Econ 4220 Midterm Makeup

This assignment has an accompanying python file instead of a SAS file. See Regression Verification.py

9.1 Consider the multiple regression model:

$$y_t = x_{t1}\beta_1 + x_{t2}\beta_2 + x_{t3}\beta_3 + e_t$$

With the nine observations on y_t, x_{t1}, x_{t2}, x_{t3} given in Table 9.4

y_t	x_{t1}	x_{t2}	x_{t3}
1	1	0	-1
-1	-1	1	0
2	1	0	0
0	0	1	0
4	1	2	0
2	0	3	0
2	0	0	1
0	1	-1	1
2	0	0	1

a) What are the elements in y and x when the model is written in the matrix notation $y = X\beta + e$ (Note: this is a special model where the first column in X is no longer a column of ones.)

$$y = \begin{bmatrix} 1 \\ -1 \\ 2 \\ 0 \\ 4 \\ 2 \\ 2 \\ 0 \\ 2 \end{bmatrix}$$

$$x = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\beta = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix}$$

$$e = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \\ e_7 \\ e_8 \\ e_9 \end{bmatrix}$$

$$y = \mathbf{X}\beta + e$$

$$\begin{bmatrix} 1 \\ -1 \\ 2 \\ 0 \\ 4 \\ 2 \\ 2 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} + \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \\ e_7 \\ e_8 \\ e_9 \end{bmatrix}$$

Not sure if this is how you wanted it stated, I tried to follow the convention from the lecture notes (Nmodel.pdf).

b) Use a hand calculator to find the following:

i) $X'X$, $X'y$ and $(X'X)^{-1}$ (Note: In this case X'X is called a diagonal matrix. The inverse of a diagonal matrix is equal to the inverse of the diagonals)

We first compute $X'X$. Since X is a 9×3 matrix will get a 3×3 matrix for $X'X$

$(9 \times 3 \cdot 3 \times 9) = (3 \times 3)$

$$X'X = \begin{bmatrix} 1 & -1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 2 & 3 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} (1+1+1+1+1) & (-1+2-1) & (-1+1) \\ (-1+2-1) & (1+1+4+9+1) & (-1) \\ (-1+1) & (-1) & (1+1+1+1) \end{bmatrix} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 16 & -1 \\ 0 & -1 & 4 \end{bmatrix}$$

$$X'y = \begin{bmatrix} 1 & -1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 2 & 3 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 2 \\ 0 \\ 4 \\ 2 \\ 2 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} (1+1+2+0+4) \\ (-1+8+6) \\ (-1+2+2) \end{bmatrix}$$

$$X'y = \begin{bmatrix} 8 \\ 13 \\ 3 \end{bmatrix}$$

$$(X'X)^{-1} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 16 & -1 \\ 0 & -1 & 4 \end{bmatrix}^{-1}$$

There is a more "efficient" way to compute the inverse of a matrix using the determinant and adjugate, but will use gaussian elimination and an identity matrix to find the inverse of $X'X$. It's simple and easy to do albeit time consuming.

$$\begin{bmatrix} 5 & 0 & 0 \\ 0 & 16 & -1 \\ 0 & -1 & 4 \end{bmatrix} \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \xrightarrow[\frac{R_1}{5}]{} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 16 & -1 \\ 0 & -1 & 4 \end{bmatrix} \begin{vmatrix} \frac{1}{5} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 16 & -1 \\ 0 & -1 & 4 \end{bmatrix} \begin{vmatrix} \frac{1}{5} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \xrightarrow[\frac{R_2}{16}]{} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{16} \\ 0 & -1 & 4 \end{bmatrix} \begin{vmatrix} \frac{1}{5} & 0 & 0 \\ 0 & \frac{1}{16} & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{16} \\ 0 & -1 & 4 \end{bmatrix} \begin{vmatrix} \frac{1}{5} & 0 & 0 \\ 0 & \frac{1}{16} & 0 \\ 0 & 0 & 1 \end{vmatrix} \xrightarrow[R_3+R_2]{} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{16} \\ 0 & 0 & \frac{63}{16} \end{bmatrix} \begin{vmatrix} \frac{1}{5} & 0 & 0 \\ 0 & \frac{1}{16} & 0 \\ 0 & \frac{1}{16} & 1 \end{vmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{16} \\ 0 & 0 & \frac{63}{16} \end{bmatrix} \begin{vmatrix} \frac{1}{5} & 0 & 0 \\ 0 & \frac{1}{16} & 0 \\ 0 & \frac{1}{16} & 1 \end{vmatrix} \xrightarrow[\frac{R_3}{\frac{63}{16}}]{} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{16} \\ 0 & 0 & 1 \end{bmatrix} \begin{vmatrix} \frac{1}{5} & 0 & 0 \\ 0 & \frac{1}{16} & 0 \\ 0 & \frac{1}{63} & \frac{16}{63} \end{vmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{16} \\ 0 & 0 & 1 \end{bmatrix} \begin{vmatrix} \frac{1}{5} & 0 & 0 \\ 0 & \frac{1}{16} & 0 \\ 0 & \frac{1}{63} & \frac{16}{63} \end{vmatrix} \xrightarrow[R_2+\frac{1}{16}R_3]{} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{vmatrix} \frac{1}{5} & 0 & 0 \\ 0 & \frac{4}{63} & \frac{1}{63} \\ 0 & \frac{1}{63} & \frac{16}{63} \end{vmatrix}$$

Because we reduced our original matrix into an identity matrix through nothing but elementary row operations, the corresponding original identity matrix is now our inverse matrix.

$$\therefore (X'X)^{-1} = \begin{bmatrix} \frac{1}{5} & 0 & 0 \\ 0 & \frac{4}{63} & \frac{1}{63} \\ 0 & \frac{1}{63} & \frac{16}{63} \end{bmatrix}$$

ii) The least squares estimator b

$$b = (X'X)^{-1}X'y$$

$$(X'X)^{-1} = \begin{bmatrix} \frac{1}{5} & 0 & 0 \\ 0 & \frac{4}{63} & \frac{1}{63} \\ 0 & \frac{1}{63} & \frac{16}{63} \end{bmatrix}$$

$$X'y = \begin{bmatrix} 8 \\ 13 \\ 3 \end{bmatrix}$$

$$b = \begin{bmatrix} \frac{1}{5} & 0 & 0 \\ 0 & \frac{4}{63} & \frac{1}{63} \\ 0 & \frac{1}{63} & \frac{16}{63} \end{bmatrix} \begin{bmatrix} 8 \\ 13 \\ 3 \end{bmatrix} = \begin{bmatrix} \frac{8}{5}+0+0 \\ 0+\frac{52}{63}+\frac{3}{63} \\ 0+\frac{13}{63}+\frac{48}{63} \end{bmatrix} = \begin{bmatrix} \frac{8}{5} \\ \frac{55}{63} \\ \frac{61}{63} \end{bmatrix}$$

iii) The least squares residual vector $\hat{e}$

$$\hat{e_i} = y_i - b_1x_{i1} - b_2x_{i2} - b_3x_{i3}$$

$$\hat{e} = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \\ e_7 \\ e_8 \\ e_9 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \\ 0 \\ 4 \\ 2 \\ 2 \\ 0 \\ 2 \end{bmatrix} - \frac{8}{5}\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} - \frac{55}{63}\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 2 \\ 3 \\ 0 \\ -1 \\ 0 \end{bmatrix} - \frac{61}{63}\begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \\ e_7 \\ e_8 \\ e_9 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \\ 0 \\ 4 \\ 2 \\ 2 \\ 0 \\ 2 \end{bmatrix} - \begin{bmatrix} \frac{8}{5} \\ -\frac{8}{5} \\ \frac{8}{5} \\ 0 \\ \frac{8}{5} \\ 0 \\ 0 \\ \frac{8}{5} \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ \frac{55}{63} \\ 0 \\ \frac{55}{63} \\ \frac{110}{63} \\ \frac{165}{63} \\ 0 \\ -\frac{55}{63} \\ 0 \end{bmatrix} - \begin{bmatrix} -\frac{61}{63} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{61}{63} \\ \frac{61}{63} \\ \frac{61}{63} \end{bmatrix}$$

$$\hat{e} = \begin{bmatrix} \frac{116}{315} \\ -\frac{86}{315} \\ \frac{2}{5} \\ -\frac{55}{63} \\ \frac{206}{315} \\ -\frac{13}{21} \\ \frac{65}{63} \\ -\frac{178}{105} \\ \frac{65}{63} \end{bmatrix}$$

iv) The variance estimate $\hat{\sigma}^2$

$$\hat{\sigma}^2 = \frac{\sum \hat{e}_i^2}{N-K} = \frac{\hat{e}'\hat{e}}{N-K}$$

$$\hat{e}'\hat{e} = \begin{bmatrix} \frac{116}{315} & -\frac{86}{315} & \frac{2}{5} & -\frac{55}{63} & \frac{206}{315} & -\frac{13}{21} & \frac{65}{63} & -\frac{178}{105} & \frac{65}{63} \end{bmatrix} \begin{bmatrix} \frac{116}{315} \\ -\frac{86}{315} \\ \frac{2}{5} \\ -\frac{55}{63} \\ \frac{206}{315} \\ -\frac{13}{21} \\ \frac{65}{63} \\ -\frac{178}{105} \\ \frac{65}{63} \end{bmatrix}$$

$$\hat{e}'\hat{e} = (\frac{116}{315})^2 + (-\frac{86}{315})^2 + (\frac{2}{5})^2 + (-\frac{55}{63})^2 + (\frac{206}{315})^2 + (-\frac{13}{21})^2 + (\frac{65}{63})^2 + (-\frac{178}{105})^2 + (\frac{65}{63})$$

$$\hat{e}'\hat{e} = \frac{13456}{99225} + \frac{7396}{99225} + \frac{4}{25} + \frac{3025}{3969} + \frac{42436}{99225} + \frac{169}{441} + \frac{4225}{3969} + \frac{31684}{11025} + \frac{4225}{3969}$$

$$\hat{e}'\hat{e} = \frac{63288}{99225} + \frac{4}{25} + \frac{11475}{3969} + \frac{169}{441} + \frac{31684}{11025}$$

$$\hat{e}'\hat{e} = \frac{63288}{99225} + \frac{15876}{99225} + \frac{286875}{99225} + \frac{38025}{99225} + \frac{285156}{99225}$$

$$\hat{e}'\hat{e} = \frac{689220}{99225} = \frac{2188}{315}$$

$N = 9$
$K = 3$

$$\hat{\sigma}^2 = \frac{\hat{e}'\hat{e}}{N-K} = \frac{\frac{2188}{315}}{6} = \frac{1094}{945}$$

v) The estimated covariance matrix $\hat{\sigma}^2(X'X)^{-1}$

$$\hat{\sigma^2}(X'X)^{-1} = \frac{1094}{945}\begin{bmatrix} \frac{1}{5} & 0 & 0 \\ 0 & \frac{4}{63} & \frac{1}{63} \\ 0 & \frac{1}{63} & \frac{16}{63} \end{bmatrix} = \begin{bmatrix} \frac{1094}{4725} & 0 & 0 \\ 0 & \frac{4376}{59535} & \frac{1094}{59535} \\ 0 & \frac{1094}{59535} & \frac{17504}{59535} \end{bmatrix}$$

vi) the standard errors for b_1, b_2 and b_3

$$SE(b) = \sqrt{var(b)}$$

We sub in along the diagonals from our COV matrix:

$$SE(b_1) = \sqrt{var(b_1)} = \sqrt{\frac{1094}{4735}} = 0.4812$$

$$SE(b_2) = \sqrt{var(b_2)} = \sqrt{\frac{4376}{59535}} = 0.2711$$

$$SE(b_3) = \sqrt{var(b_3)} = \sqrt{\frac{17504}{59535}} = 0.5422$$

vii) the coefficient of determination R^2

$$R^2 = 1 - \frac{\sum \hat{e_i}^2}{\sum(y_i - \bar{y})^2} = 1 - \frac{\hat{e}'\hat{e}}{\sum(y_i - \bar{y})^2}$$

$$y = \begin{bmatrix} 1 \\ -1 \\ 2 \\ 0 \\ 4 \\ 2 \\ 2 \\ 0 \\ 2 \end{bmatrix}$$

$$\bar{y} = \frac{(1 - 1 + 2 + 0 + 4 + 2 + 2 + 0 + 2)}{9} = \frac{12}{9} = \frac{4}{3}$$

$$y_i - \bar{y} = \begin{bmatrix} 1 \\ -1 \\ 2 \\ 0 \\ 4 \\ 2 \\ 2 \\ 0 \\ 2 \end{bmatrix} - \begin{bmatrix} \frac{4}{3} \\ \frac{4}{3} \\ \frac{4}{3} \\ \frac{4}{3} \\ \frac{4}{3} \\ \frac{4}{3} \\ \frac{4}{3} \\ \frac{4}{3} \\ \frac{4}{3} \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} \\ -\frac{7}{3} \\ \frac{2}{3} \\ -\frac{4}{3} \\ \frac{8}{3} \\ \frac{2}{3} \\ \frac{2}{3} \\ -\frac{4}{3} \\ \frac{2}{3} \end{bmatrix}$$

$$\sum(y_i - \bar{y})^2 = \frac{162}{9} = 18$$

$$R^2 = 1 - \frac{\hat{e}'\hat{e}}{\sum(y_i - \bar{y})^2} = 1 - \frac{\frac{2188}{315}}{18} = \frac{5670}{5670} - \frac{2188}{5670} = \frac{3482}{5670}$$

$$R^2 = 0.6141$$

Pretty good R^2, how about adjusted R^2?

$$\bar{R}^2 = 1 - \frac{N-1}{N-K} \cdot (1 - R^2) = 1 - \frac{9-1}{9-3} \cdot (1 - \frac{3482}{5670})$$

$$\bar{R}^2 = 1 - \frac{4}{3}(\frac{2188}{5670}) = 1 - (\frac{8752}{17010}) = \frac{8258}{17010} = 0.4853$$

C) Use a computer to verify the results you obtained in b)

Although I know you despise python, its the only language I'm proficient enough in to do matrix algebra, I thought it would be a great opportunity to practice my coding rather than using an online calculator or chatGPT.

I coded it in JupyterLab/Notebook, but I ran it in VSCODE and it works fine there too, but the outputs don't display as clearly.

See output from PDF attached in submission/email titled "Econ 4220 Output for Regression Verification"

My calculations all match the computer outputs.

16.1 In Examples 16.2 and 16.4 we presented the linear probability and probit model estimates using an example of transportation choice. The logit model for the same example is $P(\text{AUTO} = 1) = \Lambda(\gamma_1 + \gamma_2\text{DTIME})$, where $\Lambda(\bullet)$ is the logistic cdf in equation (16.7). The logit model parameter estimates and their standard errors are:

$$\tilde{\gamma}_1 + \tilde{\gamma}_2\text{DTIME} = -0.2376 + 0.5311\text{DTIME}$$

$$se \qquad (0.7505) \qquad (0.2064)$$

a. Calculate the estimated probability that a person will choose automobile transportation given that $DTIME = 1$

$$\tilde{\gamma}_1 + \tilde{\gamma}_2\text{DTIME} = -0.2376 + 0.5311\text{DTIME}$$

$$-0.2376 + 0.5311(1) = 0.2935$$

$$P(AUTO = 1|DTIME = 1) = \Lambda(0.2935)$$

Our logistic CDF is:

$$\Lambda(\bullet) = \frac{1}{1 + \frac{e^{-(x-\mu)}}{s}}$$

The location parameter $(\mu) = 0$ and our scale parameter $(s) = 1$,

$$\therefore \quad \Lambda(\bullet) = \frac{1}{1 + e^{-x}}$$

$$\Lambda(0.2935) = \frac{1}{1 + e^{-0.2935}}$$

$$\Lambda(0.2935) = \frac{1}{1.74564922286} = 0.5738$$

b. Using the probit model results in Example 16.4, calculate the estimated probability that a person will choose automobile transportation given that $DTIME = 1$. How does this result compare to the logit estimate? (Hint: Recall that Statistical Table 1 gives cumulative probabilities for the standard normal distribution)

From the Principles of Econometrics by R.C Hill 5th Edition, Example 16.4:

"The probit model is $P(AUTO = 1) = \Phi(\beta_1 + \beta_2 DTIME)$"

The maximum likelihood estimates of the parameters are:

$$\tilde{\beta}_1 + \tilde{\beta}_2 DTIME_i = -0.0644 + 0.3000 DTIME$$

$$(se) \qquad (0.3992) \qquad (0.1029)$$

Following the same procedure as before but this time using a normal CDF instead of logistic:

$$P(AUTO = 1|DTIME = 1) = \Phi(-0.0644 + 0.3000(1)) = \Phi(0.2356) = 0.5932$$

There is only about a 2% probability difference in our CDFs between the logit and probit difference. People are slightly more likely to choose automobile transportation in our probit model rather than logit.

c. Using the logit model results, compute the estimated marginal effect of an increase in travel time of 10 minutes for an individual whose travel time is currently 30 minutes longer by bus (public transportation). Using the linear probability model results, compute the same marginal effect estimate. How do they compare?

From example 16.5:

Logit Model:

$$\frac{\widehat{dp}}{dDTIME} = \Lambda(\tilde{\beta}_1 + \tilde{\beta}_2 DTIME)\tilde{\beta}_2$$
$$= \lambda(-0.2376 + 0.5311(3))0.5311$$

To find the estimated marginal increase for the logit model we must use the logistic PDF this time:

$$\lambda(1.3557) = \frac{e^{1.3557}}{(1 + e^{1.3557})^2} = \frac{3.8794756}{23.809282} = 0.1629$$
$$MER(AUTO = 1| \uparrow DTIME = 30) = 0.1629 \cdot 0.5311 = 0.086516919$$

Linear Probability Model:

Parameters for linear probability model gathered from Example 16.2:

$$\widehat{AUTO}_i = 0.4848 + 0.0703 DTIME_i$$

For the linear probability model our marginal effect is constant because the derivative of a linear function is always a constant (its slope).

$$\frac{\widehat{dp}}{dDTIME} = 0.0703$$

d. Using the logit model results, compute the estimated marginal effect of a decrease in travel time of 10 minutes for an individual whose travel time is currently 50 minutes longer by driving. Using the probit results, compute the same marginal effect estimate. How do they compare?

Logit Model:

$$\frac{\widehat{dp}}{dDTIME} = \lambda(\tilde{\beta}_1 + \tilde{\beta}_2 DTIME)\tilde{\beta}_2$$
$$= \lambda(-0.2376 + 0.5311(5))0.5311$$

Again for the logistic model we use the logistic PDF:

$$\lambda(2.4179) = \frac{e^{2.4179}}{(1 + e^{2.4179})^2} = 0.075124$$
$$MER(AUTO = 1| \uparrow DTIME = 30) = 0.075124 \cdot 0.5311 = 0.039898$$

Probit Model:

$$\frac{\widehat{dp}}{dDTIME} = \phi(\tilde{\beta}_1 + \tilde{\beta}_2 DTIME)\tilde{\beta}_2$$

For the probit model though we must use the normal PDF:

$$= \phi(-0.0644 + 0.3000(5))0.3000$$
$$\phi(1.4356) = 0.14235772$$

$$MER(AUTO=1|\uparrow DTIME=30)=0.14235772\cdot 0.3=0.042707316$$

If a random person's travel time is currently 50 minutes, and their commute time increases by 10 minutes they are marginally more likely to buy a car in the probit model than the logit model.

16.2 In Appendix 16A.1, we illustrate the calculation of a standard error for the marginal effect in a probit model of transportation, Example 16.4. In the Appendix, the calculation for the marginal effect when it currently takes 20 minutes longer to commute by bus (DTIME = 2)

a. Repeat the calculation for the probit model when $DTIME=1$ (Hint: The values of the standard normal pdf are given in Statistical Table 6.)

$$-0.0644+0.3000DTIME$$

$$\operatorname{var}\left[g(\tilde{\beta}_1,\tilde{\beta}_2)\right]\approxeq\left[\frac{\partial g(\beta_1,\beta_2)}{\partial\beta_1}\right]^2\operatorname{var}(\tilde{\beta}_1)+\left[\frac{\partial g(\beta_1,\beta_2)}{\partial\beta_2}\right]^2\operatorname{var}(\tilde{\beta}_2)+2\left[\frac{\partial g(\beta_1,\beta_2)}{\partial\beta_1}\right]\left[\frac{\partial g(\beta_1,\beta_2)}{\partial\beta_2}\right]\operatorname{cov}(\tilde{\beta}_1,\tilde{\beta}_2)$$

Step 1. Evaluate partial derivative with respect to β_1

$$\frac{\partial(\beta_1,\beta_2)}{\partial\beta_1}=-\phi(\beta_1+\beta_2x_0)\times(\beta_1+\beta_2x_0)\times\beta_2$$

$$-\phi(-0.0644+0.3(1))\times(-0.00644+0.3(1))\times 0.3$$

$$-\phi(0.2356)\times 0.2356\times 0.3=-0.3880\times 0.2356\times 0.3=-0.02742384$$

$$\frac{\partial(\beta_1,\beta_2)}{\partial\beta_1}=-0.02742384$$

Step 2. Evaluate partial derivative with respect to β_2

$$\frac{\partial g(\beta_1,\beta_2)}{\partial\beta_2}=\phi(\beta_1+\beta_2x_0)[1-(\beta_1+\beta_2x_0)\times\beta_2x_0]$$

$$\phi(0.2356)\times(1-(0.2356\times 0.3))=0.3880\times 0.92932=0.36057616$$

Step 3. Substitute the values of our covariance matrix and evaluated derivatives into the original variance equation.

$$\begin{bmatrix}\widehat{\operatorname{var}}(\tilde{\beta}_1) & \widehat{\operatorname{cov}}(\tilde{\beta}_1,\tilde{\beta}_2)\\ \widehat{\operatorname{cov}}(\tilde{\beta}_1,\tilde{\beta}_2) & \widehat{\operatorname{var}}(\tilde{\beta}_2)\end{bmatrix}=\begin{bmatrix}0.1593956 & 0.0003261\\ 0.0003261 & 0.0105817\end{bmatrix}$$

$$\operatorname{var}\left[g(\tilde{\beta}_1,\tilde{\beta}_2)\right]\approxeq(-0.02742384)^2(0.1593956)+(0.36057616)^2(0.0105817)+2(-0.02742384)(0.36057616)(0.0003261)$$

$$\operatorname{var}\left[g(\tilde{\beta}_1,\tilde{\beta}_2)\right]\approxeq 0.000119876+0.001375781-0.000006449$$

$$\operatorname{var}\left[g(\tilde{\beta}_1,\tilde{\beta}_2)\right]\approxeq 0.001489208$$

Step 4: Calculate standard error from variance.

$$se(g(\tilde{\beta}_1,\tilde{\beta}_2))=\sqrt{\operatorname{var}\left[g(\tilde{\beta}_1\right.}=0.038590255$$

b. Using the probit model, construct a 95% interval estimate for the marginal effect of a 10-minute increase in travel time by bus when $DTIME=1$

A 95% confidence interval implies a Z score of 1.96

To make a confidence interval for our marginal effect we take the marginal effect itself and add/subtract the 95% z score times our standard error:

$$CI=\frac{\widehat{dp}}{dDTIME}\pm z\cdot se(g(\tilde{\beta}_1,\tilde{\beta}_2))$$

$$\frac{\widehat{dp}}{dDTIME}=\phi(-0.00644+0.3(1))\times 0.3=\phi(0.2356)\times 0.3=0.3880\times 0.3$$

$$\frac{\widehat{dp}}{dDTIME}=0.1164$$

$$CI_{\alpha=0.05}=0.1164\pm 1.96\cdot 0.038590255$$

$$CI_{\alpha=0.05} = 0.0407631 \leq \frac{\widehat{dp}}{dDTIME} \leq 0.1920369$$

or

$$CI_{\alpha=0.05} = [0.0407631, 0.1920369]$$