Multi-Agent Decision Making
Assignment 3 – COMP4418
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Question 1
In the tournament in Figure 1, all the arcs missing from the figure are downward arcs. For the tournament in Figure 1, find the (a) the uncovered set (b) the top cycle (c) the set of Copeland winners (d) the set of Banks winners (e) the set of Condorcet winners and give arguments for your answers.

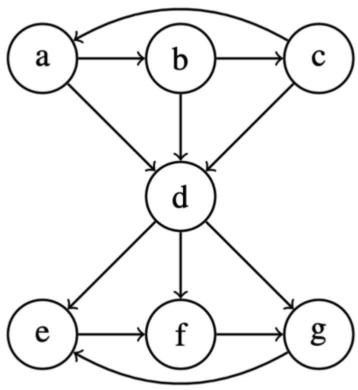


Figure 1: Tournament

Solution :

a) The Uncovered Set
The Uncovered Set of a tournament T = (V, E), denoted by UC (T), is the set of alternative that can reach every other alternative in at most two steps.

Alternatives	a	b	c	d	e	f	g
A	-	1	2	1	1	1	1
B	2	-	1	1	1	1	1
C	1	2	-	1	1	1	1
D	-	-	-	-	1	1	1
E	-	-	-	-	-	1	1
F	-	-	-	-	2	-	1
G	-	-	-	-	1	2	-

The Uncovered Set is {a,b,c} because there every alternative in this set can reach every other alternative in the tournament in at most two steps.

b) The Top Cycle
The top cycle of a tournament T = (V, E), denoted by TC (T), is the unique minimal dominant subset of V . In other words an alternative that dominates every other alternative such that there exits a path from the alternative to other alternatives.

Alternatives	Dominates	Dominated by
A	b,d,e,f,g	c
B	c,d,e,f,g	a
C	a,d,g,f,e	b
D	e,f,g	a,b,c
E	f	a,b,c,d,g
F	g	a,b,c,d,e
G	e	a,b,c,d,f

The Top Cycle is a->b->c->a or in other words the set {a,b,c} is a Top Cycle.

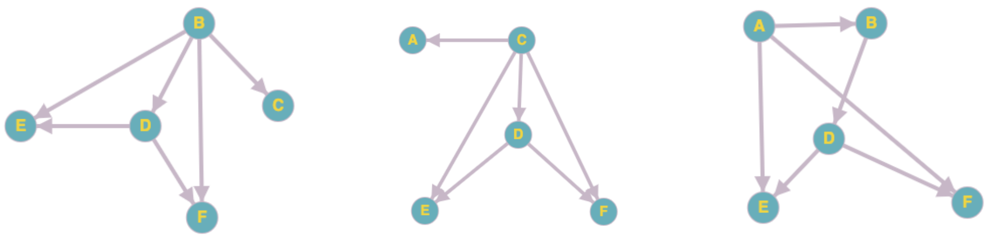
c) The Set of Copeland Winners

Alternatives	Outdegree	Copeland Score
A	b,d,e,f,g	5
B	c,d,e,f,g	5
C	a,d,g,f,e	5
D	e,f,g	3
E	f	1
F	g	1
G	e	1

The set of Copeland Winners is {a,b,c} because it has the maximum Copeland Score of 5.

d) The Set of Banks winners

An alternative x is a Banks winner if it is a top element in a maximal acyclic subgraph of the tournament T.



The set of Banks winners is {a,b,c} because each of the alternate comes on top of every other alternative in a maximal acyclic graph.

e) The set of Condorcet winners

There is no Condorcet winner in the given Tournament T because there exists no single alternative that dominates all other alternatives.
A Condorcet winner is defined as an alternative that is pairwise preferred by a majority of voters over every other alternative.

Alternatives	A	B	C	D	E	F	G
A	-	a	c	a	a	a	a
B	a	-	b	b	b	b	b
C	c	b	-	c	c	c	c
D	a	b	c	-	d	d	d
E	a	b	c	d	-	e	g
F	a	b	c	d	e	-	F
G	a	b	c	d	g	f	-

Here, there is no Condorcet winner because pairwise preferred does not ensure a majority for any given alternative.

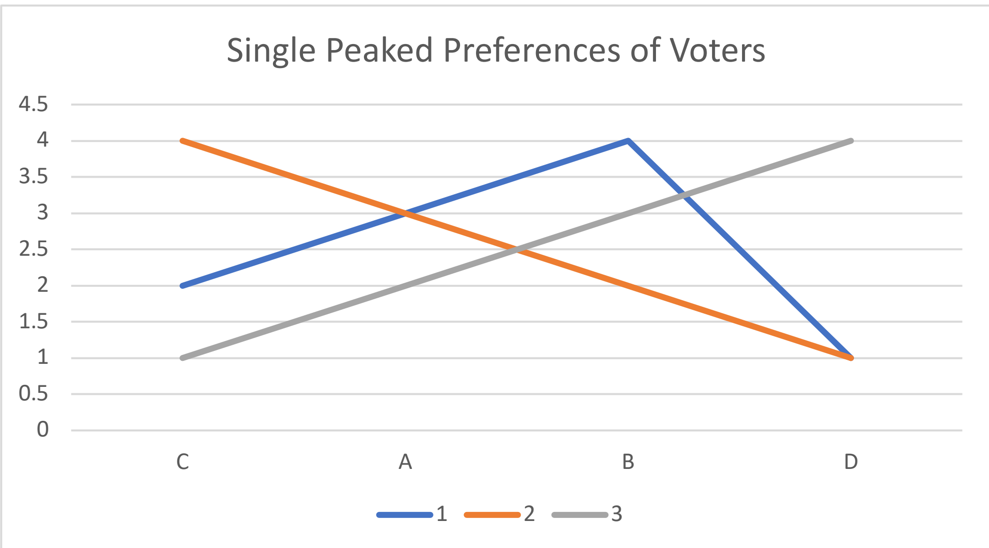
Question 2 :

Consider the following preference profile of voters.

$$1 : b \succ a \succ c \succ d$$
$$2 : c \succ a \succ b \succ d$$
$$3 : d \succ b \succ a \succ c$$

Prove or disprove that the preference profile is single-peaked. Prove or disprove that a Condorcet winner exists for the preference profile.

Solution :



Based on the majority voting rule, the order of alternatives along the X-axis is changed until a single peak can be obtained as a preference. If not the preference profile is not single peaked.

A preference profile has single-peaked preferences if there exists a left to right ordering > on the alternatives such that any voter prefers a to b if a is between b and her top alternative. Given a left-to-right ordering >, the median voter rule asks each voter for their top alternative and elects the alternative proposed by the voter corresponding to the median with respect to >. (Lecture Slides)

Here, the preference profile is single peaked.
If there exists a single peaked preference that implies that *there exists a Condorcet winner* and that alternative will get elected by the median voter rule (Black's Theorem).
The Condorcet winner by that definition would be B because B would have been the alternative proposed by the voter corresponding to the median.

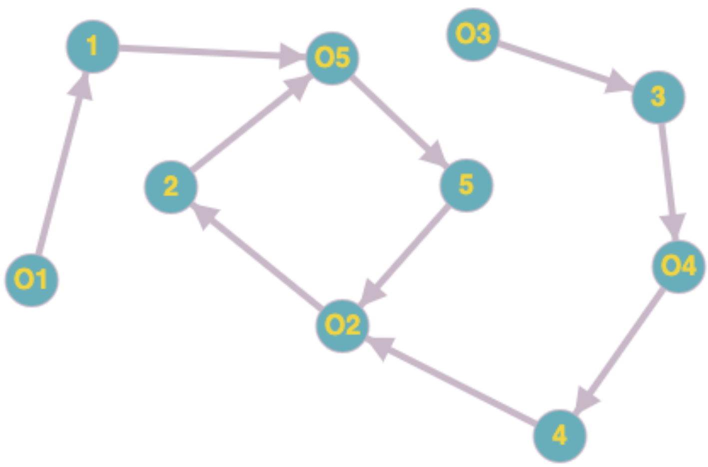
Question 3

Consider a Shapley-Scarf housing market with a set of agents N = {1,2,3,4,5}, a set of items O = {o1,o2,o3,o4,o5}, an endowment function ω : N → 2 O such that ω(i) = {oi}. The preferences of the agents are as follows from left to right in decreasing order of preference.

1 : o5,o2,o1,o3,o4
2 : o5,o4,o3,o1,o2
3 : o4,o2,o3,o5,o1
4 : o2,o1,o5,o3,o4
5 : o2,o4,o1,o5,o3

Find the outcome of the TTC (top trading cycles) algorithm. Can agent 4 misreport her preference to get a more preferred allocation? Prove or disprove that the outcome is individually rational.

Solution :



There exists a cycle between O2,2,O5 and 5. The cycle can be broken by trading one resource for another. So, O2 is allocated to 5 and O5 is allocated to 2. The first preferences for the other elements are accordingly allocated based on the available first preferences.
Allocation would therefore be :

1 -> O1
2 -> O5
3 -> O4
4 -> O3
5 -> O2

The resource O5 is allocated to 2 because O5 dominates O2.
The resource O2 is allocated to 5 because O2 dominates O5.
The resource O3 is allocated to 4 because O3 dominates O4.
The resource O4 is allocated to 3 because O4 dominates O3.
O1 does not make a trade because O5 and O2 are allocated to 2 and 5 to break the cycle and O1 dominates all other available alternatives.

Even if agent 4 misreports her preference it will not change the allocation. This is because TTC is core stable, strategy proof and individually rational. Misreporting preferences does not change individual rationality and cannot change the way resources are endowed.
Moreover, the other agents would still be endowed with their most preferred resources. Here, Agent 4 gets O3 which is a better trade than his/her original endowment.

Question 4

Consider the following school choice problem with five students 1, 2, 3, 4, 5 and five schools a, b, c, d, and e with each school having exactly one seat. The preferences of the students are as follows from left to right in decreasing order of preference.

Solution :

1: e,b,a,c,d.	a: 2,4,2,5,1
2: b,a,c,d,e.	b:3,2,4,5,1
3: a,b,c,d,e.	c:3,2,4,5,1
4:a,b,c,d,e.	d:5,2,4,3,1
5:d,b,c,a,e	e:1,2,3,4,5

1 applies to e. 2 applies to b. 3,4 applies to a. 5 applies to d.	Accepted. Accepted. 4 is accepted over 3. Accepted.	{1.e} {2,b} {4,a} {5,d}
3 applies to b.	B accepts 3. Rejects 2 over 3.	{1,e} {3,b} {4,a} {5,d}
2 applies to a.	A accepts 2 over 4. 4 is rejected.	{1,e} {3,b} {2,a} {5,d}

4 applies to b.	B rejects 4. Prefers 3 over 4.	{1,e} {3,b} {2,a} {5,d}
4 applies to c.	C accepts 4.	{1,e} {3,b} {2,a} {5,d} {4,c}

The solution set contains {{1,e},{3,b},{2,a},{5,d},{4,c}}.
An allocation X is Pareto optimal if there exists no allocation Y such that Y (i) > i X(i) for all i ∈ N and Y (i) i > X(i) for some i ∈ N (Lecture Slides). Assigning points from 5 to 1 in order of most preferred to least preferred (from the student's point of view) for each iteration of the process we get the table below :

Students/Schools	A	B	C	D	E
1	3	4	2	1	5
2	4	5	3	2	1
3	5	4	3	2	1
4	5	4	3	2	1
5	2	4	3	5	1

It is not Pareto Optimal for all students as all the students did not get the most optimal alternative in this assignment. Swapping 2 and 3 for b and a would have been better off and it doesn't make it worse for anyone. Hence, clearly this is not Pareto Optimal.

Question 5
Consider a resource allocation setting in which 2 agents have additive utilities over m divisible items. For any item, an agent can have positive or negative utility for it. Prove that there always exists an envy-free and Pareto optimal allocation of the divisible items. Design a O(m 2) or faster algorithm that computes an envy-free and Pareto optimal allocation and prove that the algorithm returns an envy-free and Pareto optimal allocation.
Solution :
Algorithm :
Based on Adjusted Winner Theorem (Brams and Taylor[1996]).
Step 1 : Agent 1 and Agent 2 are given 'y' points for equal division of items. Here there are 'm' divisible items.
Step 2 : Assign all m items to the two agents based on their order of preference. Agent 1 is given a higher preference initially.
Step 3: To ensure equitable division of all the divisible items, items must be divided and distributed such that Agent 1 uses 'a' points and Agent 2 uses 'b' points to get items.

Here, the distribution of divisible items is achieved such that
ax1/bx1 > ax2/bx2 >………………………… axy/bxy => allocation for Agent 1
axy+1/by+1 > …………… axm/bxm => allocation for Agent 1
Adjusted Winner is Pareto Optimal and envy-free that requires at most one item to be split.

Pseudo Code :

```
points = p;
agent A;
agent B;
items = m.

def assign(x,y):
        #check if the bid is the lowest for all possible items
        if x_i/y_i  <   x_m/y_m :
                continue
        else:
                i++; #increment until the lowest bid is found

 #assign m items to two agents
 for 'i' in 'm'  items :
        if A wants i:
                i -> A.
        else if B wants i:
                i -> B.
#check if they have equal points
 if p(A)==p(B):
        break;     #equal division assumed
        return success;
 else if p(A) > p(B):
        assign(m(A), m(B)) #item/items of A is given to B
 else if p(B) > p(A) :
        assign(m(B),m(A)) #item/items of B is given to A
else:
        return failure;


return;
```

This pseudo code will be within the bounds of O(m 2) time complexity. It is justified by the AW Theorem that it is Pareto Optimal and envy free.