Challenge Problem 1

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Download latex-tikz code from

https://github.com/ka−raja−babu/Matrix−Theory/tree/main/ChallengeProblem1

1 Challenge Question 1

Show that the matrix $(t\mathbf{I} - \mathbf{n}\mathbf{n}^T)$ in the given document is a rank 1 matrix for a parabola.

2 Solution

Let

$$\mathbf{V} = (t\mathbf{I} - \mathbf{n}\mathbf{n}^T) \tag{2.0.1}$$

where,

$$t = \frac{\|\mathbf{n}\|^2}{e^2} \tag{2.0.2}$$

Let rank of matrix be represented by ρ and trace of matrix be represented by tr

Now,

$$tr(\mathbf{V}) = tr(t\mathbf{I} - \mathbf{n}\mathbf{n}^T) \tag{2.0.3}$$

$$\implies tr(\mathbf{V}) = tr(t\mathbf{I}) - tr(\mathbf{n}\mathbf{n}^T) \tag{2.0.4}$$

$$\implies tr(\mathbf{V}) = 2t - \|\mathbf{n}\|^2 \tag{2.0.5}$$

$$\implies tr(\mathbf{V}) = \frac{\|\mathbf{n}\|^2 (2 - e^2)}{e^2} \tag{2.0.6}$$

Now,

$$\mathbf{V}^T = (t\mathbf{I} - \mathbf{n}\mathbf{n}^T)^T \tag{2.0.7}$$

$$\implies \mathbf{V}^T = (t\mathbf{I})^T - (\mathbf{n}\mathbf{n}^T)^T \tag{2.0.8}$$

$$\implies \mathbf{V}^T = t(\mathbf{I}^T) - ((\mathbf{n}^T)^T\mathbf{n}^T) \tag{2.0.9}$$

$$\implies \mathbf{V}^T = (t\mathbf{I} - \mathbf{n}\mathbf{n}^T) \tag{2.0.10}$$

$$\implies \mathbf{V}^T = \mathbf{V} \tag{2.0.11}$$

So,using (2.0.11),$\mathbf{V}$ is a symmetric matrix .

Theorem 2.1. *Rank of a 2x2 symmetric matrix* $\mathbf{A}$ *is given by*

$$\rho(\mathbf{A}) = \frac{(tr(\mathbf{A}))^2}{tr(\mathbf{A}^2)} \tag{2.0.12}$$

Theorem 2.2. *For a 2x2 matrix* $\mathbf{A}$

$$tr(\mathbf{A}^2) = (tr(\mathbf{A}))^2 - 2|\mathbf{A}| \tag{2.0.13}$$

Using theorem 2.2

$$tr(\mathbf{V}^2) = (\frac{\|\mathbf{n}\|^2 (2 - e^2)}{e^2})^2 - 2|\mathbf{V}| \tag{2.0.14}$$

Using theorem 2.1

$$\rho(\mathbf{V}) = \frac{(\frac{\|\mathbf{n}\|^2(2-e^2)}{e^2})^2}{(\frac{\|\mathbf{n}\|^2(2-e^2)}{e^2})^2 - 2|\mathbf{V}|} \tag{2.0.15}$$

For a parabola where e=1 and $|\mathbf{V}|$=0,

$$\rho(\mathbf{V}) = \frac{\|\mathbf{n}\|^4}{\|\mathbf{n}\|^4} \tag{2.0.16}$$

$$\implies \rho(\mathbf{V}) = 1 \tag{2.0.17}$$