Applied Mathematics I

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Chapter 7 Additional Homework

Find and solve the variational equations for $X' = F(X)$

1 $F = (x^2 + xy, x + y^3), X = (-1, 1)$

So, first we compute the Jacobian matrix of this F:

$$DF(X) = \begin{pmatrix} \partial(x^2+xy)_x & \partial(x^2+xy)_y \\ \partial(x+y^3)_x & \partial(x+y^3)_y \end{pmatrix}$$

$$DF(X) = \begin{pmatrix} 2x+y & x \\ 1 & 3y^2 \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 1 & 3 \end{pmatrix}$$

Thus, we now find the solution of:

$$\begin{pmatrix} x(t)' \\ y(t)' \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

We find eigenvalues: $(-1-a)(3-a)+1 = -3+a-3a+a^2+1 = a^2-2a-2$

Thus eigenvalues: $c = 1+\sqrt{3}$ and $d = 1-\sqrt{3}$

Now we find eigenvector for $1+\sqrt{3}$:

$-x - y = x + \sqrt{3}x$

$x + 3y = y + \sqrt{3}y$

And adding together:

$2y = x + y + \sqrt{3}x + \sqrt{3}y$

$2y - y - \sqrt{3}y = x + \sqrt{3}x$

$y(1 - \sqrt{3}) = x(1 + \sqrt{3})$

$y = x\frac{1+\sqrt{3}}{1-\sqrt{3}} = x(-2 - \sqrt{3})$

Thus, first eigenvector: $[1, -2 - \sqrt{3}]^T$

Now we work on the eigenvector for: $1 - \sqrt{3}$

$-x - y = x - \sqrt{3}x$

$x + 3y = y - \sqrt{3}y$

Adding together...

$2y = x + y - \sqrt{3}x - \sqrt{3}y$

$2y - y + \sqrt{3}y = x - \sqrt{3}x$

$y(1 + \sqrt{3}) = x(1 - \sqrt{3}) \rightarrow y = x\frac{1-\sqrt{3}}{1+\sqrt{3}} \rightarrow y = x(-2 + \sqrt{3})$

Thus second eigenvector is: $[1, -2 + \sqrt{3}]^T$

Thus our solution is: $U(t) = ae^{1+\sqrt{3}t}[1, -2 - \sqrt{3}]^T + be^{(-2+\sqrt{3})t}[1, -2 + \sqrt{3}]^T$

And our variational solution is:

$$X(t) = \begin{pmatrix} -1 + ae^{(1+\sqrt{3})t} + be^{(-2+\sqrt{3})t} \\ 1 + (-2 - \sqrt{3})ae^{(1+\sqrt{3})t} + (-2 + \sqrt{3})be^{(-2+\sqrt{3})t)} \end{pmatrix}$$

2 $x' = x^{4/3}, x(t) = 27(3-t)^{-3}$

We see that $x(t_0) = 27 * 1/3^3 = 1$

We only have one variable here, thus $DF(X_0) = \frac{4}{3}x^{1/3} = 4/3$

Thus variational problem: $x' = 4/3x \to x(t) = ae^{4/3t}$

Thus variational solution is: $X(t) = 27(3-t)^{-3} + ae^{4/3t}$