1) Given,

Base case: There is pair of positive integers n and m such that $n, m \in S$.

Constructor: If $x, y \in S$ and $x \geq y$, then $rem(x, y) \in S$.

We need to prove $x \in S$ implies that $gcd(n,m) | x$ by structural induction.

Induction step:

Assume, $x, y \in S$ and $x \geq y$ and $rem(x, y) \in S$

Now, $gcd(n,m) | rem(x,y)$ as $rem(x,y) \in S$, which means there is a number that divides $rem(x,y)$ as -

$rem(x,y) = k \times gcd(n,m)$.

$x = q \cdot y + rem(x,y)$.

$x = q \cdot y + k \times gcd(n,m)$.

Now, $q \cdot y$ can be divisible by $gcd(n,m)$ as y and $gcd(n,m)$ are coprimes and $k \times gcd(n,m)$ are also divisible with $gcd(n,m)$.

Therefore, by structural induction if $x \in S$, then $gcd(n,m) | x$.

2) Given a, b, c are positive Integers.

a, b are not relatively prime.

c is relatively prime to a and b.

$n = s \times a + t \times b$.

We need to prove 'n cannot be divisor of c'.

We will prove this by method of contradiction.

Assume, n is divisor of c, which means

$c = k \times n$

$c = k(s \times a + t \times b)$ (Substitute n)

As a, b are not relatively prime, $\gcd(a,b) = d$ $(d > 1)$.

Here, d is dividing both a & b. So, d can also divide any combination of a & b. So, d divides n.

So, d will be common divisor to n, c.

But $\gcd(a,b) = d$ contradicts that c is relatively prime to both a & b.

We have got contradiction to our assumption.

Therefore, n cannot be divisor of c.

3) a. $\text{rem}(112445^{114155123} \times 33161^{1352}, 18)$

$= \text{rem}(112445^{114155123}, 18) \times \text{rem}(33161^{1352}, 18)$.

[$\because \text{rem}(i \times j, k) = \text{rem}(i,k) \times \text{rem}(j,k)$]

$= \text{rem}((\text{rem}(112445, 18))^{114155123}, 18) \times \text{rem}(\text{rem}(33161, 18)^{1352}, 18)$

$= \text{rem}(17^{114155123}, 18) \times \text{rem}(5^{1352}, 18)$

(As $\text{rem}(112445, 18) = 17$
$\text{rem}(33161, 18) = 5$)

Now, $\text{rem}(17, 18) = 17$
$\text{rem}(17^2, 18) = 1$
$\text{rem}(17^3, 18) = 17$
$\text{rem}(17^4, 18) = 1$.

For odd powers, the remainder is '17'.

Now, $\text{rem}(5, 18) = 5$ $\quad$ $\text{rem}(5^2, 18) = 7$
$\text{rem}(5^3, 18) = 17$ $\quad$ $\text{rem}(5^4, 18) = 13$
$\text{rem}(5^5, 18) = 11$ $\quad$ $\text{rem}(5^6, 18) = 1$
$\text{rem}(5^7, 18) = 5$ $\quad$ $\text{rem}(5^8, 18) = 7$

The remainder is repeating after '6' times of powers.

So, our equation will be

$= rem(17^{114155723}, 18) \times rem(5^{1352}, 18)$

$= rem(17 \times 7, 18)$

$= rem(119, 18)$

$= \underline{11}$.

[As, 114155723 is a odd number, the remainder will be 17]

[For number 1352, it is in second position out of '6' as mentioned in rem(5^n, 18). So the value will be 7].

3) b. We need value of $\phi(18)$, $\phi \to$ Euler's function.

In the set $[0, 18) \to$ the numbers which are relatively prime to 18 are '1, 5, 7, 11, 13, 15'.
There are 6 values.
So, $\phi(18) = \underline{6}$.

3) c. We need, $\text{rem}(33161^{\phi(18)+\phi(18)}, 18)$.

As written above $\phi(18) = 6$.
and $\text{rem}(33161, 18) = 5$.
So, $\text{rem}(33161^{\phi(18)+\phi(18)}, 18)$
$= \text{rem}(5^{6+6}, 18)$
$= \text{rem}(5^{12}, 18)$ $\therefore$ [As $5^{12} = 244140625$].
$= \text{rem}(244140625, 18)$
$= \underline{1}$.

Therefore, $\text{rem}(33161^{\phi(18)+\phi(18)}, 18) = \underline{1}$.