Find the general solution of: $x' = x(1-x) - h$

We expand into: $x' = x - x^2 - h$ and rearrange: $dx/dt = -x^2 + x - h$. Now we manipulate to get:

$1/(-x^2 + x - h)dx = dt \implies 1/(x^2 - x + h) = -dt$ and we integrate: $\int(1/(x^2 - x + h))dx = -\int dt$

And we get: $\int(1/(x^2 - x + h))dx = c - t$ where c is some general constant.

Now we consider the denominator in the left integral: $x^2 - x + h$ which we observe has two solutions: $x = \frac{1 \pm \sqrt{1-4h}}{2}$

Thus we can factor the fraction as follows: $\frac{1}{x^2-x+h} = \frac{1}{(x+a)(x+b)}$ where $a = \frac{1+\sqrt{1-4h}}{2}$ and $b = \frac{1-\sqrt{1-4h}}{2}$ assuming that both a, b are real here!

We now have: $\int \frac{1}{(x+a)(x+b)} dx = c - t$ We now proceed to use partial fractions on the left integrand:

$\frac{1}{(x+a)(x+b)} = A/(x+a) + B/(x+b)$ and expanding we get:

$1 = A(x+b) + B(x+a) \implies 1 = Ax + Ab + Bx + Ba \implies 1 = x(A+B) + Ab + Ba$

We observe this can be represented as a matrix equation:

$\begin{bmatrix} x & x \\ b & a \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and we multiply by the inverse to get:

$\begin{bmatrix} A \\ B \end{bmatrix} = \frac{1}{x(a-b)} \begin{bmatrix} a & -x \\ -b & x \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{x(a-b)} \begin{bmatrix} -x \\ x \end{bmatrix} = \frac{1}{a-b} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

We thus find that: $A = -1/(a-b)$ and $B = 1/(a-b)$

Our left integral is now: $\int \frac{-1/(a-b)}{(x+a)} + \frac{1/(a-b)}{x+b} dx = \frac{-1}{a-b} \int \frac{1}{x+a} dx + \frac{1}{a-b} \int \frac{1}{x+b} dx$ and after integrating we get:

$\frac{-1}{a-b} \ln|x+a| + \frac{1}{a-b} \ln|x+b| = c_2 - t$ where we have absorbed the constant from the left integration into the one on the right to create c_2.

We now do some simplifying:

$\ln|x+b| - \ln|x+a| = (a-b)(c_2 - t) \implies \ln|\frac{x+b}{x+a}| = (a-b)(c_2 - t)$ And we place both left hand side and right hand side in exponent of base e to get:

$|\frac{x+b}{x+a}| = e^{(a-b)(c_2-t)}$ and we solve to get:

$\frac{x+b}{x+a} = \pm e^{(a-b)(c_2-t)} \implies \frac{x+b}{x+a} = de^{(a-b)(c_2-t)}$ where we have used d to absorb that irritating $\pm$

$x + b = (x+a)de^{(a-b)(c_2-t)} = xde^{(a-b)(c_2-t)} + ade^{(a-b)(c_2-t)}$

$x - xde^{(a-b)(c_2-t)} = ade^{(a-b)(c_2-t)} - b$

$x(1-de^{(a-b)(c_2-t)}) = ade^{(a-b)(c_2-t)} - b$

$x = \frac{ade^{(a-b)(c_2-t)}-b}{1-de^{(a-b)(c_2-t)}}$

Work on the constants

$a - b = \frac{1+\sqrt{1-4h}}{2} - \frac{1-\sqrt{1-4h}}{2} = \frac{1-1+\sqrt{1-4h}+\sqrt{1-4h}}{2} = \sqrt{1-4h}$

We also observe that: $b = a - \sqrt{1-4h}$

Thus plugging these in we find:

$x(t) = \frac{\frac{(1+\sqrt{1-4h})}{2}de^{\sqrt{1-4h}(c_2-t)} - \frac{1+\sqrt{1-4h}}{2} + \sqrt{1-4h}}{1-de^{\sqrt{1-4h}(c_2-t)}}$ and after simplifying the fraction we find:

$x(t) = [(1+\sqrt{1-4h})de^{\sqrt{1-4h}(c_2-t)} - 1 + \sqrt{1-4h}]/[2(1-de^{\sqrt{1-4h}(c_2-t)})]$

Work on when we have imaginary roots...

We now proceed to work on the equation when: $1 - 4h < 0$

We thus have two imaginary solutions: $x = \frac{1 \pm i\sqrt{1-4h}}{2}$ which we shall again refer to as: a, b.

We know though that there exists c, d such that: $a = c + di$ and $b = c - di$ Our fraction in the left-hand integral thus becomes:

$\frac{1}{(x+c+di)(x+c-di)} = \frac{1}{x^2+xc-xdi+xc+c^2-cdi+xdi+cdi+d^2} = 1/(x^2+2xc+c^2+d^2) = 1/((x+c)^2+d^2)$

We thus now have this situation:

$\int 1/((x+c)^2+d^2)dx = f - t$ where we have chosen f to be our constant from the right-hand integration.

We use a CAS to integrate the left-hand side.

$\arctan((x+c)/d)/d = g - t$ where we have absorbed the costant from left hand integration into f

$\arctan((x+c)/d) = d(g-t)$

$(x+c)/d = \tan(d(g-t))$

$x + c = d\tan(d(g-t))$

$x = d\tan(d(g-t)) - c$ or in expanded form: ;)

$x(t) = (\sqrt{1-4h}/2)\tan[(\sqrt{1-4h}/2)(g-t)] - 1/2$ is our second final solution.

Thus the general solution is:

$$x(t) = \begin{cases} [(1+\sqrt{1-4h})de^{\sqrt{1-4h}(c_2-t)} - 1]/[2(1-de^{\sqrt{1-4h}(c_2-t)})] \text{ when } 1-4h > 0 \\ (\sqrt{1-4h}/2)\tan[(\sqrt{1-4h}/2)(g-t)] - 1/2 \text{ when } 1-4h < 0 \end{cases}$$

where $d = \pm 1$ and c_2, g, h are constants to be determined.