Chapter 3: #2 Vance Turnewitsch

ii $\begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$

A. Find the eigenvalues and eigenvectors of A

Characteristic Equation: $(1-\lambda)(-\lambda)+1=0 \rightarrow \lambda^2-\lambda+1=0$

Thus: $\lambda = \frac{1\pm\sqrt{1-4}}{2} = \frac{1\pm i\sqrt{3}}{2}$

Thus: $\lambda_1 = \frac{1+i\sqrt{3}}{2}$ and we proceed to find the eigenvector:

$$\begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \frac{1+i\sqrt{3}}{2}\begin{bmatrix} a \\ b \end{bmatrix}$$

$$\begin{gathered} a+b=\lambda_1 a \\ -a=\lambda_1 b \end{gathered}$$

$$\begin{gathered} b=a(\lambda_1-1) \\ a=-\lambda_1 b \end{gathered}$$

Thus eigenvector:

B. Find the matrix T that puts A in conanical form

So, we take the egienvector above and split it into its real and imaginary parts:

$$V_1 = \begin{bmatrix} -1/2 \\ 1 \end{bmatrix}, V_2 = \begin{bmatrix} -\sqrt{3}/2 \\ 0 \end{bmatrix}$$

Thus we have:

$$T = \begin{bmatrix} -1/2 & -\sqrt{3}/2 \\ 1 & 0 \end{bmatrix}$$

C. Find the general solution of both X'=AX and Y'=$(T^{-1}AT)Y$

$$T^{-1}AT = \begin{bmatrix} 0 & 1 \\ -2/\sqrt{3} & -1/\sqrt{3} \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} -1/2 & -\sqrt{3}/2 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ -2/\sqrt{3} & -1/\sqrt{3} \end{bmatrix}\begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix} = \begin{bmatrix} 1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{bmatrix}$$

$$Y' = \begin{bmatrix} 1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{bmatrix} Y$$

Thus we have the solution for Y:

$$Y(t) = c_1 e^{1/2t}\begin{pmatrix} cos(\sqrt{3}/2t) \\ -sin(\sqrt{3}/2t) \end{pmatrix} + c_2 e^{1/2t}\begin{pmatrix} sin(\sqrt{3}/2t) \\ cos(\sqrt{3}/2t) \end{pmatrix}$$

$$= e^{1/2t} \begin{bmatrix} c_1 cos(\sqrt{3}/2t) + c_2 sin(\sqrt{3}/2t) \\ c_2 cos(\sqrt{3}/2t) - c_1 sin(\sqrt{3}/2t) \end{bmatrix}$$

Now we use the map T to transform this solution into the X(t)

$$X(t) = TY(t) = \begin{bmatrix} -1/2 & -\sqrt{3}/2 \\ 1 & 0 \end{bmatrix} e^{1/2t} \begin{bmatrix} c_1 cos(\sqrt{3}/2t) + c_2 sin(\sqrt{3}/2t) \\ c_2 cos(\sqrt{3}/2t) - c_1 sin(\sqrt{3}/2t) \end{bmatrix}$$

$$= e^{1/2t} \begin{bmatrix} -\frac{c_1}{2} cos(\sqrt{3}/2t) - \frac{c_2}{2} sin(\sqrt{3}/2t) - \frac{\sqrt{3}c_2}{2} cos(\sqrt{3}/2t) + \frac{\sqrt{3}c_1}{2} sin(\sqrt{3}/2t) \\ c_1 cos(\sqrt{3}/2t) + c_2 sin(\sqrt{3}/2t) \end{bmatrix}$$

$$= e^{1/2t} \begin{bmatrix} \frac{c_1}{2} \left(\sqrt{3} sin(\sqrt{3}/2t) - cos(\sqrt{3}/2t)\right) - \frac{c_2}{2} \left(\sqrt{3} cos(\sqrt{3}/2t) + sin(\sqrt{3}/2t)\right) \\ c_1 cos(\sqrt{3}/2t) + c_2 sin(\sqrt{3}/2t) \end{bmatrix}$$

$$X(t) = c_1 e^{1/2t} \begin{pmatrix} \frac{\sqrt{3}}{2} sin(\sqrt{3}/2t) - \frac{1}{2} cos(\sqrt{3}/2t) \\ cos(\sqrt{3}/2t) \end{pmatrix} + c_2 e^{1/2t} \begin{pmatrix} \frac{-\sqrt{3}}{2} cos(\sqrt{3}/2t) - \frac{1}{2} sin(\sqrt{3}/2t) \\ sin(\sqrt{3}/2t) \end{pmatrix}$$