Challenge Problem 1

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Download latex-tikz code from

https://github.com/ka−raja−babu/Matrix−Theory/tree/main/ChallengeProblem1

1 Challenge Question 1

Show that the matrix $(t\mathbf{I} - \mathbf{n}\mathbf{n}^T)$ in the given document is a rank 1 matrix for a parabola.

2 Solution

Let

$$\mathbf{V} = (t\mathbf{I} - \mathbf{n}\mathbf{n}^T) \tag{2.0.1}$$

where,

$$t = \frac{\|\mathbf{n}\|^2}{e^2} \tag{2.0.2}$$

Let rank of matrix be represented by ρ.

Theorem 2.1. *A matrix* $\mathbf{A}$ *is orthogonally diagonizable if and only if* $\mathbf{A}$ *is symmetric.*

Proof. Let us assume that $\mathbf{A}$ is orthogonally diagonizable such that

$$\mathbf{D} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P} \tag{2.0.3}$$

$$\mathbf{P}^T = \mathbf{P}^{-1} \tag{2.0.4}$$

Now,

$$\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1} \tag{2.0.5}$$

$$\implies \mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^T \tag{2.0.6}$$

Now,

$$\mathbf{A}^T = (\mathbf{P}\mathbf{D}\mathbf{P}^T)^T \tag{2.0.7}$$

$$\implies \mathbf{A}^T = (\mathbf{P}^T)^T\mathbf{D}^T\mathbf{P}^T \tag{2.0.8}$$

$$\implies \mathbf{A}^T = \mathbf{P}\mathbf{D}\mathbf{P}^T \tag{2.0.9}$$

$\therefore$ From (2.0.6) and (2.0.9),

$$\mathbf{A} = \mathbf{A}^T \tag{2.0.10}$$

Hence,$\mathbf{A}$ is orthogonally diagonizable only when $\mathbf{A}$ is symmetric. □

Lemma 2.1. *Let* $\mathbf{A}$ *be a mxn diagonizable matrix.Then,*

$$\rho(\mathbf{A}) = \rho(\mathbf{B}^{-1}\mathbf{A}\mathbf{B}) = \rho(\mathbf{D}) \tag{2.0.11}$$

Proof. Here , $\mathbf{A}$ and $\mathbf{D}$ are similar matrices such that

$$\mathbf{D} = \mathbf{B}^{-1}\mathbf{A}\mathbf{B} \tag{2.0.12}$$

Now,

$$\mathbf{B}\mathbf{D} = (\mathbf{B}\mathbf{B}^{-1})\mathbf{A}\mathbf{B} \tag{2.0.13}$$

$$\implies \mathbf{B}\mathbf{D} = \mathbf{I}\mathbf{A}\mathbf{B} \tag{2.0.14}$$

$$\implies \mathbf{B}\mathbf{D} = \mathbf{A}\mathbf{B} \tag{2.0.15}$$

So,

$$\rho(\mathbf{A}\mathbf{B}) = \rho(\mathbf{B}\mathbf{D}) \tag{2.0.16}$$

Since $\mathbf{B}$ is an invertible matrix and hence a full rank matrix.
So,

$$\rho(\mathbf{A}\mathbf{B}) = \rho(\mathbf{A}) \tag{2.0.17}$$

$$\rho(\mathbf{B}\mathbf{D}) = \rho(\mathbf{D}) \tag{2.0.18}$$

$\therefore$ Using (2.0.16),(2.0.17) and (2.0.18),

$$\rho(\mathbf{A}) = \rho(\mathbf{D}) \tag{2.0.19}$$

□

Definition 1. *Rank of a diagonal matrix is equal to the number of its non-zero eigen values.*

Let trace of matrix be represented by tr

Now,

$$tr(\mathbf{V}) = tr(t\mathbf{I} - \mathbf{n}\mathbf{n}^T) \tag{2.0.20}$$

$$\implies tr(\mathbf{V}) = tr(t\mathbf{I}) - tr(\mathbf{n}\mathbf{n}^T) \tag{2.0.21}$$

$$\implies tr(\mathbf{V}) = 2t - \|\mathbf{n}\|^2 \tag{2.0.22}$$

$$\implies tr(\mathbf{V}) = \frac{\|\mathbf{n}\|^2(2 - e^2)}{e^2} \tag{2.0.23}$$

Lemma 2.2. *Eigen values of a 2x2 matrix* $\mathbf{A}$ *are :*

$$\lambda = \frac{tr(\mathbf{A}) \pm \sqrt{tr(\mathbf{A})^2 - 4\left|\mathbf{A}\right|}}{2} \tag{2.0.24}$$

Proof. Let $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.
Then,characteristic polynomial $p(\lambda)$ is

$$p(t) = |\mathbf{A} - \lambda\mathbf{I}| \tag{2.0.25}$$
$$= (a - \lambda)(d - \lambda) - bc \tag{2.0.26}$$
$$= \lambda^2 - (a + d)\lambda + ad - bc \tag{2.0.27}$$
$$= \lambda^2 - tr(\mathbf{A})\lambda + |\mathbf{V}| \tag{2.0.28}$$

So,eigen values which are roots of $p(\lambda)$ are

$$\lambda = \frac{tr(\mathbf{A}) \pm \sqrt{tr(\mathbf{A})^2 - 4\left|\mathbf{A}\right|}}{2} \tag{2.0.29}$$

□

∴ Using (2.0.23), for a parabola where e=1,

$$tr(\mathbf{V}) = \|\mathbf{n}\|^2 \tag{2.0.30}$$

∴ Using lemma 2.2, eigen values are:

$$\lambda_1 = 0 \tag{2.0.31}$$
$$\lambda_2 = \|\mathbf{n}\|^2 \tag{2.0.32}$$

Now,

$$\mathbf{V}^T = (\|\mathbf{n}\|^2 \mathbf{I} - \mathbf{n}\mathbf{n}^T)^T \tag{2.0.33}$$
$$\implies \mathbf{V}^T = (\|\mathbf{n}\|^2 \mathbf{I})^T - (\mathbf{n}\mathbf{n}^T)^T \tag{2.0.34}$$
$$\implies \mathbf{V}^T = \|\mathbf{n}\|^2 (\mathbf{I}^T) - ((\mathbf{n}^T)^T\mathbf{n}^T) \tag{2.0.35}$$
$$\implies \mathbf{V}^T = (\|\mathbf{n}\|^2 \mathbf{I} - \mathbf{n}\mathbf{n}^T) \tag{2.0.36}$$
$$\implies \mathbf{V}^T = \mathbf{V} \tag{2.0.37}$$

So,using (2.0.37),$\mathbf{V}$ is a symmetric matrix .

∴ Using (2.0.37) and theorem 2.1, $\mathbf{V}$ is a diagonizable matrix such that

$$\mathbf{P}^{-1}\mathbf{V}\mathbf{P} = \mathbf{D} \tag{2.0.38}$$
$$\mathbf{D} = \begin{pmatrix} 0 & 0 \\ 0 & \|\mathbf{n}\|^2 \end{pmatrix} \tag{2.0.39}$$

∴ Using def.1,lemma 2.1 and (2.0.39).

$$\rho(\mathbf{V}) = \rho(\mathbf{D}) = 1 \tag{2.0.40}$$