Econ 4140 Assignment 3

Consider the market closing daily prices of TESLA in files TESLA.txt and TESLA.xlsx. Estimate the ARCH(1) and ARCH(2) models using the regressions provided in the code for Assignment 3.

1. Write the theoretical ARCH(1) and ARCH(2) models

ARCH(1)

$$\sigma_t^2 = \alpha_0 + \alpha_1 r_{t-1}^2$$
$$r_t = \sigma_t \cdot \epsilon_t$$
$$r_t^2 = (\alpha_0 + \alpha_1 r_{t-1}^2) \cdot \epsilon_t^2$$

Non Linear Interpretation:

$$r_t = \epsilon_t \sqrt{\alpha_0 + \alpha_1 r_{t-1}^2}$$

where $\alpha_0 > 0$ and $1 > \alpha_1 \geq 0$

ϵ_t i.i.d and $N(0,1)$

If $\alpha_1 = 0$ the volatility/conditional variance is constant over time. (Not an ARCH model)
If $\alpha_1 = 1$ then the volatility/condition variance is infinite which violates **weak** stationarity (still stationary, but not weakly stationary and isn't well behaved.)

ARCH(2)

$$\sigma_t^2 = \alpha_0 + \alpha_1 r_{t-1}^2 + \alpha_2 r_{t-2}^2$$
$$r_t = \sigma_t \cdot \epsilon_t$$
$$r_t^2 = (\alpha_0 + \alpha_1 r_{t-1}^2 + \alpha_2 r_{t-2}^2) \cdot \epsilon_t^2$$

Non Linear Interpretation:

$$r_t = \epsilon_t \sqrt{\alpha_0 + \alpha_1 r_{t-1}^2 + \alpha_2 r_{t-2}^2}$$

Where:

1. $\alpha_0 > 0$
2. $1 > \alpha_1 \geq 0$
3. $1 > \alpha_2 \geq 0$
4. ϵ_t i.i.d and $N(0,1)$

2. Write the formulas of the estimated two regression models.

Estimated ARCH(1):

Parameter Estimates					
Variable	DF	Parameter Estimate	Standard Error	t Value	Pr > |t|
Intercept	1	0.00152	0.00018006	8.46	<.0001
retsql1	1	0.15637	0.03614	4.33	<.0001

$$\widehat{\alpha}_0 = 0.00152$$
$$\widehat{\alpha}_1 = 0.15637$$

$$\widehat{r_t}^2 = 0.00152 + 0.15637\hat{r}_{t-1}^2$$

Estimated ARCH(2):

Parameter Estimates					
Variable	DF	Parameter Estimate	Standard Error	t Value	Pr > |t|
Intercept	1	0.00124	0.00018570	6.69	<.0001
retsql1	1	0.12793	0.03602	3.55	0.0004
retsql2	1	0.18253	0.03602	5.07	<.0001

$$\widehat{\alpha}_0 = 0.00124$$
$$\widehat{\alpha}_1 = 0.12793$$
$$\widehat{\alpha}_2 = 0.18253$$

$$\widehat{r_t}^2 = 0.00124 + 0.12793r_{t-1}^2 + 0.18253\hat{r}_{t-2}^2$$

3. Write the main difference(s) between the theoretical and estimated models

There are a few differences between the theoretical and estimated models.

1. The theoretical models use slightly different notation $(\alpha_0, \alpha_1, \alpha_2)$ vs $(\widehat{\alpha}_0, \widehat{\alpha}_1, \widehat{\alpha}_2)$
   1. We use the hat symbol $(\widehat{x})$ to denote an estimated variable vs a theoretical one.
2. ϵ_t VS $se(\alpha_n)$, R^2 and $\mathbf{RMSE}$
   1. ϵ_t denotes the error term in our theoretical model, while in our estimated model we have $se(\alpha_n)$ to denote the standard error of our parameter estimates, R^2 to estimate the explanatory power of our model and $\mathbf{RMSE}$ to give us the error. (Root Mean Squared Error)
   2. Note that theoretically, $\epsilon_t \sim N(0,1) \implies E(\epsilon_t) = 0$ and $VAR(\epsilon_t) = 1$ but we could observe heavy tails in our distribution which causes it to deviate from normality (potentially causing increased standard error, lower R^2 and higher RMSE)

4. Discuss the statistical significance of the estimated parameters.

4.1 Statistical Significance of Estimated ARCH(1) Parameters:

Parameter Estimates					
Variable	DF	Parameter Estimate	Standard Error	t Value	Pr > |t|
Intercept	1	0.00152	0.00018006	8.46	<.0001
retsql1	1	0.15637	0.03614	4.33	<.0001

a) Statistical Significance of $\widehat{\alpha_0}$ in ARCH(1)

$H_0 : \widehat{\alpha}_0 = 0$
$H_a : \widehat{\alpha}_0 \neq 0$

$$t_{\widehat{\alpha}_0} = \frac{\widehat{\alpha}_0 - 0}{se(\widehat{\alpha}_0)} = \frac{0.00152}{0.00018006} \approx 8.46 > 1.96$$

$\therefore$ We reject our null hypothesis $H_0 : \widehat{a}_0 = 0$ because our t statistic is greater than the standard normal of 1.96

Additionally, $PR_{\widehat{a}_0} > |t| = 0.0001 < 0.05$ and because of this, our statistic is inside of the rejection region and is statistically significant because we do not reject $H_a : \widehat{\alpha}_0 = 0$

b) Statistical Significance of $\widehat{\alpha}_1$ in ARCH(1)

$H_0 : \widehat{\alpha}_1 = 0$
$H_a : \widehat{\alpha}_1 \neq 0$

$$t_{\widehat{\alpha}_1} = \frac{\widehat{\alpha}_1 - 0}{se(\widehat{\alpha}_1)} = \frac{0.15637}{0.03614} \approx 4.33 > 1.96$$

Again, we look at our t value in the table and it is still greater than 1.96 ($4.33 > 1.96$), and that our $PR_{\alpha_1} = 0.0001 < 0.05$ therefore we reject our null hypothesis $H_0 : \alpha_1 = 0$ and do not reject our alternate hypothesis that $H_a : \alpha_1 \neq 0$. Therefore, α_1 under an ARCH(1) process is statistically significant.

c) Statistical Significance of $\widehat{\alpha}_0 = \widehat{\alpha}_1 = 0$ in ARCH(1)

The tests we performed before are are only separate tests for $H_0 : \alpha = 0$, not a joint test for $H_0 : \alpha_1 = \alpha_2 = 0$.

Therefore we must test:
$H_0 : \widehat{\alpha}_0 = \widehat{\alpha}_1 = 0$
$H_a : \widehat{\alpha}_0 \neq \widehat{\alpha}_1 \neq 0$

To actually test for joint statistical significant we must look at our ANOVA's F-Value/Pr > F.

Analysis of Variance					
Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Model	1	0.00039483	0.00039483	18.72	<.0001
Error	747	0.01575	0.00002109		
Corrected Total	748	0.01615			

Like our previous ARCH(1) results, our F Value is greater than 1.96 ($18.72 > 1.96$) and our $Pr > F = 0.0001 < 0.005$ therefore we reject our joint null hypothesis that $H_0 : \alpha_1 = \alpha_2 = 0$ and do not reject our alternate hypothesis that $H_a : \alpha_0 \neq \alpha_1 \neq 0$

4.2 Statistical Significance of ARCH(2) Parameters

Parameter Estimates					
Variable	DF	Parameter Estimate	Standard Error	t Value	Pr > |t|
Intercept	1	0.00124	0.00018570	6.69	<.0001
retsql1	1	0.12793	0.03602	3.55	0.0004
retsql2	1	0.18253	0.03602	5.07	<.0001

a) Statistical Significance of $\widehat{\alpha}_0$ in ARCH(2)

$H_0 : \widehat{\alpha}_0 = 0$
$H_a : \widehat{\alpha}_0 \neq 0$

$$t_{\widehat{\alpha}_0} = \frac{\widehat{\alpha}_0 - 0}{se(\widehat{\alpha}_0)} = \frac{0.00124}{0.00018570} \approx 6.69 > 1.96$$

$\therefore$ We reject our null hypothesis $H_0 : \widehat{a}_0 = 0$ because our t statistic is greater than the standard normal of 1.96
Additionally, $PR_{\widehat{\alpha}_0} > |t| = 0.0001 < 0.05$ and because of this, our statistic is inside of the rejection region and is statistically significant because we do not reject $H_a : \widehat{\alpha}_0 = 0$

b) Statistical Significance of $\widehat{\alpha}_1$ in ARCH(2)

$H_0 : \widehat{\alpha}_1 = 0$
$H_a : \widehat{\alpha}_1 \neq 0$

$$t_{\widehat{\alpha}_1} = \frac{\widehat{\alpha}_1 - 0}{se(\widehat{\alpha}_1)} = \frac{0.12793}{0.03602} \approx 3.55 > 1.96$$

$\therefore$ We reject our null hypothesis $H_0 : \widehat{a}_1 = 0$ because our t statistic is greater than the standard normal of 1.96
Additionally, $PR_{\widehat{\alpha}_1} > |t| = 0.0001 < 0.05$ and because of this, our statistic is inside of the rejection region and is statistically significant because we do not reject $H_a : \widehat{\alpha}_1 = 0$

c) Statistical Significance of $\widehat{\alpha}_2$ in ARCH(2)

$H_0 : \widehat{\alpha}_2 = 0$
$H_a : \widehat{\alpha}_2 \neq 0$

$$t_{\widehat{\alpha}_2} = \frac{\widehat{\alpha}_2 - 0}{se(\widehat{\alpha}_2)} = \frac{0.18253}{0.03602} \approx 5.07 > 1.96$$

$\therefore$ We reject our null hypothesis $H_0 : \widehat{\alpha}_2 = 0$ because our t statistic is greater than the standard normal of 1.96
Additionally, $PR_{\widehat{\alpha}_2} > |t| = 0.0001 < 0.05$ and because of this, our statistic is inside of the rejection region and is statistically significant because we do not reject $H_a : \widehat{\alpha}_2 = 0$

d) Statistical Significance of $\widehat{\alpha}_0 = \widehat{\alpha}_1 = \widehat{\alpha}_2 = 0$ in ARCH(2)

Analysis of Variance					
Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Model	2	0.00091991	0.00045995	22.50	<.0001
Error	745	0.01523	0.00002044		
Corrected Total	747	0.01615			

5. Are the squared returns on TESLA serially correlated and which result(s) in the computer output leads you to this conclusion?

Yes because:

1. Previous tests of statistical significance in Question 4.
   In question 4 we tested our ARCH models for statistical significance and because all statistics $(\alpha_0, \alpha_1, \alpha_2)$ are statistically significant including when jointly tested then we can in fact say that squared returns on TESLA are serially correlated. The goal of our ARCH model itself is to test for serial correlations in volatility from past squared returns so when we test this model and it is significant, then there has to be serial correlation.

   This means if past returns r_{t-1} is large, then the squared value r_{t-1}^2 will also be large, and therefore σ_t^2 will also be large.
   Inversely, if r_{t-1} is small, then r_{t-1}^2 is small, and σ_t^2 will also be small.

2. R^2 increasing from ARCH(1) to ARCH(2):
   Although R^2 is relatively low in both our ARCH(1) and ARCH(2), is does increase from ARCH(1) to ARCH(2). Implying that as you increase your lag, the explanatory power of your model increases. I was curious to see what an ARCH(3) model looks like, and we can also check it's R^2 was to see if I was right.

ARCH(1)

Root MSE	0.00459	R-Square	0.0244
Dependent Mean	0.00181	Adj R-Sq	0.0231
Coeff Var	254.37220		

$$\text{ARCH}(1) R^2 = 0.0244 = 2.44\%$$

ARCH(2)

Root MSE	0.00452	R-Square	0.0570
Dependent Mean	0.00180	Adj R-Sq	0.0544
Coeff Var	250.66683		

$$\text{ARCH}(2) R^2 = 0.0570 = 5.70\%$$

ARCH(3)

Root MSE	0.00451	R-Square	0.0650
Dependent Mean	0.00180	Adj R-Sq	0.0612
Coeff Var	250.11296		

$$\text{ARCH}(3) R^2 = 0.0650 = 6.50\%$$

So, as you increase your lag in your ARCH, the models exhibit increasing serial correlation not only due to statistical significance, but also increasing accuracy of our R^2.

It is important to note that increasing lag to ∞ (or the bounds of your data set e.x ARCH(751)) also results in inherent overfitting.

6. Is the volatility of TESLA stock predictable? If yes, compute the out–of–sample prediction of volatility of TESLA at time T+1, i.e. on January 1, 2022.

Yes, because the volatility/squared returns of TESLA's stock are predictable and serially correlated in both of our models we can predict the volatility at time $t+1$ by transforming the following equations:

$$\sigma_t^2 = \alpha_0 + \alpha_1 r_{t-1}^2 \tag{ARCH(1)}$$

$$\sigma_t^2 = \alpha_0 + \alpha_1 r_{t-1}^2 + \alpha_2 r_{t-2}^2 \tag{ARCH(2)}$$

into:

$$\sigma_{t+1}^2 = \alpha_0 + \alpha_1 r_t^2 \tag{ARCH(1)}$$

$$\sigma_{t+1}^2 = \alpha_0 + \alpha_1 r_t^2 + \alpha_2 r_{t-1}^2 \tag{ARCH(2)}$$

then we need to plug in our different α_1's, our α_2 from our regression, and our squared returns for December 31st 2021 (r_t^2) and December 30th 2021 (r_{t-1}^2) from our data.

	price	return	t	retsq
728	378.996674	0.0496471569	728	0.0024648402
729	381.58667	0.0068105778	729	0.000046384
730	365	-0.044440654	730	0.0019749717
731	361.533325	-0.009543131	731	0.0000910713
732	338.323334	-0.066352175	732	0.0044026111
733	336.33667	-0.005889395	733	0.000034685
734	350.583344	0.0414858119	734	0.0017210726
735	356.320007	0.0162307589	735	0.0002634375
736	334.600006	-0.062893417	736	0.0039555819
737	339.01001	0.0130938285	737	0.0001714483
738	322.136658	-0.051053776	738	0.002606488
739	319.503326	-0.008208175	739	0.0000673741
740	325.329987	0.0180723281	740	0.000326609
741	308.973328	-0.051585056	741	0.002661018
742	310.856659	0.0060769464	742	0.0000369293
743	299.980011	-0.03561606	743	0.0012685037
744	312.843323	0.0419866573	744	0.0017628794
745	336.290009	0.07227141	745	0.0052231567
746	355.666656	0.0560200229	746	0.003138243
747	364.646667	0.0249349165	747	0.0006217501
748	362.823334	-0.005012817	748	0.0000251283
749	362.063324	-0.002096908	749	4.3970227E-6
750	356.779999	-0.014699782	750	0.0002160836
751	352.26001	-0.012749775	751	0.0001625568

$$\therefore r_t^2 = 0.0001625568 \quad \text{and} \quad r_{t-1}^2 = 0.0002160836$$

We can verify that $t = 750$ and $t = 751$ correlate with December 30th and 31st 2021 because of our excel file

751	2021-12-30	356.779999
752	2021-12-31	352.26001

where the stock prices equal each other (even though the index is off, its because we lose a degree of freedom when calculating returns from stock prices)

a) ARCH(1)

$\widehat{\alpha}_0 = 0.00152$
$\widehat{\alpha}_1 = 0.15637$

$$\sigma_{t+1}^2 = \alpha_0 + \alpha_1 r_t^2 = 0.00152 + 0.15637(0.0001625568) = 0.001545419 \tag{ARCH(1)}$$

b) ARCH(2)

$\widehat{\alpha}_0 = 0.00124$
$\widehat{\alpha}_1 = 0.12793$
$\widehat{\alpha}_2 = 0.18253$

$$\sigma_{t+1}^2 = \alpha_0 + \alpha_1 r_t^2 + \alpha_2 r_{t-1}^2 = 0.00124 + 0.12793(0.0001625568) + 0.18253(0.0002160836) = 0.00130023763 \tag{ARCH(2)}$$

7. Does your finding contradict the former conclusion that returns on TESLA are white noise?

No, it is not contradictory at all. It is perfectly possible for the returns on TESLA to be white noise while the squared returns on TESLA have serial correlation. To put succinctly, movements in TESLA's returns are unpredictable but the squared returns of these movements aren't.

In real life this is due to things like earnings releases or macroeconomic shocks. E.x (Volatility increases after earnings reports, also historical volatility increased significantly recently during/after the election).

An ARCH process is defined by this and if there was a contradiction then our model wouldn't be able to predict anything.