Lecture # 3

* Gauss Divergence theorem

Gauss Divergence theorem:

Suppose V is the Volume bounded by a closed piecewise smooth surface S. Suppose $\bar{F}$ be a Vector point function of position which is continous and has continous first order partial derivatives on V. Then

$$\boxed{\iint_S \bar{F}\cdot\bar{n}\, ds = \iiint_V \operatorname{div}\bar{F}\, dv}$$

where $\bar{n}$ is the outward drawn unit normal vector to S

$$\iint_S \bar{F}\cdot\bar{n}\, ds = \iiint_V \nabla\cdot\bar{F}\, dv \quad \rightarrow (1)$$

Alternative form: Let $\bar{F} = F_1\bar{i} + F_2\bar{j} + F_3\bar{k}$.

If α, β, c are the angles which outward drawn unit normal $\bar{n}$ makes with positive directions of x, y and z-axes. then $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ are the direction cosines of $\bar{n}$ and we have

$$\bar{n} = \cos\alpha\,\bar{i} + \cos\beta\,\bar{j} + \cos\gamma\,\bar{k}$$

$$\Rightarrow \bar{F}\cdot\bar{n} = F_1\cos\alpha + F_2\cos\beta + F_3\cos\gamma$$

$$\Rightarrow \operatorname{div}\bar{F} = \nabla\cdot\bar{F} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

(1) $\Rightarrow$

$$\iiint_V \left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}\right) dx\,dy\,dz$$

$$= \iint_S \left(F_1\cos\alpha + F_2\cos\beta + F_3\cos\gamma\right) ds$$

$$= \iint_S \left[F_1\,dy\,dz + F_2\,dz\,dx + F_3\,dx\,dy\right]$$

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Note: Gauss Divergence theorem is the relation between Surface and Volume integrals.

→ Using this theorem, we evaluate Surface integral and Volume integral.

Numerical problems:

1) If $\bar{F} = ax\bar{i} + by\bar{j} + cz\bar{k}$, a, b, c are constants, show that $\iint_S \bar{F}\cdot\bar{n}\, ds = \frac{4}{3}\pi(a+b+c)$ where S is the surface of unit sphere.

proof: By divergence theorem

$$\iint_S \bar{F}\cdot\bar{n}\, ds = \iiint_V \mathrm{div}\bar{F}\, dv = \iiint_V \nabla\cdot\bar{F}\, dv$$

$$= \iiint_V \left[\frac{\partial}{\partial x}(ax) + \frac{\partial}{\partial y}(ay) + \frac{\partial}{\partial z}(az)\right] dv$$

$$= \iiint_V (a+b+c)\, dv$$

$$= (a+b+c)\iiint_V dv = (a+b+c)V$$

$$= \frac{4}{3}\pi r^3(a+b+c) = \frac{4}{3}\pi(a+b+c) \quad (\because r=1)$$

#

3. Evaluate $\iint_S x^3\,dy\,dz + x^2y\,dz\,dx + x^2z\,dx\,dy$ where S is the closed surface bounded by the planes $z=0$; $z=b$ & $x^2+y^2=a^2$.

<u>Solution</u>: From Cartesian form of Gauss divergence theorem

$$\iint_S F_1\,dy\,dz + F_2\,dx\,dz + F_3\,dx\,dy$$

$$= \iiint_V \left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}\right) dx\,dy\,dz$$

where $F_1 = x^3$: $F_2 = x^2y$, $F_3 = x^2z$

$$\Rightarrow \iint_S x^3\,dy\,dz + x^2y\,dz\,dx + x^2z\,dx\,dy$$

$$= \iiint_V (3x^2+x^2+x^2)\,dx\,dy\,dz$$

$$= \iiint_V 5x^2\,dx\,dy\,dz$$

$$= 5\int_{x=-a}^{a}\int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\int_{z=0}^{b} x^2\,dx\,dy\,dz$$

$$= 5\cdot 4\int_{x=0}^{a}\int_{y=0}^{\sqrt{a^2-x^2}}\int_{z=0}^{b} x^2\,dx\,dy\,dz$$

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$$= 20b \int_{x=0}^{a} x^2 \int_{0}^{\sqrt{a^2-x^2}} dx\, dy$$

$$= 20b \int_{x=0}^{a} x^2 \sqrt{a^2-x^2}\,.\, dx \qquad x = a\sin\theta \quad dx = a\cos\theta$$

$$= 20 \cdots$$

$$= \boxed{\frac{5}{4}\pi a^4 b}$$

3). Evaluate $\iint_S [x^2\, dy\, dz + y^2\, dz\, dx + 2z(xy-x-y)\, dx\, dy]$ where S is the Surface of the cube $0 \le x \le 1,\ 0 \le y \le 1,\ 0 \le z \le 1$ $(1/2)$

Solution: By GDT

$$\iint_S [F_1\, dy\, dz + F_2\, dx\, dz + F_3\, dx\, dy] = \iiint_V \left[\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right] dV$$

$$\iint_S [x^2\, dy\, dz + y^2\, dz\, dx + 2z(xy-x-y)\, dx\, dy]$$

$$= \iiint_V [2x + 2y + 2xy - 2x - 2y]\, dx\, dy\, dz$$

$$= \int_{x=0}^{1} \int_{y=0}^{1} \int_{z=0}^{1} 2xy\, dx\, dy\, dz = \boxed{\frac{1}{2}}$$

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4) Verify GDT for $\bar{F} = (x^3 - yz)\bar{i} - 2x^2y\bar{j} + z\bar{k}$ taken over the surface of the cube bounded by the planes $x = y = z = a$ and coordinate planes.

<u>Solution</u>: By GDT, we have

$$\iint_S \bar{F}\cdot\bar{n}\,ds = \iiint_V \operatorname{div}\bar{F}\,dv$$

Given $\bar{F} = (x^3 - yz)\bar{i} - 2x^2y\bar{j} + z\bar{k}$

$$\operatorname{div}\bar{F} = \nabla\cdot\bar{F} = \frac{\partial}{\partial x}(x^3 - yz) + \frac{\partial}{\partial y}(-2x^2y) + \frac{\partial}{\partial z}(z)$$

$$= 3x^2 - 2x^2 + 1$$

$$\therefore \operatorname{div}\bar{F} = x^2 + 1$$

<u>R.H.S</u>: $\iiint_V \operatorname{div}\bar{F}\,dv = \iiint_V (x^2+1)\,dx\,dy\,dz$

$$= \int_{x=0}^{a}\int_{y=0}^{a}\int_{z=0}^{a} (x^2+1)\,dx\,dy\,dz$$

$$= \int_{y=0}^{a} \int_{z=0}^{a} \left(\frac{x^3}{3}+x\right)_0^a dy\,dz$$

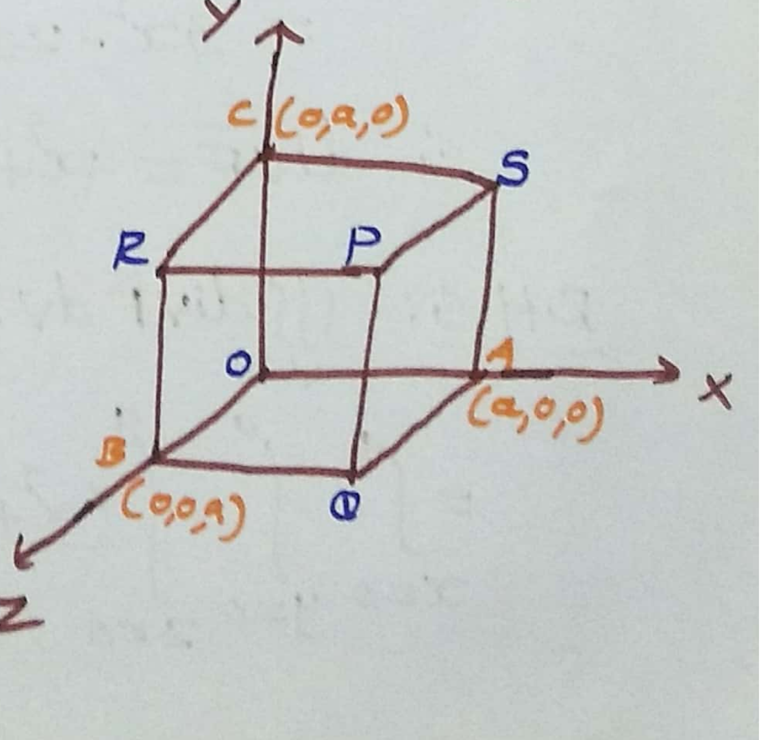
$$= \int_{y=0}^{a} \int_{z=0}^{a} \left(\frac{a^3}{3}+a\right) dy\,dz$$

$$= \left(\frac{a^3}{3}+a\right) \int_{y=0}^{a} [z]_0^a\, dy$$

$$= a\left(\frac{a^3}{3}+a\right) \int_{y=0}^{a} dy = a\,a\left(\frac{a^3}{3}+a\right)$$

$$RHS = \iiint_V \operatorname{div}\bar{F}\, dv = \frac{a^5}{3}+a^3 \longrightarrow (1)$$

<u>Verification</u>: Now we will find the value of $\iint_S \bar{F}\cdot\bar{n}\, ds$ over the 6 faces of the cube are



S_1: PQAS

S_2: OCRB

S_3: RPQB

S_4: OASC

S_5: PSCR

S_6: OAQB

$$\therefore \iint_S \bar{F}\cdot\bar{n}\,ds = \iint_{S_1} \bar{F}\cdot\bar{n}\,ds + \iint_{S_2} \bar{F}\cdot\bar{n}\,ds + \cdots\cdots + \iint_{S_6} \bar{F}\cdot\bar{n}\,ds \longrightarrow (2)$$

For S_1: PQAS: $x = a$, $ds = dy\,dz$

outward drawn unit normal vector $\bar{n} = \bar{i}$

and $y = 0$ to a : $z = 0$ to a

$$\bar{F}\cdot\bar{n} = ((x^3 - yz)\bar{i} - 2x^2y\,\bar{j} + z\bar{k})\cdot(\bar{i})$$

$$= x^3 - yz$$

$$\therefore \bar{F}\cdot\bar{n} = a^3 - yz \quad (\because x = a)$$

$$\iint_{S_1} \bar{F}\cdot\bar{n}\,ds = \int_{y=0}^{a}\int_{z=0}^{a} (a^3 - yz)\,dy\,dz = a^5 - \frac{a^4}{4} \longrightarrow (3)$$

For S_2: OCRB: $x = 0$: $ds = dy\,dz$: $\bar{n} = -\bar{i}$

$$\therefore \bar{F}\cdot\bar{n} = 0 - (x^3 - yz) = yz \quad (\because x = 0)$$

$$\therefore \iint_{S_2} \bar{F}\cdot\bar{n}\,ds = 0 \longrightarrow (3)$$

$$= \int_{y=0}^{a}\int_{z=0}^{a} yz\,dy\,dz$$

~~For S_3~~:

$$\iint_{S_2} \bar{F}\cdot\bar{n}\,ds = \frac{a^4}{4} \longrightarrow (4)$$

For S_3: RBQP: $z=a$; $ds = dx\,dy$: $\bar{n} = \bar{k}$

$x = 0$ to a : $y = 0$ to a

$$\bar{F}\cdot\bar{n} = z = a \quad (\because z = a)$$

$$\iint_{S_3} \bar{F}\cdot\bar{n}\,ds = \int_{x=0}^{a}\int_{y=0}^{a} a\,dx\,dy = a^3 \longrightarrow (5)$$

For S_4: OASC: $z=0$: $ds = dx\,dy$: $\bar{n} = -\bar{k}$

$x = 0$ to a : $y = 0$ to a

$$\bar{F}\cdot\bar{n} = z = 0 \quad (\because z = 0)$$

$$\therefore \iint_{S_4} \bar{F}\cdot\bar{n}\,ds = 0 \longrightarrow (6)$$

For S_5: PSCR: $y = a$: $ds = dx\,dz$, $\bar{n} = \bar{j}$

$x = 0$ to a : $y = 0$ to a

$$\bar{F}\cdot\bar{n} = -2x^2y$$

$$\bar{F}\cdot\bar{n} = -2ax^2 \quad (\because y = a)$$

$$\iint_{S_5} \bar{F}\cdot\bar{n}\,ds = \iint_{S_5} (-2ax^2)\,dx\,dz$$

$$= \int_{x=0}^{a}\int_{z=0}^{a} (-2ax^2)\,dx\,dz = -\frac{2a^5}{3}$$

$$\therefore \iint_{S_5} \bar{F}\cdot\bar{n}\,ds = -\frac{2}{3}a^5 \longrightarrow (7)$$

For S_6: OBRA: $y=0$: $ds = dx\,dz$, $\bar{n} = -\bar{j}$

$x = 0$ to a; $z = 0$ to a

$\bar{F}\cdot\bar{n} = 2x^2y$

$\bar{F}\cdot\bar{n} = 0$ ($\because y=0$)

$$\therefore \iint_{S_6} \bar{F}\cdot\bar{n}\,ds = 0 \longrightarrow (8)$$

Sub (3), (4), (5), (6), (7), (8) in (2)

$$\therefore \iint_S \bar{F}\cdot\bar{n}\,ds = a^5 - \frac{a^4}{4} + \frac{a^4}{4} + a^3 + 0 - \frac{2}{3}a^5 + 0$$

$$\iint_S \bar{F}\cdot n\,ds = \frac{a^5}{3} + a^3$$

$$= \iiint_V \text{div}\bar{F}\,dv \quad (\text{from } (1))$$

$$\therefore \iint_S \bar{F}\cdot\bar{n}\,ds = \iiint_V \text{div}\bar{F}\,dv$$

Hence GDT is Verified

5): Verify GDT for the vector field $\bar{F} = 4x\bar{i} - 2y^2\bar{j} + z^2\bar{k}$ taken over the region bounded by cylinder $x^2+y^2=4$ & $z=0$ and $z=3$

Solution: By GDT $\iint_S \bar{F}\cdot\bar{n}\,ds = \iiint_V \operatorname{div}\bar{F}\,dv$

$$\bar{F} = 4x\bar{i} - 2y^2\bar{j} + z^2\bar{k}$$

$$\operatorname{div}\bar{F} = 4 - 4y + 2z$$

$$RHS = \iiint_V \operatorname{div}\bar{F}\,dv$$

$$= \int_x\int_y\int_z (4-4y+2z)\,dx\,dy\,dz$$

Limits of z : $z = 0$ to 3

" " y : $y = -\sqrt{4-x^2}$ to $\sqrt{4-x^2}$

" " x : $x = -2$ to 2

$$\iiint_V \operatorname{div}\bar{F}\,dv = \int_{x=-2}^{2}\int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{z=0}^{3} (4-4y+2z)\,dx\,dy\,dz$$

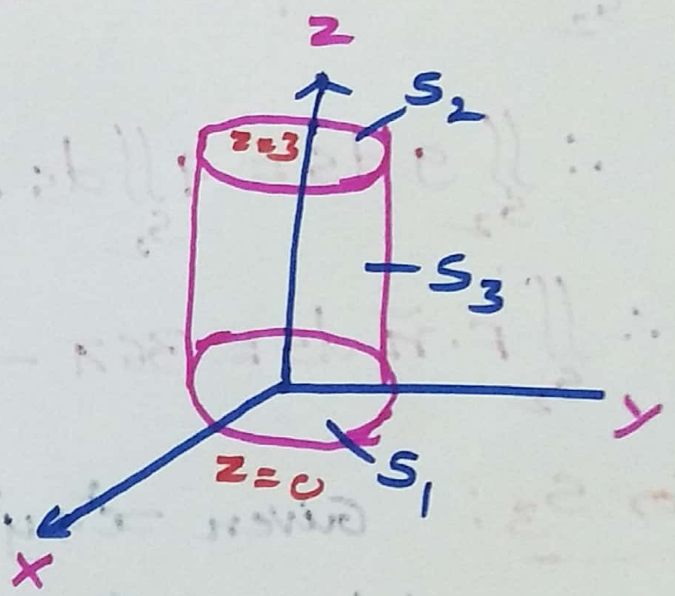
$$\therefore \iiint_V \operatorname{div}\bar{F}\,dv = 84\pi \longrightarrow (1)$$

Verification : L.H.S $\iint_S \bar{F}\cdot\bar{n}\,ds$:

Now we will calculate

$\iint_S \bar{F}\cdot\bar{n}\,ds$ over the 3

faces of cylinder S_1, S_2, S_3



$$\iint_S \bar{F}\cdot\bar{n}\,ds = \iint_{S_1} \bar{F}\cdot\bar{n}\,ds + \iint_{S_2} \bar{F}\cdot\bar{n}\,ds + \iint_{S_3} \bar{F}\cdot\bar{n}\,ds \longrightarrow (2)$$

For S_1 : $z=0$: $\bar{n} = -\bar{k}$: $ds = dx\,dy$

$\bar{F}\cdot\bar{n} = (4x\bar{i} - 2y^2\bar{j} + z^2\bar{k})\cdot(-\bar{k})$

$= -z^2$

$\bar{F}\cdot\bar{n} = 0 \quad (\because z=0)$

$\therefore \iint_{S_1} \bar{F}\cdot\bar{n} = 0 \longrightarrow (3)$

For S_2 : $z=3$ ∴ $\bar{n}=\bar{k}$

$$\bar{F}\cdot\bar{n} = z^2 = 9 \quad (\because z=3)$$

$$\iint_{S_2} \bar{F}\cdot\bar{n}\, ds = \iint_{S_2} 9\, ds = 9\iint_{S_2} ds$$

$$\iint_{S_2} ds = \text{Surface Area} = \pi r^2 \quad (\because r=2)$$
$$= 4\pi$$

$$\therefore \iint_{S_2} 9\, ds = 9\iint_{S_2} ds = 9(4\pi) = 36\pi$$

$$\therefore \iint_{S_2} \bar{F}\cdot\bar{n}\, ds = 36\pi \longrightarrow (4)$$

For S_3 : Given $x^2+y^2=4$

let $\phi = x^2+y^2-4$

$$\nabla\phi = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$$

$$\nabla\phi = 2x\bar{i} + 2y\bar{j}$$

unit normal vector $\bar{n} = \dfrac{\nabla\phi}{|\nabla\phi|}$

$$= \frac{(2x\bar{i}+2y\bar{j})}{2\sqrt{x^2+y^2}}$$

$$\bar{n} = \frac{x\bar{i}+y\bar{j}}{2}$$

$$\bar{F}\cdot\bar{n} = (4x\bar{i} - 2y^2\bar{j} + z^2\bar{k})\cdot(x\bar{i} + y\bar{j})\frac{1}{2}$$

$$= \frac{1}{2}(4x^2 - 2y^3)$$

$$\bar{F}\cdot\bar{n} = 2x^2 - y^3$$

$$\therefore \iint_{S_3} \bar{F}\cdot\bar{n}\, ds = \iint_{S_3} (2x^2 - y^3)\, ds_3$$

$$x = r\cos\theta = 2\cos\theta \;:\; y = r\sin\theta = 2\sin\theta$$

$$ds_3 = r\, d\theta\, dz = 2\, d\theta\, dz$$

Limits of θ: $\theta = 0$ to 2π

" " z: $z = 0$ to 3

$$\iint_{S_3} \bar{F}\cdot\bar{n}\, ds = \int_{\theta=0}^{2\pi}\int_{z=0}^{3} 2(2\cos\theta)^2 - (2\sin\theta)^3\, 2\, d\theta\, dz$$

$$= \int_0^{2\pi}\int_0^{3} 16(\cos^2\theta - \sin^3\theta)\, d\theta\, dz$$

$$= 16\int_{\theta=0}^{2\pi} (\cos^2\theta - \sin^3\theta)(z)_0^3\, d\theta$$

$$= 48\int_{\theta=0}^{2\pi} (\cos^2\theta - \sin^3\theta)\, d\theta$$

$$\cos^2\theta = \frac{1+\cos 2\theta}{2} \;:\; \sin^3\theta = \frac{1}{4}[3\sin\theta - \sin 3\theta]$$

$$= 48\int_{\theta=0}^{2\pi}\left[\left(\frac{1+\cos 2\theta}{2}\right)-\frac{1}{4}(3\sin\theta-\sin 3\theta)\right]d\theta$$

$$= 48\left\{\frac{1}{2}\left[\theta+\frac{\sin 2\theta}{2}\right]_{\theta=0}^{2\pi}-\frac{1}{4}\left[-3\cos\theta+\frac{1}{3}\cos 3\theta\right]_{\theta=0}^{2\pi}\right\}$$

$$= 48\left[\frac{1}{2}(2\pi)-\frac{1}{4}\left[-3+\frac{1}{3}-\left(-3+\frac{1}{3}\right)\right]\right\}$$

$$= 48[\pi+0] = 48\pi$$

$$\therefore \iint_{S_3}\bar{F}\cdot\bar{n}\,ds = 48\pi \longrightarrow (5)$$

Sub (3), (4) (5) in (2)

$$\therefore \iint_{S}\bar{F}\cdot\bar{n}\,ds = 0+36\pi+48\pi = \cancel{84\pi}$$

$$= 84\pi$$

$$= \iiint_V \operatorname{div}\bar{F}\,dv \quad (\text{from } (1))$$

$$\therefore \iint_S \bar{F}\cdot\bar{n}\,ds = \iiint_V \operatorname{div}\bar{F}\,dv$$

Hence GDT is verified.

Lecture # 4

* Green's Theorem

* Examples

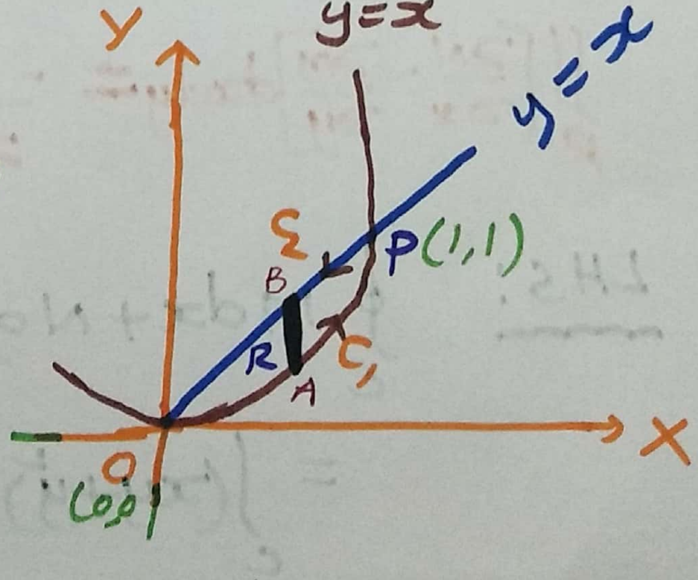
1). Verify Green's theorem in the plane for $\oint_C (xy+y^2)dx + x^2 dy$ where C is close curve of the region bounded by $y=x$ and $x^2=y$

By Green's theorem in the plane we have

$$\oint_C (M\,dx + N\,dy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

where $N(x,y) = x^2$

$M(x,y) = xy + y^2$.



the curves $y=x$ and $x^2=y$ intersects at

$x^2 - x = 0$

$x = 0, 1$

$y = 0, 1$

$O(0,0)$: $P(1,1)$

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(x^2) = 2x$$

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(xy+y^2) = x+2y$$

RHS:

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \iint_R (2x-x-2y)\,dx\,dy$$

$$\boxed{\begin{array}{l}\text{Limits of } x = 0 \text{ to } 1 \\ \text{" " } y = x^2 \text{ to } x\end{array}}$$

$$= \iint_R (x-2y)\,dx\,dy$$

$$= \int_{x=0}^{1}\int_{y=x^2}^{x} (x-2y)\,dx\,dy$$

$$= \int_{x=0}^{1}\left\{\int_{y=x^2}^{x}(x-2y)\,dy\right\}dx$$

$$= \int_{x=0}^{1}(x^4-x^3)\,dx = \left[\frac{x^5}{5}-\frac{x^4}{4}\right]_{x=0}^{1} =$$

$$\iint_R \left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] dx\,dy = -\frac{1}{20} \longrightarrow (1)$$

LHS:

$$\oint_C M\,dx + N\,dy = \oint_C (xy+y^2)\,dx + x^2\,dy$$

$$= \int_{C_1}(xy+y^2)\,dx + x^2\,dy + \int_{C_2}(xy+y^2)\,dx + x^2\,dy$$

Along C_1. $y = x^2$, $dy = 2x$; Along C_2: $y = x$, $dy = dx$

Along c_1 : $x = 0$ to 1

Along c_2 : $x = 1$ to 0

$$\oint_c [M dx + N dy]$$

$$= \int_{x=0}^{1} [(x^3 + x^4) dx + 2x^3 dx] + \int_{x=1}^{0} [2x^2 dx + x^2 dx]$$

$$= \int_{x=0}^{1} [3x^3 dx + x^4 dx] + \int_{x=1}^{0} 3x^2 dx$$

$$= \left[3\frac{x^4}{4} + \frac{x^5}{5}\right]_{x=0}^{1} + \left[3\frac{x^3}{3}\right]_{x=1}^{0}$$

$$= \frac{3}{4} + \frac{1}{5} - 1 = -\frac{1}{20}$$

$$\therefore \oint_c M dx + N dy = -\frac{1}{20}$$

$$\therefore \oint_c M dx + N dy = \iint_R \left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] dx\, dy$$

Hence Green's theorem is verified

2. Verify Green's theorem in the plane for $\int_C [(3x^2-8y^2)dx + (4y-6xy)dy]$ where C is the region bounded by $x=0$, $y=0$, $x+y=1$

Soln: By Green's theorem, we have

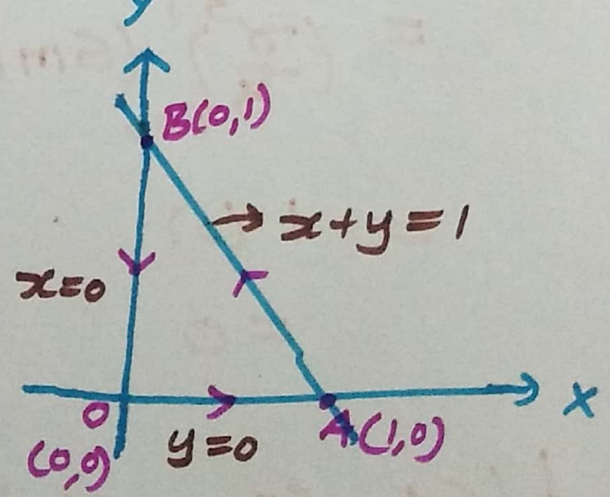
$$\oint_C [M dx + N dy] = \iint_R \left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] dx\,dy$$

Here $M = 3x^2 - 8y^2$

$N = 4y - 6xy$

$$\frac{\partial M}{\partial y} = -16y \; ; \; \frac{\partial N}{\partial x} = -6y$$

LHS: $\oint_C M dx + N dy$



Now we evaluate

$\oint_C M dx + N dy$

along OA, AB & BO

$$\therefore \oint_C M dx + N dy = \int_{OA} M dx + N dy + \int_{AB} M dx + N dy + \int_{BO} M dx + N dy \longrightarrow (1)$$

Along OA: $y=0 \Rightarrow dy=0$

x varies from 0 to 1

$$\therefore \int_{OA} M\,dx+N\,dy = \int_{x=0}^{1} (3x^2-8y^2)\,dx+(4y-6xy)\,dy$$

$$= \int_{x=0}^{1} 3x^2\,dx \quad (\because y=0, dy=0)$$

$$\therefore \int_{OA} M\,dx+N\,dy = 1 \longrightarrow (2)$$

Along AB: $x+y=1 \Rightarrow y=1-x$

$dy=-dx$

x varies from 1 to 0

$$\int_{AB} M\,dx+N\,dy = \int_{x=1}^{0} (3x^2-8y^2)\,dx+(4y-6xy)\,dy$$

$$= \int_{x=1}^{0} (-11x^2+26x-12)\,dx$$

$$= \left(-11\frac{x^3}{3}+26\frac{x^2}{2}-12x\right)_1^0$$

$$= \left[0-\left(\frac{-11}{3}+13-12\right)\right] = \frac{8}{3}$$

$$\therefore \int_{AB} M\,dx+N\,dy = \frac{8}{3} \longrightarrow (3)$$

Along BO: $x = 0 \Rightarrow dx = 0$

y varies from 1 to 0

$$\int_{BO} M\,dx + N\,dy = \int_{y=1}^{0} 4y\,dy = \left[4 \cdot \frac{y^2}{2}\right]_1^0 = 2(0-1)$$
$$= -2$$

$$\therefore \int_{BO} M\,dx + N\,dy = -2 \rightarrow (4)$$

Sub (2), (3), (4) in (1)

$$\therefore \int_{C} M\,dx + N\,dy = \frac{5}{3} \rightarrow (5)$$

Now RHS: $\iint_{R} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

$$= \int_{x=0}^{1} \int_{y=0}^{1-x} (-6y + 16y)\,dx\,dy$$

$$= \int_{x=0}^{1} \int_{y=0}^{1-x} 10y\,dx\,dy$$

$$= 10 \int_{x=0}^{1} \left(\frac{y^2}{2}\right)_0^{1-x} \cdot dx$$

$$= \frac{10}{2} \int_{x=0}^{1} (1-x)^2\,dx$$

$$= 5 \left[\frac{(1-x)^3}{-3}\right]_0^1 = \frac{-5}{3}(0-1) = 5/3.$$

$$\therefore \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = 5/3 \rightarrow (6)$$

$\therefore$ from (5) & (6) we have

$$\oint_C [M\,dx + N\,dy] = \iint_R \left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] dx\,dy$$

Hence Green's theorem is verified

3. Verify Green's th in the plane for $\oint_C (x^2 - xy^3)\,dx + (y^2 - 2xy)\,dy$ where c is a Square with Vertices $(0,0), (2,0), (2,2), (0,2)$

<u>Solution:</u> By Green's th

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

Here $M = (x^2 - xy^3)$

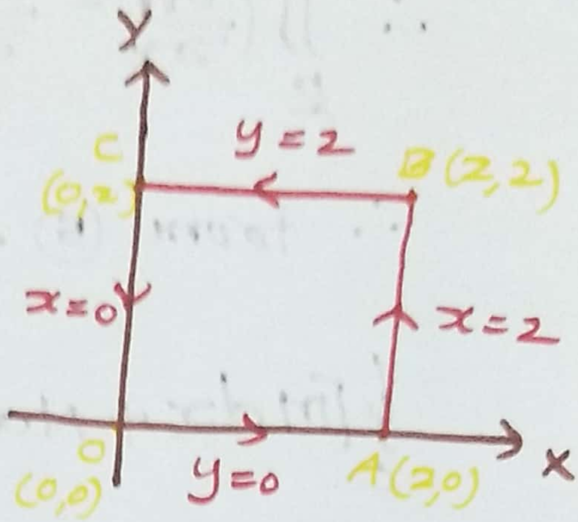
$N = y^2 - 2xy$

$$\frac{\partial M}{\partial y} = -3xy^2$$

$$\frac{\partial N}{\partial x} = -2y$$

LHS : $\int_C M\,dx + N\,dy$

Now evaluate it along OA, AB, BC and CO



$$\therefore \int_C M\,dx + N\,dy = \int_{OA} M\,dx + N\,dy + \int_{AB} M\,dx + N\,dy$$

$$+ \int_{BC} M\,dx + N\,dy + \int_{CO} M\,dx + N\,dy \quad \to (1)$$

Along OA : $\int_{OA} M\,dx + N\,dy = 8/3 \to (2)$

Along AB : $\int_{AB} M\,dx + N\,dy = -\frac{16}{3} \to (3)$

Along BC : $\int_{BC} M\,dx + N\,dy = \frac{40}{3} \to (4)$

Along CO : $\int_{CO} M\,dx + N\,dy = -8/3 \to (5)$

Sub (2), (3), (4), (5) in (1) we get

$$\int_C M\,dx + N\,dy = 8 \to (6)$$

RHS: $\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

$$= \int_{x=0}^{2}\int_{y=0}^{2} (-2y + 3xy^2)\,dx\,dy$$

$$= \int_{x=0}^{2} \left[-2\frac{y^2}{2} + 3x\frac{y^3}{3}\right]_{y=0}^{2} dx$$

$$= \int_{x=0}^{2} (-4 + 8x)\,dx$$

$$= \left(4x + 8\frac{x^2}{2}\right)_0^2 = -8 + 16 = 8$$

$$\therefore \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = 8 \longrightarrow (7)$$

from (6) & (7)

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

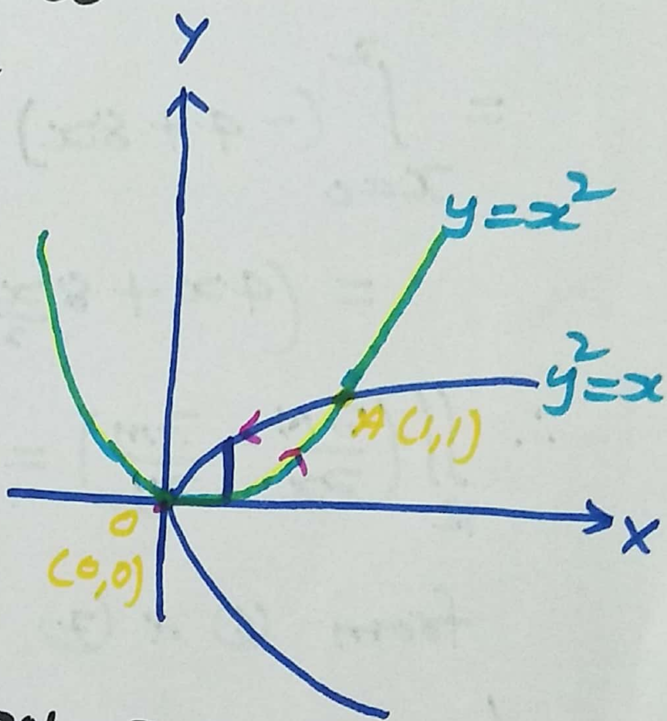
Hence Green's th is verified.

4.) Using Green's th evaluate

$\oint_c [(2xy - x^2)dx + (x^2+y^2)dy]$ where c is the closed curve of the region bounded by $y = x^2$ and $y^2 = x$

Soln: Here $M = 2xy - x^2$

$N = x^2 + y^2$



$\frac{\partial M}{\partial y} = 2x$

$\frac{\partial N}{\partial x} = 2x$

By Green's th

$$\oint_c M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx dy$$

$$\oint_c (2xy - x^2)dx + (x^2+y^2)dy = \iint_R (2x - 2x) dx dy$$

$$= \iint_R 0 \, dx dy$$

$$= \boxed{0}$$

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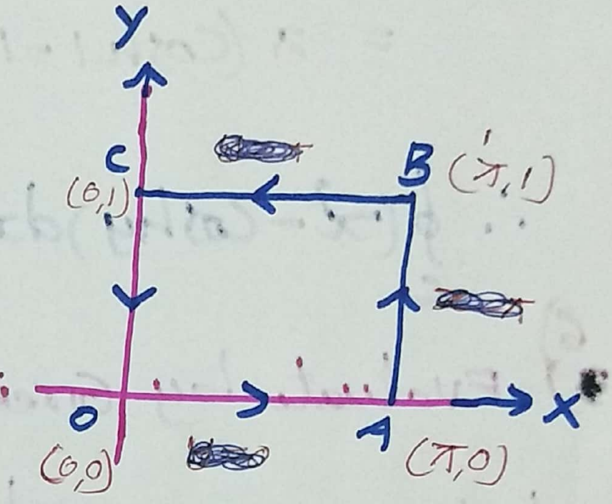
5) Evaluate by Green's theorem

$\int_C (x^2 - \cosh y)dx + (y + \sin x)dy$, where C is the rectangle with vertices $(0,0), (\pi,0)$ $(\pi,1), (0,1)$

Soln: By Green's th we have



$$\oint_C [M dx + N dy] = \iint_R \left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] dx\,dy$$

$$\int_C (x^2 - \cosh y)dx + (y + \sin x)dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

Here $M = x^2 - \cosh y \Rightarrow \frac{\partial M}{\partial y} = \cancel{2x} - \sinh y$

$N = y + \sin x \Rightarrow \frac{\partial N}{\partial x} = \cos x$

~~RHS~~ =: $\iint_R \left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] dx\,dy = \iint (\cos x + \sinh y)\,dx\,dy$

$$= \int_{x=0}^{\pi} \int_{y=0}^{1} (\cos x + \sinh y)\,dx\,dy$$

$$= \int_{x=0}^{\pi} [y \cos x + \cosh y]_{y=0}^{1}\,dx$$

$$= \int_{x=0}^{\pi} (\cos x + \cosh 1 - 1)\,dx$$

$$= \left(\sin x + x\cosh 1 - x\right)_0^{\pi}$$

$$= \pi\cosh 1 - \pi$$

$$= \pi(\cosh 1 - 1)$$

$$\therefore \oint_C (x^2 - \cosh y)\,dx + (y + \sin x)\,dy = \boxed{\pi(\cosh 1 - 1)}$$

6) Evaluate by Green.s theorem

$\oint_C (\cos x \sin y - xy)\,dx + \sin x \cos y\,dy$ where

c is the circle $x^2 + y^2 = 1$

<u>Soln</u>: Here $M = \cos x \sin y - xy$

$N = \sin x \cos y$

$$\frac{\partial M}{\partial y} = \cos x \cos y - x$$

$$\frac{\partial N}{\partial x} = \cos x \cos y$$

By Green.s theorem we, have

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$$\oint_c (\cos x \sin y - xy)\,dx + \sin x \cos y\,dy$$

$$= \iint_R (\cos x \cos y - \cos x \cos y + x)\,dx\,dy$$

$$= \iint_R x\,dx\,dy$$

$$= \int_{x=0}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} x\,dx\,dy$$

$x^2 + y^2 = 1$

$x = r\cos\theta$

$y = r\sin\theta$

$dx\,dy = r\,dr\,d\theta$

$r = 0$ to 1

$\theta = 0$ to 2π

$$= \int_{r=0}^{1} \int_{\theta=0}^{2\pi} r\cos\theta \; r\,dr\,d\theta$$

$$= \int_{r=0}^{1} r^2\,dr \int_{\theta=0}^{2\pi} \cos\theta\,d\theta$$

$$= \left(\frac{r^3}{3}\right)_{r=0}^{1} (\sin\theta)_{\theta=0}^{2\pi}$$

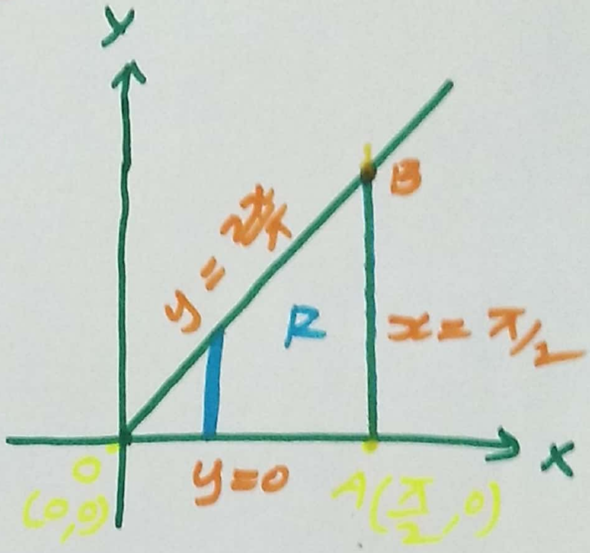
$$= \frac{1}{3}(0)$$

$$= 0$$

$$\therefore \oint_c (\cos x \sin y - xy)\,dx + \sin x \cos y\,dy = \boxed{0}$$

(7). Evaluate by Green's th

$\oint_C [(y-\sin x)\,dx + \cos x\,dy]$ where C is the triangle enclosed by the lines $y=0$, $x=\frac{\pi}{2}$, $\pi y = 2x$



Solution: Given that

$M = y - \sin x$

$\frac{\partial M}{\partial y} = 1$

$N = \cos x$

$\frac{\partial N}{\partial x} = -\sin x$

By Green's th we have

$$\oint_C M\,dx + N\,dy = \iint_R \left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] dx\,dy$$

$$\oint_C [(y-\sin x)\,dx + \cos x\,dy] = \cancel{\iint_R (1-\sin x)\,dx\,dy}$$

$$= \iint_R (-\sin x - 1)\,dx\,dy$$

$$= -\iint_R (1+\sin x)\,dx\,dy$$

$$= -\int_{x=0}^{\pi/2}\int_{y=0}^{2x/\pi}(1+\sin x)\,dx\,dy$$

$$\boxed{= -\left[\frac{\pi}{4}+\frac{2}{\pi}\right]}$$

Green's theorem in a plan

Statement: Let R be a closed region in the xy-plane whose boundary C consist of finitely many smooth curves. Let M and N be the continous functions of x and y having continous partial derivatives $\frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$ in R. then

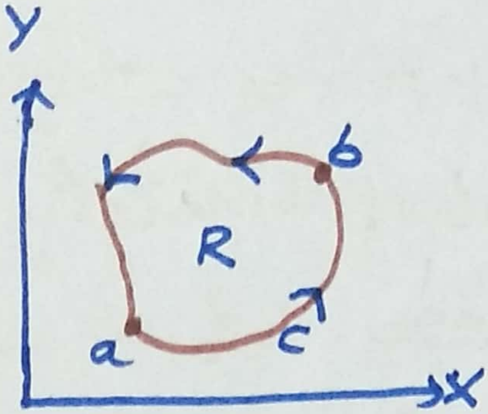
$$\boxed{\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy}$$

where the line integral being taken along the entire boundary C of R such that R is on the left sides as on advances in the directions of integration.

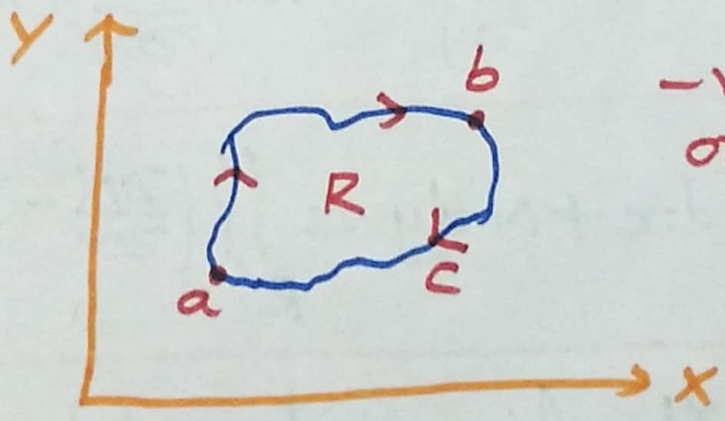
$$\oint_C \longrightarrow \iint_R$$

utilizing Green's theorem

+ve orientation



This curve runs "counter clockwise"
(the theorem applies)



-ve orientation

this curve runs "clockwise"
Switch the Sign

$$\oint_C m\,dx + N\,dy = -\iint_R \left(\frac{\partial M}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

Note: Transformation b/n Line & Surface integral

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Lecture # 5

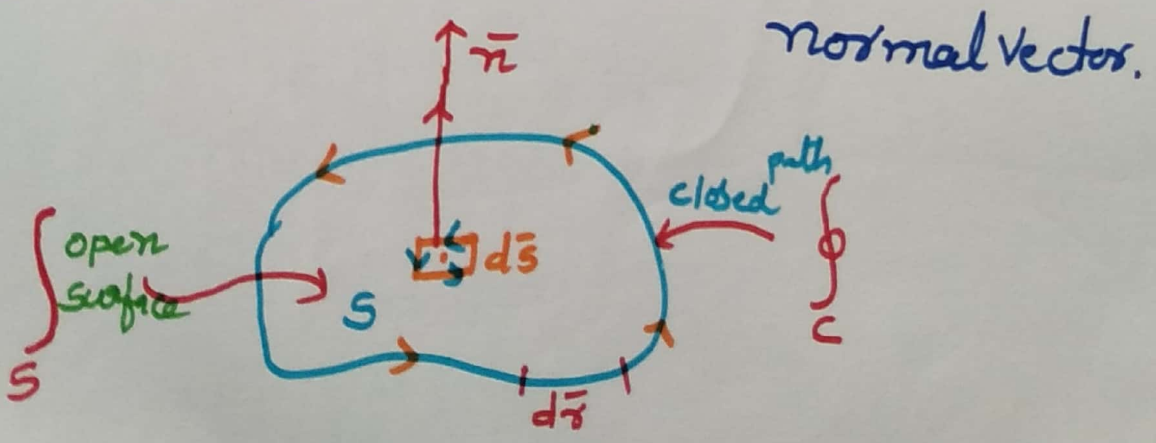
Stoke's theorem

Statement: Let 'S' be a piecewise smooth open surfaces bounded by a piecewise simple closed curve 'C'. Let $\bar{F}(x,y,z)$ be a continous vector function which has continous first order partial derivatives in a region of space which contains 'S' in its interior then

$$\boxed{\oint_C \bar{F}\cdot d\bar{r} = \iint_S \text{curl}\,\bar{F}\cdot\bar{n}\, ds = \iint_S (\nabla\times\bar{F})\cdot\bar{n}\, ds}$$

$ds \rightarrow$ projection along xy-plane $\quad ds = \dfrac{dx\,dy}{|\bar{n}\cdot\bar{k}|}$

where $\bar{n}$ is a unit outward drawn normal vector.



Alternative Form: Let $\bar{F} = F_1\bar{i} + F_2\bar{j} + F_3\bar{k}$,

$\bar{n}$ the outward drawn normal, makes angle α, β and γ with +ve directions of x, y and z axes. then

$$\bar{n} = \cos\alpha\,\bar{i} + \cos\beta\,\bar{j} + \cos\gamma\,\bar{k}$$

Therefore

$$\oint_C \left[(F_1\bar{i} + F_2\bar{j} + F_3\bar{k}) \cdot (dx\,\bar{i} + dy\,\bar{j} + dz\,\bar{k})\right]$$

$$= \iint_S (\nabla \times \bar{F}) \cdot (\cos\alpha\,\bar{i} + \cos\beta\,\bar{j} + \cos\gamma\,\bar{k})\, ds$$

$$\Rightarrow \oint_C [F_1\,dx + F_2\,dy + F_3\,dz]$$

$$= \iint_S \left[\left(\frac{\partial f_3}{dy} - \frac{\partial f_2}{\partial z}\right)\cos\alpha + \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{dx}\right)\cos\beta + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)\cos\gamma\right] ds$$

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① prove that $\oint_C \bar{r} \cdot d\bar{r} = 0$

Proof: $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$d\bar{r} = dx\,\bar{i} + dy\,\bar{j} + dz\,\bar{k}$

$$\text{curl}\,\bar{r} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \bar{0}$$

By stokes theorem

$$\oint_C \bar{F} \cdot d\bar{r} = \iint_S (\nabla \times \bar{F}) \cdot \bar{n}\, ds$$

$$\oint_C \bar{r} \cdot d\bar{r} = \iint_S (\nabla \times \bar{r}) \cdot \bar{n}\, ds$$

$$= \iint_S \bar{0} \cdot \bar{n}\, ds = 0$$

$$\therefore \boxed{\oint_C \bar{r} \cdot d\bar{r} = 0}$$

② Show that $\oint_C \phi \nabla\phi \cdot d\bar{r} = 0$

Proof: $\oint_C \phi \nabla\phi \cdot d\bar{r} = \iint_S \text{curl}(\phi \nabla\phi) \cdot \bar{n}\, ds$

$$= \iint_S [\phi\, \underbrace{\text{curl}(\nabla\phi)}_{=0} + \underbrace{\nabla\phi \times \nabla\phi}_{=0}] \cdot \bar{n}\, ds$$

$$= \iint 0\, ds = \boxed{0}$$

(3) Verify Stoke.s theorem for $\bar{F} = (x^2+y^2)\bar{i} - 2xy\,\bar{J}$, taken around the rectangle bounded by the lines $x = \pm a$, $y=0$, $y=b$

proof: By stoke.s th. we have

$$\oint_C \bar{F}\cdot d\bar{r} = \iint_S \text{curl}\bar{F}\cdot\bar{n}\, ds$$

LHS: $\oint_C \bar{F}\cdot d\bar{r}$:

Since. particle is moving along xy-plane

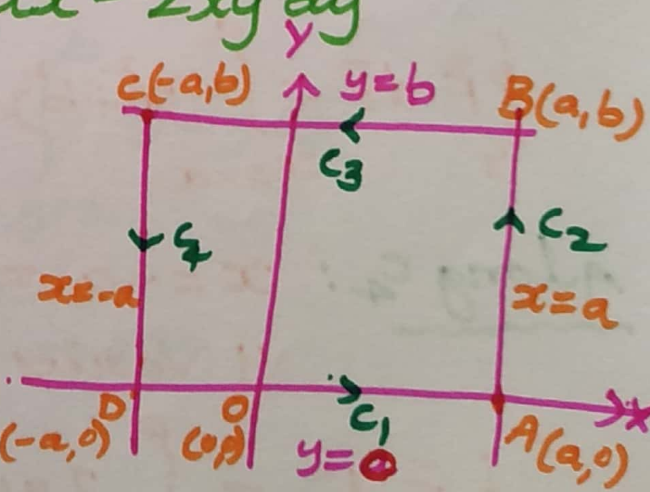
$$\bar{r} = x\bar{i} + y\bar{J}$$

$$d\bar{r} = dx\,\bar{i} + dy\,\bar{J}$$

$$\bar{F}\cdot d\bar{r} = \left[(x^2+y^2)\bar{i} - 2xy\,\bar{J}\right]\cdot(dx\,\bar{i} + dy\,\bar{J})$$

$$\bar{F}\cdot d\bar{r} = (x^2+y^2)dx - 2xy\,dy$$

Now we evaluate $\oint_C \bar{F}\cdot d\bar{r}$ along c_1, c_2, c_3 and c_4



$$\therefore \oint_C \bar{F}\cdot d\bar{r} = \oint_{c_1} \bar{F}\cdot d\bar{r} + \oint_{c_2} \bar{F}\cdot d\bar{r} + \oint_{c_3} \bar{F}\cdot d\bar{r} + \oint_{c} \bar{F}\cdot d\bar{r} \quad \to (1)$$

Along C_1: $y=0$; $dy=0$

x varies from $-a$ to a

$$\oint_{C_1} \bar{F}\cdot d\bar{r} = \int_{C_1} (x^2+y^2)dx - 2xy\,dy$$

$$= \int_{x=-a}^{a} x^2 dx = \left(\frac{x^3}{3}\right)_{-a}^{a} = 2\frac{a^3}{3}$$

$$\therefore \oint_{C} \bar{F}\cdot d\bar{r} = 2\frac{a^3}{3} \longrightarrow (2)$$

Along C_2: $x=a \Rightarrow dx=0$

y varies from 0 to b

$$\oint_{C_2} \bar{F}\cdot d\bar{r} = \int_{y=0}^{b} (-2ay)\,dy = -ab^2 \longrightarrow (3)$$

Along C_3: $y=b \Rightarrow dy=0$

x varies from a to $-a$

$$\oint_{C_3} \bar{F}\cdot d\bar{r} = \int_{x=-a}^{a} (x^2+b^2)dx = -\frac{2a^3}{3} - 2ab^2 \longrightarrow (4)$$

Along C_4: $x=-a \Rightarrow dx=0$

y varies from b to 0

$$\oint_{C_4} \bar{F}\cdot d\bar{r} = \int_{y=b}^{0} 2ay\,dy = -ab^2 \longrightarrow (5)$$

Sub (2), (3), (4), (5) in (1)

$$\text{LHS} = \oint_{C} \bar{F}\cdot d\bar{r} = -4ab^2 \longrightarrow (6)$$

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RHS : $\iint_S \text{curl}\,\bar{F}\cdot\bar{n}\,ds$

$$\text{curl}\,\bar{F} = \nabla\times\bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2+y^2 & -2xy & 0 \end{vmatrix}$$

$\therefore \text{curl}\,\bar{F} = -4y\bar{k}$

here $\bar{n} = \bar{k}$ ($\because$ xy plane)

$\text{curl}\,\bar{F}\cdot\bar{n} = (-4y\bar{k})\cdot\bar{k} = -4y$

projection in xy- plance $ds = \dfrac{dx\,dy}{|\bar{n}\cdot\bar{k}|}$

$$ds = \frac{dx\,dy}{|\bar{k}\cdot\bar{k}|} = dx\,dy$$

Limits of x are : $x = -a$ to a

" " y " : $y = 0$ to b

$$\therefore \iint_S \text{curl}\,\bar{F}\cdot\bar{n}\,ds = \int_{x=-a}^{a}\int_{y=0}^{b} y\,dy\,dx$$

$$= -4ab^2 \longrightarrow (7)$$

$\therefore$ from (6) & (7)

$$\oint_C \bar{F}\cdot d\bar{r} = \iint_S \text{curl}\,\bar{F}\cdot\bar{n}\,ds$$

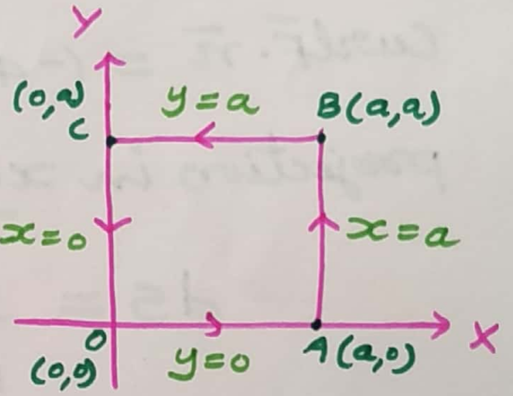
Hence Stokes theorem is Verified

(2) Verify stokes theorem for the function $\bar{F} = x^2\bar{i} + xy\bar{j}$ integrated round the Square in the plane $z=0$ and bounded by lines $x=0, y=0, x=a, y=a$.

Soln: By stoke.s theorem

$$\oint_C \bar{F}.d\bar{r} = \iint_S \text{curl}\,\bar{F}.\bar{n}\,ds$$

RHS: $\iint_S \text{curl}\,\bar{F}.\bar{n}\,ds$



$$\nabla \times \bar{F} = \text{curl}\,\bar{F}$$

$$= \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & xy & 0 \end{vmatrix} = \bar{k}y$$

$\text{curl}\,\bar{F}.\bar{n}$
$= \bar{k}y.\bar{k}$
$= y$

The plane $z=0$ means xy plane

$$\bar{n} = \bar{k}$$

$$ds = \frac{dx\,dy}{|\bar{n}.\bar{k}|} = \frac{dx\,dy}{|\bar{k}.\bar{k}|} = dx\,dy$$

$$\therefore \iint_S \text{curl}\,\bar{F}.\bar{n}\,ds = \iint_S y\,dx\,dy$$

$$= \int_{x=0}^{a}\int_{y=0}^{a} y\,dy\,dx$$

$$= \frac{a^3}{2} \longrightarrow (1)$$

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LHS: $\oint_C \bar{F}\cdot d\bar{r}$

$$\oint_0 \bar{F}\cdot d\bar{r} = \int_C (x^2\bar{i} + xy\bar{j})\cdot(\bar{i}\,dx + \bar{j}\,dy)$$

$$\oint_C \bar{F}\cdot d\bar{r} = \oint_C x^2dx + xy\,dy$$

this is calculated along OA, AB, BC, CO

$$\oint_C \bar{F}\cdot d\bar{r} = \oint_{OA} \bar{F}\cdot d\bar{r} + \oint_{AB} \bar{F}\cdot d\bar{r} + \int_{BC} \bar{F}\cdot d\bar{r} + \oint_{CO} \bar{F}\cdot d\bar{r}$$

Along AO:

Along AB:

Along BC:

Along CO:

$$\therefore \oint_C \bar{F}\cdot d\bar{r} = \oint_C [x^2dx + xy\,dy] = a^3/2 \rightarrow (2)$$

from (1) & (2)

$$\therefore \oint_C \bar{F}\cdot d\bar{r} = \iint_S \text{curl}\,\bar{F}\cdot\bar{n}\,ds$$

Hence Stoke,s theorem is Verified.

(3) Evaluate $\oint_C [(x+y)dx + (2x-z)dy + (y+z)dz]$ where 'C' is the boundary of the triangle with vertices (2,0,0), (0,3,0) and (0,0,6) using stokes theorem.

Soln: By Stokes theorem

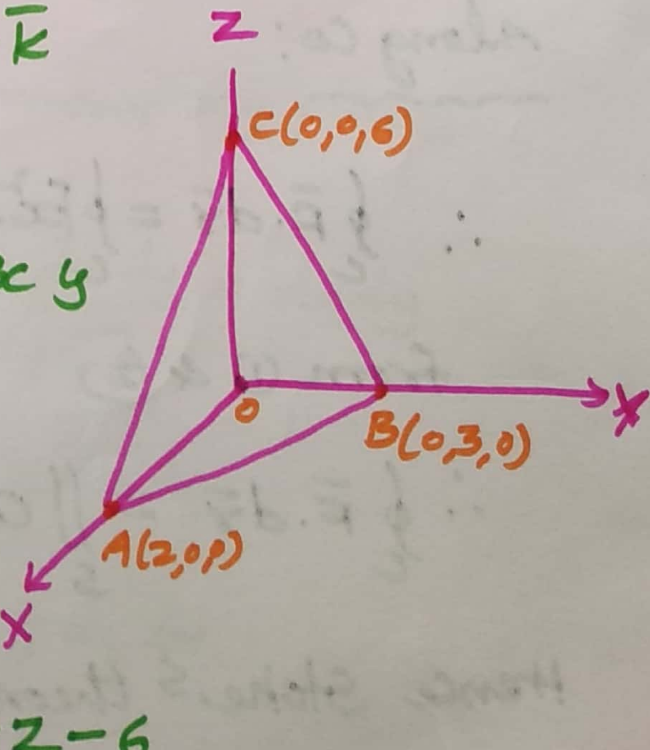
$$\oint_C \bar{F}\cdot d\bar{r} = \iint_S \text{Curl}\,\bar{F}\cdot\bar{n}\,ds$$

given $\therefore \oint_C [(x+y)dx + (2x-z)dy + (y+z)dz]$

here $\bar{F} = (x+y)\bar{i} + (2x-z)\bar{j} + (y+z)\bar{k}$

$$\text{Curl}\,\bar{F} = \nabla\times\bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y & 2x-z & y+z \end{vmatrix}$$

$$\text{Curl}\,\bar{F} = 2\bar{i} + \bar{k}$$



Now we want $\bar{n}$:

the eqn of the plane ABC is

$$\frac{x}{2} + \frac{y}{3} + \frac{z}{6} = 1$$

$$\Rightarrow 3x + 2y + z = 6$$

let $\phi = 3x + 2y + z - 6$

$$\nabla\phi = \cancel{\frac{\partial}{\partial x}(3x) + \frac{\partial}{\partial y}(2y) + \frac{\partial}{\partial z}}(z$$

$$= \bar{i}\frac{\partial}{\partial x}(3x+2y+z-6) + \bar{j}\frac{\partial}{\partial y}(3x+2y+z-6)$$
$$+ \bar{k}\frac{\partial}{\partial z}(3x+2y+z-6)$$

$$\nabla\phi = 3\bar{i} + 2\bar{j} + \bar{k}$$

$$\bar{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{(3\bar{i}+2\bar{j}+\bar{k})}{\sqrt{3^2+2^2+1^2}} = \frac{(3\bar{i}+2\bar{j}+\bar{k})}{\sqrt{14}}$$

$$\therefore \bar{n}\cdot\bar{k} = \frac{1}{\sqrt{14}}(3\bar{i}+2\bar{j}+\bar{k})\cdot\bar{k} = \frac{1}{\sqrt{14}}$$

$$ds = \frac{dx\cdot dy}{|\bar{n}\cdot\bar{k}|} = \frac{dx\,dy}{\frac{1}{\sqrt{14}}} = \sqrt{14}\,dx\,dy$$

$$\therefore \iint_S \text{curl}\,\bar{F}\,ds = \iint_S (2\bar{i}+\bar{k})\cdot\frac{(3\bar{i}+2\bar{j}+\bar{k})}{\sqrt{14}}\sqrt{14}\,dx\,dy$$

$$= \iint_R (6+1)\,dx\,dy$$

$$= 7\iint_R dx\,dy$$

$$= 7\,(\text{Area of } \Delta\, OAB)$$

$$= 7\left(\frac{1}{2}\times 2\times 3\right)$$

$$\iint_S \text{curl}\,\bar{F}\cdot n\,ds = 21$$

$$\therefore \oint_C ((x+y)dx + (2x-z)dy + (y+z)dz) = \iint_S \text{curl}\,\bar{F}\cdot\bar{n}\,ds$$

$$\boxed{=21}$$

(4) Evaluate by Stokes theorem $\oint_c (e^x dx + 2y\,dy - dz)$ where 'c' is the curve $x^2+y^2=9$ and $z=2$

Soln: let $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$$d\bar{r} = dx\,\bar{i} + dy\,\bar{j} + dz\,\bar{k}$$

$$\bar{F}\cdot d\bar{r} = e^x dx + 2y\,dy - dz$$

$$\bar{F} = e^x\bar{i} + 2y\,\bar{j} - \bar{k}$$

$$\nabla\times\bar{F} = \text{curl}\,\bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^x & 2y & -1 \end{vmatrix} = \bar{0}$$

$\therefore$ curl $\bar{F} = 0$

by Stokes theorem

$$\oint_c \bar{F}\cdot d\bar{r} = \iint_S \text{curl}\,\bar{F}\cdot\bar{n}\,ds$$

$$\oint_c (e^x dx + 2y\,dy - dz) = \iint_S \bar{0}\cdot\bar{n}\,ds$$

$$\boxed{=0}$$

⑤ Verify Stokes theorem for $\bar{F} = (2x-y)\bar{i} - yz^2\bar{j} - y^2z\bar{k}$ over the upper half surface of the sphere $x^2+y^2+z^2=1$ bounded by the projection of the xy-plane.

Soln: the boundary C of S is a circle in xy-plane. i.e. $x^2+y^2=1$; $z=0$

By stokes theorem

$$\oint_C \bar{F}\cdot d\bar{r} = \iint_S \text{curl}\,\bar{F}\cdot\bar{n}\,ds$$

LHS: $\oint_C \bar{F}\cdot d\bar{r}$:

$$\bar{F} = (2x-y)\bar{i} - yz^2\bar{j} - y^2z\bar{k}$$

$$\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$$

$$d\bar{r} = dx\,\bar{i} + dy\,\bar{j} + dz\,\bar{k}$$

$$\bar{F}\cdot d\bar{r} = (2x-y)dx \cancel{+} -(yz^2)dy - y^2z\,dz$$

xy-plane: $z=0$; $dz=0$

$$\bar{F}\cdot d\bar{r} = (2x-y)dx$$

$$\oint_C \bar{F}\cdot d\bar{r} = \oint_C (2x-y)\,dx$$

put $x = \cos\theta$: $y = \sin\theta$

$dx = -\sin\theta\, d\theta$

$\theta = 0$ to 2π

$$\oint_C \bar{F}\cdot d\bar{r} = \int_{\theta=0}^{2\pi} (2\cos\theta - \sin\theta)(-\sin\theta\, d\theta)$$

$$= -\int_{\theta=0}^{2\pi} (2\sin\theta\cos\theta - \sin^2\theta)\,d\theta$$

$$= \int_{\theta=0}^{2\pi} \sin^2\theta\, d\theta - \int_{\theta=0}^{2\pi} \sin 2\theta\, d\theta$$

$$= \int_{\theta=0}^{2\pi} \left(\frac{1-\cos 2\theta}{2}\right) d\theta - \int_{\theta=0}^{2\pi} \sin 2\theta\, d\theta$$

$$= \frac{1}{2}\left(\theta - \frac{\sin 2\theta}{2}\right)_{\theta=0}^{2\pi} + \left(\frac{\cos 2\theta}{2}\right)_{\theta=0}^{2\pi}$$

$$= \frac{1}{2}[2\pi - 0] + \frac{1}{2}[1-1]$$

$(\because \sin n\pi = 0$
$\cos n\pi = (-1)^n)$

$$= \pi$$

$$\therefore \oint_C \bar{F}\cdot d\bar{r} = \pi \longrightarrow (1)$$

RHS $\iint_S \operatorname{curl}\bar{F}\cdot\bar{n}\,ds$:

$$\operatorname{curl}\bar{F} = \nabla\times\bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x-y & -yz^2 & -y^2z \end{vmatrix} = \bar{k}4$$

$\bar{n} = \bar{k}$ ($\because$ xy-plane)

$$ds = \frac{dx\,dy}{|\bar{n}\cdot\bar{k}|} = \frac{dx\,dy}{|\bar{k}\cdot\bar{k}|} = dx\,dy$$

$$\therefore \iint_S \operatorname{curl}\bar{F}\cdot\bar{n}\,ds = \iint_R 4\bar{k}\cdot\bar{k}\,dx\,dy$$

$$= 4\iint_R dx\,dy$$

$x^2+y^2=1 \Rightarrow y = \pm\sqrt{1-x^2}$

Limits: $x = \pm 1$

$y = 0$ to $\sqrt{1-x^2}$

$x = 0$ to 1 ($\because$ in xy-plane)

$$\iint_S \operatorname{curl}\bar{F}\cdot\bar{n}\,ds = 4\int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}} dy\,dx$$

$$= 4\int_{x=0}^{1} (y)_0^{\sqrt{1-x^2}}\,dx$$

$$= 4\int_0^1 \sqrt{1-x^2}\,dx$$

$$= 4\left[\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}x\right]_0^1$$

$$= 4\left[0 + \frac{1}{2}\sin^{-1}(1) - 0\right]$$

$$= 4\ \frac{1}{2}\frac{\pi}{2} = \pi$$

$$\therefore \iint_S \text{curl}\,\bar{F}\cdot\bar{n}\,ds = \pi \rightarrow (2)$$

from (1) & (2)

$\oint_C \bar{F}\cdot$ [redacted] $= \iint_S \text{curl}\,\bar{F}\cdot\bar{n}\,ds$

Hence stoke,s theorem is verified.