Applied Mathematics I

Homework Chapter 7 #7

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Let $A(t)$ be a continuous family of $n \times n$ matrices, and let $P(t)$ be the matrix solution to the initial value problem $P' = A(t)P, P(0) = P_0$. Show that:

$\det P(t) = (\det P_0) exp(\int_0^t \mathrm{Tr} A(s) ds)$

Case i:

where $A(t) = diag(\lambda_1(t), \ldots \lambda_n(t))$

We observe that for any $n \times q$ matrix, P, we have: $P' = A(t)P \rightarrow [A(t)P_1, A(t)P_2, \ldots, A(t)P_q]$ where P_i are the columns from P. Thus we have q equations of the form:

$P_i' = A(t)P_i$ where P_i' is the ith column of the P' matrix.

We use the result from problem #2:

Let A be an $n \times n$ matrix. Then for the problem $X' = AX$ the initial value solution is: $X(t) = exp(tA)X_0$ where of course $X(0) = X_0$

Thus, we perform Picard iteration on each of the P_i equations given here to get: $P_i(t) = exp(tA)P_{i0}$

Thus: $\det P(t) = \det([exp(tA)P_{10}, exp(tA)P_{20}, \ldots, exp(tA)P_{q0}])$

$= \det(exp(tA)[P_{10}, \ldots, P_{q0}])$

$= \det(exp(tA)) \det(P_0)$

Using results from page 124 (exponential of diagonal matrix...)

$= \det(diag(e^{\lambda_1 t}, \ldots, e^{\lambda_n t})) \det(P_0) = exp((\lambda_1(t) + \lambda_2(t) + \ldots + \lambda_n(t))t) \det(P_0)$

And: $det(P_0) = det(P_0)$

Observe that: $\mathrm{Tr}(A(s)) = \lambda_1(s)+...+\lambda_n(s)$

Thus: $\int_0^t \lambda_1(t)+...+\lambda_n(t)ds = (\lambda_1(t)+...+\lambda_n(t))t$ for small values of t.

Thus: $exp(\int_0^t \mathrm{Tr}(A(s)ds = exp((\lambda_1(t)+...+\lambda_n(t))t)$

Thus: $(\det P_0)exp(\int_0^t \mathrm{Tr}A(s)ds) = \det(P_0)exp((\lambda_1(t)+...+\lambda_n(t))t)$

While: $\det P(t) = exp((\lambda_1(t) + \lambda_2(t)+...+\lambda_n(t))t)\det(P_0)$

Thus: $\det P(t) = (\det P_0)exp(\int_0^t \mathrm{Tr}A(s)ds)$ with commutation.

Case ii:

Where $n = 2$ and matrix may not be diagonal.

Let $P = \begin{bmatrix} P_{1,1} & P_{1,2} \\ P_{2,1} & P_{2,2} \end{bmatrix}$, $P' = \begin{bmatrix} P'_{1,1} & P'_{1,2} \\ P'_{2,1} & P'_{2,2} \end{bmatrix}$, $A = \begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{bmatrix}$, and $P_0 = \begin{bmatrix} P_{1,1,0} & P_{1,2,0} \\ P_{2,1,0} & P_{2,2,0} \end{bmatrix}$

Thus: $Tr(A) = A_{1,1} + A_{2,2}$ and $\exp(\int_0^t A_{1,1} + A_{2,2}ds) = \exp(\int_0^t A_{1,1}ds)\exp(\int_0^t A_{2,2}ds)$

Thus: $\det(P) = P_{1,1}P_{2,2} - P_{2,1}P_{1,2}$

Thus: $\det(P_0) = P_{1,1,0}P_{2,2,0} - P_{2,1,0}P_{1,2,0}$

Thus: $\det(P_0)\exp(\int_0^t A_{1,1} + A_{2,2}ds) = \det(P_0)\exp(\int_0^t A_{1,1}ds)\exp(\int_0^t A_{2,2}ds)$

$= (P_{1,1,0}P_{2,2,0} - P_{2,1,0}P_{1,2,0})\exp(\int_0^t A_{1,1}ds)\exp(\int_0^t A_{2,2}ds)$

$= P_{1,1,0}P_{2,2,0}\exp(\int_0^t A_{1,1}ds)\exp(\int_0^t A_{2,2}ds) - P_{2,1,0}P_{1,2,0}\exp(\int_0^t A_{1,1}ds)\exp(\int_0^t A_{2,2}ds)$

And since $P_{1,1,0}$ is just a constant, we can commute it:

$$= P_{1,1,0} \exp(\int_0^t A_{1,1} ds) \cdot P_{2,2,0} \exp(\int_0^t A_{2,2} ds) - P_{2,1,0} \exp(\int_0^t A_{1,1}) \cdot P_{1,2,0} \exp(\int_0^t A_{2,2} ds)$$

$$\det(P') = P'_{1,1} P'_{2,2} - P'_{2,1} P'_{1,2} = \det(A(t) P_0) = \det(A(t)) \det(P_0)$$

$$= (A_{1,1} A_{2,2} - A_{2,1} A_{1,2})(P_{1,1,0} P_{2,2,0} - P_{2,1,0} P_{1,2,0})$$

$$= A_{1,1} A_{2,2} P_{1,1,0} P_{2,2,0} - A_{1,1} A_{2,2} P_{2,1,0} P_{1,2,0} - A_{2,1} A_{1,2} P_{1,1,0} P_{2,2,0} + A_{2,1} A_{1,2} P_{2,1,0} P_{1,2,0}$$

$$= (A_{1,1} A_{2,2} - A_{2,1} A_{1,2}) P_{1,1,0} P_{2,2,0} - (A_{1,1} A_{2,2} - A_{2,1} A_{1,2}) P_{2,1,0} P_{1,2,0}$$