hw 2 qm

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(1)

(i)

Let vectors $h_1 \in \mathcal{H}_1$,$h_2 \in \mathcal{H}_2$, $h_3 \in \mathcal{H}_3$

Then since $\mathcal{H}_1, \mathcal{H}_2$ are subspaces of a broader vector space, $\mathcal{H}_1 \oplus \mathcal{H}_2 = \{h_1 + h_2 | h_1 \in \mathcal{H}_1, h_2 \in \mathcal{H}_2\}$. By definition of tensor product, $(\mathcal{H}_1 \oplus \mathcal{H}_2) \otimes \mathcal{H}_3 = \{(h_1 + h_2) \otimes h_3 | h_1 \in \mathcal{H}_1, h_2 \in \mathcal{H}_2, h_3 \in \mathcal{H}_3\}$. Since, tensor product distributes associatively, $= \{(h_1 \otimes h_3) + (h_2 \otimes h_3) | h_1 \otimes h_3 \in \mathcal{H}_1 \otimes \mathcal{H}_3, h_2 \otimes h_3 \in \mathcal{H}_2 \otimes \mathcal{H}_3\} = (\mathcal{H}_1 \otimes \mathcal{H}_3) \oplus (\mathcal{H}_2 \otimes \mathcal{H}_3)$ by definition of direct sum.

(ii)

$$F = (\nabla \times A) + (-\nabla \phi) \tag{1}$$

$\nabla \times A = \{v \in \mathbb{R}^3 | \nabla . v = 0\}$ $-\nabla \phi = \{u \in \mathbb{R}^3 | \nabla \times u = 0\}$. Both are subspaces in $\mathbb{R}^3$. Since the spaces are orthogonal to each other, $(\nabla \times A) \cap (-\nabla \phi) = \{0\}$. By the definition of direct sum,

$$F = (\nabla \times A) \oplus (-\nabla \phi) \tag{2}$$

(2)

(i)

$$\langle \psi | \psi \rangle = |\alpha|^2 (1 + |i|^2 + \frac{1}{4} + |2i|^2) = |\alpha|^2 (\frac{25}{4}) = 1 \Rightarrow |\alpha| = \frac{2}{5} \tag{3}$$

(ii)

$$|\langle\frac{1}{2},1|\psi\rangle|^2 = 0 \tag{4}$$

$$|\langle\frac{1}{2},0|\psi\rangle|^2 = \frac{4}{25} \tag{5}$$

$$|\langle\frac{1}{2},1|\psi\rangle|^2 = \frac{1}{25} \tag{6}$$

$$|\langle\frac{1}{2},1|\psi\rangle|^2 = 0 \tag{7}$$

$$|\langle\frac{1}{2},1|\psi\rangle|^2 = \frac{16}{25} \tag{8}$$

$$|\langle\frac{1}{2},1|\psi\rangle|^2 = \frac{4}{25} \tag{9}$$

$S_A^z \backslash S_B^z$	1	0	-1
$\frac{1}{2}$	0	$\frac{4}{25}$	$\frac{1}{25}$
$-\frac{1}{2}$	0	$\frac{16}{25}$	$\frac{4}{25}$

(iii)

Since the probability of finding $s_A = \frac{1}{2}$, the wave function will collapse to:

$$|\phi\rangle = \beta\frac{2}{5}(|\frac{1}{2};0\rangle - \frac{1}{2}|\frac{1}{2};-1\rangle) \tag{10}$$

where β is the new normalisation. Then,

$$\langle\phi|\phi\rangle = |\beta|^2(\frac{4}{25} + \frac{1}{25}) = 1 \Rightarrow |\beta| = \sqrt{5} \tag{11}$$

$$|\phi\rangle = \frac{2}{\sqrt{5}}|\frac{1}{2};0\rangle - \frac{1}{\sqrt{5}}|\frac{1}{2};-1\rangle \tag{12}$$

Subsequent measurement of S_B^z returns,

$$s_B = 1 \Rightarrow |\langle\frac{1}{2};1|\phi\rangle|^2 = 0 \tag{13}$$

$$s_B = 0 \Rightarrow |\langle\frac{1}{2};0|\phi\rangle|^2 = |\frac{2}{\sqrt{5}}|^2 = \frac{4}{5} \tag{14}$$

$$s_B = -1 \Rightarrow |\langle\frac{1}{2};-1|\phi\rangle|^2 = |\frac{1}{\sqrt{5}}| = \frac{1}{5} \tag{15}$$

(iii)

Since the probability of finding $s_A = -\frac{1}{2}$, the wave function will collapse to:

$$|\phi\rangle = \eta\frac{2}{5}(i| - \frac{1}{2};-1\rangle - 2i| - \frac{1}{2};0\rangle) \tag{16}$$

where η is the new normalisation. Then,

$$\langle\phi|\phi\rangle = |\eta|^2\frac{4}{25}(1+4) = 1 \Rightarrow |\eta| = \frac{\sqrt{5}}{2} \tag{17}$$

$$|\phi\rangle = \frac{i}{\sqrt{5}}|-\frac{1}{2};-1\rangle - \frac{2i}{\sqrt{5}}|-\frac{1}{2};0\rangle \tag{18}$$

Subsequent measurement of S_B^z returns,

$$s_B = 1 \Rightarrow |\langle\frac{1}{2};1|\phi\rangle|^2 = 0 \tag{19}$$

$$s_B = 0 \Rightarrow |\langle\frac{1}{2};0|\phi\rangle|^2 = |\frac{2}{\sqrt{5}}|^2 = \frac{4}{5} \tag{20}$$

$$s_B = -1 \Rightarrow |\langle\frac{1}{2};-1|\phi\rangle|^2 = |\frac{1}{\sqrt{5}}| = \frac{1}{5} \tag{21}$$

(iv)

$|\psi\rangle$ is not entagled since we can write it in a factorised form:

$$|\psi\rangle = \frac{2}{5}(-\frac{1}{2}|\frac{1}{2},\frac{1}{2}\rangle + i|\frac{1}{2},-\frac{1}{2}\rangle) \otimes (|1,-1\rangle - 2|1,0\rangle) \tag{22}$$

(3)

(i)

$$\hat{H}|\psi\rangle = E|\psi\rangle = \kappa\vec{S_A}.\vec{S_B}|\psi\rangle = \kappa\alpha\hbar^2|\psi\rangle \tag{23}$$

where α is dimensionless constant. Then, $E \approx \hbar\nu \Rightarrow \kappa \approx \frac{\hbar\nu}{\hbar^2} \approx \frac{\nu}{\hbar} \approx \frac{T^{-1}}{M^1L^2T^{-1}} = M^{-1}L^{-2}$

(ii)

$$\hat{H} = \frac{\kappa}{2}(L^2 - (S_A^2 + S_B^2)) \tag{24}$$

Since, $S_A^2, S_B^2 \in \mathcal{H}$, they can be related to their forms in the extension spaces as, $S_{A_{irr}}^2 \otimes \mathbb{I}^B, \mathbb{I}^A \otimes S_{B_{irr}}^2$ $s.t$ $S_{A_{irr}}^2, S_{B_{irr}}^2, \mathbb{I}_A, \mathbb{I}_B \in \mathcal{H}_{\frac{1}{2}}$. Since, $\mathcal{H} = \{h_1 \otimes h_2 | h_1 \in \mathcal{H}_{A_{\frac{1}{2}}}, h_2 \in \mathcal{H}_{B_{\frac{1}{2}}}\}$, $(S_{A_{irr}}^2 \otimes \mathbb{I}^B).(h_1 \otimes h_2) = (S_{A_{irr}}^2.h_1) \otimes (\mathbb{I}^B.h_2) = \frac{1}{2}(\frac{1}{2}+1)\hbar^2(\mathbb{I}^A.h_1) \otimes (\mathbb{I}^B.h_2) = \frac{1}{2}(\frac{1}{2}+1)\hbar^2(\mathbb{I}_{\mathbb{A}} \otimes \mathbb{I}_B).(h_1 \otimes h_2)$. Where, $\mathbb{I}_{\mathbb{A}} \otimes \mathbb{I}_B = \mathbb{I}_{\mathcal{H}} \in \mathcal{H}$. Hence, $S_A^2 = S_B^2 = \frac{1}{2}(\frac{1}{2}+1)\hbar^2\mathbb{I}_{\mathcal{H}}$. Then,

$$\hat{H} = \frac{\kappa\hbar^2}{2}(\frac{L^2}{\hbar^2} - \frac{3}{2})\mathbb{I}_{\mathcal{H}} \tag{25}$$

(iii)

Choosing the complete set of eigenkets of L^2 as a basis:$\{|e_i\rangle\} = \{|1,1\rangle, |1,0\rangle, |1,-1\rangle, |0,0\rangle\}$
Then, let $|\psi\rangle = c^i|e_i\rangle$,

$$\hat{H}|\psi\rangle = c^i\hat{H}|e_i\rangle = \frac{\kappa\hbar^2}{2}(S(S+1)-\frac{3}{2})c^i|e_i\rangle = E|\psi\rangle \; for \; S \in \{1,0\} \tag{26}$$

Then, $|\psi\rangle$ is the solution. $E = \frac{\kappa\hbar^2}{2}(S(S+1)-\frac{3}{2})$. The set of energies $\{E_i\} = \{\frac{\kappa\hbar^2}{4}, \frac{-3\kappa\hbar^2}{4}\}$. The degenracies: $d(S=1) = 2.1+1 = 3, d(S=0) = 2.0+1 = 1$

(iv)

Changing eigenbasis from $S_z^{(A)} \otimes S_z^{(B)}$ to L^2:

$$\begin{aligned}
|\psi(0)\rangle &= \frac{1}{\sqrt{2}}(|+-\rangle + |++\rangle) && (27)\\
&= \frac{1}{\sqrt{2}}(\frac{1}{\sqrt{2}}(|1,0\rangle + |0,0\rangle) + |1,1\rangle) && (28)\\
&= \frac{1}{2}|1,0\rangle + \frac{1}{2}|0,0\rangle + \frac{1}{\sqrt{2}}|1,1\rangle && (29)\\
& && (30)
\end{aligned}$$

Then,

$$\begin{aligned}
|\psi(t)\rangle &= U(t)|\psi(0)\rangle && (31)\\
&= \exp(\frac{-i\hat{H}t}{\hbar})|\psi(0)\rangle && (32)\\
&= \exp(\frac{-iE_1t}{\hbar})(\frac{1}{2}|1,0\rangle + \frac{1}{\sqrt{2}}|1,1\rangle) + \exp(\frac{-iE_0t}{\hbar})\frac{1}{2}|0,0\rangle && (33)\\
&= \exp(\frac{-i\kappa\hbar t}{4})(\frac{1}{2}|1,0\rangle + \frac{1}{\sqrt{2}}|1,1\rangle) + \exp(\frac{i3\kappa\hbar t}{4})\frac{1}{2}|0,0\rangle && (34)\\
& && (35)
\end{aligned}$$

Changing the basis back to $S_z^{(A)} \otimes S_z^{(B)}$:

$$\begin{aligned}
|\psi(t)\rangle &= \exp(\frac{-i\kappa\hbar t}{4})(\frac{1}{2\sqrt{2}}|+-\rangle + \frac{1}{2\sqrt{2}}|-+\rangle + \frac{1}{\sqrt{2}}|++\rangle) && (36)\\
&+ \exp(\frac{i3\kappa\hbar t}{4})\frac{1}{2\sqrt{2}}(|+-\rangle - |-+\rangle) && (37)\\
&= \frac{1}{2\sqrt{2}}(|+-\rangle(\exp(\frac{-i\kappa\hbar t}{4}) + \exp(\frac{i3\kappa\hbar t}{4})) && (38)\\
&+ |-+\rangle(\exp(\frac{-i\kappa\hbar t}{4}) - \exp(\frac{i3\kappa\hbar t}{4}))) + \frac{1}{\sqrt{2}}|++\rangle(\exp(\frac{-i\kappa\hbar t}{4})) && (39)
\end{aligned}$$

(v)

Let $a = \exp(\frac{-i\kappa\hbar t}{4})$ and $b = \exp(\frac{i\kappa 3\hbar t}{4})$. Then,

$$\langle S_A^z \rangle = \langle \psi | S_A^z | \psi \rangle \tag{40}$$
$$= \frac{\hbar}{16}(a+b)^*(a+b) - \frac{\hbar}{16}(a-b)^*(a-b) + \frac{\hbar}{4}(a^*a) \tag{41}$$
$$= \frac{\hbar}{16}\Bigg[(a^*a + b^*b + a^*b + b^*a) - (a^*a + b^*b - a^*b - b^*a)\Bigg] + \frac{\hbar}{4}a^*a \tag{42}$$
$$= \frac{\hbar}{16}\Bigg[2(a^*b + ab^*)\Bigg] + \frac{\hbar}{4}a^*a \tag{43}$$

Since, $a^*a = \exp(\frac{-i\kappa\hbar t}{4} + \frac{i\kappa\hbar t}{4}) = 1$

$$= \frac{\hbar}{4}\left[\frac{1}{2}(a^*b + ab^*) + 1\right] \tag{44}$$
$$= \frac{\hbar}{4}\Re(a^*b) + \frac{\hbar}{4} \tag{45}$$
$$= \frac{\hbar}{4}\Re(\exp(\frac{i\kappa\hbar t}{4}) + \frac{i\hbar 3t}{4}) + \frac{\hbar}{4} \tag{46}$$
$$= \frac{\hbar}{4}\Re(\exp(i\kappa\hbar t)) + \frac{\hbar}{4} \tag{47}$$
$$= \frac{\hbar}{4}\cos(\kappa\hbar t) + \frac{\hbar}{4} \tag{48}$$

$$\langle S_B^z \rangle = \langle \psi | S_B^z | \psi \rangle \tag{49}$$
$$= -\frac{\hbar}{16}(a+b)^*(a+b) + \frac{\hbar}{16}(a-b)^*(a-b) + \frac{\hbar}{4}(a^*a) \tag{50}$$
$$= \frac{\hbar}{16}\Bigg[-(a^*a + b^*b + a^*b + b^*a) + (a^*a + b^*b - a^*b - b^*a)\Bigg] + \frac{\hbar}{4}a^*a \tag{51}$$
$$= \frac{\hbar}{16}\Bigg[-2(a^*b + ab^*)\Bigg] + \frac{\hbar}{4}a^*a \tag{52}$$

Since, $a^*a = \exp(\frac{-i\kappa\hbar t}{4} + \frac{i\kappa\hbar t}{4}) = 1$

$$= \frac{\hbar}{4}\left[-\frac{1}{2}(a^*b + ab^*) + 1\right] \tag{53}$$
$$= -\frac{\hbar}{4}\Re(a^*b) + \frac{\hbar}{4} \tag{54}$$
$$= -\frac{\hbar}{4}\Re(\exp(\frac{i\kappa\hbar t}{4}) + \frac{i\hbar 3t}{4}) + \frac{\hbar}{4} \tag{55}$$
$$= -\frac{\hbar}{4}\Re(\exp(i\kappa\hbar t)) + \frac{\hbar}{4} \tag{56}$$
$$= -\frac{\hbar}{4}\cos(\kappa\hbar t) + \frac{\hbar}{4} \tag{57}$$