Fourier Transform Project

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1 Fast Fourier Transform

1.1 DFT

The discrete Fourier transform of a vector $\mathbf{y} = (y_0, y_1, ..., y_{N-1})$ is given by:

$$c_k = \sum_{n=0}^{N-1} y_n exp\left(\frac{-2\pi ikn}{N}\right) = \sum_{n=0}^{N-1} y_n \left[cos\left(\frac{2\pi kn}{N}\right) - i\,sin\left(\frac{2\pi kn}{N}\right)\right]. \quad (1)$$

I will implement the right-most expression above in my code to avoid complex exponential problems.

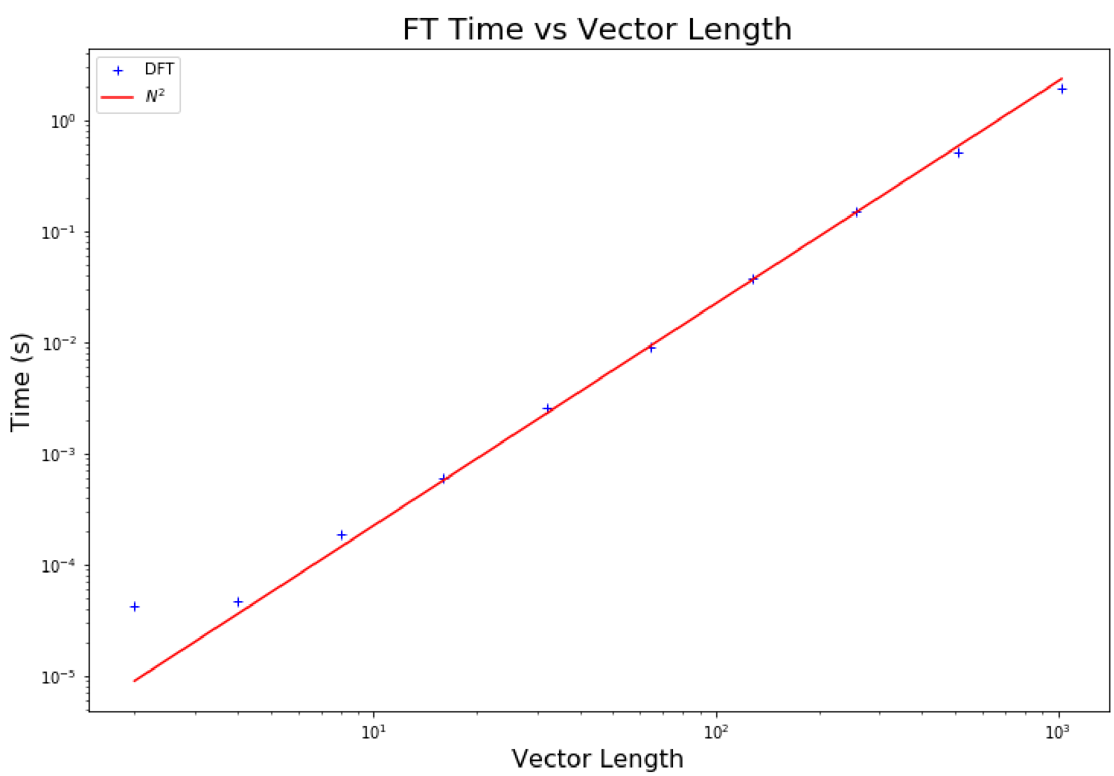
```
In [1]: from fft_class import fourier as ft
        import numpy as np
        import numpy.fft as fftw
        import matplotlib.pyplot as plt
        import random as rnd
        import time as tm
        %matplotlib inline

In [2]: rnd.seed(0)
        v = [rnd.random() for  i in range(2)]
        vec = v.copy()
        time = 0.
        N = 0
        StepList = list()
        DFTList = list()
        print("Started Fourier Transforming")
        while time < 1.:
            N = len(vec)
            f = ft(vec)
            dft = f.dft()
            time = f.t
            vec += vec # double size of vector for next iteration
            StepList.append(N)
            DFTList.append(time)
            print("dft:\tN = ",N,"\tDFT = {:.5f} seconds".format(time))
```

```
Started Fourier Transforming
dft:        N =  2         DFT = 0.00004 seconds
dft:        N =  4         DFT = 0.00005 seconds
dft:        N =  8         DFT = 0.00018 seconds
dft:        N =  16         DFT = 0.00061 seconds
dft:        N =  32         DFT = 0.00253 seconds
dft:        N =  64         DFT = 0.00914 seconds
dft:        N =  128         DFT = 0.03689 seconds
dft:        N =  256         DFT = 0.14836 seconds
dft:        N =  512         DFT = 0.51438 seconds
dft:        N =  1024         DFT = 1.93328 seconds
```

In [3]:
```
StepList = np.array(StepList)
mid = int(len(StepList)/2)
fig,axes = plt.subplots(figsize=(12,8))
axes.loglog(StepList,DFTList,'b+',label='DFT')
axes.loglog(StepList,(StepList**2)*(DFTList[mid]/(StepList[mid]**2)),'r',label='$N^2$')
axes.legend()
axes.set_title("FT Time vs Vector Length",fontsize=20)
axes.set_xlabel("Vector Length",fontsize=16)
axes.set_ylabel("Time (s)",fontsize=16)
```

Out[3]: <matplotlib.text.Text at 0x116d33cf8>



We can see above that the DFT scales as N^2 as expected. The largest vector transformed in less than a second had 512 terms.

1.2 FFT

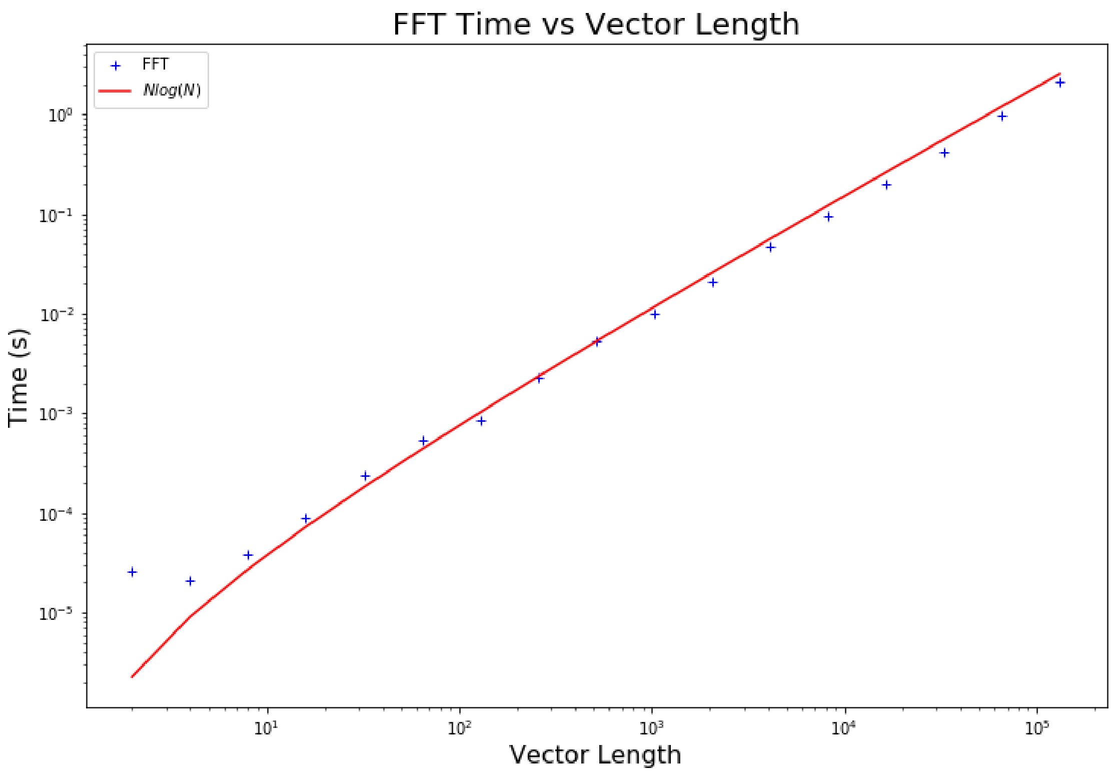
In [4]:
```
vec = v.copy()
time = 0.
N = 0
StepList = list()
FFTList = list()
print("Started Fourier Transforming")
while time < 1.:
    N = len(vec)
    f = ft(vec)
    dft = f.fft()
    time = f.t
    vec += vec # double size of vector for next iteration
    StepList.append(N)
    FFTList.append(time)
    print("fft:\tN = ",N,"\tFFT = {:.6f} seconds".format(time))
```

```
Started Fourier Transforming
fft:        N =  2         FFT = 0.000026 seconds
fft:        N =  4         FFT = 0.000021 seconds
fft:        N =  8         FFT = 0.000038 seconds
fft:        N =  16         FFT = 0.000089 seconds
fft:        N =  32         FFT = 0.000241 seconds
fft:        N =  64         FFT = 0.000529 seconds
fft:        N =  128         FFT = 0.000838 seconds
fft:        N =  256         FFT = 0.002300 seconds
fft:        N =  512         FFT = 0.005227 seconds
fft:        N =  1024         FFT = 0.009883 seconds
fft:        N =  2048         FFT = 0.021281 seconds
fft:        N =  4096         FFT = 0.046574 seconds
fft:        N =  8192         FFT = 0.095953 seconds
fft:        N =  16384         FFT = 0.197139 seconds
fft:        N =  32768         FFT = 0.418729 seconds
fft:        N =  65536         FFT = 0.957751 seconds
fft:        N =  131072         FFT = 2.144419 seconds
```

In [5]:
```
StepList = np.array(StepList)
mid = int(len(StepList)/2)
fig,axes = plt.subplots(figsize=(12,8))
axes.loglog(StepList,FFTList,'b+',label='FFT')
axes.loglog(StepList,\
            (StepList*np.log(StepList))*(FFTList[mid]/(StepList[mid]*np.log(StepList[mid
            'r',label='$Nlog(N)$')
```

```
axes.legend()
axes.set_title("FFT Time vs Vector Length",fontsize=20)
axes.set_xlabel("Vector Length",fontsize=16)
axes.set_ylabel("Time (s)",fontsize=16)
```

Out[5]: <matplotlib.text.Text at 0x11735f8d0>



The Cooley-Tukey algorithm for a fast Fourier transform was covered in class. Essentially it uses the symmetry of the Fourier transform to break it into even and odd pieces that are half the length of the original transform. This can be applied to the sub-transforms as well. As we can see above, this algorithm significantly reduces the computation time. The computation follows the expected $Nlog(N)$ trend as expected. The largest vector transformed in less than 1 second had 65,536 terms.

1.3 FFTW

I used

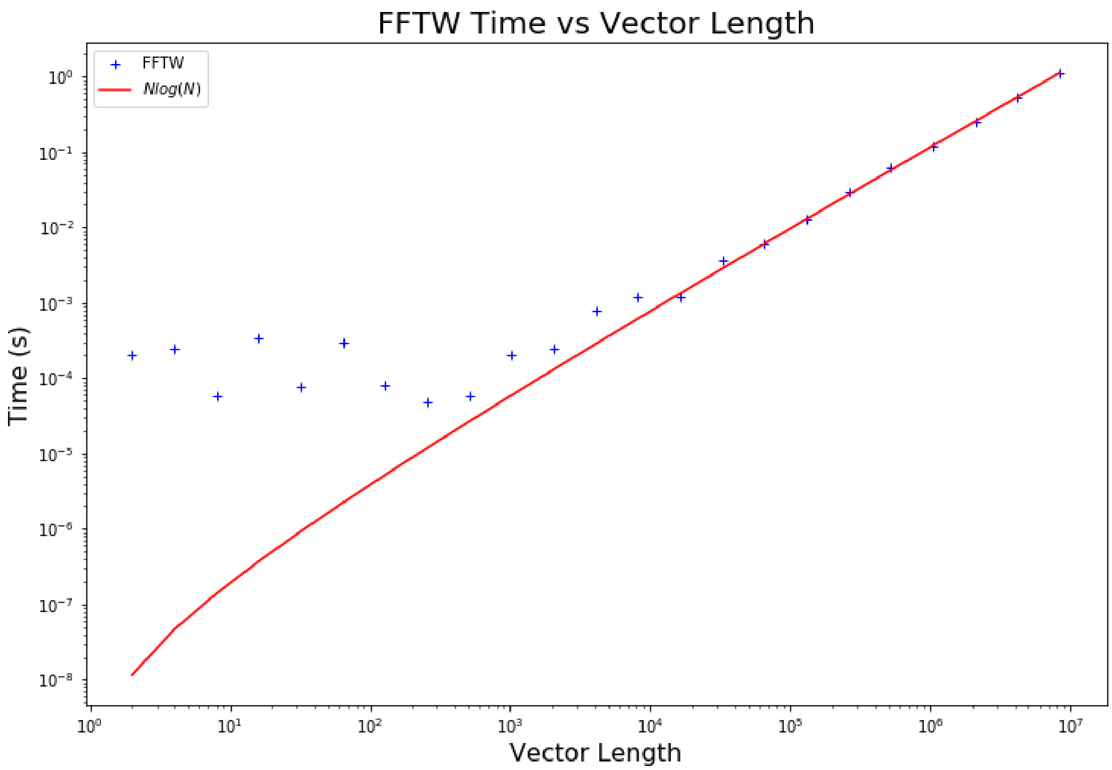
In [8]:
```
vec = v.copy()
time = 0.
N = 0
StepList = list()
FTList = list()
print("Started Fourier Transforming")
```

```
        while time < 1.:
            N = len(vec)
            start = tm.time()
            fftw.fft(vec)
            time = tm.time()-start
            vec += vec # double size of vector for next iteration
            StepList.append(N)
            FTList.append(time)
            print("fftw:\tN = ",N,"\tFFTW = {:.6f} seconds".format(time))
```

```
Started Fourier Transforming
fftw:        N =  2          FFTW = 0.000227 seconds
fftw:        N =  4          FFTW = 0.000189 seconds
fftw:        N =  8          FFTW = 0.000159 seconds
fftw:        N =  16          FFTW = 0.000056 seconds
fftw:        N =  32          FFTW = 0.000207 seconds
fftw:        N =  64          FFTW = 0.000147 seconds
fftw:        N =  128          FFTW = 0.000156 seconds
fftw:        N =  256          FFTW = 0.000074 seconds
fftw:        N =  512          FFTW = 0.000238 seconds
fftw:        N =  1024          FFTW = 0.000495 seconds
fftw:        N =  2048          FFTW = 0.000444 seconds
fftw:        N =  4096          FFTW = 0.000896 seconds
fftw:        N =  8192          FFTW = 0.001048 seconds
fftw:        N =  16384          FFTW = 0.001749 seconds
fftw:        N =  32768          FFTW = 0.003704 seconds
fftw:        N =  65536          FFTW = 0.009382 seconds
fftw:        N =  131072          FFTW = 0.019811 seconds
fftw:        N =  262144          FFTW = 0.039992 seconds
fftw:        N =  524288          FFTW = 0.073107 seconds
fftw:        N =  1048576          FFTW = 0.113759 seconds
fftw:        N =  2097152          FFTW = 0.287594 seconds
fftw:        N =  4194304          FFTW = 0.559900 seconds
fftw:        N =  8388608          FFTW = 1.091886 seconds
```

```
In [7]: StepList = np.array(StepList)
        mid = int(len(StepList)/2)
        fig,axes = plt.subplots(figsize=(12,8))
        axes.loglog(StepList,FTList,'b+',label='FFTW')
        axes.loglog(StepList,\
                    (StepList*np.log(StepList))*(FTList[-1]/(StepList[-1]*np.log(StepList[-1]))),
                    'r',label='$Nlog(N)$')
        axes.legend()
        axes.set_title("FFTW Time vs Vector Length",fontsize=20)
        axes.set_xlabel("Vector Length",fontsize=16)
        axes.set_ylabel("Time (s)",fontsize=16)

Out[7]: <matplotlib.text.Text at 0x1139faba8>
```



The plot of Numpy's FFT algorithm scaling is above. The largest vector transformed in under a second was 4,194,304 terms long. Once the computation time became non-negligible, we can see that it follows the $Nlog(N)$ scaling of the Cooley-Tukey algorithm.