Pre-calculus

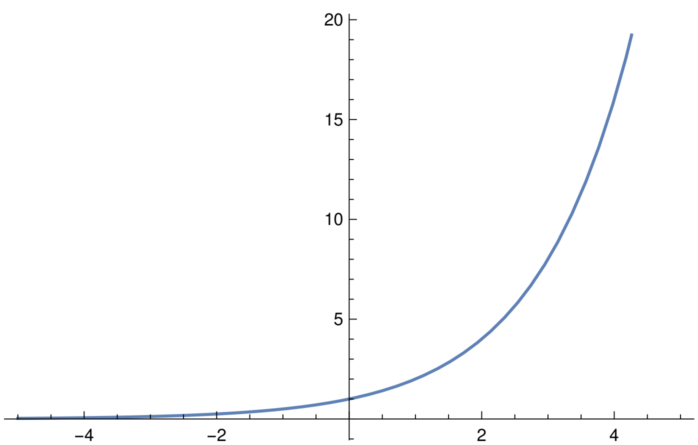
exponentials and logarithms

Not all exercises were implemented. I find it not interesting to work on items which I know that I can find a solution just by looking at them. I only do something if I do it for the first time or if it’s sufficiently different from previous exercises.
This notebook is a prerequisite for learning Calculus and also a nice process for me to get my head around Mathematica.

sketch the graph of *f*

$f(x) = 2^x$

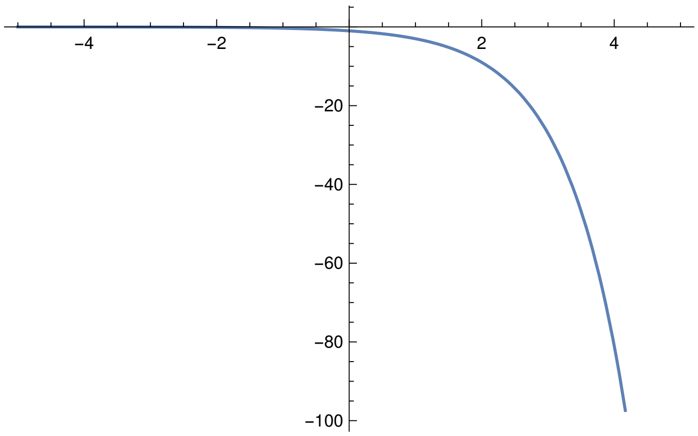
In[◦]:=
```
Plot[2^x, {x, -5, 5}]
```



$f(x) = -3^x$

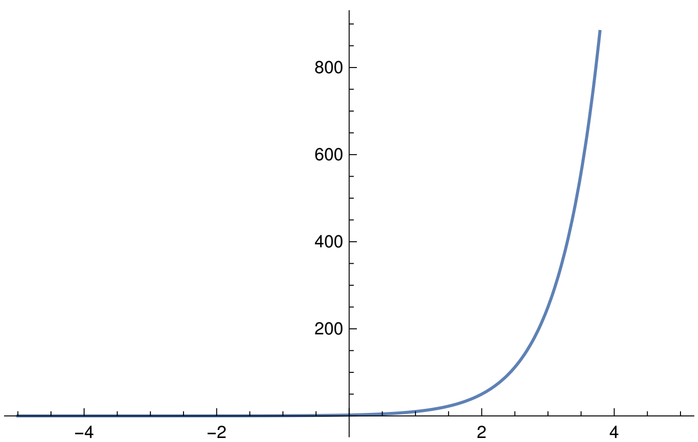
In[◦]:= `Plot[-3^x, {x, -5, 5}]`

Out[◦]=



$f(x) = 2\,(5)^x$

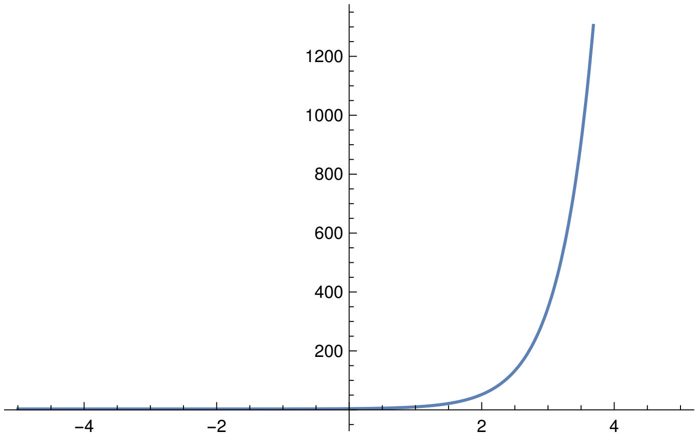
In[◦]:= `Plot[2 (5)^x, {x, -5, 5}]`



$f(x) = 7^x + 3$

In[∘]:=
```
Plot[7^x + 3, {x, -5, 5}]
```

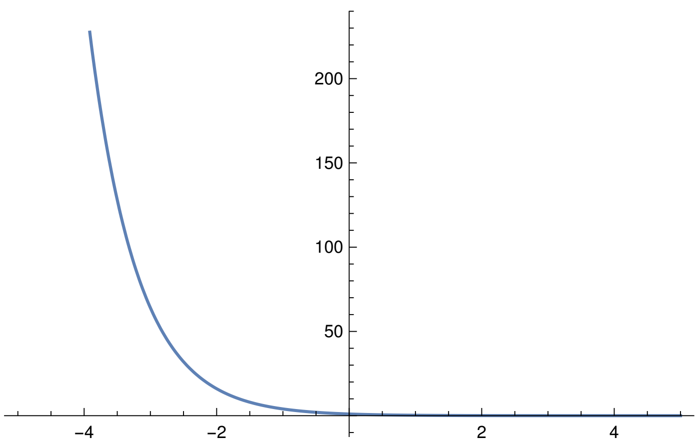
Out[∘]=



$f(x) = 4^{-x}$

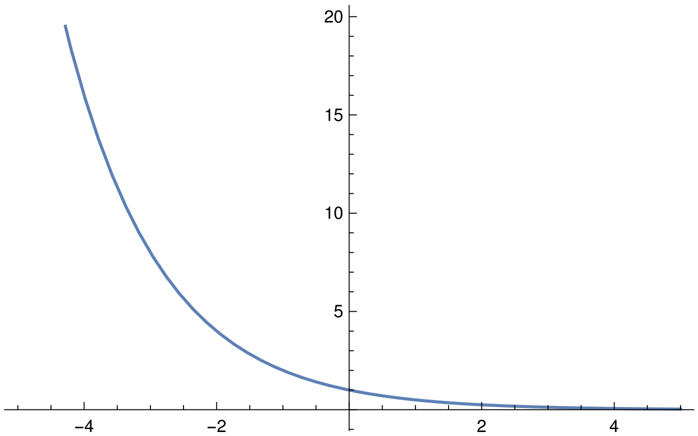
In[∘]:=
```
Plot[4^-x, {x, -5, 5}]
```

Out[∘]=



$f(x) = \left(\frac{1}{2}\right)^x$

In[•]:=
```
Plot[(1/2)^x, {x, -5, 5}]
```

Out[•]=


Solve the equations

$5^{x+8} = 5^{3x-2}$

$5^{x+8} = 5^{3x-2}$
$x + 8 = 3x - 2$
$x - 3x = -2 - 8$
$-2x = -10$
$2x = 10$
$x = 5$

```
Solve[5^(x + 8) == 5^(3 x - 2), x, Reals]
```

Out[•]= {{x → 5}}

$8^{7-x} = 8^{2x+1}$

$8^{7-x} = 8^{2x+1}$
$7 - x = 2x + 1$
$6 = 3x$
$x = 2$

In[•]:=
```
Solve[8^(7-x) == 8^(2 x+1), x, Reals]
```

Out[•]= {{x → 2}}

$5^{(x^2)} = 5^{2x+3}$

$5^{x^2} = 5^{2x+3}$

$x^2 = 2x + 3$

$x^2 - 2x - 3 = 0$

$(x + 1)(x - 3) = 0$

$x_1 = -1,\ x_2 = 3$

In[ ]:=
```
Solve[5^(x^2) == 5^(2 x+3), x, Reals]
```

Out[ ]= {{x → -1}, {x → 3}}

$25^{(x^2)} = 5^{3x+2}$

$5^{2x^2} = 5^{3x+2}$

$2x^2 = 3x + 2$

$2x^2 - 3x - 2 = 0$

I wasn't sure how to factor it, so just used Mathematica to factor

$(x - 2)(2x + 1) = 0$

$x_1 = \mathbf{2},\ x_2 = \mathbf{-\left(\frac{1}{2}\right)}$

In[ ]:=
```
Solve[25^(x^2) == 5^(3 x + 2), x, Reals]
```

Out[ ]= $\left\{\left\{x \to -\frac{1}{2}\right\}, \{x \to 2\}\right\}$

Solve word problems

A colony of an endangered species originally numbered 1,000 was predicted to have a population N after t years given by the equation $N(t) = 1000\,(0.9)^t$. Estimate population after:

a) 1 year

b) 5 years

c) 10 years

In[ ]:=
```
1000 (0.9)^1
1000 (0.9)^5
1000 (0.9)^10
```

Out[ ]= 900.

Out[ ]= 590.49

Out[ ]= 348.678

The number of bacteria in a certain culture increased from 600 to 1800 between

8 am and 10 am. Assuming the growth is exponential, the number *f(t)* of bacteria *t* hours after 8 am is given by $f(t) = 600\,(3)^{t/2}$.

a) Estimate the number of bacteria at 9 am, 11 am, and noon.
b) Sketch the graph of *f*

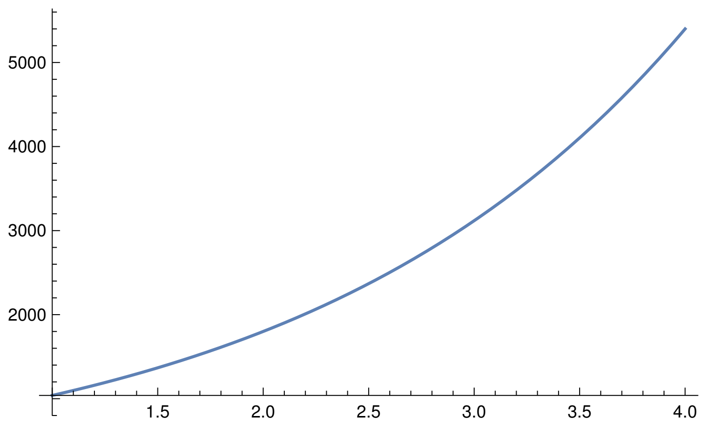
In[•]:=
```
600 (3)^(1/2) // N
600 (3)^(3/2) // N
600 (3)^2
Plot[600 (3)^(t/2), {t, 1, 4}]
```

Out[•]= 1039.23

Out[•]= 3117.69

Out[•]= 5400

Out[•]=

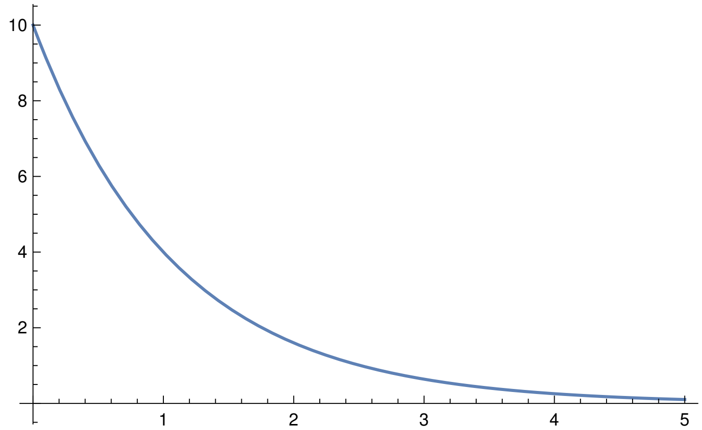


An important problem in oceanography is to determine the amount of light that can penetrate to various ocean depths. The Beer - Lambert law asserts that the exponential function given by $I(x) = I_0 c^x$ is a model for this phenomenon. For a certain location, $I(x) = 10(0.4)^x$ is the amount of light (in calories / cm^2 / sec) reaching a depth of x meters.
a) Find the amount of light at a depth of 2 m.
b) Sketch the graph of I for $0 \leq x \leq 5$.

In[∘]:=
```
10 (0.4)^2
Plot[10 (0.4)^x, {x, 0, 5}]
```

Out[∘]= 1.6

Out[∘]=



Change to logarithmic form

$5^3 = 125$

$x = \log_5 125$

$3^x = 7 + t$

$x = \log_3(7 + t)$

$5^{-3} = \frac{1}{125}$

$x = \log_5\left(\frac{1}{125}\right)$

Change to exponential form

$\log_2 32 = 5$

$2^5 = 32$

$\log_7 m = 5x + 3$

$7^{5x+3} = m$

$\log_2\left(\frac{1}{64}\right) = -6$

$2^{-6} = \frac{1}{64}$

Solve for *t* using logarithms with base *a*

$2a^{t/5} = 5$

$\frac{t}{5} = \log_{2a} 5$

$t = 5\log_{2a} 5$

Let's check the solution for $a = 4$

In[∘]:=
```
t = 5 Log[8, 5] // N;
8^(t/5) == 5
```

Out[∘]=
```
True
```

$5a^{3t} = 63$

$3t = \log_{5a}(63)$

$t = \frac{\log_{5a}(63)}{3}$

$A = \mathrm{Ba}^{\mathrm{Ct}} + D$

$\mathrm{Ba}^{\mathrm{Ct}} = A - D$

$\mathrm{Ct} = \mathrm{Log}_{\mathrm{Ba}}(A - D)$

$t = \frac{\mathrm{Log}_{\mathrm{Ba}}(A - D)}{C}$

Find the numbers, if possible

$\text{Log}_9(-3)$

Yes, it's possible, LOL.

```
In[•]:= Log[9, -3]
```

$$\text{Out[•]=}\quad \frac{i\,\pi + \text{Log}[3]}{\text{Log}[9]}$$

Solve the logarithmic equation

$\log_4 x = \log_4(8 - x)$

$x = 8 - x$
$2x = 8$
$x = 4$

Word problems II

The loudness of a sound, as experienced by the human ear, is based on its intensity level. The intensity level α (in decibels) that corresponds to a sound intensity $\boldsymbol{I}$ is $\alpha = 10\log\left(\frac{I}{I_0}\right)$, where $\boldsymbol{I_0}$ is a special value of $\boldsymbol{I}$ agreed to be the weakest sound that can be detected by the human ear under certain conditions. Find α if
a) $\boldsymbol{I}$ is 10 times as great as $\boldsymbol{I_0}$
b) $\boldsymbol{I}$ is 1000 times as great as $\boldsymbol{I_0}$

a) $\alpha = 10\log 10$; $\alpha = 10$;
b) $\alpha = 10\log 1000$; $\alpha = 30$;

A sound intensity level of 140 decibels produces pain in the average human ear. Approximately how many times greater than I_0 must I be in order for α to reach this level?

$\alpha = 10\log\left(\frac{I}{I_0}\right)$
$140 = 10\log\left(\frac{I}{I_0}\right)$
$\log\left(\frac{I}{I_0}\right) = 14$
$\left(\frac{I}{I_0}\right) = 10^{14}$

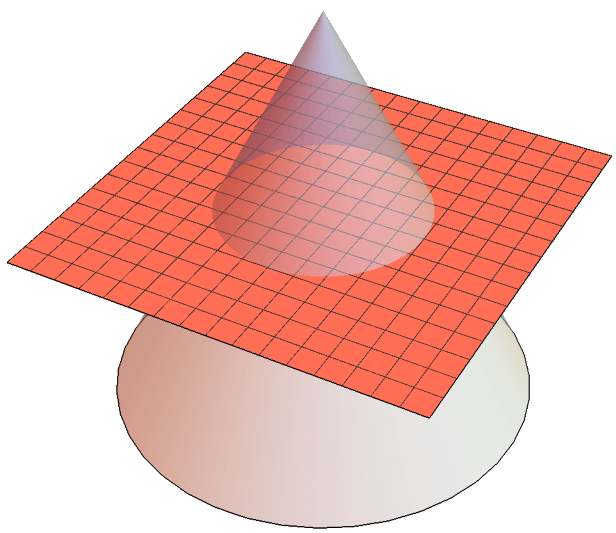
Conic sections

First of all, some conic section plots

Circle section

In[ ]:=
```
gcone = Graphics3D[{Opacity[0.5], Cone[]}, Boxed → False];
gplane = ContourPlot3D[0 x + 0 y + 10 z == 0,
   {x, -1, 1}, {y, -1, 1}, {z, -1, 1}, ContourStyle → {Pink}];
Show[gcone, gplane]
```

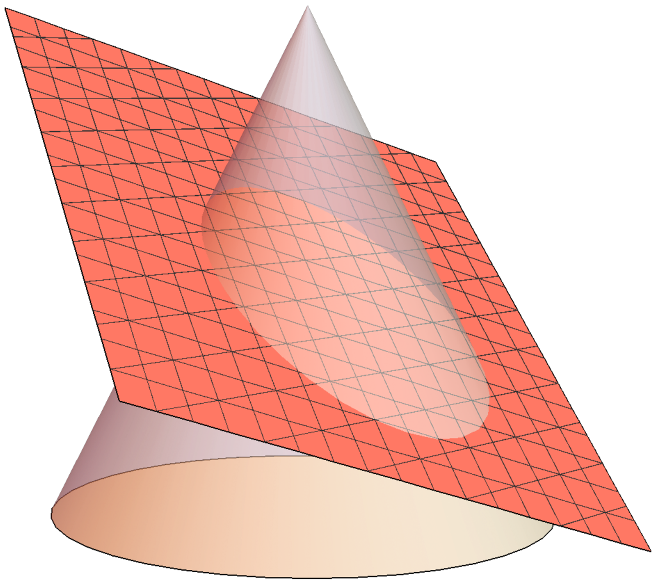
Out[ ]=



Ellipse section

In[•]:=
```
gcone = Graphics3D[{Opacity[0.5], Cone[]}, Boxed → False];
gplane = ContourPlot3D[-1 x + 2 y - 3 z == 0,
   {x, -1, 1}, {y, -1, 1}, {z, -1, 1}, ContourStyle → {Pink}];
Show[gcone, gplane]
```

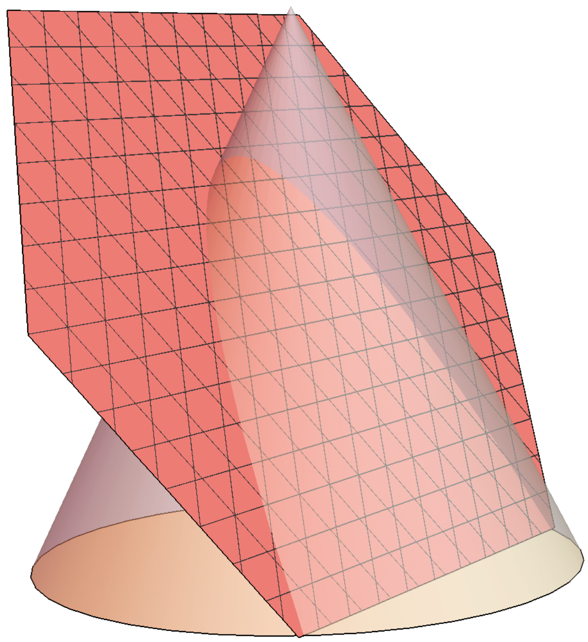
Out[•]=



Parabola section

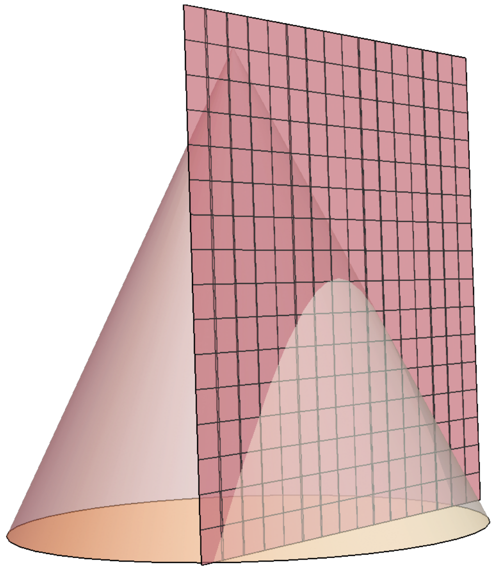
In[∘]:=
```
gcone = Graphics3D[{Opacity[0.5], Cone[]}, Boxed → False];
gplane = ContourPlot3D[-1 x + y - 0.8 z == 0,
   {x, -1, 1}, {y, -1, 1}, {z, -1, 1}, ContourStyle → {Pink}];
Show[gcone, gplane]
```

Out[∘]=



Hyperbola section

In[∘]:=
```
gcone = Graphics3D[{Opacity[0.5], Cone[]}, Boxed → False];
gplane = ContourPlot3D[x - 0.16 y - 0 z == 0.5, {x, -1, 1},
   {y, -1, 1}, {z, -1, 1}, ContourStyle → {Pink, Opacity[0.7]}];
Show[gcone, gplane]
```



Find the vertex, the focus and the directrix of the parabola. Sketch its graph, showing the focus and the directrix

$y = -\frac{1}{12}x^2$

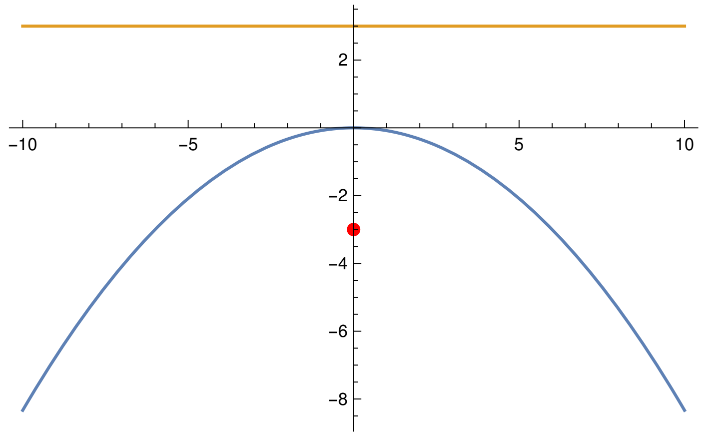
This equation has the form $y = ax^2$ with $a = -\frac{1}{12}$.

$a = \frac{1}{4p}$ so $p = \frac{1}{4a}$; $p = \frac{1}{4\left(-\frac{1}{12}\right)} = \frac{1}{-\frac{4}{12}} = \frac{1}{-\frac{1}{3}} = -3$;

Vertex = (0, 0); Focus = (0, −3); Directrix is $y = 3$;

In[•]:=
```
focus = Graphics[{PointSize[Large], Red, Point[{0, -3}]}];
parabola = Plot[{-1/12 x^2, y = 3}, {x, -10, 10}];
Show[parabola, focus]
```

Out[•]=



$2y^2 = -3x$

We can rewrite this as $-3x = 2y^2$

Divide both ends by -3

$x = -\frac{2}{3}y^2$

This equation has the form $x = ay^2$ with $a = -\frac{2}{3}$

$a = \frac{1}{4p}$, so $p = \frac{1}{4a}$

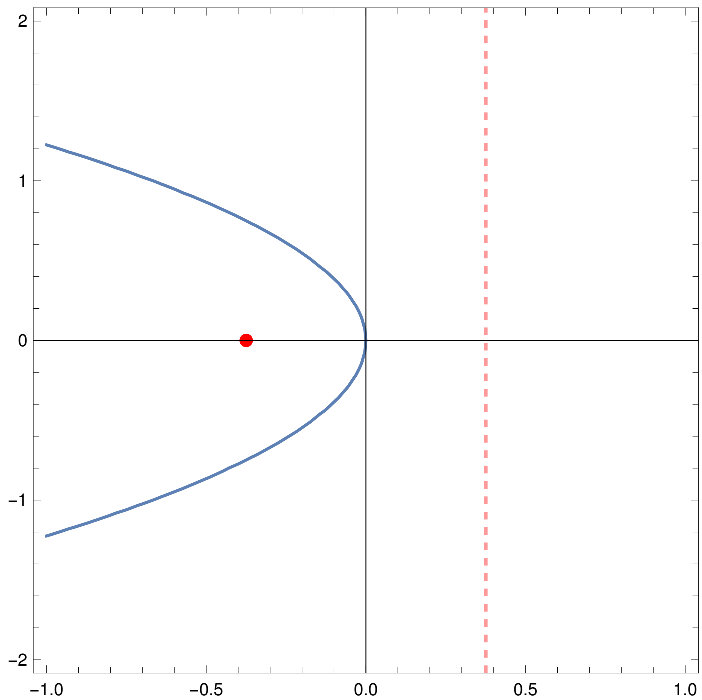
$p = \frac{1}{4\left(-\frac{2}{3}\right)} = \frac{1}{-\frac{8}{3}} = -\frac{3}{8}$

Vertex = (0, 0); Focus = $\left(-\frac{3}{8},\ 0\right)$; Directrix is $x = \frac{3}{8}$

In[∘]:=
```
focus = Graphics[{PointSize[Large], Red, Point[{-3/8, 0}]}];
parabola = ContourPlot[{2 y^2 == -3 x}, {x, -1, 1}, {y, -2, 2},
   Axes → True, GridLines → {{{3/8, {Red, Dashed, Thick}}}, None}];

Show[parabola, focus]
```

Out[∘]=



$y^2 = -100\,x$

We can rewrite this as $-100\,x = y^2$

Divide both ends by -100

$x = -\frac{1}{100} y^2$

This equation has the form $x = ay^2$ with $a = -\frac{1}{100}$

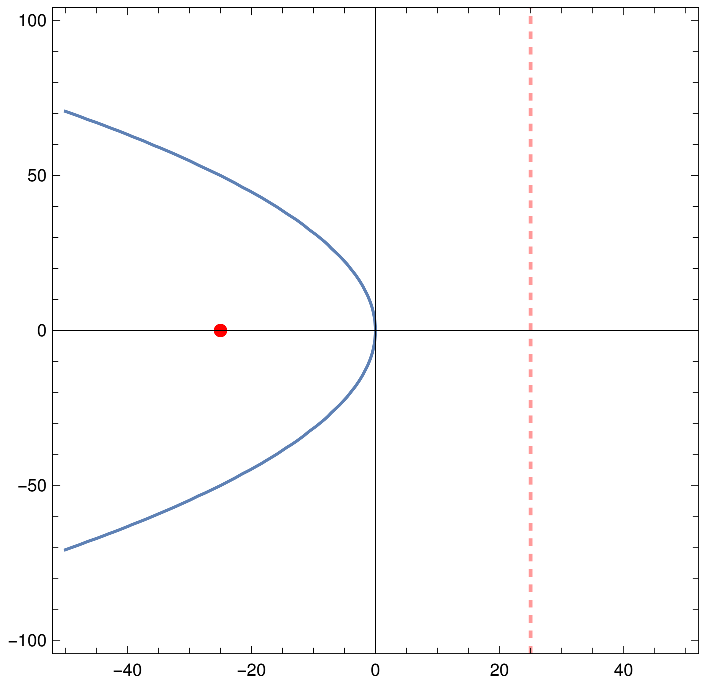
$a = \frac{1}{4p}$, so $p = \frac{1}{4a}$

$p = \frac{1}{4\left(-\frac{1}{100}\right)} = \frac{1}{-\frac{4}{100}} = \frac{-100}{4} = -25$

Vertex = (0, 0); Focus = (−25, 0); Directrix is $x = 25$

In[•]:=
```
focus = Graphics[{PointSize[Large], Red, Point[{-25, 0}]}];
parabola = ContourPlot[{y² == -100 x}, {x, -50, 50}, {y, -100, 100},
   Axes → True, GridLines → {{{25, {Red, Dashed, Thick}}}, None}];

Show[parabola, focus]
```

Out[•]=


Find the vertex and the focus of the parabola. Sketch its graph, showing the focus.

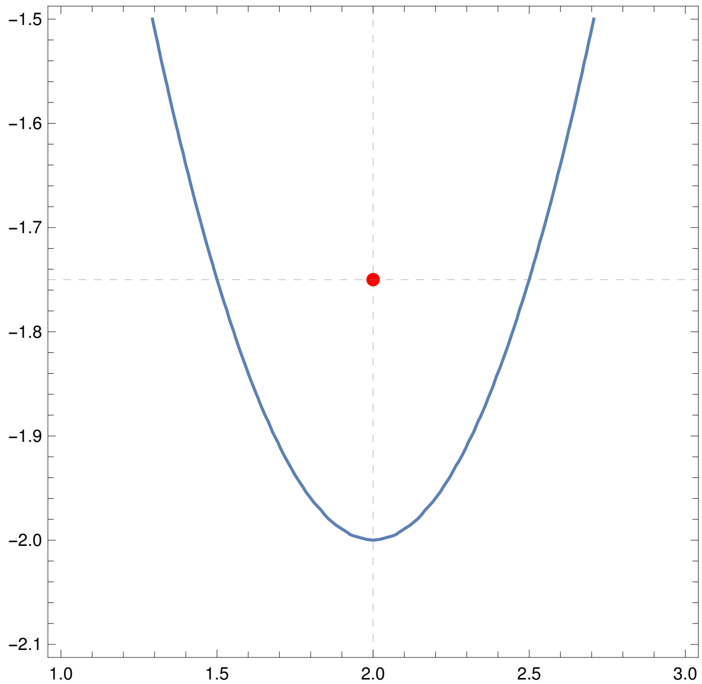
$y = x^2 - 4x + 2$

Completing the square for $x^2 - 4x + 2$:
$(x^2 - 4x + 4) + 2 - 4$
$(x^2 - 4x + 4) - 2$
$(x - 2)^2 - 2 = y$
$(x - h)^2 = 4p(y - k)$
$(x - 2)^2 = y + 2$
$(x - 2)^2 = 4\left(\frac{1}{4}\right)(y - (-2))$
$h = 2,\ k = -2,\ p = \frac{1}{4}$
Vertex = (2, -2), F = $\left(2, -2 + \frac{1}{4}\right) = \left(2,\ -1\frac{3}{4}\right)$

In[∘]:=
```
focus = Graphics[{PointSize[Large], Red, Point[{2, -1.75}]}];
parabola = ContourPlot[{y == x^2 - 4 x + 2}, {x, 1, 3}, {y, -2.1, -1.5},
   Axes → True, GridLines → {{{2, Dashed}}, {{-1.75, Dashed}}}];

Show[parabola, focus]
```

Out[∘]=



$y = 8x^2 + 16x + 10$

Completing the square for $\frac{y}{8} = x^2 + 2x + \frac{5}{4}$swokowski

$(x^2 + 2x + 1) + \frac{5}{4} - 1$

$(x^2 + 2x + 1) + \frac{1}{4}$

$(x + 1)^2 + \frac{1}{4} = \frac{y}{8}$

$(x + 1)^2 = \frac{y}{8} - \frac{1}{4}$

$(x - h)^2 = 4p(y - k)$

$(x - (-1))^2 = \frac{y-2}{8} = \frac{1}{8}(y - 2)$

$(x - (-1))^2 = 4\left(\frac{1}{32}\right)(y - 2)$

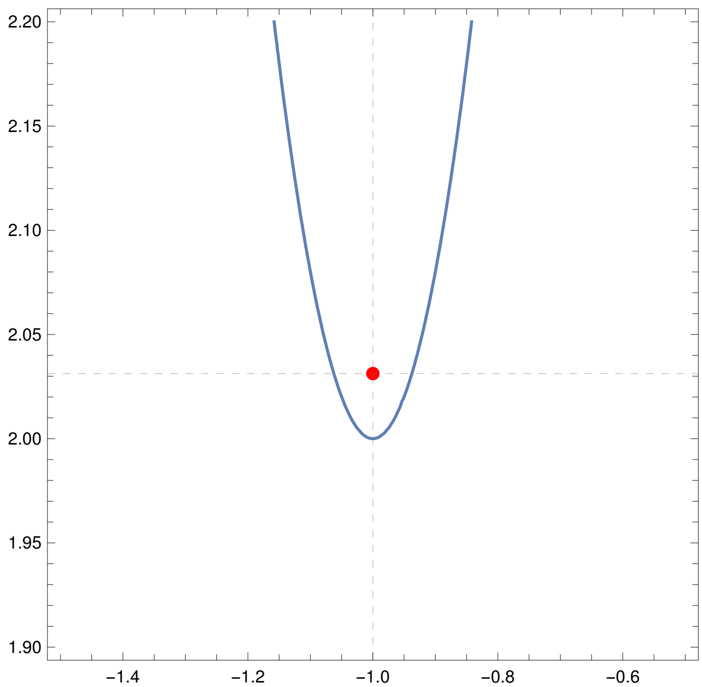
$h = -1, \; k = 2, \; p = \frac{1}{32}$

Vertex = $(-1, 2)$; Focus = $\left(-1, \; 2\,\frac{1}{32}\right)$

In[∘]:=
```
(* some helper method to use a mixed fraction input *)
(* https://mathematica.stackexchange.com/a/184782/54824 *)
CurrentValue[EvaluationNotebook[], {InputAliases, "mf"}] =
  TemplateBox[{"■", "□", "□"}, "MixedFraction",
   DisplayFunction → (RowBox[{#1, FractionBox[#2, #3]}] &),
   InterpretationFunction → (RowBox[{#1, "+", FractionBox[#2, #3]}] &)];
focus = Graphics[{PointSize[Large], Red, Point[{-1, 2 1/32}]}];
parabola = ContourPlot[{y == 8 x^2 + 16 x + 10}, {x, -1.5, -0.5}, {y, 1.9, 2.2},
   Axes → True, GridLines → {{{-1, Dashed}}, {{2 1/32, Dashed}}}];

Show[parabola, focus]
```

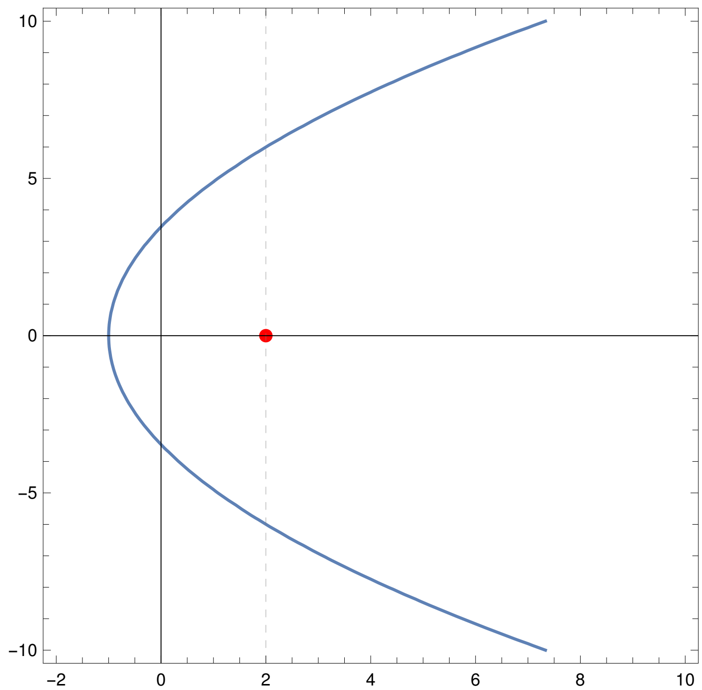
Out[∘]=



$y^2 - 12 = 12\,x$

$y^2 = 12\,x + 12$
$(y-k)^2 = 4\,p(x-h)$
$(y-0)^2 = 12\,x + 12$
$12\,x + 12 = 12\,(x+1) = 4 * 3\,(x+1)$
$(y-0)^2 = 4 * 3\,(x-(-1))$
$k = 0,\ h = -1,\ p = 3$
Vertex $= (-1,\ 0)$, Focus $= (2,\ 0)$

In[ ]:=
```
focus = Graphics[{PointSize[Large], Red, Point[{2, 0}]}];
parabola = ContourPlot[{y^2 == 12 x + 12}, {x, -2, 10},
   {y, -10, 10}, Axes → True, GridLines → {{{2, Dashed}}, None}];
Show[parabola,
 focus]
```

Out[ ]=


$y^2 - 20y + 100 = 6x$

$6x = y^2 - 20y + 100$

Completing the square for $y^2 - 20y + 100$

$(y^2 - 20y) + 100$

$(y^2 - 20y + 100)$

$(y - 10)^2 = 6x$

$(y - k)^2 = 4p(x - h)$

$4p(x - h) = 4\left(\frac{3}{2}\right)(x - 0)$

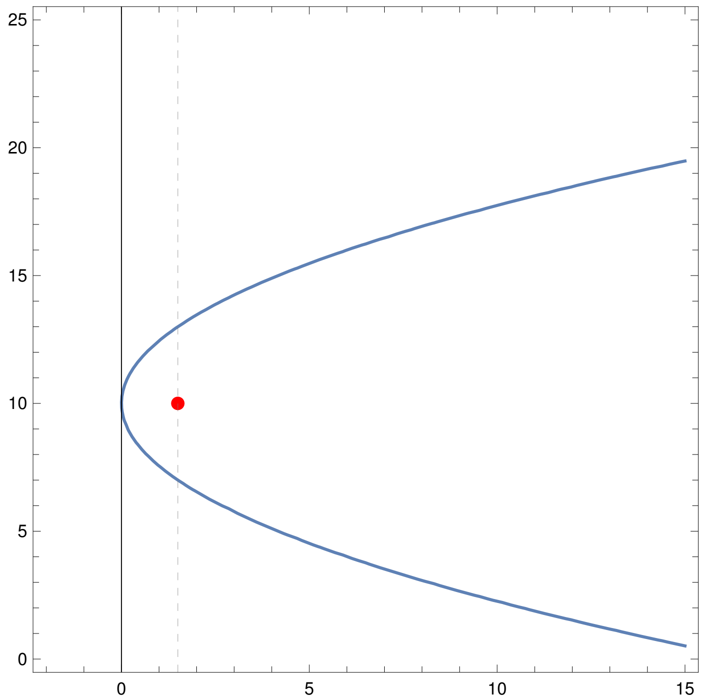
$(y - 10)^2 = 4\left(\frac{3}{2}\right)(x - 0)$

$k = 10, \ h = 0, \ p = \frac{3}{2}$

Vertex = (0, 10), Focus = $\left(\frac{3}{2}, 10\right)$

```
focus = Graphics[{PointSize[Large], Red, Point[{3/2, 10}]}];
parabola = ContourPlot[{y^2 - 20 y + 100 == 6 x}, {x, -2, 15},
   {y, 0, 25}, Axes → True, GridLines → {{{3/2, Dashed}}, None}];
Show[parabola,
 focus]
```

Out[ ]=



$y^2 - 4y - 2x - 4 = 0$

$2x = y^2 - 4y - 4$

Completing the square for $y^2 - 4y - 4$

$(y^2 - 4y) - 4$

$(y^2 - 4y + 4) - 4 - 4$

$(y-2)^2 - 8 = 2x$

$(y-k)^2 = 4p(x-h)$

$(y-2)^2 = 2x + 8$

$(2x+8) = 2(x+4)$

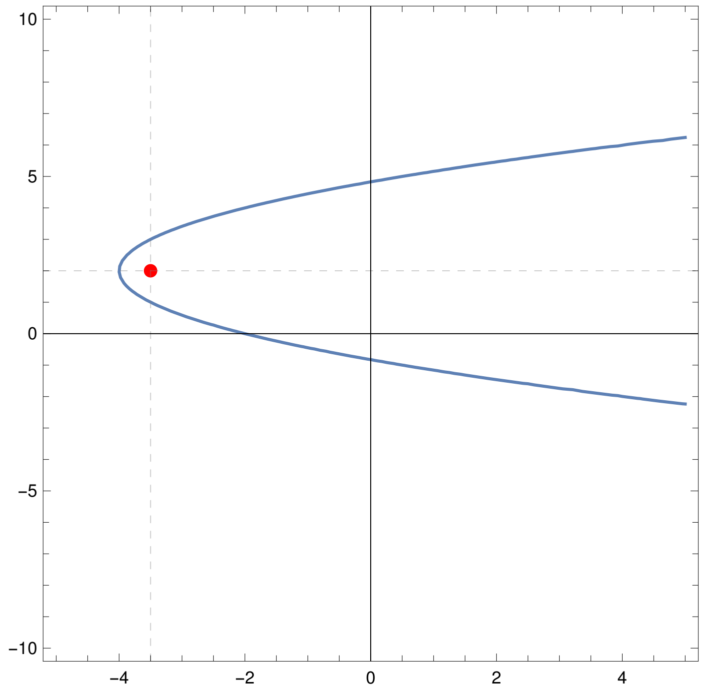
$(y-2)^2 = 2\left(\frac{1}{2}\right)(x-(-4))$

$k = 2,\ h = -4,\ p = \frac{1}{2}$

Vertex $= (-4, 2)$, Focus $= (-3.5, 2)$

In[∘]:=
```
focus = Graphics[{PointSize[Large], Red, Point[{-3.5, 2}]}];
parabola = ContourPlot[{2 x == y^2 - 4 y - 4}, {x, -5, 5}, {y, -10, 10},
   Axes → True, GridLines → {{{-3.5, Dashed}}, {{2, Dashed}}}];
Show[parabola,
 focus]
```

Out[∘]=



$y^2 + 14y + 4x + 45 = 0$

$-4x = y^2 + 14y + 45$

Completing the square for $y^2 + 14y + 45$

$(y^2 + 14y + 49) + 45 - 49$

$(y^2 + 14y + 49) - 4$

$(y + 7)^2 - 4 = -4x$

$(y + 7)^2 = -4x + 4$

$(y - k)^2 = 4p(x - h)$

$(y - (-7))^2 = -4(x - 1)$

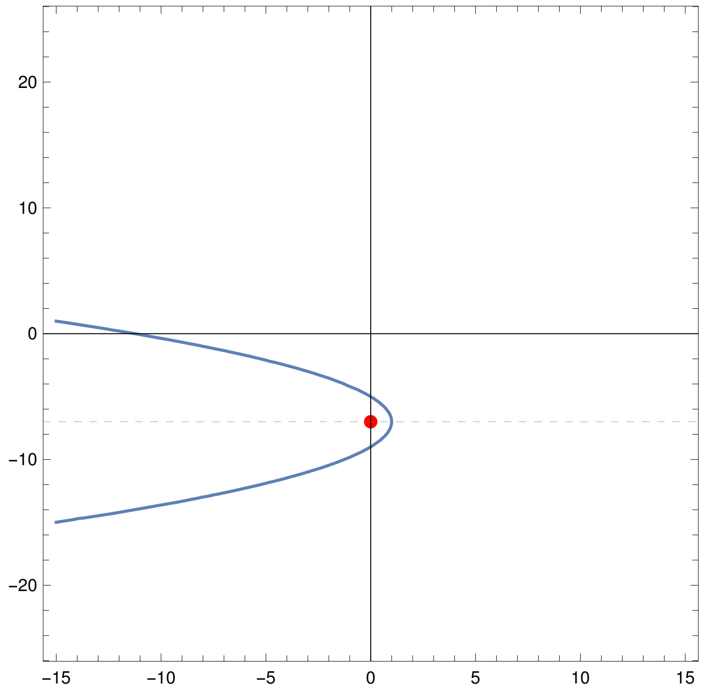
$(y - (-7))^2 = 4(-1)(x - 1)$

$k = -7,\ h = 1,\ p = -1$

Vertex $= (1,\ -7)$, Focus$(0,\ -7)$

In[∘]:=
```
focus = Graphics[{PointSize[Large], Red, Point[{0, -7}]}];
parabola = ContourPlot[{y^2 + 14 y + 4 x + 45 == 0}, {x, -15, 15},
   {y, -25, 25}, Axes → True, GridLines → {None, {{-7, Dashed}}}];
Show[parabola,
 focus]
```

Out[∘]=



$4x^2 + 40x + y + 106 = 0$

$-y = 4x^2 + 40x + 106$

Completing the square for $4x^2 + 40x + 106$

$4(x^2 + 10x) + 106$

$4(x^2 + 10x + 25) + 106 - 100$

$4(x+5)^2 + 6 = -y$

$4(x+5)^2 = -y - 6$

$(x+5)^2 = -\frac{1}{4}y - \frac{3}{2}$

$(x-h)^2 = 4p(y-k)$

$(x-(-5))^2 = -\frac{1}{4}(y+6)$

$(x-(-5))^2 = 4\left(-\frac{1}{16}\right)(y-(-6))$

$h = -5,\ k = -6,\ p = -\frac{1}{16}$

Vertex $= (-5,\ -6)$, Focus $= \left(-5,\ -6\,\frac{1}{16}\right)$

In[ ]:=
```
focus = Graphics[{PointSize[Large], Red, Point[{-5, -6 1/16}]}];
parabola = ContourPlot[{-y == 4 x^2 + 40 x + 106}, {x, -6, -4}, {y, -5, -7},
   Axes → True, GridLines → {{{-5, Dashed}}, {{-97/16, Dashed}}}];
Show[parabola,
 focus]
```

Out[ ]=