Chapter 3: #2 Vance Turnewitsch

ii $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

A. Find the eigenvalues and eigenvectors of A

Characteristic equation: $(1-\lambda)(0-\lambda)-1=0 \rightarrow -\lambda+\lambda^2-1=0 \rightarrow \lambda^2-\lambda-1=0$

Eigenvalues: $\lambda_2 = \frac{1+\sqrt{5}}{2}, \lambda_1 = \frac{1-\sqrt{5}}{2}$

Eigenvectors: $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \frac{1+\sqrt{5}}{2} \begin{bmatrix} a \\ b \end{bmatrix}$

Thus: $a+b=\lambda_1 a$ and $a=\lambda_1 b$

Second eigenvector: $\begin{bmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{bmatrix}$

Now calculate first Eigen vector: $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \frac{1-\sqrt{5}}{2} \begin{bmatrix} a \\ b \end{bmatrix}$

Thus: $a = \frac{1-\sqrt{5}}{2} b$

And second eigenvector: $\begin{bmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{bmatrix}$

B. Find the matrix T that puts A in canonical form

Thus: $T = \begin{bmatrix} \frac{1-\sqrt{5}}{2} & \frac{1+\sqrt{5}}{2} \\ 1 & 1 \end{bmatrix}$

C. Find the general solution of both $X^{'} = AX$ and $Y^{'} = (T^{-1}AT)Y$

Our $T^{-1}AT = \frac{1}{-\sqrt{5}} \begin{bmatrix} 1 & \frac{-1-\sqrt{5}}{2} \\ -1 & \frac{1-\sqrt{5}}{2} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1-\sqrt{5}}{2} & \frac{1+\sqrt{5}}{2} \\ 1 & 1 \end{bmatrix}$

$= \frac{1}{-\sqrt{5}} \begin{bmatrix} 1 & \frac{-1-\sqrt{5}}{2} \\ -1 & \frac{1-\sqrt{5}}{2} \end{bmatrix} \begin{bmatrix} \frac{3-\sqrt{5}}{2} & \frac{3+\sqrt{5}}{2} \\ \frac{1-\sqrt{5}}{2} & \frac{1+\sqrt{5}}{2} \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} \frac{-5+\sqrt{5}}{2} & 0 \\ 0 & \frac{5+\sqrt{5}}{2} \end{bmatrix}$

Thus for the system: $Y^{'} = (T^{-1}AT)Y = \frac{1}{\sqrt{5}} \begin{bmatrix} \frac{-5+\sqrt{5}}{2} & 0 \\ 0 & \frac{5+\sqrt{5}}{2} \end{bmatrix} Y$

which is the case where $\lambda_1 < 0 < \lambda_2$ and leads to the solution:

$Y(t) = \alpha e^{\frac{-5+\sqrt{5}}{2}t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta e^{\frac{5+\sqrt{5}}{2}t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and multiplying by T we can easily convert to the solution of X:

$X(t) = T \cdot Y(T) = \begin{bmatrix} \frac{1-\sqrt{5}}{2} & \frac{1+\sqrt{5}}{2} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \alpha e^{\frac{-5+\sqrt{5}}{2}t} \\ \beta e^{\frac{5+\sqrt{5}}{2}t} \end{bmatrix} = \begin{bmatrix} \frac{\alpha e^{\frac{-5+\sqrt{5}}{2}t} - \sqrt{5}\alpha e^{\frac{-5+\sqrt{5}}{2}t} + \beta e^{\frac{5+\sqrt{5}}{2}t} + \sqrt{5}\beta e^{\frac{5+\sqrt{5}}{2}t}}{2} \\ \alpha e^{\frac{-5+\sqrt{5}}{2}t} + \beta e^{\frac{5+\sqrt{5}}{2}t} \end{bmatrix} =$

$X(t) = \alpha e^{\frac{-5+\sqrt{5}}{2}t} \begin{pmatrix} 1-\sqrt{5} \\ 1 \end{pmatrix} + \beta e^{\frac{5+\sqrt{5}}{2}t} \begin{pmatrix} 1+\sqrt{5} \\ 1 \end{pmatrix}$