AM5630: Foundations of Computational Fluid Dynamics
Computer Assignment 3
Lid Driven Cavity

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1 Simulate Lid Driven Cavity Problem using Navier Stokes Equation

1.1 Problem Definition

Dimensions:

Number of X grid points : 51

Number of Y grid points : 51

Velocity of lid : 1 m/s

Length (l) : 1m

Width (w) : 1m

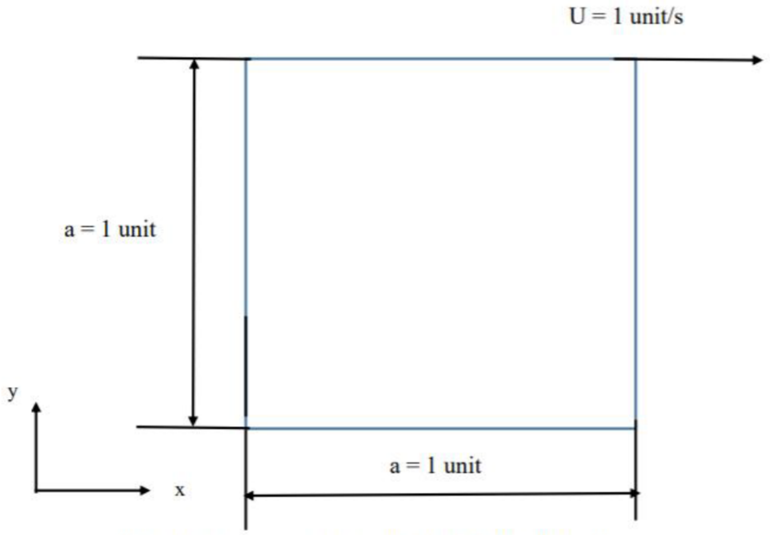


Figure 1: Computational domain of Lid Driven Cavity Problem

The given Lid Driven Cavity Problem is a 2 Dimensional Problem.

The Governing Navier-Stokes equations are for 2D problem are :

u momentum equation :

$$\frac{\partial u}{\partial t} + u * \frac{\partial u}{\partial x} + v * \frac{\partial u}{\partial y} = -\frac{\partial P}{\rho \partial x} + \nu * \nabla^2 u \tag{1}$$

v momentum equation :

$$\frac{\partial v}{\partial t} + u * \frac{\partial v}{\partial x} + v * \frac{\partial v}{\partial y} = -\frac{\partial P}{\rho \partial y} + \nu * \nabla^2 v \tag{2}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{3}$$

Using the given dimensions of the cavity and properties of the fluid we can write :

$$\nu = \frac{1}{Re}$$

1.2 Numerical Formulation

The Lid Driven Cavity Problem is solved using ***Stream Function - Vorticity Approach***

The Governing Navier-Stokes equations are used to derive the appropriate equations for the Stream function-Vorticity Approach

- Voriticity transport equation :

$$\frac{\partial \omega}{\partial t} + u\frac{\partial \omega}{\partial x} + v\frac{\partial \omega}{\partial y} = \nu\nabla^2\omega \tag{4}$$

- Velocities in terms of Stream Function :

$$u = \frac{\partial \psi}{\partial y} \tag{5}$$

$$v = -\frac{\partial \psi}{\partial x} \tag{6}$$

- Stream Function- Voriticity Relationship :

$$\nabla^2\psi = -\omega \tag{7}$$

- Pressure Poisson Equation:

$$\nabla^2 p = 2 * \rho * [\frac{\partial u}{\partial x} * \frac{\partial v}{\partial y} - \frac{\partial v}{\partial x} * \frac{\partial u}{\partial y}] \tag{8}$$

1.3 Boundary conditions:

1.3.1 Velocity Boundary conditions :

The Boundary Conditions are as follows :

$$u(x,0) = 0 \quad u(x,1) = 1 \quad u(0,y) = 0 \quad u(1,y) = 0$$

$$v(x,0) = 0 \quad v(x,1) = 1 \quad v(0,y) = 0 \quad v(1,y) = 0$$

Since the flow is parallel to walls of the cavity, walls may be considered as streamlines. Thus $\psi = c$, where c is an arbitrary constant, which may be set equal to zero.

Since the Stream Function is constant along the wall, all the derivatives of stream function ***along*** *the wall* will vanish.

The stream function-vorticity relationship reduces to :

$$\frac{\partial^2\psi}{\partial^2 n}_{wall} = -\omega_{wall}$$

where n is the normal direction.

The Discretized form of Boundary conditions for the Lid Driven Cavity Problem are :

- **Left Wall :**

$$\omega_{1,j} = -2(\frac{\partial^2\psi}{\partial x^2})_{1,j}$$

Using Taylor Series Expansion :

$$\psi_{2,j} = \psi_{1,j} + \frac{\partial\psi}{\partial x}_{1,j} Ah + \frac{\partial^2\psi}{\partial x^2}_{1,j} \frac{Ah^2}{2!} + ...$$

Along left wall its known :

$$\frac{\partial\psi}{\partial x}_{1,j} = -v_{1,j}$$

Therefore the taylor series can be rearranged as :

$$\frac{\partial^2\psi}{\partial x^2}_{1,j} = \frac{2(\psi_{2,j} - \psi_{1,j})}{Ah^2} + \frac{2v_{1,j}}{Ah}$$

$$\omega_{1,j} = \frac{2(\psi_{1,j} - \psi_{2,j})}{Ah^2} - \frac{2v_{1,j}}{Ah}$$

Applying the Velocity boundary condition we can obtain:

$$\omega_{1,j} = -2\frac{(\psi_{2,j})}{Ah^2}$$

Similarly we can find the other boundary conditions as well :

- **Top Wall :**

$$\omega_{i,ny} = -2\frac{\psi i, ny - 1}{h^2} - 2\frac{u_{wall}}{h}$$

- **Bottom Wall :**

$$\omega_{i,1} = -2\frac{\psi_{i,2}}{h^2}$$

- **Right Wall :**

$$\omega_{nx,j} = -2\frac{\psi_{nx-1,j}}{h^2}$$

1.3.2 Pressure Boundary Conditions:

$$\frac{\partial P}{\partial n} = 0$$

Where n is the direction normal to the wall.

The above equation can be discretized using backward differencing method and the boundary condidion can be incorporated into the algorithm.

1.4 Discretisation

- **Vorticity Transport Equation:**

  Substitute the equations (5) and (6) in (4)

$$\frac{\partial \omega}{\partial t} = -\frac{\partial \psi}{\partial y}\frac{\partial \omega}{\partial x} + \frac{\partial \psi}{\partial x}\frac{\partial \omega}{\partial y} + \frac{1}{Re}[\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2}]$$

  Assumption: The spatial spacing in both directions is same i.e., ***dx=dy=h***

  The above equation can be discretized as follows :

$$\begin{aligned}\omega_{i,j}^{n+1} = \omega_{i,j}^{n} + \frac{dt}{h^2}[-\frac{(\psi_{i,j+1}^n - \psi_{i,j-1}^n)(\omega_{i+1,j}^n - \omega_{i-1,j}^n)}{4} - \frac{(\psi_{i+1,j}^n - \psi_{i-1,j}^n)(\omega_{i,j+1}^n - \omega_{i,j-1}^n)}{4} \\ + \frac{\omega_{i+1,j}^n + \omega_{i-1,j}^n + \omega_{i,j+1}^n + \omega_{i,j-1}^n - 4\omega_{i,j}^n}{Re}]\end{aligned} \tag{9}$$

  where ***dt*** is the time spacing

- **Stream Function-Vorticity Relationship:**

$$-\omega_{i,j}^n = \frac{\psi_{i+1,j}^n + \psi_{i-1,j}^n + \psi_{i,j+1}^n + \psi_{i,j-1}^n - 4\psi_{i,j}^n}{h^2}$$

  The stream function values can be found out as follows :

$$\psi_{i,j}^{n+1} = \frac{\psi_{i+1,j}^n + \psi_{i-1,j}^n + \psi_{i,j+1}^n + \psi_{i,j-1}^n + h^2\omega_{i,j}^n}{4} \tag{10}$$

- **Pressure Poisson Equation:**

  The RHS of the Pressure poisson equation is discretized as follows

$$RHS_{i,j} = \frac{2\rho}{4h^2} * [((u_{i+1,j} - u_{i-1,j}) * (v_{i,j+1} - v_{i,j-1})) - ((v_{i+1,j} - v_{i-1,j}) * (u_{i,j+1} - u_{i,j-1}))] \tag{11}$$

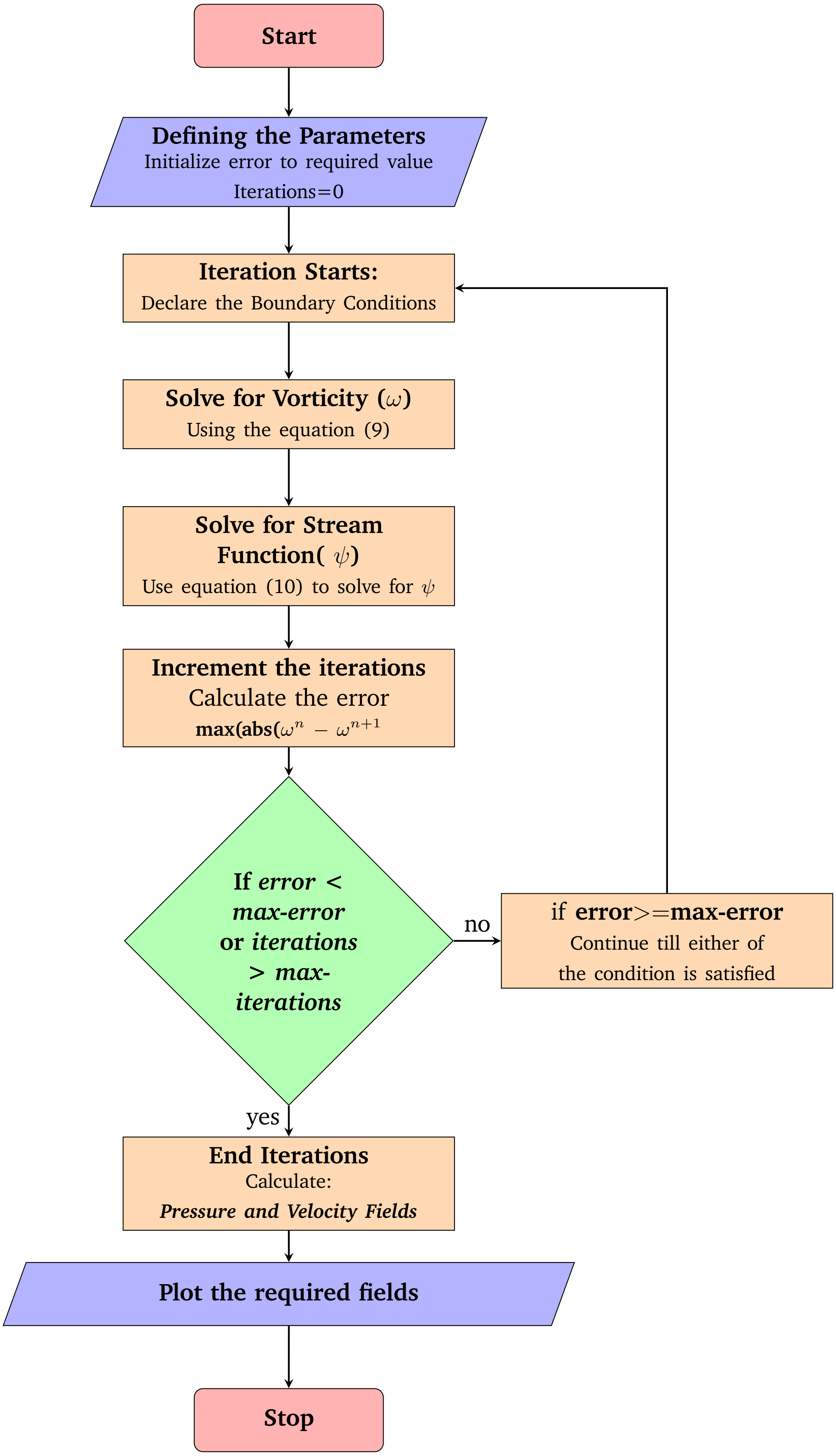
  The LHS of the Pressure poisson equation is discretized as follows

$$LHS = \frac{p_{i+1,j} - 2p_{i,j} + p_{i-1,j}}{h^2} + \frac{p_{i,j+1} - 2p_{i,j} + p_{i,j-1}}{h^2} \tag{12}$$

$$p_{i,j} = 0.25 * (p_{i+1,j} + p_{i-1,j} + p_{i,j+1} + p_{i,j-1}) - (0.25 * RHS * h^2) \tag{13}$$

  The Pressure poisson equation hence can be solved iteratively to find the value of Pressure at each nodes.

1.5 Pseudo-code



1.6 Results

- As we increase the grid points from 51 grid points to 101 grid points, the Streamlines depicting the recirculation at the bottom of the cavity become prominent.
- As we increase the grid points from 51 grid points to 101 grid points, the Vorticity countours are more prominent.
- The Centerline u-velocity and Centerline v-velocity plots are in good agreement with data from Ghia-et al (reference data).
- The pressure distribution plot reveals a low pressure region in the center of the major vortex which is expected.
- The top right corner is a stagnation point for x component of velocity(u-velocity) and is characterized by significant pressure buildup.
- The number of iterations to reach steady state and time elapsed for 51 and 101 grid points is tabulated below

Number of Grid points	**Number of Iterations to Steady State**	**Time Elapsed**
51	16326	11.82s
101	22709	40.52s

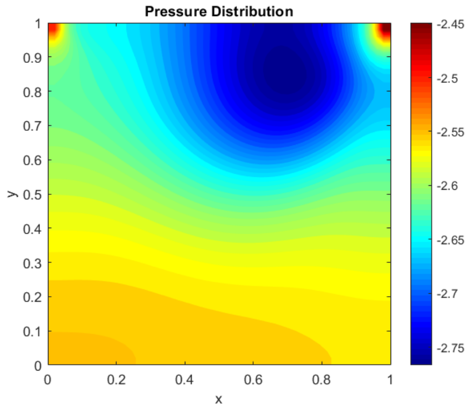
Table 1: Performance of code (rough estimate)
Time step $\triangle$t=0.001

- The maximum iterations for a time step of dt=0.0001 is set to 1000000.
- The streamlines depicting the minor vortex became even more prominent with reduced time step and increased number of iterations.

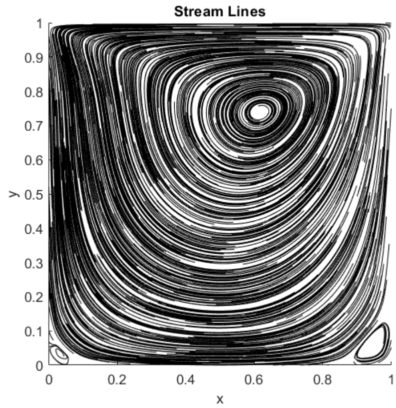
Number of Grid points	**Number of Iterations to Steady State**	**Time Elapsed**
51	13305	81.00s
101	134012	394.20s

Table 2: Performance of code (rough estimate)
Time step $\triangle$t=0.0001

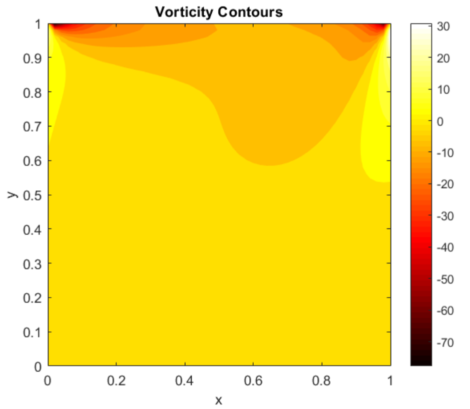
1.7 Plots and Distributions



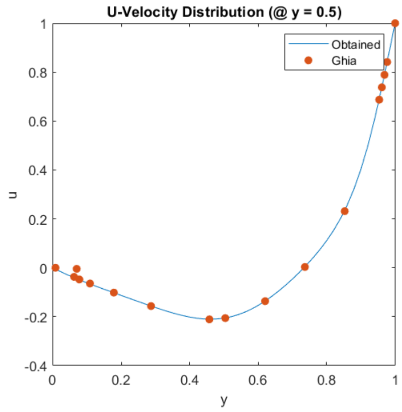
(a) Pressure Distribution



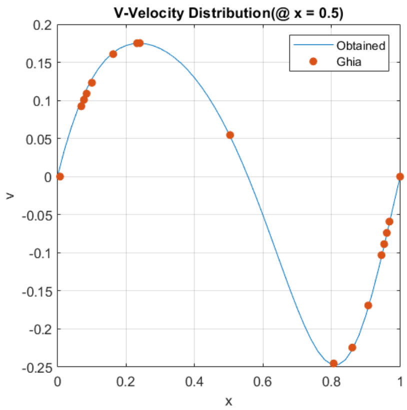
(b) Streamlines



(c) Vorticity Distribution

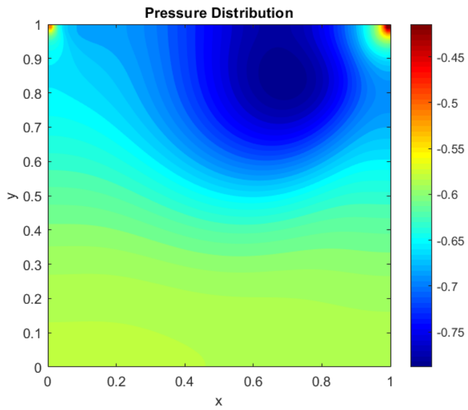


(d) u velocity at y= $0.5m$

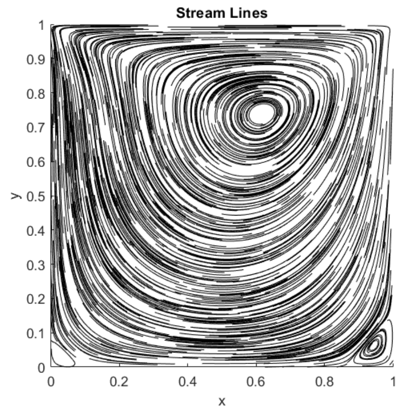


(e) v velocity at x= $0.25m$

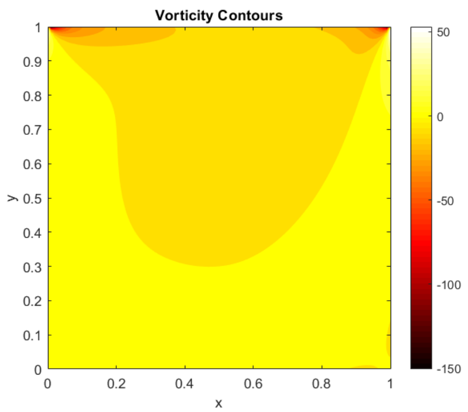
Figure 2: Plots at 51 Grid points Time step : $\triangle$t=0.001



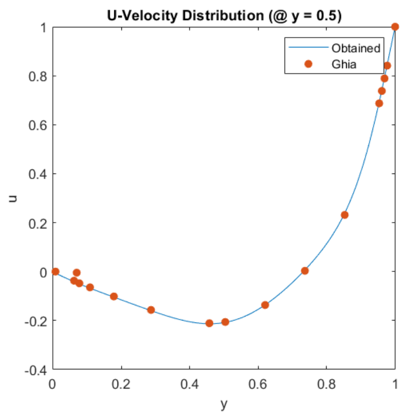
(a) Pressure Distribution



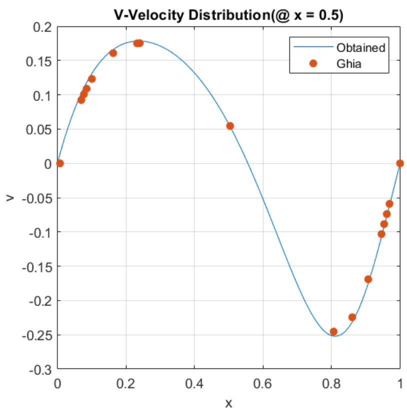
(b) Streamlines



(c) Vorticity Distribution

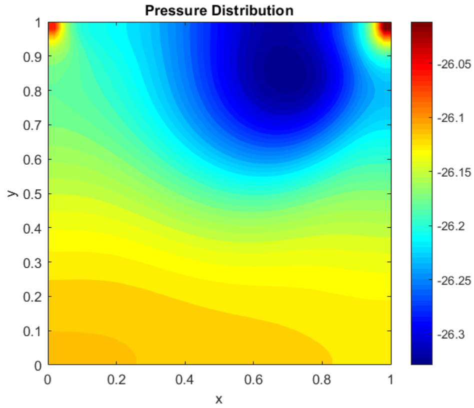


(d) u velocity at y= $0.5m$

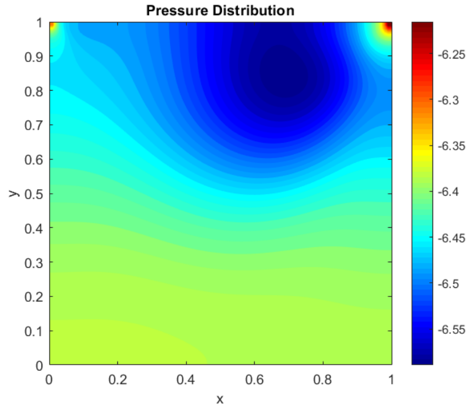


(e) v velocity at x= $0.25m$

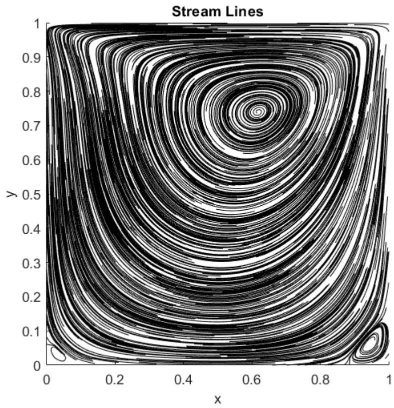
Figure 3: Plots at 101 Grid points Time step : $\triangle$t=0.001



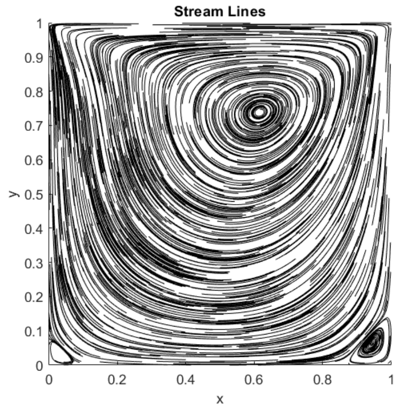
(a) Pressure Distribution 51 grid points



(b) Pressure Distribution 101 grid points



(c) Stream lines 51 grid points

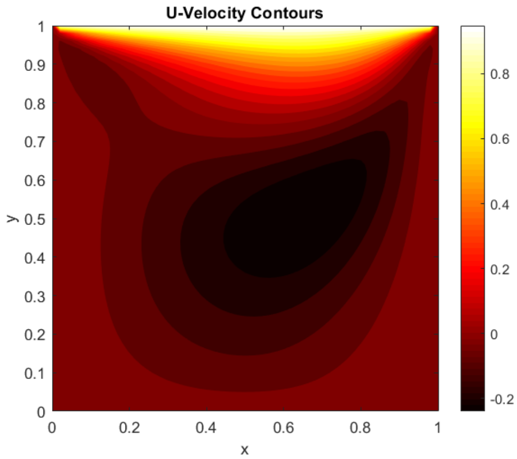


(d) Stream lines 101 grid points

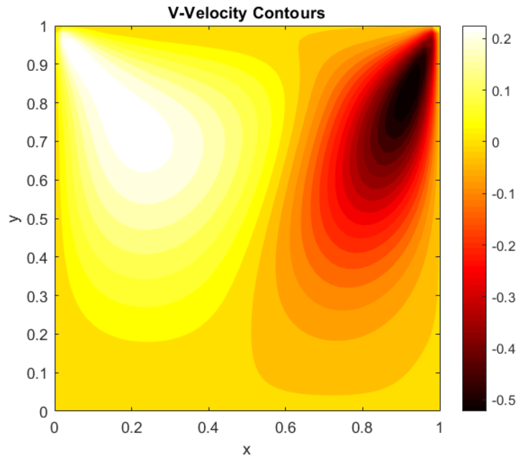
Figure 4: Plots at 51 and 101 Grid points
Time step : $\triangle$t=0.0001

1.8 Additional Results

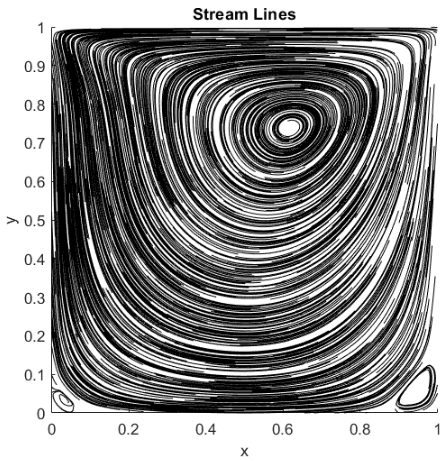
- The Lid Driven Cavity Problem is simulated in Ansys-Fluent using SIMPLE Algorithm for the same boundary conditions.
- It can be seen that velocity contours and Stream lines obtained from Stream Function-Vorticity Approach and Ansys-Fluent Results are identical.



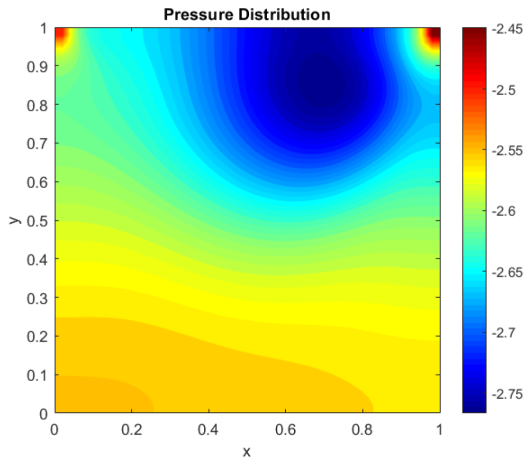
(a) u velocity contours



(b) v velocity contours

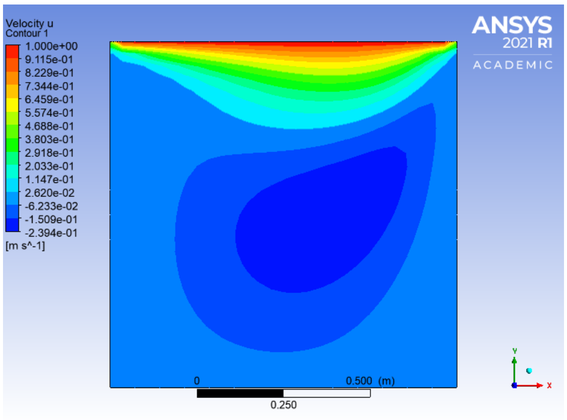


(c) Streamlines

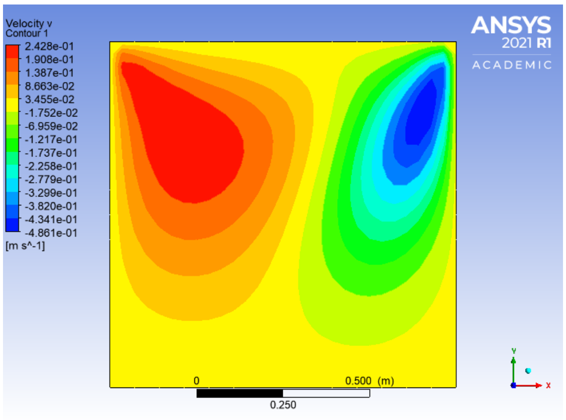


(d) Pressure Distribution

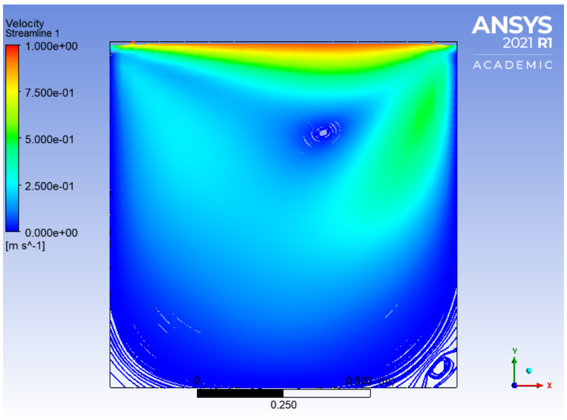
Figure 5: Results from Stream Function-Vorticity Approach
dt=0.001 nx=ny=51



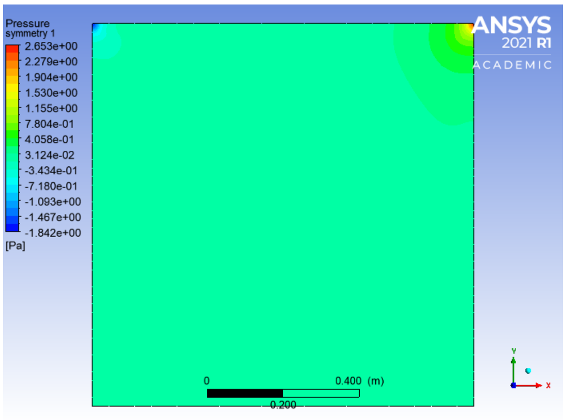
(a) u velocity contours



(b) v velocity contours



(c) Streamlines



(d) Pressure Distribution

Figure 6: Ansys-Fluent Results

2 References:

Ghia, U., Ghia, K.N. and Shin, C.T. (1982) High-Re Solutions for Incompressible Flow Using the Navier-Stokes Equations and a Multigrid Method. Journal of Computational Physics, 48, 387-411. http://dx.doi.org/10.1016/0021-9991(82)90058-4