \quad (3.0.1)$$

$$\Rightarrow y^2 - 8x = 0 \quad (3.0.2)$$

Vector form of given parabola is

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} + 2 \begin{pmatrix} -4 & 0 \end{pmatrix} \mathbf{x} + 0 = 0 \quad (3.0.3)$$

$\therefore$

$$\mathbf{V} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \mathbf{u} = \begin{pmatrix} -4 \\ 0 \end{pmatrix}, f = 0 \quad (3.0.4)$$

$\therefore |\mathbf{V}| = 0$  and  $\lambda_1 = 0$  i.e. it is in standard form  
 $\therefore$

$$\mathbf{P} = \mathbf{I} \Rightarrow \mathbf{p}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (3.0.5)$$

$$\eta = \mathbf{u}^T \mathbf{p}_1 = -4 \quad (3.0.6)$$

The vertex  $\mathbf{c}$  is given by

$$\begin{pmatrix} -8 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{c} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad (3.0.7)$$

$$\Rightarrow \mathbf{c} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (3.0.8)$$

The focal length  $\beta$  is given by

$$\beta = \frac{1}{4} \left| \frac{2\eta}{\lambda_2} \right| = \frac{1}{4} \left| \frac{-8}{1} \right| = 2 \quad (3.0.9)$$

The focus  $\mathbf{F}$  is given by

$$\mathbf{F} = \mathbf{c} + \frac{-2\eta \begin{pmatrix} 1 & 0 \end{pmatrix}^T}{4} \quad (3.0.10)$$

$$\Rightarrow \mathbf{F} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \end{pmatrix} \quad (3.0.11)$$

$$\Rightarrow \mathbf{F} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \quad (3.0.12)$$

Axis of parabola is given by

$$k(\mathbf{V}\mathbf{c} + \mathbf{u})^T \mathbf{x} = 0 \quad (k \in \mathbb{R}) \quad (3.0.13)$$

$$\Rightarrow k \begin{pmatrix} -4 & 0 \end{pmatrix} \mathbf{x} = 0 \quad (3.0.14)$$

$$\Rightarrow \begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} = 0 \quad (3.0.15)$$

Directrix of parabola is given by

$$(\mathbf{V}\mathbf{c} + \mathbf{u})^T (\mathbf{x} + \beta) + \mathbf{u}^T \mathbf{c} + f = 0 \quad (3.0.16)$$

$$\Rightarrow \begin{pmatrix} -4 & 0 \end{pmatrix} (\mathbf{x} + 2) = 0 \quad (3.0.17)$$

$$\Rightarrow \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = -2 \quad (3.0.18)$$

Latus rectum of parabola is given by

$$(\mathbf{V}\mathbf{c} + \mathbf{u})^T (\mathbf{x} - \beta) + \mathbf{u}^T \mathbf{c} + f = 0 \quad (3.0.19)$$

$$\Rightarrow \begin{pmatrix} -4 & 0 \end{pmatrix} (\mathbf{x} - 2) = 0 \quad (3.0.20)$$

$$\Rightarrow \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = 2 \quad (3.0.21)$$

End points of latus rectum are

$$\mathbf{u}^T \kappa = -\frac{(\kappa^T \mathbf{V} \kappa + f)}{2} \quad (3.0.22)$$

$$\Rightarrow \begin{pmatrix} -4 & 0 \end{pmatrix} \kappa = -\frac{\kappa^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \kappa + 0}{2} \quad (3.0.23)$$

$$\Rightarrow \kappa = \begin{pmatrix} 2 \\ \pm 4 \end{pmatrix} \quad (3.0.24)$$

Length of latus rectum  $l$  is

$$l = \|\beta(\mathbf{V}\mathbf{c} + \mathbf{u})^T\| \quad (3.0.25)$$

$$\Rightarrow l = \|2 \begin{pmatrix} -4 & 0 \end{pmatrix}\| \quad (3.0.26)$$

$$\Rightarrow l = 8 \quad (3.0.27)$$

Plot of given parabola

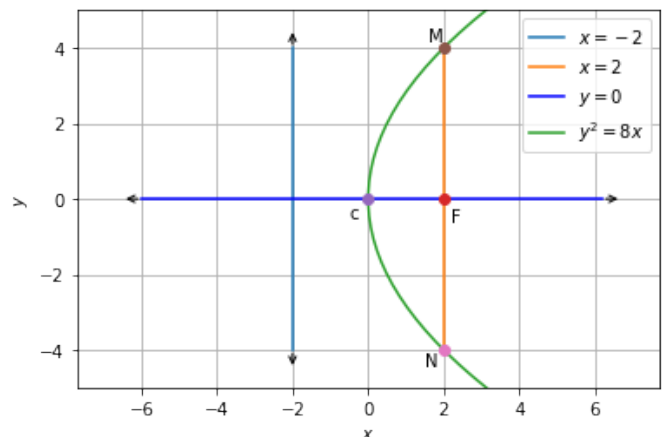


Fig. 3.1: Parabola  $y^2 = 8x$

## 4 GENERALISATION

## 4.1 Circle

## 4.1.1 Property:

$$\mathbf{V} = \mathbf{D} = \mathbf{P} = \mathbf{I} \quad (4.1.1)$$

## 4.1.2 Standard Form: From (1.0.2),

$$\mathbf{x}^T \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (4.1.2)$$

## 4.1.3 Centre: From (1.0.14),

$$\mathbf{c} = -\mathbf{u} \quad (4.1.3)$$

## 4.1.4 Radius: From (1.0.10),

$$\mathbf{r} = \sqrt{\mathbf{u}^T \mathbf{u} - f} \quad (4.1.4)$$

## 4.2 Ellipse

## 4.2.1 Property:

$$|\mathbf{V}| > 0 \quad (4.2.1)$$

$$\lambda_1 > 0, \lambda_2 < 0 \quad (4.2.2)$$

## 4.2.2 Standard Form: From (1.0.10),

$$\frac{\mathbf{y}^T \mathbf{D} \mathbf{y}}{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f} = 1 \quad (4.2.3)$$

## 4.2.3 Centre: From (1.0.14),

$$\mathbf{c} = -\mathbf{V}^{-1} \mathbf{u} \quad (4.2.4)$$

## 4.2.4 Axes: From (1.0.10),

$$\begin{cases} \sqrt{\frac{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}{\lambda_1}} \\ \sqrt{\frac{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}{\lambda_2}} \end{cases} \quad (4.2.5)$$

## 4.3 Hyperbola

## 4.3.1 Property:

$$|\mathbf{V}| < 0 \quad (4.3.1)$$

$$\lambda_1 > 0, \lambda_2 < 0 \quad (4.3.2)$$

## 4.3.2 Standard Form: From (1.0.10),

$$\frac{\mathbf{y}^T \mathbf{D} \mathbf{y}}{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f} = 1 \quad (4.3.3)$$

## 4.3.3 Centre: From (1.0.14),

$$\mathbf{c} = -\mathbf{V}^{-1} \mathbf{u} \quad (4.3.4)$$

## 4.3.4 Axes: From (1.0.10),

$$\begin{cases} \sqrt{\frac{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}{\lambda_1}} \\ \sqrt{\frac{f - \mathbf{u}^T \mathbf{V}^{-1} \mathbf{u}}{\lambda_2}} \end{cases} \quad (4.3.5)$$

## 4.4 Parabola

## 4.4.1 Property:

$$|\mathbf{V}| = 0 \quad (4.4.1)$$

$$\lambda_1 = 0 \quad (4.4.2)$$

## 4.4.2 Standard Form: From (1.0.11),

$$\mathbf{y}^T \mathbf{D} \mathbf{y} = -2\eta \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{y} \quad (4.4.3)$$

## 4.4.3 Centre: From (1.0.15),

$$\begin{pmatrix} \mathbf{u}^T + \eta \mathbf{p}_1^T \\ \mathbf{V} \end{pmatrix} \mathbf{c} = \begin{pmatrix} -f \\ \eta \mathbf{p}_1 - \mathbf{u} \end{pmatrix} \quad (4.4.4)$$

## 4.4.4 Focal length: From (1.0.13),

$$\beta = \frac{1}{2} \left| \frac{\mathbf{u}^T \mathbf{p}_1}{\lambda_2} \right| \quad (4.4.5)$$

## 4.4.5 Focus: From (1.0.16),

$$\mathbf{F} = \begin{pmatrix} -f \\ \eta \mathbf{p}_1 - \mathbf{u} \end{pmatrix} \begin{pmatrix} \mathbf{u}^T + \eta \mathbf{p}_1^T \\ \mathbf{V} \end{pmatrix}^{-1} + \frac{-2\eta \begin{pmatrix} 1 & 0 \end{pmatrix}^T}{4} \quad (4.4.6)$$

## 4.4.6 Axis: From (1.0.20),

$$k(\mathbf{V} \mathbf{c} + \mathbf{u})^T \mathbf{x} = 0 \quad (4.4.7)$$

## 4.4.7 Directrix: From (1.0.24),

$$(\mathbf{V} \mathbf{c} + \mathbf{u})^T (\mathbf{x} + \beta) + \mathbf{u}^T \mathbf{c} + \mathbf{f} = 0 \quad (4.4.8)$$

## 4.4.8 Latus Rectum: From (1.0.26),

$$(\mathbf{V} \mathbf{c} + \mathbf{u})^T (\mathbf{x} - \beta) + \mathbf{u}^T \mathbf{c} + \mathbf{f} = 0 \quad (4.4.9)$$

## 4.4.9 End points of latus rectum: From (1.0.28),

$$\mathbf{u}^T \kappa = -\frac{(\kappa^T \mathbf{V} \kappa + f)}{2} \quad (4.4.10)$$

## 4.4.10 Length of latus rectum: From (1.0.31),

$$l = \|\beta(\mathbf{V} \mathbf{c} + \mathbf{u})^T\| \quad (4.4.11)$$

Conic	Property	Standard Form	Standard Parameters
Circle	$\mathbf{V} = \mathbf{D} = \mathbf{P} = \mathbf{I}$	$\mathbf{x}^T \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0$	1)Centre : $\mathbf{c} = -\mathbf{u}$ 2)Radius : $\mathbf{r} = \sqrt{\mathbf{u}^T \mathbf{u} - f}$
Ellipse	$ \mathbf{V}  > 0$ $\lambda_1 > 0, \lambda_2 < 0$	$\frac{\mathbf{y}^T \mathbf{D} \mathbf{y}}{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f} = 1$	1)Centre : $\mathbf{c} = -\mathbf{V}^{-1} \mathbf{u}$ 2)Axes : $\begin{cases} \sqrt{\frac{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}{\lambda_1}} \\ \sqrt{\frac{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}{\lambda_2}} \end{cases}$
Hyperbola	$ \mathbf{V}  < 0$ $\lambda_1 > 0, \lambda_2 < 0$	$\frac{\mathbf{y}^T \mathbf{D} \mathbf{y}}{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f} = 1$	1)Centre : $\mathbf{c} = -\mathbf{V}^{-1} \mathbf{u}$ 2)Axes : $\begin{cases} \sqrt{\frac{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}{\lambda_1}} \\ \sqrt{\frac{f - \mathbf{u}^T \mathbf{V}^{-1} \mathbf{u}}{\lambda_2}} \end{cases}$
Parabola	$ \mathbf{V}  = 0$ $\lambda_1 = 0$	$\mathbf{y}^T \mathbf{D} \mathbf{y} = -2\eta(1 \ 0)\mathbf{y}$	1)Centre: $\begin{pmatrix} \mathbf{u}^T + \eta \mathbf{p}_1^T \\ \mathbf{V} \end{pmatrix} \mathbf{c} = \begin{pmatrix} -f \\ \eta \mathbf{p}_1 - \mathbf{u} \end{pmatrix}$ 2)Focal Length: $\beta = \frac{1}{2} \left  \frac{\mathbf{u}^T \mathbf{p}_1}{\lambda_2} \right $ 3)Focus: $\mathbf{F} = \begin{pmatrix} -f \\ \eta \mathbf{p}_1 - \mathbf{u} \end{pmatrix} \begin{pmatrix} \mathbf{u}^T + \eta \mathbf{p}_1^T \\ \mathbf{V} \end{pmatrix}^{-1} + \frac{-2\eta(1 \ 0)^T}{4}$ 4)Axis: $k(\mathbf{V}\mathbf{c} + \mathbf{u})^T \mathbf{x} = 0$ 5)Directrix: $(\mathbf{V}\mathbf{c} + \mathbf{u})^T (\mathbf{x} + \beta) + \mathbf{u}^T \mathbf{c} + \mathbf{f} = 0$ 6)Latus Rectum: $(\mathbf{V}\mathbf{c} + \mathbf{u})^T (\mathbf{x} - \beta) + \mathbf{u}^T \mathbf{c} + \mathbf{f} = 0$ 7)End points of latus rectum : $\mathbf{u}^T \kappa = -\frac{(\kappa^T \mathbf{V} \kappa + f)}{2}$ 8)Length of latus rectum: $l = \ \beta(\mathbf{V}\mathbf{c} + \mathbf{u})^T\ $

TABLE 4.1: Generalisation of conic