Statement: A statement is any declarative sentence which is either true or false. All the declarative sentences to which it is possible to assign one and only one of the two possible truth values are called statements. The symbols, which are used to represent statements, are called statement letters usually the letters P, Q, R…, p, q, r, etc. are used.

Proposition: A Proposition or a statement or logical sentence is a declarative sentence which is either true or false.

Example1: The following statements are all propositions:

- Jawaharlal Nehru is the first prime minister of India.
- It rained Yesterday.
- If x is an integer, then x2 is a +ve integer.

Example2: The following statements are not propositions:

- Please report at 11 a.m. sharp
- What is your name?
- x^2=13

Note: Proposition is a declarative statement declaring some fact.

It is either true or false but not both.

The examples of propositions are-

- 7 + 4 = 10
- Apples are black.
- Narendra Modi is president of India.
- Two and two makes 5.
- 2016 will be the lead year.
- Delhi is in India.

All these statements are propositions.

This is because they are either true or false but not both.

Statements That Are Not Propositions-

Following kinds of statements are not propositions-

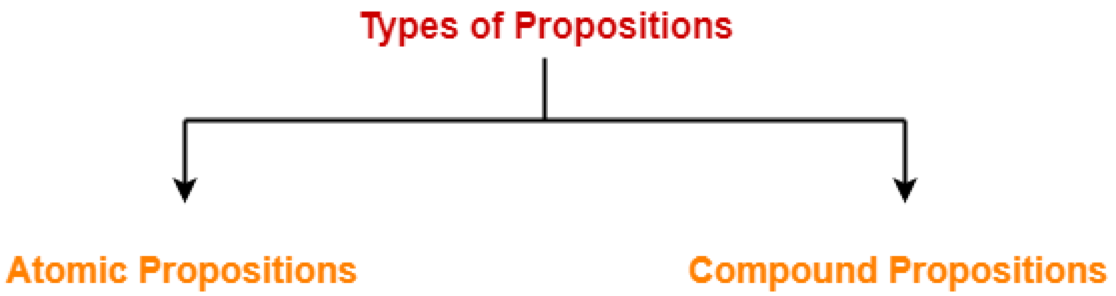
- **Command**
- **Question**
- **Exclamation**
- **Inconsistent**
- **Predicate or Proposition Function**

Following statements are not propositions-

- Close the door. (Command)
- Do you speak French? (Question)
- What a beautiful picture! (Exclamation)
- I always tell lie. (Inconsistent)
- P(x): x + 3 = 5 (Predicate)

Types Of Propositions-

In propositional logic, there are two types of propositions-



1.

Atomic Propositions-

Atomic propositions are those propositions that cannot be divided further.

Small letters like p, q, r, s etc are used to represent atomic propositions.

The examples of atomic propositions are-

p: Sun rises in the east.

q: Sun sets in the west.

r: Apples are red.

s: Grapes are green.

Compound Propositions-

Compound propositions are those propositions that are formed by combining one or more atomic propositions using connectives.

In other words, compound propositions are those propositions that contain some connective.

Capital letters like P, Q, R, S etc are used to represent compound propositions.

Examples-

P: Sun rises in the east and Sun sets in the west.

Q: Apples are red and Grapes are green.

Logical Connectives-Connectives are the operators that are used to combine one or more propositions.

Name of Connective	Connective Word	Symbol
Negation	Not	⌉ or ~ or ‘or –

Conjunction	And	∧
Disjunction	Or	∨
Conditional	If-then	→
Biconditional	If and only if	↔

Negation: If p is a proposition, then negation of p is a proposition which is-

True when p is false

False when p is true.

Truth Table-

p	**~p**
F	T
T	F

Example-

If p: It is raining outside.

Then, Negation of p is-

~p: It is not raining outside.

Conjunction-

If p and q are two propositions, then conjunction of p and q is a proposition which is-

True when both p and q are true

False when both p and q are false

Truth Table-

p	**q**	**p ∧ q**
F	F	F
F	T	F

T	F	F	
T	T	T	

Example-

If p and q are two propositions where-

p: 2 + 4 = 6

q: It is raining outside.

Then, conjunction of p and q is-

$p \wedge q$: 2 + 4 = 6 and it is raining outside

Disjunction-

If p and q are two propositions, then disjunction of p and q is a proposition which is-

True when either one of p or q or both are true

False when both p and q are false

Truth Table-

p	**q**	**$p \vee q$**	
F	F	F	
F	T	T	
T	F	T	
T	T	T	

Example-

If p and q are two propositions where-

p: 2 + 4 = 6

q: It is raining outside.

Then, disjunction of p and q is-

$p \vee q$: 2 + 4 = 6 or it is raining outside

Conditional-If p and q are two propositions, then-

Proposition of the type “If p then q” is called a conditional or implication proposition.

It is true when both p and q are true or when p is false.

It is false when p is true and q is false.

Truth Table-

p	q	$p \rightarrow q$	
F	F	T	
F	T	T	
T	F	F	
T	T	T	

Examples-

If a = b and b = c then a = c.

If I will go to Australia, then I will earn more money.

Biconditional-

If p and q are two propositions, then-

Proposition of the type “p if and only if q” is called a biconditional or bi-implication proposition.

It is true when either both p and q are true or both p and q are false.

It is false in all other cases.

Truth Table-

p	q	$p \leftrightarrow q$	
F	F	T	
F	T	F	
T	F	F	

T	T	T

Example:

He goes to play a match if and only if it does not rain.

Birds fly if and only if sky is clear.

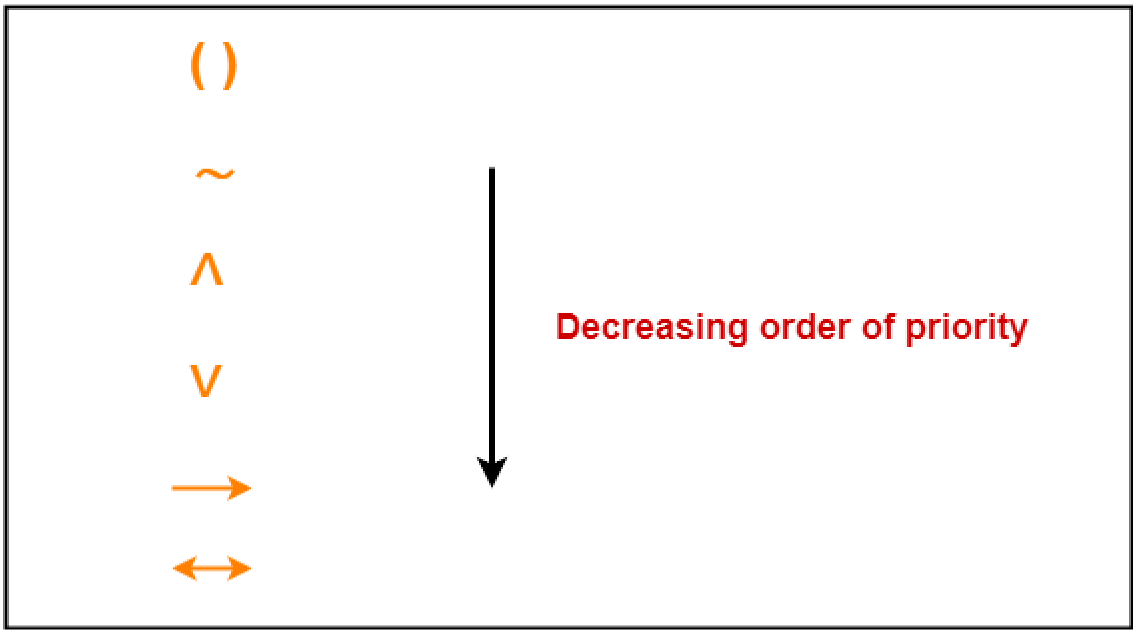
Note: Negation ≡ NOT

Conjunction ≡ AND Gate

Disjunction ≡ OR Gate

Biconditional = EX-NOR Gate

Priority:



Statement formula /Proposition formula: In propositional logic, a propositional formula is a type of syntactic formula which is well formed and has a truth value. If the values of all variables in a propositional formula are given, it determines a unique truth value. A propositional formula may also be called a propositional expression, a sentence, or a sentential formula.

A **propositional formula** is constructed from simple propositions, such as "five is greater than three" or propositional variables such as p and q, using connectives or logical operators such as NOT, AND, OR, or IMPLIES; for example:

(p AND NOT q) IMPLIES (p OR q).

Truth functional rule:

A formula is a **truth-functional tautology** if and only if the final column of its truthtable is all T s.

A formula is a **truth-functional contradiction** if and only if the final column of its truth-table is all F s.

A formula is **truth-functionally contingent** if and only if the final column of its truthtable contains at least one T and at least one F.

An argument is **truth-functionally valid** if and only if its truth-table contains no row

with all true premises and a false conclusion.

An argument is **truth-functionally invalid** if and only if its truth-table contains at

least one row with all true premises and a false conclusion.

Two formulas are **truth-functionally equivalent** if and only if the final columns on

their respective truth-tables match.

Tautology:

A compound proposition is called tautology if and only if it is true for all possible truth values of its propositional variables.

It contains only T (Truth) in last column of its truth table.

Example − Example: Prove that the statement $(p \to q) \leftrightarrow (\sim q \to \sim p)$ is a tautology.

Solution: Make the truth table of the above statement

p	q	$p \to q$	$\sim q$	$\sim p$	$\sim q \to \sim p$	$(p \to q) \leftrightarrow (\sim q \to \sim p)$
T	T	T	F	F	T	T
T	F	F	T	F	F	T
F	T	T	F	T	T	T
F	F	T	T	T	T	T

As the final column contains all T's, so it is a tautology.

Contradiction:

A statement that is always false is known as a contradiction.

Example: Show that the statement $p \wedge \sim p$ is a contradiction.

p	$\sim p$	$p \wedge \sim p$
T	F	F
F	T	F

Since, the last column contains all F's, so it's a contradiction.

Contingency:A statement that can be either true or false depending on the truth values of its variables is called a contingency.

p	q	p →q	p∧q	(p →q) ⟶ (p∧q)
T	T	T	T	T
T	F	F	F	T
F	T	T	F	F
F	F	T	F	F

Propositional Equivalences:

Two statements X and Y are logically equivalent if any of the following two conditions hold .

The truth tables of each statement have the same truth values.

Ex. Show that p →q and its contrapositive ~q→~p is logically equivalent

p	q	~p	~q	p →q	~q→~p
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

Converse, Contrapositive, and Inverse:

q → p is the converse of p → q

¬q → ¬p is the contrapositive of p → q

¬p → ¬q is the inverse of p → q

Example: Find the converse, inverse, and contrapositive of

“It is raining is a sufficient condition for my not going to town.”

Solution:

converse: If I do not go to town, then it is raining.

inverse: If it is not raining, then I will go to town.

contrapositive: If I go to town, then it is not raining.

Functionally complete set: A switching function is expressed by binary variables, the logic operation symbols, and constants 0 and 1. When every switching function can be expressed by means of operations in it, then only a set of operation is said to be functionally complete.

Ex:

1.The set (AND, OR, NOT) is a functionally complete set.

2.The set (AND, NOT) is said to be functionally complete.

3.The set (OR, NOT) is also said to be functionally complete.

The set (AND, NOT) is said to be functionally complete as (OR) can be derived using ‘AND’ and ‘NOT’ operations.

Example:

$(X + Y) = (X'.Y')'$

X'= compliment of X.

Y'= compliment of Y.

Ex: The set (OR, NOT) is said to be functionally complete as (AND) can be derived using ‘OR’ and ‘NOT’ operations.

Example: $(X.Y) = (X' + Y')'$

Note: A function can be fully functionally complete, or partially functionally complete or, not at all functionally complete.

Example-1:

If a function, $f(X, Y, Z)= (X' + YZ')$ then check whether its functionally complete or not?

Put Z = Y in the above function,

Therefore,

$f(X, Y, Y)= (X' + YY')$

$= (X' + 0)$ since, $Y.Y'=0$

$= X'$ (It is compliment i.e., NOT)

Again, put X= X’ and Z= Y’ in the above function,

Therefore,

$f(X', Y, Y')= (X')'+ Y(Y')'$

$= (X + Y.Y)$ since, $(X')'= X$ and $(Y')'= Y$

$= (X + Y)$ since, $Y.Y= Y$ (It is OR operator)

Thus, you are able to derive NOT and OR operators from the above function so this function is fully functionally complete.

Normal Form: There are two such forms:

Disjunctive Normal Form (DNF): A formula which is equivalent to a given formula and which consists of a sum of elementary products is called a disjunctive normal form of given formula.

Example:

$(P \wedge \sim Q) \vee (Q \wedge R) \vee (\sim P \wedge Q \wedge \sim R)$

The DNF of formula is not unique.

Conjunctive Normal Form: A formula which is equivalent to a given formula and which consists of a product of elementary products is called a conjunctive normal form of given formula.

Example:

$(P\sim \vee Q) \wedge (Q \vee R) \wedge (\sim P \vee Q \vee \sim R)$

The CNF of formula is not unique.

If every elementary sum in CNF is tautology, then given formula is also tautology.

Question: Transform the following formula into CNF.

$$\neg (p \rightarrow q) \vee (r \rightarrow p)$$

Express implication by disjunction and negation.

$$\neg (\neg p \vee q) \vee (\neg r \vee p)$$

Push negation inwards by De Morgan's laws and double negation.

$$(p \wedge \neg q) \vee (\neg r \vee p)$$

Convert to CNF by associative and distributive laws.

$$(p \vee \neg r \vee p) \wedge (\neg q \vee \neg r \vee p)$$

Optionally simplify by commutative and idempotent laws.

$$(p \vee \neg r) \wedge (\neg q \vee \neg r \vee p)$$

and by commutative and absorbtion laws

$$(p \vee \neg r)$$

Theory of statement calculus: An argument is a sequence of statements. The last statement is the conclusion and all its preceding statements are called premises (or hypothesis). The symbol "∴", (read therefore) is placed before the conclusion. A valid argument is one where the conclusion follows from the truth values of the premises.

Argument - Argument is a statement or premise which ends with a conclusion.

Validity - A argument is a valid if and only if argument is true and conclusion can never be false.

Fallacy - An incorrect reasoning resulting to invalid arguments.

Consistency: A formula is consistent iff it is true under at least one valuation;

A formula is inconsistent iff it is not made true under any valuation.

Argument Structure:

An argument structure is defined as using Premises and Conclusion.

Premises - p1, p2, p3, ….pn

Conclusion - q

Note: If $p1 \wedge p2 \wedge p3 \wedge, \cdots \wedge pn \rightarrow q$ is a tautology then the argument is considered as valid otherwise it is termed as invalid.

Note: The given argument is valid can be checked by truth table or using inference rule.

Question: To check the validity using table.

P1: If set of Complex Numbers are given then we will derive set of Real numbers.

P2: If set of Complex Numbers are given.

……………………………………………………………………………………….

C: we will derive set of Real numbers.

Let p: If set of Complex Numbers are given.

q: we will derive set of Real numbers.

we can express following argument in symbolic form as

Premise1: $p \rightarrow q$

Premises2: p

…………………………………………………………

Conclusion q

We shall construct the truth table for the statement

$$[(p \rightarrow q) \wedge p] \rightarrow q$$

p	q	$p \rightarrow q$	$(p \rightarrow q) \wedge p$	$[(p \rightarrow q) \wedge p] \rightarrow q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

Since last column contains only T’s, hence the given argument is valid.

Rule of Inference:

Rule of Inference	Tautology	Name
p p→ q ——— $\therefore q$	$(p \wedge (p \rightarrow q)) \rightarrow q$	Modus Ponens
$\neg q$ p→ q ——— $\therefore \neg p$	$(\neg q \wedge (p \rightarrow q)) \rightarrow \neg p$	Modus Tollens
p→ q q→ r ——— $\therefore p \rightarrow r$	$((p \rightarrow q) \wedge (q \rightarrow r)) \rightarrow (p \rightarrow r)$	Hypothetical syllogism
$\neg p$ p∨q ——— $\therefore q$	$(\neg p \wedge (p \vee q)) \rightarrow q$	Disjunctive Syllogism
p ——— $\therefore (p \vee q)$	$p \rightarrow (p \vee q)$	Addition
$(p \wedge q) \rightarrow r$ ——— $\therefore p \rightarrow (q \rightarrow r)$	$((p \wedge q) \rightarrow r) \rightarrow (p \rightarrow (q \rightarrow r))$	Exportation
p∨q $\neg p \vee r$ ——— $\therefore q \vee r$	$((p \vee q) \wedge (\neg p \vee r)) \rightarrow q \vee r$	Resolution

Using Rule of inference:

Premise1**: p → q**

Premises2**: p**

Using rule modus ponens Therefore q.

Question: Socrate says:

“If I’m guilty, I must be punished;

I’m guilty. Thus, I must be punished.”

Is the argument logically, correct?

Solution. The argument is logically correct: if p means “I’m guilty” and q means “I must be punished”, then:

$(p \rightarrow q) \wedge p \models q$ (modus ponens)

Question: Socrate says:

“If I’m guilty, I must be punished;

I’m not guilty. Thus, I must not be punished.”

Is the argument logically, correct?

Solution. The argument is not logically correct.

Algebraic laws of preposition:

Domination laws: $p \lor T \equiv T$, $p \land F \equiv F$

Identity laws: $p \land T \equiv p$, $p \lor F \equiv p$

Idempotent laws: $p \land p \equiv p$, $p \lor p \equiv p$

Double negation law: $\neg(\neg p) \equiv p$

Negation laws: $p \lor \neg p \equiv T$, $p \land \neg p \equiv F$

The first of the Negation laws is also called “law of excluded middle”.

Commutative laws: $p \land q \equiv q \land p$, $p \lor q \equiv q \lor p$

Associative laws: $(p \land q) \land r \equiv p \land (q \land r)$

$(p \lor q) \lor r \equiv p \lor (q \lor r)$

Distributive laws: $p \lor (q \land r) \equiv (p \lor q) \land (p \lor r)$

$p \land (q \lor r) \equiv (p \land q) \lor (p \land r)$

Absorption laws: $p \lor (p \land q) \equiv p$, $p \land (p \lor q) \equiv p$

Question: To prove that to prove: $\neg (p \lor (\neg p \land q)) \equiv \neg p \land \neg q$

$\neg (p \lor (\neg p \land q)) \equiv \neg p \land \neg (\neg p \land q)$ by De Morgan’s 2nd law

$\equiv \neg p \land (\neg(\neg p) \lor \neg q)$ by De Morgan’s first law

$\equiv \neg p \land (p \lor \neg q)$ by the double negation law

$\equiv (\neg p \land p) \lor (\neg p \land \neg q)$ by the 2nd distributive law

$\equiv F \lor (\neg p \land \neg q)$ because $\neg p \land p \equiv F$

$\equiv (\neg p \land \neg q) \lor F$ by commutativity of disj.

$\equiv \neg p \land \neg q$ by the identity law for F

Predicates: A predicate is an expression of one or more variables defined on some specific domain. A predicate with variables can be made a proposition by either assigning a value to the variable or by quantifying the variable.

The following are some examples of predicates −

Let E (x, y) denote "x = y"

Let X (a, b, c) denote "a + b + c = 0"

Let M (x, y) denote "x is married to y"

Quantification: The variable of predicates is quantified by quantifiers. There are two types of quantifiers in predicate logic − Universal Quantifier and Existential Quantifier.

Universal Quantifier:

Universal quantifier states that the statements within its scope are true for every value of the specific variable. It is denoted by the symbol ∀.

$\forall xP(x)$ is read as for every value of x, P(x) is true.

Example − "All Man is mortal" can be transformed into the propositional form $\forall xP(x)$ where P(x) is the predicate which denotes x is mortal and the universe of discourse is all men.

Existential Quantifier:

Existential quantifier states that the statements within its scope are true for some values of the specific variable. It is denoted by the symbol ∃.

$\exists xP(x)$ is read as for some values of x, P(x) is true.

Example − "Some people are dishonest" can be transformed into the propositional form $\exists xP(x)$ where P(x) is the predicate which denotes x is dishonest and the universe of discourse is some people.

Inference rule for Quantification: (Predicate calculus)

Rule of Inference	Name
$\frac{\forall x P(x)}{\therefore P(c)}$	Universal instantiation
$\frac{\text{P(c) for an arbitrary c}}{\therefore \forall x P(x)}$	Universal generalization
$\frac{\exists x P(x)}{\therefore P(c)\ for\ some\ c}$	Existential instantiation
$\frac{\text{P(c) for some c}}{\therefore \exists x P(x)}$	Existential generalization

:

Example:

Consider the following argument popularly known as "**Socrates argument**".

- All men are mortal
- Socrates is a man
- Therefore, Socrates is mortal

Predicate formulae the above statements:

- **H(x): x is man**
- **M(x): x is mortal.**
- **s: Socrates**.

Now the above statements can be represented as −

- **All men are mortal - $(x)(H(x)\rightarrow M(x))$**
- **Socrates is a man - H(s)**
- **Socrates is mortal - M(s)**

As a statement, we need to conclude −

$$\mathbf{(x)(H(x)\rightarrow M(x))\land H(s)\Rightarrow M(s)}$$

Solution:

(1) $(x)(H(x)\rightarrow M(x))$ - Hypotheses

(2) $H(s)\rightarrow M(s)$ - Rule US using (1)

(3) $H(s)$ - Hypotheses

(4) $M(s)$ - Simplification

DE Morgan's law:

De Morgan's First Law It states that the complement of the union of any two sets is equal to the intersection of the complement of that sets.

$$(a \land b)' = a' \lor b'$$

De Morgan's Second Law

It states that the complement of the intersection of any two sets is equal to the union of the complement of that sets.

$$(a \lor b)' = a' \land b'$$

Informal and formal proof:

Direct Proof:

The simplest (from a logic perspective) style of proof is a direct proof. Direct proofs are especially useful when proving implications. The general format to prove $P\rightarrow Q$ is this:

Assume P Explain, explain, …, explain. Therefore Q.

Proof by Contrapositive:

It gives a direct proof of the contrapositive of the implication. This is enough because the contrapositive is logically equivalent to the original implication.

The skeleton of the proof of $P\rightarrow Q$

by contrapositive will always look roughly like this:

Assume $\neg Q$. Explain, explain, … explain. Therefore $\neg P$.

Proof by Contradiction:

There might be statements which really cannot be rephrased as implications.

Ex. Prove that $\sqrt{2}$ is irrational.

Suppose $\sqrt{2}$ is rational.

That is, $\sqrt{2} = \frac{p}{q}$ for some $p \in \mathbb{Z}$ and $q \in \mathbb{N}$.

We can assume the fraction is in lowest terms.

That is, p and q share no common factors.

Then $\sqrt{2}q = p$

$2q^2 = p^2$

So p^2 is a multiple of 2,

therefore p must be a multiple of 2.

Write $p = 2m$.

Then $2q^2 = (2m)^2$

$2q^2 = 4m^2$

$q^2 = 2m^2$.

So q^2 is a multiple of 2,

therefore q is a multiple of 2.

Thus p and q share a common factor.

This is a contradiction!

Thus $\sqrt{2}$ is irrational.

Like this also proof square root of any prime no. is irrational.

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