Cover page for answers.pdf CSE512 Fall 2019 - Machine Learning - Homework 3

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1.1.1

(1.1.1) As we know that the optimal Bayes risk for data point x is

$$r^*(x) = \min_i \left\{ \sum_{i=1}^{M} \hat{n}_i(x) L(i,j) \right\}$$

$L(i,j)$ is the loss value when a data point is predicted as 'j' instead of actual class 'i'.

It is given for us that

		0	1
			True class i
Assigned j	0	0	α
	1	1	0

0 - positive
1 - Negative

Loss matrix.

$\therefore L(0,1) = 1$ & $L(1,0) = \alpha$.

when $j=0$

$$r^*(x) = \sum_{i=0}^{1} \hat{n}_i(x) \cdot L(i,j)$$

$$= \hat{n}_0(x) \cdot L(0,0) + \hat{n}_1(x) \cdot L(1,0)$$

$$= \hat{n}_0(x) \cdot (0) + (1-\eta(x)) \cdot \alpha$$

$$= (1-\eta(x))\alpha.$$

$\eta(x)$ is prob. that x is positive
So $(1-\eta(x))$ is prob. that x is -ve

when $j=1$

$$= \sum_{i=0}^{1} \hat{n}_i(x) \cdot L(i,1)$$

$$= \hat{n}_0(x) \cdot L(0,1) + \hat{n}_1(x) \cdot L(1,1)$$

$$= \eta(x) \cdot (1) + 0 = \eta(x).$$

So $\boxed{r^*(x) = \min\{ \eta(x) + (1-\eta(x)) \cdot \alpha \}.}$

1.1.2

(1.1.2) Asymptotic risk

$r(x)$ can be given by

$$r(x) = P(Y=1)\cdot P(Y_{neighbor}=0) + \alpha \cdot P(Y=0)\cdot P(Y_{neighbor}=1)$$

$$= \eta(x)\cdot\left[1-\eta(x)\right] + \alpha\cdot(1-\eta(x))\cdot\eta(x)$$

hence

$$\cancel{r(x) = \eta(x) - \eta(x)\cdot\eta(x) + \alpha\,\eta(x)}$$

$$r(x) = \eta(x)\cdot\left[1-\eta(x)+\alpha-\alpha\cdot\eta(x)\right]$$

$$= \eta(x)\cdot\left[(1-\eta(x))\cdot(1+\alpha)\right]$$

$$\boxed{r(x) = \eta(x)\cdot(1-\eta(x))\cdot(1+\alpha)}$$

1.1.3

(1.1.3) We know that

$$\gamma^*(x) = \min\{\eta(x), (1-\eta(x))\cdot\alpha\}.$$

case:1

consider $\eta(x)$ is maximum in above two terms.

i.e. $\eta(x) \geq \alpha(1-\eta(x))$

$$\eta(x) + \alpha\eta(x) \geq \alpha.$$

$$\eta(x) \geq \frac{\alpha}{1+\alpha}.$$

we know that $\eta(x) \leq 1$

hence

$$\frac{\alpha}{1+\alpha} \leq \eta(x) \leq 1. \quad \text{— (1)}$$

$$\gamma^*(x) = \alpha(1-\eta(x)).$$

$$\eta(x) = 1 - \frac{\gamma^*(x)}{\alpha} \quad \text{— (2)}$$

substituting (2) in (1)

$$\frac{\alpha}{1+\alpha} \leq 1 - \frac{\gamma^*(x)}{\alpha} \leq 1$$

$$\frac{\gamma^*(x)}{\alpha} \leq \frac{1}{1+\alpha}.$$

$$\frac{\gamma^*(x)}{\alpha}\cdot(1+\alpha) \leq 1$$

$$\gamma^*(x)\left(1+\frac{1}{\alpha}\right) - 1 \leq 0 \quad \text{— (3)}.$$

from 1.1.2 we've

$$\gamma(x) = (1+\alpha)\eta(x)(1-\eta(x))$$

$$= (1+\alpha)\left(1-\frac{\gamma^*(x)}{\alpha}\right)\cdot\left(\cancel{1}-\cancel{1}+\frac{\gamma^*(x)}{\alpha}\right) \quad \text{from (2).}$$

$$= (1+\alpha)\left(1-\frac{\alpha\gamma^*(x)}{\alpha}\right)\cdot\left(\frac{\gamma^*(x)}{\alpha}\right) \quad \text{— (4)}$$

Now consider

$$\left(1-\frac{\gamma^*(x)}{\alpha}\right)\cdot\left(\frac{\gamma^*(x)}{\alpha}\right)-(1-\gamma^*(x))\cdot\gamma^*(x)$$

$$=\left(\frac{1}{\alpha}-\frac{\gamma^*(x)}{\alpha^2}-1+\gamma^*(x)\right)\cdot\gamma^*(x)$$

$$=\left(\left(\frac{1}{\alpha}-1\right)+\gamma^*(x)\left(1-\frac{1}{\alpha^2}\right)\right)\gamma^*(x)$$

$$=\left(\gamma^*(x)\left(1+\frac{1}{\alpha}\right)-1\right)\cdot\gamma^*(x)\cdot\left(1-\frac{1}{\alpha}\right)$$

here $A \cdot B$.

$\gamma^*(x)\left(1+\frac{1}{\alpha}\right)-1$ is ≤ 0 from 3.

and $\gamma^*(x)\cdot\left(1-\frac{1}{\alpha}\right)>0$ since $1-\frac{1}{\alpha}>0$.

hence The above multiplication is ≤ 0.

$$\left(1-\frac{\gamma^*(x)}{\alpha}\right)\left(\frac{\gamma^*(x)}{\alpha}\right)-(1-\gamma^*(x))(\gamma^*(x))\leq 0.$$

$$\left(1-\frac{\gamma^*(x)}{\alpha}\right)\left(\frac{\gamma^*(x)}{\alpha}\right)\leq(1-\gamma^*(x))\gamma^*(x). \quad \text{— 5}$$

from 4 & 5

we get

$$\gamma(x)\leq(1+\alpha)(1-\gamma^*(x))\gamma^*(x) \quad \text{— 5}.$$

Case 2:

Now consider

$$\eta(x)<\alpha(1-\eta(x))$$

$$\eta(x)+\alpha\eta(x)<\alpha.$$

$$\eta(x)<\frac{\alpha}{1+\alpha}.$$

So $\gamma^*(x)=\eta(x)$ i.e. $\min\{\eta(x),\alpha(1-\eta(x))\}$.

Substituting in 1.1.2.

we get

$$\gamma(x) = (1+\alpha)\eta(x)(1-\eta(x))$$

$$= (1+\alpha)\gamma^*(x)(1-\gamma^*(x)) \quad \text{--- } 6 .$$

from 5 & 6 it is clelar that.

$$\boxed{\gamma(x) \leq (1+\alpha)\gamma^*(x)(1-\gamma^*(x))}$$

1.1.4

(1.1.4) Let R be the asymptotic risk of the 1-NN classifier & R^* be the bayes risk.

since

$$R(x) = E(r(x))$$

from 1.1.3

$$r(x) \le (1+\alpha) r^*(x)(1-r^*(x))$$

$$E(r(x)) \le E\left[(1+\alpha) r^*(x)(1-r^*(x))\right]$$

$$\le (1+\alpha) E\left[r^*(x) - r^*(x)^2\right]$$

$$\le (1+\alpha)\left[E(r^*(x)) - E(r^*(x)^2)\right]$$

$$\le (1+\alpha)\left[E(r^*(x)) - \left[var(r^*(x)) + [E(r^*(x))]^2\right]\right]$$

since $E(x^2) = var(x) + [E(x)]^2$

$$\le (1+\alpha)\left[E(r^*(x))(1-E(r^*(x))) - var(r^*(x))\right]$$

$$\le (1+\alpha)\left[R^*(1-R^*) - var(r^*(x))\right]$$

$$\le (1+\alpha) R^*(1-R^*) - (1+\alpha) var(r^*(x))$$

hence. $\boxed{R \le (1+\alpha) R^*(1-R^*)}$ since $1+\alpha \ge 0$, $var(r^*(x)) \ge 0$

1.2.1

(1.2)

(1.2.1) As we know that

Asymptotic risk in K-NN classifier

$$r(x) = P(Y=1) \cdot P(Y_{majority\text{-}neighbors} = 0) + P(Y=0) \cdot P(Y_{majority\text{-}neighbors} = 1)$$

given ~~from~~ in the data that

$$P(Y=1) = \eta(x), \quad P(Y_{majority\text{-}neighbors} = 1) = g(n,k)$$

hence $P(Y=0) = 1-\eta(x)$, $P(Y_{majority\text{-}neighbors} = 0) = 1-g(n,k)$

Substituting in the above eqn.

$$\boxed{r(x) = \eta(x) \cdot [1-g(n,k)] + (1-\eta(x)) \cdot g(n,k)}$$

1.2.2

(1.2.2) As we know

$$\gamma^*(x) = \min\{\eta(x), 1-\eta(x)\}$$

from the above derivation we've

$$\gamma(x) = \eta(x)\cdot(1-g(\eta,k)) + (1-\eta(x))\cdot g(\eta,k) \quad \text{—} \ (1)$$

So when $\eta(x) \le \frac{1}{2}$

$$\gamma^*(x) = \min\{\eta(x), 1-\eta(x)\}$$

$$= \eta(x)$$

i.e. $\eta(x) = \gamma^*(x)$

Substituting in (1)

$$\gamma(x) = \gamma^*(x)(1-g(\eta,k)) + (1-\gamma^*(x))g(\eta,k)$$

$$= \cancel{\gamma^*(x) - \gamma^*(x)\cdot g(\eta,k) +}$$

$$= \gamma^*(x) - \gamma^*(x)\cdot g(\eta,k) + (1-\gamma^*(x))\cdot g(\gamma^*(x),k)$$

$$\gamma(x) = \gamma^*(x) + (1-2\gamma^*(x))\cdot g(\gamma^*(x),k) \quad \text{—} \ (2)$$

when $\eta(x) \ge \frac{1}{2}$

$$\gamma^*(x) = 1-\eta(x) \Rightarrow \eta(x) = 1-\gamma^*(x)$$

Substituting in (1)

$$\gamma(x) = (1-\gamma^*(x))\cdot(1-g(\eta,k)) + (\cancel{1}-\cancel{1}+\gamma^*(x))\cdot g(\eta,k)$$

$$= 1-\gamma^*(x) + [\gamma^*(x) - 1 + \gamma^*(x)]\cdot g(\eta,k)$$

$$= 1-\gamma^*(x) + [2\gamma^*(x) - 1]\cdot g(1-\gamma^*(x),k)$$

$$= 1-\gamma^*(x) + (2\gamma^*(x)-1)\cdot[1-g(\gamma^*(x),k)]$$

$$\gamma(x) = \gamma^*(x) + (1-2\gamma^*(x))\cdot g(\gamma^*(x),k) \quad \text{—} \ (3)$$

1.2.3

So from ②, ③, it is clear for us that in any case

$$\boxed{\gamma(x) = \gamma^*(x) + (1-2\gamma^*(x))\cdot g(\gamma^*(x), K)}$$

(1.2.3) Hoeffding's inequality ~~pro~~ is

$$P(H(n)) \geq (p+\epsilon)n \leq \exp(-2\epsilon^2 n).$$

$H(n) \rightarrow$ no. of points that are positive

$p \rightarrow$ probability that a point is positive

n = total points.

$$g(\gamma^*(x), K) = \begin{cases} P(H(n) \geq \frac{K}{2}) & \text{when } n \text{ is even} \\ P(H(n) > \frac{K}{2} & \text{when } n \text{ is odd.} \end{cases}$$

So $P(H \geq \frac{K}{2}) = P(H \geq (\gamma^*(x) + 0.5 - \gamma^*(x))K)$.

observing with Hoeffdings inequality it's clear that $p = \gamma^*(x)$

$\epsilon = 0.5 - \gamma^*(x)$

$n = K.$

So we'll get

$$g(\gamma^*(x), K) = \boxed{P(H \geq \frac{K}{2}) \leq \exp(-2(0.5-\gamma^*(x))^2 K)}$$

1.2.4

(1.2.4)

We need to prove

$$\gamma(x) \le \gamma^*(x) + \frac{1}{\sqrt{2K}}.$$

from 1.2.2, we know that

$$\gamma(x) = \gamma^*(x) + (1-2\gamma^*(x))\, g(\gamma^*(x), K)$$

rewriting the equation as follows

$$\gamma(x) = \gamma^*(x) + \frac{1}{\sqrt{2K}} \cdot \left[\sqrt{2K}(1-2\gamma^*(x))\, g(\gamma^*(x), K)\right] \quad — (7)$$

from 1.2.3

$$g(\gamma^*(x), K) \le \exp\left(-2(0.5-\gamma^*(x))^2 K\right) \quad — (8)$$

substitute (8) in (7).

$$\gamma(x) \le \gamma^*(x) + \frac{1}{\sqrt{2K}} \left[\sqrt{2K}(1-2\gamma^*(x)) . \exp\left(-2(0.5-\gamma^*(x))^2 K\right)\right]$$

$$\le \gamma^*(x) + \frac{1}{\sqrt{2K}} \left[2\sqrt{2K}(0.5-\gamma^*(x)) . \exp\left(-2(0.5-\gamma^*(x))^2 K\right)\right]$$

Let $Z = \sqrt{2K}(0.5-\gamma^*(x))$

the the above eqn. will be of form

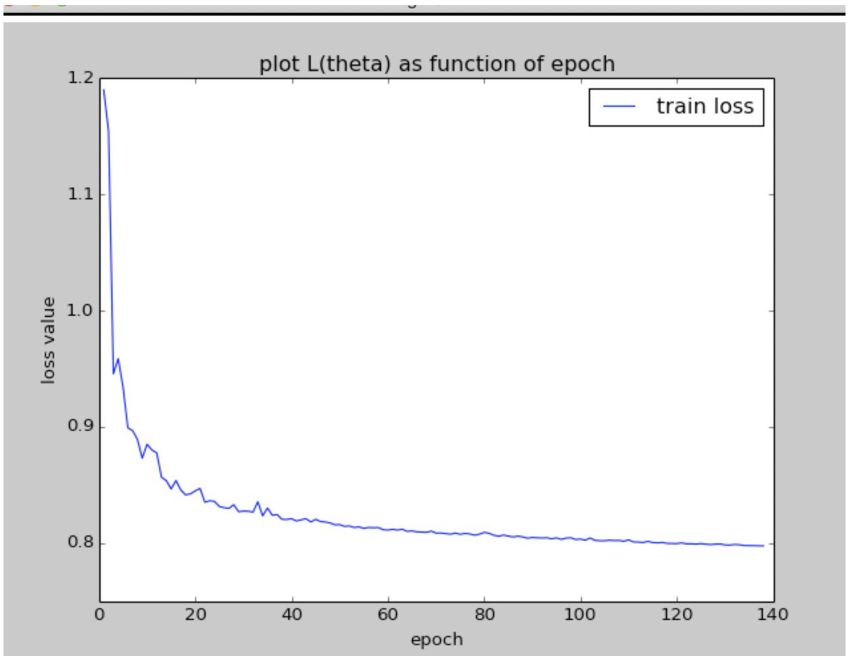
$$\le \gamma^*(x) + \frac{1}{\sqrt{2K}}\left[2Z\exp(-Z^2)\right]$$

The values for $2Z\exp(-Z^2)$ lies between $[-0.858, 0.858]$ which is ≤ 1.

hence $\boxed{\gamma(x) \le \gamma^*(x) + \frac{1}{\sqrt{2K}}}$

2.3.1

a. Number of epochs before exiting = 138
b. Plot showing L(theta) as a function of epoch

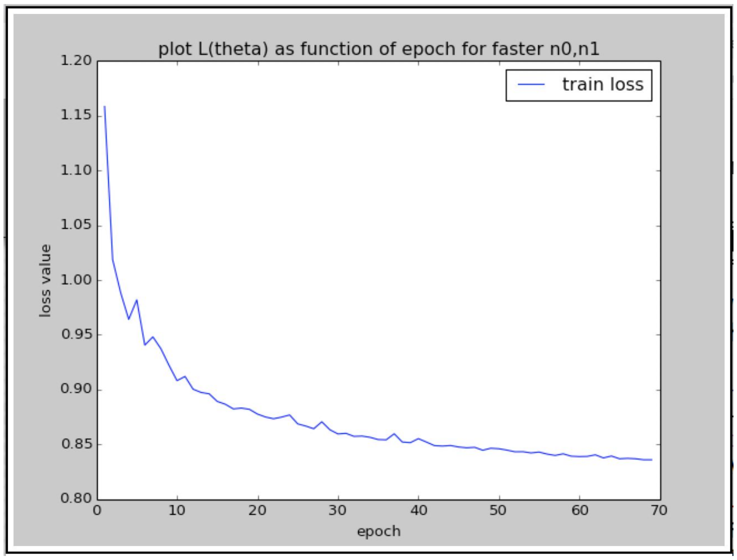


c. Final value of L(theta) after the optimization = 0.8

2.3.2.

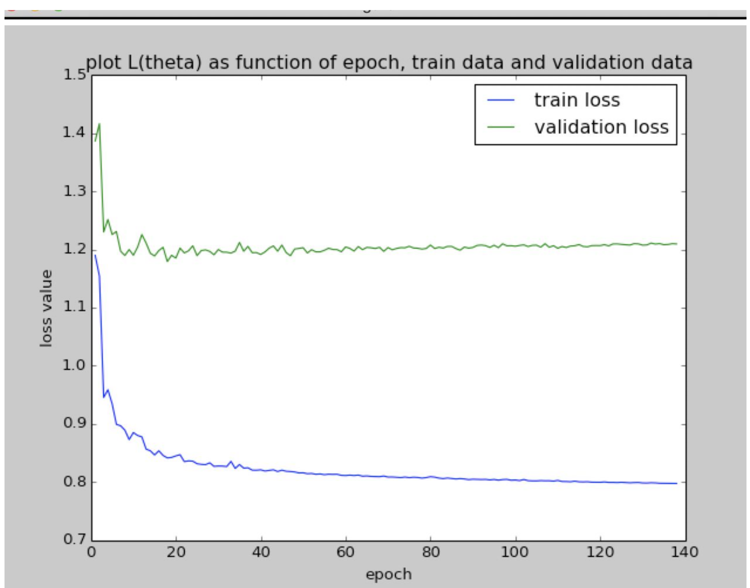
a. The pair of n0,n1 that leads to faster convergence is 0.1, 5.
   The no of epochs taken is 70.
   Final value of L(theta) = 0.85

b.

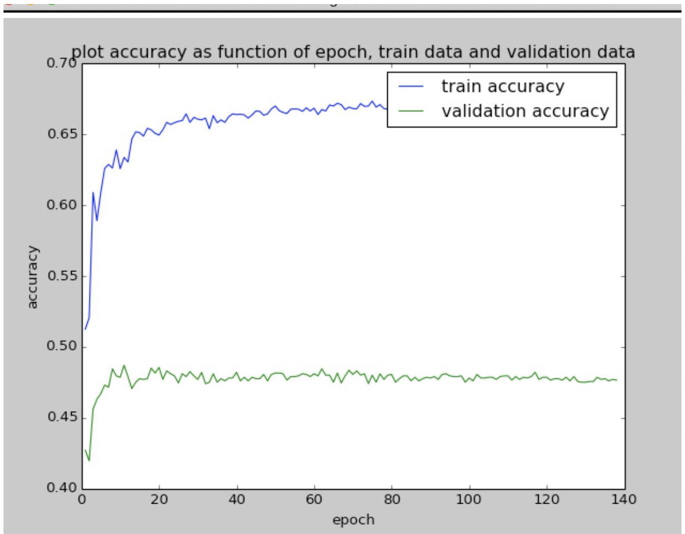

plot L(theta) as function of epoch for faster n0,n1
train loss
loss value
epoch
1.20
1.15
1.10
1.05
1.00
0.95
0.90
0.85
0.80
0
10
20
30
40
50
60
70

2.3.3

a.


plot L(theta) as function of epoch, train data and validation data
train loss
validation loss
loss value
epoch
1.5
1.4
1.3
1.2
1.1
1.0
0.9
0.8
0.7
0
20
40
60
80
100
120
140

b.



2.3.4. Confusion matrix

On Train Data

		Predicted Values			
		1	2	3	4
	1	536	185	28	34
Actual	2	139	911	114	105
Values	3	40	231	353	192
	4	29	113	93	897

On Validation Data

		Predicted Values			
		1	2	3	4
	1	169	94	26	27
Actual	2	133	359	77	91
Values	3	39	197	77	125
	4	25	116	90	355

2.4

1. Best accuracy obtained on test data on kaggle - 46%