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Phys 426: Reading 3

Q1: The meaning of vorticity

Vorticity ω is defined as being twice the average angular velocity of a fluid particle. Simply put, we can immediately recognise this as:

$$\boldsymbol{\omega} = 2\left(\frac{d\theta}{dt}\right)$$

Equivalently, we could take the average of all the particle speeds and double it. We can also find this from the equation for uniform distribution for plane-normal vorticity:

$$u_\theta = \frac{\boldsymbol{\omega} r}{2}, \qquad u_\theta = r\frac{d\theta}{dt}$$

Q2: Conservation of Vorticity

$$\frac{D\omega}{Dt} = (\omega \cdot \nabla)\boldsymbol{u} + \upsilon\nabla^2\omega$$

If we assume that υ is negligible, we can ignore the term on the right and expand the dot product:

$$\frac{D\omega}{Dt} = \left(\omega_x\frac{\partial}{\partial x} + \omega_y\frac{\partial}{\partial y} + \omega_z\frac{\partial}{\partial z}\right)(u, v, w)$$

We have no flow in either the x or y direction, so the final product is:

$$\frac{D\omega}{Dt} = \omega_z\frac{\partial w}{\partial z} > 0$$

Basically, the rate of change of vorticity in the vertical direction is nonzero. Therefore, the material derivative measuring the rate of change in vorticity that a section of water will undergo will also be nonzero.

Q3: Kelvin's Circulation Theorem

Kelvin's circulation theorem:

$$\frac{D\boldsymbol{\Gamma}}{Dt} = 0$$

We can time differentiate this equation and expand Kelvin's circulation theorem in terms of body force potential:

$$\frac{D\boldsymbol{\Gamma}}{Dt} = \frac{D}{Dt}\int_C u_i dx_i = \int_C \frac{Du_i}{Dt}dx_i + \int_C u_i\frac{D}{Dt}(dx_i)$$

$$\int_C \frac{Du_i}{Dt}dx_i = \int_C\left(-\frac{1}{\rho}\frac{\partial p}{\partial x_i} + g_i + \frac{1}{\rho}\frac{\partial \tau_{ij}}{\partial x_j}\right)dx_i = -\int_C\frac{1}{\rho}dp - \int_C d\Phi + \int_C\left(\frac{1}{\rho}\frac{\partial \tau_{ij}}{\partial x_j}\right)dx_i$$

What we end up with is an integral over a closed contour depending on the shear stress tensor:

$$\frac{D\boldsymbol{\Gamma}}{Dt} = \int_C\left(\frac{1}{\rho}\frac{\partial \tau_{ij}}{\partial x_j}\right)dx_i$$

This will be zero for both irrotational and a solid body vortex as both of these have closed circular streamlines meaning the integral will ultimately be zero over a closed contour since viscous forces will cancel out.