Matrix:

▷ A rectangular array (arrangement) of mn numbers (real or complex) in m rows and n columns is called a matrix of order m by n written as

$m \times n$

▷ $A_{m \times n}$ matrix is usually written as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}_{m \times n}$$

→ This matrix is denoted in a simple form as

$$A = [a_{ij}]_{m \times n}$$

Where a_{ij} is the element in the ith row and j^{th} column.

Type of matrices:

1. Row Matrix:

A matrix which has only one row is called a row matrix or row vector i.e

$[1 \; -1 \; 2 \; 3]_{1 \times 4}$ $[2 \; 5]_{1 \times 2}$

2. Column Matrix: A Matrix which as only one column is called a column matrix or column vector.

i.e $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}_{3\times1}$ $\begin{bmatrix} -1 \\ 1 \\ 0 \\ 4 \end{bmatrix}_{4\times1}$

3. Square Matrix:

A matrix in which the number of rows is equal to the number of columns is called square matrix.

$\begin{bmatrix} 1 & -1 & 2 \\ 3 & 1 & 4 \\ 5 & 0 & 6 \end{bmatrix}_{3\times3}$ $\begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}_{2\times2}$

4. Null or Zero Matrix:

A Matrix in which each element is equal to zero is called a null matrix and is denoted by 0

i.e $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

5. Diagonal Matrix:

A square Matrix is called a diagonal matrix if all its non diagonal

i.e $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}$

6. Scalar Matrix:

A diagonal matrix whose all diagonal elements are equal is called a scalar matrix

i.e $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$

7. Identity or Unit Matrix:

A diagonal matrix whose all diagonal elements are unity (1) is called a unit or identity matrix and is denoted by I.

i.e $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

8. Upper Triangular Matrix:

A square matrix in which all the entries below the diagonal are zero is called an upper triangular matrix.

i.e $\begin{bmatrix} 4 & -3 & 2 \\ 0 & 3 & 5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}$

9. Lower Triangular Matrix:

A square matrix in which all the entries above the diagonal are zero is called an lower triangular matrix.

i.e

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & -6 & 0 \\ 2 & 3 & 3 \end{bmatrix}$$

10. Trace of a square Matrix:

The sum of all diagonal elements of a square matrix is called the trace of a matrix.

If $A = \begin{bmatrix} -1 & 2 & 7 & 0 \\ 3 & 5 & -8 & 4 \\ 1 & 2 & 7 & -3 \\ 4 & -2 & 1 & 0 \end{bmatrix}$ then

Trace of $A = -1+5+7+0 = 11$

→ Note: If $A = [a_{ij}]_{n \times n}$ then

Trance of $A = a_{11} + a_{22} + a_{33} + \cdots + a_{nn} = \sum_{i=1}^{n} a_{ii}$

11. Transpose of a matrix:

The matrix obtained by interchanging the rows and columns of a given matrix A is called the transpose of A and is denoted by A^T or A'

i.e

IF $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}_{2 \times 3}$ then $A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}_{3 \times 2}$

⇒ Properties:

I. $(A^T)^T = A$

II $(AB)^T = B^T A^T$

III $(A+B)^T = A^T + B^T$

IV $(KA)^T = KA^T$ where K is any scalar.

12. Determinant of a Matrix:

If A is a square Matrix then determinant of A is represented as ~~[A]~~ $|A|$ or $\det(A)$

i.e IF $A = \begin{bmatrix} 2 & 1 & -3 \\ 4 & 3 & 5 \\ 5 & 4 & 9 \end{bmatrix}$ then $\det(A) = \begin{bmatrix} 2 & 1 & -3 \\ 4 & 3 & 5 \\ 5 & 4 & 9 \end{bmatrix}$

13. Singular and Non-singular Matrix:

A square Matrix A is called singular if $\det(A) = 0$ and non singular or invertible if $\det(A) \neq 0$

i.e

Let $A = \begin{bmatrix} 2 & 1 & -3 \\ 4 & 3 & 5 \\ 5 & 4 & 9 \end{bmatrix}$ then $\det(A) = \begin{bmatrix} 2 & 1 & -3 \\ 4 & 3 & 5 \\ 5 & 4 & 9 \end{bmatrix}$

$\Rightarrow |A| = 2(27-20) - 1(36-25) - 3(16-15)$

$= 14 - 11 - 3 = 0$

Hence, A is a singular matrix.

SOME SPECIAL MATRIX

1. Symmetric Matrix:

A square matrix $A = a_{ij}]_{n \times n}$ is called Symmetric matrix

If $a_{ij} = a_{ji}$ $\forall i \& j$ OR $A = A^T$.

Example: $\begin{bmatrix} 7 & -3 \\ -3 & 5 \end{bmatrix}$ $\begin{bmatrix} 5 & 4 & 5 \\ 4 & 3 & 0 \\ 5 & 0 & 7 \end{bmatrix}$

2. Skew Symmetric Matrix:

A square matrix $A = [a_{ij}]_{n \times n}$ is called Skew Symmetric Matrix if $a_{ij} = -a_{ji}$ $\forall i \& j$ OR $A = -A$

$$2a_{ii} = 0$$

$$a_{ii} = 0 \quad \forall i$$

Thus the diagonal element of a skew symmetric matrix are all zero

i.e $\begin{bmatrix} 0 & 1 & 4 \\ -1 & 0 & -3 \\ -4 & 3 & 0 \end{bmatrix}$ $\begin{bmatrix} 0 & h & g \\ -h & 0 & f \\ -g & -f & 0 \end{bmatrix}$

3. orthogonal Matrix:

A square Matrix A is called orthogonal if

$A^T = A^{-1}$ or $AA^T = A^TA = 1$

→ Complex Matrix:

If all the element of a matrix are real numbers then it is called a real matrix

If at least one element of a matrix is a complex number $a+ib$ where a, b are real number's and $i = \sqrt{-1}$ then the matrix is i.e

$$\begin{vmatrix} 2+i & 0 & -5 \\ 3 & -i & 4+2i \end{vmatrix}$$

→ conjugate of a matrix:

The matrix obtained from any given matrix A by replacing its element by the corresponding complex conjugates is called the conjugate of A and is denoted by $\bar{A}$.

i.e

$$A = \begin{bmatrix} 2+3i & -4i \\ 5 & 7-2i \end{bmatrix} \Rightarrow \bar{A} = \begin{bmatrix} 2-3i & 4i \\ 5 & 7+2i \end{bmatrix}$$

→ Transposed conjugate of a matrix:

The transpose of the conjugate matrix of a matrix A is called the transposed conjugate of A and is denoted by A^{θ}

i.e $A^{\theta} = (\bar{A})^T = \overline{(A^T)}$

Hermitian Matrix:

A square matrix $A = [a_{ij}]$ is said to be Hermitian matrix if $a_{ij} = \overline{a_{ji}}$ $\forall i \& j$

or

i.e $A = A^{\theta}$ where $A^{\theta} = (\bar{A})^T$

$$\begin{bmatrix} 1 & 2-3i & 4+i \\ 2+3i & 3 & 1+2i \\ 4-i & 1-2i & 7 \end{bmatrix}$$

Skew Hermitian Matrix:

A square matrix $A = [a_{ij}]$ is said to be skew Hermitian matrix if

$$a_{ij} = -\overline{a_{ji}} \quad \forall i \& j$$

or

$A = -A^{\theta}$ where $A^{\theta} = (\bar{A})^T$

i.e. $\begin{bmatrix} 2i & 3+i & 5 \\ -3+i & 0 & -1-i \\ -5 & 1-i & -i \end{bmatrix}$

Note: In a skew Hermitian matrix, the diagonal element's are zero or Pulerly imaginary numbers.

→ Unitary Matrix: A square matrix A is said to be unitary if

$$A^{\theta} = A^{-1} \text{ or } A^{\theta}A = AA^{\theta} = I$$

where $A^{\theta} = (\bar{A})$

→ Sub matrix: A matrix obtained by deleting some rows or columns or both of a given matrix is called sub matrix.

→ Ex: Let $A = \begin{bmatrix} 1 & 5 & 6 & 7 \\ 8 & 9 & 10 & 5 \\ 3 & 4 & 5 & -1 \end{bmatrix}$

Then $\begin{bmatrix} 1 & 5 & 6 \\ 8 & 9 & 10 \end{bmatrix}$ is a sub matrix of A obtained by deleting third row and fourth column form A

→ Minor of a matrix: Let A be an $m \times n$ matrix. The determinant of square sub matrix of A is called a minor matrix. If the order of the square sub matrix is 't' then its determinet is called a minor of order t.

1. Minor of an element of a Determinant:

If $\det(A) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ then

Minor of $a_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$

Minor of $a_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$

Minor of $a_{32} = \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$

2. Cofactor of an element of a Determinant:

If $\det(A) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ then

cofactor of $a_{11} = (-1)^{1+1} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$

cofactor of $a_{22} = (-1)^{2+2} \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$

cofactor of $a_{32} = (-1)^{3+2} \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$

Adjoint of square Matrix:

→ The transpose of the matrix of the cofactors is called the adjoint of the matrix.

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

The matrix formed by the cofactors of the element of A is

$$C_A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

Then, $\text{adj } A = (C_A)^T = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$

→ Invertible Matrix:

If A is a square matrix and if a matrix B of the same order can be found such that $AB = BA = I$

then A is said to be invertible and B is called an inverse of A i.e $B = A^{-1}$.

→ Rank of Matrix:

A matrix is said to be of rank r if it satisfies the following properties.

I. There is at least one minor of order r which is not zero

II. Every minor of order $(r+1)$ is zero.

→ Rank of Matrix A is denoted by $\rho(A)$

⇒ There are three methods for finding the rank of a matrix

1. Rank of a matrix by Determine method
2. Rank of a matrix by row Echelon form.
3. Rank of a matrix by Normal form.

→ Note:

i. If A is zero matrix then $\rho(A)=0$.

ii. If A is not a zero matrix then $\rho(A)\geq 1$

iii. If A is non-singular $n\times n$ matrix then

$\rho(A)=n$

iv. $\rho(I_n)=n$, where I_n is $n\times n$ unit matrix

v. If A is $m\times n$ matrix then $\rho(A)\leq \min\{m,n\}$.

→ Rank of the matrix by Determinant Method:

→ Example: Find the rank $A=\begin{bmatrix}-1 & 0 & 6\\ 3 & 6 & 1\\ -5 & 1 & 3\end{bmatrix}$ of order 3x3

→ solution: Let the given matrix $A=\begin{bmatrix}-1 & 0 & 6\\ 3 & 6 & 1\\ -5 & 1 & 3\end{bmatrix}$

Then $\det A=\begin{vmatrix}-1 & 0 & 6\\ 3 & 6 & 1\\ -5 & 1 & 3\end{vmatrix}=-1(18-1)-0(9+5)+6(3+30)$

$=-17-0+6(33)=181\neq 0$

since the minor of order $3\neq 0$

$\rho(A)=3$

→ Example: Find the rank $A=\begin{bmatrix}1 & 2 & 3\\ 3 & 4 & 5\\ 4 & 5 & 6\end{bmatrix}$ of order 3x3

→ solution: Let the given matrix $A=\begin{bmatrix}1 & 2 & 3\\ 3 & 4 & 5\\ 4 & 5 & 6\end{bmatrix}$

Then $\det A = \begin{vmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{vmatrix} = 1(24-25) - 2(18-20) + 3(15-16)$

$= -1 + 4 - 3 = 0$

$\therefore \rho(A) \neq 3$ so it must be less than 3

Consider a minor of order 2 $= \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4 - 6 = -2 \neq 0$

Hence there is at least a minor of order 2 which is not zero $\therefore \rho(A) = 2$.

→ Example: Find the rank $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 8 & 7 & 0 & 5 \end{bmatrix}_{3\times 4}$

→ Solution: Here the matrix $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 8 & 7 & 0 & 5 \end{bmatrix}$ is of order 3×4

Its rank $\leq \min(3,4) = 3$

$\therefore$ Highest order of the minor will be 3

Let us consider the sub matrix $= \begin{bmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \\ 8 & 7 & 0 \end{bmatrix}$

Then the minor $= \begin{vmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \\ 8 & 7 & 0 \end{vmatrix} = 1(0-49) - 2(0-56) + (35-48)$

$= 24 \neq 0$

Hence the rank of the given matrix $A = 3$

Rank of the matrix by Echelon form

→ ECHELON FORM OF A MATRIX:

A matrix is said to be in Echelon form if it has the following properties.

(i) zero rows, if any are below any non-zero row

(ii) The number of zero's before the first non zero element in a row is less than the number of such zeros in the next row.

Note: Let A be any matrix using row operation Reducing into Echelon form then the number of non-zero rows in the echlon form is called the Rank A.

Ex: $\begin{bmatrix} 1 & -5 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 1 & 7 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\begin{bmatrix} 1 & 7 & 0 & 1 \\ 0 & 3 & 0 & 2 \\ 0 & 0 & 0 & 3 \end{bmatrix}$

→ Example: Reduce the matrix $A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$ or order 4×4 into echelon form and hence find the rank.

→ solution: Let the given matrix $A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$

Appling $R_2 \to R_2 - 2R_1$; $R_3 \to R_3 - 3R_1$, $R_4 \to R_4 -$

$$A \sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -8 & 3 \\ 0 & -4 & -11 & 5 \end{bmatrix}$$

Appling $R_2 \leftrightarrow R_3$.

$$A \sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -11 & 5 \end{bmatrix}$$

Appling $R_4 \to R_4 - R_2$

$$A \sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & -3 & 2 \\ 0 & 0 & -3 & 2 \end{bmatrix}$$

Appling $R_4 \to R_4 - R_3$

$$A \sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & -3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This is in Echelon form and the number of non zero rows is 3.

$\therefore$ Rank $(A) = n(A) = 3$

→ Example: Reduce the matrix $A = \begin{bmatrix} 1 & 4 & 3 & -2 & 1 \\ -2 & -3 & -1 & 4 & 3 \\ -1 & 6 & 7 & 2 & 9 \\ -3 & 3 & 6 & 6 & 12 \end{bmatrix}$ of order 4×5 in echelon form and hence find the rank.

→ solution: Let the given matrix $A = \begin{bmatrix} 1 & 4 & 3 & -2 & 1 \\ -2 & -3 & -1 & 4 & 1 \\ -1 & 6 & 7 & 2 & 3 \\ -3 & 3 & 6 & 6 & 9 \\ & & & & 12 \end{bmatrix}$

Appling $R_2 \to R_2 + 2R_1$, $R_3 \to R_3 + R_1$, $R_4 \to R_4 + 3R_1$

$$\begin{bmatrix} 1 & 4 & 3 & -2 & 1 \\ 0 & 5 & 5 & 0 & 5 \\ 0 & 10 & 10 & 0 & 10 \\ 0 & 15 & 15 & 0 & 15 \end{bmatrix}$$

Appling $\frac{R_2}{5}$, $\frac{R_3}{10}$, $\frac{R_4}{15}$

$$A \sim \begin{bmatrix} 1 & 4 & 3 & -2 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix}$$

Appling $R_3 \to R_3 - R_2$, $R_4 \to R_4 - R_2$

$$A \sim \begin{bmatrix} 1 & 4 & 3 & -2 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

This is in Echelon form and the number of nonzero rows is 2

$\therefore$ Rank $(A) = \rho(A) = 2$

→ solution: Let the given matrix $A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 2 & 4 & 3 & 0 \\ 1 & 0 & 2 & -8 \end{bmatrix}$

Appling $R_2 \to R_2 + 2R_1$, $R_3 \to R_3 - R_1$

$$A \sim \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 8 & 5 & 0 \\ 0 & -2 & 1 & -8 \end{bmatrix}$$

Appling $R_3 \to 4R_3 + R_2$

$$A \sim \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 8 & 5 & 0 \\ 0 & 0 & 9 & -32 \end{bmatrix}$$

This is in Echelon form and the number of non zero rows is 3 $\therefore$ Rank $(A) = \rho(A) = 3$

→ Example: Find value of k such that the rank of $A = \begin{bmatrix} 1 & 1 & -1 & 1 \\ 1 & -1 & k & -1 \\ 3 & 1 & 0 & 1 \end{bmatrix}$ is 2

→ solution: Let the given matrix $A = \begin{bmatrix} 1 & 1 & -1 & 1 \\ 1 & -1 & k & -1 \\ 3 & 1 & 0 & 1 \end{bmatrix}$

Appling $R_2 \to R_2 - R_1$, $R_3 \to R_3 - 3R_1$

$$A \sim \begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & -2 & k+1 & -2 \\ 0 & -2 & 3 & -2 \end{bmatrix}$$

Appling $R_3 \to R_3 - R_2$

$$A \sim \begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & -2 & k+1 & -2 \\ 0 & 0 & 2-k & 0 \end{bmatrix}$$

Since the rank of A is 2 there will be only two non-zero rows

$\therefore$ Third row must be a zero row

$\Rightarrow 2-k=0 \Rightarrow k=2$

→ Example: Find the value of k such that the rank of $A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & k & 0 \end{bmatrix}$ is 2

→ solution: Let the given matrix $A = \begin{bmatrix} 0 & 1 & -3 & \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & k & 0 \end{bmatrix}$

Appling $R_1 \leftrightarrow R_2$

$$A \sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & k & 0 \end{bmatrix}$$

Appling $R_3 \to R_3 - 3R_1$, $R_4 \to R_4 - R_1$

$$A \sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & k-1 & -1 \end{bmatrix}$$

For the rank A to be equal to 2 we must have 3 rows identical

$\therefore k-1=-3 \Rightarrow k=-2$.

Rank of the matrix by Normal Form (Canonical Form)

→ Reduce into Normal Form:

- Let A be given matrix of order $m \times n$
- Using elementary rows and columns operations change matrix A into the following form.
- The form I_r. $[I_r, 0]$, $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$

The above form is called Normal form or canonical form of the matrix where I_r is the r rowed unit Matrix.

Ex: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Example: Reduce the matrix $A = \begin{bmatrix} 0 & 1 & 2 & -2 \\ 4 & 0 & 2 & 6 \\ 2 & 1 & 3 & 1 \end{bmatrix}$ into Normal form and hence find the rank

→ Solution: let the given matrix $A = \begin{bmatrix} 0 & 1 & 2 & -2 \\ 4 & 0 & 2 & 6 \\ 2 & 1 & 3 & 1 \end{bmatrix}$

Appling $C_1 \leftrightarrow C_2$

$A \sim \begin{bmatrix} 1 & 0 & 2 & -2 \\ 0 & 4 & 2 & 6 \\ 1 & 2 & 3 & 1 \end{bmatrix}$

Appling $R_3 \to R_3 - R_1$

$$A \sim \begin{bmatrix} 1 & 0 & 2 & -2 \\ 0 & 4 & 2 & 6 \\ 0 & 2 & 1 & 3 \end{bmatrix}$$

Appling $C_3 \to C_3 - 2C_1$, $C_4 \to C_4 + 2C_1$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & 2 & 6 \\ 0 & 2 & 1 & 3 \end{bmatrix}$$

Appling $R_3 \to 2R_3 - R_2$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & 2 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Appling $C_3 \to 2C_3 - C_2$, $C_4 \to 4C_4 - 6C_2$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Appling $C_2 \to \frac{C_2}{4}$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This in the normal form of $\begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$.

Hence the Rank $(A) = \rho(A) = 2$

→ Example: Reduce the matrix $A = \begin{bmatrix} 1 & -1 & 0 & 3 \\ 4 & 2 & 0 & 2 \\ 2 & -2 & 0 & 6 \\ 1 & -2 & 1 & 2 \end{bmatrix}$ into Normal form and hence find the rank

solution: Let the given matrix $A = \begin{bmatrix} 1 & -1 & 0 & 3 \\ 4 & 2 & 0 & 2 \\ 2 & -2 & 0 & 6 \\ 1 & -2 & 1 & 2 \end{bmatrix}$

Appling $R_2 \to R_2 - 4R_1$, $R_3 \to R_3 - 2R_1$, $R_4 \to R_4 - R_1$

$$A \sim \begin{bmatrix} 1 & -1 & 0 & 3 \\ 0 & 6 & 0 & -10 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & -1 \end{bmatrix}$$

Appling $C_2 \to C_2 + C_1$, $C_4 \to C_4 - 3C_1$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 6 & 0 & -10 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & -1 \end{bmatrix}$$

Appling $R_3 \to 6R_3 + R_2$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 6 & 0 & -10 \\ 0 & 0 & 6 & -16 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Appling $C_2 \to \frac{C_2}{6}$, $C_3 \to \frac{C_3}{6}$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -10 \\ 0 & 0 & 1 & -16 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Appling $C_4 \to C_4 + 10C_2$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -16 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Appling $C_4 \to C_4 + 16C_3$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This is the Normal form of $\begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}$

Hence the Rank $(A) = \rho(A) = 3$

→ Example:- Reduce the matrix $A = \begin{bmatrix} 1 & -1 & 3 & -2 \\ 0 & 1 & 2 & 3 \\ 5 & 1 & -1 & 2 \\ 4 & 3 & 2 & 1 \end{bmatrix}$ into Normal form and hence find the rank

solution: Let the given Matrix $A = \begin{bmatrix} 1 & -1 & 3 & -2 \\ 0 & 1 & 2 & 3 \\ 5 & 1 & -1 & 2 \\ 4 & 3 & 2 & 1 \end{bmatrix}$

Appling $R_3 \to R_3 - 5R_1$, $R_4 \to R_4 - 4R_1$

$$A \sim \begin{bmatrix} 1 & -1 & 3 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 6 & -16 & 12 \\ 0 & 7 & -10 & 9 \end{bmatrix}$$

Appling $C_2 \to C_2 + C_1$, $C_3 \to C_3 - 3C_1$, $C_4 \to C_4 + 2$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 6 & -16 & 12 \\ 0 & 7 & -10 & 9 \end{bmatrix}$$

Appling $R_3 \to R_3 - 6R_2$, $R_4 \to R_4 - 7R_2$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & -28 & -6 \\ 0 & 0 & -24 & -12 \end{bmatrix}$$

Appling $C_3 \to C_3 - 2C_2$, $C_4 \to C_4 - 3C_2$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -28 & -6 \\ 0 & 0 & -24 & -12 \end{bmatrix}$$

Appling $R_3 \to \frac{R_3}{-2}$, $R_4 \to \frac{R_4}{-12}$

$$A \sim \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 14 & 3 \\ 0 & 0 & 2 & 1 \end{vmatrix}$$

Appling $R_3 \leftrightarrow R_4$

$$A \sim \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 14 & 3 \end{vmatrix}$$

Appling $R_4 \to R_4 - 7R_3$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & -4 \end{bmatrix}$$

Appling $C_4 \to 2C_4 - C_3$

$$A \sim \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -4 \end{vmatrix}$$

Appling $C_3 \to \frac{C_3}{2}$, $C_4 \to \frac{C_4}{-4}$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

This is the normal form of $I_r = I_4$

Hence the Rank $(A) = \rho(A) = 4$.

→ Note:

If A is an $m \times n$ matrix of rank r, there exists non singular matrices P and Q such that

$$PAQ = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

→ Elementary Matrix:

It is a matrix obtained from a unit matrix by a single elementary transformation

For example $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

are the elementary matrices obtained from I_3 by applying elementary operations $C_1 \leftrightarrow C_2$, $R_3 \to 2R_3$ and $R_1 \to R_1 + 2R_2$ respectively.

→ Example: Find the non-singular matrices P and Q such that the normal form of A is PAQ where $A = \begin{bmatrix} 1 & 3 & 6 & -1 \\ 1 & 4 & 5 & 1 \\ 1 & 5 & 4 & 3 \end{bmatrix}_{3\times 4}$. Hence find the rank

solution: We can find two non-singular matrices P and Q such that $PAQ = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ For this we use following procedure

we write $A = I_m A I_n = I_3 A I_4$

i.e $\begin{bmatrix} 1 & 3 & 6 & -1 \\ 1 & 4 & 5 & 1 \\ 1 & 5 & 4 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

We will apply elementary row operations and column on A in L.H.S to reduce to a normal form $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$.

Every row operations will also be applied

to prefactor I_3 of the product on R.H.S and every column operations will also be applied to post factor I_4 of the product on R.H.S.

$$\begin{bmatrix} 1 & 3 & 6 & -1 \\ 1 & 4 & 5 & 1 \\ 1 & 5 & 4 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Appling $R_2 \to R_2$, $R_3 \to R_3 - R_1$

$$\begin{bmatrix} 1 & 3 & 6 & -1 \\ 0 & 1 & -1 & 2 \\ 0 & 2 & -2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Appling $C_2 \to C_2 - 3C_1$, $C_3 \to C_3 - 6C_1$, $C_4 \to C_4 + C_1$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 2 \\ 0 & 2 & -2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -3 & -6 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Applying $R_3 \to R_3 - 2R_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -3 & -6 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Appling $C_3 \to C_3 + C_2$, $C_4 \to C_4 - 2C_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -3 & -9 & 7 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix} = PAQ$$

where $P = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix}$ and $Q = \begin{bmatrix} 1 & -3 & -9 & 7 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Since $A \sim \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$ therefore, Rank of A is 2

→ Example: Find the non-singular matrices P and Q such that the normal of A is PAQ where $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix}_{3\times 3}$ Hence find the rank

solution: We can find two non-singular matrices P and Q such that $PAQ = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ For this we use following procedure

we write $A = I_m A I_n = I_3 A I_3$

i.e $\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

we will apply elementary row operations and column on A in L.H.S to reduce to a normal form $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$

Every row operations will also be applied to prefactor I_3 of the product on R.H.S

and every column operations will also be applied to fast-factor I_4 of the product on R.H.S

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Applying $R_2 \to R_2 - R_1$

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Applying $C_2 \to C_2 - C_1$, $C_3 \to C_3 - 2C_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Applying $R_3 \to R_3 + R_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Applying $C_3 \to C_3 - C_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix} = PAQ$$

where $P = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}$ and $Q = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

since $A \sim \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$ therefore Rank of A is 2

→ Invertible Matrix

If A is a square matrix and if a matrix B of the same order can be found such that

$$AB = BA = I$$

then A is said to be Invertible and B is called an inverse of A i.e $B = A^{-1}$

→ Inverse of a matrix by Elementary Transformation:

→ Gauss - Jordan Method:

Let A be a given non-singular matrix of order n. Let I_n be the unit matrix of order n.

To find A^{-1} take the matrix $[A : I_n]$ and reduce it with the help of a series of row operations to the form $[I_n : B]$, then $B = A^{-1}$

→ Example: Find the inverse of $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$ by using Gauss - Jordan Method.

→ solution: let the given matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$

By using Gauss - Jordan Method

Let $[A \mid I_3] = \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right]$

Appling $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - R_1$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & 1 & -2 & 0 \\ 0 & -2 & 5 & 1 & -1 & 0 \end{array}\right]$$

Applying $R_1 \to R_1 - 2R_2$, $R_3 \to R_3 + 2R_2$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 9 & 1 & 5 & -2 \\ 0 & 1 & -3 & 1 & -2 & 0 \\ 0 & 0 & -1 & 1 & -5 & 2 \end{array}\right]$$

Applying $R_3 \to R_3(-1)$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 9 & 1 & 5 & -2 \\ 0 & 1 & -3 & 1 & -5 & 0 \\ 0 & 0 & 1 & 1 & 5 & -2 \end{array}\right]$$

Applying $R_1 \to R_1 - 9R_3$, $R_2 \to R_2 + 3R_3$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -4 & 0 & 16 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$$

Hence $[A \mid I_3]$ reduce $[I_3 \mid B]$ then $B = A^{-1}$

$$B = A^{-1} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}$$

→ Example: Find the inverse of $A = \begin{bmatrix} 1 & 3 & 4 \\ -2 & 1 & 3 \\ 0 & 2 & -1 \end{bmatrix}$ by using Gauss-Jordan Method.

→ solution: Let the given matrix $A = \begin{bmatrix} 1 & 3 & 4 \\ -2 & 1 & 3 \\ 0 & 2 & -1 \end{bmatrix}$

By using Gauss-Jordan Method

let $[A|I_3] = \left[\begin{array}{ccc|ccc} 1 & 3 & 4 & 1 & 0 & 0 \\ -2 & 1 & 3 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right]$

Applying $R_2 \to R_2 + 2R_1$

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 4 & 1 & 0 & 0 \\ 0 & 7 & 11 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right]$$

Applying $R_2 \to R_2\left(\frac{1}{7}\right)$

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 4 & 1 & 0 & 0 \\ 0 & 1 & \frac{11}{7} & \frac{2}{7} & \frac{1}{7} & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right]$$

Applying $R_1 \to R_1 - 3R_2$, $R_3 \to 2R_2$.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & \frac{-5}{7} & \frac{1}{7} & \frac{-3}{7} & 0 \\ 0 & 1 & \frac{11}{7} & \frac{2}{7} & \frac{1}{7} & 0 \\ 0 & 0 & \frac{-29}{7} & \frac{-4}{7} & \frac{-2}{7} & 1 \end{array}\right]$$

Applying $R_2 \to R_3\left(\frac{-7}{29}\right)$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & \frac{-5}{7} & \frac{1}{7} & \frac{-5}{7} & 0 \\ 0 & 1 & \frac{11}{7} & \frac{2}{7} & \frac{1}{7} & 0 \\ 0 & 0 & 1 & \frac{4}{29} & \frac{2}{29} & \frac{-7}{29} \end{array}\right]$$

Applying $R_1 \to R_1 + \frac{5}{7} R_3$. $R_2 \to R_2 - \frac{11}{7} R_3$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{7}{29} & \frac{-11}{29} & \frac{-5}{29} \\ 0 & 1 & 0 & \frac{2}{29} & \frac{1}{29} & \frac{11}{29} \\ 0 & 0 & 1 & \frac{4}{29} & \frac{2}{29} & \frac{-7}{29} \end{array}\right]$$

Hence $[A|I_3]$ reduce to $[I_3|B]$ then $B = A^{-1}$

$$B = A^{-1} \begin{bmatrix} \frac{7}{29} & \frac{-11}{29} & \frac{-5}{29} \\ \frac{2}{29} & \frac{1}{29} & \frac{11}{29} \\ \frac{4}{29} & \frac{2}{29} & \frac{-7}{29} \end{bmatrix}$$

→ system of Non-Homogenenous Linear Equations:

A system of m linear equations in n unknowns can be written as

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

The above system can be written in a matrix form as

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{12} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

or simply $A \quad x = \quad B$

where $A = \left[\begin{array}{cccc:c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$

is called coefficient matrix of order $m \times n$

$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ and $B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$ is any vector of order $n \times 1$ & $m \times 1$

respectively.

$$[A : B] = \left[\begin{array}{cccc:c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

is the argumented matrix of the given system of linear equations.

→ solution of system of Linear equations:

For a system of m linear equations in n unknows, there are three possibilities of the solutions of the system

1. The system has unique solution
ii. The system has infinite solutions;
iii. The solution has no solution.

→ when the systems of linear Equations has one or more solutions, the system is said to be consistent otherwise it is inconsistent

→ For Non-Homogeneous systems:

The system is consistent i.e. it has a solution (unique or infinite) if and only if
rank of A = rank of [A|B].

i) If rank of A = rank of [A|B] = r < number of unknowns the system is consistent, but there exists infinite number of solutions. Giving arbitrary values to n−r of the unknowns we may express the other r unknowns in terms of these.

ii) If rank of A = rank of [A|B] = r = n the system has A unique solution.

iii) If rank of A ≠ rank of [A|B] the system is inconsistent. It has no solution

To obtain solutions set (n−r) variables any arbitrary value and solve for the remaining unknowns.

Thus irrespective of m<n or m=n, or m>n we have decided about the existence and uniquenesss of a system of linear

simultaneous equation $Ax = B$

-> Example: Find whether the following equations are consistent or not. If it is consistent solve them $x+y+z=4$, $2x+5y-2z=3$, $x+7y-7z=5$

-> Solution:

The given Equations can be written in the matrix form $Ax=B$ we have

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 5 & -2 \\ 1 & 7 & -7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix} \quad \ldots (1)$$

consider Argument matrix $[A|B] = \begin{bmatrix} 1 & 1 & 1 & 4 \\ 2 & 5 & -2 & 3 \\ 1 & 7 & -7 & 5 \end{bmatrix}$

Applying $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - R_1$

$$[A|B] \sim \begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & 3 & -4 & -5 \\ 0 & 6 & -8 & 1 \end{bmatrix}$$

Applying $R_3 \to R_3 - 2R_2$

$$[A|B] \sim \begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & 3 & -4 & -5 \\ 0 & 0 & 0 & 11 \end{bmatrix}$$

The number of non zero rows in $[A|B]$ is 3

$\therefore$ The rank of argument matrix $[A|B] = 3$

and the number of non zero rows in $[A]$ is 2

$$\rho(A) = 2$$

Hence we have $\rho(A) \neq \rho[A|B]$

$\therefore$ The given system is not consistent.

→ Example: Find whether the following equations are consistent or not. If it is consistent solve them $x+y+2z=4$, $2x-y+3z=9$, $3x-y-z=2$.

→ solution

The given equation can be given written in the matrix form $AX=B$, we have

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & -1 & 3 \\ 3 & -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 2 \end{bmatrix} \quad \ldots\ldots (1)$$

Applying $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$

$$[A|B] \sim \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & -3 & -1 & 1 \\ 0 & -4 & -7 & -10 \end{bmatrix}$$

Applying $R_3 \to 3R_3 - 4R_2$

$$[A|B] \sim \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & -3 & -1 & 1 \\ 0 & -4 & -7 & -10 \end{bmatrix}$$

The number of non zero rows in $[A|B]$ is 3

$\therefore$ The rank of argument matrix $[A|B] = 3$

and the number of non zero rows in $[A]$ is 3

$$\rho(A) = 3$$

Hence we have $\rho(A) = \rho[A|B] = n$

$\therefore$ The given system is consistent. so it unique

Then $\begin{bmatrix} 1 & 1 & 2 \\ 0 & -3 & -1 \\ 0 & 0 & -17 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ -34 \end{bmatrix}$

write into Equation form

$x+y+2z=4$ --(a), $-3y-z=1$ --(b),

$-17z=-34$ --(c) from (c) $z=2$

Substituting $z=2$ in (b) $\Rightarrow -3y-2=1 \Rightarrow y=-1$

substituting $z=2, y=-1$ in (a) $\Rightarrow x-1+4=4 \Rightarrow x=$

$\therefore x=1, y=1, z=2$ is the solution

$\Rightarrow$ Example: Find whether the following Equation are consistent or not. If it is consistent solve them $x+y+2z=2$, $3x-2y-z=5$,

$2x-5y+3z=-4$, $x+4y+6z=0$

$\Rightarrow$ solution

The given equations can be written in the matrix form $Ax=B$, we have

$$\begin{bmatrix} 1 & 1 & 2 \\ 3 & -2 & -1 \\ 2 & -5 & 3 \\ 1 & 4 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ -4 \\ 0 \end{bmatrix} \quad \ldots\ldots (1)$$

consider Argument matrix $[A|B] = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 3 & -2 & -1 & 5 \\ 2 & -5 & 3 & -4 \\ 1 & 4 & 6 & 0 \end{bmatrix}$

Applying $R_2 \to R_2 - 3R_1$, $R_3 \to R_3 - 2R_1$, $R_4 \to R_4 - R_1$

$$[A|B] \sim \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & -8 & -7 & -1 \\ 0 & -9 & -1 & -8 \\ 0 & 2 & 4 & -2 \end{bmatrix}$$

Applying $R_3 \to 8R_3 - 9R_2$, $R_4 \to 4R_4 + R_2$

$$[A|B] \sim \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & -8 & -7 & -1 \\ 0 & 0 & 55 & -55 \\ 0 & 0 & 9 & -9 \end{bmatrix}$$

Applying $R_3 \to \frac{1}{55} R_3$, $R_4 \to \frac{1}{9} R_4$

$$[A|B] \sim \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & -8 & -7 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

Applying $R_4 \to R_4 - R_3$

$$[A|B] \sim \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & -8 & -7 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The number of non zero rows in $[A|B]$ is 3

$\therefore$ The rank of argument matrix $[A|B] = 3$

and the number of non zero rows in $[A]$ is 3

$$\rho(A) = 3$$

Hence we have $\rho(A) = \rho[A|B] = n = 3$

$\therefore$ The given system is consistent, so it has unique solution.

Then $\begin{bmatrix} 1 & 1 & 2 \\ 0 & -8 & -7 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -1 \\ 0 \end{bmatrix}$

write into equation form

$x+y+2z=2$ -- (a), $-8y-7z=-1$ -- (b),

$z=-1$ -- (c). from (c) $z=-1$

substituting $z=-1$ in (b) $\Rightarrow -8y+7=-1 \Rightarrow y=1$

substituting $z=-1$ $y=1$ in (a) $\Rightarrow x+1-2=2 \Rightarrow x=3$

$\therefore x=3 \cdot y=1, z=-1$ is the solution.

Example: Find whether the following equations are consistent or not. If it is consistent solve them $3x-y+4z=3, x+2y-3z=-2,$

$6x+5y-5z=-3$

→ solution:

The given equations can be written in the matrix form $AX=B$, we have

$$\begin{bmatrix} 3 & -1 & 4 \\ 1 & 2 & -3 \\ 6 & 5 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ -3 \end{bmatrix} \quad \ldots (1)$$

consider Argument matrix $[A|B] = \begin{bmatrix} 3 & -1 & 4 & 3 \\ 1 & 2 & -3 & -2 \\ 6 & 5 & -5 & -3 \end{bmatrix}$

Applying $R_2 \to R_2 - R_1$, $R_3 \to R_3 - 2R_1$

$$[A|B] \sim \begin{bmatrix} 3 & -1 & 4 & 3 \\ 0 & 7 & -13 & -9 \\ 0 & 7 & -13 & -9 \end{bmatrix}$$

Applying $R_3 \to R_3 - R_2$

$$[A|B] \sim \begin{bmatrix} 3 & -1 & 4 & 3 \\ 0 & 7 & -13 & -9 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The number of non zero rows in $[A|B]$ is 2

$\therefore$ The rank of argument matrix $[A|B] = 2$

and the number of non zero rows in $[A]$ is 2

$$\rho(A) = 2$$

Hence we have $\rho(A) = \rho[A|B] \neq n$

$\therefore$ The given system is consistent and it has infinite number of solution.

To obtain solutions, set $(n-r)$ variables any arbitrary value and solve for the remaining unknowns Here $n-r = 3-2 = 1$

Then $\begin{bmatrix} 3 & -1 & 4 \\ 0 & 7 & -13 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -9 \\ 0 \end{bmatrix}$

write into equation form

$3x - y + 4z = 3$ --(a), $7y - 13z = -9$ --(b)

let $z=k$

substituting $z=k$ in (b) $\Rightarrow 7y-13k=-9 \Rightarrow y=\frac{13k-9}{7}$

substituting $z=k$ $y=\frac{13k-9}{7}$ in (a) $\Rightarrow 3x-\frac{13k-9}{7}+4k=3$

$\Rightarrow x=\frac{4-5k}{7}$

$\therefore x=\frac{4-5k}{7}, y=\frac{13k-9}{7}, z=k$

This will given infinte number of the solution

→ Example: Find whether the following Equations are consistent or not. If it is consistent solve them.

$5x+3y+7z=4, 3x+26y+2z=9,$

$7x+2y+10z=5$

→ solution:

The given Equations can be written in the matrix form $AX=B$, we have

$$\begin{bmatrix} 5 & 3 & 7 \\ 3 & 26 & 2 \\ 7 & 2 & 10 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 4 \\ 9 \\ 5 \end{bmatrix} \quad \ldots (1)$$

consider Argument matrix $[A|B]=\begin{bmatrix} 5 & 3 & 7 & 4 \\ 3 & 26 & 2 & 9 \\ 7 & 2 & 10 & 5 \end{bmatrix}$

Applying $R_2 \to 5R_2-3R_1, R_3 \to 5R_3-7R_1$

$$[A|B] \sim \begin{bmatrix} 5 & 3 & 7 & 4 \\ 0 & 121 & -11 & 33 \\ 0 & -11 & 1 & -3 \end{bmatrix}$$

Applying $R_3 \to \frac{R_3}{11}$

$$[A|B] \sim \begin{bmatrix} 5 & 3 & 7 & 4 \\ 0 & 11 & -1 & 3 \\ 0 & -11 & 1 & -3 \end{bmatrix}$$

Applying $R_2 \to R_3 + R_2$

$$[A|B] \sim \begin{bmatrix} 5 & 3 & 7 & 4 \\ 0 & 11 & -1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The number of non zero rows in $[A|B]$ is 2

$\therefore$ The rank of argument matrix $[A|B] = 2$

and the number of non zero rows in $[A]$ is 2

$$\rho(A) = 2$$

Hence we have $\rho(A) = \rho[A|B] \neq n$

$\therefore$ The given system is consistent and it has infinite number of solution.

To obtain solutions, set $(n-r)$ variables any arbitary value and solve for the remaining unknowns. Here $n-r = 3-2 = 1$

Then $\begin{bmatrix} 5 & 3 & 7 \\ 0 & 11 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix}$

write into equation form

$5x + 3y + 7z = 4$ --(a), $11y - z = 3$ --(b)

let $z = k$

substituting $z = k$ in (b) $\Rightarrow 11y - k = 3 \Rightarrow y = \frac{3+k}{11}$

substituting $z = k, y = \frac{3+k}{11}$ in (a) $\Rightarrow 5x + 3\frac{3+k}{11} + 7k$

$= 4 \Rightarrow x = \frac{53 - 80k}{55}$

$\therefore x = \frac{53-80k}{55}, y = \frac{3+k}{11}, z = k$

This will give infinite number of the solution

→ Example: Find the values of a and b for which the equations $x + y + z = 3$, $x + 2y + 2z = 6$, $x + 2y + az = b$.

(i) No solution ii) A unique solution (iii) infinite number of solutions.

→ solution

The given equations can be given written in the matrix form $AX = B$, we have

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & a \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ b \end{bmatrix} \quad \text{---(1)}$$

consider Argument matrix $[A|B] = \begin{bmatrix} 1 & 1 & 1 & 3 \\ 1 & 2 & 3 & 6 \\ 1 & 2 & a & b \end{bmatrix}$

Applying $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$

$$[A|B] \sim \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & 3 \\ 0 & 8 & a-1 & b-3 \end{bmatrix}$$

Applying $R_3 \to R_3 - 8R_2$

$$[A|B] \sim \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & a-17 & b-27 \end{bmatrix}$$

Case I: Let $a = 17$, $b \neq 27$

The number of non zero rows in $[A|B]$

$\therefore$ The rank of argument matrix $\rho[A|B] =$

and the number of non zero rows in $[A]$

$\rho(A) = 2$

Hence we have $\rho(A) \neq \rho[A|B]$

$\therefore$ The system is non consistent and it has no solution.

Case II: Let $a \neq 17$, $b \neq 27$

The number of non zero rows in $[A|B]$

$\therefore$ The rank of argument matrix $\rho[A|B] = 3$

and the number of non rows in $[A]$ is 3

$\rho(A) = 3$

Hence we have $\rho(A) = \rho[A|B] = n$

$\therefore$ The system is consistent and it has a unique solution.

Case III : Let $a = 17, b = 27$

The number of non zero rows in $[A|B]$ is 2

$\therefore$ The rank of argument matrix $\rho[A|B] = 2$

and the number of non zero rows in $[A]$ is 2

$$\rho(A) = 2$$

Hence we have $\rho(A) = \rho[A|B] \neq n$

$\therefore$ The system is consistent and it has infinite Number of solution.

$\rightarrow$ System of Homogeneous Linear Equations:

A system of m linear equations in n unknows can be written as

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = 0$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = 0$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = 0$$

The above system can be written in a matrix form as

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

or simply $A \cdot X = 0$

where $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$

is called coefficient matrix of order $m \times n$

$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ and $O = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ is any vector of order $n \times 1$ & $m \times 1$

respectively

$$[A:O] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

is the argumented matrix and argumented matrix is always is Equal to matrix A.

→ For Homogeneous system:

The system is always consistent i.e has a solution (unique or infinite rank of = rank of [A|B]

i) If rank of A = rank of [A|B] = $r = n$ the system has A unique solution and it has trivial solution (zero solution)

(ii) If rank of A = rank of [A|B] = $r <$ number of unknowns, the solution consistent

but there exists infinite number of solutions. Giving arbitrary values to $n-r$ of the unknowns, we may express the order r unknowns in terms of these.

To obtain solutions, set $(n-r)$ varibles any abitrary value and solve for the remaining unknowns.

→ Example: solve the system of equations

$x+2z-2w=0$, $2x-y-w=0$, $x+2z-w=0$

$4x-y+3z-w=0$.

→ solution

The given equations can be written in the matrix form $Ax=0$ we have

$$\begin{bmatrix} 1 & 0 & 2 & -2 \\ 2 & -1 & 0 & -1 \\ 1 & 0 & 2 & -1 \\ 4 & -1 & 3 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \ldots (1)$$

Consider Argument matrix $[A|B] = [A|0] = [A] =$

$$\begin{bmatrix} 1 & 0 & 2 & -2 \\ 2 & -1 & 0 & -1 \\ 1 & 0 & 2 & -1 \\ 4 & -1 & 3 & -1 \end{bmatrix}$$

Applying $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - R_1$, $R_4 \to R_4 - 4R_1$

$$[A] \sim \begin{bmatrix} 1 & 0 & 2 & -2 \\ 0 & -1 & -4 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & -1 & -5 & 7 \end{bmatrix}$$

Applying $R_3 \leftrightarrow R_4$

$$[A] \sim \begin{bmatrix} 1 & 0 & 2 & -2 \\ 0 & -1 & -4 & 3 \\ 0 & 0 & -5 & 7 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The number of non zero rows in $[A|O]$ is

$\therefore$ The rank of argument matrix $[A|O] = 4$

and the number of non zero rows in $[A]$ is

$$\rho(A) = 4$$

Hence we have $\rho(A) = \rho[A|B] = n = 4$

$\therefore$ The given system is consistent

So it has unique solution and solution is trivial solution.

i.e $x=0, y=0, z=0, w=0$

$\Rightarrow$ Example: solve the ~~following~~ system of Equations

$x+y-3z+2w=0, 2x-y+2z-3w=0$

$2x-2y+z-4w=0, -4x+y+w=0$

$\rightarrow$ solution

The given equations can be written in the matrix form $AX=0$, we have

$$\begin{bmatrix} 1 & 1 & -3 & 2 \\ 2 & -1 & 2 & -3 \\ 3 & -2 & 1 & -4 \\ -4 & 1 & -3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \ldots (1)$$

consider Argument matrix $[A|B] = [A|0] = [A]$ =

$$\begin{bmatrix} 1 & 1 & -3 & 2 \\ 2 & -1 & 2 & -3 \\ 3 & -2 & 1 & -4 \\ -4 & 1 & -3 & 1 \end{bmatrix}$$

Applying $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$, $R_4 \to R_4 + 4R_1$

$$[A] \sim \begin{bmatrix} 1 & 1 & -3 & 2 \\ 0 & -3 & 8 & -7 \\ 0 & -5 & 10 & -10 \\ 0 & 5 & -15 & 9 \end{bmatrix}$$

Applying $R_3 \to 3R_3 - 5R_2$, $R_4 \to 3R_4 + 5R_2$

$$[A] \sim \begin{bmatrix} 1 & 1 & -3 & 2 \\ 0 & -3 & 8 & -7 \\ 0 & 0 & -10 & 5 \\ 0 & 0 & -5 & 52 \end{bmatrix}$$

Applying $R_4 \to 2R_4 - R_3$

$$[A] \sim \begin{bmatrix} 1 & 1 & -3 & 2 \\ 0 & -3 & 8 & -7 \\ 0 & 0 & -10 & 5 \\ 0 & 0 & 0 & 99 \end{bmatrix}$$

The number of non zero rows in $[A|0]$ is 4

$\therefore$ The rank of argument matrix $[A|O] = 4$

and the number of nonzero rows in $[A]$ is 4

$$\rho(A) = 4$$

Hence we have $\rho(A) = \rho[A|B] = n = 4$

$\therefore$ The given system is consistent

so it has unique solution and solution is trival solution i.e $x=0, y=0, z=0, w=0$

→ Example: solve the system equations

$$x+3y-2z=0, \quad 2x-y+4z=0,$$
$$x-11y+14z=0.$$

→ solution

The given equations can be written in the matrix form $AX=0$, we have

$$\begin{bmatrix} 1 & 3 & -2 \\ 2 & -1 & 4 \\ 1 & -11 & 14 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \ldots (1)$$

consider Argument matrix $[A|B] = [A|O] = [A] =$

$$\begin{bmatrix} 1 & 3 & -2 \\ 2 & -1 & 4 \\ 1 & -11 & 14 \end{bmatrix}$$

Applying $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - R_1$

$$[A] \sim \begin{bmatrix} 1 & 3 & -2 \\ 0 & -7 & 8 \\ 0 & -14 & 16 \end{bmatrix}$$

Applying $R_3 \to R_3 - 2R_2$

$$[A] \sim \begin{bmatrix} 1 & 3 & -2 \\ 0 & -7 & 8 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus the matrix is in echelon form

$\therefore$ The rank of argument matrix $[A|0] = 2$

and the number of non zero rows in $[A]$ is 2

$$\rho(A) = 2$$

Hence we have $\rho(A) = \rho[A \| B] \neq n$

$\therefore$ The given system is consistent and it has infinite number of solution

To obtain solutions, set $(n-r)$ variables any arbitrary value and solve for the remaining unknowns, Here $n-r = 3-2 = 1$

Then $\begin{bmatrix} 3 & -2 \\ -7 & 8 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Write into equation form

$x + 3y - 2z = 0$ --- (a), $-7y + 8z = 0$ --- (b)

let $z = k$

substituting $z=k$ in (b) $\Rightarrow -7y+8k=0 \Rightarrow y=\frac{8k}{7}$

substituting $z=k,\ y=\frac{8k}{7}$ in (a) $\Rightarrow x+3\frac{8k}{7}-2k=0$

$\Rightarrow x=\frac{-10k}{7}$

$$\therefore x=\frac{-10k}{7},\ y=\frac{8k}{7},\ z=k$$

This will give infinite number of the solution

→ Example: solve the system of equations,

$x+y-2z+3w=0,\ x-2y+z-w=0$

$4x+y-5z+8w=0,\ 5x-7y+2z-w=0$

→ solution

The given equations can be written in the matrix form $Ax=0$ we have

$$\begin{bmatrix} 1 & 1 & -2 & 3 \\ 1 & -2 & 1 & -1 \\ 4 & 1 & -5 & 8 \\ 5 & -7 & 2 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \dots (1)$$

consider Argument $[A|B]=[A|0]=[A]=$

$$\begin{bmatrix} 1 & 1 & -2 & 3 \\ 1 & -2 & 1 & -1 \\ 4 & 1 & -5 & 8 \\ 5 & -7 & 2 & -1 \end{bmatrix}$$

Applying $R_2 \to R_2 - R_1,\ R_3 \to R_3 - 4R_1,$

$R_4 \to R_4 - 5R_1$

$$[A] \sim \begin{bmatrix} 1 & 1 & -2 & 3 \\ 0 & -3 & 3 & -4 \\ 0 & -3 & 3 & -4 \\ 0 & -12 & 12 & -16 \end{bmatrix}$$

Applying $R_4 \to \frac{R_4}{4}$

$$[A] \sim \begin{bmatrix} 1 & 1 & -2 & 3 \\ 0 & -3 & 3 & -4 \\ 0 & -3 & 3 & -4 \\ 0 & -3 & 3 & -4 \end{bmatrix}$$

Applying $R_3 \to R_3 - R_2$, $R_4 \to R_4 - R_3$

$$[A] \sim \begin{bmatrix} 1 & 1 & -2 & 3 \\ 0 & -3 & 3 & -4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus the matrix is in echelon form

$\therefore$ The rank of argument matrix $[A|B] = 2$

and the number of non zero rows in $[A]$ is 2

$\rho(A) = 2$

Hence we have $\rho(A) = \rho[A|B] \neq n$

$\therefore$ The given system is consistent and it has infinite number of solution.

To obtain solutions, set $(n-r)$ variables any arbitrary and solve for the remaining unknowns Here $r = 4-2 = 2$

Then $\begin{bmatrix} 1 & 1 & -2 & 3 \\ 0 & -3 & 3 & -4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

write into equation form

$x+y-2z+3w=0$ -- (a) $-3y+3z-4w=0$ -- (b)

let $w=k_1, z=k_2$

substituting $w=k_1, z=k_2$ in (b) $\Rightarrow -3y+3k_2-4k_1$

$\Rightarrow y=k_2-\frac{4k_1}{3}$

substituting $w=k_1, z=k_2$ $y=k_2-\frac{4k_1}{3}$ in (a)

$\Rightarrow x+k_2-\frac{4k_1}{3}-2k_2+3k_1=0 \Rightarrow x=k_2-\frac{5k_1}{3}$

$\therefore x=k_2-\frac{5k_1}{3}, y=k_2-\frac{4k_2}{3}, z=k_2,$

$w=k_1$

This will give infinite number of the solution.

→ Example: solve the system of equations

$x+2y+4z+3w=0, 4x+3y+6z+7w=0$

$x+2z+w=0$

→ solution

The given equations can be written in the matrix form $AX=0$, we have

$$\begin{bmatrix} 1 & 2 & 4 & 3 \\ 4 & 3 & 6 & 7 \\ 0 & 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \ldots (1)$$

Consider Argument matrix $[A|B] = [A|0] = [A] =$

$$\begin{bmatrix} 1 & 2 & 4 & 3 \\ 4 & 3 & 6 & 7 \\ 0 & 1 & 2 & 1 \end{bmatrix}$$

Applying $R_2 \to R_2 - 4R_1$

$$[A] \sim \begin{bmatrix} 1 & 2 & 4 & 3 \\ 0 & -5 & -10 & -5 \\ 0 & 1 & 2 & 1 \end{bmatrix}$$

Applying $R_2 \to \frac{R_2}{-5}$

$$[A] \sim \begin{bmatrix} 1 & 2 & 4 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 1 & 2 & 1 \end{bmatrix}$$

Applying $R_3 \to R_3 - R_1$

$$[A] \sim \begin{bmatrix} 1 & 2 & 4 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus the matrix is in echelon form

$\therefore$ The rank of argument matrix $[A|0] = 2$

and the number of non zero rows in $[A] = 2$

$\rho(A) = 2 \qquad n = 4$

Hence we have $\rho(A) = \rho[A|B] \neq n$

$\therefore$ The given system is consistent and it has infinite number of solution.

To obtain solutions, set $(n-r)$ variables any arbitrary value and solve for the remaining unknowns, Here

$n-r = 4-2 = 2$

Then
$$\begin{bmatrix} 1 & 2 & 4 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

write into equation form

$x + 2y + 4z + 3w = 0$ — (a),

$y + 2z + w = 0$ — (b)

(Wrong)

let $w = k_1, z = k_2$.

Substituting $w = k_1, z = k_2, y = -2k_2 - k_1$ in (a)

$\Rightarrow x + 2(-2k_2 - k_1) + 4k_2 + 3k_1 = 0 \Rightarrow x = -k_1$

$\therefore x = -k_1, y = -2k_2 - k_1, z = k_2, w = k_1$

This will give infinite number of the solution.

→ solution of Linear systems – Direct Method

→ Gauss Elimination Method:

This Method of solving a system of n linear equations in n unknowns consists of eliminating the coefficients in such a way that the system reduces to upper triangular system which may be solved by backward substitution.

We discuss the method here for $n = 3$

consider the system

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

Consider Argument matrix of this system is

$$[A|B] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{bmatrix} \quad -(1)$$

Here we assume $a_{11} \neq 0$

Applying $R_2 \to R_2 - \frac{a_{21}}{a_{11}} R_1$, $R_3 \to R_3 \to \frac{a_{31}}{a_{11}} R_1$

we get

$$[A|B] \sim \begin{bmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ 0 & a_{22} & a_{23} & \beta_2 \\ 0 & a_{32} & a_{33} & \beta_3 \end{bmatrix}$$

Where $a_{22} = a_{22} - a_{21} - a_{12}\left(\frac{a_{22}}{a_{11}}\right)$, $a_{23} = a_{23} - a_{13}\left(\frac{a_{21}}{a_{11}}\right)$

$a_{32} = a_{12}\left(\frac{a_{31}}{a_{11}}\right)$, $a_{33} = a_{33} - a_{13}\left(\frac{a_{31}}{a_{11}}\right)$

We call $\frac{-a_{21}}{a_{11}}, \frac{-a_{31}}{a_{11}}$ as multipliers for the first stage a_{11} is called first pivot.

$$[A|B] \sim \begin{bmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ 0 & a_{22} & a_{23} & \beta_2 \\ 0 & a_{32} & a_{33} & \beta_3 \end{bmatrix}$$

here we assume $a_{22} \neq 0$

Applying $R_3 \to R_3 - \frac{a_{32}}{a_{22}} R_2$, we get

$$[A|B] \sim \begin{bmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ 0 & a_{22} & a_{23} & \beta_2 \\ 0 & 0 & \gamma_{33} & \Delta_3 \end{bmatrix} \quad \text{---} (2)$$

where $\gamma_{33} = \alpha_{33} - \left(\frac{\alpha_{32}}{\alpha_{22}}\right)\alpha_{23}$. $\Delta_3 = \beta_3 - \left(\frac{\alpha_{32}}{\alpha_{22}}\right)\beta_2$

we call $\frac{-\alpha_{32}}{\alpha_{22}}$ as multipliers for the second stage α_{22} is called new pivot.

The augmeted matrix (2) corresponds to an upper triangular system which can be solved by back ward substitution. The solution obtained is exact.

Note: If one of the elements $a_{11}, a_{22}, \alpha_{33}$ are zero, the method is modified by rearraging the rows. so that pivot is non zero.

UNIT - 2

Eigan Values and Eigan vectors and quadatic forms.

⇒ A non zero vector 'x' is said to be a characterstics vector of Eigan vector of a square matrix 'A'. If there exists a number λ such that

$$Ax = \lambda x$$

$$Ax = \lambda I x$$

$$Ax - \lambda I x = 0$$

$$(A - \lambda I)x = 0$$

$$[A - \lambda I]x = 0$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$ is any column vector

⇒ $[A - \lambda I]$ characterstics matrix of 'A' where I is the unit matrix of order 'n'.

→ $|A - \lambda I| = 0$ is called the Eigan values of the matrix 'A'

→ The set of all Eigen vectors is called the Eigen space of "A" corresponding to the Eigen values

Note-1: The characteristics eqn of matrix A of order 2 is $\boxed{\lambda^2 - s_1\lambda + s_2 = 0}$

s_1 = sum of the principle diagonal elements of A s_2 = det A (or) $|A|$

Note-2: The characteristics eqn of matrix 'A' of order 3 is $\boxed{\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0}$

s_1 = sum of the principle diagonal elements of A

s_3 = det or $|A|$.

s_2 = sum of the minors of the principle diagonal elements of 'A'.

Note-3: The sum of the eigen values of a matrix is equal to the sum of the its principle diagonal elements

Note-4: The product of the eigen values to the sum det of matrix 'A'.

1. Find the eigenvalues of the following matrix

$$A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$$

Given matrix $A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$

characteristic matrix $A = \lambda I$

$$A - \lambda I = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{bmatrix}$$

characteristic eqn of matrix A is

$$|A - \lambda I| = 0$$

$$\begin{bmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{bmatrix} = 0$$

Charateristic eqn of matrix of order 2

$$\lambda^2 - S_1\lambda + S_2 = 0$$

S_1 = sum of Principle diagonal elements.

$$S_1 = 7$$

$$S_2 = \det A$$

$$S_2 = 6$$

$$\lambda^2 - 7\lambda + 6 = 0$$

$$\lambda^2 - \lambda - 6\lambda + 6 = 0$$

$$\lambda(\lambda - 1) - 6(\lambda - 1) = 0$$

$$\lambda = 1, 6$$

1. Find the eigen values and eigen vector of the matrix $\begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{bmatrix}$

Given matrix A is

$$A = \begin{bmatrix} 4 & 6 & 6 \\ 1 & 2 & 2 \\ -1 & -4 & -3 \end{bmatrix}$$

Characteristic eqn of the matrix A is

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 4-\lambda & 6 & 6 \\ 1 & 3-\lambda & 2 \\ -1 & -4 & -3-\lambda \end{vmatrix} = 0$$

The characteristic eqn of matrix of order 3

$$-\lambda^3 + S_1\lambda^2 + S_2\lambda - S_3 = 0$$

S_1 = sum of triple diagonal elements

$$S_1 = 4$$

S_2 = sum of the minors of the principle diagonal elements $= -1 + (-6) + 6$

$$S_2 = -1$$

$$S_3 = \begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{bmatrix}$$

$$= 4[-9+8] - 6[-3+2] + 6[-4+3]$$

$$= -4 + \cancel{6} - \cancel{6}$$

$$S_3 = -4$$

$\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$

$\lambda^3 - 4\lambda^2 - \lambda + 4 = 0$ — (1)

for $\lambda = 1$ is one of the root Eqn (1)

By using synthetic division find roots

$$\begin{array}{r|rrrr} & 1 & -4 & -1 & 4 \\ 1 & 0 & 1 & -3 & -4 \\ \hline & 1 & -3 & -4 & \boxed{0} \end{array}$$

$\lambda^2 - 3\lambda - 4 = 0$

$\lambda^2 + \lambda - 4\lambda - 4 = 0$

$\lambda = 4, \lambda = -1$

Eigen values of Given matrix $\lambda = -1, 1, 4$

To Find the Eigen vectors:

Case-1: $\lambda = 1$ in $[A - \lambda I]X = 0$

$$\begin{bmatrix} 4-\lambda & 6 & 6 \\ 1 & 3-\lambda & 2 \\ -1 & -4 & -3-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 3 & 6 & 6 \\ 1 & 2 & 2 \\ -1 & -4 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

first two rows are identical

take one eqn from identical and the remaining

$x + 2y + 2z = 0$ — (1)

$-x - 4y - 4z = 0$ — (2)

solve (1) & (2) by Cramer's rule.

$$\begin{array}{ccc} x & y & z \end{array}$$
$$\begin{vmatrix} 2 & 2 & 1 & 2 \\ -4 & -4 & -1 & -4 \end{vmatrix}$$

$$\frac{x}{-8+6} = \frac{y}{-2+4} = \frac{z}{-4+2} = t$$

$$\frac{x}{0} = \frac{y}{2} = \frac{z}{-2} = 1$$

$$x = 0 \quad y = t \quad z = -t$$

$$X_1 \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ t \\ -t \end{bmatrix} \Rightarrow t \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = t \times 1.$$

X_1 is eigen vector corresponding to $\lambda = 1$

<u>case-2</u> $\lambda = -1$ in $[A - \lambda I] X = 0$

$$\begin{bmatrix} 4-\lambda & 6 & 6 \\ 1 & 3-\lambda & 2 \\ -1 & -4 & -3-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 5 & 6 & 6 \\ 1 & 4 & 2 \\ -1 & -4 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$5x + 6y + 6z = 0 \quad \text{--- (1)}$$

$$x + 4y + 2z = 0 \quad \text{--- (2)}$$

using cramer's rule solve (1) & (2)

$$\begin{vmatrix} x & y & z \\ 6 & 6 & 5 & 6 \\ 4 & 2 & 1 & 4 \end{vmatrix}$$

$$\frac{x}{12-24} = \frac{y}{6-10} = \frac{z}{20-6} = t$$

$$\frac{x}{6} = \frac{y}{2} = \frac{z}{-7} = t$$

$x = 6t \quad y = 2t \quad z = -7t$

$$x_2 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6t \\ 2t \\ -7t \end{bmatrix} \Rightarrow t \begin{bmatrix} 6 \\ 2 \\ -7 \end{bmatrix} = t x_2$$

where x_2 is the eigen vector corresponding the eigen values $\lambda = -1$

case (iii) $\lambda = 4$ in $[A - \lambda I]X = 0$

$$\begin{bmatrix} 4-\lambda & 6 & 6 \\ 1 & 3-\lambda & 2 \\ -1 & -4 & -3-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\begin{bmatrix} 0 & 6 & 6 \\ 1 & -1 & 2 \\ -1 & -4 & -7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$6y + 6z = 0$ — (1) $\quad x - y + 2z = 0$ — (2)

$-x - 4y - 7z = 0$ — (3)

solve (1) & (2)

$$\left| \begin{matrix} x & y & z \\ 6 & 6 & 0 \\ 1 & -1 & 2 \\ -1 & 2 & 1 \end{matrix} \right|$$

$$\frac{x}{12+6} = \frac{y}{6-0} = \frac{z}{-6} = t$$

$$\frac{x}{18} = \frac{y}{6} = \frac{z}{-6} = t$$

$x = 3t \quad y = t \quad z = -t$

$$x_3 \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3t \\ t \\ -t \end{bmatrix} \Rightarrow t \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} = t x_3$$

Where x_3 is the eigen vector corresponding to the eigen value $\lambda = 4$.

*

(2) Find the eigen values and eigen vector of the matrix $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

characteristic eqn of matrix A is $[A - \lambda I] = 0$

$$\begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0$$

characteristic eqn of matrix A of order 3

det of matrix $[A - \lambda I] = 0$

$\Rightarrow 8-\lambda\left[21 - 10\lambda + \lambda^2 - 16\right] + 6\left[-18 + 6\lambda + 8\right] + 2\left[24 - 14 + 2\lambda\right] = 0$

$\Rightarrow 8-\lambda\left[\lambda^2 + 5 - 10\lambda\right] + 6\left[6\lambda - 10\right] + 2\left[10 + 2\lambda\right] = 0$

$\Rightarrow 5\lambda^2 + 40 - 80\lambda - \lambda^3 - 5\lambda + 10\lambda^2 + 36\lambda - 60 + 20 + 4\lambda = 0$

$-\lambda^3 + 18\lambda^2 - 45\lambda = 0$

$\lambda = 3$ is one of the root

$$\begin{array}{c|ccc} & -1 & 18 & -45 \\ 3 & 0 & -3 & 45 \\ \hline & -1 & 15 & 0 \end{array}$$

$\Rightarrow \lambda^2 + 15 = 0$

$\lambda = 15, 0$

Eigen values of given matrix are $\lambda = 0, 3, 15$

To Find the Eigen Vectors:

case (i) $\lambda = 0$ in $[A - \lambda I]x = 0$

$$\begin{bmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$3x - 6y + 2z = 0$ — (1) $-6x + 7y - 4z = 0$ — (2)

$2x - 4y + 3y = 0$ — (3)

solve (1) & (2)

$$\begin{vmatrix} x & y & z \\ -6 & 2 & 8 & -6 \\ 7 & -4 & -6 & 7 \end{vmatrix}$$

$$\frac{x}{24-14} = \frac{y}{-12+32} = \frac{z}{56-36} = t$$

$$\frac{x}{10} = \frac{y}{20} = \frac{z}{20} = t$$

$x = t \quad y = 2t \quad z = 2t$

$$x_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} t \\ 2t \\ 2t \end{bmatrix} \Rightarrow t \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = t x_1$$

x_1 is the eigen vector corresponding to eigen value $\lambda = 0$

Case (ii) $\lambda = 3$

$$\begin{bmatrix} 8-3 & -6 & 2 \\ -6 & 7-3 & -4 \\ 2 & -4 & 8-3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\begin{bmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$5x - 6y + 2z = 0$ — ① $\quad -6x + 4y - 4z = 0$ — ②

$2x - 4y = 0$ — ③

solve ② & ③

$$\begin{vmatrix} x & y & z \\ 4 & -4 & -6 & 4 \\ -4 & 0 & 2 & -4 \end{vmatrix}$$

$$\frac{x}{-16} = \frac{y}{-8} = \frac{z}{24-8} = t$$

$$\frac{x}{-16} = \frac{y}{-8} = \frac{z}{16} = t$$

$$\frac{x}{2} = \frac{y}{1} = \frac{z}{-2} = t$$

$x = 2t \quad y = t \quad z = -2t$

$$x_2 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2t \\ t \\ -2t \end{bmatrix} \Rightarrow t \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} = t X_2$$

x_2 is the Eigen vector corresponding to Eigen value $\lambda = 3$

case (iii) $\quad \lambda = 15$

$$\begin{bmatrix} 8-15 & -6 & 2 \\ -6 & 7-15 & -4 \\ 2 & -4 & 3-15 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\begin{bmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$-7x - 6y + 2z = 0$ — (1) $\quad -6x - 8y - 4z = 0$ — (2)

$2x - 4y - 12z = 0$ — (3)

solve (2) & (3)

$$\begin{vmatrix} x & y & z \\ -8 & -4 & -6 \\ -4 & -12 & 2 \end{vmatrix} \begin{matrix} -8 \\ -4 \end{matrix}$$

$$\frac{x}{96-16} = \frac{y}{-8-72} = \frac{z}{24+16} = t$$

$$\frac{x}{80} = \frac{y}{-80} = \frac{z}{40} = t$$

$$\frac{x}{2} = \frac{y}{-2} = \frac{z}{1} = t$$

$x = 2t \quad y = -2t \quad z = t$

$$X_3 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} t \\ -t \\ 2t \end{bmatrix} \rightarrow t \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} = t X_3$$

where X_3 is the eigen vector corresponding to eigen value $\lambda = 15$

(3) Find the Eigen Values and Eigen vectors of the matrix $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

characteristics Eqn of matrix A

$$|A - \lambda I| = 0$$

$$\begin{bmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{bmatrix} = 0$$

characteristics Eqn of matrix order 3

$$\Rightarrow -2-\lambda[-\lambda+\lambda^2-12] - 2[-2\lambda-6] - 3[-4+1-\lambda] = 0$$

$$-2\lambda - 2\lambda^2 + 24 + \lambda^2 - \lambda^3 + 12\lambda + 4\lambda + 12\lambda + 12 + 9 + 3\lambda = 0$$

$$\Rightarrow -\lambda^3 - \lambda^2 + 21\lambda + 45 = 0$$

$\lambda = -3$ is ~~com~~ one of the root

$$\begin{array}{r|rrrr} -3 & -1 & -1 & 21 & 45 \\ & 0 & 3 & -6 & -45 \\ \hline & -1 & 2 & 15 & \boxed{0} \end{array}$$

$$-\lambda^2 + 2\lambda + 15 = 0$$

$$\lambda^2 - 2\lambda - 15 = 0$$

$$\lambda^2 + 3\lambda - 5\lambda - 15 = 0$$

$$\lambda(\lambda+3) - 5(\lambda+3) = 0$$

$$\lambda = -3 \quad \lambda = 5$$

$\therefore$ Eigen values of given matrix are

$$\lambda = -3, -3, 5$$

To find Eigen vectors

<u>case (i)</u> $\lambda = -3$ in $[A - \lambda I]X = 0$

$$\begin{bmatrix} -2-\lambda & 2 & -3 \\ 2 & 4-\lambda & -6 \\ -1 & -2 & -\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$x + 2y - 3z = 0$ — ① $z = t_1$, $y = t_2$ in ①

$x - 2t_2 - 3t_3 = 0$

$x = 3t_1 - 2t_2$

$$X_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3t_1 - 2t_2 \\ t_2 \\ t_1 \end{bmatrix}$$

$$X_1 = \begin{bmatrix} 3t_1 - 2t_2 \\ 0 + t_2 \\ t_1 + 0 \end{bmatrix}$$

$$X_1 = t_1 \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} + t_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = t_1 X_1 + t_2 X_2$$

where X_1 is the eigen vector corresponding to eigen value $\lambda = -3$ where X_1, X_2 are linearly independent vectors

case (ii): $\lambda = 5$

$$\Rightarrow \begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$-7x + 2y - 3z = 0$ — ①

$2x - 4y - 6z = 0$ — ②

$-x - 2y - 5z = 0$ — ③

solving ② + ③

$$\rightarrow \begin{vmatrix} x & y & z \\ -4 & -6 & 2 & -4 \\ -2 & -5 & -1 & -2 \end{vmatrix}$$

$$\frac{x}{20-12} = \frac{y}{6+10} = \frac{y}{-4+4} = t$$

$$\frac{x}{8} = \frac{y}{16} = \frac{y}{-8} = t$$

$$x = t \quad y = 2t \quad z = -t$$

$$x_3 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} t \\ 2t \\ -t \end{bmatrix} \rightarrow t \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = t \times_2$$

where x_2 is the eigen vector corresponding to eigen value $\lambda = 5$

4. find the eigen values and eigen vector of the matrix $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 3 \end{bmatrix} \rightarrow$ Given matrix

characteristic eqn of matrix A

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 0-\lambda & 1 & 0 \\ 0 & 0-\lambda & 1 \\ 1 & -3 & 3-\lambda \end{vmatrix} = 0$$

Cleaterstic Eqn of matrix of order 3

$$\Rightarrow -\lambda[-3\lambda + \lambda^2 + 3] - 1[-1] + 0[\lambda]$$

$$\Rightarrow 3\lambda^2 - \lambda^3 - 3\lambda + 1 = 0$$

$\Rightarrow -\lambda^3+3\lambda^2-3\lambda+1=0$

$\lambda=1$ is one of the root of A

$$\begin{array}{c|cccc} 1 & -1 & 3 & -3 & 1 \\ & 0 & -1 & 2 & -1 \\ \hline & -1 & 2 & -1 & \boxed{0} \end{array}$$

$-\lambda^2+2\lambda-1=0$

$\lambda^2-2\lambda+1=0$

$\lambda^2-\lambda-\lambda+1=0$

$\lambda(\lambda-1)-(\lambda-1)=0$

$\lambda=1,1$

Eigen values are $\lambda=1,1,1$

To find the Eigen vectors

Case (i) : $\lambda=1$ in $(A-\lambda I)x=0$

$$\begin{vmatrix} 0-\lambda & 1 & 0 \\ 0 & 0-\lambda & 1 \\ 1 & -3 & 3-\lambda \end{vmatrix} \begin{vmatrix} x \\ y \\ z \end{vmatrix} = 0$$

$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & -3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$-x+y=0$

$-y+z=0$

$x-3y+2z=0$

solve ① + ②

$$\begin{vmatrix} x & y & z \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{vmatrix}$$

$$\frac{x}{1} = \frac{y}{1} = \frac{z}{1} = t$$

$x = t \quad y = t \quad z = t$

$x = t \quad y = t \quad z = t$

$$x_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} t \Rightarrow x_1 t$$

x_1 is the eigen vector corresponding to eigen value $\lambda = 1$

NOTE:- The eigen vectors corresponding to distinct eigen values of a real symmetric matrix (or) orthogonal.

i.e $x_1^T x_2 = 0$

$x_2^T x_3 = 0$

$\lambda_3^T x_1 = 0$

1. Find the orthogonal eigen vectors for the matrix

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$$ - Given matrix

characteristics Eqn of matrix order 3

$\Rightarrow 1-\lambda \left[\cancel{36} - 4\lambda - 9\lambda + \lambda^2 - \cancel{36}\right] - 2\left[\cancel{18} - 2\lambda - \cancel{18}\right] + 3\left[\cancel{12} - \cancel{12} + 3\lambda\right]$

$\Rightarrow 1-\lambda\left[\lambda^2-13\lambda\right]-2\left[-2\lambda\right]+3\left[3\lambda\right]$

$= \lambda^2-13\lambda-\lambda^3+13\lambda^2+4\lambda+9\lambda$

$\rightarrow -\lambda^3+14\lambda^2-\cancel{18}\lambda=0$

$\lambda=14$ is one of the root

$$\begin{array}{r|rr} 14 & -1 & 14 \\ & 0 & -14 \\ \hline & -1 & \boxed{0} \end{array}$$

$\lambda = 0,0$

Eigen values are $\lambda=0,0,14$

To find the Eigen vectors :-

$\lambda=0$ in $|A-\lambda I|x=0$

$$\begin{bmatrix} 1-\lambda & 2 & 3 \\ 2 & 4-\lambda & 6 \\ 3 & 6 & 9-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=0$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=0$$

$x+2y+3y=0 \quad z=t_1 \; y=t_2$

$x+2t_2+3t_1=0$

$x=-2t_2-3t_1$

$\therefore x_1=\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} -2t_2-3t_1 \\ t_2 \\ t_1 \end{bmatrix} \Rightarrow t_2\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}+t_2\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$

x_1, x_2 is the corresponding Eigen vector for Eigen value $\lambda=0$

$\Rightarrow t_1x_1 + t_2x_2$ {where x_1, x_2 are linearly independent eigen vectors corresponding to $\lambda = 0$ }

Case (ii)

$\lambda = 14$ in $|A - \lambda I| x = 0$

$$\begin{vmatrix} 1-\lambda & 2 & 3 \\ 2 & 4-\lambda & 6 \\ 3 & 6 & 9-\lambda \end{vmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\begin{bmatrix} -13 & 2 & 3 \\ 2 & -10 & 6 \\ 3 & 6 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$-13x + 2y + 3z = 0$ — (1) $\quad 2x - 10y + 6z = 0$ — (2)

$3x + 6y - 5z = 0$ — (3)

solve (2) + (3)

$$\begin{bmatrix} x & y & z \\ -10 & 6 & 2 \\ 6 & -5 & 3 \end{bmatrix} \begin{matrix} -10 \\ 6 \end{matrix}$$

$$\frac{x}{50-36} = \frac{y}{18+10} = \frac{z}{12+30} = t$$

$x = t, \; y = 2t, \; z = 3t$

$$x_3 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} t \\ 2t \\ 3t \end{bmatrix} \Rightarrow t \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \Rightarrow x_3 t$$

where x_3 is the corresponding eigen vector for eigen value $\lambda = 14$.

since x_1, x_2 are not orthogonal, we choose x_2 such that x_1, x_2, x_3 are orthogonal

$$x_2 = \begin{bmatrix} l \\ m \\ n \end{bmatrix}$$

$$x_1^T x_2 = 0$$

$$\begin{bmatrix} -2 & 1 & 0 \end{bmatrix} \begin{bmatrix} l \\ m \\ n \end{bmatrix} = 0$$

$$\Rightarrow -2l + m + 0(n) = 0 \quad \text{--- (1)}$$

$$x_3^T x_2 = 0$$

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} l \\ m \\ n \end{bmatrix} = 0$$

$$l + 2m + 3n = 0 \quad \text{--- (2)}$$

solve (1) & (2)

$$\begin{array}{ccc} l & m & n \\ \end{array}$$
$$\begin{vmatrix} 1 & 0 & -2 & 1 \\ 2 & 3 & 1 & 2 \end{vmatrix}$$

$$\frac{l}{3} = \frac{m}{6} = \frac{n}{-4-1} = t$$

$$\frac{l}{3} = \frac{m}{6} = \frac{n}{-5} = t$$

$$l = 3t \quad m = 6t \quad n = -5t$$

$$x_2 = \begin{bmatrix} l \\ m \\ n \end{bmatrix} = \begin{bmatrix} 3t \\ 6t \\ -5t \end{bmatrix} \Rightarrow t \begin{bmatrix} 3 \\ 6 \\ -5 \end{bmatrix} = t \times 2$$

$$\lambda = 0$$

$\therefore x_1, x_2, x_3$ are mutually orthogonal

verify: $x_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ $x_2 = \begin{bmatrix} 3 \\ 6 \\ -5 \end{bmatrix}$ $x_3 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$x_1^T x_2 = 0 \quad x_2^T x_3 = 0$

$$\begin{bmatrix} -2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 6 \\ -5 \end{bmatrix} = 0 \qquad \begin{bmatrix} 3 & 6 & -5 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 0$$

$-\cancel{6} + \cancel{6} = 0$ $\qquad$ $3 + \cancel{12} - \cancel{15} = 0$

$0 = 0$ $\qquad$ $0 = 0$

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = 0$$

$-\cancel{2} + \cancel{2} = 0$

$0 = 0$

6. Find the eigen values and eigen vectors of the following matrices $\begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$

Given matrix $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$

characteristic eqn of matrix A

$$|A - \lambda I| = 0$$

$$= \begin{bmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{bmatrix} = 0$$

characteristics eqn of matrix of order 3

$$\Rightarrow (1-\lambda)\left[6 - 2\lambda - 3\lambda + \lambda^2 - 2\right] - 1\left[2 - 4 + 2\lambda\right] = 0$$

$\Rightarrow (1-\lambda)[\lambda^2-5\lambda+4]-1[2\lambda-2]=0$

$\Rightarrow \lambda^2-5\lambda+4-\lambda^3+5\lambda^2-4\lambda-2\lambda+2=0$

$-\lambda^3+6\lambda^2-11\lambda+6=0$

$\lambda=2$ is one of the root

$$\begin{array}{c|cccc} 2 & -1 & 6 & -11 & 6 \\ & 0 & -2 & 8 & -6 \\ \hline & -1 & 4 & -3 & \boxed{0} \end{array}$$

$-\lambda^2+4\lambda-3=0$

$\lambda^2-4\lambda+3=0$

$\lambda^2-\lambda-3\lambda+3=0$

$\lambda(\lambda-1)-3(\lambda-1)=0$

$\lambda=3,\ \lambda=1$

$\therefore$ Eigen values of Given matrix $\lambda=1,2,3$

To find Eigen vectors:

case (i) $\lambda=1$ $[A-\lambda I]X=0$

$$\begin{bmatrix} 0 & 0 & -1 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=0$$

$x+y+z=0$ — ①

$-z=0$ — ②

solve ① & ②

$$\begin{array}{ccc} x & y & z \\ 1 & 1 & 1 \\ 0 & -1 & 0 \end{array}$$

$\frac{x}{-1}=\frac{y}{1}=\frac{z}{0}=t$

$x = -t$

$$x_1 \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -t \\ t \\ 0 \end{bmatrix} \Rightarrow t \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \rightarrow t x_1$$

$\therefore x_1$ is eigen vector corresponding to eigen value

$\lambda = 1$

<u>case (ii)</u> $\lambda = 2$ in $[A - \lambda I]x = 0$

$$\begin{bmatrix} -1 & 0 & -1 \\ 1 & 0 & 1 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$x + z = 0$ — ①

$2x + 2y + z = 0$ — ②

solve ① & ②

$$\begin{array}{ccc} x & y & z \end{array}$$
$$\begin{vmatrix} 0 & 1 & 1 & 0 \\ 2 & 1 & 2 & 2 \end{vmatrix}$$

$$\frac{x}{-2} = \frac{y}{1} = \frac{z}{2} = t$$

$x = -2t \quad y = t \quad z = 2t$

$$x_2 \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2t \\ t \\ 2t \end{bmatrix} \Rightarrow t \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} \Rightarrow t x_2$$

x_2 is eigen vector corresponding to eigen value

$\lambda = 2$

case (iii) $\lambda = 3$

$$\begin{bmatrix} -2 & 0 & -1 \\ 1 & -1 & 1 \\ 2 & 2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$-2x - 3 = 0$ — ①

$x - y + 3 = 0$ — ②

$2x + 2y = 0$ — ③

solve ① & ③

$$\begin{vmatrix} x & y & 3 \\ 0 & 1 & 0 \\ 2 & 0 & 2 \end{vmatrix}$$

$$\frac{x}{2} = \frac{y}{-2} = \frac{3}{-4} = t$$

$$-1 \qquad -2$$

$x = 1 \quad y = -t \quad 3 = 2t$

$$x_3 = \begin{bmatrix} x \\ y \\ 3 \end{bmatrix} = \begin{bmatrix} t \\ -t \\ -2t \end{bmatrix} \Rightarrow t \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix} \rightarrow t x_3$$

$\therefore x_3$ is Eigen vector corresponding to Eigan value $\lambda = 3$

8. $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

characteristics matrix eqn of A

$$[A - \lambda I] = 0$$

$$\begin{bmatrix} 0-\lambda & 1 & 1 \\ 1 & 0-\lambda & 1 \\ 1 & 1 & 0-\lambda \end{bmatrix}$$

characteristics matrix eqn of matrix order 3.

$\Rightarrow -\lambda[\lambda^2-1]-1[-\lambda-1]+1[1+\lambda]=0$

$-\lambda^3+\lambda+1+\lambda+\lambda+1=0$

$\rightarrow -\lambda^3+3\lambda+2=0$

$\lambda=2$ is one of the root

$$\begin{array}{c|cccc} 2 & -1 & 0 & 3 & 2 \\ & 0 & -2 & -4 & 2 \\ \hline & -1 & -2 & -1 & \boxed{0} \end{array}$$

$\lambda^2+2\lambda+1=0$

$\lambda^2+\lambda+\lambda+1=0$

$\lambda(\lambda+1)+1(\lambda+1)=0$

$\lambda=-1 \quad \lambda=-1$

$\therefore$ Eigen values are $\lambda=2,-1,-1$

To find Eigen vectors.

case (i) $\lambda=-1$

$[A-\lambda I]X=0$

$$\rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=0$$

$x+y+z=0$ — ① $\quad z=t_2 \quad y=t_1$ in ①

$x+t_1+t_2=0$

$x=-t_1-t_2$

$$X_1=\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} -t_1-t_2 \\ t_1 \\ t_2 \end{bmatrix} \Rightarrow t_1\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}+t_2\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$x_1t_1+x_2t_2$

where x_1 is the eigen vector corresponding to eigen value $\lambda = -1$

where x_1, x_2 are linearly independent vectors

case (ii) $\lambda = 2$

$$[A - \lambda I]x = 0$$

$$\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$-2x + y + z = 0$ — (1) $\quad x + y - 2z = 0$ — (3)

$x - 2y + z = 0$ — (2)

$$\begin{array}{ccc} x & y & z \end{array}$$

$$\begin{vmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \end{vmatrix}$$

$$\frac{x}{-1} = \frac{y}{1+2} = \frac{z}{1+2} = t$$

$$\frac{x}{3} = \frac{y}{3} = \frac{z}{3} = t$$

$$x = t, \; y = t, \; z = t$$

$$x_3 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \Rightarrow \begin{bmatrix} t \\ t \\ t \end{bmatrix} \Rightarrow t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \Rightarrow x_3 t$$

$\therefore x_3$ is the eigen vector corresponding to eigen value $\lambda = 2$

Cayley - Hamilton theorem :

statement : Every square matrix satisfies its own characteristics equation

1. Verify cayley - Hamilton theorem for the following matrix and values hence find A^{-1}, A^4

$$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

Given matrix $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

characteristic Eqn of matrix of order 3

$$\Rightarrow 2-\lambda\left[4-2\lambda-2\lambda+\lambda^2-1\right]+1\left[-2+\lambda+1\right]+1\left[1-2+\lambda\right]=0$$

$$\Rightarrow 2-\lambda\left[\lambda^2-4\lambda+3\right]+1\left[\lambda-1\right]+1\left[\lambda-1\right]=0$$

$$\Rightarrow 2\lambda^2-8\lambda+6-\lambda^3+4\lambda^2-3\lambda+\lambda-1+\lambda-1=0$$

$$\Rightarrow -\lambda^3+6\lambda^2-9\lambda+4=0$$

$$\lambda^3-6\lambda^2+9\lambda-4=0$$

The characteristics eqn to the matrix A is

$$A^3-6A^2+9A-4I=0 \quad \text{--- (1)}$$

$$A^2 = A \cdot A$$

$$= \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 4+1+1 & -2-2-1 & 2+1+2 \\ -2-2-1 & 1+4+1 & -1-2-2 \\ 2+1+2 & -1-2-2 & 1+1+4 \end{bmatrix} \Rightarrow \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$$

$$A^3 = A^2 A$$

$$= \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 12+5+5 & -6-10-5 & 6+5+10 \\ -10-6-5 & 5+12+5 & -5-6-10 \\ 10+6+5 & -5-10-6 & 5+5+12 \end{bmatrix} \Rightarrow \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix}$$

from ①

$$A^3 - 6A^2 + 9A - 4I = 0$$

$$LHS = A^3 - 6A^2 + 9A - 4I$$

$$\Rightarrow \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix} - 6\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} + 9\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix} - \begin{bmatrix} 36 & -30 & 30 \\ -30 & 36 & -30 \\ 30 & -30 & 36 \end{bmatrix} + \begin{bmatrix} 18 & -9 & 9 \\ -9 & 18 & -9 \\ 9 & -9 & 18 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 22-36+18-4 & -21+30-9+0 & 21-30+9+0 \\ -21+30-9+0 & 22-36+18-4 & -21-30-9+0 \\ 21-30+9+0 & -21+30-9+0 & 22-36+18-4 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 4+1+1 & -2-2-1 & 2+1+2 \\ -2-2-1 & 1+4+1 & -1-2-2 \\ 2+1+2 & -1-2-2 & 1+1+4 \end{bmatrix} \Rightarrow \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$$

$$A^3 = A^2 A$$

$$= \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 12+5+5 & -6-10-5 & 6+5+10 \\ -10-6-5 & 5+12+5 & -5-6-10 \\ 10+6+5 & -5-10-6 & 5+5+12 \end{bmatrix} \Rightarrow \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix}$$

from ①

$$A^3 - 6A^2 + 9A - 4I = 0$$

$$LHS = A^3 - 6A^2 + 9A - 4I$$

$$\Rightarrow \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix} - 6\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} + 9\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix} - \begin{bmatrix} 36 & -30 & 30 \\ -30 & 36 & -30 \\ 30 & -30 & 36 \end{bmatrix} + \begin{bmatrix} 18 & -9 & 9 \\ -9 & 18 & -9 \\ 9 & -9 & 18 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 22-36+18-4 & -21+30-9+0 & 21-30+9+0 \\ -21+30-9+0 & 22-36+18-4 & -21-30-9+0 \\ 21-30+9+0 & -21+30-9+0 & 22-36+18-4 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The matrix A satisfies its own characteristics

Cayley - Hamilton theorem is verified.

To find A^{-1}:

we multiply eqn by A^{-1}

$$A^{-1}(A^3 - 6A^2 + 9A - 4I) = 0$$

$$A^2 - 6A + 9A^0 - 4A^{-1} = 0$$

$$A^2 - 4A^{-1} - 6A + 9I = 0$$

$$4A^{-1} = A^2 - 6A + 9I$$

$$A^{-1} = \frac{1}{4}\left[A^2 - 6A + 9I\right]$$

$$A^{-1} = \frac{1}{4}\left[\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} - 6\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} + \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}\right]$$

$$A^{-1} = \frac{1}{4}\left[\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}\right] \Rightarrow A^{-1} = \begin{bmatrix} 3/4 & 1/4 & -1/4 \\ 1/4 & 3/4 & 1/4 \\ -1/4 & 1/4 & 3/4 \end{bmatrix}$$

$$A^4 = A^3 \cdot A$$

$$A^4 = \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 44+21+21 & -22-42-21 & 22+21+42 \\ -22-42-21 & 44+21+21 & -22-42-21 \\ 22+21+42 & -22-42-21 & 44+21+21 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 86 & -85 & 85 \\ -85 & 86 & -85 \\ 85 & -85 & 86 \end{bmatrix}$$

2. verify Cayley-Hamilton and also find A^3, A^4

$$\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

characterist Eqn of matrix A $[A-\lambda I] = 0$

$$\begin{bmatrix} 1-\lambda & 2 & -2 \\ -1 & 3-\lambda & 0 \\ 0 & -2 & 1-\lambda \end{bmatrix} = 0$$

characteristics Eqn of matrix of order 3

$$1-\lambda\,[3-3\lambda-\lambda+\lambda^2] - 2[-1+\lambda] - 2[2] = 0$$

$$\Rightarrow 1-\lambda\,[\lambda^2-4\lambda+3] - 2[\lambda-1] - 4 = 0$$

$$\Rightarrow \lambda^3 - 4\lambda + 3 - \lambda^3 + 4\lambda^2 - 3\lambda - 2\lambda + 2 - 4 = 0$$

$$\Rightarrow -\lambda^3 + 5\lambda^2 - 9\lambda + 1 = 0$$

The characterise Eqn of matrix A is

$$A^3 - 5A^2 + 9A - I = 0 \quad \text{--- (1)}$$

$$A^2 = A \cdot A$$

$$A^2 = \begin{bmatrix} -1 & 10 & -4 \\ -4 & 7 & 8 \\ 2 & -8 & 1 \end{bmatrix}$$

$$A^3 = A^2 \cdot A$$

$$= \begin{bmatrix} -1 & 12 & -4 \\ -4 & 7 & 2 \\ 2 & -8 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} -13 & 42 & -2 \\ -11 & 9 & 10 \\ 10 & -22 & -3 \end{bmatrix}$$

from ①

$$A^3 - 5A^2 + 9A - I = 0$$

$$\Rightarrow \begin{bmatrix} -13 & 42 & -2 \\ -11 & 9 & 10 \\ 10 & -22 & -3 \end{bmatrix} - 5\begin{bmatrix} -1 & 12 & -4 \\ -4 & 7 & 2 \\ 2 & -8 & 1 \end{bmatrix} + 9\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} =$$

$$\Rightarrow \begin{bmatrix} -13 & 42 & -2 \\ -11 & 9 & 10 \\ 10 & -22 & -3 \end{bmatrix} - \begin{bmatrix} -5 & 60 & -20 \\ -20 & 35 & 10 \\ 10 & -40 & 5 \end{bmatrix} + \begin{bmatrix} 9 & 18 & -18 \\ -9 & 27 & 0 \\ 0 & -18 & 9 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} =$$

$$\Rightarrow \begin{bmatrix} -13+5+9-1 & 42-60+18+0 & -2+20-18+0 \\ -11+20-9+0 & 9-35+27-1 & 10-10+0+0 \\ 10-10+0 & -22+40-18+0 & -3-5+9+1 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ The matrix A satisfies its own characteristics eqn

$\therefore$ Cayley - Hamiton theorem is verified

To find A^{-1}

we multiply eqn by A^{-1}

$$A^{-1}(A^3 - 5A^2 + 9A - I) = 0 \rightarrow A^2 - 5A + 9A^0 - A^{-1} = 0$$

$$\rightarrow A^2 - 5A + 9I = A^{-1}$$

$$\begin{bmatrix} -1 & 12 & -4 \\ -4 & 7 & 2 \\ 2 & -8 & 1 \end{bmatrix} - 5\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix} + 9\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = A^{-1}$$

$$A^{-1} = \begin{bmatrix} -1 & 12 & -4 \\ -4 & 7 & 2 \\ 2 & -8 & 1 \end{bmatrix} - \begin{bmatrix} 5 & 10 & -10 \\ -5 & 15 & 0 \\ 0 & -10 & 5 \end{bmatrix} + \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 3 \end{bmatrix}$$

To find A^4

$A^3 A$

$$\Rightarrow \begin{bmatrix} -13 & 42 & -2 \\ -4 & 9 & 10 \\ 10 & -22 & -3 \end{bmatrix} \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -13-42+0 & -26+126+4 & 26-2+0 \\ -11-9+0 & -22+27-20 & 22+10+0 \\ 10+22+0 & 20-66+6 & -20+0-3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -55 & 104 & 24 \\ -20 & -15 & 32 \\ 32 & -40 & -23 \end{bmatrix}$$

(or)

$$A(A^3 - 5A^2 + 9A - I) = 0$$

$$A^4 - 5A^3 + 9A^2 - A = 0$$

$$A^4 = 5A^3 - 9A^2 + A$$

6. show that the matrix $A = \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}$ satisfies cayley Hamilton theorem. find A^{-1} if it exists.

characteristics eqn of matrix of order 3

$$\Rightarrow -\lambda(\lambda^2 + a^2) - c(c\lambda - ab) - b(ac + b\lambda) = 0$$

$\Rightarrow -\lambda^3 - \lambda a^2 - \lambda b^2 - \lambda c^2 = 0$

$\Rightarrow -\lambda^3 - \lambda(a^2+b^2+c^2) = 0$

$\Rightarrow \lambda^3 + \lambda(a^2+b^2+c^2) = 0$

characteristics eqn of A

$$A^3 + Aa^2 + Ab^2 + Ac^2 = 0 \quad \text{--- (1)}$$

$A^2 = A \cdot A$

$$= \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix} \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} -c^2-b^2 & ab & ac \\ ab & -c^2-a^2 & bc \\ ac & bc & -b^2-a^2 \end{bmatrix}$$

$A^3 = A^2 \cdot A$

$$= \begin{bmatrix} -c^2-b^2 & ab & ac \\ ab & -c^2-a^2 & bc \\ ac & bc & -b^2-a^2 \end{bmatrix} \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -c^3-b^2c-a^2c & bc^2+b^3+a^2b \\ c^3+a^2c+b^2c & 0 & -ab^2-ac^2-a^3 \\ -bc^2-b^3-a^2b & ac^2+ab^2+c^3 & 0 \end{bmatrix}$$

$$A^3 = -(a^2+b^2+c^2) \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}$$

from (1)

$$A^3 + A(a^2+b^2+c^2) = 0$$

$$-(a^2+b^2+c^2)\begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix} + \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}(a^2+b^2+c^2)$$

$$= 0$$

Hence matrix A satisfies Cayley - Hamilton theorem. To find A^{-1}

we multiply Eqn (1) by A^{-1}.

as $|A| = 0$ so A^{-1} does not exist.

$$A \begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix}$$

characteristic Eqn of matrix A

$$[A - \lambda I] = 0$$

$$\Rightarrow \begin{bmatrix} 2-\lambda & 0 & -1 \\ 0 & 2-\lambda & 0 \\ -1 & 0 & 2-\lambda \end{bmatrix}$$

characteristic Eqn of matrix of order 3

$$\Rightarrow 2-\lambda[4-2\lambda-2\lambda+\lambda^2] + 0[0] - 1[2-\lambda] = 0$$

$$\Rightarrow 2-\lambda[\lambda^2-4\lambda+4] - 1[2-\lambda] = 0$$

$$\Rightarrow 2\lambda^2 - 8\lambda + 8 - \lambda^3 + 4\lambda^2 - 4\lambda - 2 + \lambda = 0$$

$$\Rightarrow -\lambda^3 + 6\lambda^2 - 11\lambda + 6 = 0$$

$$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0 \quad \text{--- (1)}$$

The characteristic Eqn of matrix A

$$A^3 - 6A^2 + 11A - 6I = 0$$

$A^2 = A \cdot A$

$$\begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4+0+1 & 0+0+0 & -2+0-2 \\ 0+0+0 & 0+4+0 & 0+0+0 \\ -2+0-2 & 0+0+0 & 1+0+4 \end{bmatrix}$$

$A^3 = A^2 \cdot A$

$$= \begin{bmatrix} 5 & 0 & -4 \\ 0 & 4 & 0 \\ -4 & 0 & 5 \end{bmatrix} \begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 10+0+4 & 0+0+0 & -5+0-8 \\ 0+0+0 & 0+8+0 & 0+0+0 \\ -8+0-5 & 0+0+0 & 4+0+10 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 14 & 0 & -13 \\ 0 & 8 & 0 \\ -13 & 0 & 14 \end{bmatrix}$$

$A^3 - 6A^2 + 11A - 6I = 0$

$$\rightarrow \begin{bmatrix} 14 & 0 & -18 \\ 0 & 8 & 0 \\ -13 & 0 & 14 \end{bmatrix} - 6\begin{bmatrix} 5 & 0 & -4 \\ 0 & 4 & 0 \\ -4 & 0 & 5 \end{bmatrix} + 11\begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix} - 6\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 0$$

$$\rightarrow \begin{bmatrix} 14 & 0 & -17 \\ 0 & 8 & 0 \\ -13 & 0 & 14 \end{bmatrix} - \begin{bmatrix} 30 & 0 & -24 \\ 0 & 24 & 0 \\ -24 & 0 & 30 \end{bmatrix} + \begin{bmatrix} 22 & 0 & -11 \\ 0 & 22 & 0 \\ -11 & 0 & 22 \end{bmatrix} - 6\begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix} = 0$$

$$A^T = \begin{bmatrix} -44 & -33 & -46 \\ -24 & -13 & -24 \\ -22 & -14 & -8 \end{bmatrix} + \begin{bmatrix} 252 & 54 & 144 \\ 216 & 162 & -36 \\ 26 & 72 & 252 \end{bmatrix} + \begin{bmatrix} 40 & 80 & 120 \\ 80 & -40 & 160 \\ 120 & 40 & -40 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 248 & 101 & 218 \\ 272 & 109 & 50 \\ 104 & 98 & 204 \end{bmatrix}$$

Diagonalization:

characteristics Eqn of matrix A

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{vmatrix} = 0$$

characteristics eqn of matrix of order 3

$$\Rightarrow 3-\lambda\left[15-5\lambda-3\lambda+\lambda^2-1\right] + 1\left[-3+\lambda+1\right] + 1\left[1-5+\lambda\right] = 0$$

$$\Rightarrow 3\lambda^2 - 24\lambda + 42 - \lambda^3 + 8\lambda^2 - 14\lambda + \lambda - 2 + \lambda - 4 = 0$$

$$\Rightarrow -\lambda^3 + 11\lambda^2 - 36 + 36 = 0$$

$\lambda - 2$ is one of the root

$$\begin{array}{c|cccc} 2 & -1 & 11 & -36 & 36 \\ & 0 & -2 & 18 & -36 \\ \hline & -1 & 9 & -18 & \boxed{0} \end{array}$$

$$\lambda^2 - 9\lambda + 18 = 0$$

$$\lambda(\lambda - 6) - 3(\lambda - 6) = 0$$

Eigen values are $\lambda = 2, 3, 6$

To find Eigen vectors:

$$\rightarrow \begin{bmatrix} 1 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$x - y + z = 0$ — ① $-x + 3y - z = 0$ — ②

solve ① & ②

$$\begin{array}{ccc} x & y & z \\ -1 & 1 & 1 \\ 3 & -1 & -1 \end{array}$$

$$\frac{x}{1-3} = \frac{y}{-1+1} = \frac{z}{3-1} = t$$

$x = t \quad y = 0 \quad z = -t$

$$X_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} t \\ 0 \\ -t \end{bmatrix} \rightarrow t \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = t X_1$$

X_1 is the Eigen vector corresponding to Eigen value $\lambda = 2$

Case (ii) $\lambda = 3 \quad [A - \lambda I] X = 0$

$$\begin{bmatrix} 0 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$-y + z = 0$ — ① $-x + 2y - z = 0$ — ②

$x - y = 0$ — ③

$$\begin{vmatrix} x & y & z \\ -1 & 1 & 0 & -1 \\ -1 & 0 & 1 & -1 \end{vmatrix} \Rightarrow \frac{x}{1} = \frac{y}{1} = \frac{z}{1} = t$$

$x = t \quad y = t \quad z = t$

$$x_2 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = x_2 t$$

$\therefore x_2$ is the eigen vector corresponding to eigen value $\lambda = 3$

case (iii) $\lambda = 6 \quad (A - \lambda I)x = 0$

$$\Rightarrow \begin{bmatrix} -3 & -1 & 1 \\ -1 & -1 & -1 \\ 1 & -1 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$-3x - y + z = 0$ — ① $\quad -x - y - z = 0$ — ② $\quad x - y - 3y = 0$ — ③

solve ① & ②

$$\begin{vmatrix} x & y & z \\ -1 & 1 & -3 & -1 \\ -1 & 1 & -1 & -1 \end{vmatrix}$$

$$\frac{x}{1+1} = \frac{y}{-1-3} = \frac{z}{3-1} = t$$

$x = t \quad y = -2t \quad z = t$

$$x_3 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} t \\ -2t \\ t \end{bmatrix} \rightarrow t \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = x_3 t$$

② Determine diagonal matrix orthogonally similar to the real symmetric matrix

$$\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$ & also find modal matrix.

characteristics Eqn of matrix A is $|A-\lambda I|=0$

$$\Rightarrow \begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix}$$

characteristic Eqn of matrix A of order 3 is

$$\Rightarrow 8-\lambda\,[21-10\lambda+\lambda^2-16]+6\,[-18+6\lambda+8]+2\,[24-14+2\lambda]=0$$

$$\Rightarrow -\lambda^3+18\lambda^2-45\lambda=0$$

$\lambda=3$ is one of the root

$$\begin{array}{c|ccc} 3 & -1 & 18 & -45 \\ & 0 & -3 & 45 \\ \hline & -1 & 15 & \boxed{0} \end{array}$$

$$\lambda=15,0$$

$\therefore$ Eigen values are $\lambda=0,3,15$

To find the Eigen vectors

case (i) $\lambda=0$ in $|A-\lambda I|X=0$

$$\begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=0$$

$$\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=0$$

$8x-6y+2z=0$ — ① $\quad -6x+7y-4z=0$ — ②

$2x - 4y + 3z = 0$ — (3)

solve (1) & (2)

$$\begin{array}{c} x \quad y \quad z \\ \left| \begin{matrix} -6 & 2 & 8 & -6 \\ 7 & -4 & -6 & 7 \end{matrix} \right| \end{array}$$

$$\frac{x}{24-14} = \frac{y}{-12+32} = \frac{z}{36-36} = t$$

$x = t, \; y = 2t, \; z = 2t$

$$x_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} t \\ 2t \\ 2t \end{bmatrix} \Rightarrow t \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \Rightarrow x_1 t$$

x_1 is the eigen vector corresponding to $\lambda = 0$

case (ii) $\lambda = 3$

$$[A - \lambda I]x = 0$$

$$\begin{bmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$5x - 6y + 2z = 0$ — (1) $\quad -6x + 4y - 4z = 0$ — (2)

$2x - 4y + z = 0$ — (3)

solve (1) & (3)

$$\begin{array}{c} x \quad y \quad z \\ \left| \begin{matrix} -6 & 2 & 5 & -6 \\ -4 & 0 & 2 & -4 \end{matrix} \right| \end{array}$$

$$\frac{x}{-8} = \frac{y}{-20} = \frac{z}{-20+12} = t$$

$x = 2t, \; y = t, \; z = -2t$

$$X_2 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \Rightarrow t \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} \Rightarrow tX_2$$

X_2 is the eigan vector, corresponding to $\lambda = 3$

case (iii) $\lambda = 15$

$$|A - \lambda I| X = 0$$

$$\Rightarrow \begin{bmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$-7x - 6y + 2z = 0$ — ① $\quad -6x - 8y - 4z = 0$ — ②.

$2x - 4y - 12y = 0$ — ③

solve ① & ③

$$\begin{vmatrix} x & y & z \\ -6 & -2 & -7 & -6 \\ -4 & -12 & 2 & -4 \end{vmatrix}$$

$$\frac{x}{72+8} = \frac{y}{4-84} = \frac{z}{28+12} = t$$

$x = 2t \quad y = -2t \quad z = t$

$$X_3 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2t \\ -2t \\ t \end{bmatrix} \Rightarrow t \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} \Rightarrow tX_3$$

X_3 is the eigan vector corresponding to eigan value $\lambda = 15$

To find modal matrix P:

The modal matrix P having the normalized eigan vectors as its coloumn vectors

length of eigen vector $x_1 = \sqrt{1+4+4} = 3$

length of eigen vector $x_2 = \sqrt{4+1+4} = 3$

length of eigen vector $x_3 = \sqrt{4+4+1} = 3$

The normalized eigen vectors are

$$x_1 = \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix} \quad x_2 = \begin{bmatrix} 2/3 \\ 1/3 \\ -2/3 \end{bmatrix} \quad x_3 = \begin{bmatrix} 2/3 \\ -2/3 \\ 1/3 \end{bmatrix}$$

$$P = \begin{bmatrix} \frac{x_1}{\|x_1\|} & \frac{x_2}{\|x_2\|} & \frac{x_3}{\|x_3\|} \end{bmatrix}$$

$$P = \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix}$$

$$P^T = \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix}$$

$$D = P^T A P$$

$$D = \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix} \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix}$$

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{bmatrix}$$

(3.) Determine a diagonal matrix orthogonally similar to real symmetric matrix $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ also find modal matrix A is

$$[A - \lambda I] = 0$$

$$\Rightarrow \begin{bmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{bmatrix} = 0$$

characteristics eqn of matrix A of order 3 is

$$\Rightarrow 6-\lambda[9-3\lambda-3\lambda+\lambda^2-1]+2[-6+2\lambda+2]+2[2-6+2\lambda]=0$$

$$\Rightarrow 6\lambda^2-36\lambda+48-\lambda^3+6\lambda^2-8\lambda+4\lambda-8+4\lambda-8=0$$

$$\Rightarrow -\lambda^3+12\lambda^2-36\lambda+32=0$$

$\lambda = 2$ is one of the root

$$\begin{array}{c|cccc} 2 & -1 & +12 & -36 & 32 \\ & 0 & -2 & 20 & -32 \\ \hline & -1 & 10 & -16 & \boxed{0} \end{array}$$

$$\lambda^2-10\lambda+16=0$$

$$\lambda(\lambda-2)-8(\lambda-2)=0$$

$$\lambda-8 \quad \lambda=2$$

$\therefore$ Eigen values are $\lambda = 2, 2, 8$

To find Eigen vectors:

Case (i) $\lambda=2$ in $[A-\lambda I]X=0$

$$\Rightarrow \begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow 2x+y-z=0$ — (1) $\quad 2x-y+z=0$ — (2)

$x-2y+z=0$

$z = t_2$ $y = t_1$

$2x - t_1 + t_2 = 0$

$2x = t_1 - t_2$

$x = \frac{t_1 - t_2}{2}$

$$X_2 = \begin{bmatrix} \frac{t_1}{2} - \frac{t_2}{2} \\ t_1 \\ t_2 \end{bmatrix} \Rightarrow t_1 \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} -1/2 \\ 0 \\ 1 \end{bmatrix}$$

$\Rightarrow x_2 t_1 + x_3 t_2$

where X_2 is the eigen vector corresponding no eigen value $\lambda = 2$ x_2, x_3 are linearly independent eigen which are not mutually orthogonal eigen vector

case (ii)

$\lambda = 3$

$[A - \lambda I]X = 0$

$$\Rightarrow \begin{bmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$-2x - 2y + 3z = 0$ — (1) $-2x - 5y - z = 0$ — (2)

$2x - 1y - 5z = 0$ — (3)

solve (1) & (2)

$$\begin{vmatrix} x & y & z \\ -2 & 2 & -2 & -2 \\ -5 & -1 & -2 & -5 \end{vmatrix}$$

$\frac{x}{2+10} = \frac{y}{-4-2} = \frac{z}{10-4} = t$

$$\frac{x}{12} = \frac{y}{-6} = \frac{z}{6} = t$$

$$\frac{x}{2} = \frac{y}{-1} = \frac{z}{1} = t$$

$x = 2t \quad y = -t \quad z = t$

$$x_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2t \\ -t \\ t \end{bmatrix} \Rightarrow t\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \Rightarrow x_1 t$$

where x_1 is the eigen vector corresponding to eigen value $\lambda = 8$

The matrix A has three linearly independent eigen value $\lambda = 8$

The matrix A has three linearly independent eigen vectors which is same as its order matrix A

$\therefore$ The matrix A is diagonalized

let $x_3 = \begin{bmatrix} l \\ m \\ n \end{bmatrix}$

$$x_1^T x_3 = 0$$

$$\begin{bmatrix} 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} l \\ m \\ n \end{bmatrix} = 0$$

$$2l - m + n = 0 \quad \text{--- (1)}$$

$$x_2^T x_3 = 0$$

$$\begin{bmatrix} 1/2 & 1 & 0 \end{bmatrix} \begin{bmatrix} l \\ m \\ n \end{bmatrix} = 0$$

$$l/2 + m + 0(n) = 0 \quad \text{--- (2)}$$

solve ① & ②

$$\begin{vmatrix} l & m & n \\ -1 & 1 & 2 \\ 1 & 0 & 1/2 \end{vmatrix}$$

$$\frac{l}{-1} = \frac{m}{1/2} = \frac{n}{5/2} = t$$

$$\frac{l}{2} = \frac{m}{-1} = \frac{n}{-5} = t$$

$$x_3 = \begin{bmatrix} l \\ m \\ n \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -5 \end{bmatrix}$$

x_1, x_2, x_3 are orthogonal eigen vectors,

verify

$x_1^T x_2 = 0$

$$\begin{bmatrix} 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} = 0$$

$\cancel{x - x = 0}$

$x_2^T x_3 = 0$

$$\begin{bmatrix} 1/2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1/2 \\ 0 \\ 1 \end{bmatrix} \rightarrow 0 + 0 = 0$$

To find modal matrix P :

'P' having normalized eigen vectors as its column vectors $P = \left[\frac{x_1}{\|x_1\|}, \frac{x_2}{\|x_2\|}, \frac{x_3}{\|x_3\|} \right]$

length of eigen vector $\|x_1\| = \sqrt{4+1+1} = \sqrt{6}$

length of eigen vector $x_2 = \sqrt{1/4 + 1} = \frac{\sqrt{5}}{2}$

length of eigen vector $x_3 = \sqrt{4+1+25} = \sqrt{20}$

The normalized eigen vectors

$$x_1 = \begin{bmatrix} 2/\sqrt{6} \\ -1/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix} \quad x_2 = \begin{bmatrix} 1/\sqrt{5} \\ 2\sqrt{5} \\ 0 \end{bmatrix} \quad x_3 = \begin{bmatrix} 2/\sqrt{30} \\ -1/\sqrt{30} \\ -5/\sqrt{30} \end{bmatrix}$$

$$P = \begin{bmatrix} 2\sqrt{6} & 1/\sqrt{5} & 2/\sqrt{30} \\ -1/\sqrt{6} & 2/\sqrt{5} & -1/\sqrt{30} \\ 1/\sqrt{6} & 0 & -5/\sqrt{30} \end{bmatrix}$$

$P^T = P^{-1}$ (A is real symmetric matrix)

$$P^T = \begin{bmatrix} 2/\sqrt{6} & -1/\sqrt{6} & 1/\sqrt{6} \\ 1/\sqrt{5} & 2/\sqrt{5} & 0 \\ 2/\sqrt{30} & -1/\sqrt{30} & -5/\sqrt{30} \end{bmatrix}$$

$D = P^{-1}AP$

$D = P^TAP$

$$D = \begin{bmatrix} 2\sqrt{6} & -1/\sqrt{6} & 1/\sqrt{6} \\ 1/\sqrt{5} & 2\sqrt{5} & 0 \\ 2\sqrt{30} & -1/\sqrt{30} & -5/\sqrt{30} \end{bmatrix} \begin{bmatrix} 6 & -2 & -2 \\ -2 & 3 & -1 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} 2/\sqrt{6} & 1/\sqrt{5} & 2/\sqrt{30} \\ -1/\sqrt{6} & 2\sqrt{5} & -1/\sqrt{30} \\ 1/\sqrt{6} & 0 & -5/\sqrt{30} \end{bmatrix}$$

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

A. Determine the modal matrix to the matrix.

$$A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$ verify $P^{-1}AP$ is diagonal matrix

5. Determine the diagonal matrix orthogonal similar to the following real symmetric matrices and also find the modal matrix.

i) $\begin{bmatrix} 7 & 4 & -4 \\ 4 & -8 & -1 \\ -4 & -1 & -8 \end{bmatrix}$

characteristics eqn of matrix A is $|A-\lambda I|=0$

$\Rightarrow \begin{vmatrix} 7-\lambda & 4 & -4 \\ 4 & -8-\lambda & -1 \\ -4 & -1 & -8-\lambda \end{vmatrix}=0$

characteristics eqn of matrix of order 3

$\Rightarrow 7-\lambda\left[\lambda^2+16\lambda+64-1\right]-4\left[-32-4\lambda-4\right]-4\left[-4-32-4\lambda\right]$

$\Rightarrow 7\lambda^2+112\lambda+441-\lambda^3-63\lambda-4\left[-4\lambda-36\right]-4\left[-4\lambda-36\right]=0$

$\rightarrow -\lambda^3-9\lambda^2+18\lambda+729=0$

$\lambda=9$ is one of the root

$$\begin{array}{r|rrrr} 9 & -1 & -9 & 81 & 729 \\ & 0 & -9 & -162 & -729 \\ \hline & -1 & -18 & -81 & \boxed{0} \end{array}$$

$\lambda^2+18\lambda+81=0$

$\lambda=9, -9, -9$ are the eigen vectors

To find eigen vectors:

case (i) $\lambda=9$ $(A-\lambda I)X=0$

$\begin{bmatrix} 7-9 & 4 & -4 \\ 4 & -8-9 & -1 \\ -4 & -1 & -17 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=0$

$\Rightarrow \begin{bmatrix} -2 & 4 & -4 \\ 4 & -17 & -1 \\ -4 & -1 & -17 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=0$

$-2x+4y-4z=0$ — (1) $\quad -4x-y-17z=0$ — (2)

solve (1) & (2)

$$\begin{vmatrix} x & y & z \\ 4 & -4 & -2 & 4 \\ -1 & -17 & -4 & -1 \end{vmatrix}$$

$$\frac{x}{-68-4} = \frac{y}{16-34} = \frac{z}{2+16} = t$$

$$\frac{x}{-72} = \frac{y}{18} = \frac{z}{18} = t$$

$x=4t \quad y=1t \quad z=-t$

$$x_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \Rightarrow t\begin{bmatrix} 4 \\ 1 \\ -1 \end{bmatrix} \Rightarrow x_1 t$$

$\therefore$ x_1 is the eigen value corresponding to eigen value case (ii) $\lambda=-9$ $[A-\lambda I]x=0$

$$= \begin{bmatrix} 16 & 4 & -4 \\ 4 & 1 & -1 \\ -4 & -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=0 \qquad z=t_2 \quad y=t_1$$

$4x+y-z=0$ — (1)

$4x+t_1-t_2=0$

$4x=t_2-t_1$

$$x=\frac{t_2-t_1}{4}$$

$$x_2 \begin{bmatrix} x \\ y \\ z \end{bmatrix} \Rightarrow \begin{bmatrix} \frac{t_2-t_1}{4} \\ t_1 \\ t_2 \end{bmatrix} \Rightarrow t_1\begin{bmatrix} 1/4 \\ 1 \\ 0 \end{bmatrix} + t_2\begin{bmatrix} 1/4 \\ 0 \\ 1 \end{bmatrix}$$

$x_2 t_1 + x_3 t_2$

x_2 is the eigen vector corresponding to eigen value $\lambda = -9$

x_1, x_3 are two not mutually orthogonal vectors corresponding to $\lambda = -9$.

The matrix A has three linearly independent eigen vector which is same as its order matrix A

$\therefore$ The matrix A is diagonalized

let $x_3 = \begin{bmatrix} l \\ m \\ n \end{bmatrix}$

$$x_1^T x_3 = 0$$

$$\begin{bmatrix} 4 & 1 & -1 \end{bmatrix} \begin{bmatrix} l \\ m \\ n \end{bmatrix} = 0$$

$$4l + m - n = 0 \quad \text{--- (1)}$$

$$x_2^T x_3 = 0$$

$$\begin{bmatrix} -1/4 & 1 & 0 \end{bmatrix} \begin{bmatrix} l \\ m \\ n \end{bmatrix}$$

$$\Rightarrow -\frac{1}{4} l + m = 0 \quad \text{--- (2)}$$

solve (2) & (1)

$$\begin{array}{ccc} l & m & n \end{array}$$
$$\begin{vmatrix} 1 & 1 & -4 & 1 \\ 1 & 0 & -1/4 & 1 \end{vmatrix}$$

$$\frac{l}{1} = \frac{m}{1/4} = \frac{n}{4 + 1/4} = t$$

$l = t$ mate/u $n = 17/4$

$$x_3 = \begin{bmatrix} l \\ m \\ n \end{bmatrix} = \begin{bmatrix} 1 \\ 1/4 \\ 17/4 \end{bmatrix}$$

x_1, x_2, x_3 are orthogonal eigen vectors

To find: modal matrix P

'P' having normalized eigen vectors as its column vectors

length of eigen vector $x_1 = \sqrt{16+1+1} = \sqrt{18}$

length of eigen vector $x_2 = \sqrt{1/6+1} = \sqrt{17}/4$

length of eigen vector $x_3 = \sqrt{1+\frac{1}{16}+\frac{289}{16}} = \sqrt{\frac{306}{4}}$

The normalized eigen vectors.

$$x_1 = \begin{bmatrix} 4/\sqrt{18} \\ 1/\sqrt{18} \\ -1/\sqrt{18} \end{bmatrix} \quad x_2 = \begin{bmatrix} -1/\sqrt{17} \\ 4/\sqrt{17} \\ 0 \end{bmatrix} \quad x_3 = \begin{bmatrix} 4/\sqrt{306} \\ 1/\sqrt{306} \\ 17/\sqrt{306} \end{bmatrix}$$

$$P = \begin{bmatrix} 4/\sqrt{18} & -1/\sqrt{17} & 4/\sqrt{306} \\ 1/\sqrt{18} & 4/\sqrt{17} & 1/\sqrt{306} \\ -1/\sqrt{18} & 0 & 17/\sqrt{306} \end{bmatrix}$$

$$P^T = \begin{bmatrix} 4/\sqrt{18} & 1/\sqrt{18} & -1/\sqrt{18} \\ -1/\sqrt{17} & 4/\sqrt{17} & 0 \\ 4/\sqrt{306} & 1/\sqrt{306} & 17/\sqrt{306} \end{bmatrix}$$

$$P = \begin{bmatrix} 4/\sqrt{18} & 1/\sqrt{18} & -1/\sqrt{18} \\ -1/\sqrt{17} & 4/\sqrt{17} & 0 \\ 4/\sqrt{306} & 1/\sqrt{306} & 17/\sqrt{306} \end{bmatrix}$$

$D = P^T A P$

$$D = \begin{bmatrix} 4/\sqrt{18} & 1/\sqrt{18} & -1/\sqrt{18} \\ -1/\sqrt{17} & 4/\sqrt{17} & 0 \\ 4/\sqrt{306} & 1/\sqrt{306} & 17/\sqrt{306} \end{bmatrix} \begin{bmatrix} 7 & 4 & -4 \\ -4 & -8 & -1 \\ 4 & -1 & -8 \end{bmatrix}$$

$$\begin{bmatrix} 4/\sqrt{18} & -1/\sqrt{12} & 4\sqrt{306} \\ 1/\sqrt{18} & 4/\sqrt{17} & 1/\sqrt{306} \\ -1/\sqrt{18} & 0 & 17/\sqrt{306} \end{bmatrix}$$

$$D = \begin{bmatrix} 9 & 0 & 0 \\ 0 & -9 & 0 \\ 0 & 0 & -9 \end{bmatrix}$$

Quadratic form (Q.F):

A homogenous of second degree in 'n' variables is called the quadratic form

matrix of an Quadratic form:

→ Quadratic form corresponding to a real symmetric matrix A can be written as $Q = X^T A X$

where $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$

linear transformation:

let $Q = X^T A X$ be a quadratic form $X = PY$ be a non-similar linear transformation then $Q = (PY)^T A (PY)$

$Q = Y^T P^T A P Y$

$Q = Y^T (P^T A P) Y$

$= Y^T D Y$

The form $Y^T D Y$ is called a linear transformation of the quadratic form

Rank of a quadratic form

→ The no. of the terms in canonical form is called the rank of quadratic form.

→ The no. of +ve terms to canonical form is called the index of the quadratic form. It is denoted by 'p'.

Singnature:

→ The diff b/w the no. of +ve and -ve eigan values in the canonical form is called the called the signature of quadratic form. It is denoted by s

Nature of Quadratic form by eigan values

Eigan value	Nature of eigan value
1. +ve definite	1. +ve eigan values ($\lambda = 1, 2, 3$)
2 +ve semi definite	2. +ve eigan values and atleast $\lambda = 0$
3 -ve definite	3. -ve eigan value
4 -ve semi definite	4. -ve eigan values and atleast one is '0'.
5. Indefinite.	5. +ve as well as -ve eigan values.

Excercise the following quadratic forms in matrix notation

1. $x^2 - 6xy + y^2$

2. $2x^2 + 3y^2 - 5z^2 - 2xy + 6xz - 10yz$

3. $x_1^2 + 2x_2^2 + 3x_3^2 + x_4^2 - 2x_1x_2 + 4x_1x_3 - 2x_1x_4 + 4x_2x_3 - 6x_2x_4 - 8x_3x_4$

1. Given Quadratic form $x^2 - 6xy + y^2 = Q$

$$Q = X^T A X$$

$$= \begin{array}{c} \\ [x\ y] \end{array} \times \begin{array}{c} \begin{array}{cc} x & y \end{array} \\ \begin{bmatrix} 1 & -3 \\ -3 & 1 \end{bmatrix} \end{array} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & -3 \\ -3 & 1 \end{bmatrix}$$

2. $Q = X^T A X$

$$= [x\ y\ z] \times \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{bmatrix} 2 & -1 & 3 \\ -1 & 3 & -5 \\ 3 & -5 & -5 \end{bmatrix} \end{array} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & -1 & 3 \\ -1 & 3 & -5 \\ 3 & -5 & -5 \end{bmatrix}$$

3. $Q = X^T A X$

$$[x_1\ x_2\ x_3\ x_4] \begin{array}{c} \\ x_1 \\ x_4 \\ x_3 \\ x_4 \end{array} \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_2 & x_4 \end{array} \\ \begin{bmatrix} 1 & -1 & 2 & -1 \\ -1 & 2 & 2 & -3 \\ 2 & 2 & 3 & -4 \\ -1 & -3 & -4 & 1 \end{bmatrix} \end{array} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$$\Rightarrow A = \begin{bmatrix} 1 & -1 & 2 & -1 \\ -1 & 2 & 2 & -3 \\ 2 & 2 & 3 & -4 \\ -1 & -3 & -4 & 1 \end{bmatrix}$$

II Determine the name, index, signature, rank of the following Quadratic forms

i) $x_1^2 + 5x_2^2 + x_3^2 + 2x_2x_3 + 6x_3x_1 + 2x_1x_2$

(ii) $6x_1^2 + 3x_2^2 + 3x_3^2 - 4x_1x_2 - 2x_2x_3 + 4x_3x_1$

(iii) $x_1^2 + 4x_2^2 + x_3^2 - 4x_1x_2 + 2x_3x_1 - 4x_2x_3$

i) Given Quadratic form

$$Q = x_1^2 + 5x_2^2 + x_3^2 + 2x_2x_3 + 6x_3x_1 + 2x_1x_2$$

$$Q = x^T A x$$

$$Q = [x_1\ x_2\ x_3] \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix} \end{array} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$$

charact Eqn is $|A - \lambda I| = 0$

$$\rightarrow \begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{vmatrix} = 0$$

$$\rightarrow (1-\lambda)[(5-\lambda)(1-\lambda) - 1] - 1[1(1-\lambda) - 3] + 3[1 - 3(5-\lambda)] = 0$$

$$\lambda^3 - 7\lambda^2 + 36 = 0 \quad \text{--- (1)}$$

$$\lambda = -2 \quad \begin{array}{|cccc} 1 & -7 & 0 & 36 \\ 0 & -2 & 18 & -36 \\ \hline 1 & -9 & 18 & \boxed{0} \end{array}$$

$$\lambda^2 - 9\lambda + 18 = 0$$

$$\lambda(\lambda - 6) - 3(\lambda - 6) = 0$$

$$\lambda = -2, 3, 6$$

Rank of = 3 (no. of non zero values)

Inder (p) = 2 (+ve eigen values)

Signature (s) = 1 (diff b/w +ve & -ve eigen values)

Nature = Indefinite (have +ve as well as -ve

eigan values)

ii) Given Of

$$Q = 6x_1^2 + 3x_2^2 + 3x_3^2 - 4x_1x_2 - 2x_2x_3 + 4x_3x_1$$

$$Q = x^T A x$$

$$= [x_1\ x_2\ x_3] \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} \end{array} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

charac eqn $[A - \lambda I] = 0$

$$\begin{bmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{bmatrix} = 0$$

$$\Rightarrow 6-\lambda\left[9 - 3\lambda - 3\lambda + \lambda^2 - 1\right] + 2\left[-6 + 2 + 2\right] + 2\left[2 - 6 + 2\lambda\right] = 0$$

$$\Rightarrow 6\lambda^2 - 36\lambda + 48 - \lambda^3 + 6\lambda^2 - 8\lambda + 4\lambda - 8 + 4\lambda - 8 = 0$$

$$-\lambda^3 + 12\lambda^2 - 36\lambda + 32 = 0$$

$\lambda = 2$ is one of the root

$$\begin{array}{c|cccc} 2 & -1 & 12 & -36 & 32 \\ & 0 & -2 & 20 & -32 \\ \hline & -1 & 10 & -16 & \boxed{0} \end{array}$$

$$\lambda^2 - 10\lambda + 16 = 0$$

$$\lambda = 2, 8$$

Rank $= 3$

Index $(p) = 3$

signature (s) = 3

Nature = +ve definite

iii) Given of

$$Q = x_1^2 + 4x_2^2 + x_3^2 - 4x_1x_2 + 2x_3x_1 - 4x_2x_3$$

$$Q = x^T A x$$

$$= [x_1 \; x_2 \; x_3] \begin{array}{c} x_1 \; x_2 \; x_3 \\ \begin{bmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{bmatrix} \end{array} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{bmatrix}$$

charact eqn $[A - \lambda I] = 0$

$$\Rightarrow \begin{bmatrix} 1-\lambda & -2 & 1 \\ -2 & 4-\lambda & -2 \\ 1 & -2 & 1-\lambda \end{bmatrix} = 0$$

$$\Rightarrow 1-\lambda\left[4 - 4\lambda - \lambda + \lambda^2 - 4\right] + 2\left[-2 + 2\lambda + 2\right] + 1\left[4 - 4 + \lambda\right] = 0$$

$$= \lambda^2 - 5\lambda - \lambda^3 + 5\lambda^2 + 4\lambda + \lambda = 0$$

$$-\lambda^3 + 6\lambda^2 = 0$$

$\lambda = 0, 0, 6$ are the eigen values

Rank : No of non zero values = 1

Index : 1

Signature : 1

nature : +ve semi definite

III. Reduce the quadratic form $3x^2+5y^2+3z^2+2yz-2zx-2xy$ to the canonical form by orthogonal form by transformation & also find Index, rank, signature, nature of the quadratic form.

Given Q.F is

$$Q = 3x^2+5y^2+3z^2-2yz+2zx-2xy$$

$$Q = x^T A X$$

$$Q = \begin{array}{c} \\ [x\ y\ z] \end{array} \times \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix} \end{array} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

characteristics of A is $|A-\lambda I|=0$

$$\begin{vmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & -3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow 3-\lambda\left[15-5\lambda-3\lambda+\lambda^2-1\right]+1\left[-3+\lambda+1\right]+1\left[1-5+\lambda\right]=0$$

$$-\lambda^3+11\lambda^2-36\lambda+36=0$$

$\lambda = 2$ is one of the root

$$\begin{array}{c|cccc} 2 & -1 & 11 & -36 & 36 \\ & 0 & -2 & 18 & -36 \\ \hline & -1 & 9 & -18 & 0 \end{array}$$

$$\lambda^2-9\lambda+18=0$$

$$\lambda = 3, 6$$

$\therefore \lambda = 2, 3, 6$ are eigan values (from diagonalization problem ①

$$D = P^T A P$$

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

To find linear transformation :-

$$x = PY$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ 0 & 1/\sqrt{3} & -2/\sqrt{6} \\ -1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$x = 1/\sqrt{2}\, y_1 + \frac{1}{\sqrt{3}} y_2 + 1\sqrt{6}\, y_3$$

$$y = \frac{-1}{\sqrt{3}} y_2 - 2\sqrt{6}\, y_3$$

$$z = -\frac{1}{\sqrt{8}} y_1 + \frac{1}{\sqrt{3}} y_2 + \frac{1}{\sqrt{6}} y_3$$

The linear transformation

$$Q = Y^T D Y$$

$$= \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$= \begin{bmatrix} 2y_1 & 3y_2 & 6y_3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$Q = 2y_1^2 + 3y_2^2 + 6y_3^2$$

Rank $= 3$

Index $(p) = 3$

signature $(s) = 3$

signature (s) = 3

Nature = +ve definte

II Reduce the Q.D form $3x^2+2y^2+3z^2-2xy-2xy-2yz$ to the normal for (canonical form) by orthogonal transformation and find rank, index, signature, of the Q.F

$$Q = 3x^2+2y^2+3z^2-2xy-2yz$$

$$Q = X^T A X$$

$$Q = [x\,y\,z] \begin{array}{c|ccc} & x & y & z \\ \hline x & 3 & -1 & 0 \\ y & -1 & 2 & -1 \\ z & 0 & -1 & 3 \end{array} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$A = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix}$$

characteristics of A is $|A-\lambda I|=0$

$$\begin{vmatrix} 3-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow 3-\lambda\left[6-2\lambda-3\lambda+\lambda^2-1\right]+1\left[-3+\lambda\right]=0$$

$$\Rightarrow 3-\lambda\left[\lambda^2-5\lambda+5\right]-3+\lambda=0$$

$$\Rightarrow -\lambda^3+3\lambda^2-19\lambda+12=0$$

$$\lambda=3$$

$$\begin{array}{c|cccc} 3 & -1 & 8 & -19 & 12 \\ & 0 & -3 & 15 & -12 \\ \hline & -1 & 5 & -4 & \boxed{0} \end{array}$$

$$\lambda^2-5\lambda+4=0$$

$$\lambda(\lambda-1)-4(\lambda-1)=0$$

$\lambda = 1, 4$

$\therefore \lambda = 1, 3, 4$

To find Eigen vectors:

case (i) $\lambda = 1$ in $[A - \lambda I]X = 0$

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$2x - y = 0$ — (1) $\quad -x + y - z = 0$ — (2)

$-y + 2z = 0$ — (3)

solve (1) & (2)

$$\begin{vmatrix} -1 & 0 & 2 & -1 \\ 1 & -1 & -1 & 1 \end{vmatrix}$$

$\frac{x}{1} = \frac{y}{2} = \frac{z}{1} = t$

$x = t \quad y = 2t \quad z = t$

$$X_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \Rightarrow X_1 t$$

$\therefore X_1$ is the Eigen vector corresponding to Eigen value $\lambda = 1$

case (ii) $\lambda = 3$

$$= \begin{bmatrix} 0 & -1 & 0 \\ -1 & -1 & -1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$-x - y - z = 0$ — ① $\quad -y = 0$ — ②

solve ① & ②

$$\begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{vmatrix} -1 & -1 & -1 \\ -1 & 0 & 0 \\ & & -1 \end{vmatrix} \end{array}$$

$x = t \quad y = 0 \quad z = -t$

$$x_2 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} x_2 t$$

$\therefore x_3$ is the eigen vector corresponding to eigen value $\lambda = 4$ the matrix A has three linearly independent eigen vectors which is same as its order. The matrix A is diagonalized.

To find matrix P:

The modal matrix P having the normalized eigen vectors as its column vectors

length of eigen vector $x_1 = \sqrt{1+4+1} = \sqrt{6}$

length of eigen vector $x_2 = \sqrt{1+1} = \sqrt{2}$

length of eigen vector $x_3 = \sqrt{1+1+1} = \sqrt{3}$

The normalized eigen vectors are

$$x_1 = \begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix} \quad x_2 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix} \quad x_3 = \begin{bmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

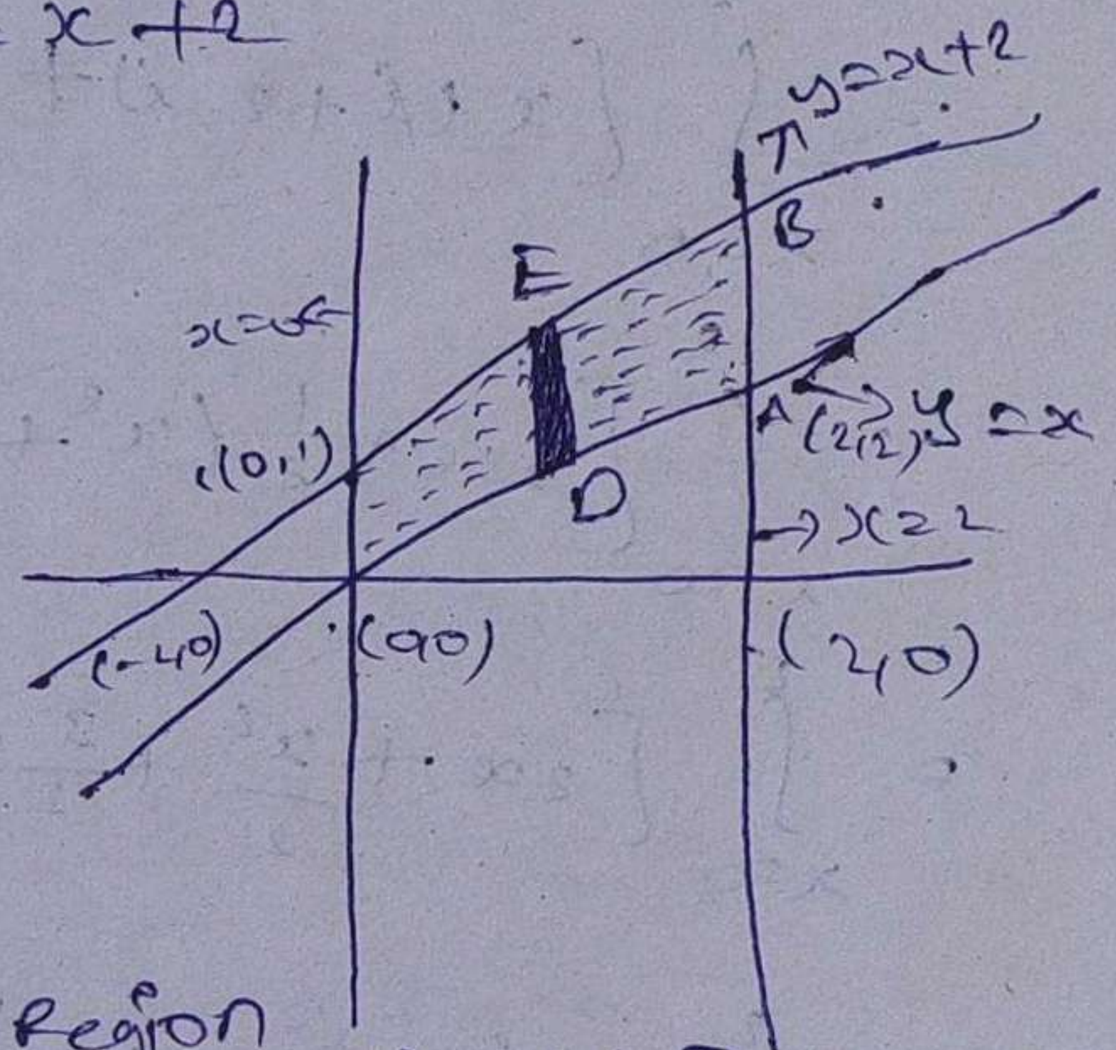
$P =$

* Evaluate $\iint_R (x+y)\,dy\,dx$ where R is the region bounded $x=0$, $x=2$, $y=x$, $y=x+2$

Sol Given integral is $\iint_R (x+y)\,dy\,dx$

and the given lines are $x=0$, $x=2$

$y=x$ and $y=x+2$



Here the common region is OABC and DE is the vertical strip in common region which is parallel to y-axis

The integrating points of $y=x$ & $x=2$ is $(2,2)$

and integrating point of $y=x+2$ of $x=2$ is $(2,4)$

and integrating point of $y=x+2$ & $x=0$ is $(0,2)$

$\Rightarrow$ The curve $y=x+2$ is intersecting on x-axis y-axis at $(-2,0)$ and $(0,2)$ respectively.

The limits of y are $y=x$ to $y=x+2$

The limits of x are $x=0$ to $x=2$

then $\iint (x+y)\,dy\,dx = \int_{x=0}^{2}\int_{y=x}^{x+2} (x+y)\,dy\,dx$

$$= \int_{x=0}^{2} \left[\int_{y=x}^{x+2} (x+y)\, dy \right] dx$$

$$= \int_{x=0}^{2} \left[xy + \frac{y^2}{2} \right]_{y=x}^{x+2} dx$$

$$= \int_{x=0}^{2} \left[x(x+2-x) + \frac{1}{2}\left((x+2)^2 - x\right) \right] dx$$

$$= \int_{x=0}^{2} \left[2x + \frac{1}{2}\left(x^2 + 4x + 4 - x\right) \right] dx$$

$$= \int_{x=0}^{2} \left[2x + \frac{x^2}{2} + \frac{3}{2}x + 2 \right) dx$$

$$= \int_{x=0}^{2} \left(\frac{x^2}{2} + \frac{7}{2}x + 2 \right) dx$$

$$= \left[\frac{x^3}{6} + \frac{7x^2}{4} + 2x \right]_{x=0}^{2}$$

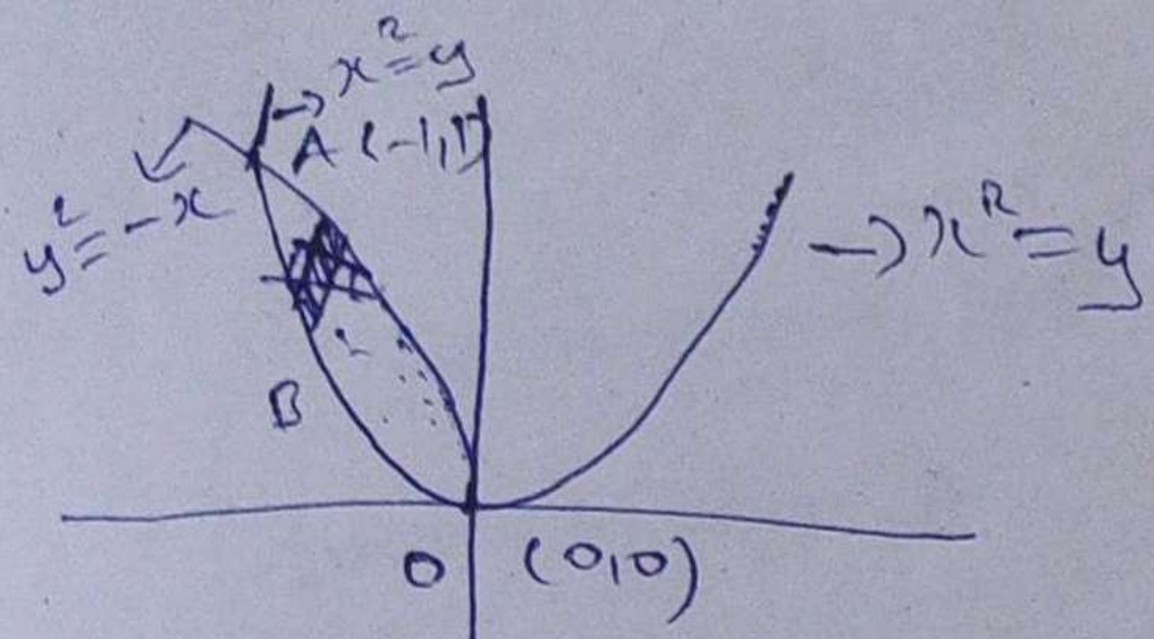
$$= \frac{8}{6} + 7 + 4$$

**

Evaluate $\iint xy\,(x+y)\, dx\, dy$. $x^2 = y$, $y^2 = -x$

<u>sol</u> Given integral is $\iint_R xy\,(x+y)\, dx\, dy$

and to given curves are $x^2 = y$, $y^2 = -x$



Here The common Region is OA and BC is the vertic common region, which is parallel to y-axis

The intersecting points of $x^2=y$, $y^2=-x$

$\Rightarrow x^2=y$, $y^2=-x$

Put $x=-y^2$ in $x^2=y$

$\Rightarrow (-y^2)^2=y$

$\Rightarrow y^4=y$

$\Rightarrow y^4-y=0$

$= y(y^3-1)=0$

$\Rightarrow y=0$; $y^3-1=0$

$\Rightarrow y^3=(1)^3$

$\Rightarrow y=0$, $y=1$

when $y=0$, $x=0$; $(0,0)$

when $y=1$; $x=-1$; $(-1,1)$

The limits of $x=-\sqrt{y}$ to $x=-y^2$

The limits of y are $y=0$ to $y=1$

Then $\iint_R xy(x+y)\,dx\,dy = \int_{y=0}^{1}\int_{x=-\sqrt{y}}^{-y^2} xy(x+y)\,dx\,dy$

$= \int_{y=0}^{1}\left[\int_{x=-\sqrt{y}}^{-y^2} xy(x+y)\,dx\right]dy$

Unit - V

Multivariables and calculus

(Partial Differentiation & Applications)

⇒ Partial derivatives : A partial derivative of a function of several variables is the ordinary derivative with respect to one of the variables, when all the remaining variables are kept constant.

⇒ consider a function $u(x,y)$ (mean's u is function two independented variables x and y) The partial derivative of $u(x,y)$ with wrt x is the derivative of u wrt, x is the ordinary derivative of u wrt. x, keeping y constant.

⇒ It is denoted by $\frac{\partial u}{\partial x}$ (or) u_x is known as first order partial derivative of u with respect to x

⇒ Similarly the partial derivative of $u(x,y)$ wrt y is the ordinary derivative u wrt to y treating x as constant. It is denoted by $\frac{\partial u}{\partial y}$ or u_y and is know as first order partial derivation of u wrt,

Higher order partial derivatives

Partial derivatives of higher order of function $u(x,y)$ are obtained by partial diversion of 1st order partial derivative.

1. $\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)$

2. $\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)$

3. $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)$

4. $\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)$

Are called second order partial derivatives similarly other higher order derivatives can also be obtained.

$\rightarrow \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$

Ex: let $u = x^2 y^2$

$\Rightarrow \frac{\partial u}{\partial x} = 2xy^2$; $\frac{\partial u}{\partial y} = 2x^2 y$

and $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)$

$\Rightarrow \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}(2x^2 y) = 4xy$

$\Rightarrow \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)$

$\Rightarrow \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}(2xy^2) = 4xy$

$\therefore \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$

*→ $u = e^x$ Then find $\frac{\partial^2 u}{\partial y \partial x}$

<u>sol</u> Given that $u = e^x$

$\Rightarrow \frac{\partial u}{\partial x} = e^x$ and $\frac{\partial u}{\partial y} = 0$

then $\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial y}(e^x) = 0$

*→ If $f = x^2 + y^2$ then find $\frac{\partial^2 f}{\partial x \partial y}$

<u>soln</u> Given that $f = x^2 + y^2$

Now diff 'f' w.r.to x & y partially

$\Rightarrow \frac{\partial f}{\partial x} = 2x$; $\frac{\partial f}{\partial y} = 2y$

then $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(2y) = 0$

*→ If $u = e^{ax} \sin by$ then prove that $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$

<u>soln</u>: Given that $u = e^{ax} \sin by$

Now diff 'u' w.r.to x & y partially

let $\frac{\partial u}{\partial x} = ae^{ax} \sin by + e^{ax}(0) = ae^{ax} \sin by$

and $\frac{\partial u}{\partial y} = be^{ax} \cos by + \sin by(0) = be^{ax} \cos by$

consider $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}(be^{ax} \cos by) = b$

$[ae^{ax} \cos by + e^{ax}(0)]$

$\Rightarrow \frac{\partial^2 u}{\partial x \partial y} = abe^{ax} \cos by$

and $\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial y}$

If $u = \log(x^3+y^3+z^3-3xyz)$ then find $x\frac{\partial u}{\partial x}$, $y\frac{\partial u}{\partial y}$, $z\frac{\partial u}{\partial z}$

Given that $u = \log(x^3+y^3+z^3-3xyz)$

now diff 'u' w.r.to x, y, z respectively

let $\frac{\partial u}{\partial x} = \frac{1}{x^3+y^3+z^3-3xyz} \times (3x^2-3yz)$

and $\frac{\partial x}{\partial y} = \frac{1}{x^3+y^3+z^3-3xyz} \times (3y^2-3xz)$

and $\frac{\partial u}{\partial z} = \frac{1}{x^3+y^3+z^3-3xyz} \times (3z^2-3xy)$

Then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = x\left(\frac{3x^2-3yz}{x^3+y^3+z^3-3xyz}\right) + y\left[\frac{3y^2-3xz}{x^3+y^3+z^3-3xyz}\right] + z\left[\frac{3z^2-3xy}{x^3+y^3+z^3-3xyz}\right]$

$$= \frac{3x^3 - 3xyz + 3y^3 - 3xyz + 3z^3 - 3xyz}{x^3+y^3+z^3-3xyz}$$

$$= \frac{3(x^3+y^3+z^3) - 9xyz}{x^3+y^3+z^3-3xyz}$$

$$= \frac{3\cancel{[x^3+y^3+z^3-3xyz]}}{\cancel{x^3+y^3+z^3-3xyz}}$$

$$= 3 //.$$

<u>Total derivative</u>: Let u is function of two independent variables x, y an x, y are function in terms t than

$$u \to x, y$$
$$x, y \to t$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial t}$$

<u>Chain rule</u>: (composit function of two variables)

let $u(x, y)$ is function of two independed variables x, y and (x, y) are function of two independed variables (r, θ) then

$$u \to x, y$$
$$x, y \to r, \theta$$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial r}$$

and $$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta}$$

Find $\frac{du}{dt}$, if $u = \frac{x}{y}$ where $x = e^t, y =$

$\sqrt{x}$ //

If $u = x^3 + y^3$ where $x = a\cos t$, $y = b\sin t$,

find $\frac{du}{dt}$

If $u = \frac{x}{y}$ where $x = e^t$; $y = \log t$

<u>soln</u> Given $u = \frac{x}{y}$

where $x = e^t$; $y = \log t$.

then $\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \frac{dy}{dt}$.

$= \frac{1}{y} \cdot e^t + \left(-\frac{x}{y^2}\right)\left(\frac{1}{t}\right)$

$\Rightarrow \frac{du}{dt} = \frac{e^t}{y} - \frac{x}{ty^2}$ //

If $u = x^3 + y^3$; where $x = a\cos t$; $y = b\sin t$

<u>soln</u> :- Given that $u = x^3 + y^3$

where $x = a\cos t$; $y = b\sin t$

Then $\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}$

$= 3x^2(-a\sin t) + 3y^2(b\cos t)$

$\Rightarrow \frac{du}{dt} = -3ax^2\sin t + 3by^2\cos t$ //

If $z = f(x, y)$, $x = uv$, $y = \frac{u+v}{u-v}$, prove that

$u\frac{\partial z}{\partial u} + v \cdot \frac{\partial z}{\partial v} = 2x\frac{\partial z}{\partial x}$

Soln Given $z = f(x, y)$

$x = uv \; ; \; y = \frac{u+v}{u-v}$

let $\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u}$ — ①

and $\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v}$ — ②

Here $\frac{\partial x}{\partial u} = v$, $\frac{\partial x}{\partial v} = u$

and $\frac{\partial y}{\partial x} = \frac{(u-v)(1) - (u+v)(1)}{(u-v)^2}$

$\Rightarrow \frac{\partial y}{\partial u} = \frac{-2v}{(u-v)^2}$

and $\frac{\partial y}{\partial v} = \frac{(u-v)(1) - (u+v)(-1)}{(u-v)^2}$

$\Rightarrow \frac{\partial y}{\partial v} = \frac{2u}{(u-v)^2}$

eqn ① $\Rightarrow \frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot v + \frac{\partial z}{\partial y}\left(\frac{-2v}{(u-v)^2}\right)$

eqn ② $\Rightarrow \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}(u) + \frac{\partial z}{\partial y}\left(\frac{2u}{(u-v)^2}\right)$

Then $u\frac{\partial z}{\partial u} + v\frac{\partial z}{\partial v} = u\left[\frac{\partial z}{\partial x} v + \frac{\partial z}{\partial y}\left(\frac{-2v}{(u-v)^2}\right)\right]$

$+ v\left[\frac{\partial z}{\partial x}(u) + \frac{\partial z}{\partial y}\left(\frac{2u}{(u-v)^2}\right)\right]$

$= \frac{\partial z}{\partial x}\left[uv + uv\right] + \frac{\partial z}{\partial y}\left[\frac{-2uv + 2uv}{(u-v)^2}\right]$

$= 2uv\frac{\partial z}{\partial x}$

$\Rightarrow u\frac{\partial z}{\partial x} + v\frac{\partial z}{\partial v} = 2x\frac{\partial z}{\partial x}$