Assignment 8

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Download all python codes from

https://github.com/ka−raja−babu/Matrix−Theory/tree/main/Assignment8/Codes

and latex-tikz codes from

https://github.com/ka−raja−babu/Matrix−Theory/tree/main/Assignment8

1 Question No. 2.78

Find the equation of all lines having slope 2 which are tangents to the curve $\frac{1}{x-3}$, $x \neq 3$.

2 Solution

Given curve

$$y = \frac{1}{x-3}, x \neq 3 \tag{2.0.1}$$
$$\implies xy - 3y - 1 = 0 \tag{2.0.2}$$

$\therefore$

$$\mathbf{V} = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{2.0.3}$$
$$\mathbf{u} = \frac{-3}{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2.0.4}$$
$$f = -1 \tag{2.0.5}$$

$\because$

$$|\mathbf{V}| = \frac{-1}{4} \tag{2.0.6}$$
$$\implies |\mathbf{V}| < 0 \tag{2.0.7}$$

$\therefore$ (2.0.1) represents a hyperbola .
Now,the characteristic equation of $\mathbf{V}$ is

$$|\mathbf{V} - \lambda\mathbf{I}| = \begin{vmatrix} -\lambda & \frac{1}{2} \\ \frac{1}{2} & -\lambda \end{vmatrix} = 0 \tag{2.0.8}$$
$$\implies \lambda^2 - \frac{1}{4} = 0 \tag{2.0.9}$$

$\therefore$ Eigen values are

$$\lambda_1 = \frac{1}{2}, \lambda_2 = \frac{-1}{2} \tag{2.0.10}$$

Eigen vector $\mathbf{p}$ is

$$\mathbf{Vp} = \lambda\mathbf{p} \tag{2.0.11}$$
$$\implies (\mathbf{V} - \lambda\mathbf{I})\mathbf{p} = 0 \tag{2.0.12}$$

Eigen vector $\mathbf{p}_1$ corressponding to λ_1 can be obtained as

$$(\mathbf{V} - \lambda_1\mathbf{I}) = \begin{pmatrix} \frac{-1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{-1}{2} \end{pmatrix} \xleftrightarrow[R_1 \leftarrow -2R_1]{R_2 = R_1 + R_2} \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \tag{2.0.13}$$
$$\implies \mathbf{p_1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{2.0.14}$$

Similarly,

$$\mathbf{p_2} = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \end{pmatrix} \tag{2.0.15}$$

$\therefore$

$$\mathbf{P} = \begin{pmatrix} \mathbf{p_1} & \mathbf{p_2} \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \tag{2.0.16}$$
$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{-1}{2} \end{pmatrix} \tag{2.0.17}$$

Now,

$$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} \tag{2.0.18}$$
$$= -\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ \frac{-3}{2} \end{pmatrix} \tag{2.0.19}$$
$$= \begin{pmatrix} 3 \\ 0 \end{pmatrix} \tag{2.0.20}$$

and

$$\sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f}{\lambda_1}} = \sqrt{2} \tag{2.0.21}$$
$$\sqrt{\frac{f - \mathbf{u}^T\mathbf{V}^{-1}\mathbf{u}}{\lambda_2}} = \sqrt{2} \tag{2.0.22}$$

$\therefore$ Equation of standard hyperbola can be expressed as

$$\frac{x^2}{2} - \frac{y^2}{2} = 1 \tag{2.0.23}$$

Now,direction vector of tangent with slope = 2 is

$$\mathbf{m} = \begin{pmatrix} 1 \\ m \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \tag{2.0.24}$$

and,normal vector of same tangent is

$$\mathbf{m}^T \mathbf{n} = 0 \tag{2.0.25}$$

$$\implies \mathbf{n} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \tag{2.0.26}$$

Now,

$$\kappa = \pm \sqrt{\frac{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}{\mathbf{n}^T \mathbf{V}^{-1} \mathbf{n}}} \tag{2.0.27}$$

$$= \pm \sqrt{\frac{1}{-8}} \tag{2.0.28}$$

$\therefore$ Real value of κ does not exist and hence points of contacts of tangent $\mathbf{q_1}, \mathbf{q_2}$ also does not exist.
Hence,there exists no tangent to the curve having slope = 2 .

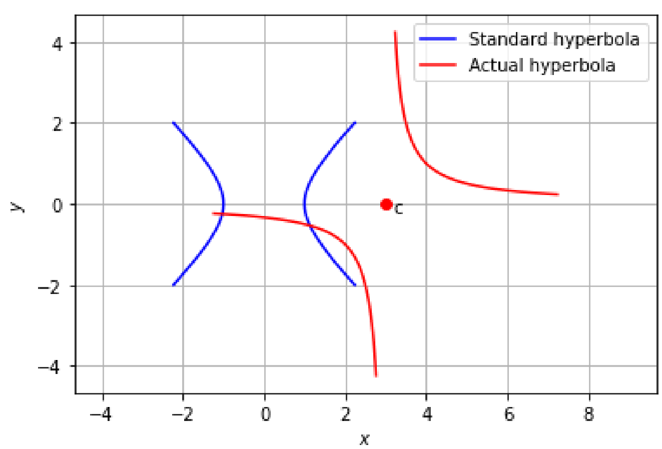

Fig. 2.1: Standard and actual hyperbola