Numerically Efficient Fourier-Based Technique for Calculating Error Probabilities with Intersymbol Reference

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Introduction

Here we propose a numerically efficient technique for calculating of the probability the symbol error for arbitary coherent modulation schemes in the presence of Intersymbol Reference and additive noise.

Abbreviations and Notations

ISI - Intersymbol Interference

CAD - Computer-Aided Design

M-PSK - M-ary Phase-Shift Keying

M-QAM - M-ary Quadrature Amplitude Modulation

QPSK - Quadrature Phase Shift Keying

BPSK - Binary Phase Shift Keying

Outline

- The effects of Intersymbol Interference(ISI) can degrade the probability of error performance of the receivers.
- For determining the performance of the receiver P_e is evaluated over a particular sequence of ISI symbols and then averaged.
- But this technique is computationally expensive so cosiderable work is done for more efficient techniques.
- Here we present a method that can be applied to 1 and 2-D coherent signaling formats.
- The probability of correctly decoding a particular symbol is the inverse Fourier transform of the windowed characteristic function.
- Sampling theorem , truncation and Appropriate spectral windowing are used for numerical evaluation of the integral.

Problem Statement and General Solution

Receiver Output

Sample solution of a coherent receiver y in the presence of ISI and noise

$$y = \sum_{k=-N_{k\neq 0}}^{N} b_k a_k + b_0 a_0 + n \tag{1}$$

where

- b_k's represent the transmitted data sequence in which each symbol is one of the M-possible equally probable symbols.
- a_k's are ISI coefficients.
- b_0 current symbol to be detected.

Probability of correct decision

The probability of making a correct decision when j^{th} element s_j is sent is where

$$P_{cj} = \Pr\left(y \in D_j | b_0 = s_j\right) \tag{2}$$

- D_j is the decision region in the complex plane corresponding to s_j.

The probability of the error is

$$P_E = 1 - \frac{1}{M}\sum_{j=1}^{M} P - c_j. \tag{3}$$

Here y is written in terms of in-phase and quadrature components as $y = y_I + iy_Q$, $h_Y(\omega_I, \omega_Q)$ is it's bivariate characteristic notation,

$$h_Y(\omega_I, \omega_Q) = h_{ISI}(\omega_I, \omega_Q) h_N(\omega_I, \omega_Q) e^{i(\omega_I Res_j a_0 + \omega_Q Ims_j a_0)} \tag{4}$$

Probability Density Function

The Probability density function of y is given by

$$f_Y(y_I, y_Q) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h_Y(\omega_I, \omega_Q) e^{-i(\omega_i y_I + \omega_Q y_Q)} \, d\omega_I \, d\omega_Q \tag{5}$$

As,

$$P_{cj} = \int \int D_j f_Y(y_I, y_Q) \, dy_i \, dy_Q \tag{6}$$

We combine (5) and (6) to

$$P_{c_j} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} K_j(\omega_I, \omega_Q) h_{ISI}(\omega_I, \omega_Q) h_N(\omega_I, \omega_Q) e^{i(\omega_I Re\{s_j a_0\} + \omega_Q Im\{s_j a_o\})} \tag{7}$$

$$d\omega_I d\omega_Q \tag{8}$$

Fourier Equation

Here $K_j(\omega_I, \omega_Q)$ is called the Fourier form and the equation is called the Inverse Fourier Transformation.

- The probability and the spectral domains are equivalently stated with the relations

$$P_j(t_I, t_Q) = F^{-1}\{k_j(\omega_I, \omega_Q)h_{ISI}(\omega_I, \omega_Q)h_N(\omega_I, \omega_Q)\} \tag{9}$$

Or equivalently

$$P_j(t_I, t_Q) = D_j(t_I, t_Q)f_ISI(t_I, t_Q)f_N(t_I, t_Q) \tag{10}$$

- Where f_{ISI} and f_N are the probability density functions of the random variables representing ISI and Noise.
- Sampling theorem and Equations(8) and (9) are used to compute the Fourier Integrals.

Poisson's Sum

With rectangular grid of Spectral Resolution $\delta w_I = \delta w_Q = \frac{2\pi}{T}$, the Poisson's sum is

$$\sum_{m_Q=-\infty}^{\infty} \sum_{m_Q=-\infty}^{\infty} P_j(t_I - m_I T, t_Q - m_Q T) = \frac{1}{T^2} \sum_{t_Q=-\infty}^{\infty} \sum_{t_Q=-\infty}^{\infty} K_j(\frac{2\pi}{T} l_I, \frac{2\pi}{T} l_Q) \tag{11}$$

$$.h_{ISI}(\frac{2\pi}{T} l_I, \frac{2\pi}{T} l_Q).h_N(\frac{2\pi}{T} l_I, \frac{2\pi}{T} l_Q).e^{i(l_I t_I + l_Q t_Q)\frac{2\pi}{T}} \tag{12}$$

But $eq - (11)$ is not defined in some regions because the decision region D_f can be of infinite duration in the (t_I, t_Q)n plane in infinite P_j. To correct this decision region is modified with a ideal brick-wall window.

Corrected Probability

After truncation the calculated probability of correctly detecting the j^{th} symbol is

$$\mathbf{P_{cj}} = \frac{1}{T^2} \sum_{t_Q=-P}^{P} \sum_{t_Q=-P}^{P} \mathbf{K_j} \prod_{k=-N}^{k=N} h_{ISIk}(_I, \omega_Q).h_N(_{I,Q})e^{i(_I Re\{s_j a_0\}+_Q Im\{s_j a_o\})} \quad (13)$$

$$d\omega_I d\omega_Q \quad (14)$$

ANd $P_{cj} = \mathbf{P_{cj}}$+error(error=$\epsilon_1 -_2 + \epsilon_3$)
The first component ϵ_1 is the result of the modified design region and can be written as

$$\epsilon_1 = (D_j(t_I, t_Q) - (D_J)(t_I, t_Q)) * f_{ISI}(t_I, t_Q) * f_N(t_I, t_Q) \quad (15)$$

evaluated at $t_I = a_{0I} - a_{0Q}$ and $t_Q = a_{0I} + a_{0Q}$

errors

The second component ϵ_2 is due to aliasing and is determined by applying $\mathbf{D_j}$ to eq(9) and using eq(10) as,

$$\epsilon_2 = \sum \sum \mathbf{P_j}(t_I - m_I T, t_Q - m_Q T) \tag{16}$$

Finally ϵ_3 is due to spectral truncation

$$|\epsilon_3| < \frac{1}{T^2} |\mathbf{k_j}(\omega l_I, \omega l_Q)||h_{ISI}(\omega l_I, \omega l_Q)||h_N(\omega l_I, \omega l_Q)| \tag{17}$$

Note that error should be much lesser than P_{cj}

Error Probability for QPSK

QPSK

The characteristic function of the k^{th} interfernce component where b_k's represent intake and quadratic components ($b_k = \pm 1 \pm i$) is given by

$$h_{ISIk}(\omega_I, \omega_Q) = cos(\omega_I a_{kI} + \omega_Q a_{kQ}) cos(\omega_I a_{kQ} - \omega_Q a_{kI}) \tag{18}$$

For noise both inphase and quadratic components are independent and identically distributed

$$h_N(\omega_I, \omega_Q) = e^{-\frac{(\omega_I^2 + \omega_Q^2)\sigma^2}{2}} \tag{19}$$

The fourier transformation is

$$(K_1)(\omega_I, \omega_Q) = \frac{T^2}{4} sinc \frac{\omega_I T}{4\pi} sinc \omega_Q T 4\pi e^{-i(\omega_I + \omega_Q)\frac{t}{4}} \tag{20}$$

where sinc(x)=sin$\pi * x \frac{}{\pi * x}$

parameters

ϵ is a user-specified parameter where γ and ϕ and P the spectral truncation function are

$$\epsilon = \phi(\frac{\frac{-T}{2} + \gamma}{\sigma}) \tag{21}$$

$$\gamma = \sum_{K=-N}^{N} (|a_{kI}| + a_{kQ}|) \tag{22}$$

$$\phi = (\sqrt{2\pi})^{\frac{-1}{2}} \sum_{-\infty}^{x} e^{\frac{-y^2}{2}dy} \tag{23}$$

$$P = (\frac{LT}{2\pi}) \tag{24}$$

The total numerical error is

$$|error| < 30\epsilon \tag{25}$$

T

he probability of error is $P_E = 1 - P_{C1}$. The probability of error for QPSK is

$$\mathbf{P_E} = 1 - \frac{1}{4}\sum_{l_Q=P}^{P}\sum_{l_Q=P}^{P} sinc\frac{l_I}{2} sinc\frac{l_Q}{2}\prod_{k=-N} Nh_{ISIk}(\omega l_I, \omega_Q).h_N(\omega l_I, \omega l_Q) \quad (26)$$

$$.cos(\omega l_I(a_{0I} - a_{0Q} - \frac{T}{4}) + \omega l_Q(a_{0I} + a_{0Q} - \frac{T}{4} \quad (27)$$

. The probability of bit error for BPSK is $P_B = 1 - \sqrt{1 - P_E}$. Table 1 contains the calculated probability probability for symbol error for r=0(induces crosstalk between in-phase and quadrature channels) and for various values of N and table 2 is for r=0.25.
This is a simple numerically efficient techniques for calculating probability of error with ISI and additive noise.

TABLE I

QPSK ERROR PROBABILITIES FOR $t/T_S = 0.1$, SNR $= 14.98235$ dB, $r = 0$, AND $L = 7.2$

N	T	$\hat{P}_E$	Relative Error	Relative Error Bound
1	4.8111003	2.6654431×10^{-6}	6.2562056×10^{-7}	3.7517237×10^{-5}
2	5.0083197	6.8222171×10^{-6}	4.9091744×10^{-8}	1.1335020×10^{-5}
3	5.1396165	1.4551146×10^{-5}	8.5758927×10^{-9}	6.8723108×10^{-6}
4	5.2380412	1.9005067×10^{-5}	2.3658970×10^{-9}	5.2617547×10^{-6}
10	5.5708367	3.1407996×10^{-5}		
100	6.4594314	4.2844639×10^{-5}		
1000	7.3636225	4.4205987×10^{-5}		

TABLE II

QPSK ERROR PROBABILITIES FOR $t/T_S = 0.1$, SNR $= 14.98235$ dB, $r = 0.25$, AND $L = 7.2$

N	T	$\hat{P}_E$	Relative Error	Relative Error Bound
1	5.4022729	3.8765087×10^{-4}	$6.9651912 \times 10^{-10}$	2.5796408×10^{-7}
2	5.6487971	7.4307113×10^{-4}	$4.4673606 \times 10^{-11}$	1.3457662×10^{-7}
3	5.8129181	9.5683016×10^{-4}	$4.6412543 \times 10^{-12}$	1.0451176×10^{-7}
4	5.9359489	1.0923858×10^{-3}	$1.9310245 \times 10^{-12}$	9.1542749×10^{-8}
10	6.3519433	1.3991347×10^{-3}		
100	7.4626867	1.6308787×10^{-3}		
1000	8.5929256	1.6563448×10^{-3}		

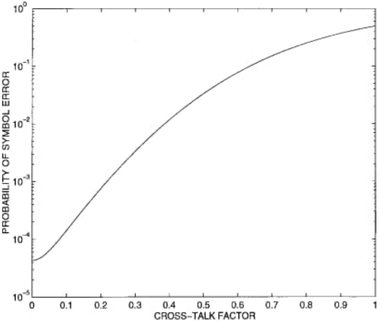

Fig. 1. QPSK $\hat{P}_E$ versus crosstalk factor r.

Figure: Table 1 and 2 P_E vs r curve