UNIT – IV

VECTOR DIFFERENTIATION

Vector point function :- Consider a region in three-dimensional space. If to each point $p(x, y, z)$ we associate a unique vector $\bar{f}(x, y, z)$, then $\bar{f}$ is called a vector point function.

~~Vector~~ **scalar point function :** If to each point $p(x, y, z)$, suppose we associate a unique real number (called scalar) say ϕ. This $\phi(x, y, z)$ is called a scalar point function defined on the region.

Differentation of a vector point function :-

Let $\bar{F}$ be a vector point function then

If $\lim_{t \to a} \frac{\bar{F}(t) - \bar{F}(a)}{t - a}$ exist then it is called derivative of $\bar{f}$ and it is denoted by $\frac{d\bar{f}}{dt}$.

Vector differential operator :-

the vector differential operator is defined as del (∇)

$$\nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$$

gradient of a scalar point function :-

let $\phi(x, y, z)$ be a scalar point function of position defined in some region of space. then the vector function $i\frac{\partial \phi}{\partial x} + j\frac{\partial \phi}{\partial y} + k\frac{\partial \phi}{\partial z}$ is know as the gradient of ϕ. and is denoted by "grad ϕ" (or) $\nabla \phi$

$$\nabla \phi = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\phi$$

$$\Rightarrow \nabla \phi = \bar{i}\frac{\partial \phi}{\partial x} + j\frac{\partial \phi}{\partial y} + k\frac{\partial \phi}{\partial z}$$

lly if $f(x, y, z)$ be a scalar point function then

$$\nabla f = i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z}$$

"gradient of $\bar{f}$" (or) "grad $\bar{f}$"

(or) "∇f"

properties

If f and g are two scalar point function

(i) $\text{grad}\,(f \pm g) = \text{grad} f \pm \text{grad}\, g$

(ii) $\text{grad}\,(fg) = f\,(\text{grad}\, g) + g\,(\text{grad} f)$

(iii) $\text{grad}\left(\dfrac{f}{g}\right) = \dfrac{g\,(\text{grad} f) - f\,(\text{grad}\, g)}{g^2}$

iv) If 'c' is constant,
$\text{grad}\,(cf) = c\,(\text{grad} f)$

(v) let $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$
then $d\bar{r} = (dx)\,\bar{i} + (dy)\,\bar{j} + (dz)\,\bar{k}$

Unit Normal Vector:- If $f(x, y, z)$ is any scalar point function defined on surface then, unit outward normal to surface.

$$\hat{n} = \frac{\nabla f}{|\nabla f|} \quad (\text{or}) \quad \bar{e} = \frac{\nabla f}{|\nabla f|}$$

(or)

$$\hat{n} = \frac{\nabla \phi}{|\nabla \phi|} \quad (\text{or}) \quad \bar{e} = \frac{\nabla \phi}{|\nabla \phi|}$$

'ϕ' is any scalar point function.

Directional Derivative :- the directional derivative of a scalar point function ϕ at a point $p(x,y,z)$ in the direction of a unit vector $\bar{e}$ is equal to $\bar{e}\cdot \text{grad}\phi = \bar{e}\cdot\nabla\phi$.

$$\text{Directional Derivative} = \bar{e}\cdot\nabla\phi = \hat{n}\cdot\nabla\phi$$

$$= \nabla\phi\cdot\frac{\nabla f}{|\nabla f|} \quad \left(\hat{n} = \bar{e} = \frac{\nabla f}{|\nabla f|}\right)$$

(or)

→ Directional derivative of a scalar point function f at a point $p(x,y,z)$ in the direction of a unit vector $\bar{e}$ is equal to

$$\bar{e}\cdot\text{grad}f = \bar{e}\cdot\nabla f$$

$$\text{Directional Derivative} = \bar{e}\cdot\nabla f = \hat{n}\cdot\nabla f$$

$$= \nabla f\cdot\frac{\nabla\phi}{|\nabla\phi|} \Rightarrow \left(\hat{n} = \bar{e} = \frac{\nabla\phi}{|\nabla\phi|}\right)$$

NOTE If $f_1(x,y,z)$ and $f_2(x,y,z)$ are any two surfaces defined on a scalar field then, angle between them

$$\cos\theta = \frac{\nabla f_1\cdot\nabla f_2}{|\nabla f_1|\,|\nabla f_2|}$$

$$\theta = \cos^{-1}\left[\frac{\nabla f_1\cdot\nabla f_2}{|\nabla f_1|\,|\nabla f_2|}\right]$$

(1) Find $\nabla\phi$ if $\phi = \log(x^2+y^2+z^2)$

Sol: Given that $\phi = \log(x^2+y^2+z^2)$

$$\nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$$

$$\nabla\phi = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\phi$$

$$= i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$$

$$= i\frac{\partial}{\partial x}\log(x^2+y^2+z^2) + j\frac{\partial}{\partial y}\log(x^2+y^2+z^2) + k\frac{\partial}{\partial z}\log(x^2+y^2+z^2)$$

$$= i\left[\frac{1}{x^2+y^2+z^2}\frac{\partial}{\partial x}(x^2+y^2+z^2)\right] + j\left[\frac{1}{x^2+y^2+z^2}\frac{\partial}{\partial y}(x^2+y^2+z^2)\right] + k\left[\frac{1}{x^2+y^2+z^2}\frac{\partial}{\partial z}(x^2+y^2+z^2)\right]$$

$$= i\left[\frac{2x+0+0}{x^2+y^2+z^2}\right] + j\left[\frac{0+2y+0}{x^2+y^2+z^2}\right] + k\left[\frac{0+0+2z}{x^2+y^2+z^2}\right]$$

$$\nabla\phi = \frac{2x\bar{i} + 2y\bar{j} + 2z}{x^2+y^2+z^2}$$

$$\nabla\phi = \frac{2(x\bar{i}+y\bar{j}+z\bar{k})}{x^2+y^2+z^2} = \frac{2\bar{r}}{x^2+y^2+z^2}$$

$(\bar{r} = x\bar{i}+y\bar{j}+z\bar{k})$

(2) Find $\nabla(x^2+y^2z)$

sol: $\nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$

$\nabla(x^2+y^2z) = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)(x^2+y^2z)$

$= i\frac{\partial}{\partial x}(x^2+y^2z) + j\frac{\partial}{\partial y}(x^2+y^2z) + k\frac{\partial}{\partial z}(x^2+y^2z)$

$= i(2x+0) + j(0+2yz) + k(0+y^2)$

$\nabla(x^2+y^2z) = 2x\bar{i} + 2yz\bar{j} + y^2\bar{k}$

(3) Find ∇f at $(1,-1,1)$ $f = 3x^2 - yz$

sol Given that $f = 3x^2 - yz$

$\nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$

$\nabla f = i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z}$

$= i\frac{\partial}{\partial x}(3x^2 - yz) + j\frac{\partial}{\partial y}(3x^2 - yz)$

$+ k\frac{\partial}{\partial z}(3x^2 - yz)$

$= i(6x-0) + j(0-z) + k(0-y)$

$\nabla f = 6x\bar{i} - z\bar{j} - y\bar{k}$

∇f at $(1,-1,1)$ $= 6(1)\bar{i} - (1)\bar{j} - (-1)\bar{k}$

$= 6\bar{i} - \bar{j} + \bar{k}$

(4) prove that $\nabla r^n = n r^{n-2} \bar{r}$ (4)

sol: let $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$$|\bar{r}| = r = \sqrt{x^2+y^2+z^2}$$

$$\boxed{r^2 = x^2+y^2+z^2}$$

$$r^2 = x^2+y^2+z^2$$

Diff wrto 'x' on b/s

$$\frac{\partial}{\partial x}(r^2) = \frac{\partial}{\partial x}(x^2+y^2+z^2)$$

$$2r\frac{\partial r}{\partial x} = 2x+0+0 \Rightarrow \cancel{2}r\frac{\partial r}{\partial x} = \cancel{2}x$$

$$\boxed{\frac{\partial r}{\partial x} = \frac{x}{r}} \quad \text{lly} \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\nabla r^n = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right) r^n$$

$$= i\frac{\partial}{\partial x}r^n + j\frac{\partial}{\partial y}r^n + k\frac{\partial}{\partial z}r^n$$

$$= i\, n.r^{n-1}\frac{\partial r}{\partial x} + j\, n.r^{n-1}\frac{\partial r}{\partial y} + k\, n.r^{n-1}\frac{\partial r}{\partial z}$$

$$= i\, n.r^{n-1}\left(\frac{x}{r}\right) + j\, n.r^{n-1}\left(\frac{y}{r}\right) + k\, n.r^{n-1}\left(\frac{z}{r}\right)$$

$$= i\, n.r^{n-1}.r^{-1}.x + j\, n\, r^{n-1}.r^{-1}y + k.\, n\, r^{n-1}.r^{-1}z$$

$$= (xi+yj+zk)(n.r^{n-2})$$

$$= n r^{n-2}.\bar{r} \qquad (\bar{r} = xi+yj+zk)$$

⑤ Find the greatest value of the directional derivative of the function.

$f = x^2yz^3$ at $(2,1,-1)$

sol: Given that

$$f = x^2yz^3$$

$\therefore$ Greatest value of the directional derivative of $f = |\nabla f|$.

$$\nabla f = i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z}$$

$$= i\frac{\partial}{\partial x}(x^2yz^3) + j\frac{\partial}{\partial y}(x^2yz^3) + k\frac{\partial}{\partial z}(x^2yz^3)$$

$$\nabla f = i(2xyz^3) + j(x^2z^3) + k(3x^2yz^2)$$

$$\nabla f \text{ at } (2,1,-1) = -4\bar{i} - 4\bar{j} + 12\bar{k}$$

$$|\nabla f| = \sqrt{16+16+144}$$

$\therefore$ Greatest value of directional derivative of $f = |\nabla f|$

$$= \sqrt{176}$$

$$= 4\sqrt{11} \;//$$

(6) Find the Unit normal vector to given surface $x^2y + 2xz = 4$ at $(2, -2, 3)$

Sol. Given that surface

let $f = x^2y + 2xz - 4$

Unit normal vector $\hat{n} = \bar{e} = \frac{\nabla f}{|\nabla f|}$

$\nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$

$\nabla f = i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z}$

$\nabla f = i\frac{\partial}{\partial x}(x^2y + 2xz - 4) + j\frac{\partial}{\partial y}(x^2y + 2xz - 4)$
$+ k\frac{\partial}{\partial z}(x^2y + 2xz - 4)$

$= i(2xy + 2z - 0) + j(x^2 + 0 - 0) + k(0 + 2x - 0)$

$\nabla f = (2xy + 2z)\bar{i} - x^2\bar{j} + 2x\bar{k}$

∇f at $(2, -2, 3)$ $= [2(2)(-2) + 2(3)]\bar{i} - (2)^2\bar{j} + 2(2)\bar{k}$

$\nabla f = (-8 + 6)\bar{i} - 4\bar{j} + 4\bar{k}$

$\nabla f = -2\bar{i} - 4\bar{j} + 4\bar{k}$

$|\nabla f| = \sqrt{(-2)^2 + (-4)^2 + 4^2} = \sqrt{4 + 16 + 16} = \sqrt{36} = 6$

$\therefore$ Unit normal vector $= \hat{n} = \frac{\nabla f}{|\nabla f|}$

$= \frac{-2\bar{i} - 4\bar{j} + 4\bar{k}}{6} = \frac{\cancel{2}(-\bar{i} - 2\bar{j} + 2\bar{k})}{\cancel{6}3}$

$\hat{n} = \frac{-\bar{i} - 2\bar{j} + 2\bar{k}}{3}$

(7) find a unit normal vector to the surface $z = x^2+y^2$ at $(-1,-2,5)$

sol: Given that the surface

let $f = x^2+y^2-z = \phi$

Unit normal vector $\hat{n} = \bar{e} = \frac{\nabla f}{|\nabla f|} = \frac{\nabla\phi}{|\nabla\phi|}$

$\nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$

$\nabla\phi = i\frac{\partial}{\partial x}(x^2+y^2-z) + j\frac{\partial}{\partial y}(x^2+y^2-z)$
$+k\frac{\partial}{\partial z}(x^2+y^2-z)$

$\nabla\phi = i(2x+0-0) + j(0+2y-0) + k(0+0-1)$

$\nabla\phi = 2x\bar{i} + 2y\bar{j} - \bar{k}$

$\nabla\phi$ at $(-1,-2,-5) = 2(-1)\bar{i} + 2(-2)\bar{j} - \bar{k}$

$\nabla\phi = -2\bar{i} - 4\bar{j} - \bar{k}$

$\rightarrow |\nabla\phi| = \sqrt{(-2)^2+(-4)^2+(-1)^2} = \sqrt{4+16+1}$
$= \sqrt{21}$

Unit normal vector $\hat{n} = \bar{e} = \frac{\nabla\phi}{|\nabla\phi|}$

$\hat{n} = \frac{-2\bar{i}-4\bar{j}-\bar{k}}{\sqrt{21}}$

(8) Find a unit normal vector to the surface $x^2+y^2+2z^2=26$ at the point $(2,2,3)$ (7)

sol: Given that the surface

let $f = x^2+y^2+2z^2-26$

Unit normal vector $\hat{n} = \bar{e} = \frac{\nabla f}{|\nabla f|}$

$$\nabla f = i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z}$$

$$\nabla f = i\frac{\partial}{\partial x}(x^2+y^2+2z^2-26) + j\frac{\partial}{\partial y}(x^2+y^2+2z^2-26) + k\frac{\partial}{\partial z}(x^2+y^2+2z^2-26)$$

$$= i(2x+0+0-0) + j(0+2y+0-0) + k(0+0+4z-0)$$

$$\nabla f = 2x\bar{i} + 2y\bar{j} + 4z\bar{k}$$

$$\nabla f \text{ at } (2,2,3) = 2(2)\bar{i} + 2(2)\bar{j} + 4(3)\bar{k}$$

$$\nabla f = \cancel{6}4\bar{i} + 4\bar{j} + 12\bar{k}$$

$\therefore$ Unit normal vector $\hat{n} = \frac{\nabla f}{|\nabla f|} = \frac{4i+4j+12k}{\sqrt{(6)^2+(4)^2+(12)^2}}$

$$= \frac{4i+4j+12k}{\sqrt{36+16+144}}$$

$$= \frac{4i+4j+12k}{\sqrt{196}} = \frac{\cancel{4}(i+j+3k)}{\cancel{4}\sqrt{11}}$$

$$= \frac{i+j+3k}{\sqrt{11}}$$

(9) Find the angle between intersection of spheres $x^2+y^2+z^2=29$, $x^2+y^2+z^2+4x-6y-8z-47=0$ at $(4,-3,2)$

Sol: Given that two spheres.

let $f_1 = x^2+y^2+z^2-29$

$f_2 = x^2+y^2+z^2+4x-6y-8z-47$

Angle b/w two spheres is

$$\cos\theta = \frac{\nabla f_1 \cdot \nabla f_2}{|\nabla f_1|\,|\nabla f_2|}$$

$\rightarrow \nabla f_1 = i\frac{\partial f_1}{\partial x} + j\frac{\partial f_1}{\partial y} + k\frac{\partial f_1}{\partial z}$

$= i\frac{\partial}{\partial x}(x^2+y^2+z^2-29) + j\frac{\partial}{\partial y}(x^2+y^2+z^2-29)$

$+ z\frac{\partial}{\partial x}(x^2+y^2+z^2-29)$

$= i(2x) + j(2y) + z(2z)$

$\nabla f_1 = 2x\bar{i} + 2y\bar{j} + 2z\bar{k}$

∇f_1 at $(4,-3,2) = 2(4)\bar{i} + 2(-3)\bar{j} + 2(2)\bar{k}$

$\nabla f_1 = 8\bar{i} - 6\bar{j} + 4\bar{k}$

$\rightarrow |\nabla f_1| = \sqrt{64+36+16} = \sqrt{116} = 2\sqrt{29}$

$\rightarrow \nabla f_2 = i\frac{\partial f_2}{\partial x} + j\frac{\partial f_2}{\partial y} + k\frac{\partial f_2}{\partial z}$

$= i\frac{\partial}{\partial x}(x^2+y^2+z^2+4x-6y-8z-47)$

$+ j\frac{\partial}{\partial y}(x^2+y^2+z^2+4x-6y-8z-47)$

$+ k\frac{\partial}{\partial z}(x^2+y^2+z^2+4x-6y-8z-47)$

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$$\nabla f_2 = i(2x+4) + j(2y-6) + k(2z-8)$$

$$\nabla f_2 \text{ at } (4,-3,2) = i[2(4)+4] + j[2(-3)-6] + k[2(2)-8]$$

$$\nabla f_2 = 12\bar{i} - 12\bar{j} - 4k$$

$$\rightarrow |\nabla f_2| = \sqrt{144+144+16} = \sqrt{304} = 4\sqrt{19}$$

$$\cos\theta = \frac{\nabla f_1 \cdot \nabla f_2}{|\nabla f_1|\,|\nabla f_2|}$$

$$\cos\theta = \frac{(8i-6j+4k)\cdot(12i-12j-4k)}{\sqrt{116}\,\sqrt{304}}$$

$$\cos\theta = \frac{(12\times 8)+(6\times 12)-(4\times 4)}{\sqrt{116}\,\sqrt{304}}$$

$$\cos\theta = \frac{96+72-16}{2\sqrt{29}\;4\sqrt{19}}$$

$$\cos\theta = \frac{152}{8\sqrt{29}\sqrt{19}}$$

$$\theta = \cos^{-1}\left[\frac{152}{8\sqrt{29}\;\sqrt{19}}\right]$$

$i \cdot i = 1$
$j \cdot j = 1$
$k \cdot k = 1$
otherwise $= 0$
ie $i \cdot j = i \cdot k = 0$
$j \cdot i = j \cdot k = 0$
$k \cdot i = k \cdot j = 0$

(10) Evaluate the angle between the normals to the surface $xy = z^2$ at the points $(4,1,2)$, & $(3,3,-3)$

Sol: Given that the surface is

$$f = xy - z^2$$

$$\nabla f = i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z}$$

$$\nabla f = i\frac{\partial}{\partial x}(xy - z^2) + j\frac{\partial}{\partial y}(xy - z^2) + k\frac{\partial}{\partial z}(xy - z^2)$$

$$\nabla f = i(y-0) + j(x-0) + k(0-2z)$$

$$\nabla f = y\bar{i} + x\bar{j} - 2z\bar{k}$$

Let us consider ∇f_1 at $(4,1,2)$

$$\nabla f_1\,(4,1,2) = (1)\bar{i} + 4\bar{j} - 2(2)\bar{k}$$

$$\nabla f_1 = i + 4j - 4k$$

$$|\nabla f_1| = \sqrt{1+16+16} = \sqrt{33}$$

Let us consider ∇f_2 at $(3,3,-3)$

$$\nabla f_2 \text{ at } (3,3,-3) = 3\bar{i} + 3\bar{j} - 6\bar{k}$$

$$|\nabla f_2| = \sqrt{9+9+36} = \sqrt{54}$$

angle between two surfaces

$$\cos\theta = \frac{\nabla f_1 \cdot \nabla f_2}{|\nabla f_1| \, |\nabla f_2|}$$

$$\cos\theta = \frac{(i + 4j - 4k) \cdot (3i + 3j + 6k)}{\sqrt{33} \, \sqrt{54}}$$

$$\cos\theta = \frac{3 + 12 - 24}{\sqrt{33} \, \sqrt{55}} = \frac{-9}{\sqrt{33} \, \sqrt{55}}$$

$$\theta = \cos^{-1}\left[\frac{-9}{\sqrt{33} \, \sqrt{55}}\right]$$

(11) Find the angle between the normals to the surface $x^2 = yz$ at the points $(1,1,1)$ & $(2,4,1)$

sol Given that the surface.

$$f = x^2 - yz$$

$$\nabla f = i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z}$$

$$= i\frac{\partial}{\partial x}(x^2 - yz) + j\frac{\partial}{\partial y}(x^2 - yz) + k\frac{\partial}{\partial z}(x^2 - yz)$$

$$= i(2x - 0) + j(0 - z) + k(0 - y)$$

$$\nabla f = 2x\bar{i} - z\bar{j} - y\bar{k}$$

let us consider ∇f_1 at $(1,1,1)$

$$\nabla f_1 = 2(1)i - (1)j - 1k = 2i - j - k$$

$\rightarrow |\nabla f_1| = \sqrt{4+1+1} = \sqrt{6}$

let us consider ∇f_2 at $(2,4,1)$

$\nabla f_2 = 4i - j - 4k$

$\rightarrow |\nabla f_2| = \sqrt{16+1+16} = \sqrt{33}$

$\therefore$ let 'θ' be the angle b/w two normals.

$$\cos\theta = \frac{\nabla f_1 \cdot \nabla f_2}{|\nabla f_1|\,|\nabla f_2|}$$

$$\cos\theta = \frac{(2i-j-k)\cdot(4i-j-4k)}{\sqrt{6}\ \sqrt{33}}$$

$$\cos\theta = \frac{8+1+4}{\sqrt{6}\ \sqrt{33}}$$

$$\theta = \cos^{-1}\left[\frac{13}{\sqrt{198}}\right]$$

Q2) Find the angle between the surfaces $x^2+y^2+z^2=9$, $z=x^2+y^2-3$ at the point $(2,-1,2)$

Sol:- Given that two surfaces.

$f_1 = x^2+y^2+z^2-9$

$f_2 = x^2+y^2-z-3$

$$\rightarrow \nabla f_1 = i\frac{\partial f_1}{\partial x} + j\frac{\partial f_1}{\partial y} + k\frac{\partial f_1}{\partial z}$$

$$= i\frac{\partial}{\partial x}(x^2+y^2+z^2-9) + j\frac{\partial}{\partial y}(x^2+y^2+z^2-9)$$
$$+ k\frac{\partial}{\partial z}(x^2+y^2+z^2-9)$$

$$\nabla f_1 = i(2x) + j(2y) + k(2z)$$

$$\nabla f_1 = 2x\bar{i} + 2y\bar{j} + 2z\bar{k}$$

$$\nabla f_1 \text{ at } (2,-1,2) = 2(2)\bar{i} + 2(-1)\bar{j} + 2(2)\bar{k}$$

$$\rightarrow \nabla f_1 = 4\bar{i} - 2\bar{j} + 4\bar{k}$$

$$|\nabla f_1| = \sqrt{16+4+16} = \sqrt{36} = 6$$

$$\rightarrow \nabla f_2 = i\frac{\partial f_2}{\partial x} + j\frac{\partial f_2}{\partial y} + k\frac{\partial f_2}{\partial z}$$

$$= i\frac{\partial}{\partial x}(x^2+y^2-z-3) + j\frac{\partial}{\partial y}(x^2+y^2-z-3)$$
$$+ k\frac{\partial}{\partial z}(x^2+y^2-z-3)$$

$$= i(2x) + j(2y) + k(-1)$$

$$\nabla f_2 = 2xi + 2yj - k$$

$$\nabla f_2 \text{ at } (2,-1,2) = 2(2)i + 2(-1)j - k$$

$$\nabla f_2 = 4i - 2j - k$$

$$|\nabla f_2| = \sqrt{16+4+1} = \sqrt{21}$$

$$\cos\theta = \frac{\nabla f_1 \cdot \nabla f_2}{|\nabla f_1|\ |\nabla f_2|}$$

$$\cos\theta = \frac{(4i - 2j + 4k)\cdot(4i - 2j - k)}{6\cdot\sqrt{21}}$$

$$\cos\theta = \frac{16 + \cancel{4} - \cancel{4}}{6\sqrt{21}}$$

$$\cos\theta = \frac{16}{6\sqrt{21}} \Rightarrow \theta = \cos^{-1}\left(\frac{8}{3\sqrt{21}}\right)$$

(13) Find the constants a & b so that the surface $ax^2 - byz = (a+2)x$ will be orthogonal to the surface $4x^2y + z^3 = 4$ at the point $(1, -1, 2)$

sol: Given that the two surfaces are

$$f_1 = ax^2 - byz - ax - 2x \quad \text{--- (1)}$$

$$f_2 = 4x^2y + z^3 - 4 \quad \text{--- (2)}$$

Given that two surfaces meet at the point $(1, -1, 2)$

put in (1)

$$ax^2 - byz - ax - 2x = 0$$

$$\cancel{a(1)^2 + b(1)(2) - a(1) - 2(1) = 0}$$

$$\cancel{a + 2b + \cancel{b} - 2 = 0}$$

$$ax^2 - byz - ax - 2x = 0$$

$$a(1)^2 - b(-1)(2) - a(1) - 2(1) = 0$$

$$\cancel{a} + 2b - \cancel{a} - 2 = 0$$

$$2b = 2 \Rightarrow \boxed{b = 1}$$

$$\nabla f_1 = i\frac{\partial}{\partial x} f_1 + j\frac{\partial}{\partial y} f_1 + k\frac{\partial f_1}{\partial z}$$

$$= i\frac{\partial}{\partial x}(ax^2 - byz - ax - zx) + j\frac{\partial}{\partial y}(ax^2 - byz - ax - zx)$$

$$+ k\frac{\partial}{\partial z}(ax^2 - byz - ax - zx)$$

$$= i(2ax - 0 - a - z) + j(0 - bz - 0 - 0)$$
$$+ k(0 - by - 0 - 0)$$

$$\nabla f_1 = (2ax - a - z)i + (-bz)j + (-by)k$$

$$\nabla f_1 \text{ at } (1,-1,2) = (2(1).a - a - 2)i + (-b2)j + (-b(-1))k$$
$$= (2a - a - 2)i - 2bj + bk$$

$$\rightarrow \nabla f_1 = (a-2)\bar{i} - 2b\bar{j} + b\bar{k}$$

$$\nabla f_2 = i\frac{\partial f_2}{\partial x} + j\frac{\partial f_2}{\partial y} + k\frac{\partial f_2}{\partial z}$$

$$= i\frac{\partial}{\partial x}(4x^2y + z^3 - 4) + j\frac{\partial}{\partial y}(4x^2y + z^3 - 4)$$
$$+ k\frac{\partial}{\partial z}(4x^2y + z^3 - 4)$$

$$\nabla f_2 = i(8xy) + j(4x^2) + k(3z^2)$$

$$\nabla f_2 \text{ at } (1,-1,2) = i[8(1)(-1)] + j[4(1)]$$
$$+ k(3(2)^2)$$

$$\nabla f_2 = -8\bar{i} + 4\bar{j} + 12\bar{k}$$

Given that $\theta = 90^\circ$

$$\cos\theta = \frac{\nabla f_1 \cdot \nabla f_2}{|\nabla f_1|\,|\nabla f_2|}$$

$$\cos 90^\circ = \frac{\nabla f_1 \cdot \nabla f_2}{|\nabla f_1|\,|\nabla f_2|}$$

$$0 = \frac{\nabla f_1 \cdot \nabla f_2}{|\nabla f_1|\,|\nabla f_2|}$$

$$\Rightarrow \nabla f_1 \cdot \nabla f_2 = 0$$

$$\Rightarrow \left[(a-2)\bar{i} - 2\bar{j} + \bar{k}\right] \cdot \left[-8\bar{i} + 4\bar{j} + 12\bar{k}\right] = 0$$

$$\Rightarrow (a-2)(-8) + (-2)(4) + (1)(12) = 0$$

$$\Rightarrow -8a + 16 - 8 + 12 = 0$$

$$\Rightarrow -8a + 28 - 8 = 0$$

$$\Rightarrow -8a + 20 = 0$$

$$\Rightarrow 8a = 20$$

$$a = \frac{\cancel{20}\,5}{\cancel{8}\,2}$$

$$a = 5/2$$

$\therefore$ Have $a = 5/2,\ b = 1$

UNIT – IV

VECTOR DIFFERENTIATION

*** (1) Find the Directional Derivative of $x^2yz+4xz^2$ at point $(1,-2,-1)$ in the direction of Normal to surface $x\log z - y^2$ at $(-1,2,1)$

Sol:-

Directional Derivative $= \nabla\phi \cdot \hat{n} = \nabla\phi \cdot \frac{\nabla f}{|\nabla f|}$

let $\phi = x^2yz+4xz^2$

$f = x\log z - y^2$

$\nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$

$= i\frac{\partial}{\partial x}(x^2yz+4xz^2) + j\frac{\partial}{\partial y}(x^2yz+4xz^2)$
$\quad + k\frac{\partial}{\partial z}(x^2yz+4xz^2)$

$= i(2xyz+4z^2) + j(x^2z+0) + k(x^2y+8xz)$

$\nabla\phi = i(2xyz+4z^2) + j(x^2z) + k(x^2y+8xz)$

$\nabla\phi_{at(1,-2,-1)} = i[2(1)(-2)(-1)+4(-1)^2]$
$\quad + j[(1)^2(-1)] + k[(1)^2(-2)+8(1)(-1)]$

$\nabla\phi_{at(1,-2,-1)} = 8i - j - 10k.$

$f = x \log z - y^2$

$\nabla f = i \frac{\partial f}{\partial x} + j \frac{\partial f}{\partial y} + k \frac{\partial f}{\partial z}$

$\nabla f = i \frac{\partial}{\partial x}(x \log z - y^2) + j \frac{\partial}{\partial y}(x \log z - y^2)$

$+ k \frac{\partial}{\partial z}(x \log z - y^2)$

$\nabla f = i(\log z) + j(-2y) + k\left(\frac{x}{z}\right)$

∇f at $(-1, 2, 1)$ $= i \log(1) + j(-2(2)) + k\left(\frac{-1}{1}\right)$

$\rightarrow \nabla f$ at $(-1, 2, 1)$ $= -4j - k$

$|\nabla f| = \sqrt{(-4)^2 + (-1)^2} = \sqrt{16+1} = \sqrt{17}$

$\therefore$ Directional Derivative

$= \nabla \phi \cdot \frac{\nabla f}{|\nabla f|} = (8i - j - 10k) \cdot \frac{(-4j - k)}{\sqrt{17}}$

$= \frac{0 + 4 + 10}{\sqrt{17}} = \frac{14}{\sqrt{17}}$

**

(2) Find the Directional Derivative of $f = xy + yz + zx$ in the Direction of vector $i + 2j + 2k$ at $(1, 2, 0)$

sol:-

Directional Derivative $= \nabla f \cdot \hat{n}$

$= \nabla f \cdot \frac{\nabla \phi}{|\nabla \phi|}$.

let $f = xy + yz + zx$

$\nabla \phi = i + 2j + 2k$.

$\nabla f = i \frac{\partial f}{\partial x} + j \frac{\partial f}{\partial y} + k \frac{\partial f}{\partial z}$

$= i \frac{\partial}{\partial x}(xy + yz + zx) + j \frac{\partial}{\partial y}(xy + yz + zx)$

$+ k \frac{\partial}{\partial z}(xy + yz + zx)$

$\nabla f = i(y+z) + j(x+z) + k(x+y)$

$\rightarrow \nabla f$ at $(1,2,0) = 2\hat{i} + \hat{j} + 3k$

Given that $\nabla \phi = \hat{i} + 2\hat{j} + 2k$

$|\nabla \phi| = \sqrt{(1)^2 + (2)^2 + (2)^2} = \sqrt{9} = 3$

Directional Derivative $= \nabla f \cdot \hat{n} = \nabla f \cdot \frac{\nabla \phi}{|\nabla \phi|}$

$= (2\hat{i} + j + 3k) \cdot \frac{(\hat{i} + 2j + 2k)}{\sqrt{9}}$

$= \frac{2 + 2 + 6}{3} = 10/3$

(3) Find the Directional Derivative of $\phi = x^2yz + 4xz^2$ at $(1,-2,-1)$ in the Direction of vector $2\hat{i} - \hat{j} - 2k$

sol:-

Directional Derivative $= \nabla\phi \cdot \hat{n}$

$$= \nabla\phi \cdot \frac{\nabla f}{|\nabla f|}$$

let $\phi = x^2yz + 4xz^2$

in the direction of vector

$\nabla f = 2i - j - 2k$

$$\nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$$

$$= i\frac{\partial}{\partial x}(x^2yz + 4xz^2) + j\frac{\partial}{\partial y}(x^2yz + 4xz^2)$$

$$+ k\frac{\partial}{\partial z}(x^2yz + 4xz^2)$$

$$\nabla\phi = i(2xyz + 4z^2) + j(x^2z) + k(x^2y + 8xz)$$

$\rightarrow \nabla\phi$ at $(1, -2, -1) = 8i - j - 10k$

Given that $\nabla f = 2i - j - 2k$

$$|\nabla f| = \sqrt{(2)^2 + (-1)^2 + (-2)^2} = \sqrt{4+1+4} = 3$$

$\therefore$ Directional Derivative

$$= \nabla\phi \cdot \frac{\nabla f}{|\nabla f|}$$

$$= (8i - j - 10k) \cdot \frac{(2i - j - 2k)}{3}$$

$$= \frac{16 + 1 + 20}{3}$$

$$= 37/3$$

** (4) Find the Directional Derivative of function $f = x^2 - y^2 + 2z^2$ at the point $(1, 2, 3)$ in the direction of line PQ. where $Q = (5, 0, 4)$

sol: Directional Derivative $= \nabla f \cdot \hat{n}$

$$= \nabla f \cdot \frac{\nabla \phi}{|\nabla \phi|}$$

let $f = x^2 - y^2 + 2z^2$

$$\nabla f = i \frac{\partial f}{\partial x} + j \frac{\partial f}{\partial y} + k \frac{\partial f}{\partial z}$$

$$\nabla f = i \frac{\partial}{\partial x}(x^2 - y^2 + 2z^2) + j \frac{\partial}{\partial y}(x^2 - y^2 + 2z^2)$$
$$+ k \frac{\partial}{\partial z}(x^2 - y^2 + 2z^2)$$

$$\nabla f = i(2x) + j(-2y) + k(4z)$$

$$\rightarrow \nabla f_{at\, P(1,2,3)} = 2i - 4j + 12k.$$

Given that P, Q are position vectors w.r.t to origin.

$\overline{OQ} = 5i + 4k$, $\overline{OP} = i + 2j + 3k$

$$\overline{PQ} = \overline{OQ} - \overline{OP} = (5i + 4k) - (i + 2j + 3k)$$

$$\overline{PQ} = 4i - 2j + k$$

let $\overline{PQ} = 4i - 2j + k = \nabla \phi$

$$|\nabla\phi| = \sqrt{16+4+1} = \sqrt{21}$$

$\therefore$ Directional Derivative $= \nabla f \cdot \dfrac{\nabla\phi}{|\nabla\phi|}$

$$= (2\hat{i} - 4\hat{j} + 12k) \cdot \frac{(4\hat{i} - 2\hat{j} + k)}{\sqrt{21}}$$

$$= \frac{8+8+12}{\sqrt{21}} = \frac{28}{\sqrt{21}}$$

***

(5) prove that

$r^n \bar{r}$ is solenoidal if $n = -3$

sol:- let $\bar{f} = r^n \bar{r}$

let $\bar{r} = x\hat{i} + y\hat{j} + z\bar{k}$

$$|\bar{r}| = r = \sqrt{x^2+y^2+z^2}$$

$$r^2 = x^2+y^2+z^2$$

To PT $\bar{f}$ is solenoidal if $n=-3$

ie $\nabla \cdot \bar{f} = 0$ if $n = -3$

$$\bar{f} = r^n \bar{r} = r^n (x\hat{i} + y\hat{j} + zk)$$

$$\bar{f} = r^n x\hat{i} + r^n y\bar{j} + r^n z k$$

$$\nabla \cdot \bar{f} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right) \cdot \left(r^n x\hat{i} + r^n y\hat{j} + r^n z k\right)$$

(5) find directional derivative of $f = zx^2 - xyz$ (15)
at $(1,3,1)$ in direction of vector $3i - 2j + k$

sol: Directional Derivative $= \nabla f \cdot \hat{n}$

$$= \nabla f \cdot \frac{\nabla \phi}{|\nabla \phi|}$$

let $f = zx^2 - xyz$

$$\nabla f = i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z}$$

$$= i\frac{\partial}{\partial x}(zx^2 - xyz) + j\frac{\partial}{\partial y}(zx^2 - xyz)$$
$$+ k\frac{\partial}{\partial z}(zx^2 - xyz)$$

$$= i(2xz - yz) + j(0 - xz) + k(x^2 - xy)$$

$$\nabla f = (2xz - yz)\bar{i} + (-xz)\bar{j} + (x^2 - xy)\bar{k}$$

$$\nabla f \text{ at } (1,3,1) = [2(1)(1) - 3(1)]\bar{i} + (-1\cdot 1)\bar{j}$$
$$+ (1 - 1\times 3)\bar{k}$$

$$\nabla f = (2-3)\bar{i} + (-1)\bar{j} + (-2)\bar{k}$$

$$\nabla f = -\bar{i} - \bar{j} - 2\bar{k}$$

$$|\nabla f| = \bar{i} - \bar{j} - 2\bar{k}$$

Given that

in the direction of vector is

consider $\nabla\phi = 3i - 2j + k$

$$|\nabla\phi| = \sqrt{(3)^2+(-2)^2+(1)^2} = \sqrt{9+4+1} = \sqrt{14}$$

$\therefore$ Directional Derivative $= \nabla f \cdot \frac{\nabla\phi}{|\nabla\phi|}$

$$= \frac{-i-j-2k \cdot (3i-2j+k)}{\sqrt{14}}$$

$$= \frac{(-3)+(2)+(-2)}{\sqrt{14}} = \frac{\cancel{-3}+\cancel{2}-\cancel{2}}{\sqrt{15}} = \frac{-3}{\sqrt{14}}$$

(6) Find the directional derivative of $xyz^2 + zx$ at $(1,1,1)$ in direction of normal to surface $3xy^2 + y = z$ at $(0,1,1)$

sol: Directional Derivative $= \nabla\phi \cdot \hat{n} = \nabla\phi \cdot \frac{\nabla f}{|\nabla f|}$

let $\phi = xyz^2 + zx$

$$\nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$$

$$= i\frac{\partial}{\partial x}(xyz^2+zx) + j\frac{\partial}{\partial y}(xyz^2+zx) + k\frac{\partial}{\partial z}(xyz^2+zx)$$

$$\nabla\phi = i(yz^2+z) + j(xz^2) + k(2xyz+x)$$

$\nabla\phi$ at $(1,1,1) = 2i + j + 3k$

Given that

Pu the direction of normal to surface

$f = 3xy^2 + y - z$

$\nabla f = i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z}$

$= i\frac{\partial}{\partial x}(3xy^2 + y - z) + j\frac{\partial}{\partial y}(3xy^2 + y - z)$

$+ k\frac{\partial}{\partial z}(3xy^2 + y - z)$

$= i(3y^2) + j(6xy + 1) + k(-1)$

$\nabla f = (3y^2)\bar{i} + (6xy + 1)\bar{j} + (-1)\bar{k}$

∇f at $(0,1,1) = i(3) + j(1) + k(-1)$

$\nabla f = 3\bar{i} + \bar{j} - \bar{k}$

$\rightarrow |\nabla f| = \sqrt{9+1+1} = \sqrt{11}$

$\therefore$ Directional Derivative $= \nabla\phi \cdot \frac{\nabla f}{|\nabla f|}$

$= (2i + j + 3k) \cdot \frac{(3\bar{i} + \bar{j} - \bar{k})}{\sqrt{11}}$

$= \frac{6+1-3}{\sqrt{11}} = \frac{7-3}{\sqrt{11}} = \frac{4}{\sqrt{11}}$ //

(7) Find the directional derivative of the function $xy^2 + yz^2 + zx^2$ along the tangent to the curve $x = t,\ y = t^2,\ z = t^3$ at the point $(1,1,1)$

Sol Directional Derivative $= \nabla\phi \cdot \hat{n} = \nabla\phi \cdot \frac{\nabla f}{|\nabla f|}$

let $\phi = xy^2 + yz^2 + zx^2$

$\nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$

$\nabla\phi = i\frac{\partial}{\partial x}(xy^2 + yz^2 + zx^2) + j\frac{\partial}{\partial y}(xy^2 + yz^2 + zx^2)$
$+ k\frac{\partial}{\partial z}(xy^2 + yz^2 + zx^2)$

$= i(y^2 + 0 + 2xz) + j(2xy + z^2 + 0)$
$+ k(0 + 2yz + x^2)$

$\nabla\phi = (y^2 + 2xz)\bar{i} + (2xy + z^2)\bar{j} + (x^2 + 2yz)\bar{k}$

$\nabla\phi$ at $(1,1,1)$ $= 3\bar{i} + 3\bar{j} + 3\bar{k}$

let $\bar{r}$ be the position vector of any point on the curve $x = t,\ y = t^2,\ z = t^3$

then

$\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$\bar{r} = t\bar{i} + t^2\bar{j} + t^3\bar{k}$

let $\nabla f = \frac{d\bar{r}}{dt}$ is the vector along the tangent to the curve

$\nabla f = \frac{d}{dt}(t\bar{i} + t^2\bar{j} + t^3 k) = \bar{i} + 2t\bar{j} + 3t^2 k$

$\nabla f = i + 2t\bar{j} + 3t^2\bar{k}$

∇f at $(1,1,1)$ $= \bar{i} + 2\bar{j} + 3k$

$|\nabla f| = \sqrt{1+4+9} = \sqrt{14}$

$\therefore$ Directional Derivative $= \nabla\phi \cdot \frac{\nabla f}{|\nabla f|}$

$= (3\bar{i} + 3\bar{j} + 3k) \cdot \frac{(\bar{i} + 2\bar{j} + 3k)}{\sqrt{14}}$

$= \frac{3+6+9}{\sqrt{14}} = \frac{18}{\sqrt{14}}$ //

(8) Find the directional derivative of $\frac{1}{r}$ at $(1,1,2)$ in the direction of $\bar{r} = xi + yj + zk$

sol: Directional Derivative $= \nabla f \cdot \hat{n} = \nabla f \cdot \frac{\nabla\phi}{|\nabla\phi|}$

let $f = \frac{1}{r}$

Given that $\bar{r} = x\bar{i} + y\bar{j} + zk$

$|\bar{r}| = r = \sqrt{x^2+y^2+z^2}$

$r^2 = x^2+y^2+z^2$

Diff w.r.t. x on b.s

$\frac{\partial}{\partial x}(r^2) = \frac{\partial}{\partial x}(x^2+y^2+z^2)$

$$2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{2x}{2r} \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$$

$$\text{lly } \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \text{lly } \frac{\partial r}{\partial z} = \frac{z}{r}$$

Now $\nabla f = \nabla\left(\frac{1}{r}\right)$

$$= \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\frac{1}{r}$$

$$= i\frac{\partial}{\partial x}\left(\frac{1}{r}\right) + j\frac{\partial}{\partial y}\left(\frac{1}{r}\right) + k\frac{\partial}{\partial z}\left(1/r\right)$$

$$= i\left(\frac{-1}{r^2}\right)\frac{\partial r}{\partial x} + j\left(\frac{-1}{r^2}\right)\frac{\partial r}{\partial y} + k\left(\frac{-1}{r^2}\right)\frac{\partial r}{\partial z}$$

$$= -\left[i\frac{1}{r^2}\left(\frac{x}{r}\right) + j\frac{1}{r^2}\frac{y}{r} + k\frac{1}{r^2}\frac{z}{r}\right)$$

$$\nabla f = -\frac{(xi + yj + zk)}{r^3}$$

$$\nabla f = -\frac{(xi + yj + zk)}{r^3}$$

$$\nabla f \text{ at } (1,1,2) = \frac{-(i + j + 2k)}{(x^2 + y^2 + z^2)^{3/2}}$$

$$= \frac{-(i + j + k)}{(1 + 1 + 4)^{3/2}} = -\frac{(i + j + k)}{(6)^{3/2}}$$

→ Given that in the direction vector

$$\nabla\phi = \bar{r} = xi + yj + zk$$

$$\nabla\phi = \bar{r} = xi + yj + zk$$

~~[illegible]~~

$\nabla\phi = \bar{r} = xi + yj + zk$

$\nabla\phi_{\text{at } (1,1,2)} = i + j + 2k$

$|\nabla\phi| = \sqrt{1+1+4} = \sqrt{6}$

$\therefore$ Directional Derivative $= \nabla f \cdot \frac{\nabla\phi}{|\nabla\phi|}$

$= -\frac{(i+j+k)}{(6)^{3/2}} \cdot \frac{(i+j+2k)}{\sqrt{6}}$

$= \frac{-[1+1+2]}{6^{3/2+1/2}} = \frac{-(4)}{36} = -1/9$

$\therefore$ Hence the directional derivative of $(\frac{1}{r})$ in the direction of $\bar{r}$ at $(1,1,2)$ is $-\frac{1}{9}$ //

(9) Find the directional derivative of $\nabla \cdot \nabla\phi$ at the point $(1,-2,1)$ in the direction of the normal to the surface $xy^2z = 3x + z^2$ where $\phi = 2x^3y^2z^4$

Sol: Directional Derivative $= \nabla f \cdot \hat{n} = \nabla f \cdot \frac{\nabla g}{|\nabla g|}$

let $f = \nabla \cdot \nabla\phi$

when $\phi = 2x^3y^2z^4$

$f = \nabla \cdot \nabla \phi = \nabla^2 \phi$

$f = \left(i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z}\right) \cdot \left(i \frac{\partial \phi}{\partial x} + j \frac{\partial \phi}{\partial y} + k \frac{\partial \phi}{\partial z}\right)$

$f = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = \nabla^2 \phi$

$f = \frac{\partial^2}{\partial x^2}(2x^3y^2z^4) + \frac{\partial^2}{\partial y^2}(2x^3y^2z^4)$
$+ \frac{\partial^2}{\partial z^2}(2x^3y^2z^4)$

$f = 12xy^2z^4 + 4x^3z^4 + 24x^3y^2z^2$

$\nabla f = i \frac{\partial f}{\partial x} + j \frac{\partial f}{\partial y} + k \frac{\partial f}{\partial z}$

$= i \frac{\partial}{\partial x}(12xy^2z^4 + 4x^3z^4 + 24x^3y^2z^2)$
$+ j \frac{\partial}{\partial y}(12xy^2z^4 + 4x^3z^4 + 24x^3y^2z^2)$
$+ k \frac{\partial}{\partial z}(12xy^2z^4 + 4x^3z^4 + 24x^3y^2z^2)$

$= (12y^2z^4 + 12x^2z^4 + 72x^2y^2z^2)\,i + (24xyz^4 + 48x^3yz^2)\,j$
$+ (48xy^2z^3 + 16x^3z^3 + 48x^3y^2z)\,k$

∇f at $(1, -2, 1) = 348\,\bar{i} - 144\,\bar{j} + 400\,k$

Given that in the direction normal to the surface

$g = x^2y^2z - 3x - z^2$

$g = xy^2z - 3x - z^2$

$$\nabla g = i\frac{\partial g}{\partial x} + j\frac{\partial g}{\partial y} + k\frac{\partial g}{\partial z}$$

$$= i\frac{\partial}{\partial x}(xy^2z - 3x - z^2) + j\frac{\partial}{\partial y}(xy^2z - 3x - z^2)$$
$$+ k\frac{\partial}{\partial z}(xy^2z - 3x - z^2)$$

$$\nabla g = \bar{i}(y^2z - 3)i + \bar{j}(2xyz)j + k(xy^2 - 2z)\bar{k}$$

$$\nabla g \text{ at } (1,-2,1) = \bar{i} - 4\bar{j} + 2\bar{k}$$

$$|\nabla g| = \sqrt{1+16+4} = \sqrt{21}$$

$\therefore$ Directional Derivative $= \nabla f \cdot \frac{\nabla g}{|\nabla g|}$

$\Rightarrow$ ~~∇f~~ $(f = \nabla \cdot \nabla\phi)$

$$= (348i - 144j + 400k) \cdot \frac{(i - 4j + 2k)}{\sqrt{21}}$$

$$= \frac{1724}{\sqrt{21}}$$

(10) If $a = x+y+z$, $b = x^2+y^2+z^2$, $c = xy+yz+zx$ PT $[\text{grad } a, \text{grad } b, \text{grad } c] = 0$

sol Given that

$$a = x+y+z$$

$$\text{grad } a = \nabla a = i\frac{\partial a}{\partial x} + j\frac{\partial a}{\partial y} + k\frac{\partial a}{\partial z}$$

$$= i\frac{\partial}{\partial x}(x+y+z) + j\frac{\partial}{\partial y}(x+y+z) + k\frac{\partial}{\partial z}(x+y+z)$$

$$= i + j + k$$

$\rightarrow \quad grad\, a = \nabla a = i + j + k$

$\rightarrow \quad grad\, b = \nabla b = i \frac{\partial b}{\partial x} + j \frac{\partial b}{\partial y} + k \frac{\partial b}{\partial z}$

$= i \frac{\partial}{\partial x}(x^2+y^2+z^2) + j \frac{\partial}{\partial y}(x^2+y^2+z^2) + k \frac{\partial}{\partial z}(x^2+y^2+z^2)$

$\nabla b = 2xi + 2yj + 2zk$

$\rightarrow \quad grad\, c = \nabla c = i \frac{\partial c}{\partial x} + j \frac{\partial c}{\partial y} + k \frac{\partial c}{\partial z}$

$= i \frac{\partial}{\partial x}(xy+yz+zx) + j \frac{\partial}{\partial y}(xy+yz+zx)$

$\quad + k \frac{\partial}{\partial z}(xy+yz+zx)$

$\nabla c = i(y+z) + j(z+x) + k(x+y)$

$[\nabla a, \nabla b, \nabla c] = [grad\, a, grad\, b, grad\, c]$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ y+z & z+x & x+y \end{vmatrix}$$

$= 1[(2y)(x+y) - 2z(z+x)] - 1\left(2x(x+y) - 2z(y+z)\right)$
$\quad + 1[2x(z+x) - 2y(y+z)]$

$= 2\cancel{xy} + 2\cancel{y^2} - \cancel{2z^2} - 2\cancel{zx} + \cancel{2x^2} - \cancel{2xy}$
$\quad + 2y\cancel{z} + 2\cancel{z^2} + 2\cancel{xz} + 2\cancel{x^2} - 2y^2 + 2yz$
$= 0.$

DIVERGENCE OF A VECTOR :-

let $\bar{f}$ be any vector point function than divergence of $\bar{f}$ is "$\nabla \cdot \bar{f}$" (or) "div $\bar{f}$"

$$\nabla \cdot \bar{f} = \text{div}\,\bar{f} = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right) \cdot \bar{f}$$

where $\bar{f} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$

$$= \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right) \cdot \left(f_1\bar{i} + f_2\bar{j} + f_3\bar{k}\right)$$

$$\nabla \cdot \bar{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} = \text{div}\,\bar{f}$$

$$\Rightarrow \therefore \boxed{\text{divergence of } \bar{f} = \text{div}\,\bar{f} = \nabla \cdot \bar{f}}$$

SOLENOIDAL VECTOR :-

A vector point functions $\bar{f}$ is said to be solenoidal if div $\bar{f} = 0$ (or) $\nabla \cdot \bar{f} = 0$ (or) divergence of $\bar{f} = 0$

$\nabla \cdot \bar{f} = 0$ div $\bar{f} = 0 \Rightarrow \bar{f}$ is solenoidal vector.

(1) If $\bar{f} = xy^2\bar{i} + 2x^2yz\bar{j} - 3yz^2\bar{k}$ Find $\operatorname{div}\bar{f}$ at $(1,-1,1)$

Sol:- Given that $\bar{f} = xy^2\bar{i} + 2x^2yz\bar{j} - 3yz^2\bar{k}$

To Find $\operatorname{div}\bar{f}$ at $(1,-1,1)$

$$\nabla\cdot f = \operatorname{div}\bar{f} = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\cdot\left(xy^2\bar{i} + 2x^2yz\bar{j} - 3yz^2\bar{k}\right)$$

$$= \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\cdot\left(xy^2\bar{i} + 2x^2yz\bar{j} - 3yz^2\bar{k}\right)$$

$$= \frac{\partial}{\partial x}(xy^2) + \frac{\partial}{\partial y}(2x^2yz) + \frac{\partial}{\partial z}(-3yz^2)$$

$$\nabla\cdot\bar{f} = y^2 + 2x^2z + (-6yz)$$

$$\nabla\cdot\bar{f} = \operatorname{div}\bar{f}_{\text{at}(1,-1,1)} = (-1)^2 + 2(1)^2 1 + (-6(-1)\cdot 1)$$

$$\Rightarrow \operatorname{div}\bar{f} = 1 + 2 + 6 = 9$$

** (2) Find $\operatorname{div}\bar{f}$ when $\bar{f} = \operatorname{grad}(x^3 + y^3 + z^3 - 3xyz)$

Sol.

$$\bar{f} = \operatorname{grad}(x^3 + y^3 + z^3 - 3xyz)$$

$$\bar{f} = \nabla(x^3 + y^3 + z^3 - 3xyz)$$

$$\bar{f} = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)(x^3 + y^3 + z^3 - 3xyz)$$

$$\bar{f} = i\frac{\partial}{\partial x}(x^3 + y^3 + z^3 - 3xyz) + j\frac{\partial}{\partial y}(x^3 + y^3 + z^3 - 3xyz) + k\frac{\partial}{\partial z}(x^3 + y^3 + z^3 - 3xyz)$$

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$\bar{f} = \bar{i}(3x^2 - 3yz) + \bar{j}(3y^2 - 3zx) + \bar{k}(3z^2 - 3xy)$

To find $\operatorname{div} \bar{f}$

$$\operatorname{div} \bar{f} = \nabla \cdot \bar{f} = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right) \cdot \left[\bar{i}(3x^2 - 3yz) + \bar{j}(3y^2 - 3zx) + \bar{k}(3z^2 - 3xy)\right]$$

$$= \frac{\partial}{\partial x}(3x^2 - 3yz) + \frac{\partial}{\partial y}(3y^2 - 3zx) + \frac{\partial}{\partial z}(3z^2 - 3xy)$$

$$= (6x - 0) + (6y - 0) + (6z - 0)$$

$$\Rightarrow \nabla \cdot \bar{f} = \operatorname{div} \bar{f} = 6x + 6y + 6z = 6(x + y + z)$$

(3) If $\bar{f} = (x + 3y)\bar{i} + (y - 2z)\bar{j} + (x + pz)\bar{k}$ is solenoidal, find p

sol: Given that $\bar{f}$ is solenoidal

i.e. $\nabla \cdot \bar{f} = \operatorname{div} \bar{f} = 0$

$$\left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right) \cdot \left[(x + 3y)i + (y - 2z)\bar{j} + (x + pz)\bar{k}\right] = 0$$

$$\frac{\partial}{\partial x}(x + 3y) + \frac{\partial}{\partial y}(y - 2z) + \frac{\partial}{\partial z}(x + pz) = 0$$

$$(1 + 0) + (1 - 0) + (0 + p) = 0$$

$$1 + 1 + p = 0$$

$$2 + p = 0$$

$$\boxed{p = -2}$$

(4) Find div $\bar{r}$ where $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

sol: Given that $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$$\text{div}\,\bar{r} = \nabla \cdot \bar{r} = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\cdot\left(x\bar{i} + y\bar{j} + z\bar{k}\right)$$

$$= \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 1+1+1 = 3$$

***
***

(5) prove that $r^n\bar{r}$ is solenoidal if $n = -3$

(or)

Find div $\bar{f}$ where $\bar{f} = r^n\bar{r}$. Find n if it is solenoidal?

(or)

prove that $\text{div}\,(r^n\bar{r}) = (n+3)r^n$

sol:- let $\bar{f} = r^n\bar{r}$

To PT $r^n\bar{r}$ is solenoidal if $n = -3$

ie $\text{div}\,\bar{f} = 0$ if $n = -3$, $(\because \bar{f} = r^n\bar{r})$

(or)

$\text{div}(r^n\bar{r}) = 0$ if $n = -3$

let $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$$|\bar{r}| = r = \sqrt{x^2 + y^2 + z^2}$$

$$r^2 = x^2 + y^2 + z^2$$

$r^2 = x^2+y^2+z^2$ (22)

P.D. Diff w.r.t. 'x' on b.s

$$\frac{\partial}{\partial x}(r^2) = \frac{\partial}{\partial x}(x^2+y^2+z^2)$$

$$2r\frac{\partial r}{\partial x} = 2x+0+0 \Rightarrow \cancel{2}r\frac{\partial r}{\partial x} = \cancel{2}x$$

$$\boxed{\frac{\partial r}{\partial x} = \frac{x}{r}} \quad \text{lly} \quad \boxed{\frac{\partial r}{\partial y} = \frac{y}{r}} \quad \text{lly} \quad \boxed{\frac{\partial r}{\partial z} = \frac{z}{r}}$$

Two $r^n\bar{r} = r^n(xi+yj+zk)$

$$r^n\bar{r} = r^n x\bar{i} + r^n y\bar{j} + r^n z\bar{k}$$

To PT $\operatorname{div}(r^n\bar{r}) = 0$ if $n=-3$

$$\operatorname{div}(r^n\bar{r}) = \nabla\cdot(r^n\bar{r})$$

$$=\left(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right)\cdot\left(r^n x\bar{i}+r^n y\bar{j}+r^n z\bar{k}\right)$$

$$=\frac{\partial}{\partial x}r^n x+\frac{\partial}{\partial y}r^n y+\frac{\partial}{\partial z}r^n z$$

$$=\left[r^n\frac{\partial}{\partial x}\cdot x+x\cdot\frac{\partial}{\partial x}r^n\right]+\left[r^n\cdot\frac{\partial}{\partial y}y+y\frac{\partial}{\partial y}r^n\right]$$

$$+\left[r^n\cdot\frac{\partial}{\partial z}\cdot z+z\cdot\frac{\partial}{\partial z}r^n\right]$$

$$=\left[r^n\cdot 1+x\cdot n\cdot r^{n-1}\frac{\partial r}{\partial x}\right]+\left[r^n\,1+y\cdot n\,r^{n-1}\frac{\partial r}{\partial y}\right]$$

$$+\left[r^n\cdot 1+z\cdot n\cdot r^{n-1}\frac{\partial r}{\partial z}\right]$$

$$=\left[r^n+x\cdot n\,r^{n-1}\cdot\frac{x}{r}\right]+\left[r^n+y\cdot n\,r^{n-1}\cdot\frac{y}{r}\right]$$

$$+\left[r^n+z\cdot n\cdot r^{n-1}\frac{z}{r}\right]$$

$= r^n + x \cdot n r^{n-1} \cdot x \cdot r^{-1} + r^n + y \cdot n r^{n-1} \cdot y \cdot r^{-1}$

$+ r^n + z \cdot n \cdot r^{n-1} \cdot z \cdot r^{-1}$

$= 3r^n + x^2 n \cdot r^{n-2} + y^2 n r^{n-2} + z^2 n \cdot r^{n-2}$

$= 3r^n + n r^{n-2} (x^2 + y^2 + z^2)$

$= 3r^n + n \cdot r^{n-2} (r^2) \qquad (\because r^2 = x^2 + y^2 + z^2)$

$= 3r^n + n \cdot r^{n-2+2}$

$= 3r^n + n \cdot r^n = r^n (n+3)$

$\therefore \operatorname{div} (r^n \bar{r}) = (n+3) r^n$

$r^n \bar{r}$ is solenoidal if $n = -3$ //

if $n = -3$

$\operatorname{div} (r^n \bar{r}) = \nabla \cdot (r^n \bar{r}) = \nabla \cdot \bar{f} = 0$

***

(6) Show that $\frac{\bar{r}}{r^3}$ is solenoidal

(or)

Evaluate $\nabla \cdot \left(\frac{\bar{r}}{r^3}\right)$ where $\bar{r} = xi + yj + zk$

$r = |\bar{r}|$

Sol: To show that $\frac{\bar{r}}{r^3}$ is solenoidal

ie $\nabla \cdot \frac{\bar{r}}{r^3} = 0$

$\bar{r} = xi + yj + zk$

$|\bar{r}| = r = \sqrt{x^2 + y^2 + z^2}$

$r^2 = x^2 + y^2 + z^2$

$r^2 = x^2 + y^2 + z^2$

$\frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$

$\frac{\bar{r}}{r^3} = \bar{r}\, r^{-3} = (xi + yj + zk)(r^{-3})$

$\frac{\bar{r}}{r^3} = r^{-3} x\bar{i} + r^{-3} y\bar{j} + r^{-3} z\bar{k}$

To show that $\frac{\bar{r}}{r^3}$ is solenoidal

ie $\nabla \cdot \left(\frac{\bar{r}}{r^3}\right) = 0$

LHS $= \nabla \cdot \left(\frac{\bar{r}}{r^3}\right) = \nabla \cdot \left(r^{-3} x\bar{i} + r^{-3} y\bar{j} + r^{-3} z\bar{k}\right)$

$= \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right) \cdot \left(r^{-3} x\bar{i} + r^{-3} y\bar{j} + r^{-3} z\bar{k}\right)$

$= \frac{\partial}{\partial x}(x r^{-3}) + \frac{\partial}{\partial y}(r^{-3} y) + \frac{\partial}{\partial z}(r^{-3} \cdot z)$

$= \left[r^{-3} \cdot \frac{\partial}{\partial x} x + x \frac{\partial}{\partial x} r^{-3}\right] + \left[r^{-3} \cdot \frac{\partial}{\partial y} y + y \frac{\partial}{\partial y} r^{-3}\right]$

$+ \left[r^{-3} \frac{\partial}{\partial z} z + z \frac{\partial}{\partial z} r^{-3}\right]$

$= \left[r^{-3} \cdot 1 + x \cdot (-3) r^{-3-1} \frac{\partial r}{\partial x}\right] + \left[r^{-3} \cdot 1 + y \cdot (-3) r^{-3-1} \frac{\partial r}{\partial y}\right]$

$+ \left[r^{-3} \cdot 1 + z \cdot (-3) r^{-3-1} \frac{\partial r}{\partial z}\right]$

$= \left[r^{-3} - 3x \cdot r^{-4} \cdot \frac{x}{r}\right] + \left[r^{-3} - 3y\, r^{-4} \frac{y}{r}\right]$

$+ \left[r^{-3} - 3z\, r^{-4} \frac{z}{r}\right]$

$$= \bar{r}^{3} - 3x \cdot r^{n-4} \cdot x \cdot \bar{r}^{1} + \bar{r}^{3} - 3y\, r^{n-4} \cdot y \cdot \bar{r}^{1}$$

$$+ \bar{r}^{3} - 3z\, r^{n-4} \cdot z\, \bar{r}^{1}$$

$$= 3\bar{r}^{n} - 3x^{2} r^{n-5} - 3y^{2} r^{n-5} - 3z^{2} r^{n-5}$$

$$= 3\bar{r}^{3} - 3r^{n-5}(x^{2}+y^{2}+z^{2})$$

$$= 3\bar{r}^{3} - 3r^{n-5}(r^{2})$$

$$= 3\bar{r}^{3} - 3r^{n-5+2} = 3\bar{r}^{3} - 3r^{-3} = 0$$

$$= 0$$

$$\therefore \operatorname{div}\left(\frac{\bar{r}}{r^{3}}\right) = 0$$

$$\nabla \cdot \left(\frac{\bar{r}}{r^{3}}\right) = \operatorname{div}\left(\frac{\bar{r}}{r^{3}}\right) = 0$$

$\Rightarrow \frac{\bar{r}}{r^{3}}$ is solenoidal vector.

CURL OF A VECTOR :- If $\bar{f}$ be a vector point function than curl $\bar{f}$ is defined as $\nabla \times f$

$$\nabla \times \bar{f} = \text{curl}\,\bar{f} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

where $\bar{f} = f_1 i + f_2 j + f_3 k$

IRROTATIONAL VECTOR :- A vector $\bar{f}$ is said to be irrotational if curl $\bar{f} = 0$ (or) $\nabla \times \bar{f} = 0$ than $\bar{f}$ is called irrotational vector.

SCALAR POTENTIAL :-

If $\bar{f}$ is irrotational vector, there will always exist a scalar function $\phi(x, y, z)$ such that

$\bar{f} = \text{grad}\,\phi$ (or) $\bar{f} = \nabla\phi$. this called scalarpotential of $\bar{f}$.

$$\bar{f} = \text{grad}\,\phi = \nabla\phi.$$

$$f_1 i + f_2 j + f_3 k = i\frac{\partial \phi}{\partial x} + j\frac{\partial \phi}{\partial y} + k\frac{\partial \phi}{\partial z}.$$

① If $\bar{f} = xy^2\bar{i} + 2x^2yz\bar{j} - 3yz^2\bar{k}$

Find curl $\bar{f}$ at the point $(1,-1,1)$

sol:- Given that

$$\bar{f} = xy^2\bar{i} + 2x^2yz\bar{j} - 3yz^2\bar{k}$$

$$\text{curl}\bar{f} = \nabla\times\bar{f} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2 & 2x^2yz & -3yz^2 \end{vmatrix}$$

$$\left(\nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z} \right)$$

$$= i\begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x^2yz & -3yz^2 \end{vmatrix} - j\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ xy^2 & -3yz^2 \end{vmatrix}$$

$$+ k\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ xy^2 & 2x^2yz \end{vmatrix}$$

$$= i\left[\frac{\partial}{\partial y}(-3yz^2) - \frac{\partial}{\partial z}(2x^2yz)\right]$$

$$-j\left[\frac{\partial}{\partial x}(-3yz^2) - \frac{\partial}{\partial z}(xy^2)\right] + k\left[\frac{\partial}{\partial x}(2x^2yz) - \frac{\partial}{\partial y}(xy^2)\right]$$

$$\nabla\times f = i(-3z^2 - 2x^2y) - j(0-0) + k(4xyz - 2xy)$$

$$\nabla\times f \text{ at } (1,-1,1) = i(-3(1) - 2(1)(-1)) + k[4(1)(-1)(1) - 2(1)(-1)]$$

$$\nabla\times\bar{f} = -i - 2\bar{k} = \text{curl}\bar{f}$$

at $(1,-1,1)$

② Find curl $\bar{f}$ where $\bar{f} = \text{grad}(x^3+y^3+z^3-3xyz)$

sol: Given that

$$\bar{f} = \text{grad}(x^3+y^3+z^3-3xyz)$$

$$\bar{f} = \nabla(x^3+y^3+z^3-3xyz)$$

$$\bar{f} = \left(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right)(x^3+y^3+z^3-3xyz)$$

$$\bar{f} = i\frac{\partial}{\partial x}(x^3+y^3+z^3-3xyz)+j\frac{\partial}{\partial y}(x^3+y^3+z^3-3xyz)$$
$$+k\frac{\partial}{\partial z}(x^3+y^3+z^3-3xyz)$$

$$\bar{f} = \bar{i}(3x^2-3yz)+\bar{j}(3y^2-3xz)$$
$$+\bar{k}(3z^2-3xy)$$

To Find curl $\bar{f}$ (or) $\nabla\times\bar{f}$

$$\nabla\times\bar{f} = \text{curl}\,\bar{f} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (3x^2-3yz) & (3y^2-3xz) & (3z^2-3xy) \end{vmatrix}$$

$$= i\left[\frac{\partial}{\partial y}(3z^2-3xy)-\frac{\partial}{\partial z}(3y^2-3xz)\right]$$
$$-j\left[\frac{\partial}{\partial x}(3z^2-3xy)-\frac{\partial}{\partial z}(3x^2-3yz)\right]$$
$$+k\left[\frac{\partial}{\partial x}(3y^2-3xz)-\frac{\partial}{\partial y}(3x^2-3yz)\right]$$

$\nabla \times \bar{f} = \text{curl}\, \bar{f} = i(-3x+3x) - j(-3y+3y) + k(-3z+3z)$

$= 0$

$\therefore$ Here $\nabla \times \bar{f} = 0$ (or) $\text{curl}\, \bar{f} = 0$

$\Rightarrow \bar{f}$ is irrotational vector

(3) If $\bar{F} = (x+y+1)\bar{i} + \bar{j} - (x+y)\bar{k}$ then

show that $\bar{F} \cdot \text{curl}\, \bar{F} = 0$

sol: Given that

$\bar{F} = (x+y+1)\bar{i} + \bar{j} - (x+y)\bar{k}$

$$\text{curl}\, \bar{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y+1 & 1 & -x-y \end{vmatrix}$$

$$= i\left[\frac{\partial}{\partial y}(-x-y) - \frac{\partial}{\partial z}(1)\right] - j\left[\frac{\partial}{\partial x}(-x-y) - \frac{\partial}{\partial z}(x+y+1)\right] + k\left[\frac{\partial}{\partial x}(1) - \frac{\partial}{\partial y}(x+y+1)\right]$$

$= i(-1-0) - j(-1-0) + k(0-1)$

$\text{curl}\, \bar{F} = -i + j - k$

$\bar{F} \cdot \text{curl}\, \bar{F} = \left[(x+y+1)\bar{i} + \bar{j} + (-x-y)\bar{k}\right] \cdot \left[-i + j - k\right]$

$= (x+y+1)(-1) + 1 + (-x-y)(-1)$

$\bar{F} \cdot \text{curl}\, \bar{F} = \cancel{-x} \cancel{-y} \cancel{-1} \cancel{+1} \cancel{+x} \cancel{+y} = 0$

(4) prove that $\bar{F} = yz\bar{i} + zx\bar{j} + xy\bar{k}$ (25)
is irrotational

sol:- Given that

$$\bar{F} = yz\bar{i} + zx\bar{j} + xy\bar{k}$$

$$\operatorname{curl}\bar{F} = \nabla\times\bar{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & zx & xy \end{vmatrix}$$

$$= i\left[\frac{\partial}{\partial y}(xy) - \frac{\partial}{\partial z}(zx)\right] - j\left[\frac{\partial}{\partial x}(xy) - \frac{\partial}{\partial z}(yz)\right]$$
$$+ k\left[\frac{\partial}{\partial x}(zx) - \frac{\partial}{\partial y}(yz)\right]$$

$$= i(x-x) - j(y-y) + k(z-z)$$

$$= 0\bar{i} + 0.\bar{j} + 0.\bar{k} = \bar{0}$$

$\therefore \nabla\times\bar{F} = \operatorname{curl}\bar{F} = \bar{0} \Rightarrow \bar{F}$ is irrotational vector

(5) If $\bar{F} = xi - y^2j + z^3k$ Find $\operatorname{curl}\bar{F}$

sol: Given that

$$F = xi - y^2j + z^3k$$

$$\operatorname{curl}\bar{F} = \nabla\times\bar{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & -y^2 & z^3 \end{vmatrix}$$

$$= i\left[\frac{\partial}{\partial y}(zy) - \frac{\partial}{\partial z}(-y^2)\right] - j\left[\frac{\partial}{\partial x}(zy) - \frac{\partial}{\partial z}(xy)\right]$$

$$+k\left[\frac{\partial}{\partial x}(-y^2) - \frac{\partial}{\partial y}(xy)\right]$$

$$= i(0-0) - j(0-0) + k(0-0) = 0i + 0j + 0k$$

$$= \bar{0}$$

⑥ If $\bar{A}$ is irrotational vector, evaluate div $(\bar{A} \times \bar{r})$ where $\bar{r} = xi + yj + zk$

sol:- Given that $\bar{r} = xi + yj + zk$

Given that $\bar{A}$ is an irrotational vector

ie $\nabla \times \bar{A} = 0$ —①

Now $\text{div}(\bar{A} \times \bar{r}) = \nabla \cdot (\bar{A} \times \bar{r})$

$$= \bar{r} \cdot (\nabla \times \bar{A}) - \bar{A} \cdot (\nabla \times \bar{r})$$

$$= \bar{r} \cdot (0) - \bar{A} \cdot (\nabla \times \bar{r}) = -\bar{A} \cdot (\nabla \times \bar{r})$$ —②

Now
$$\nabla \times \bar{r} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$$

$$= i\left(\frac{\partial}{\partial y} z - \frac{\partial}{\partial z} y\right) - j\left(\frac{\partial}{\partial x} z - \frac{\partial}{\partial z}(x)\right)$$

$$+k\left(\frac{\partial}{\partial x} y - \frac{\partial}{\partial y} x\right)$$

$$= i(0-0) - j(0-0) + k(0-0) = \bar{0}$$

$\therefore \bar{A} \cdot (\nabla \times \bar{r}) = 0$

$\therefore \text{div}(\bar{A} \times \bar{r}) = 0$ //

(7) *** prove that if $\bar{r}$ is the position (26) vector any point in space, then $r^n\bar{r}$ is irrotational vector.

(or)

show that $\text{curl}(r^n\bar{r}) = 0$

(or)

show that $\nabla \times (r^n\bar{r}) = 0$

Sol:- let $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$|\bar{r}| = r = \sqrt{x^2+y^2+z^2}$

$r^2 = x^2+y^2+z^2$

'p' Diff w.r.to 'x' on b.s

$\frac{\partial}{\partial x}(r^2) = \frac{\partial}{\partial x}(x^2+y^2+z^2)$

$2r\frac{\partial r}{\partial x} = 2x \Rightarrow \boxed{\frac{\partial r}{\partial x} = \frac{x}{r}}$

lly $\boxed{\frac{\partial r}{\partial y} = \frac{y}{r}}$ lly $\boxed{\frac{\partial r}{\partial z} = \frac{z}{r}}$

$\rightarrow r^n\bar{r} = r^n(x\hat{i}+y\hat{j}+z\hat{k})$

$r^n\bar{r} = xr^n\hat{i} + yr^n\hat{j} + zr^n\hat{k}$

To show that $r^n\bar{r}$ is irrotational

i.e $\nabla \times (r^n\bar{r}) = 0$

(or)

$\text{curl}(r^n\bar{r}) = 0$

LHS $=$ $\text{curl}(r^n \bar{r}) = \nabla \times (r^n \bar{r})$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x r^n & y r^n & z r^n \end{vmatrix}$$

$$= i\left[\frac{\partial}{\partial y}(z r^n) - \frac{\partial}{\partial z}(y r^n)\right] - j\left[\frac{\partial}{\partial x}(z r^n) - \frac{\partial}{\partial z}(x r^n)\right] + k\left[\frac{\partial}{\partial x}(y r^n) - \frac{\partial}{\partial y}(x r^n)\right]$$

$$= i\left[z \cdot n r^{n-1}\frac{\partial r}{\partial y} - y \cdot n r^{n-1}\frac{\partial r}{\partial z}\right] - j\left[z \cdot n r^{n-1}\frac{\partial r}{\partial x} - x \cdot n r^{n-1}\frac{\partial r}{\partial z}\right] + k\left[y \cdot n r^{n-1}\frac{\partial r}{\partial x} - x n r^{n-1}\frac{\partial r}{\partial y}\right]$$

$$= i\left[z \cdot n r^{n-1} \cdot \frac{y}{r} - y \cdot n r^{n-1} \cdot \frac{z}{r}\right] - j\left[z n r^{n-1} \frac{x}{r} - x \cdot n r^{n-1} \frac{z}{r}\right] + k\left[y n r^{n-1} \frac{x}{r} - x n r^{n-1} \frac{y}{r}\right]$$

$$= i(0-0) - j(0-0) + k(0-0)$$

$$= \bar{0}$$

$\therefore$ $\text{curl}(r^n \bar{r}) = 0$

$\Rightarrow$ $r^n \bar{r}$ is irrotational vector.

***

(8) Show that the vector $(x^2-yz)\bar{i}+(y^2-zx)\bar{j}+(z^2-xy)\bar{k}$ is irrotational and find its scalar potential (27)

sol: Given that

$$\bar{f}=(x^2-yz)\bar{i}+(y^2-zx)\bar{j}+(z^2-xy)\bar{k}$$

To show that $\bar{f}$ is irrotational

ie $\nabla\times\bar{f}=0$ (or) curl $\bar{f}=0$

$$\nabla\times\bar{f}=\text{curl}\,\bar{f}=\begin{vmatrix} \bar{i} & \bar{j} & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2-yz & y^2-zx & z^2-xy \end{vmatrix}$$

$$=i\left[\frac{\partial}{\partial y}(z^2-xy)-\frac{\partial}{\partial z}(y^2-zx)\right]-j\left[\frac{\partial}{\partial x}(z^2-xy)-\frac{\partial}{\partial z}(x^2-yz)\right]+k\left[\frac{\partial}{\partial x}(y^2-zx)-\frac{\partial}{\partial y}(x^2-yz)\right]$$

$$=i\left[-x+x\right]-j(-y+y)+k(-z+z)$$

$$=i(0)-j(0)+k(0)=\bar{0}$$

$\therefore$ curl $\bar{F}=\nabla\times\bar{f}=0$

$\Rightarrow$ $\bar{f}$ is irrotational vector

$\therefore$ $\bar{f}$ is irrotational. then there exist a scalar function 'ϕ' such that

$$\bar{f} = \nabla\phi = \text{grad}\,\phi$$

$$\Rightarrow (x^2-yz)\bar{i} + (y^2-zx)\bar{j} + (z^2-xy)\bar{k}$$

$$= i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$$

comparing i, j, k co-efficients on b.s

$$\Rightarrow x^2-yz = \frac{\partial\phi}{\partial x} \Rightarrow \partial\phi = (x^2-yz)\,\partial x$$

IBS.

$$\Rightarrow \phi = \int(x^2-yz)\,dx = \frac{x^3}{3} - xyz + c_1 \quad -①$$

$$\Rightarrow (y^2-zx) = \frac{\partial\phi}{\partial y} \Rightarrow \partial\phi = (y^2-zx)\,\partial y$$

IBS

$$\Rightarrow \phi = \int(y^2-zx)\,dy = \frac{y^3}{3} - xyz + c_2 \quad -②$$

$$\Rightarrow (z^2-xy) = \frac{\partial\phi}{\partial z} \Rightarrow \partial\phi = (z^2-xy)\,\partial z$$

IBS

$$\Rightarrow \phi = \int(z^2-xy)\,dz = \frac{z^3}{3} - xyz + c_3 \quad -③$$

fr ①, ②, ③

$$\phi = \frac{x^3}{3} - xyz + y^3/3 - xyz + \frac{z^3}{3} - xyz + c.$$

$$= \frac{1}{3}(x^3+y^3+z^3) - 3xyz + c$$

(9) Find constants a, b, c so that the vector (28)

$\bar{f} = (x + 2y + az)\bar{i} + (bx - 3y - z)\bar{j} + (4x + cy + 2z)\bar{k}$

is irrotational. Also find find ϕ such that

$\bar{f} = \nabla\phi$ (i.e. find its scalar potential) ...

sol: Given that

$\bar{f}$ is irrotational vector

i.e. $\nabla \times \bar{f} = \text{curl } \bar{f} = 0$

$\bar{f} = (x + 2y + az)\bar{i} + (bx - 3y - z)\bar{j}$
$+ (4x + cy + 2z)\bar{k}$

$\nabla \times \bar{f} = \text{curl } \bar{F} = 0$

$$\Rightarrow \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x + 2y + az & bx - 3y - z & 4x + cy + 2z \end{vmatrix} = 0$$

$$\Rightarrow i\left[\frac{\partial}{\partial y}(4x + cy + 2z) - \frac{\partial}{\partial z}(bx - 3y - z)\right]$$
$$- j\left[\frac{\partial}{\partial x}(4x + cy + 2z) - \frac{\partial}{\partial z}(x + 2y + az)\right]$$
$$+ k\left[\frac{\partial}{\partial x}(bx - 3y - z) - \frac{\partial}{\partial y}(x + 2y + az)\right] = 0$$

$\Rightarrow \quad \bar{i}(c+1) + \bar{j}(a-4) + \bar{k}(b-2) = \bar{0}$

$\Rightarrow \quad (c+1)\bar{i} + (a-4)\bar{j} + (b-2)\bar{k} = 0\bar{i} + 0\bar{j} + 0\bar{k}$

comparing both sides i, j, k co-efficients

$c+1 = 0 \Rightarrow c = -1$

$a-4 = 0 \Rightarrow a = 4$

$b-2 = 0 \Rightarrow b = 2$

Now $\quad \bar{f} = (x+2y+az)\bar{i} + (bx-3y-z)\bar{j} + (4x+cy+2z)\bar{k}$

$\bar{f} = (x+2y+4z)\bar{i} + (2x-3y-z)\bar{j} + (4x-y-2z)\bar{k}$

To Find $\quad \bar{f} = \nabla\phi$

$\Rightarrow \quad (x+2y+4z)\bar{i} + (2x-3y-z)\bar{j} + (4x-y-2z)\bar{k}$
$= i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$

comparing both sides

$\rightarrow \quad \frac{\partial\phi}{\partial x} = x+2y+4z \Rightarrow \partial\phi = x+2y+4z\,\partial x$

IBS

$\Rightarrow \phi = \frac{x^2}{2} + 2xy + 4zx + c_1$

$\rightarrow \quad \frac{\partial\phi}{\partial y} = 2x-3y-z \Rightarrow \partial\phi = (2x-3y-z)\,\partial y$

IBS

$\Rightarrow \phi = 2xy - \frac{3y^2}{2} - yz + c_2$

$\rightarrow \quad \frac{\partial\phi}{\partial z} = 4x-y-2z \Rightarrow \partial\phi = (4x-y-2z)\,\partial z$

$\Rightarrow \phi = 4xz - yz + z^2 + c_3$

$\phi = \frac{x^2}{2} - \frac{3y^2}{2} + z^2 + 2xy - yz + 4zx$

(b)

(10)

(10). Find whether the function

$\bar{F} = (x^2-y^3)\bar{i} + (y^2-3x)\bar{j} + (z^2-xy)\bar{k}$

is irrotational and hence find scalar potential function corresponding to it.

Sol: Given that

$$\bar{F} = (x^2-y^3)\bar{i} + (y^2-3x)\bar{j} + (z^2-xy)\bar{k}$$

$$\nabla \times \bar{F} = \text{curl}\,\bar{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2-y^3 & y^2-3x & z^2-xy \end{vmatrix}$$

$$= i\left[\frac{\partial}{\partial y}(z^2-xy) - \frac{\partial}{\partial z}(y^2-3x)\right]$$

$$+ j\left[\frac{\partial}{\partial x}(z^2-xy) - \frac{\partial}{\partial z}(x^2-y^3)\right]$$

$$+ k\left[\frac{\partial}{\partial x}(y^2-3x) - \frac{\partial}{\partial y}(x^2-y^3)\right]$$

$$= i(0-x-0) - j(-y-0) + k(-3+3y^2)$$

$$\nabla \times \bar{f} = -x\bar{i} + y\bar{j} + k(3y^2-3)$$

$\nabla \times \bar{f} \neq 0$

$\bar{F}$ is irrotational.

(11) prove that
$\bar{F} = (y^2\cos x + z^3)\bar{i} + (2y\sin x - 4)\bar{j} + 3xz^2 k$ is irrotational and find its scalar potential

Sol: Given that

$$\bar{F} = (y^2\cos x + z^3)\bar{i} + (2y\sin x - 4)\bar{j} + 3xz^2\bar{k}$$

$$\nabla\times\bar{F} = \text{curl}\,\bar{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2\cos x + z^3 & 2y\sin x - 4 & 3xz^2 \end{vmatrix}$$

$$= i\left[\frac{\partial}{\partial y}(3xz^2) - \frac{\partial}{\partial z}(2y\sin x - 4)\right] - j\left[\frac{\partial}{\partial x}(3xz^2) - \frac{\partial}{\partial z}(y^2\cos x + z^3)\right]$$
$$+ k\left[\frac{\partial}{\partial x}(2y\sin x - 4) - \frac{\partial}{\partial y}(y^2\cos x + z^3)\right]$$

$$= i[0-0] - j[3z^2 - 3z^2] + k(2y\cos x - 2y\cos x)$$

$$= i(0) - j(0) + k(0) = \bar{0}$$

$\therefore \nabla\times\bar{F} = \text{curl}\,\bar{F} = 0 \Rightarrow \bar{F}$ is irrotational

Now $\bar{f} = \nabla\phi$

$$(y^2\cos x + z^3)\bar{i} + (2y\sin x - 4)\bar{j} + 3xz^2\bar{k}$$
$$= i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$$

comparing i, j, k coefficients on both

$\rightarrow \frac{\partial \phi}{\partial x} = y^2 \cos x + z^3 \Rightarrow \partial\phi = (y^2 \cos x + z^3)\,\partial x$ (30)

I B S

$\Rightarrow \phi = y^2 \sin x + z^3 x + c_1$ — ①

$+ \frac{\partial \phi}{\partial y} = 2y \sin x - 4 \Rightarrow \partial\phi = (2y \sin x - 4)\,\partial y$

I B S

$\Rightarrow \phi = \cancel{2} \sin x \cdot \frac{y^2}{\cancel{2}} - 4y + c_2$

$\Rightarrow \phi = y^2 \sin x - 4y + c_2$ — ②

$+ \frac{\partial \phi}{\partial z} = 3x z^2 \Rightarrow \partial\phi = (3x z^2)\,\partial z$

I B S

$\Rightarrow \phi = 3xz + z^3/3 + c_3$ — ③

$\Rightarrow \phi = y^2 \sin x + z^3 x + y^2 \sin x - 4y$
$+ 3xz + z^3/3 + c.$

(12) Find constants a, b, c if the vector

$\bar{f} = (2x + 3y + az)\bar{i} + (bx + 2y + 3z)\bar{j}$
$+ (2x + cy + 3z)\bar{k}$ is irrotational

sol: Given that

$\bar{f}$ is irrotational vector

ie $\nabla \times \bar{f} = \text{curl } \bar{f} = 0$

$$\Rightarrow \nabla \times \bar{f} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x+3y+az & bx+2y+3z & 2x+cy+3z \end{vmatrix} = \bar{0}$$

$$\Rightarrow i\left[\frac{\partial}{\partial y}(2x+cy+3z) - \frac{\partial}{\partial z}(bx+2y+3z)\right]$$
$$- j\left[\frac{\partial}{\partial x}(2x+cy+3z) - \frac{\partial}{\partial z}(2x+3y+az)\right]$$
$$+ k\left[\frac{\partial}{\partial x}(bx+2y+3z) - \frac{\partial}{\partial y}(2x+3y+az)\right]$$
$$= \bar{0}$$

$$\Rightarrow i\,[c-3) - j(2-a) + k(b-3) = \bar{0}$$

$$\Rightarrow \bar{i}(c-3) - \bar{j}(2-a) + \bar{k}(b-3) = 0\bar{i} + 0\bar{j} + 0\bar{k}$$

$\Rightarrow$ comparing i, j, k co-efficients on b/s

$$c - 3 = 0 \Rightarrow c = 3$$
$$2 - a = 0 \Rightarrow a = 2$$
$$b - 3 = 0 \Rightarrow b = 3$$

Laplacian operator ∇^2 :

→ $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

→ $\nabla^2\phi = \nabla \cdot \nabla\phi = \text{div}(\text{grad}\,\phi)$

→ If $\nabla^2\phi = 0$ than ϕ is said to be satisfy laplacian equation. than ϕ is called a harmonic function.

(1) prove that $\text{div}(\text{grad}\, r^m) = m(m+1)r^{m-2}$

(or)

prove that $\nabla^2(r^m) = m(m+1)r^{m-2}$

(or)

prove that $\nabla^2(r^n) = n(n+1)r^{n-2}$

sol:- $\nabla^2\phi = \nabla \cdot \nabla\phi$

To P.T $\text{div}(\text{grad}\, r^m) = \nabla \cdot (\nabla r^m)$

let $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$|\bar{r}| = r = \sqrt{x^2+y^2+z^2}$

$r^2 = x^2+y^2+z^2$

P Diff w.r.to 'x' on b.s

$\frac{\partial}{\partial x}(r^2) = \frac{\partial}{\partial x}(x^2+y^2+z^2)$

$$2r\frac{\partial r}{\partial x} = 2x + 0 + 0 \Rightarrow \cancel{2}r\frac{\partial r}{\partial x} = \cancel{2}x$$

$$\boxed{\frac{\partial r}{\partial x} = \frac{x}{r}}$$

$$\text{IIy } \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \text{IIy } \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\rightarrow \nabla r^M = \left(i\frac{\partial}{\partial x}r^M + j\frac{\partial}{\partial y}r^M + k\frac{\partial}{\partial z}r^M\right)$$

$$= i\,M r^{M-1}\frac{\partial r}{\partial x} + j\,M r^{M-1}\frac{\partial r}{\partial y} + k\,M r^{M-1}\frac{\partial r}{\partial z}$$

$$= i\,M r^{M-1}\frac{x}{r} + j\,M r^{M-1}\frac{y}{r} + k\cdot M r^{M-1}\frac{z}{r}$$

$$= i\,x\,M\cdot r^{M-1}\cdot r^{-1} + j\,y\,M\cdot r^{M-1}\cdot r^{-1} + k\cdot M\cdot z\,r^{M-1}\cdot r^{-1}$$

$$\rightarrow \nabla r^M = x\,\bar{i}\,M r^{M-2} + y\,\bar{j}\,M r^{M-2} + z\,\bar{k}\,n\,r^{M-2}$$

NOW P.T $\nabla\cdot\nabla r^M = M(M+1)r^{M-2}$

$$\text{LHS} = \nabla\cdot\nabla r^M$$

$$= \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\cdot\left[i\,x\,M r^{M-2} + j\,y\,M r^{M-2} + k\,z\,M r^{M-2}\right]$$

$$= \frac{\partial}{\partial x}x\,M r^{M-2} + \frac{\partial}{\partial y}y\,M r^{M-2} + \frac{\partial}{\partial z}k\,M r^{M-2}$$

$$= M\left[\left[x\cdot\frac{\partial}{\partial x}r^{M-2} + r^{M-2}\cdot\frac{\partial}{\partial x}\right] + \left[y\left[\frac{\partial}{\partial y}r^{M-2} + r^{M-2}\frac{\partial}{\partial y}y\right]\right.\right.$$
$$\left.\left. + \left[\cancel{z}\frac{\partial}{\partial z}r^{M-2} + r^{M-2}\frac{\partial}{\partial z}z\right]\right]\right]$$

(32)

$$= M\Big[\Big[x(M-2)\,r^{M-2-1}\cdot\frac{\partial r}{\partial x} + r^{M-2}\cdot 1\Big] + \Big[y(M-2)\,r^{M-2-1}\cdot\frac{\partial r}{\partial y} + r^{M-2}\Big] + \Big[z(M-2)\,r^{M-2-1}\cdot\frac{\partial r}{\partial z} + r^{M-2}\Big]\Big]$$

$$= M\Big[x(M-2)\,r^{M-3}\cdot\frac{x}{r} + r^{M-2} + y(M-2)\,r^{M-3}\frac{y}{r} + r^{M-2} + z(M-2)\,r^{M-3}\frac{z}{r} + r^{M-2}\Big]$$

$$= M\Big[x^2(M-2)\,r^{M-4} + r^{M-2} + y^2(M-2)\,r^{M-4} + r^{M-2} + z^2(M-2)\,r^{M-4} + r^{M-2}\Big]$$

$$= M\Big[3r^{M-2} + (M-2)\,r^{M-4}(x^2+y^2+z^2)\Big]$$

$$= M\Big[3r^{M-2} + (M-2)\,r^{M-4}(r^2)\Big]$$

$$= M\Big[3r^{M-2} + (M-2)\,r^{M-2}\Big]$$

$$= M\Big[r^{M-2}(3+M-2)\Big]$$

$$= M\Big[r^{M-2}(M+1)\Big]$$

$$= M(M+1)\,r^{M-2} = \text{RHS}$$

$\therefore$ Hence proved

$$\Rightarrow \nabla^2(r^M) = \nabla\cdot\nabla(r^M) = \operatorname{div}(r^M) = M(M+1)\,r^{M-2}$$

(2) show that $(\bar{a}\cdot\nabla)\phi = \bar{a}\cdot\nabla\phi$

sol: let $\bar{a} = a_1\bar{i} + a_2\bar{j} + a_3\bar{k}$

$$\bar{a}\cdot\nabla = (a_1\bar{i} + a_2\bar{j} + a_3\bar{k})\cdot\left[i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right]$$

$$= a_1\frac{\partial}{\partial x} + a_2\frac{\partial}{\partial y} + a_3\frac{\partial}{\partial z}$$

$$(\bar{a}\cdot\nabla)\phi = \left(a_1\frac{\partial}{\partial x} + a_2\frac{\partial}{\partial y} + a_3\frac{\partial}{\partial z}\right)\phi$$

$$= a_1\frac{\partial\phi}{\partial x} + a_2\frac{\partial\phi}{\partial y} + a_3\frac{\partial\phi}{\partial z}$$

$$= (\bar{a}\cdot\nabla\phi) = \text{RHS.}$$

(3) show that $(\bar{a}\cdot\nabla)\bar{r} = \bar{a}$

sol

$$\bar{r} = xi + yj + zk$$

$$\bar{a} = a_1 i + a_2 j + a_3 k$$

$$(\bar{a}\cdot\nabla) = (a_1 i + a_2 j + a_3 k)\cdot\left[i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)$$

$$= \left(a_1\frac{\partial}{\partial x} + a_2\frac{\partial}{\partial y} + a_3\frac{\partial}{\partial z}\right)$$

$$(\bar{a}\cdot\nabla)\bar{r} = \left(a_1\frac{\partial}{\partial x} + a_2\frac{\partial}{\partial y} + a_3\frac{\partial}{\partial z}\right)(xi + yj + zk)$$

$$= a_1\frac{\partial}{\partial x}(xi + yi + zk) + a_2\frac{\partial}{\partial y}(xi + yi + zk)$$
$$+ a_3\frac{\partial}{\partial z}(xi + yi + zk)$$

$$= a_1(i)(1) + a_2(j)1 + a_3(k)1 = a_1\bar{i} + a_2\bar{j} + a_3\bar{k}$$

$$= \bar{a} = \text{RHS.}$$

(4) Show that

$$\nabla^2[f(r)] = \frac{\partial^2 f}{\partial r^2} + \frac{2}{r}\frac{\partial f}{\partial r} = f''(r) + \frac{2}{r}f'(r)$$

where $r = |\bar{r}|$

sol: grad $f(r) = \nabla f(r)$

$$\nabla f(r) = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right) f(r)$$

$$= \Sigma\, i \frac{\partial}{\partial x} f(r) = \Sigma\, i\, f'(r)\cdot\frac{\partial r}{\partial x}$$

$$\nabla f(r) = \Sigma\, i\, f'(r)\cdot\frac{x}{r}$$

$$\nabla^2 f(r) = \nabla\cdot\nabla f(r) = \operatorname{div}[\text{grad } f(r)]$$

$$(\nabla^2\phi = \nabla\cdot\nabla\phi \quad \text{Laplacian})$$

$$\nabla^2 f(r) = \nabla\cdot\nabla f(r)$$

$$= \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\cdot\left(\Sigma\, i\, f'(r)\frac{x}{r}\right)$$

$$= \Sigma\, i\frac{\partial}{\partial x}\cdot\Sigma\, i\, f'(r)\frac{x}{r}$$

$$= \Sigma\left(i\frac{\partial}{\partial x}\right)\cdot\left(i\, f'(r)\frac{x}{r}\right)$$

$$= \Sigma\frac{\partial}{\partial x}\left(f'(r)\cdot\frac{x}{r}\right)$$

$$= \Sigma\left[\frac{r\frac{\partial}{\partial x}\left(f'(r)\cdot\frac{x}{r}\right) - f'(r)\cdot\frac{x}{r}\cdot\frac{\partial}{\partial x}(r)}{r^2}\right]$$

By $\frac{U}{V}$ Rule (Quotient rule)

$$= \sum \left[\frac{r\left(f''(r)\cdot\frac{\partial r}{\partial x}\cdot x + f'(r)\right) - f'(r)\cdot r\cdot\frac{x}{r}}{r^2} \right]$$

using product rule (uv Rule)

$$= \sum \frac{r f''(r)\cdot\frac{x}{r}\cdot x + r f'(r) - x^2 f'(r)\cdot\frac{1}{r}}{r^2}$$

$$= \sum \frac{r f''(r)\cdot\frac{x^2}{r} + r f'(r) - x^2 f'(r)\cdot\frac{1}{r}}{r^2}$$

$$= \sum \frac{\cancel{r} f''(r)\cdot\frac{x^2}{\cancel{r}}}{r^2} + \sum \frac{r f'(r)}{r^2} - \sum \frac{x^2 f'(r)\cdot\frac{1}{r}}{r^2}$$

$$= \frac{f''(r)}{r^2}\sum x^2 + \frac{\cancel{r}}{r\cancel{r}}\sum f'(r) - \frac{1}{r^2}\cancel{f}(r)\cdot x^2$$

$$= \frac{f''(r)}{r^2}\sum x^2 + \frac{f'(r)}{r}\sum 1 - \frac{1}{r^3} f'(r)\sum x^2$$

$$= \frac{f''(r)}{r^2}(x^2+y^2+z^2) + \frac{f'(r)}{r}(1+1+1)$$
$$-\frac{1}{r^3} f'(r) (x^2+y^2+z^2$$

$$= \frac{f''(r)}{\cancel{r^2}}\cdot\cancel{r^2} + \frac{3}{r} f'(r) - \frac{1}{\cancel{r^3}} f'(r)\cdot\cancel{r^2}$$
$$= f''(r) + 3/r\, f'(r) - \frac{1}{r} f'(r)$$
$$= f''(r) + \frac{2}{r} f'(r) = RHS$$

VECTOR IDENTITIES :-

(1) If $\bar{a}$ is a differentiable function and ϕ is a differentiable scalar function, then prove that $div(\phi\bar{a}) = (grad\,\phi).\bar{a} + \phi\, div\,\bar{a}$

(or)

$$\nabla.(\phi\bar{a}) = (\nabla\phi).\bar{a} + \phi(\nabla.\bar{a})$$

sol: show that

$$div(\phi\bar{a}) = \nabla.(\phi\bar{a}) = (\nabla\phi).\bar{a} + \phi(\nabla.\bar{a})$$

$$LHS = div(\phi\bar{a}) = \nabla.(\phi\bar{a})$$

$$= \nabla.(\phi\bar{a})$$

$$= (i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}).(\phi\bar{a})$$

$$= \overline{\sum(i\frac{\partial}{\partial x}).\phi\bar{a}}$$

$$= \sum \bar{i}.\frac{\partial}{\partial x}(\phi\bar{a})$$

$$= \sum i.\left[\frac{\partial\phi}{\partial x}\bar{a} + \phi\frac{\partial\bar{a}}{\partial x}\right]$$

$$= \sum(i\frac{\partial\phi}{\partial x}).\bar{a} + \sum(i\frac{\partial}{\partial x}.\bar{a})\phi$$

$$= \sum(i\frac{\partial\phi}{\partial x}).\bar{a} + \sum(i\frac{\partial}{\partial x}.\bar{a})\phi$$

$$= (\nabla\phi).\bar{a} + (\nabla.\bar{a})\phi$$

$$= RHS$$

$$\therefore \nabla.(\phi\bar{a}) = (\nabla\phi).\bar{a} + \phi(\nabla.\bar{a})$$

(2) prove that

$$\text{curl}(\phi\bar{a}) = (\text{grad}\,\phi)\times\bar{a} + \phi\,\text{curl}\,\bar{a}$$

(or)

$$\nabla\times(\phi\bar{a}) = (\nabla\phi)\times\bar{a} + \phi(\nabla\times\bar{a})$$

sol:- Show that

$$\nabla\times(\phi\bar{a}) = (\nabla\phi)\times\bar{a} + \phi(\nabla\times\bar{a})$$

$$LHS = \nabla\times(\phi\bar{a})$$

$$= \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\times(\phi\bar{a})$$

$$= \sum i\frac{\partial}{\partial x}\times(\phi\bar{a})$$

$$= \sum i\times\frac{\partial}{\partial x}(\phi\bar{a})$$

$$= \sum i\times\left[\frac{\partial\phi}{\partial x}\bar{a} + \phi\frac{\partial\bar{a}}{\partial x}\right]$$

$$= \sum\left(i\frac{\partial\phi}{\partial x}\right)\times\bar{a} + \sum\left(i\frac{\partial}{\partial x}\times\bar{a}\right)\phi$$

$$= (\nabla\phi)\times\bar{a} + (\nabla\times\bar{a})\phi$$

$$= RHS.$$

$$\therefore \nabla\times(\phi(\bar{a})) = (\nabla\phi)\times\bar{a} + (\nabla\times\bar{a})\phi$$

$$\text{curl}(\phi\bar{a}) = (\text{grad}\,\phi)\times\bar{a} + \phi\,\text{curl}\,\bar{a} \quad //$$

(3) prove that

$$\operatorname{div}(\bar{a}\times\bar{b}) = \bar{b}\,\operatorname{curl}\bar{a} - \bar{a}\cdot\operatorname{curl}\bar{b}$$

(or)

$$\nabla\cdot(\bar{a}\times\bar{b}) = \bar{b}(\nabla\times\bar{a}) - \bar{a}\cdot(\nabla\times\bar{b})$$

sol:- PT $\nabla\cdot(\bar{a}\times\bar{b}) = \bar{b}(\nabla\times\bar{a}) - \bar{a}\cdot(\nabla\times\bar{b})$

$$LHS = \nabla\cdot(\bar{a}\times\bar{b})$$

$$= \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\cdot(\bar{a}\times\bar{b})$$

$$= \Sigma\, i\frac{\partial}{\partial x}\cdot(\bar{a}\times\bar{b})$$

$$= \Sigma\, i\cdot\frac{\partial}{\partial x}(\bar{a}\times\bar{b})$$

$$= \Sigma\, i\cdot\left[\frac{\partial\bar{a}}{\partial x}\times\bar{b} + \bar{a}\times\frac{\partial\bar{b}}{\partial x}\right]$$

$$= \Sigma\, i\cdot\left(\frac{\partial\bar{a}}{\partial x}\times\bar{b}\right) + \Sigma\, i\cdot\left(\bar{a}\times\frac{\partial\bar{b}}{\partial x}\right)$$

$$= \Sigma\left(i\times\frac{\partial\bar{a}}{\partial x}\right)\cdot\bar{b} - \Sigma\left(i\times\frac{\partial\bar{b}}{\partial x}\right)\cdot\bar{a}$$

$$= (\nabla\times\bar{a})\cdot\bar{b} - (\nabla\times\bar{b})\cdot\bar{a}$$

$$= RHS$$

(4) prove that

$$\operatorname{curl}(\bar{a}\times\bar{b}) = \bar{a}\operatorname{div}\bar{b} - b\operatorname{div}\bar{a} + (\bar{b}\cdot\nabla)\bar{a} - (\bar{a}\cdot\nabla)\bar{b}$$

sol:- prove that

$$\operatorname{curl}(\bar{a}\times\bar{b}) = \bar{a}\operatorname{div}\bar{b} - b\operatorname{div}\bar{a} + (\bar{b}\cdot\nabla)\bar{a} - (\bar{a}\cdot\nabla)\bar{b}$$

(or)

$$\nabla\times(\bar{a}\times\bar{b}) = \bar{a}(\nabla\cdot\bar{b}) - b(\nabla\cdot\bar{a}) + (\bar{b}\cdot\nabla)\bar{a} - (\bar{a}\cdot\nabla)\bar{b}$$

$$\text{LHS} = \nabla\times(\bar{a}\times\bar{b})$$

$$= \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\times(\bar{a}\times\bar{b})$$

$$= \Sigma\, i\frac{\partial}{\partial x}\times(\bar{a}\times\bar{b})$$

$$= \Sigma\, i\times\frac{\partial}{\partial x}(\bar{a}\times\bar{b})$$

$$= \Sigma\, i\times\left[\frac{\partial\bar{a}}{\partial x}\times\bar{b} + \bar{a}\times\frac{\partial\bar{b}}{\partial x}\right]$$

$$= \Sigma\, i\times\left[\frac{\partial\bar{a}}{\partial x}\times\bar{b} + \bar{a}\times\frac{\partial\bar{b}}{\partial x}\right]$$

$$= \Sigma\, i\times\left[\frac{\partial\bar{a}}{\partial x}\times\bar{b}\right] + \Sigma\, i\times\left[\bar{a}\times\frac{\partial\bar{b}}{\partial x}\right]$$

$$[\Rightarrow \quad \bar{a}\times(\bar{b}\times\bar{c}) = (\bar{a}\cdot\bar{c})\bar{b} - (\bar{a}\cdot\bar{b})\bar{c}\,]$$

$$= \left[\Sigma\,(i\cdot\bar{b})\frac{\partial\bar{a}}{\partial x} - \left(i\cdot\frac{\partial\bar{a}}{\partial x}\right)\bar{b}\right] + \left[\Sigma\left(i\cdot\frac{\partial\bar{b}}{\partial x}\right)\bar{a} - (i\cdot\bar{a})\frac{\partial\bar{b}}{\partial x}\right]$$

(36)

$$= \left[\Sigma (\bar{i}\cdot\bar{b}) \frac{\partial \bar{a}}{\partial x} - (\bar{i}\cdot\frac{\partial \bar{a}}{\partial x})\bar{b} \right] + \left[\Sigma (\bar{i}\cdot\frac{\partial \bar{b}}{\partial x})\bar{a} - (\bar{i}\cdot\bar{a})\frac{\partial \bar{b}}{\partial x} \right]$$

$$= \Sigma (\bar{b}\cdot\bar{i})\frac{\partial \bar{a}}{\partial x} - \Sigma (\bar{i}\cdot\frac{\partial \bar{a}}{\partial x})\bar{b} + \Sigma (\bar{i}\cdot\frac{\partial \bar{b}}{\partial x})\bar{a} - (\bar{a}\cdot\Sigma \bar{i}\frac{\partial}{\partial x})\bar{b}$$

$$= (\bar{b}\cdot\nabla)\bar{a} - (\nabla\cdot\bar{a})\bar{b} + (\nabla\cdot\bar{b})\bar{a} - (\bar{a}\cdot\nabla)\bar{b}$$

$$= (\nabla\cdot\bar{b})\bar{a} - (\nabla\cdot\bar{a})\bar{b} + (\bar{b}\cdot\nabla)\bar{a} - (\bar{a}\cdot\nabla)\bar{b}$$

$$= \bar{a}\,\mathrm{div}\,\bar{b} - \bar{b}\,\mathrm{div}\,\bar{a} + (\bar{b}\cdot\nabla)a - (\bar{a}\cdot\nabla)\bar{b}$$

$$= \text{RHS.}$$

(5) prove that

$$\mathrm{grad}\,(\bar{a}\cdot\bar{b}) = (\bar{b}\cdot\nabla)\bar{a} + (\bar{a}\cdot\nabla)\bar{b} + \bar{b}\times \mathrm{curl}\,\bar{a} + \bar{a}\times \mathrm{curl}\,\bar{b}$$

(or)

$$\nabla(\bar{a}\cdot\bar{b}) = (\bar{b}\cdot\nabla)\bar{a} + (\bar{a}\cdot\nabla)\bar{b} + \bar{b}\times(\nabla\times\bar{a}) + \bar{a}\times(\nabla\times\bar{b})$$

sol:- consider $\bar{a}\times(\nabla\times\bar{b})$

$$\bar{a}\times(\nabla\times\bar{b}) = \bar{a}\times\Sigma \bar{i}\frac{\partial}{\partial x}\times\bar{b}$$

$$= \bar{a}\times\Sigma \bar{i}\times\frac{\partial \bar{b}}{\partial x}$$

$$= \bar{a} \times \sum i \times \frac{\partial \bar{b}}{\partial x}$$

$$= \sum \bar{a} \times \left(i \times \frac{\partial \bar{b}}{\partial x} \right)$$

$$\left(\bar{a} \times (\bar{b} \times \bar{c}) = (\bar{a} \cdot \bar{c}) \bar{b} - (\bar{a} \cdot \bar{b}) \bar{c} \right)$$

$$= \sum \left(\bar{a} \cdot \frac{\partial \bar{b}}{\partial x} \right) \bar{i} - (\bar{a} \cdot \bar{i}) \frac{\partial \bar{b}}{\partial x}$$

$$= \sum \left(\bar{a} \cdot \frac{\partial \bar{b}}{\partial x} \right) i - \sum (\bar{a} \cdot \bar{i}) \frac{\partial \bar{b}}{\partial x}$$

$$= \sum i \left(\bar{a} \cdot \frac{\partial \bar{b}}{\partial x} \right) - \bar{a} \cdot \left(\sum i \frac{\partial}{\partial x} \right) \bar{b}$$

$$\bar{a} \times \operatorname{curl} \bar{b} = \sum i \left(\bar{a} \cdot \frac{\partial \bar{b}}{\partial x} \right) - (\bar{a} \cdot \nabla) \bar{b} \quad \text{--- (1)}$$

$$\text{lly} \quad \bar{b} \times \operatorname{curl} \bar{a} = \sum i \left(\bar{b} \cdot \frac{\partial \bar{a}}{\partial x} \right) - (\bar{b} \cdot \nabla) \bar{a} \quad \text{--- (2)}$$

(1) + (2) gives

$$\Rightarrow \bar{a} \times \operatorname{curl} \bar{b} + \bar{b} \times \operatorname{curl} \bar{a}$$

$$= \sum i \left(\bar{a} \cdot \frac{\partial \bar{b}}{\partial x} \right) - (\bar{a} \cdot \nabla) \bar{b}$$

$$+ \sum i \left(\bar{b} \cdot \frac{\partial \bar{a}}{\partial x} \right) - (\bar{b} \cdot \nabla) \bar{a}$$

$$\Rightarrow \bar{a} \times \operatorname{curl} \bar{b} + \bar{b} \times \operatorname{curl} \bar{a} + (\bar{a} \cdot \nabla) \bar{b} + (\bar{b} \cdot \nabla) \bar{a}$$

$$= \sum i \left(\bar{a} \cdot \frac{\partial \bar{b}}{\partial x} \right) + \sum i \left(\bar{b} \cdot \frac{\partial \bar{a}}{\partial x} \right)$$

$$= \sum i \left[\bar{a} \cdot \frac{\partial \bar{b}}{\partial x} + \bar{b} \cdot \frac{\partial \bar{a}}{\partial x} \right]$$

$$= \sum i \left(\frac{\partial}{\partial x} (\bar{a} \cdot \bar{b}) \right)$$

$$= \nabla (\bar{a} \cdot \bar{b}) = \text{LHS}.$$

(6) prove that curl grad $\phi = 0$

sol: PT curl grad $\phi = 0$

(or)

$$\nabla \times (\nabla \phi) = 0$$

$$\nabla \phi = i\frac{\partial \phi}{\partial x} + j\frac{\partial \phi}{\partial y} + k\frac{\partial \phi}{\partial z}$$

$$\text{LHS} = \nabla \times (\nabla \phi)$$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \end{vmatrix}$$

$$= i\left[\frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{\partial \phi}{\partial y}\right)\right] - j\left[\frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{\partial \phi}{\partial x}\right)\right] + k\left[\frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial x}\right)\right]$$

$$= i\left[\cancel{\frac{\partial^2 \phi}{\partial y \partial z}} - \cancel{\frac{\partial^2 \phi}{\partial z \partial y}}\right] - j\left[\cancel{\frac{\partial^2 \phi}{\partial x \partial z}} - \cancel{\frac{\partial^2 \phi}{\partial z \partial x}}\right] + k\left[\cancel{\frac{\partial^2 \phi}{\partial x \partial y}} - \cancel{\frac{\partial^2 \phi}{\partial y \partial x}}\right]$$

$$= 0.$$

$$= \text{RHS.}$$

(7) prove that div curl $\bar{f} = 0$

sol: P.T div curl $\bar{f} = 0$

(or)

P.T $\nabla \cdot (\nabla \times f) = 0$

$$\nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$$

$$\bar{f} = f_1\bar{i} + f_2\bar{j} + f_3 k$$

$$\rightarrow \nabla \times \bar{f} = \text{curl } \bar{f} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$\nabla \times \bar{f} = i\left[\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right] - \bar{j}\left[\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z}\right] + k\left[\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right]$$

Now P.T. $\nabla \cdot (\nabla \times \bar{f}) = 0$

LHS $= \nabla \cdot (\nabla \times f)$

$$= \left[i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right] \cdot \left[i\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) - j\left(\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z}\right) + k\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)\right]$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) - \frac{\partial}{\partial y}\left(\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z}\right) + \frac{\partial}{\partial z}\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)$$

$$= \cancel{\frac{\partial^2 f_3}{\partial x \partial y}} - \cancel{\frac{\partial^2 f_2}{\partial x \partial z}} - \cancel{\frac{\partial f_3}{\partial y \partial x}} + \cancel{\frac{\partial f_1}{\partial y \partial z}}$$

$$+ \cancel{\frac{\partial^2 f_2}{\partial z \partial x}} - \cancel{\frac{\partial^2 f_1}{\partial z \partial y}}$$

$$= 0$$

$$\therefore \quad \nabla \cdot (\nabla \times \bar{f}) = 0$$

$\Rightarrow \nabla \times \bar{f}$ is soleniodal vector

(8) prove that $(\nabla f \times \nabla g)$ is solenoidal

(or)

if ϕ and ψ are scalar functions, then prove that $\nabla \phi \times \nabla \psi$ is solenoidal

sol:- We know that

$$\nabla \cdot (\bar{a} \times \bar{b}) = \bar{b}(\nabla \times \bar{a}) - \bar{a}(\nabla \times \bar{b})$$

let $\bar{a} = \nabla f, \quad \bar{b} = \nabla g$

than $\nabla \cdot (\nabla f \times \nabla g) = \nabla g \cdot \text{curl } \nabla f - \nabla f \cdot \text{curl } \nabla g$

$$= \nabla g \cdot \text{curl } \nabla f - \nabla f \cdot \text{curl } \nabla g$$

$$= 0$$

$$(\because \text{curl}(\nabla f) = \bar{0}, \quad \text{curl } \nabla g = \bar{0})$$

$\therefore \nabla f \times \nabla g$ is solenoidal vector.

Assignment problems

(1) If $\bar{a}$ is constant vector then PT

$$\text{grad}\,(\bar{a}\cdot\bar{r}) = \bar{a}$$

(or) $\nabla(\bar{a}\cdot\bar{r}) = \bar{a}$

(2) If $f(r)$ is differentiable, show that

$$\text{curl}\,[\bar{r}\,f(r)] = \bar{0} \qquad \bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$$

(3) If $\bar{a} = (x+y+1)\bar{i} + \bar{j} - (x+y)\bar{k}$

ST. $\bar{a}\cdot\text{curl}\,\bar{a} = 0$

(4) PT $\nabla\left[\nabla\cdot\dfrac{\bar{r}}{r}\right] = -\dfrac{2}{r^3}\bar{r}$

(5) Find $(A\times\nabla)\phi$ if

$$A = yz^2 i - 3xz^2 \bar{j} + 2xyz\bar{k}$$

and $\phi = xyz$

(Ans $= -x^2yz^2\bar{i} + xy^2z^2\bar{j} + 4xyz^3\bar{k}$)

(6) Evaluate $\nabla\cdot\left[r\,\nabla\left(\dfrac{1}{r^3}\right)\right]$ where $r = \sqrt{x^2+y^2+z^2}$

(Ans $= 3/r^4$)

(7) Find $(A\cdot\nabla)\phi$ of $(1,-1,1)$

$A = 3xyz^2 i + 2xy^3 \bar{j} - x^2yz k \qquad \phi = 3x^2 - yz$

(Ans $= -15$)

(8) If $\phi_1 = x^2y$, $\phi_2 = xz + y^2$

Find $\nabla\times(\nabla\phi_1\times\nabla\phi_2)$

(A $= 8xy\,\bar{i} + (2xz - 4y^2)\bar{j} - 4xy\bar{k}$)