Challenge Problem 1

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Download latex-tikz code from

https://github.com/ka−raja−babu/Matrix−Theory/tree/main/ChallengeProblem1

1 Challenge Question 1

Show that the matrix $(t\mathbf{I} - \mathbf{n}\mathbf{n}^T)$ in the given document is a rank 1 matrix for a parabola.

2 Solution

Let

$$\mathbf{V} = (t\mathbf{I} - \mathbf{n}\mathbf{n}^T) \tag{2.0.1}$$

For any conic, $ax^2+2bxy+cy^2+2dx+2ey+f = 0$,

$$\mathbf{V} = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \tag{2.0.2}$$

So,$\mathbf{V}$ is a symmetric matrix such that

$$\mathbf{V} = \mathbf{V}^T \tag{2.0.3}$$

Theorem 2.1. *A matrix* $\mathbf{A}$ *is orthogonally diagonizable if and only if* $\mathbf{A}$ *is symmetric.*

Proof. Let us assume that $\mathbf{A}$ is orthogonally diagonizable such that

$$\mathbf{D} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P} \tag{2.0.4}$$
$$\mathbf{P}^T = \mathbf{P}^{-1} \tag{2.0.5}$$

Now,

$$\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1} \tag{2.0.6}$$
$$\implies \mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^T \tag{2.0.7}$$

Now,

$$\mathbf{A}^T = (\mathbf{P}\mathbf{D}\mathbf{P}^T)^T \tag{2.0.8}$$
$$\implies \mathbf{A}^T = (\mathbf{P}^T)^T\mathbf{D}^T\mathbf{P}^T \tag{2.0.9}$$
$$\implies \mathbf{A}^T = \mathbf{P}\mathbf{D}\mathbf{P}^T \tag{2.0.10}$$

$\therefore$ From (2.0.7) and (2.0.10),

$$\mathbf{A} = \mathbf{A}^T \tag{2.0.11}$$

Hence,$\mathbf{A}$ is orthogonally diagonizable only when $\mathbf{A}$ is symmetric. □

$\therefore$ Using (2.0.3) and theorem 2.1, $\mathbf{V}$ is a diagonizable matrix such that

$$\mathbf{P}^{-1}\mathbf{V}\mathbf{P} = \mathbf{D} \tag{2.0.12}$$
$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \tag{2.0.13}$$

Let rank of matrix be represented by ρ.

Lemma 2.1. *Let* $\mathbf{A}$ *be a mxn diagonizable matrix.Then,*

$$\rho(\mathbf{A}) = \rho(\mathbf{B}^{-1}\mathbf{A}\mathbf{B}) = \rho(\mathbf{D}) \tag{2.0.14}$$

Proof. Here , $\mathbf{A}$ and $\mathbf{D}$ are similar matrices such that

$$\mathbf{D} = \mathbf{B}^{-1}\mathbf{A}\mathbf{B} \tag{2.0.15}$$

Now,

$$\mathbf{B}\mathbf{D} = (\mathbf{B}\mathbf{B}^{-1})\mathbf{A}\mathbf{B} \tag{2.0.16}$$
$$\implies \mathbf{B}\mathbf{D} = \mathbf{I}\mathbf{A}\mathbf{B} \tag{2.0.17}$$
$$\implies \mathbf{B}\mathbf{D} = \mathbf{A}\mathbf{B} \tag{2.0.18}$$

So,

$$\rho(\mathbf{A}\mathbf{B}) = \rho(\mathbf{B}\mathbf{D}) \tag{2.0.19}$$

Since $\mathbf{B}$ is an invertible matrix and hence a full rank matrix.
So,

$$\rho(\mathbf{A}\mathbf{B}) = \rho(\mathbf{A}) \tag{2.0.20}$$
$$\rho(\mathbf{B}\mathbf{D}) = \rho(\mathbf{D}) \tag{2.0.21}$$

$\therefore$ Using (2.0.19),(2.0.20) and (2.0.21),

$$\rho(\mathbf{A}) = \rho(\mathbf{D}) \tag{2.0.22}$$

□

Definition 1. *Rank of a diagonal matrix is equal to the number of its non-zero eigen values.*

Now,in case of parabola,

$$\lambda_1 = 0 \tag{2.0.23}$$
$$\lambda_2 \neq 0 \tag{2.0.24}$$

And,in case of ellipse,hyperbola and circle,

$$\lambda_1 \neq 0 \tag{2.0.25}$$
$$\lambda_2 \neq 0 \tag{2.0.26}$$

$\therefore$ Using def.1,lemma 2.1 and values of λ_1 and λ_2 ,
For a parabola,

$$\rho(\mathbf{V}) = \rho(\mathbf{D}) = 1 \tag{2.0.27}$$

For ellipse,hyperbola and circle,

$$\rho(\mathbf{V}) = \rho(\mathbf{D}) = 2 \tag{2.0.28}$$