The thermal conductivity of insulating materials is very low. This is because many building and insulating materials have a porous structure with some fluid, mostly air, trapped in. Since air is a bad conductor of heat so the thermal conductivity of the air filled porous materials is low.

The thermal conductivity of *liquids* and *gases* is smaller than that of solids because their intermolecular spacing is much larger and so there is less effective transport of energy. The thermal conductivity of a gas increases with increasing temperature and decreasing molecular weight whereas it generally decreases with increasing temperature for non-metallic liquids.

There are some materials which have a very high thermal conductivity and very low temperatures. These are called *super conductors* for example, thermal conductivity of aluminium at 10 K is of the order of 20000 W/mK which is more than 100 times its value at 20°C.

Appendix A gives the thermal conductivity and other physical properties of some of the most commonly used substances. A comprehensive list of the thermal conductivities of materials can be obtained in the handbooks and databooks on Heat and Mass Transfer (Rosenhow and Hartnet: (1973); Kothandaraman and Subramanyan: (1977); Kumar: (1976); IFI: (1969).

Example 1.1

A stainless steel plate 2 cm thick is maintained at a temperature of 550°C at one face and 50°C on the other. The thermal conductivity of stainless steel at 300°C is 19.1 W/mK. Compute the heat transferred through the material per unit area.

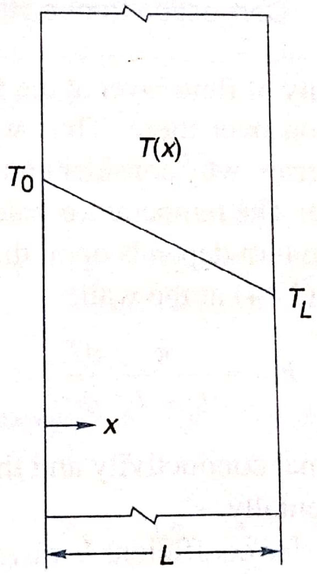

Fig. Ex. 1.1

Solution

This is the case of a plane wall as shown in Fig. Ex. 1.1. Using Eqn. (1.12).

$$Q_x = \frac{kA}{L}(T_0 - T_L)$$

or

$$\frac{Q_x}{A} = q_x = \frac{k}{L}(T_0 - T_L) = \frac{(19.1)(550 - 50)}{2 \times 10^{-2}} = 477.5 \text{ kW/m}^2$$

2.4 HEAT CONDUCTION EQUATION IN CYLINDRICAL COORDINATES

The heat conduction equation derived in the previous section can be used for solids with rectangular boundaries like slabs, cubes, etc. But then there are bodies like cylinders, tubes, cones, spheres to which Cartesian coordinates system is not applicable. A more suitable system will be one in which the coordinate surfaces coincide with the boundary surfaces of the region. For cylindrical bodies, a cylindrical coordinate system should be used. The heat conduction equation in cylindrical coordinates can be obtained by doing an energy balance over a differential element (Fig. 2.2), a procedure similar to that described previously. The equation could also be obtained by doing a coordinate transformation from Fig. 2.2:

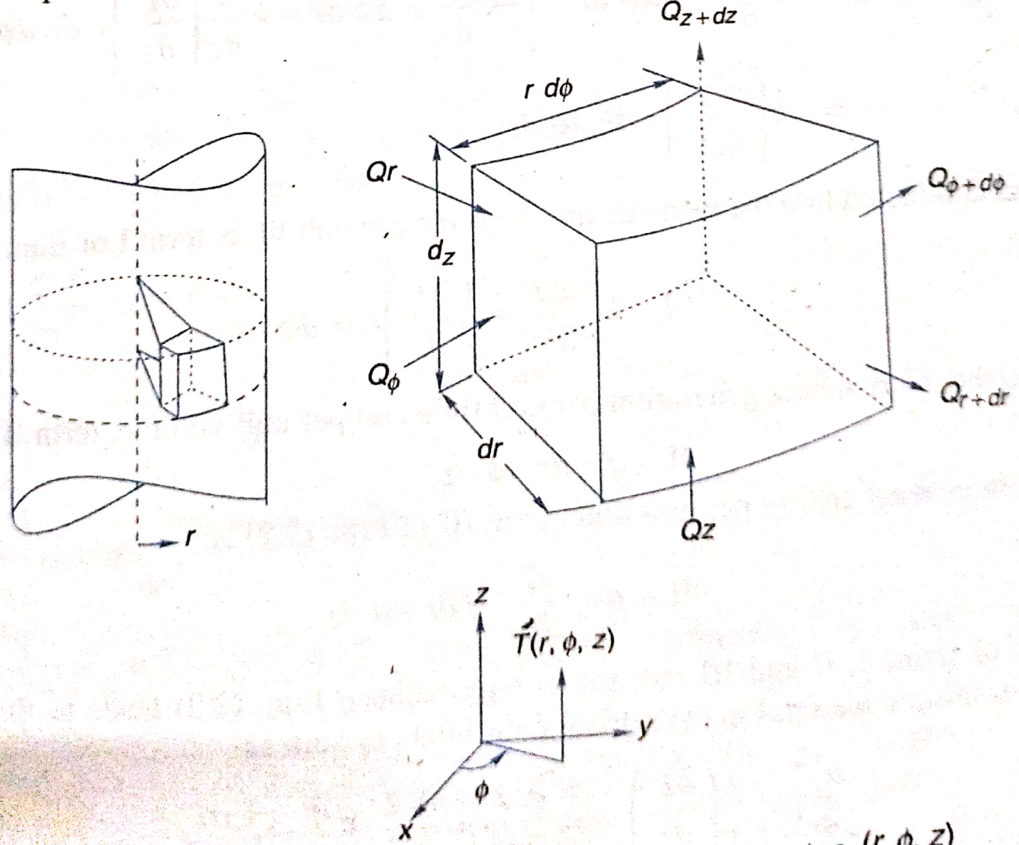

Fig. 2.2 Heat Conduction in Cylindrical Coordinates (r, ϕ, z)

Consider a small volume element having sides dr, dz and $r\,d\phi$ as shown in Fig. 2.2. Assuming the material to be isotropic, the rate of heat flow into the element in r-direction is:

$$Q_r = -k\frac{\partial T}{\partial r}\,r\,d\phi\,dz$$

The rate of heat flow out of the element in r-direction at $r + dr$ is:

$$Q_{r+dr} = Q_r + \frac{\partial Q_r}{\partial r}\,dr$$

Then, the net rate of heat entering the element in r-direction is given by

$$Q_r - Q_{r+dr} = k\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)dr\,d\phi\,dz$$

$$= k\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r}\right)dr\,d\phi\,dz$$

Similarly,

$$Q_\phi - Q_{\phi+d\phi} = -k\frac{\partial T}{r\,\partial\phi}\,dr\,dz - \left[-k\frac{\partial T}{r\,\partial\phi}\,dr\,dz - \frac{k\partial}{r\,d\phi}\left(\frac{\partial T}{r\,\partial\phi}\right)\cdot r\,d\phi\,dr\,dz\right]$$

$$= k\left(\frac{1}{r^2}\frac{\partial^2 T}{\partial\phi^2}\right)r\,dr\,d\phi\,dz$$

$$Q_z - Q_{z+dz} = -k\frac{\partial T}{\partial z}\cdot r\,d\phi\,dr - \left[-k\frac{\partial T}{\partial z}\,r\,d\phi\,dr - k\frac{\partial}{\partial z}\left(\frac{\partial T}{\partial z}\right)r\,dr\,d\phi\,dz\right]$$

$$= -k\left(\frac{\partial^2 T}{\partial z^2}\right)r\,dr\,d\phi\,dz$$

The net heat conducted into the element $dr \cdot r\,d\phi\,dz$ per unit time, term I of Eqn. (2.2) is:

$$\mathrm{I} = k\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r}\right)r\,dr\,d\phi\,dz$$

Taking $\dot{q}$ as the internal heat generation per unit time and per unit volume, term II of Eqn. (2.2) is

$$\mathrm{II} = \dot{q}\,r\,dr\,d\phi\,dz$$

The change in internal energy per unit time, term III of Eqn. (2.2) is:

$$\mathrm{III} = \rho c_p\frac{\partial T}{\partial t}\cdot r\,dr\,d\phi\,dz$$

Substitution of terms I, II and III into the energy balance Eqn. (2.2) leads to three-dimensional equation for an isentropic material in cylindrical coordinate system as

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\left(\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 T}{\partial\phi^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{q}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{2.17}$$

Alternatively, the above equation could also be obtained by doing a coordinate transformation as described below.

$$x = r\cos\phi$$
$$y = r\sin\phi$$
$$z = z$$
$$\phi = \tan^{-1}(y/x)$$

Using the chain rule, we may carry out partial differentiation of T w.r.t. r and ϕ as:

$$\frac{\partial T}{\partial r} = \frac{\partial T}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial T}{\partial y}\frac{\partial y}{\partial r} = \cos\phi\frac{\partial T}{\partial x} + \sin\phi\frac{\partial T}{\partial y} \tag{2.18}$$

$$\frac{\partial T}{\partial \phi} = \frac{\partial T}{\partial x}\frac{\partial x}{\partial \phi} + \frac{\partial T}{\partial y}\frac{\partial y}{\partial \phi} = -r\sin\phi\frac{\partial T}{\partial x} + r\cos\phi\frac{\partial T}{\partial y} \tag{2.19}$$

or
$$\cos\phi\frac{\partial T}{\partial r} = \cos^2\phi\frac{\partial T}{\partial x} + \sin\phi\cos\phi\frac{\partial T}{\partial y}$$

$$\frac{\sin\phi}{r}\frac{\partial T}{\partial \phi} = -\sin^2\phi\frac{\partial T}{\partial x} + \sin\phi\cos\phi\frac{\partial T}{\partial y}$$

$$\frac{\partial T}{\partial x} = \cos\phi\frac{\partial T}{\partial r} - \frac{\sin\phi}{r}\frac{\partial T}{\partial \phi}$$

Similarly
$$\frac{\partial T}{\partial y} = \sin\phi\frac{\partial T}{\partial r} + \frac{\cos\phi}{r}\frac{\partial T}{\partial \phi}$$

It we put $\partial T/\partial x$ for T in this equation, we get

$$\frac{\partial}{\partial x}\left(\frac{\partial T}{\partial x}\right) = \frac{\partial^2 T}{\partial x^2} = \cos\phi\frac{\partial}{\partial r}\left(\frac{\partial T}{\partial x}\right) - \frac{\sin\phi}{r}\frac{\partial}{\partial \phi}\left(\frac{\partial T}{\partial x}\right)$$

$$= \cos\phi\frac{\partial}{\partial r}\left[\cos\phi\frac{\partial T}{\partial r} - \frac{\sin\phi}{r}\frac{\partial T}{\partial \phi}\right] - \frac{\sin\phi}{r}\frac{\partial}{\partial \phi}\left[\cos\phi\frac{\partial T}{\partial r} - \frac{\sin\phi}{r}\frac{\partial T}{\partial \phi}\right]$$

or
$$\frac{\partial^2 T}{\partial x^2} = \left[\cos^2\phi\frac{\partial^2 T}{\partial r^2} + \frac{\cos\phi\sin\phi}{r^2}\frac{\partial T}{\partial \phi} + \frac{\sin^2\phi}{r}\frac{\partial T}{\partial r} + \frac{\sin^2\phi}{r^2}\frac{\partial^2 T}{\partial \phi^2} + \frac{\sin\phi\cos\phi}{r^2}\frac{\partial T}{\partial \phi}\right]$$

similarly
$$\frac{\partial^2 T}{\partial y^2} = \left[\sin^2\phi\frac{\partial^2 T}{\partial r^2} + \frac{\cos^2\phi}{r}\frac{\partial T}{\partial r} - \frac{\cos\phi\sin\phi}{r^2}\frac{\partial T}{\partial \phi} + \frac{\cos\phi}{r^2}\frac{\partial^2 T}{\partial \phi^2} + \left(-\frac{\cos\phi\sin\phi}{r^2}\frac{\partial T}{\partial \phi}\right)\right]$$

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = \frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \phi^2} \tag{2.20}$$

or
$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 T}{\partial \phi^2}$$

Substituting Eqn. (2.20) into Eqn. (2.11) leads us to the following general heat conduction equation in the cylindrical coordinate system for a constant conductivity material.

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\left(\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 T}{\partial \phi^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{q}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{2.21}$$

where $T = T(r, \phi, z, t)$

For the special case when temperature varies in r-direction only, Eqn. (2.21) becomes, for $T = T(r, t)$.

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{\dot{q}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}. \tag{2.22}$$

2.5 HEAT CONDUCTION EQUATION IN SPHERICAL COORDINATE SYSTEM

The heat conduction equation in spherical coordinates can also be derived by following the procedures outlined in the previous sections. One way of obtaining it is by doing an energy balance across a differential element in spherical coordinates as illustrated in Fig. 2.3. The other method is by making a coordinate transformation for the Laplacian operator $\nabla^2 T$ noting that from Fig. 2.3.

$$x = r \sin\psi \cos\phi$$
$$y = r \sin\psi \sin\phi$$
$$z = r \cos\psi \tag{2.23}$$

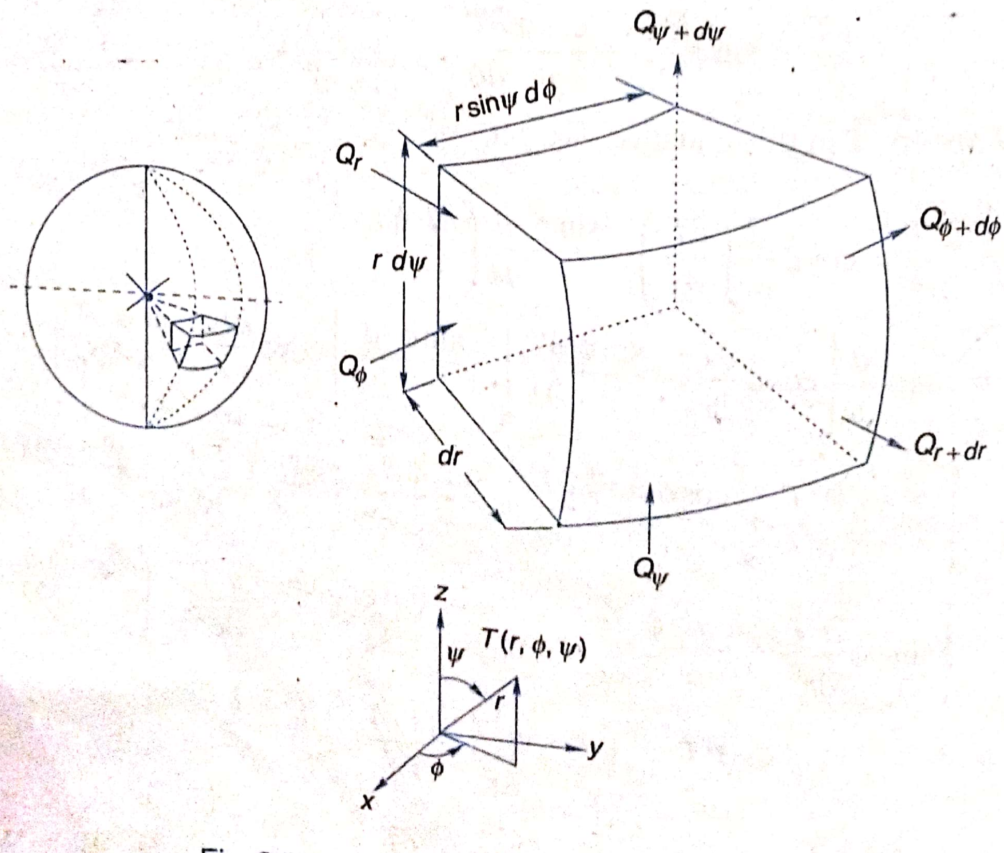

Fig. 2.3 Spherical Coordinates (r, ϕ, ψ)

The net heat conducted into the element $dr \cdot r\,d\psi \cdot r\sin\psi\,d\phi$ per unit time, term I of Eqn. (2.2) is

$$I = k\left[\frac{\partial^2 T}{\partial r^2} + \frac{2}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2 \sin\psi}\frac{\partial}{\partial \psi}\left(\sin\psi \frac{\partial T}{\partial \psi}\right) + \frac{1}{r^2 \sin^2\psi}\frac{\partial^2 T}{\partial \phi^2}\right] \cdot (dr \cdot r\,d\psi \cdot r\sin\psi\,d\phi)$$

Taking $\dot{q}$ as the internal heat generation per unit time and per unit volume, term II of Eqn. (2.2) is:

$$\text{II} = \dot{q}\,dr \cdot r\,d\psi \cdot r\sin\psi\,d\phi$$

The change in internal energy per unit time, term III of Eqn. (2.2) is:

$$\text{III} = \rho c_p \frac{\partial T}{\partial t} \cdot dr \cdot r\,d\psi \cdot r\sin\psi\,d\phi$$

Substitution of terms, I, II and III into the energy balance equation, Eqn. (2.2) leads to the general equation for an isotropic material in spherical coordinate system as:

$$\frac{\partial^2 T}{\partial r^2} + \frac{2}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2 \sin\psi}\frac{\partial}{\partial \psi}\left(\sin\psi \frac{\partial T}{\partial \psi}\right) + \frac{1}{r^2 \sin^2\psi}\frac{\partial^2 T}{\partial \phi} + \frac{\dot{q}}{k}$$

$$= \frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{2.24}$$

For steady state one dimensional heat conduction in radial direction without heat generation, the above equation takes the form:

$$\frac{\partial^2 T}{\partial r^2} + \frac{2}{r}\frac{\partial T}{\partial r} = 0$$

or

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial T}{\partial r}\right) = 0. \tag{2.25}$$

2.6 INITIAL AND BOUNDARY CONDITIONS

The set of equations derived so far describe a whole class of conduction phenomena in the most general form. The temperature distribution in a medium of given form and size can be determined by solving an appropriate differential equation from the set of Eqns. (2.10) to (2.25) subject to given constraints, called the *initial* and *boundary* conditions of the problem. The initial conditions describe the temperature distribution in a medium at the initial moment of time, and these are needed only for the time dependent (transient) problems. In general, these can be expressed as

$$t = 0, T = T(x, y, z) \tag{2.26}$$

A simple but typical form of Eqn. (2.26) for a uniform initial temperature distribution is

$$t = 0, T = T_0 = \text{constant}. \tag{2.27}$$

The boundary conditions specify the temperature or the heat flow at the surface of the body. Boundary conditions may be prescribed in a number of ways.

(i) Boundary condition of the first kind (prescribed surface temperature)

The temperature distribution, T_s, is given at a boundary surface for each moment of time.

$$T_s = T(x, y, z, t) \tag{2.28}$$

A typical example of boundary condition of the first kind for a slab is shown in Fig. 2.4.

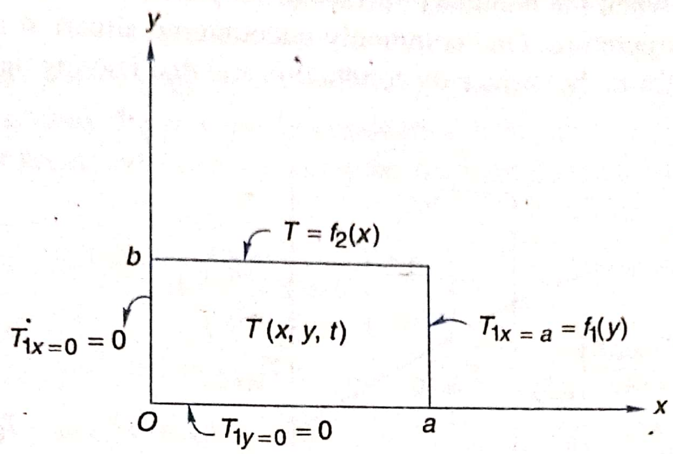

Fig. 2.4 Boundary Condition of the First Kind

$$\begin{aligned}
T(x, y, t) &= 0 && \text{at } x = 0 \\
T(x, y, t) &= 0 && \text{at } y = 0 \\
T(x, y, t) &= f_1(y) && \text{at } x = a \\
T(x, y, t) &= f_2(x) && \text{at } y = b
\end{aligned} \tag{2.29}$$

(ii) Boundary condition of the second kind (prescribed heat flux)

In this case, the heat flux at a boundary is prescribed, this can be expressed as (see Fig. 2.5).

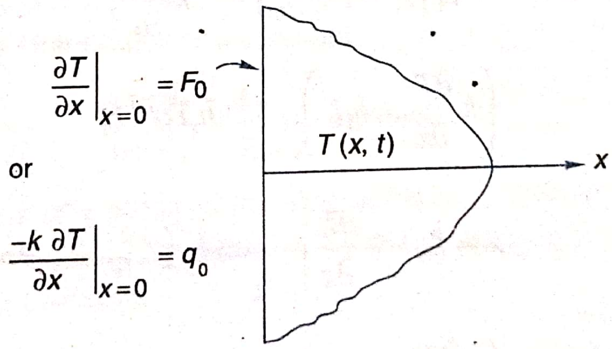

Fig. 2.5 Boundary Condition of the Second Kind

$$-k\frac{\partial T(x,t)}{\partial x} = q_0 \qquad \text{at } x = 0$$

or

$$+\frac{\partial T}{\partial x} = -\frac{q_0}{k} = F_0 \quad \text{at } x = 0 \tag{2.30}$$

$$Q = \frac{kA(T_1 - T_2)}{L} = \frac{(0.70)\,(5 \times 4)\,(110 - 40)}{0.25}$$

$$= 3920 \text{ W or } 3.92 \text{ kW}$$

At $x = 0.20$, T, by Eqn. (3.2), is

$$T = \frac{(T_2 - T_1)}{L} \cdot x + T_1$$

$$= \frac{(40 - 110)}{0.25} \cdot (0.20) + 110 = -56 + 110 = 54°\text{C}.$$

3.2.2 Radial Heat Conduction through Cylindrical Systems

Consider a long cylinder of inside radius r_i, outside radius r_o, and length L (Fig. 3.2). We consider the cylinder to be long so that the end losses are negligible. The inside and outside surfaces are kept at constant temperatures T_i and T_o respectively. A steam pipe in a room can be taken as an example of a long hollow cylinder.

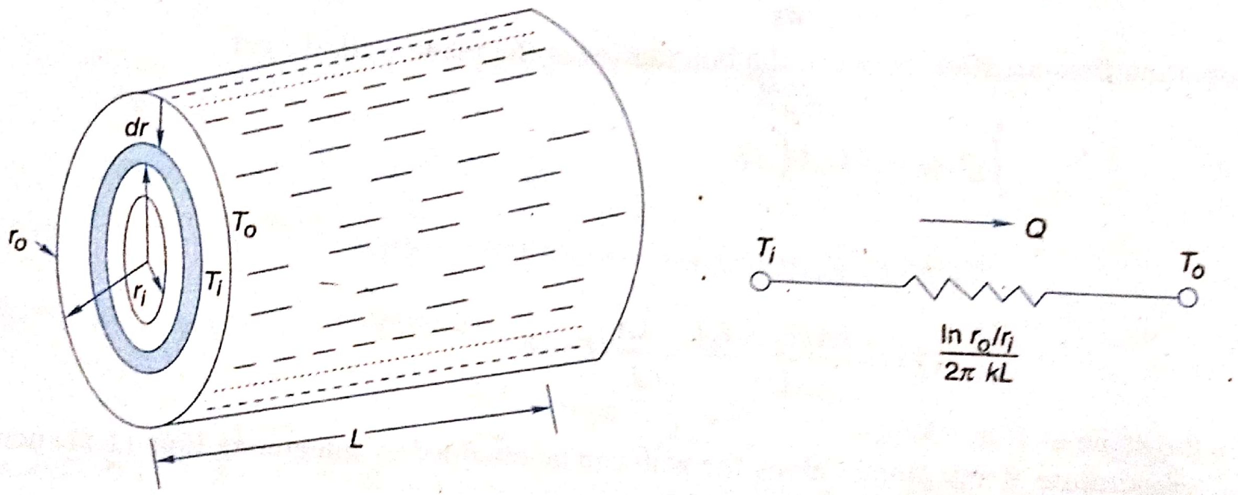

Fig. 3.2 Steady State Conduction through a Hollow Cylinder

The general heat conduction equation in cylindrical coordinates is

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\left(\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 T}{\partial \phi^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{q}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{2.21}$$

Assuming that heat flows only in a radial direction, the above equation under steady state (without heat generation) takes the form:

Subject to the boundary conditions,

$$T = T_i \text{ at } r = r_i$$
$$T = T_o \text{ at } r = r_o$$

Integrating Eqn. (3.4) twice we get

$$T = C_1 \ln r + C_2 \tag{3.5}$$

Using the boundary conditions

at $r = r_i,\ T = T_i;\ T_i = C_1 \ln r_i + C_2$

at $r = r_o,\ T = T_o;\ T_o = C_1 \ln r_o + C_2$

$$C_1 = \frac{T_i - T_o}{\ln \frac{r_i}{r_o}} = \frac{T_o - T_i}{\ln \frac{r_o}{r_i}}$$

$$C_2 = T_i - \frac{T_o - T_i}{\ln \frac{r_i}{r_o}} \ln r_i = \frac{T_i \ln r_o - T_o \ln r_i}{\ln \frac{r_o}{r_i}}$$

Substituting the values of C_1 and C_2 in Eqn. (3.5),

$$T = \frac{(T_o - T_i)}{\ln \frac{r_o}{r_i}} \ln r + \frac{T_i \ln r_o - T_o \ln r_i}{\ln \frac{r_o}{r_i}}$$

$$Q = -kA_r \left.\frac{dT}{dr}\right|_{r = r_i} = -k \cdot 2\pi r_i L \cdot \frac{C_1}{r_i}$$

$$= -k \cdot 2\pi r_i L \cdot (T_o - T_i) \cdot \frac{1}{r_i \ln \frac{r_o}{r_i}} = \frac{2\pi k L (T_i - T_o)}{\ln \frac{r_o}{r_i}} \tag{3.6}$$

Equation (3.6) can alternatively be derived as follows:

$$Q = -kA \frac{dT}{dr}, \quad \text{where } A = 2\pi r L$$

or

$$Q \frac{dr}{r} = -2\pi k L\, dT$$

Integration of this equation gives

$$Q \int_{r_i}^{r_o} \frac{dr}{r} = -2\pi k L \int_{T_i}^{T_o} dT$$

$$Q \ln\left(\frac{r_o}{r_i}\right) = -2\pi k L (T_o - T_i)$$

or $$Q = \frac{2\pi kL(T_i - T_o)}{\ln \frac{r_o}{r_i}}$$ (3

The thermal resistance for the hollow cylinder would be

$$R_{th} = \frac{\ln\left(\frac{r_o}{r_i}\right)}{2\pi kL}.$$ (3

Example 3.2

A hollow cylinder 5 cm I.D. and 10 cm O.D. has an inner surface temperature of 200°C and an outer surface temperature of 100°C. Determine the temperature of the point half way between the inner and the outer surfaces. It the thermal conductivity of the cylinder material of 70 W/mK determine the heat flow through the cylinder per linear metre.

Solution

Equation (3.7) gives

$$Q = \frac{2\pi kL(T_i - T_o)}{\ln \frac{r_o}{r_i}} = \frac{(6.28)\,(70)\,(1)\,(200 - 100)}{\ln \frac{5}{2.5}}$$

$$= 63420.9 \text{ W/m} = 63.42 \text{ kW/m}.$$

At half way between r_i and r_o, radius $r' = \frac{(5 + 2.5)}{2} = 3.75$ cm. Since Q remains the same,

$$Q = \frac{2\pi kL(T_i - T_o)}{\ln \frac{r_o}{r_i}} = \frac{2\pi kL(T_i - T')}{\ln \frac{r'}{r_i}}$$

$$\therefore \quad \frac{T_i - T_o}{\ln \frac{r_o}{r_i}} = \frac{T_i - T'}{\ln \frac{r'}{r_i}}$$

or $$T_i - T' = (T_i - T_o)\frac{\ln \frac{r'}{r_i}}{\ln \frac{r_o}{r_i}} = (T_i - T_o)\ln \frac{r'}{r_o} = \frac{(100)\ln\left(\frac{3.75}{2.50}\right)}{\ln\left(\frac{5.00}{2.5}\right)} = 58.5$$

$$\therefore \quad T' = T_i - 58.5 = 141.5°\text{C}$$

Logmean Area: Let us now consider a cylinder and a slab, both made of the same material. Let T_i and T_o be the temperatures maintained on the two sides of the plane slab and also on the inside and outside of the cylinder respectively. Let r_i and r_o be the inside and outside radii of the cylinder, and let the thickness of the slab be equal to $(r_o - r_i)$.

Heat flow through cylinder, by Eqn. (3.7) $= \dfrac{2\pi kL(T_i - T_o)}{\ln \dfrac{r_o}{r_i}}$

Heat flow through slab, by Eqn. (3.3) $= \dfrac{kA_m(T_i - T_o)}{(r_o - r_i)}$

where A_m is taken in such a way that the heat flows through the cylinder and the slab are equal for the same temperature difference across them.

Then
$$\frac{2\pi kL(T_i - T_o)}{\ln \dfrac{r_o}{r_i}} = \frac{kA_m(T_i - T_o)}{(r_o - r_i)}$$

$$A_m = \frac{2\pi L(r_o - r_i)}{\ln \dfrac{r_o}{r_i}} = \frac{2\pi Lr_o - 2\pi Lr_i}{\ln \dfrac{2\pi Lr_o}{2\pi Lr_i}}$$

or
$$A_m = \frac{A_o - A_i}{\ln \dfrac{A_o}{A_i}} \tag{3.9}$$

where A_i and A_o are the inside and outside surface areas of the cylinder.

A_m is called the *logmean area* of the cylinder, and can be used to transform a cylinder into an equivalent slab. It so turns out that if $A_o/A_i < 2$, then A_m can be taken as an average area $= (A_o + A_i)/2$

$$R_{th} = \frac{(r_o - r_i)}{kA_m} = \frac{\text{Lcyl}}{kA_m}, \text{ where Lcyl} = r_o - r_i \tag{3.10}$$

The thermal resistance of a hollow cylinder is of exactly the same form as that for a slab except that the *logarithmic mean area* is used for the cylinder.

3.2.3 Radial Heat Conduction through Spherical Systems

Consider a hollow sphere (Fig. 3.3) whose inside and outside surfaces and held at constant temperatures T_i and T_o respectively. If the temperature variation is only in the radial direction, then for steady state conditions with no heat generation, the heat conduction Eqn. (2.25) is

$$\frac{d}{dr}\left(r^2 \frac{dT}{dr}\right) = 0 \tag{3.11}$$

Integration of Eqn. (3.11) gives

$$dT = C_1 \frac{dr}{r^2}$$

Integrating again, we get

$$T = \frac{-C_1}{r} + C_2$$

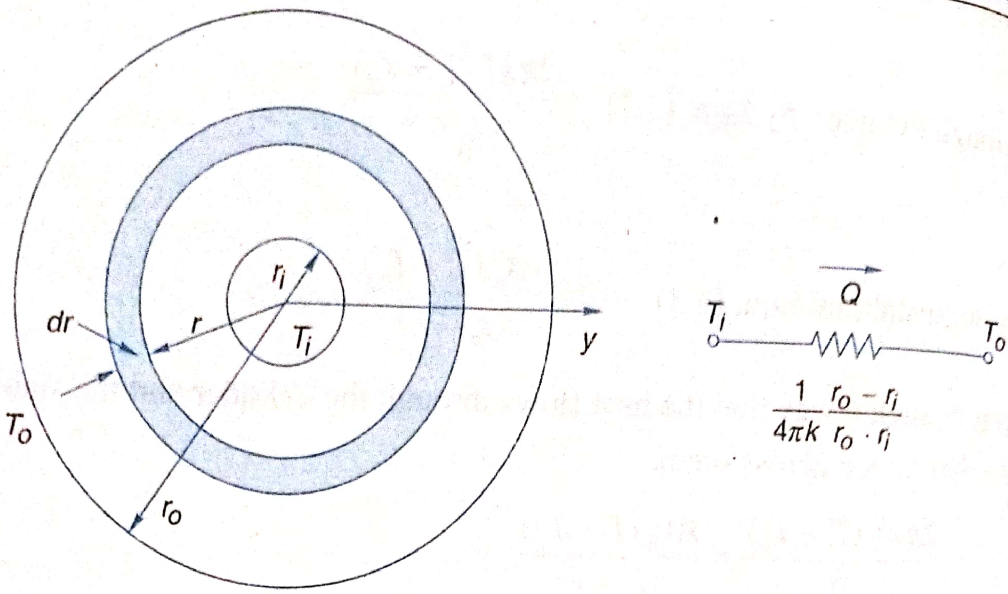

Fig. 3.3 Steady State Conduction through a Hollow Sphere

Applying the boundary conditions

$$T = T_i \quad \text{at} \quad r = r_i; \; T_i = \frac{-C_1}{r_i} + C_2$$

$$T = T_o \quad \text{at} \quad r = r_o; \; T_o = \frac{-C_1}{r_o} + C_2$$

Solving for C_1 and C_2

$$C_1 = \frac{T_i - T_o}{\left[\frac{1}{r_o} - \frac{1}{r_i}\right]}$$

$$C_2 = T_i + \frac{1}{r_i}\frac{(T_i - T_o)}{\left[\frac{1}{r_o} - \frac{1}{r_i}\right]}$$

$$\therefore \quad T = T_i - \frac{(T_i - T_o)}{\left[\frac{1}{r_o} - \frac{1}{r_i}\right]}\left[\frac{1}{r} - \frac{1}{r_i}\right] = \frac{r_o}{r}\left(\frac{r - r_i}{r_o - r_i}\right)(T_o - T_i) + T_i \quad (3$$

Knowing that $\quad Q = -kA_r \left.\frac{dT}{dr}\right|_{r = r_i}$, where $A_r = 4\pi r^2$.

We can show that

$$Q = \frac{4\pi r_o r_i \cdot k(T_i - T_o)}{(r_o - r_i)} \quad (3.1$$

Equation (3.13) can also be obtained by integration of Fourier's equation as follows:

$$Q = -kA\frac{dT}{dr} = -4\pi k r^2 \frac{dT}{dr}$$

$$Q\int_{r_i}^{r_o}\frac{dr}{r^2} = -4\pi k\int_{T_i}^{T_o} dT$$

$$-Q\left[\frac{1}{r_o} - \frac{1}{r_i}\right] = -4\pi k (T_o - T_i)$$

or

$$-Q\frac{(r_i - r_o)}{r_o r_i} = -4\pi k (T_o - T_i)$$

and

$$Q = \frac{4\pi r_o r_i \cdot k(T_i - T_o)}{(r_o - r_i)} \tag{3.13}$$

Eqn. (3.13) can also be put as

$$Q = \frac{T_i - T_o}{(1/4\pi k)\,[(r_o - r_i)/r_o r_i]} = \frac{T_i - T_o}{R_{sph}}$$

where the thermal resistance of a sphere is defined as

$$R_{sph} = \frac{1}{4\pi k}\frac{r_o - r_i}{r_o r_i} \tag{3.14}$$

Geometric Mean Area: R_{sph} can be rearranged as

$$R_{sph} = \frac{1}{4\pi k}\frac{r_o - r_i}{r_o r_i} = \frac{r_o - r_i}{k\sqrt{(4\pi r_o^2)(4\pi r_i^2)}} = \frac{L_{sph}}{k\sqrt{A_o \cdot A_i}} = \frac{L_{sph}}{A_g \cdot k} \tag{3.15}$$

where A_i and A_o are the areas of the inner and outer surfaces of the sphere.

$$L_{sph} = r_o - r_i = \text{thickness of sphere}$$

$$A_g = \sqrt{A_o A_i} \tag{3.16}$$

is called the *geometric mean area* of the sphere. The thermal resistance of a sphere given by Eqn. (3.15) is similar to that of a plane wall except that the area is replaced by the geometric mean area.

Example 3.3

A hollow sphere 10 cm I.D. and 30 cm O.D. of a material having thermal conductivity 50 W/mK is used as a container for a liquid chemical mixture. Its inner and outer surface temperatures are 300°C and 100°C respectively. Determine the heat flow rate through the sphere. Also estimate the temperature at a point a quarter of the way between the inner and outer surfaces.

Solution

Referring to Eqn. (3.13)

$$Q = \frac{4\pi r_o r_i\, k(T_i - T_o)}{(r_o - r_i)} = \frac{(12.56)\,(0.15)\,(0.05)\,(50)\,(300 - 100)}{(0.15 - 0.05)} = 9.42 \text{ kW}$$

The value of r at one-fourth way of the inner and outer surfaces is $5 + \frac{1}{4}(15 - 5) = 7.5$ cm.

Referring to Eqn. (3.12), temperature at $r = 7.5$ cm is.

$$T = \frac{r_o}{r}\left(\frac{r - r_i}{r_o - r_i}\right)(T_o - T_i) + T_i$$

$$= \frac{(0.15)}{0.075}\left(\frac{0.075 - 0.05}{0.15 - 0.05}\right)(100 - 300) + 300 = 200°\text{C}.$$

3.3 SYSTEMS WITH VARIABLE THERMAL CONDUCTIVITY

In all the cases considered in Sec. 3.2, the thermal conductivity k, has been assumed as constant. This assumption is probably satisfactory for materials involving small temperature differences across them. In practice, the thermal conductivity of most material is temperature dependent, and it would be necessary to include in the analysis the variation of thermal conductivity with temperature. Under normal circumstances, for limited ranges of temperature, it is sufficiently accurate to use the linear expression for k, *i.e.*,

$$k = k_0 (1 + \beta T) \tag{3.17}$$

where k_0 is the value of the thermal conductivity at $T = 0$, and β is the temperature coefficient of thermal conductivity. The effect of variable conductivity on heat flow for one-dimensional systems can be analysed in a straight forward manner described as follows:

3.3.1 Plane Wall (Slab) with Variable k

Consider a plane wall (Fig. 3.1) in the region $0 \le x \le L$ having boundary surfaces at $x = 0$ and $x = L$ kept at uniform temperatures T_1 and T_2. The problem can be formulated as

$$\frac{d}{dx}\left[\left(k(T)\frac{dT}{dx}\right)\right] = 0 \text{ in } 0 \le x \le L$$

$$T = T_1 \text{ at } x = 0$$

$$T = T_2 \text{ at } x = L$$

where $k(T) = k_0 (1 + \beta T)$

$$\frac{d}{dx}\left[k_0 (1 + \beta T)\frac{dT}{dx}\right] = 0 \tag{3.18}$$

The integration of Eqn. (3.18) w.r.t. x gives

$$k_0\left[(1 + \beta T)\frac{dT}{dx}\right] = C_1$$

e being the specific energy (J/kg)

By energy balance,

$$\frac{\partial}{\partial t}\oint_v \rho c T \, dV = \oint_v q_G \, dV - \oint_v \text{div}\, \bar{q} \, dV \tag{2.12}$$

This is the energy equation for the entire CV. Writing the energy equation for the elemental volume dV within the CV

$$\rho c \frac{\partial T}{\partial t} dV = q_G \, dV - \text{div}\, \bar{q} \, dV$$

Since dV is now independent, it can be removed from the above equation. Therefore,

$$\rho c \frac{\partial T}{\partial t} = q_G - \text{div}\, \bar{q} \tag{2.13}$$

Now,

$$\text{div}\, \bar{q} = \nabla \cdot \bar{q}$$

and from Eq. (2.4)

$$\bar{q} = -k\nabla T$$

$\therefore$

$$\text{div}\, \bar{q} = \nabla \cdot (-k\,\nabla T) = -k\,\nabla^2 T, \text{ for constant } k$$

Substituting in Eq. (2.13)

$$\rho c \frac{\partial T}{\partial t} = q_G + k\nabla^2 T$$

Therefore, $\nabla^2 T + \dfrac{q_G}{k} = \dfrac{1}{\alpha}\dfrac{\partial T}{\partial t}$

which is the same as Eq. (2.10).

2.1.2 Cylindrical Coordinates

For a general transient three-dimensional heat conduction problem in the cylindrical coordinates with $T = T(r, \theta, z, t)$, let us consider an elementary volume $dV = dr r\, d\theta\, dz$ (Fig. 2.6).

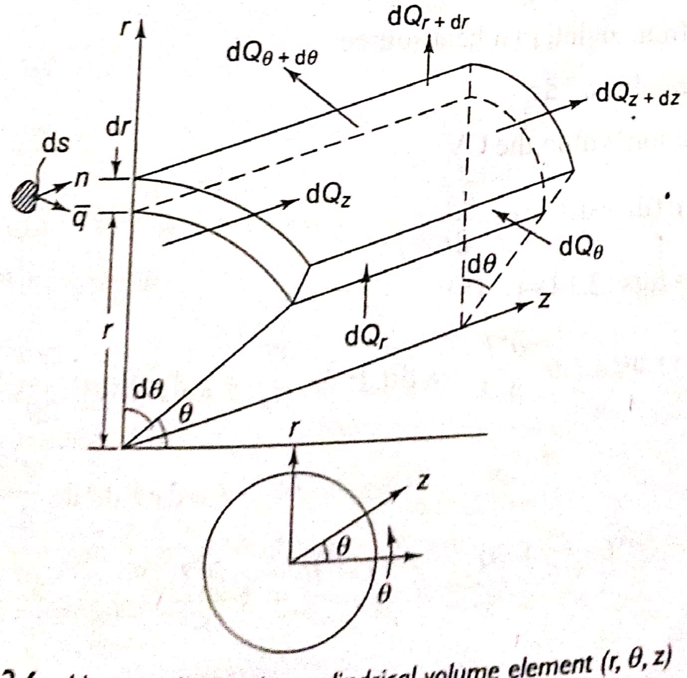

Fig. 2.6 *Heat conduction in a cylindrical volume element (r, θ, z)*

By Fourier's law,

$$dQ_r = -k(r\,d\theta\,dz)\frac{\partial T}{\partial r}$$

$$dQ_{r+dr} = dQ_r + \frac{\partial}{\partial r}(dQ_r)\,dr$$

$$dQ_r - dQ_{r+dr} = -\frac{\partial}{\partial r}\left(-kr\,d\theta\,dz\frac{\partial T}{\partial r}\right)dr$$

$$= kr\,d\theta\,dz\,dr\frac{\partial^2 T}{\partial r^2} + k\,d\theta\,dz\,dr\frac{\partial T}{\partial r} \tag{2.14}$$

Similarly,

$$dQ_\theta = -k\,dz\,dr\frac{\partial T}{r\partial\theta}$$

$$dQ_{\theta+d\theta} = dQ_\theta + \frac{\partial}{r\partial\theta}(dQ_\theta)\,r\,d\theta$$

$$dQ_\theta - dQ_{\theta+d\theta} = -\frac{\partial}{\partial\theta}\left(-k\,dz\,dr\frac{\partial T}{r\partial\theta}\right)d\theta$$

$$= k\,dz\,dr\,d\theta\frac{1}{r}\frac{\partial^2 T}{\partial\theta^2} \tag{2.15}$$

$$dQ_z = -k\,dr\,r\,d\theta\frac{\partial T}{\partial z}$$

$$dQ_{z+dz} = dQ_z + \frac{\partial}{\partial z}(dQ_z)\,dz$$

$$dQ_z - dQ_{z+dz} = -\frac{\partial}{\partial z}\left(-k\,dr\,r\,d\theta\frac{\partial T}{\partial z}\right)dz$$

$$= k\,dr\,r\,d\theta\,dz\frac{\partial^2 T}{\partial z^2} \tag{2.16}$$

Rate of heat generation from an internal heat source

$$= q_G\,dr\,r\,d\theta\,dz \tag{2.17}$$

Rate of energy accumulation within the CV

$$= \rho(dr\,r\,d\theta\,dz)\,c\frac{\partial T}{\partial t} \tag{2.18}$$

By energy balance, from Eqs (2.14)–(2.18),

$$\rho\,dr\,r\,d\theta\,dz\,c\frac{\partial T}{\partial t} = kr\,d\theta\,dz\,dr\frac{\partial^2 T}{\partial r^2} + k\,d\theta\,dz\,dr\frac{\partial T}{\partial r} + k\,dz\,dr\,d\theta\frac{\partial^2 T}{\partial\theta^2} + k\,dr\,r\,d\theta\,dz\frac{\partial^2 T}{\partial z^2} + q_G\,dr\,r\,d\theta\,dz$$

$$\rho c\frac{\partial T}{\partial t} = k\frac{\partial^2 T}{\partial r^2} + k\frac{1}{r}\frac{\partial T}{\partial r} + k\frac{1}{r^2}\frac{\partial^2 T}{\partial\theta^2} + k\frac{\partial^2 T}{\partial z^2} + q_G$$

$$\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2}+\frac{\partial^2 T}{\partial z^2}+\frac{q_G}{k}=\frac{1}{\alpha}\frac{\partial T}{\partial t}$$

or

$$\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial T}{\partial r}+\frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2}+\frac{\partial^2 T}{\partial z^2}+\frac{q_G}{k}=\frac{1}{\alpha}\frac{\partial T}{\partial t}$$

or (2.19)

This is the general heat conduction equation in cylindrical coordinates. If we compare this equation with Eq. (2.10), the Laplacian is

$$\nabla^2 T=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2}+\frac{\partial^2 T}{\partial z^2} \tag{2.20}$$

If heat flows only in radial direction, $T = T(r, t)$, Eq. (2.19) reduces to

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)+\frac{q_G}{k}=\frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{2.21}$$

If the temperature distribution does not vary with time, then at steady state,

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right)+\frac{q_G}{k}=0 \tag{2.22}$$

In this case the equation for the temperature contains only a single variable r and is therefore an ordinary differential equation.

When there is no volumetric energy generation and the temperature is a function of the radius only, the steady-state conduction for cylindrical coordinates is

$$\frac{d}{dr}\left(r\frac{dT}{dr}\right)=0 \tag{2.23}$$

2.1.3 Spherical Coordinates

For spherical coordinates, as shown in Fig. 2.7(a), the temperature is a function of the three space

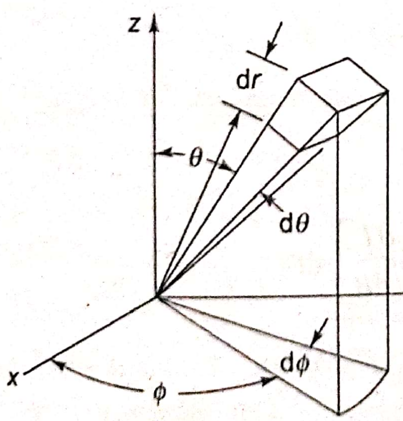

Fig. 2.7(a) *Spherical coordinate system for the general conduction equation*

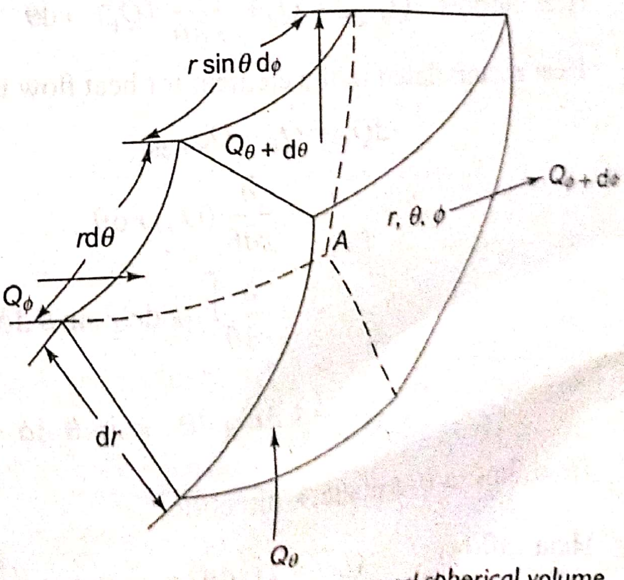

Fig. 2.7(b) *Elemental spherical volume*

coordinates r, θ and ϕ and time t, i.e., $T = T(r, \theta, \phi, t)$. The general form of the conduction equation in spherical coordinates can be found as

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial\phi^2}+\frac{q_G}{k}=\frac{1}{\alpha}\frac{\partial T}{\partial t} \quad (2.2$$

where the Laplacian includes the first three terms of the above equation in spherical coordinates.

Let us consider an elemental volume having the coordinates (r, ϕ, θ) for three dimensional he conduction analysis [Fig. 2.7(b)].

The volume of the element $= dr \cdot rd\theta \cdot r\sin\theta\, d\phi$.

(A) Net heat accumulated in the element due to conduction of heat from all the coordinate directions

Heat flow through $r-\theta$ plane, ϕ-direction:

Heat inflow, $$Q_\phi = -k\, dr \cdot rd\theta \frac{\partial T}{r\sin\theta\,\partial\phi}\, dt$$

Heat outflow, $$Q_{\phi+d\phi} = Q_\phi + \frac{\partial Q_\phi}{r\sin\theta\,\partial\phi}\, r\sin\theta\, d\phi$$

∴ Heat accumulated in the element for heat flow in ϕ direction

$$dQ_\phi = Q_\phi - Q_{\phi+d\phi} = -\frac{1}{r\sin\theta}\frac{\partial Q_\phi}{\partial\phi}\, r\sin\theta\, d\phi$$

$$= -\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left[-k dr\, rd\theta\frac{1}{r\sin\theta}\frac{\partial T}{\partial\phi}\cdot dt\right] r\sin\theta\, d\phi$$

$$= k(dr\cdot rd\theta\cdot r\sin\theta\cdot d\phi)\frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial\phi^2}\, dt \quad (2.2$$

Heat flow in r-ϕ plane, θ-direction:

Heat inflow, $$Q_\theta = -k(dr\times r\sin\theta\, d\phi)\frac{\partial T}{r\partial\theta}\cdot dt$$

Heat outflow, $$Q_{\theta+d\theta} = Q_\theta + \frac{\partial}{r\partial\theta}(Q_\theta)\cdot rd\theta$$

Heat accumulated in the element for heat flow in θ-direction:

$$dQ_\theta = Q_\theta - Q_{\theta+d\theta}$$

$$= -\frac{\partial}{r\partial\theta}(Q_\theta)\, rd\theta$$

$$= -\frac{\partial}{r\partial\theta}\left[-k\, dr.r\sin\theta\, d\phi\frac{\partial T}{r\partial\theta}.dt\right] rd\theta$$

$$= k\, dr\cdot rd\theta\cdot r\sin\theta\, d\phi\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial T}{\partial\theta}\right] dt \quad (2.26$$

Heat flow in θ-ϕ plane, r-direction:

Heat inflow, $$Q_r = -k(rd\theta\cdot r\sin\theta\, d\phi)\frac{\partial T}{\partial r}\, dt$$

Heat outflow, $$Q_{r+dr} = Q_r + \frac{\partial}{\partial r}(Q_r)\cdot dr$$

2.2 STEADY HEAT CONDUCTION IN SIMPLE GEOMETRIES

We will now derive solutions to the conduction equations as obtained in the previous section for simp geometrical systems with and without heat generation.

2.2.1 Plane Wall

(a) Without Heat Generation

In the first chapter we saw that the temperature distribution for one-dimensional steady conduction throu a wall is linear. We will verify this result by simplifying the more general equation [Eq. (2.10)]

$$\nabla^2 T + \frac{q_G}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

For steady state, $\partial T/\partial t = 0$. Since T is only a function of x, $\partial T/\partial y = 0$ and $\partial T/\partial z = 0$. There is no inter heat generation, $q_G = 0$. Therefore, the above equation reduces to

$$\frac{d^2T}{dx^2} = 0 \qquad (2.3$$

Integrating the ordinary differential equation twice yields the linear temperature distribution

$$T(x) = C_1 x + C_2 \qquad (2.32)$$

For a wall (Fig. 2.8), at $x = 0$, $T = T_1$ and at $x = b$, $T = T_2$

$$T = -\frac{T_1 - T_2}{b}x + T_1 \qquad (2.33)$$

which agrees with the linear temperature distribution deduced by integrating Fourier's law, $Q_x = -kA\dfrac{dT}{dx}$.

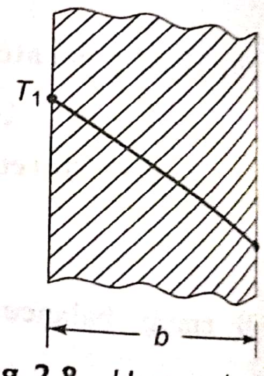

Fig. 2.8 Heat conductio through a wall

(b) With Heat Generation

Let us now consider a heat source generating heat throughout the system. If the thermal conductivity constant and the heat generation is uniform, Eq. (2.10) reduces to

$$\frac{d^2T}{dx^2} + \frac{q_G}{k} = 0$$

On integration,

$$\frac{dT}{dx} = -\frac{q_G x}{k} + C_1 \qquad (2.3$$

A second integration gives

$$T(x) = -\frac{q_G}{2k}x^2 + C_1 x + C_2 \qquad (2.35$$

where C_1 and C_2 are constants.

At $x = 0$, $T = T_1$ and at $x = b$, $T = T_2$ substituting in Eq. (2.35),

$$T_1 = C_2$$

$$T_2 = -\frac{q_G}{2k}b^2 + C_1 b + T_1$$