Functional Programming

2021-2022

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Higher-order functions

Lecture 5

Outline

- Functions as first-class citizens
- Functions as arguments
- Functions as results
- Folds (and scans) and unfolds
- Summary

Functions as first-class citizens

- functional programming concerns functions (of course!)
- functions are *first-class citizens* of the language
  - functions have all the rights of other types:
    - may be passed as arguments
    - may be returned as results
    - may be stored in data structures
- functions that manipulate functions are *higher-order*

Functions as arguments

- we have already seen examples of higher-order operators encapsulating patterns of computation:

```
map :: (a⟶b)⟶[a]⟶[b]
map f []     = []
map f (x:xs) = f x : map f xs

filter :: (a⟶Bool)⟶[a]⟶[a]
filter p []   = []
filter p (x:xs)
    | p x       = x : filter p xs
    | otherwise = filter p xs
```

- each is a parameterizable program scheme
- parameterization improves modularity, and hence understanding, modification, and reuse

Functions as arguments — continued

- `quickSort` relies on predefined ordering (< and ≤):

```
quickSort :: Ord a ⇒ [a] → [a]
quickSort [ ]      = [ ]
quickSort (x : xs) = quickSort littles ++ [x] ++ quickSort bigs
  where littles = [a | a ← xs, a < x]
        bigs    = [a | a ← xs, x ≤ a]
```

Functions as arguments — continued

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```

- abstract away from ordering

```
quickSortBy :: (a → a → Bool) → [a] → [a]
quickSortBy cmp [ ]    = [ ]
quickSortBy cmp (x:xs) = quickSortBy cmp lefts ++[x]++ quickSortBy cmp rights
  where lefts  = [a | a ← xs, not (x `cmp` a)]
        rights = [a | a ← xs, x `cmp` a]
```

- more flexible/general e.g. `quickSortBy (>)`

Functions as results

- functions may also be returned as results

```
data Op = Add | Sub | Mul | Div
op :: Op ⟶ (Integer ⟶ Integer ⟶ Integer)
op Add = (+)
op Sub = (-)
op Mul = (*)
op Div = div
```

- partial application
- currying
- function composition

Partial application

- consider `add' x y = x + y`
- type `Integer` ⟶ `Integer` ⟶ `Integer`; takes two integers and returns an integer (e.g. `add' 3 4 = 7`)
- another view: type `Integer` ⟶ `(Integer` ⟶ `Integer)` (⟶ associates to the right); takes a single Integer and returns an `Integer` ⟶ `Integer` function (sometimes called an `Integer`-transformer)
  - e.g. `add' 3` is the `Integer`-transformer that adds three
- need not apply function to all its arguments at once: *partial application*; result will then be a function, awaiting remaining arguments (e.g. `op`, `op Add`, `op Add 47`; the expression `op Add 47 11` is a full application)
- *sectioning* (`(3+)`, `(+)`, `(<4)`) is partial application of binary operators
  - what's the type of `(<4)`?

Currying

- in Haskell, every function takes *exactly* one argument
- a function taking pair of arguments can be transformed into a function taking two successive arguments, and vice versa

```
add :: (Integer, Integer) ⟶ Integer
add (x, y) = x + y
add’ :: Integer ⟶ Integer ⟶ Integer
add’ x y = x + y
```

- add’ is called the curried version of add
- named after logician Haskell B. Curry (like the language)
- thus, pair-consuming functions are unnecessary
- in fact, curried functions are the norm in Haskell

Currying — continued

- transformations are implementable as higher-order operations

```
curry :: ((a,b) → c) → (a → b → c)

uncurry :: (a → b → c) → ((a,b) → c)
```

Currying — continued

- transformations are implementable as higher-order operations

```
curry :: ((a,b) ⟶ c) ⟶ (a ⟶ b ⟶ c)
curry f a b = f (a,b)
uncurry :: (a ⟶ b ⟶ c) ⟶ ((a,b) ⟶ c)
```

Currying — continued

- transformations are implementable as higher-order operations

```
curry :: ((a,b) → c) → (a → b → c)
curry f a b = f (a,b)
uncurry :: (a → b → c) → ((a,b) → c)
uncurry f (a,b) = f a b
```

- e.g. `add’ = curry add`
- a related higher-order operation: flip arguments of binary function
  - (later: `reverse = foldl (flip (:)) [ ]`)

Currying — continued

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- e.g. `add’ = curry add`
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```
flip :: (a → b → c) → (b → a → c)
```

Currying — continued

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- e.g. `add’ = curry add`
- a related higher-order operation: flip arguments of binary function
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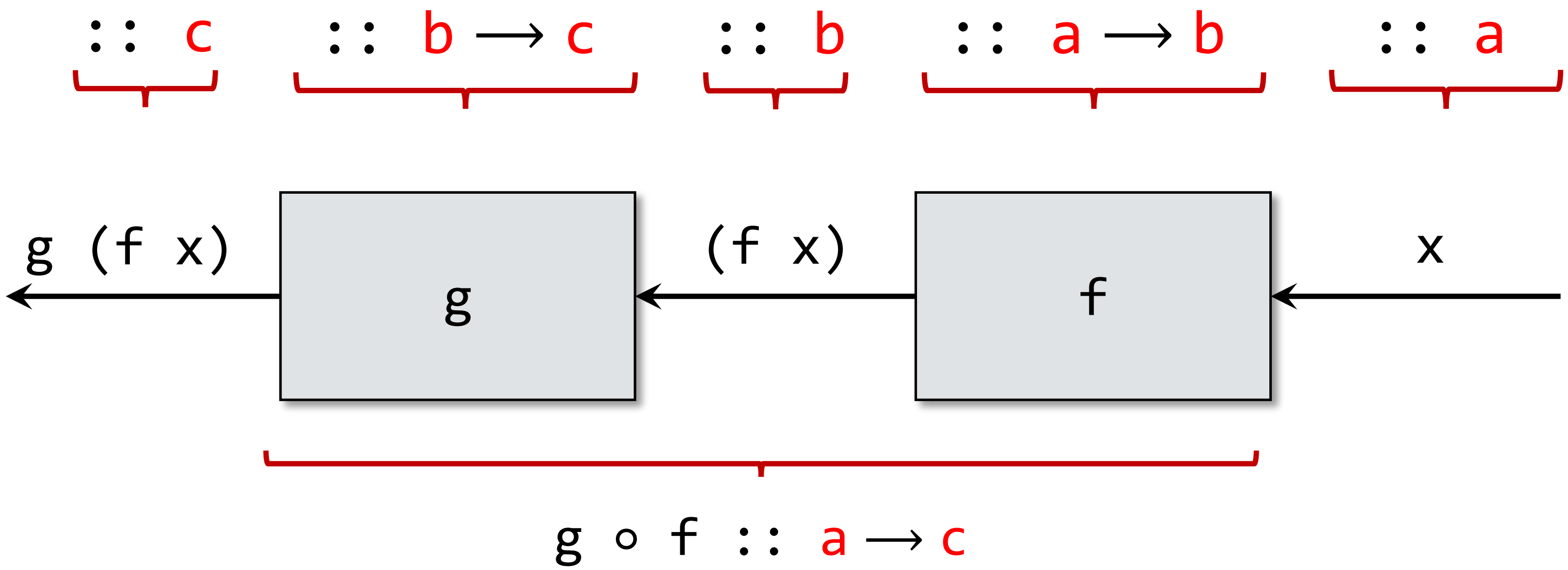
```
flip :: (a → b → c) → (b → a → c)
flip f b a = f a b
```

Function composition



$(\circ)$ :: (b → c) → (a → b) → (a → c)

g ∘ f = \x → g (f x)

- takes two functions that 'meet in the middle' and glues them together to form a third

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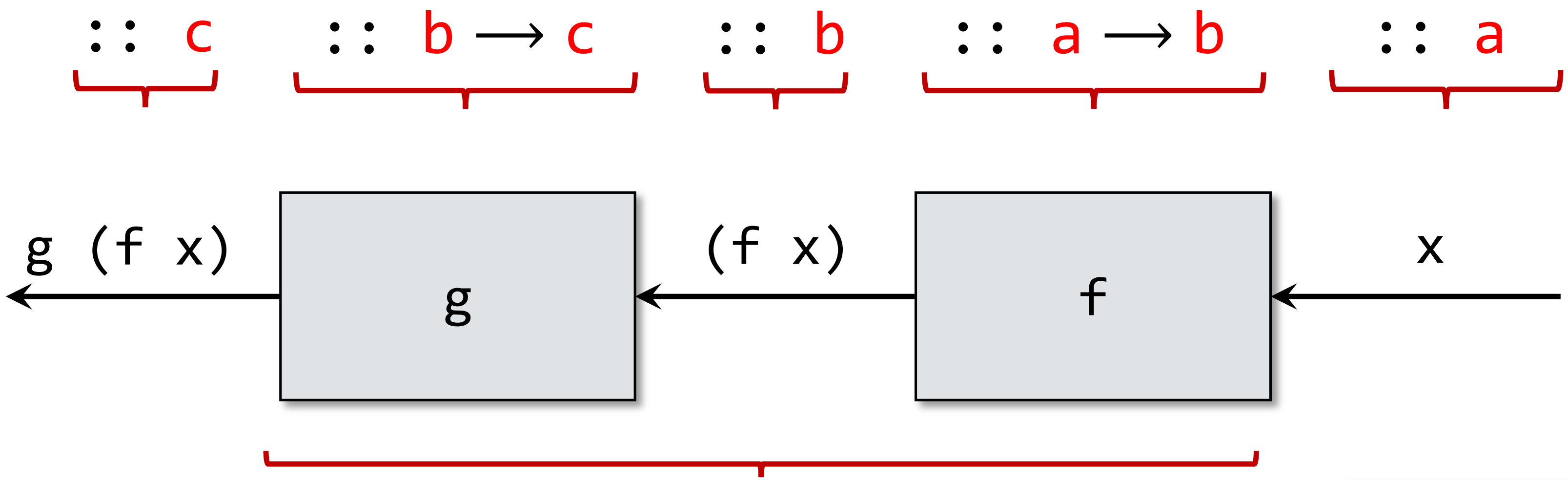

`g ∘ f :: a → c`

- mathematical convention
- pronounce: *g* <u>after</u> *f*

`(∘) :: (b → c) → (a → b) → (a → c)`

`g ∘ f = \x → g (f x)`

- takes two functions that 'meet in the middle' and glues them together to form a third

Folds

- many recursive definitions on lists share a pattern of computation
- capture that pattern as a function (abstraction, conciseness, general properties, familiarity, . . .)
- `map` and `filter` are two common patterns
- *folds* capture many more

A common pattern: foldr

```
sum :: Num p ⇒ [p] ⟶ p
sum []       = 0
sum (x : xs) = x + sum xs

product :: Num p ⇒ [p] ⟶ p
product []       = 1
product (x : xs) = x * product xs

and :: [Bool] ⟶ Bool
and []       = True
and (x : xs) = x && and xs

or :: [Bool] ⟶ Bool
or []       = False
or (x : xs) = x || or xs
```

A common pattern: foldr

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sum :: Num p ⇒ [p] ⟶ p
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A common pattern: foldr

```
foldr :: (a ⟶ b ⟶ b) ⟶ b ⟶ [a] ⟶ b
foldr st ba []       = ba
foldr st ba (x:xs) = x `st` foldr st ba xs
```

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foldr st ba []       = ba
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product :: Num p ⇒ [p] ⟶ p
product []         = 1
product (x : xs)   = x * product xs
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```
and :: [Bool] ⟶ Bool
and []         = True
and (x : xs)   = x && and xs
```

```
sum     = foldr (+)  0
product = foldr (*)  1
and     = foldr (&&) True
or      = foldr (||) False
```

```
or :: [Bool] ⟶ Bool
or []         = False
or (x : xs)   = x || or xs
```

foldr — continued

```
foldr :: (a ⟶ b ⟶ b) ⟶ b ⟶ [a] ⟶ b
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foldr — continued

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list  = 1 : ( 2 : ( 3 : ( 4 : ( 5 : [] ) ) ) ) )
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foldr — continued

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foldr st ba []     = ba
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```
           list   = 1 : ( 2 : ( 3 : ( 4 : ( 5 : [] ) ) ) ) )
foldr ⊕ e list   = 1 ⊕ ( 2 ⊕ ( 3 ⊕ ( 4 ⊕ ( 5 ⊕ e  ) ) ) ) )
```

foldr examples

```
length []       = 0
length (x : xs) = 1 + length xs

id []       = []
id (x : xs) = x : id xs

map f []       = []
map f (x : xs) = f x : map f xs

concat []       = []
concat (x : xs) = x ++ concat xs

reverse []       = []
reverse (x : xs) = reverse xs ++ [x]

filter p []    = []
filter p (x:xs)
    | p x       = x : filter p xs
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[]       ++ ys = ys
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```haskell
length = foldr (\x y → y + 1) 0
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       = foldr (const (+1)) 0
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            snoc x xs = xs ++ [x]
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reverse = foldr snoc [ ] where
              snoc x xs = xs ++ [x]
        = foldr (flip (++).(:[])) []
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xs ++ ys = foldr (:) ys xs
```

List design pattern and foldr

- task: define a function `f :: [P]⟶S`
- step 1: solve the problem for the empty list

```
f []        = ...
```

- step 2: solve the problem for non-empty lists;
- assume that you already have the solution for `xs` at hand; extend the intermediate solution to a solution for `x:xs`

```
f []     = ...
f (x:xs) = ... x ... xs ... f xs ...
```

List design pattern and foldr

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- suppose we don't need `xs` in step 2

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$$
\begin{aligned}
&\texttt{f []} &&= \texttt{...} \\
&\texttt{f (x:xs)} &&= \texttt{... x ...}\ \cancel{\texttt{xs}}\ \texttt{... f xs ...}
\end{aligned}
$$

- suppose we don't need `xs` in step 2

List design pattern and foldr (2)

- task: define a function `f :: [P]⟶S`
- step 1: solve the problem for the empty list

```
f []     = nil
```

- step 2: solve the problem for non-empty lists;
- assume that you already have the solution for `xs` at hand; extend the intermediate solution to a solution for `x:xs`

```
f []     = nil
f (x:xs) = step x (f xs)
```

List design pattern and foldr (2)

- task: define a function `f :: [P]⟶S`
- step 1: solve the problem for the empty list

```
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- step 2: solve the problem for non-empty lists;
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```
f []       = nil
f (x:xs) = step x (f xs)
```

- Then `f = foldr step nil`

Sorting: insertion sort

- given

```
insert :: (Ord a) ⇒ a ⟶ [a] ⟶ [a]
insert x [ ]    = [x]
insert x (y : ys)
    | x ≤ y     = x : y : ys
    | otherwise = y : insert x ys
```

- we have

```
insertionSort :: (Ord a) ⇒ [a] ⟶ [a]
insertionSort []     = []
insertionSort (x:xs) = insert x (insertionSort xs)
```

Sorting: insertion sort

- given

```
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insert x [ ]    = [x]
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```

- hence

```
insertionSort = foldr insert [ ]
```

foldl

foldl

```
         list   = 1 : ( 2 : ( 3 : ( 4 : ( 5 : [] ) ) ) ) )
foldr ⊕ e list   = 1 ⊕ ( 2 ⊕ ( 3 ⊕ ( 4 ⊕ ( 5 ⊕ e  ) ) ) ) )
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foldl

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```

- not every list function is a `foldr`
  - e.g. `decimal [1,2,3] = 123`
- efficient algorithm using *Horner's rule*:
  - `decimal [1,2,3] = ((0 * 10 + 1) * 10 + 2) * 10 + 3`
- left-to-right computation— hence `foldl`

foldl

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              list   = 1 : ( 2 : ( 3 : ( 4 : ( 5 : [] ) ) ) ) )
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```
             list   = 1 : ( 2 : ( 3 : ( 4 : ( 5 : [] ) ) ) ) )
foldl ⊗ a list   = ( ( ( ( ( a ⊗ 1 ) ⊗ 2 ) ⊗ 3 ) ⊗ 4 ) ⊗ 5 )
```

foldl

```
           list   = 1 : ( 2 : ( 3 : ( 4 : ( 5 : [] ) ) ) ) )
foldr ⊕ e list    = 1 ⊕ ( 2 ⊕ ( 3 ⊕ ( 4 ⊕ ( 5 ⊕ e  ) ) ) ) )
```

- not every list function is a `foldr`
  - e.g. `decimal [1,2,3] = 123`
- efficient algorithm using *Horner's rule*:
  - `decimal [1,2,3] = ((0 * 10 + 1) * 10 + 2) * 10 + 3`
- left-to-right computation— hence `foldl`

```
           list   = 1 : ( 2 : ( 3 : ( 4 : ( 5 : [] ) ) ) ) )
foldl ⊗ a list    = ( ( ( ( ( a ⊗ 1 ) ⊗ 2 ) ⊗ 3 ) ⊗ 4 ) ⊗ 5 )
```

- definition of decimal

```
decimal :: [Integer]⟶Integer
decimal = foldl (\a n ⟶ a * 10 + n) 0
```

foldl — continued

```
foldl :: (b ⟶ a ⟶ b) ⟶ b ⟶ [a] ⟶ b
foldl op ac []     = ac
foldl op ac (x:xs) = foldl op (ac `op` x) xs
```

- example: `reverse`

```
reverse :: [a]⟶[a]
reverse  = foldr (\x xs⟶xs ++ [x]) [ ]
```

- another (more efficient) definition

```
reverse’ :: [a]⟶[a]
reverse’ = foldl (flip (:)) [ ]
```

reverse’ = foldl (flip (:)) [ ]

```
                 list = 1 : ( 2 : ( 3 : ( 4 : ( 5 : [] ) ) ) ) )
foldl ⊗         e  list = ( ( ( ( ( e ⊗ 1 ) ⊗ 2 ) ⊗ 3 ) ⊗ 4 ) ⊗ 5 )
foldl (flip (:)) [] list =
```

reverse’ = foldl (flip (:)) [ ]

```
                    list = 1 : ( 2 : ( 3 : ( 4 : ( 5 : [] ) ) ) ) )
foldl ⊗          e  list = ( ( ( ( ( e ⊗ 1 ) ⊗ 2 ) ⊗ 3 ) ⊗ 4 ) ⊗ 5 )
foldl (flip (:)) [] list =
 ((((([] (flip (:)) 1) (flip (:)) 2) (flip (:)) 3) (flip (:)) 4) (flip (:)) 5) =
```

reverse’ = foldl (flip (:)) [ ]

```
                list = 1 : ( 2 : ( 3 : ( 4 : ( 5 : [] ) ) ) ) )
foldl ⊗         e   list = ( ( ( ( ( e ⊗ 1 ) ⊗ 2 ) ⊗ 3 ) ⊗ 4 ) ⊗ 5 )
foldl (flip (:)) [] list =
 (((((([] (flip (:)) 1) (flip (:)) 2) (flip (:)) 3) (flip (:)) 4) (flip (:)) 5) =
 (((((1:[])             (flip (:)) 2) (flip (:)) 3) (flip (:)) 4) (flip (:)) 5) =
```

reverse’ = foldl (flip (:)) [ ]

```
                 list = 1 : ( 2 : ( 3 : ( 4 : ( 5 : [] ) ) ) ) )
foldl ⊗        e  list = ( ( ( ( ( e ⊗ 1 ) ⊗ 2 ) ⊗ 3 ) ⊗ 4 ) ⊗ 5 )
foldl (flip (:)) [] list =
 ((((([] (flip (:)) 1) (flip (:)) 2) (flip (:)) 3) (flip (:)) 4) (flip (:)) 5) =
 (((((1:[])            (flip (:)) 2) (flip (:)) 3) (flip (:)) 4) (flip (:)) 5) =
 ((((2:1:[])                         (flip (:)) 3) (flip (:)) 4) (flip (:)) 5) =
```

reverse’ = foldl (flip (:)) [ ]

```
                  list = 1 : ( 2 : ( 3 : ( 4 : ( 5 : [] ) ) ) ) )
foldl ⊗         e  list = ( ( ( ( ( e ⊗ 1 ) ⊗ 2 ) ⊗ 3 ) ⊗ 4 ) ⊗ 5 )
foldl (flip (:)) [] list =
 (((((([] (flip (:)) 1) (flip (:)) 2) (flip (:)) 3) (flip (:)) 4) (flip (:)) 5) =
 (((((1:[])             (flip (:)) 2) (flip (:)) 3) (flip (:)) 4) (flip (:)) 5) =
 ((((2:1:[])                          (flip (:)) 3) (flip (:)) 4) (flip (:)) 5) =
 (((3:2:1:[])                                       (flip (:)) 4) (flip (:)) 5) =
```

reverse’ = foldl (flip (:)) [ ]

```
                   list = 1 : ( 2 : ( 3 : ( 4 : ( 5 : [] ) ) ) ) )
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 ((((2:1:[])                          (flip (:)) 3) (flip (:)) 4) (flip (:)) 5) =
 (((3:2:1:[])                                       (flip (:)) 4) (flip (:)) 5) =
 ((4:3:2:1:[])                                                    (flip (:)) 5) =
```

reverse’ = foldl (flip (:)) [ ]

```
                 list = 1 : ( 2 : ( 3 : ( 4 : ( 5 : [] ) ) ) ) )
foldl ⊗        e  list = ( ( ( ( ( e ⊗ 1 ) ⊗ 2 ) ⊗ 3 ) ⊗ 4 ) ⊗ 5 )
foldl (flip (:)) [] list =
 (((((([] (flip (:)) 1) (flip (:)) 2) (flip (:)) 3) (flip (:)) 4) (flip (:)) 5) =
 (((((1:[])             (flip (:)) 2) (flip (:)) 3) (flip (:)) 4) (flip (:)) 5) =
 ((((2:1:[])                          (flip (:)) 3) (flip (:)) 4) (flip (:)) 5) =
 (((3:2:1:[])                                       (flip (:)) 4) (flip (:)) 5) =
 ((4:3:2:1:[])                                                    (flip (:)) 5) =
 (5:4:3:2:1:[])
```

scanl

- sometimes convenient to apply `foldl` to every initial segment (i.e. prefix) of a list
  - `scanl (⊗) e [x,y,z] = [e,e ⊗ x,(e ⊗ x) ⊗ y,((e ⊗ x) ⊗ y) ⊗ z]`
- e.g. `scanl (+) 0` computes running totals (prefix sums)
- e.g. `scanl (*) 1 [1..n]` computes first n + 1 factorials
- start with specification

```
scanl :: (b⟶a⟶b)⟶b⟶([a]⟶[b])
scanl op a = map (foldl op a) . inits
```

- inefficient, as quadratically many applications of `op`
- has more efficient implementation

```
scanl op a []       = [a]
scanl op a (x : xs) = a : scanl op (a `op' x) xs
```

Fibonacci sequences

- the Fibonacci sequence goes as 1, 1, 2, 3, 5, 8, 13,…

```
fib 0 = 1
fib 1 = 1
fib (n+1) = fib n + fib (n-1)
```

- the Fibonacci sequence using scanl

```
fibSeq :: [Integer]
fibSeq = scanl (+) 1 (0:fibSeq)
```

scanr

- dually

```
scanr :: (a→ans→ans)→ans→([a]→[ans])
scanr op e = map (foldr op e) . tails
```

- has a more efficient implementation

```
scanr op e = foldr (\x ys → (x `op` head ys) : ys) [e]
```

Producers

- so far we have focused on *consumers* (this seems to be close to the spirit of the time)
  - foldr consumes a list to provide a single value.
- *producers* are important too
- producers (unfolds) are *dual* to consumers (folds)
- `unfoldr` creates a list out of some seed.

Producers: some examples

```
-- repeat x is an infinite list, with x the value of every element.
repeat :: a ⟶ [a]
repeat x = x : repeat x

-- iterate f x == [x, f x, f (f x), ...]
iterate :: (a ⟶ a) ⟶ a ⟶ [a]
iterate f x =  x : iterate f (f x)

-- [n..] = [n,n+1,…] = enumFrom n
enumFrom :: Int ⟶ [Int]
enumFrom n = n : enumFrom (n+1)

-- replicate n x is a list of length n with x the value of every element.
replicate :: Int ⟶ a ⟶ [a]
replicate 0 x =  []
replicate n x =  x : replicate (n-1) x

-- [n..m] = [n,n+1,..,m] = enumFromTo n m
enumFromTo :: Int ⟶ Int ⟶ [Int]
enumFromTo n m
  | n <= m    = n : enumFromTo (n+1) m
  | otherwise = []
```

Common pattern: `unfoldr`

```
unfoldr :: (s ⟶ Maybe (a, s)) ⟶ s ⟶ [a]
unfoldr grow seed = case grow seed of
   Just (a, new_seed) ⟶ a : unfoldr grow new_seed
   Nothing            ⟶ []
```

`unfoldr` examples

```
repeat  x       = x : repeat x
repeatU x       = unfoldr (\s ⟶ Just (s,s)) x
```

`unfoldr` examples

```
repeat   x        = x : repeat x
repeatU  x        = unfoldr (\s ⟶ Just (s,s)) x

iterate  f x      = x : iterate f (f x)
iterateU f x      = unfoldr (\s ⟶ Just (s, f s)) x
```

unfoldr examples

```
repeat   x        = x : repeat x
repeatU  x        = unfoldr (\s ⟶ Just (s,s)) x

iterate  f x      = x : iterate f (f x)
iterateU f x      = unfoldr (\s ⟶ Just (s, f s)) x

enumFrom n        = n : enumFrom (n+1)
enumFromU n       = unfoldr (\s ⟶ Just (s,s+1)) n
```

`unfoldr` examples

```
repeat   x        = x : repeat x
repeatU x         = unfoldr (\s ⟶ Just (s,s)) x

iterate  f x      = x : iterate f (f x)
iterateU f x      = unfoldr (\s ⟶ Just (s, f s)) x

enumFrom n        = n : enumFrom (n+1)
enumFromU n       = unfoldr (\s ⟶ Just (s,s+1)) n

replicate 0 x     =  []
replicate n x     =  x : replicate (n-1) x
replicateU n x    = unfoldr (\n ⟶ if n==0 then Nothing else Just (x, n-1)) n
```

`unfoldr` examples

```
repeat   x        = x : repeat x
repeatU  x        = unfoldr (\s ⟶ Just (s,s)) x

iterate  f x      = x : iterate f (f x)
iterateU f x      = unfoldr (\s ⟶ Just (s, f s)) x

enumFrom n        = n : enumFrom (n+1)
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replicate 0 x     =  []
replicate n x     =  x : replicate (n-1) x
replicateU n x    = unfoldr (\n ⟶ if n==0 then Nothing else Just (x, n-1)) n

enumFromTo n m
  | n <= m        = n : enumFromTo (n+1) m
  | otherwise     = []

enumFromToU n m = unfoldr (\n ⟶ if n<=m then Just (n, n+1) else Nothing) n
```

unfoldr examples

```
repeat   x        = x : repeat x
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enumFromTo n m
  | n <= m        = n : enumFromTo (n+1) m
  | otherwise     = []

enumFromToU n m = unfoldr (\n ⟶ if n<=m then Just (n, n+1) else Nothing) n

mapU :: (a ⟶ b) ⟶ [a] ⟶ [b]

mapU f = unfoldr (\l ⟶ case l of [] ⟶ Nothing; x:xs ⟶ Just (f x, xs))
```

Sorting: selection sort

- given

```
select :: Ord a ⇒ [a] → (a,[a])
select [x]    = (x,[])
select (x:xs) = let (m,ys) = select xs in
                if x<m then (x,m:ys) else (m,x:ys)
```

- we have

```
selectionSort :: Ord a ⇒ [a] → [a]
selectionSort [] = []
selectionSort xs = let (m,ys) = select xs in m : selectionSort ys
```

`select` as a `foldr`

```
select :: Ord a ⟹ [a] ⟶ (a,[a])
select [x]    = (x,[])
select (x:xs) = step x (select xs)
   where
      step x (m,ys) = if x<m then (x,m:ys) else (m,x:ys)
```

- `select` using `foldr`

```
selectF :: Ord a ⟹ [a] ⟶ (a,[a])
selectF (x:xs) = foldr step (x,[]) xs
   where
      step x (m,ys) = if x<m then (x,m:ys) else (m,x:ys)
```

`selectionSort` as an `unfoldr`

```
selectionSort :: Ord a ⇒ [a] ⟶ [a]
selectionSort [] = []
selectionSort xs = let (m,ys) = select xs in m : selectionSort ys
```

- `selectionSort` using `unfoldr`

```
selectionSortU :: Ord a ⇒ [a] ⟶ [a]
selectionSortU = unfoldr (\l ⟶ case l of
                                      [] ⟶ Nothing
                                      xs ⟶ Just (selectF xs))
```

folding and unfolding trees

- externally-labelled binary trees (*leaf trees*)

```
data LTree a = Tip a | Bin (LTree a) (LTree a)
```

folding and unfolding trees

- externally-labelled binary trees (*leaf trees*)
  `data LTree a = Tip a | Bin (LTree a) (LTree a)`
- `LTree` design pattern: define a function `f :: LTree P ⟶ S`
  - step 1: solve the problem for a leaf
    ```
    f (Tip n)           = … n …
    ```
  - step 2: solve the problem for internal nodes; assume that you already have the solutions for `l`, `r` at hand; extend the intermediate solution to a solution for `Bin l r`
    ```
    f (Tip n)    = … n …
    f (Bin l r) = … l … r … f l … f r …
    ```

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    ```

- suppose you don't need `l`,`r` in step 2. Hence

folding and unfolding trees

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    f (Tip n)   = … n …
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`folding and unfolding trees`

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    `f (Tip n)    = … n …`

    `f (Bin l r) = …` ~~`l`~~ `…` ~~`r`~~ `… f l … f r …`

- suppose you don't need `l`,`r` in step 2. Hence

```
f (Tip n)    = tip n
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```

folding and unfolding trees

- externally-labelled binary trees (*leaf trees*)

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    f (Tip n) = … n …

    f (Bin l r) = … ~~l~~ … ~~r~~ … f l … f r …

- suppose you don't need l,r in step 2. Hence

  f (Tip n) = tip n

  f (Bin l r) = bin (f l) (f r)

- Then f t = foldLTree bin tip t

foldLTree

- definition

```
foldLTree :: (b ⟶ b ⟶ b) ⟶ (a ⟶ b) ⟶ LTree a ⟶ b
foldLTree bin tip = consume where
   consume (Tip e)   = tip e
   consume (Bin l r) = bin (consume l) (consume r)
```

foldLTree

- definition

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foldLTree bin tip = consume where
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```

- examples

```
size :: (Num a) ⟹ LTree a -> a
size  = foldLTree (+) (\_ ⟶ 1)

sum :: (Num a) ⟹ LTree a ⟶ a
sum   = foldLTree (+) id

depth :: (Ord a, Num a) ⟹ LTree a ⟶ a
depth = foldLTree (\l r ⟶ max l r + 1) (\_ ⟶ 1)
```

Either

- the Either type

```
data Either a b = Left a | Right b
```

- like Maybe often used for modeling exceptions

```
head :: [a] ⟶ Either String a
head []     = Left "head: empty list"
head (x:_) = Right x
```

unfoldLTree

- unfolding a leaf tree

```
unfoldLTree :: (s → Either a (s, s)) → s → LTree a
unfoldLTree grow seed = produce seed where
   produce seed = case grow seed of
                    Left e                  → Tip e
                    Right (lseed, rseed) → Bin (produce lseed) (produce rseed)
```

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```

- a map for LTrees

```
mapLTree :: (a → b) → LTree a → LTree b
mapLTree f lt = unfoldLTree go lt where
-- go :: LTree a -> Either b (LTree a, LTree a)
```

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   go (Tip e)   = Left  (f e)
   go (Bin l r) = Right (l,r)
```

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   go (Bin l r) = Right (l,r)
```

- growing a balanced tree from a list

```
list2Tree :: [a] ⟶ LTree a
list2Tree = unfoldLTree (\s ⟶ case s of
                                  [e
                                  xs
```

unfoldLTree

- unfolding a leaf tree

```
unfoldLTree :: (s ⟶ Either a (s, s)) ⟶ s ⟶ LTree a
unfoldLTree grow seed = produce seed where
   produce seed = case grow seed of
                       Left e                ⟶ Tip e
                       Right (lseed, rseed)  ⟶ Bin (produce lseed) (produce rseed)
```

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   go (Bin l r) = Right (l,r)
```

- growing a balanced tree from a list

```
list2Tree :: [a] ⟶ LTree a
list2Tree = unfoldLTree (\s ⟶ case s of
                              [e] ⟶ Left e
                              xs  ⟶ Right (splitAt (length xs `div` 2) xs))
```

The art of functional programming

- question: what are the three most important concepts in programming?
- answer: abstraction, abstraction, abstraction!
- higher-order functions (HOFs) allow you to capture control structures, in particular, common patterns of recursion