HW 8

1 (i) $\psi(x) = A_\lambda \left(x^\lambda \Theta\left(\frac{L}{2}-x\right) \Theta(x) + (L-x)^\lambda \Theta\left(x-\frac{L}{2}\right) \Theta(L-x) \right)$

(ii) $\frac{d\psi(x)}{dx} = A_\lambda \left(\lambda x^{\lambda-1} \Theta\left(\frac{L}{2}-x\right) \Theta(x) - \lambda (L-x)^{\lambda-1} \Theta\left(x-\frac{L}{2}\right) \Theta(L-x) \right)$

$\left(\because \frac{d\psi}{dx} \text{ is continous} \right)$

(iii) $\frac{d^2\psi}{dx} = A_\lambda \Big[\Big(\lambda(\lambda-1) x^{\lambda-2} \Theta\left(\frac{L}{2}-x\right) \Theta(x)$

$- \lambda x^{\lambda-1} \delta\left(\frac{L}{2}-x\right) \Theta(x)$

$+ \lambda x^{\lambda-1} \Theta\left(\frac{L}{2}-x\right) \delta(x) \Big)$

$-$

$\Big(-\lambda(\lambda-1)(L-x)^{\lambda-2} \Theta\left(x-\frac{L}{2}\right) \Theta(L-x)$

$+ \lambda (L-x)^{\lambda-1} \delta\left(x-\frac{L}{2}\right) \Theta(L-x)$

$- \lambda (L-x)^{\lambda-1} \Theta\left(x-\frac{L}{2}\right) \delta(L-x) \Big) \Big]$

next page (iv) part

(iv) First doing the $\left[x^{\lambda}\,\Theta(\tfrac{L}{2}-x)\,\Theta(x)\right]$ integral.

$$= \int_0^{L/2} dx\left(\lambda(\lambda-1)x^{2\lambda-2}\right) + \lambda\left(\frac{L}{2}\right)^{2\lambda-1} \left[\because \int_0^{L/2} dx\, f(x)\,\delta(x-\tfrac{L}{2}) = f(\tfrac{L}{2}) \quad \int_0^{L/2} dx\, f(x)\,\delta(x) = 0.\right]$$

$$+$$

$$\int_0^{L/2} dx\, \cancel{f(x;\lambda)}^{\,0}\,\Theta(x-\tfrac{L}{2})\,\Theta(L-x) \quad [\because \text{out of bounds}]$$

$$+$$

$$\int_0^{L/2} dx\, \cancel{f(x;\lambda)}^{\,0}\,\delta(x-\tfrac{L}{2})\,\Theta(L-x) \quad [\because \text{out of bounds}]$$

$$+\,0$$

$$\int_0^{L/2} dx\, \cancel{f(x;\lambda)}\,\Theta(x-\tfrac{L}{2})\,\delta(L-x) \quad [\because \text{out of bounds}]$$

Scond part using the same reasoning with convolution,

$$= \int_{L/2}^{L} dx\,(L-x)^{2\lambda-2}\,\lambda(\lambda-1) + \lambda\left(\frac{L}{2}\right)^{2\lambda-1}$$

then, $$= \lambda(\lambda-1)\left[\int_0^{L/2} dx\; x^{2\lambda-2} + \int_{L/2}^{L} dx\,(L-x)^{2\lambda-2}\right]$$

$$= \frac{2\lambda(\lambda-1)}{2\lambda-1}\left[\left(\frac{L}{2}\right)^{2\lambda-1}\right] \qquad \left[\lambda > \tfrac{1}{2} \text{ for convergence}\right]$$

$$\langle\psi|p^2\psi\rangle = \frac{2\lambda+1}{2}\left(\frac{2}{L}\right)^{2\lambda+1}\cdot\frac{2\lambda^2}{2\lambda-1}\left(\frac{L}{2}\right)^{2\lambda-1}\cdot\hbar^2$$

$$\frac{1}{m}\langle\psi|p^2\psi\rangle = \boxed{\left(\frac{2\hbar}{L}\right)^2 \frac{1}{2m} \lambda^2 \frac{2\lambda+1}{2\lambda-1}}$$ matches lecture.

2.

$$\langle H\rangle = \frac{-\hbar^2}{2m}\langle\nabla^2\rangle + \langle V\rangle$$

First Normalise : $|A\lambda|^2 = \frac{1}{\int_{-\infty}^{\infty} dx\, e^{-\frac{2x^2}{\lambda^2}}} = \boxed{\frac{1}{\lambda}\sqrt{\frac{2}{\pi}}}$

then $\langle\nabla^2\rangle$

$$= \frac{1}{\lambda}\sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} dx\, e^{-\frac{x^2}{\lambda^2}} \frac{d^2}{dx^2}\left(e^{-\frac{x^2}{\lambda^2}}\right)$$

$$= \frac{1}{\lambda}\sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} dx\, e^{-\frac{2x^2}{x^2}} \left[\frac{4x^2}{\lambda^4} - \frac{2}{\lambda^2}\right] = \boxed{-\frac{1}{\lambda^2}}$$

then $\langle V\rangle = A\frac{1}{\lambda}\sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} dx\, x^4\, e^{-\frac{2x^2}{\lambda^4}} = \frac{1}{\cancel{\lambda}}\sqrt{\frac{2}{\cancel{\pi}}}\, \frac{3}{16}\sqrt{\frac{\cancel{\pi}}{2}}\, \lambda^{\cancel{5}\,4}$

$$= \boxed{\frac{A\,3}{16}\lambda^4}$$

$$E(\lambda) = \frac{A3}{16}\lambda^4 + \frac{\hbar^2}{2m}\frac{1}{\lambda^2}$$

using mathematica solving $E'(\lambda) = 0$

using the condition $E''(\lambda) > 0$ and removing imaginary roots,

which gives,

$$\lambda = \pm \frac{(2\hbar)^{1/3}}{(3AM)^{1/6}}$$, Plugging it into $E(\lambda)$ gives:

$$E_{\text{lowerbound}} = \boxed{\frac{3(3A)^{1/3}\hbar^{4/3}}{4(2M)^{2/3}}}$$

3

$$D = a_0 \int d^3r \frac{\psi_0(r)^2}{|r-R|} = a_0 \int d^3r \frac{\psi_0(|r-R|)^2}{r}$$

$$= \frac{2\pi a_0}{\pi a_0^{3\,2}} \int_0^\infty r\,dr \left[\int_0^\pi \sin\theta\, d\theta\, e^{-2\frac{\sqrt{r^2+R^2-2rR\cos\theta}}{a_0}}\right]$$

$$\left[u = \sqrt{r^2+R^2-2rR\cos\theta},\ \frac{du}{d\theta} = \frac{1}{u}2rR\sin\theta\right]$$

$$= \frac{2}{a^2}\int_0^\infty \frac{dr}{R}\left[\int_{|r-R|}^{r+R} du\, e^{\frac{-2u}{a_0}} u\right] = \text{next page}$$

$$= \frac{2}{a_0^2}\int_0^\infty \frac{dr}{R}\left[-\frac{a_0}{2}\left(e^{-2(r+R)/a_0}\left(r+R-\frac{a_0}{2}\right)-e^{-2|r-R|/a_0}\left(|r-R|+\frac{a_0}{2}\right)\right]$$

$$= -\frac{1}{a_0 R}\left[\int_0^\infty dr\, e^{-2(r+R)/a_0}\left(r+R-\frac{a}{2}\right)\right.$$

$$\left. + \int_0^R dr\, e^{-2(R-r)/a_0}\left(R-r+\frac{a_0}{2}\right)+\int_R^\infty e^{-2(r-R)/a_0}\left(r-R+\frac{a_0}{2}\right)\right]$$

$$= \boxed{\frac{1}{\lambda}-\left(1+\frac{1}{\lambda}\right)e^{-2\lambda}}$$ [same as textbook]

$$X = \frac{2}{a_0^2}\int_0^\infty dr\, r e^{-r/a_0}\left[\int_0^\pi d\theta \sin\theta\, e^{-\sqrt{r^2+R^2-2rR\cos\theta}}\right]$$

$$= \frac{2}{a_0^2 R}\int_0^\infty dr\, e^{-r/a_0}\left[-\frac{a_0}{2}\left(e^{-(r+R)/a_0}(r+R+a_0)-e^{|r-R|/a_0}(|r-R|+a_0)\right)\right]$$

$$= -\frac{2}{a_0 R}\int_0^\infty dr\, e^{-R/a_0}\, e^{-2r/a_0}(r+R+a_0)$$

$$- e^{-R/a_0}\int_0^R dr(R-r+a_0)e^{2r/a_0} - e^{R/a_0}\int dr e^{-2r/a_0}(r-R+a_0)$$

next page

$$= \boxed{e^{-\lambda}(1+\lambda)}$$ matches textbook

4. (a) $H = \begin{bmatrix} E_a & h \\ h & E_b \end{bmatrix}$ $P(\lambda) = (E_a - \lambda)(E_b - \lambda) - h^2 = 0$

$$\lambda_\pm = \frac{1}{2}\left(E_a + E_b \pm \sqrt{(E_a - E_b)^2 + 2ih^2}\right)$$

(b) $E_a(h) = E_a^0 + \langle \psi_a | H' | \psi_a \rangle + \frac{|\langle \psi_b | H' | \psi_a \rangle|^2}{E_a - E_b}$

$$= E_a^0 + 0 + \frac{h^2}{E_a - E_b}$$

$$E_b(h) = E_b^0 + 0 + \frac{h^2}{E_b - E_a}.$$

(c) $\langle H \rangle = \langle \cos\phi\,\psi_a + \sin\phi\,\psi_b | H^0 | \cos\phi\,\psi_a + \sin\phi\,\psi_b \rangle$

$+$

$\langle \cos\phi\,\psi_a + \sin\phi\,\psi_b | H' | \cos\phi\,\psi_a + \sin\phi\,\psi_b \rangle$

$\langle H^0 \rangle = \cos\phi^2 E_a + \sin^2\phi E_b$ $\langle H' \rangle = 2h\cos\phi\sin\phi$

$E(\phi) = \langle H \rangle = \cos^2\phi\, E_a + \sin^2\phi\, E_b + 2h\cos\phi\sin\phi$

$$E'(\phi) = -2\cos\phi\sin\phi\, E_a + 2\sin\phi\cos\phi\, E_b$$
$$+ \quad 2h(\cos^2\phi - \sin^2\phi)$$
$$= (E_b - E_a)\sin 2\phi + 2h\cos 2\phi = 0$$

$$= (E_b - E_a)\tan 2\phi + 2h = 0$$

using mathematica; $\phi_n = \frac{1}{2}\left(\tan^{-1}\left[\frac{2h}{E_a - E_b}\right] + \pi n\right)$

$E''(\phi) = 2(E_b - E_a)\sec^2(2\phi) > 0$ for all $n \in \mathbb{Z}$

Thus plugging $\phi = \frac{1}{2}\left(\tan^{-1}\left(\frac{2h}{E_a - E_b}\right)\right)$ in $E(\phi)$

$$E_{min} = \frac{1}{2}\left[E_b + E_a - \sqrt{(E_b - E_a)^2 + 4h^2}\right]$$

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6 (i) $A = \sqrt{\dfrac{1}{\int d^3r\, e^{-2\beta r/r_0}}} = \sqrt{\dfrac{1}{4\pi \int_0^\infty r^2 dr\, e^{-2\beta r/r_0}}} = \dfrac{1}{2\sqrt{\pi}}\left(\dfrac{4\beta}{r_0}\right)^{3/2}$

(ii) $\langle H \rangle = -\dfrac{\hbar^2}{2M}\langle \nabla^2 \rangle + \langle V \rangle$

$$\frac{\langle \nabla^2 \rangle}{|A|^2} = 4\pi \int_0^\infty r^2 dr\, e^{-\beta r/r_0} \frac{d^2}{dr^2}\left(e^{-\beta r/r_0}\right) = \boxed{\frac{\pi r_0}{\beta}}$$

$$\frac{\langle V \rangle}{|A|^2} = -4\pi \int_0^\infty dr\, r\, r_0 V_0\, e^{-\frac{r}{r_0}(2\beta + 1)} = \boxed{\frac{-4\pi r_0^3 V_0}{(1+2\beta)^2}}$$

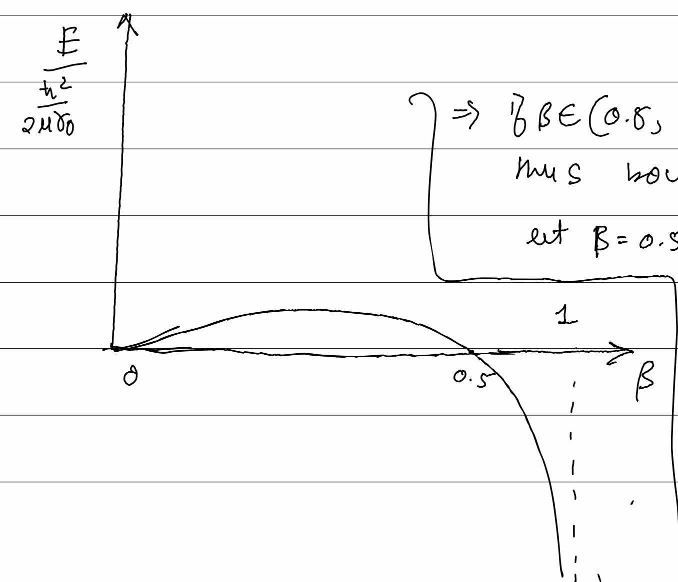
$$E = \langle H \rangle = \frac{\hbar^2}{2\mu r_0^2}\beta^2\left[1 - \frac{4\gamma\beta}{(1+2\beta)^2}\right]$$

(c) $$E'(\beta) = \frac{\hbar^2}{\mu r_0^2}\beta\left(1 - \frac{2\beta(3+2\beta)\gamma}{(1+2\beta)^3}\right) = 0$$

$$\gamma = \frac{(1+2\beta)^3}{2\beta(3+2\beta)}$$

Pluggin gamma in $$E_{min}(\beta) = \frac{\hbar^2}{2\mu r_0^2}\frac{\beta^2(1-2\beta)}{(3+2\beta)}$$

(d) setting $\frac{\hbar^2}{2\mu r_0^2} = 1$ and plotting E_{min} vs β



$\Rightarrow$ for $\beta \in (0.5, 1)$ we can have $-E$ thus bound states.

let $\beta = 0.5 + \varepsilon$ where $0 < \varepsilon << 0.5$

$$\gamma = \frac{2(1+\varepsilon)^3}{(0.5+\varepsilon)(2+\varepsilon)} \approx 2.$$

$$V_0 = \frac{\hbar^2}{2\mu r_0^2}\,2 = \frac{\hbar^2}{\frac{2(m_p m_n)}{(m_p+m_n)}\frac{\hbar^2}{m_\pi^2 c^2}}\,2.$$

$m_n \approx m_p \quad m_\pi = \frac{15}{100} m_p$

$$= 2 \cdot \left(\frac{15}{100}\right)^2 10^3 \text{ MeV} = \frac{225}{10} \times 2 = \boxed{\cancel{4}48.0\,\text{MeV}}$$

very close to

52 MeV