hw1

1 By definition,

1a) Show that $1^k+2^k+3^k+....+n^k$ is $O(n^{k+1})$, where k is a positive integer?

g(n) is an asymptotic upper bound for f(n) if there exist constants C and n_0 such that $0<=f(n)<=cg(n)$ for $n>=n_0$

Here $f(n)= 1^k+2^k+3^k+....+n^k$, $g(n)=n^{k+1}$

$1^k+2^k+3^k+....+n^k <= c.n^{k+1}$

As k,n are positive integers, replace $1^k,2^k,3^k$ with n^k

$n^k+n^k+n^k+.......+n^k <=c.n^{k+1}$

There are n terms

$n.n^k <= c.n^{K+1}$

$n^{k+1} <=c.n^{k+1}$ => This is true for all $c>=1$ and $n_0>=1$

Therefore, $1^k+2^k+3^k+....+n^k$ is $O(n^{k+1})$

Solution: $1^k+2^k+3^k+....+n^k = O(n^{k+1})$

1b) Show that $(n^3+2n)/(2n+1)$ is $O(n^2)$

g(n) is an asymptotic upper bound for f(n) if there exist constants C and n_0 such that $0<=f(n)<=cg(n)$ for $n>=n_0$

Here $f(n) = (n^3+2n)/(2n+1)$, $g(n) = n^2$

$(n^3+2n)/(2n+1) <= c. n^2$

$\frac{n^2}{2}-\frac{n}{4}+\frac{9}{8}-\frac{9}{8(2n+1)} <= c.n^2$

By Ignoring the lower order terms we get

- ⇨ $n^2/2 <= c. n^2$ => This is true for all $c>=1$ and $n_0>=1$
- ⇨ $O(n^2)$

Solution: $(n^3+2n)/(2n+1) = O(n^2)$

1c) Prove that $(n+3)^3=\Theta(n^3)$

g(n) is an asymptotic tight bound for f(n) if there exist constants c_1, c_2 and n_0 such that $0<= c_1.g(n)<= f(n)<=c_2.g(n)$ for $n>=n_0$

Here, $f(n)= (n+3)^3$, $g(n)= n^3$

$(n+3)^3 <= c.n^3$

$c_1.n^3 <= n^3+9n^2+27n+27 <= c_2.n^3$

$c_1.n^3 <= n^3+9n^3+27n^3+27n^3 <= c_2.n^3$

$c_1.n^3 <= 64n^3 <= c_2.n^3$ => This is true for $c_1<=63$, $c_2=>65$ $n_0>=1$

- ⇨ $\Theta(n^3)$
- ⇨ Therefore, $(n+3)^3 = \Theta(n^3)$

Solution: $(n+3)^3=\Theta(n^3)$

2.

2a) Is $2^{n+1} = O(2^n)$? Why?

g(n) is an asymptotic upper bound for f(n) if there exist constants C and n_0 such that $0<=f(n)<=cg(n)$ for $n>=n_0$

Here $f(n)=2^{n+1}$, $g(n)=2^n$

$2^{n+1} <= c. 2^n$

$2^n.2 <=c.2^n$ => This is true for all $c>=2$, $n_0>=0$

Solution: $O(2^n)$

2b) Is $2^{2n} = O(2^n)$

g(n) is an asymptotic upper bound for f(n) if there exist constants C and n_0 such that $0<=f(n)<=cg(n)$ for $n>=n_0$,

Here, $f(n)=2^{2n}$, $g(n)=2^n$

$2^{2n} <= c.2^n$

$(2^n)^2<=c.2^n$

$2^n.2^n <= c.2^n$

$2^n<=c \Rightarrow n$ is not bounded by c.

Therefore f(n) is not $O(2^n)$

Solution: $2^{2n} != O(2^n)$

---

3. Order the following functions into a list such that if f(n) comes before g(n) in the list then f(n) = O(g(n)). If any two (or more) of the same asymptotic order, indicate which.

3a) Start with these basic functions

n, 2^n, nlgn, n^3, lgn, n-n^3+7n^5, n^2+lgn

lgn – logarithmic

n- linear

nlgn – linear multiplied by log

n^2+logn – By ignoring lower order terms we get n^2 (Polynomial)

n^3 – polynomial with higher degree than n^2

n-n^3+7n^5 - By ignoring lower order terms we get 7n^5 (Polynomial with higher degree than n^2, n^3)

solution: lgn ≤ n ≤ nlgn ≤ n^2+lgn ≤ n^3 ≤ n-n^3+7n^5

3b) Combine the following functions into your answer for part (a). Assume that 0 < ε < 1.

e^n, √n, 2^{n-1}, lglgn, $(\sqrt{2})^{lgn}$, lnn, $(lgn)^2$, n!, $n^{1+\varepsilon}$, 1

- $(\sqrt{2})^{lgn}$ => $(2^{1/2})^{lgn}$ => $(2)^{\log_2 \sqrt{n}}$ =>√n
  - √n, $(\sqrt{2})^{logn}$ has same asymptotic order.
- 2^{n-1} => $\frac{1}{2}2^n$
  - 2^{n-1},2^n has same asymptotic order
- lgn^2 => 2.lgn ; lnn => $log_2e(log_2n)$
  - lgn^2, lnn, lgn has same asymptotic order

- lgn, lnn, lgn^2 -> Same asymptotic order
- √n, $(\sqrt{2})^{logn}$ -> Same asymptotic order
- 2^{n-1}, 2^n – Same asymptotic order

Solution:

1 ≤ lglgn ≤lgn ≤ lnn ≤ $(lgn)^2$≤ {$(\sqrt{2})^{lgn}$,$\sqrt{n}$}≤ n ≤nlgn ≤ $n^{1+\varepsilon}$ ≤n^2+lgn ≤ n^3 ≤n-n^3+7n^5 ≤2^{n-1} ≤2^n ≤ e^n ≤ n!

4. Find the solution for each of the following recurrences, and then give tight bounds (i.e., in Θ(·)) for T (n).

4a) T(n)=T(n-1)+1/n with T(0)=0

T(n-1)=T(n-2)+1/(n-1)

T(n-2)=T(n-3)+1/(n-2)

T(n-3)=T(n-4)+1/(n-3)

T(n)=T(n-1)+1/n

T(n)=T(n-2)+[1/(n-1)] +[1/n]

T(n)=T(n-3)+[1/(n-2)] +[1/(n-1)]+[1/n]

T(n)=T(n-k)+ [1/n-(k-1)]+ [1/n-(k-2)].........+[1/(n-2)]+[1/(n-1)]+1/n

Put n-k=0

T(n)=T(0)+1+(1/2)+(1/3)+.........(1/n)

T(n)=T(0)+$\sum_{k=1}^{n} \frac{1}{n}$

T(n)=0+log(n)

T(n)=Θ(log(n)

4b) T(n)=T(n-1)+c^n with T(0)=1, where c>1

T(n-1)=T(n-2)+c^{n-1}

T(n-2)=T(n-3)+c^{n-2}

T(n-3)=T(n-4)+c^{n-3}

T(n)=T(n-1)+c^n

- ⇨ T(n-2)+c^{n-1}+c^n
- ⇨ T(n-3)+c^{n-2}+c^{n-1}+c^n

T(n)=T(n-k)+$c^{n-(k-1)}$+ $c^{n-(k-2)}$+.........+ c^{n-2}+c^{n-1}+c^n

Put n-k=0

T(n)= T(0)+c^1+c^2+......c^n

T(n)=1+ c^1+c^2+......c^n

$T(n)=c^0+c^1+c^2+\ldots\ldots c^n$

$T(n)=\frac{c^{n+1}-1}{c-1}$

g(n) is an asymptotic tight bound for f(n) if there exist constants k_1, k_2 and n_0 such that $0<=k_1.g(n)<=f(n)<=k_2.g(n)$ for $n>=n_0$

let $g(n) = c^n$

$0 <= k_1.c^n <=\frac{c^{n+1}-1}{c-1} <= k_2.c^n$

- ⇨ $0 <= (c-1).k_1.c^n <=c^{n+1}-1<=(c-1).\ k_2.c^n$
- ⇨ $0 <= ((c-1).k_1.c^n) +1<=c^{n+1}<=((c-1).\ k_2.c^n)+1$ this is true for $k_1=1$, $k_2=2$, $n_0=1$

Solution: $T(n)=\Theta(c^n)$

4c) T(n)=2T(n-1)+1 with T(0)=1

$T(n-1)=2T(n-2)+1$

$T(n-2)=2T(n-3)+1$

$T(n-3)=2T(n-4)+1$

$T(n)=2T(n-1)+1$

- ⇨ $2[2T(n-2)+1]+1$
- ⇨ $4T(n-2)+2+1$
- ⇨ $4[2T(n-3)+1]+2+1$
- ⇨ $8T(n-3)+4+2+1$
- ⇨ $2^k.T(n-k)+2^{k-1}+2^{k-2}+2^{k-3}+\ldots 4+2+1$

Put n-k=0

- ⇨ $2^kT(0)+\sum_{k=0}^{n-1} 2^k$

n=k

- ⇨ $2^n+(2^n-1)$
- ⇨ $2^{n+1}-1$

g(n) is an asymptotic tight bound for f(n) if there exist constants c_1, c_2 and n_0 such that $0<=c_1.g(n)<=f(n)<=c_2.g(n)$ for $n>=n_0$

let $g(n) = 2^n$

$0 <= c_1.2^n <= 2^{n+1}-1 <= c_2.2^n$

- ⇨ $0 <= c_1.2^n+1 <= 2^{n+1} <= c_2.2^n+1$
- ⇨ $0<= \frac{c_1.2^n+1}{2} <= 2^n <= \frac{c_2.2^n+1}{2}$ this is true for $c_1=1$, $c_2=4$, $n_0=1$

$T(n) = \Theta(2^n)$

---

5. Find the solutions of the following recurrence relations

5a) $f_n = f_n-1 + f_n-2$ with $f_0= 0$ and $f_1 = 1$.

$f_n= f_n-1+f_n-2$

$f_n- f_n-1+f_n-2=0$

The characteristic equation is

$x^2-x-1=0$

Roots are:

- ⇨ $x_1 =\frac{1+\sqrt{5}}{2}$; $x_2=\frac{1-\sqrt{5}}{2}$

x_1, x_2 are distinct real roots,

$f_n=c_1(x_1)^n+c_2(x_2)^n$

Substitute n=0

- ⇨ $f_0=c_1+c_2$
- ⇨ $c_1 = -c_2$

Substitute n=1

$f_1=c_1[\frac{1+\sqrt{5}}{2}]+c_2[\frac{1-\sqrt{5}}{2}]$

- ⇨ $c_1[\frac{1+\sqrt{5}}{2}]-c_1[\frac{1-\sqrt{5}}{2}]$
- ⇨ $c_1[\frac{1+\sqrt{5}}{2}]+c_1[\frac{\sqrt{5}-1}{2}]$
- ⇨ $c_1[\frac{2\sqrt{5}}{2}]$
- ⇨ $c_1.\sqrt{5}$

$f_1=c_1. \sqrt{5}$

$1=c_1. \sqrt{5}$ (Since $f_1=1$)

$c_1=1/\sqrt{5}$

$c_2= -1/\sqrt{5}$

$f_n=\frac{1}{\sqrt{5}}(x_1)^n-\frac{1}{\sqrt{5}}(x_2)^n$

$\mathbf{f_n=\frac{1}{\sqrt{5}}[\frac{1+\sqrt{5}}{2}]^n-\frac{1}{\sqrt{5}}[\frac{1-\sqrt{5}}{2}]^n}$

5b) $a_n=6a_{n-1}-9a_{n-2}$ with $a_0=1$ and $a_1=6$

$a_n-6a_{n-1}-9a_{n-2}=0$

The

$x^2-6x+9=0$

Roots are :

- ⇨ $x_1=3, x_2=3$

$a_n=c_1(x_1)^n+c_2(x_2)^n.n$

$a_n=c_1(3)^n+c_2(3)^n.n$

Substitute n=0 in $a_n=c_1(3)^n+c_2(3)^n.n$

$a_0=c_1(3)^0+c_2(3)^0.0$

$a_0=c_1+0$

$a_0=c_1$

- ⇨ $c_1=1$

substitute n=1 in $a_n=c_1(3)^n+c_2(3)^n.n$

$a_0=c_1(3)^1+c_2(3)^1.1$

$6=3c_1+3c_2$

$6=3(1)+3c_2$

$6=3(1+c_2)$

- ⇨ $c_2=1$

$a_n=c_1(3)^n+c_2(3)^n.n$

- ⇨ $c_1(3)^n+c_2(3)^n.n$
- ⇨ $(3)^n+(3)^n.n$

⇨ $3^n(1+n)$

$a_n=3^n(1+n)$

6.

T(n)= 3.T(n/2)+cn

T(n/2) = 3.T(n/4)+cn/2

T(n/4)=3.T(n/8)+cn/4

T(n)=3.T(n/2)+cn

T(n)=3[3.T(n/4)+cn/2]+cn

- ⇨ $3^2.T(n/4)+3cn/2+cn$
- ⇨ $3^2[3.T(n/8)+cn/4]+3cn/2+cn$
- ⇨ $3^3.T(n/8)+3^2cn/4+3cn/2+cn$
- ⇨ $3^kT(n/2^k)+\sum_{k=1}^{n}\frac{3^{k-1}.c.n}{2^{k-1}}$
- ⇨ $3^kT(n/2^k)+\sum_{k=1}^{n}\frac{3^{k-1}.c.n}{2^{k-1}}$
- ⇨ $3^kT(n/2^k)+c.n.\sum_{k=1}^{n}\frac{3^{k-1}}{2^{k-1}}$
- ⇨ $3^kT(n/2^k)+c.n.\sum_{k=1}^{n}(3/2)^{k-1}$
- ⇨ $3^kT(n/2^k)+c.n.\frac{1-(3/2)^{k-1}}{1-3/2}$
- ⇨ $3^kT(n/2^k)+c.n.\frac{(3/2)^{k-1}-1}{\left(\frac{3}{2}\right)-1}$
- ⇨ $3^kT(n/2^k)+c.n.\frac{(3/2)^{k-1}-1}{\left(\frac{3}{2}\right)-1}$
- ⇨ $3^kT(n/2^k)+c.n.\frac{(3/2)^{k-1}-1}{1/2}$
- ⇨ $3^kT(n/2^k)+2.c.n.((3/2)^{k-1}-1)$

  Plugging in k=logn we get,
- ⇨ $3^kT(n/2^k)+2.c.n.((3/2)^{k-1}-1)$
- ⇨ $3^{logn}T(1)+2.c.n.((3/2)^{logn-1}-1)$
- ⇨ $n^{log3}+4c.n.\left(\frac{\frac{3^{logn}}{n}-\frac{3}{2}}{3}\right)$
- ⇨ $n^{log3}+\frac{4}{3}.c(n^{log3}-(3n/2))$
- ⇨ **$\Theta(n^{log3})$**

Solution:

a) **General k^{th} term = $3^kT(n/2^k)+2.c.n.((3/2)^{k-1}-1)$**

b) k=logn (k should be logn)

---

7. Use the master theorem to give tight asymptotic bounds for the following recurrences.

a) $T(n)=2T(n/2)+\sqrt{n}$

It is in the form of $T(n)=aT(n/b)+f(n)$ where $a>=1$, $b>1$, $f(n)>0$

Here $a=2, b=2$, $\log_b a = \log_2 2=1$, $f(n)= \sqrt{n}$

Compare $n^{\log_2 2}$ with $f(n)= \sqrt{n}$

Case 1:

$f(n)=O(n^{(\log_2 2-\varepsilon)})$

$f(n)= O(n^{(1-\varepsilon)})$ for some $\varepsilon>0$

Solution: $T(n)= \Theta(n)$

b) $T(n)= 3T(n/2)+ cn$

It is in the form of $T(n)=aT(n/b)+f(n)$ where $a>=1$, $b>1$, $f(n)>0$

Here $a=3$, $b=2$, $f(n)= n$, $\log_2 3=1.58$

Compare $n^{\log_2 3}$ with $f(n) = n$

Case 1:

$f(n) = O(n^{\log_b a-\varepsilon})$

$f(n)= O(n^{1.58-\varepsilon})$

Solution: $T(n) = \Theta(n^{\log_2 3})$

c) $T(n)=27T(n/3)+cn^3$

It is in the form of $T(n)=aT(n/b)+f(n)$ where $a>=1$, $b>1$, $f(n)>0$

Here $a=27$, $b=3$, $f(n) = n^3$, $\log_3 27 = 3$

Compare $n^{\log_3 27}$ with $f(n)=n^3$

Case 2:

f(n) $= \Theta(n^{\log_b a}.logn)$

f(n)=$\Theta(n^{\log_3 27}.logn)$

T(n)= Θ(n^3logn)

d) T(n)=5T(n/4)+cn^2

It is in the form of T(n)=aT(n/b)+f(n) where a>=1, b>1,f(n)>0

Here, a=5, b=4, $\log_b a = \log_4 5 = 1.16$, f(n)=$n^2$

Compare f(n) with $n^{\log_b a}$

Case 3: f(n) = $\Omega(n^{\log_b a+\varepsilon})$ for some ε>0

f(n)= $\Omega(n^{\log_4 5+\varepsilon})$

Regularity condition:

a.f(n/b) <= k.f(n) for some k<1,

5f(n/4) <= k.n^2

5.n^2/16 <=k.n^2

5/16 <= k

⇨ k=0.31 is a solution (k<1)

Therefore, T(n)= **Θ**(n^2)

Solution: T(n)= Θ(n^2)

---

8. Use the master theorem to give tight asymptotic bounds for the following recurrences

8a) T(n) = 2T(n/4)+1

It is in the form of T(n) = aT(n/b)+f(n)

Where a=2, b=4, $\log_b a = \log_4 2 = ½$

Compare n^$\log_4 2$ with f(n) = 1

Case 1: f(n)= O(n^($\log_b a$-ε))

f(n)= $O(n^{(1/2-\varepsilon)})$

$f(n) = O(n^{\log_b a})$

- ⇨ $O(n^{\log_4 2})$
- ⇨ $O(\sqrt{n})$

Solution: $T(n) = \Theta(\sqrt{n})$

8b) $T(n) = 2T(n/4) + \sqrt{n}$

It is in the form of $T(n) = aT(n/b)+f(n)$ where as $a>=1$, $b>1$ and $f(n)>0$

Where $a=2$, $b=4$, $\log_b a = \log_4 2 = ½$

Compare $n^{\log_4 2}$ with $f(n) = \sqrt{n}$

Case 2 :

$f(n) = \Theta(n^{\log_b a}.logn)$

- ⇨ $\Theta((n^{\log_4 2}).logn)$
- ⇨ $\Theta(\sqrt{n}.logn)$

Solution: $T(n) = \Theta(\sqrt{n}logn)$

8c) $T(n) = 2T(n/4)+n$

It is in the form of $T(n) = aT(n/b)+f(n)$ where as $a>=1$, $b>1$ and $f(n)>0$

Here, $a=2$, $b=4$, $f(n) = n$, $\log_b a = \log_4 2 = ½$

Compare $n^{\log_4 2}$ with $f(n)=n$

Case 3: $f(n) = \Omega(n^{1+\varepsilon})$

Verify regularity condition:

$a.f(n/b) <= c.f(n)$ for some $c<1$

$2.f(n/4) <= c.n$

$=> n/2 <= c.n => c >=1$

Solution: $T(n) = \Theta(n)$

8d) $T(n) = 2T(n/4)+n^2$

It is in the form of $T(n) = aT(n/b)+f(n)$

Here, $a=2$, $b=4$, $f(n)= n^2$, $\log_b a= \log_4 2 => ½$

Compare $n^{\log_4 2}$ with $f(n)= n^2$

Case 3: $f(n) = \Omega(n^{1+\varepsilon})$

Verify regularity condition:

$a.f(n/b) <= c.f(n)$ for some $c<1$

$2.f(n/4) <= c.n^2$

$=> n^2/8 <= c.n^2$ for $c >=1$

Solution: $T(n)= \Theta(n^2)$
