UNIT1-MECHANICS

Introduction

Mechanics is a branch of physics concerned with the state of rest or motion of bodies that are subjected to the action of forces.It is the area of mathematics and physics concerned with the relationships between force, matter, and motion among physical objects.Forces applied to objects result in displacements, or changes of an object's position relative to its environment. One of the central concepts in mechanics is a"particle" or "point mass".

Multiparticle dynamics:

Multi-particle dynamics (or) dynamics in a system of particles, is the study of motion in respect of a group of particles in which the space between the particles will be very small 1.e., the distance between them will be negligible.

The dynamics of a particle system consists of the application of Newton's laws of motion to a set of particles, which can be discrete (the particles can be counted) or form part of an extended object, in this case the system it is continuous.To explain the motion of a system of particles, it is inconvenient to analyze each one separately and see what forces act on it. Instead, a representative point of the set is defined, called the center of mass .

Examples

The Earth and the Moon

Fig-Illustration of Earth and Moon

A set of discrete particles m_1, m_2, m_3 … that eventually moves with respect to the origin of a coordinate system due to some resultant force acting on them is a good example of a particle system.

The Earth can be considered as one particle and the Moon another, so both constitute a system of 2 particles under the action of the Sun's force of gravity.

Extended objects

A person, an animal or any object in the environment can also be considered as a system of particles, only that these are so small that they cannot be counted one by one. This is acontinuous system, but taking into account certain considerations, its treatment is the same as for a discrete system.

Centre of Mass(CM)

Centre of mass is the point in the body or a system of particles where the mass of the whole body seems to be concentrated. It is found by taking the weighted average position of the mass. Centre of mass is the point at which the distribution of mass is equal in all directions and does not depend on gravitational field. Centre of Mass is an important property of any rigid body system. Usually, these systems contain more than one particle. It becomes essential to analyse these systems as a whole. To perform calculations of mechanics, these bodies must be considered as a single point mass. The Centre of mass denotes such a point. Often the mechanical systems move in a transitory or a rotatory manner. In that case, the Centre of mass also moves and acquires some velocity and acceleration. Let's see how to calculate these metrics for such systems in detail. A lot of problems can be simplified if it is assumed that the mass of the object is located at one particular point. If the correct position is chosen, then equations of forces and motion behave the same way as they behave if applied when mass is spread out. This special location is termed the centre of Mass.Its position is defined relative to an object or the system of objects whose Centre of mass is to be calculated. Usually for uniform shapes, it's their centroid. For the shapes that are symmetrical and uniform, their centre of mass is located at their centroid. For a ring, its centre of mass lies inside the ring, which means it is not necessary that the Center of mass of a body lies in the body itself.

Definition

When we consider the motion of a system consisting of a large number of particles, say, N, there is one point in it which behaves as though the entire mass of the system (i.e., the sum of the masses of all the N individual particles) were concentrated there and its motion is the same as would ensue if the external forces acting on the system were applied directly to it. This point is called the centre of mass of the system. As we shall see as we proceed along, this concept of centre of mass has proved most useful in tackling many is problem and, in particular, those concerned with collision of particles.

Formula for finding the position of the (CM)of the system

Now, it is clear that bodies that are uniform and symmetrical have their Center of masses at their centroid. But for bodies that are not symmetrical and uniform, the answer is not that simple. The Center of mass for such bodies can be anywhere.To work out the Center of mass of a complex object. a weighted average of the locations of each mass of the body is taken. Taking first the simple case of a system of just two particles of masses m_1 and m_2 and of a total mass M, with r_1 and $r_2$as their position vectors with respect to some origin, if $\vec{R}$

be the position vector of the centre of mass, we have

$$\vec{R} = \frac{m_1 \vec{r_1} + m_2 \vec{r_2}}{m_1 + m_2} = \frac{m_1 \vec{r_1} + m_2 \vec{r_2}}{M}$$

Or,
$$M\vec{R} = m_1 \vec{r_1} + m_2 \vec{r_2},$$

i.e., the product of the total mass of the system and the position vector of the centre of mass (c.m.) is equal to the sum of the products of the individual masses and their respective position vectors.The position of the c.m. of the system may thus be easily obtained.

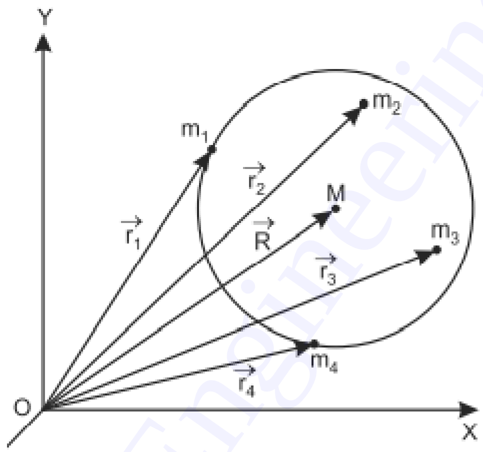
The center of mass(CM)of a system of particles

To begin the study of a particle system, we must find the center of mass (CM), which is the point where all the mass of the system is concentrated.

For the discrete system,if we have a system consisting of N particles, of masses m_1, m_2, …m_n, withr_1,r_2,r_3……..r_n as their position vectors at a given instant, the position vector R of the centre of mass of the system (Figure 1)) at that instant is given by the relation

$$M\vec{R} = m_1\vec{r}_1 + m_2\vec{r}_2 + \ldots\ldots m_n\vec{r}_n,$$

where M is the total mass of the system



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(Figure 1.A system of particles in the xyz reference frame.)

$$\text{Clearly, } M = m_1 + m_2 + \ldots m_n = \sum_{k=1}^{N} m_k,$$

wherem_k is the mass of the k^{th} particle. If, therefore, $k_r$be the position vector of the kt^h particle, we have,

$$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + \ldots m_N\vec{r}_N}{m_3 + m_2 + \ldots m_N}$$

$$= \frac{\sum_{k=1}^{N} m_k\vec{r}_k}{\sum_{k=1}^{N} nl_k} = \frac{\sum_{k=1}^{N} m_k\vec{r}_k}{M},$$

where $\sum m_k \vec{r_k}$ is called the *first moment of* *mass for the system.*

Now, $\vec{r_k} = r_k\hat{i} + y_k\hat{j} + z_k\hat{k}$ and $\vec{R} = x\hat{i} + y\hat{j} + z\hat{k}$. So that, if X, Y and Z be the Cartesian coordinates of the centre of mass, we have

$$X = \frac{m_1x_1 + m_2x_2 + \ldots m_Nx_N}{m_1 + m_2 + \ldots m_N} = \frac{\sum_{k=1}^{N} m_k x_k}{\sum_{k=1}^{N} m_k} = \frac{\sum_{k=1}^{N} m_k x_k}{M}.$$

Or, $$X = \frac{1}{M}\sum_{k=1}^{N} m_k x_k,$$

$$Y = \frac{m_1y_1 + m_2y_2 + \ldots m_Ny_N}{m_1 + m_2 + \ldots m_N} = \frac{\sum_{k=1}^{N} m_k y_k}{\sum_{k=1}^{N} m_k} = \frac{\sum_{k=1}^{N} m_k y_k}{M}.$$

Or, $$Y = \frac{1}{M}\sum_{k=1}^{N} m_k y_k,$$

and $$Z = \frac{m_1z_1 + m_2z_2 + \ldots + m_Nz_N}{m_1 + m_2 + \ldots m_N} = \frac{\sum_{k=1}^{N} m_k z_k}{\sum_{k=1}^{N} m_k} = \frac{\sum_{k=1}^{N} m_k z_k}{M}.$$

Or, $$Z = \frac{1}{M}\sum_{k=1}^{N} m_k z_k,$$

CM movement

Once the location of the centre of mass is known, the known equations of motion apply. The velocity of the CM is the first derivative of the position with respect to time:

$$\mathbf{v}_{CM} = \frac{d\mathbf{r}_{CM}}{dt}$$

In this case, the system has a total momentum **P** which is calculated as the product of the total mass of the system and the velocity of the center of mass:

$P = M \cdot v_{CM}$

Alternatively, the total momentum of the system can be calculated directly:

$P = m_1 v_1 + m_2 v_2 + m_3 v_3 + ... = \sum m_i v_i$

While the acceleration of the CM is the derivative of the velocity:

$$\mathbf{a}_{CM} = \frac{d\mathbf{v}_{CM}}{dt}$$

Force on CM

The forces acting on a system of particles can be:

- Internal forces, due to interactions between the same particles.
- External forces, caused by agents external to the system.

As the internal forces are presented in pairs, of equal magnitude and direction, but opposite directions, according to Newton's third law, it is true that:

$\sum F_{int} = 0$

Therefore, the internal forces do not alter the movement of the whole, but they are very important to determine the internal energy.

If the system is isolated and there are no external forces, according to Newton's first law ,thecenter of mass is at rest or moves with uniform rectilinear motion. Otherwise, the center of mass experiences an acceleration given by:

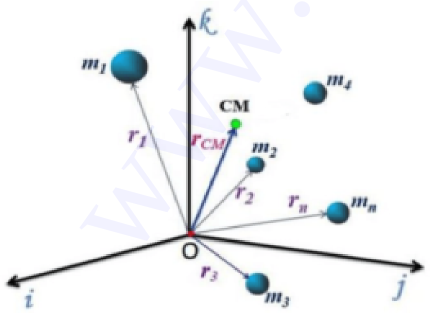
$\sum_{F\,ext} = M \cdot a_{CM}$

Where M is the total mass of the system. The above equation can be written like this:

$$\sum \mathbf{F}_{ext} = M\frac{d\mathbf{v}_{\mathbf{CM}}}{dt} = \frac{d(M\mathbf{v}_{\mathbf{CM}})}{dt} = \frac{d\mathbf{P}}{dt}$$

And it means that the external force is equivalent to the temporal variation in the momentum, another way of expressing Newton's second law .

Motion of velocity of Center of mass for a multiparticle system



Consider a system of multiparticles. Each particle of that system is moving at a different velocity. How would someone assign a velocity to the system as a whole? Let us consider a system of particles m_1, m_2, m_3 …and so on. The initial position vectors of these particles are r_1, r_2, r_3 …r_n. Now, these particles start moving in the directions of their position vectors. The goal is to find the velocity and the direction of the velocity of the Center of mass of the system.

From the definition of the Center of mass,

$Mr_{cm} = m_1r_1 + m_2r_2 + \ldots + m_nr_n$

Since the particles are in motion, they are changing their position vectors. Differentiating the equation from both sides.

$$M\frac{d\vec{R}}{dt} = m_1\frac{d\vec{r_1}}{dt} + m_2\frac{d\vec{r_2}}{dt} + \ldots + m_i\frac{d\vec{r_i}}{dt} + \ldots + m_n\frac{d\vec{r_n}}{dt}$$

But $\frac{d\vec{R}}{dt} = \vec{V}$ the velocity of centre of mass

and $\frac{d\vec{r_1}}{dt} = \vec{v_1}, \frac{d\vec{r_2}}{dt} = \vec{v_2} \ldots \frac{d\vec{r_i}}{dt} = \vec{v_i}$ and $\frac{d\vec{r_n}}{dt} = \vec{v_n}$

which represent the velocities of individual particles.

$$M\vec{V} = m_1\vec{v_1} + m_2\vec{v_2} + \ldots + m_i\vec{v_i} + \ldots + m_n\vec{v_n} = \sum m_i\vec{v_i}$$

The velocity of centre of mass is given by

$$\vec{V} = \frac{m_1\vec{v_1} + m_2\vec{v_2} + \ldots + m_i\vec{v_i} + \ldots + m_n\vec{v_n}}{M} = \frac{\sum m_i\vec{v_i}}{M}$$

Similarly, if the particles are under accelerationthe equation given above can be differentiated again to find the acceleration of the Center of mass of the body.

The acceleration of the centre of mass is given by

$$\vec{a} = \frac{m_1\vec{a_1} + m_2\vec{a_2} + \ldots + m_i\vec{a_i} + \ldots + m_n\vec{a_n}}{M} = \frac{\sum m_i\vec{a_i}}{M}$$

Center of mass for a continuous bodies:

What is the center of mass of a uniform sphere? That isn't too bad. By symmetry it's at the sphere's center. But what if the sphere is not of uniform density? The top hemisphere is made out of balsa wood and the bottom hemisphere is made out of roquefort cheese? That's a bit more tricky, and to solve this sort of problem, we'll have to formulate it in terms of integration. Let's start off with the one dimensional case. Suppose that you have a rod that has a mass per unit$\lambda(x)$ that can vary with position x. We choose coordinates so that the left hand side of the rod is at 0 and the right hand side is at L. To obtain the center of mass, we break up the rod into millions of tiny section each of lengthΔx. Then the center of mass is the sum of all the individual masses m_i times their corresponding position x_i. So

$$x_c = \frac{m_1x_1 + m_2x_2 + \ldots + m_Nx_N}{m_1 + m_2 + \ldots + m_N}$$

Now one of these little masses

$$m_i = \Delta x \lambda(x_i)$$

So,

$$x_c = \frac{\Delta x \sum_{i=1}^{N} i \lambda(x_i) x_i}{\Delta x \sum_{i=1}^{N} \lambda(x_i)}$$

,

You might remember from calculus expressions of this kind. If we take the limit as Δx tends to 0 then these become integrals

$$x_c = \frac{\int^L \lambda(x) x dx}{\int^L \lambda(x) dx}$$

Since λ(x)dx = dm , an infinitessimal of mass, this integral can be more elegantly, but less usefully written as

$$x_c = \frac{\int x dm}{\int dm}$$

Kinetic energy of the system of particles

Let there are n number of particles in a n particle system and these particles possess some motion. The motion of the i'th particle of this system would depend on the external force Fi acting on it. Let at any time if the velocity of i'th particle be vi then its kinetic energy would be

$$E_{Ki} = \frac{1}{2} m v_i^2$$

$$E_{Ki} = \frac{1}{2} m(\boldsymbol{v}_i . \boldsymbol{v}_i) \qquad (1)$$

Let r_i be the position vector of the i^{th} particle w.r.t. O and r'_ibe the position vector of the centre of mass w.r.t. r_i ,as shown below in the figure , then

$\mathbf{r}_i = \mathbf{r'}_i + \mathbf{R}_{cm}$ (2)

where $\mathbf{R}_{cm}$ is the position vector of centre of mass of the system w.r.t. O.

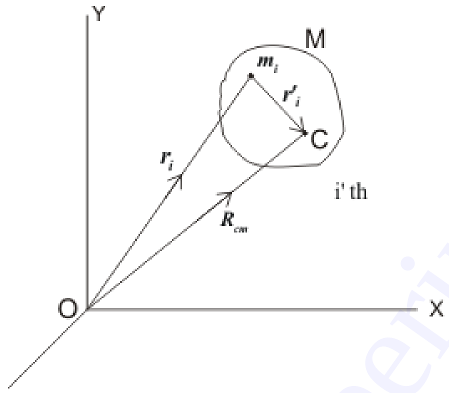
Differentiating equation 2 we get

$$\frac{d\boldsymbol{r}_i}{dt} = \frac{d\boldsymbol{r}'_i}{dt} + \frac{d\boldsymbol{R}_{cm}}{dt}$$

or,

$$\boldsymbol{v}_i = \boldsymbol{v}_i' + \boldsymbol{V}_{cm} \qquad (3)$$

where vi is the velocity of i'th particle w.r.t. centre of mass and Vcm is the velocity of centre of mass of system of particle. Putting equation 3 in 1 we get,



. r'_i Is the position vector of center of mass w.r.t. r_i

$$E_{Ki} = \frac{1}{2} m_i \left[(\boldsymbol{v}_i' + \boldsymbol{V}_{cm}).(\boldsymbol{v}_i' + \boldsymbol{V}_{cm}) \right] = \frac{1}{2} m_i \left[(\boldsymbol{v}_i^{2}{}' + 2\boldsymbol{v}'_i . \boldsymbol{V}_{cm} + \boldsymbol{V}_{cm}^2) \right]$$

$$E_{Ki} = \frac{1}{2} m_i \boldsymbol{v}_i'^2 + m_i \boldsymbol{v}'_i . \boldsymbol{V}_{cm} + \frac{1}{2} m_i \boldsymbol{V}_{cm}^2 \qquad (4)$$

Sum of Kinetic energy of all the particles can be obtained from equation 4

$$E_K = \sum_{i=1}^{n} E_{Ki} = \sum_{i=1}^{n} \left[\frac{1}{2} m_i \boldsymbol{v}_i'^2 + m_i \boldsymbol{v}'_i . \boldsymbol{V}_{cm} + \frac{1}{2} m_i \boldsymbol{V}_{cm}^2 \right]$$

$$E_K = \sum_{i=1}^{n} \frac{1}{2} m_i \boldsymbol{v}_i'^2 + \sum_{i=1}^{n} m_i \boldsymbol{v}'_i . \boldsymbol{V}_{cm} + \sum_{i=1}^{n} \frac{1}{2} m_i \boldsymbol{V}_{cm}^2$$

$$E_K = \frac{1}{2} \boldsymbol{V}_{cm}^2 \sum_{i=1}^{n} m_i + \sum_{i=1}^{n} \frac{1}{2} m_i \boldsymbol{v}_i'^2 + \boldsymbol{V}_{cm} \sum_{i=1}^{n} m_i \boldsymbol{v}'_i$$

$$E_K = \frac{1}{2} \boldsymbol{V}_{cm}^2 M + \sum_{i=1}^{n} \frac{1}{2} m_i \boldsymbol{v}_i'^2 + \boldsymbol{V}_{cm} \frac{d}{dt} \sum_{i=1}^{n} m_i \boldsymbol{r}'_i \qquad (5)$$

Now last term in above equation which is

$$\sum_{i=1}^{n} m_i \boldsymbol{r}'_i = 0$$

would vanish as it defines the null vector because

$$\sum_{i=1}^{n} m_i \boldsymbol{r'}_i = \sum_{i=1}^{n} m_i (\boldsymbol{r}_i - \boldsymbol{R}_{cm}) = M\boldsymbol{R}_{cm} - M\boldsymbol{R}_{cm} = 0$$

Therefore kinetic energy of the system of particles is,

$$E_K = \frac{1}{2} M V_{cm}^2 + \frac{1}{2} \sum_{i=1}^{n} m_i v_i'^2 = E_{Kcm} + E_K' \qquad (6)$$

where,

$$E_{Kcm} = \frac{1}{2} V_{cm}^2 M$$

is the kinetic energy obtained as if all the mass were concentrated at the centre of mass and

$$E'_K = \sum_{i=1}^{n} \frac{1}{2} m_i v_i'^2$$

is the kinetic energy of the system of particle w.r.t. the centre of mass.

- Hence it is clear from equation 6 that kinetic energy of the system of particles consists of two parts: the kinetic energy obtained as if all the mass were concentrated at the centre of mass plus the kinetic energy of motion about the centre of mass.
- If there were no external force acting on the particle system then the velocity of the centre of mass of the system will remain constant and Kinetic Energy of the system would also remain constant.

Rotation of a Rigid Bodies

Rigid Body

A body is said to be rigid, if the relative positions of any two particles do not change under the action of the forces acting on it. In Fig. 1.1 (a), point A and B are the original positions in a body. After the application of forces F1, F2, F3, the body takes the position as shown in Fig. 1.1(b). A′and B′ are the new positions of A and B. If the body is treated as rigid, the relative position of A′B′and AB are the same i.e.A′B′ = AB.Many engineering problems can be solved by assuming bodies rigid.

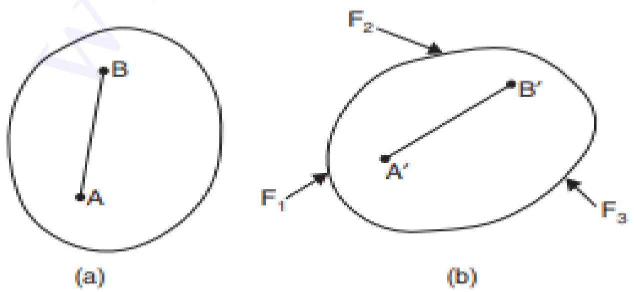


Fig. 1.1

A rigid body (also known as a rigid object) is a solid body in which deformation is zero or so small it can be neglected. The distance between any two given points on a rigid body remains constant in time regardless of external forces exerted on it. A rigid body is usually considered as a continuous distribution of mass.It can be also defined as a collection of particles with the property that the distance between particles remains unchanged during the course of motions of the body.

Or you may say the body which does not deform under the influence of forces is known as a rigid body.But, in real life, there would be some force under which the body starts to deform. For example, a bridge does not deform under the weight of a single man but it may deform under the load of a truck or ten trucks. However, the deformation is small.

Rotational Motion

When a rigid body is in pure rotational motion, all particles in the body rotate through the same angle during the same time interval. Thus, all particles have the same angular velocity and the same angular acceleration. Rotational motion exists everywhere in the universe. The motion of electrons about an atom and the motion of the moon about the earth are examples of rotational motion.

Objects cannot be treated as particles when exhibiting rotational motion since different parts of the object move with different velocities and accelerations. Therefore, it is necessary to treat the object as a system of particles.

Rigid Body Rotation

Rigid body rotation is featured prominently in science, sports, and engineering. Theoretically, it is a collection of particles that are at a fixed distance from one another. In an ideal scenario, these bodies do not change their shape or deform. These kinds of bodies are usually a continuous distribution of mass. For analyzing the motion of such bodies, the position of the center of mass is required. When forces are applied to such bodies, it produces an angular acceleration and linear acceleration. It becomes essential to study how these forces affect rigid bodies and produce an angular and linear acceleration in them. Let's see them in the detail about the rotational motion of a rigid body.Rotational motion is more complicated than linear motion, and only the motion of rigid bodies will be considered here. A rigid body is an object with a mass that holds a rigid shape.Many of the equations for the mechanics of rotating objects are similar to the motion equations for linear motion.Rigid body rotation is a motion that occurs when a solid body moves in a circular path around something. The rotational motion can be broken down into two types of rotation – Rotation about a fixed axis and rotation about a fixed point. Rotation about a fixed axis is said to be when the body is rotating about an axis that has a fixed location and orientation relative to the body. Example of such rotations includes – hinged door. The second type of rotational motion involves the rotation of the body around a point. A child's spinning top is one example of this type of rotational motion.

Angular velocity

The angular displacement of a rotating wheel is the angle between the radius at the beginning and the end of a given time interval. The SI units are radians. The average angular velocity (ω, Greek letter omega), measured in radians per second, is

$$\omega = \frac{\text{angular displacement}}{\text{elapsed time}} = \frac{\theta}{t}$$

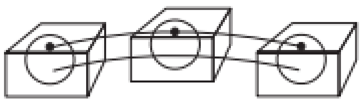
Angular acceleration

The angular acceleration (α, Greek letter alpha) has the same form as the linear quantity

$$\alpha = \frac{\text{change in angular velocity}}{\text{elapsed time}} = \frac{\omega_f - \omega_o}{t}$$

Translational motion of a rigid body

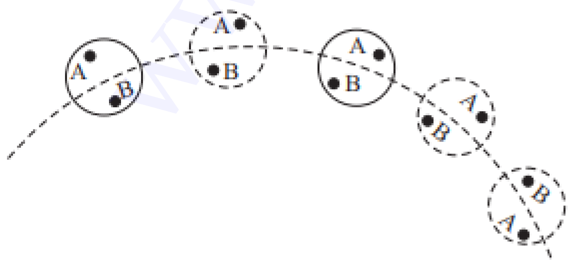
When a rigid body moves in such a way that all its particles move along parallelpaths as shown in given figure, its motion is called translational motion.



(Fig- a wooden block moving along the floor performs translational motion)

Rotational motion of a rigid body

The motion of a rigid body in which all its constituent particles describe concentric circular paths is known as rotational motion.The rotational motion can be obtained by keeping a point of the body fixed so that it cannot have any translational motion. For the sake of mathematical convenience, this point is taken to be the CM. The rotation is then about an axis passing through the CM. A good example of rotational motion is the earth’s rotation about its own axis shown in given figure.



(Fig - Rotation of the earth)

Rotational kinematics

The kinematics of rotational motion describes the relationships between the angle of rotation, angular velocity, angular acceleration, and time. It only describes motion—it does not include any forces or masses that may affect rotation

The kinematical quantities in rotationalmotion, angular displacement (θ), angular velocity (ω) and angular acceleration (α)respectively are analogous to kinematic quantities in linear motion, displacement (x),velocity (v) and acceleration (a).

We know thekinematical equations of linear motion withuniform (i.e. constant) acceleration**:**

$$v = v_0 + at \qquad \text{(a)}$$

$$x = x_0 + v_0 t + \frac{1}{2}at^2 \qquad \text{(b)}$$

$$v^2 = v_0^2 + 2ax \qquad \text{(c)}$$

where x_0 = initial displacement and v_0= initialvelocity. The word 'initial' refers to values of the quantities at t = 0 .

The corresponding kinematic equations forrotational motion with uniform angularacceleration are:

$$\omega = \omega_0 + \alpha t$$

$$\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$$

$$\text{and } \omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0)$$

where θ_0= initial angular displacement of therotating body, and ω_0 = initial angular velocity of the body.

Rotational kinetic energy and moment of inertia

Let C be the centre of mass of a rigid body. Suppose it rotates about an axis through this point (Fig-shown below). Suppose particles of masses m_1 , m_2 , m_3 ...are located at distances r_1 , r_2 , r_3 ...from the axis of rotation and are moving with speeds v_1, v_2 , $v_3$respectively.Then particle 1 has kinetic energy

$$(½)\ m_1 v_1^2 .$$

Similarly, the kinetic energy of particle of mass m_2 is

$$(½)\ m_2 v_2^2 .$$

By adding the kinetic energies of all the particles, we get the total energy of the body. If T denotes the total kinetic energy of the body, we can write

$$T = (½)\ m_1 v_1^2 + (½)\ m_2 v_2^2 + \ldots$$

$$= \sum_{i=1}^{i=1} \left(\frac{1}{2}\right) m_i v_i^2$$

where ,

$$\sum_{i=1}^{i=n}$$

indicates the sum over all the particles of the body.

The angular speed (ω) is related to linear speed (v) through the equation v = r ω.

Using this result in Eqn. (7.11), we get

$$T = \sum_{i=1}^{i=n}\left(\frac{1}{2}\right)m_i(r_i\omega)^2$$

Note that we have not put the subscript i with ω because all the particles of a rigid body have the same angular speed. Eqn. (7.12) can now be rewritten as

$$T = \frac{1}{2}\left(\sum_{i=1}^{i=n} m_i\, r_i^2\right)\omega^2$$

$$= \frac{1}{2}\, I\, \omega^2$$

The quantity

$$I = \sum m_i\, r_i^2$$

is called the moment of inertia of the body.

It is important to remember that moment of inertia is defined with reference to an axis of rotation. Therefore, whenever you mention moment of inertia, the axis of rotation must also be specified.The moment of inertia is expressed in kg m^2 . The moment of inertia of a rigid body is often written as

$I = M K^2$

where M is the total mass of the body

and K is called the radius of gyrationof the body.

The radius of gyration is that distance from the axis of rotation where the whole mass of the body can be assumed to be placed to get the same moment of inertia which the body actually has. It is important to remember that the moment of inertia of a body about an axis depends on the distribution of mass around that axis. If the distribution of mass changes, the moment of inertia will also change.

Physical significance of moment of inertia

The physical significance of moment of inertia is that it performs the same role in rotational motion that the mass does in linear motion. Just as the mass of a body resists change in its state of linear motion, the moment of inertia resists a change in its rotational motion. This property of the moment of inertia has been put to a great practical use. Most machines, which produce rotational motion have as one of their components a disc which has a very large moment of inertia.

Examples of such machines are the steam engine and the automobile engine. The disc with a large moment of inertia is called a flywheel. To understand how a flywheel works, imagine that the driver of the engine wants to suddenly increase the speed. Because of its large moment of inertia, the flywheel resists this attempt. It allows only a gradual increase in speed. Similarly, it works against the attempts to suddenly reduce the speed, and allows only a gradual decrease in the speed. Thus , theflywheel, with its large moment of inertia, prevents jerky motion and ensures a smooth ride for the passengers.

Theorems of moment of inertia

There are two theorems of moment of inertia:

(1) Perpendicular axis theorem, and

(2) Parallel axis theorem.

These are explained and proved below.

Perpendicular Axis Theorem

The moment of inertia of an area about an axis pependicular to its plane (polar moment of inertia) at any point O is equal to the sum of moments of inertia about any two mutually perpendicular axis through the same point O and lying in the plane of the area.

Referring to Fig. 4.40, if z-z is the axis normal to the plane of paper passing through point O, as per this theorem,Izz = Ixx + Iyy ... (1)

The above theorem can be easily proved. Let us consider an elementalarea dA at a distance r from O. Let the coordinates of dA be x and y.Then from definition:

$$
\begin{aligned}
I_{zz} &= \Sigma r^2 dA \\
&= \Sigma (x^2 + y^2) dA \\
&= \Sigma x^2 dA + \Sigma y^2 dA \\
I_{zz} &= I_{xx} + I_{yy}
\end{aligned}
$$

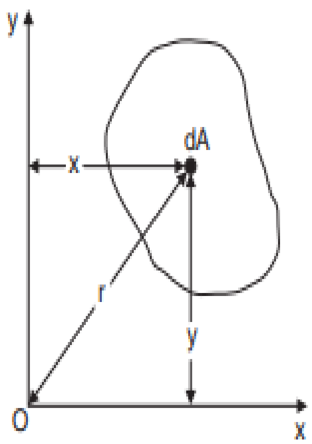


Fig. 4.40

Parallel Axis Theorem

Moment of inertia about any axis in the plane of an area is equal to the sum of moment of inertia about a parallel centroidal axis and the product of area and square of the distance between the two parallel axis. Referring to Fig. 4.41 the above theorem means:

$$I_{AB} = I_{GG} + A\, y_c^2$$

where

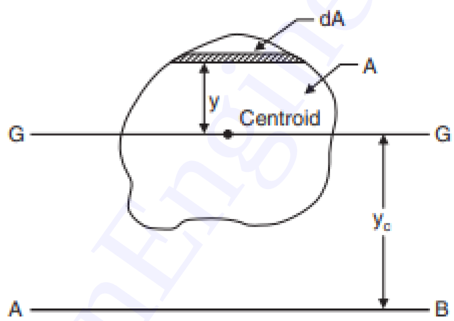
I_{AB} = moment of inertia about axis AB

I_{GG} = moment of inertia about centroidal axis GGparallel to AB.

A = the area of the plane figure given and

y_c = the distance between the axis AB and the parallel centroidalaxis GG.

Proof: Consider an elemental parallel strip dA at a distance y from the centroidal axis (Fig. below).



Then

$$I_{AB} = \Sigma(y + y_c)^2 dA$$

$$= \Sigma(y^2 + 2y\,y_c + y_c^2)dA$$

$$= \Sigma y^2 dA + \Sigma 2y\,y_c\, dA + \Sigma\, y_c^2 dA$$

$$\Sigma y^2 dA = \text{Moment of inertia about the axis } GG$$

$$= I_{GG}$$

Then, $I_{AB} = \Sigma(y + y_c)^2 dA$

$$= \Sigma(y^2 + 2y\, y_c + y_c^2)dA$$

$$= \Sigma y^2 dA + \Sigma 2y\, y_c\, dA + \Sigma\, y_c^2 dA$$

Now, $\Sigma y^2 dA$ = Moment of inertia about the axis GG

$$= I_{GG}$$

$$\Sigma 2yy_c\, dA = 2y_c\, \Sigma y\, dA$$

$$= 2y_c A\, \frac{\Sigma ydA}{A}$$

In the above term $2y_c A$ is constant and $\frac{\Sigma ydA}{A}$ is the distance of centroid from the reference axis GG. Since GG is passing through the centroid itself $\frac{ydA}{A}$ is zero and hence the term $\Sigma 2yy_c dA$ is zero.

Now, the third term,

$$\Sigma y_c^2 dA = y_c^2 \Sigma dA$$

$$= Ay_c^2$$

$$\therefore \quad I_{AB} = I_{GG} + Ay_c^2$$

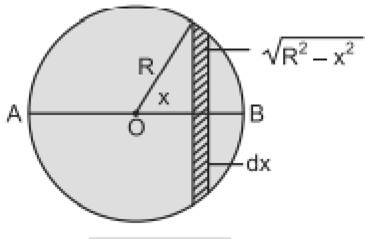
Note: The above equation cannot be applied to any two parallel axis. One of the axis (GG) must be centroidal axis only.

Moment of inertia of continuous bodies

Moment of Inertia of a solid sphere

(i)About a diameter(Fig – 1) represents a section, through the centre, of a solid sphere of radius R and mass M, whosemoment of inertia is to be determined about a diameter AB, say, its value being the same about any other diameter.

Since the volume of sphere = $4\pi R^3/3$, its mass per unit volume (or density) = $M/4/3\,\pi R^3$ = $3M/4\pi R^3$



(Fig-1)

Considering a thin circular slice of the sphere at a distance x from its centre O and of thicknessdx, we have surface area of the slice (which is obviously a disc of radius

$\sqrt{R^2-x^2} = \pi(R^2-x^2)$

and its Volume = Area × Thickness= $\pi(R^2 - x^2)$ dx and hence its mass = volume × density

= $\pi(R^2 - x^2)dx \times 3\, M/4\pi R^3$

And therefore M.I. of this slice or disc about AB (i.e., an axis passing through its centre and perpendicular to its plane) = its mass × $(\text{radius})^2/2$

$$= \frac{3M(R^2-x^2)}{4R^3}dx \cdot \frac{R^2-x^2}{2} = \frac{3M(R^2-x^2)^2}{8R^3}dx.$$

Hence M. I. of the whole sphere about its diameter (AB) is equal to twice the integral of the above expression between the limits x = 0 and x = R, i.e.,

$$I = 2\int_0^R \frac{3M(R^2-x^2)^2}{8R^3}dx = \frac{2\times 3M}{8R^3}\int_0^R (R^2-x^2)^2 dx$$

$$= \frac{3}{4}\frac{M}{R^3}\int_0^R (R^4 - 2R^2x^2 + x^4)dx$$

Or,

$$I = \frac{3M}{4R^3}\left[R^4x - 2R^2\frac{x^3}{3} + \frac{x^5}{5}\right]_0^R = \frac{3M}{4R^3}\left(R^2 - \frac{2}{3}R^5 + \frac{1}{5}R^5\right)$$

$$= \frac{3M}{4R^3}\times\frac{8R^5}{15} = \frac{2}{5}MR^2.$$

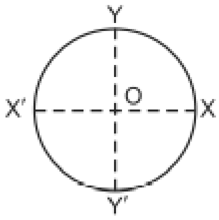
(ii) About a tangent. A tangent drawn to the sphere at any point will clearly be parallel to one or the other diameter of it (i.e., an axis passing through its centre or centre or mass) and at a distance equal to the radius of the sphere, R, from it. We therefore have, by the principle of parallel axes, moment of inertia of the sphere about a tangent, i.e. ,

$$I = \frac{2}{5}MR^2 + MR^2 = \frac{7}{5}MR^2.$$

. Moment of inertia of a hoop or a thin circular ring

(i) About an axis through its centre and perpendicular to its plane.

Let the radius of the hoop or the thin circular ring be R and its mass M, shown in below (Fig-1).



(Fig-1)

Consider a particle of mass m of the hoop or the ring. Clearly, its M.I. about an axis passing through the centre O of the hoop or the ring and perpendicularto its plane = mR^2.

Therefore,M.I. of the entire hoop or ring about this axis passing through its centre and perpendicular to its plane, $I = \Sigma mR^2 = MR^2$ [$\Sigma m = M$, the mass of the hoop or ring.]

(ii) About its diameter. Obviously, due to symmetry, the M.I. of the hoop or the ring will be the same about one diameter as about another. Thus, if I be its M.I. about the diameter XOX′ (Fig -1), it will also be I about the diameter YOY′ perpendicular to XOX′.

By the principle of perpendicular axes, therefore, the M.I. of the hoop or the ring about the axis through its centre O and perpendicular to its plane is equal to the sum of its moments of inertia about the perpendicular axes XOX′ and YOY′ in its own plane and intersecting at O,

i.e.,

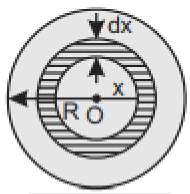
$I + I = MR^2$. Or, $2I = MR^2$, hence, $I = MR^2/2$.

Moment of inertia of a circular lamina or disc:

(i)About an axis through its centre and perpendicular to its plane.

Let M be the mass of the disc and R, its radius, so that its mass per unitarea is equal to $M/\pi R^2$.

Considering a ring of the disc, of width dx.and distant x from the axis passing through O and perpendicular to the plane of the disc (Fig. 1shown below), we have



(Fig-1)

Area of the ring = Circumference × Width = $2\pi x dx$, and hence

its mass = $(M/\pi R^2) \times 2\pi x dx$ = 2 Mxax/R^2

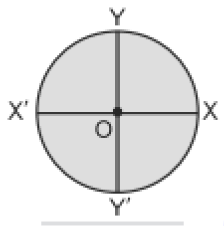
And, therefore, its M.I. about the perpendicular axis through O

$$= \frac{2Mxdx}{R^2}x^2 = \frac{2Mx^3dx}{R^2}.$$

Since the whole disc may be supposed to be made up of such like concentric rings of radii ranging from O to R, the M.I. of the whole disc about the axis through O and perpendicular to its plane, i.e., I, is obtained by integrating the above expression for the M.I. of the ring, between the limits x = 0 and x = R.. Thus

$$I = \int_0^R \frac{2M}{R^2} x^3 dx = \frac{2M}{R^2}\left[\frac{x^4}{4}\right]_0^R = \frac{2M}{R^2}\frac{R^4}{4} = MR^2/2.$$

(ii) About a diameter.Here, again, due to symmetry, the M.I. of the disc about one diameter is the same as about another. So that, if I be the M.I. of thedisc about each of the perpendicular diameters XOX′ and YOY′, (Fig. 2 shown below), we have, by the principle of perpendicular axes.



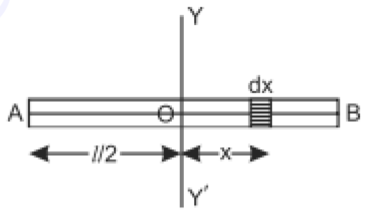
(Fig-2)

I + I = M. I. of the disc about an axis through O and perpendicular to its plane,

i.e., 2I = $MR^2/2$, whence, I = $MR^2/4$.

Moment of inertia of a uniform rod

(i) about an axis through its centre and perpendicular to its length. Let AB be a thin uniform rod, of length l and mass M, free to rotate about an axis YOY′ passing through its centre O and perpendicular to its length, as shown in(below Fig.1) Since the rod is uniform, its mass per unit length is clearly M/l.



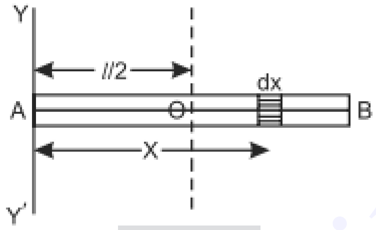
Considering a small element of the rod, of length dx at a distance x from the axis through O, we have mass of the element = (M/l).dx and, therefore, its M.I. about the axis (YOY′) through O = (M/l) dx. x^2

The moment of inertia (I) of the whole rod about the axis YOY′ is thus given by the integral of the above expression between the limits x = – l/2 and x = + l/2 or by twice its integral between the limits x = 0 and x = l/2, i.e.

$$I = 2\int_0^{l/2} \frac{M}{l} x^2 dx = \frac{2M}{l}\left[\frac{x^2}{3}\right]_0^{l/2} = \frac{2M}{l}\cdot\frac{l^3}{24} = \frac{Ml^2}{12}.$$

(ii)about an axis through one end of the rod and perpendicular to its length

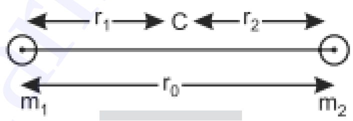
Proceeding as in case (i) above, we obtain the moment of inertia of the rod about the axis, now passing through one end A of the rod, by integrating the expression for the M.I. of the element dx of the rod, between the limits x = 0 at A and x = l at B, i.e.,



$$I = \int_0^{l} \frac{M}{l} x^2 dx = \frac{M}{l}\left[\frac{x^3}{3}\right]_0^{l} = \frac{Ml^2}{3}$$

Momentof inertia of a diatomic molecule

In a diatomic molecule, in its stable equilibrium position, the two atoms are a certain distance r_0 apart, where r_0 is called its internuclear distance or bond length. For our present purpose, however, we may imagine it to consist of two tiny spheres at either end of a thin weightless rod, as shown in Fig. below.



Let C be the centre of mass of the molecule and r_1 and r_2, the respective distances of the two atoms from it.Then, clearly $r_1 + r_2 = r_0$................(i)

and $m_1r_1 = m_2r_2$, ……...(ii)

where, m_1 and m_2 are the masses of the two atoms respectively

From relations (i) and (ii), therefore, we have

$$r_1 = \frac{m_2}{m_1 + m_2} r_0 \quad \text{and} \quad r_2 = \frac{m_1}{m_1 + m_2} r_0$$

Now, the moment of inertia of the molecule (i.e., of the two atoms) about an axis passing through the centre of mass C and perpendicular to r is clearly.

$$I = m_1 r_1^2 + m_2 r_2^2.$$

Or, substituting the values of r_1 and r_2, we have

$$I = m_1\left(\frac{m_2}{m_1+m_2}\right)^2 r_0^2 + m_2\left(\frac{m_1}{m_1+m_2}\right)^2 r_0^2.$$

Or,
$$I = \left[\frac{m_1m_2^2 + m_2m_1^2}{(m_1+m_2)^2}\right] r_0^2 = \left[\frac{m_1m_2(m_1+m_2)}{(m_1+m_2)^2}\right] r_0^2 = \frac{m_1m_2}{m_1+m_2} r_0^2.$$

But, as we know, $m_1m_2/(m_1+m_2) = \mu$, the *reduced mass* of the molecule. So that,

$$I = \mu\, r_0^2.$$

Or, M.I. of diatomic molecule = (reduced mass of the molecule) × (internuclear distance or bond length)^2

Torque

Torque is the measure of the force that can cause an object to rotate about an axis. Force is what causes an object to accelerate in linear kinematics. Similarly, torque is what causes an angular acceleration. Hence, torque can be defined as the rotational equivalent of linear force. The straight line about which the object rotates is called the axis of rotation.

In physics, torque is simply the tendency of a force to turn or twist. Different terminologies such as moment or moment of force are interchangeably used to describe torque. The distance of the point of application of force from the axis of rotation is sometimes called the moment arm or lever arm.

Types of Torque

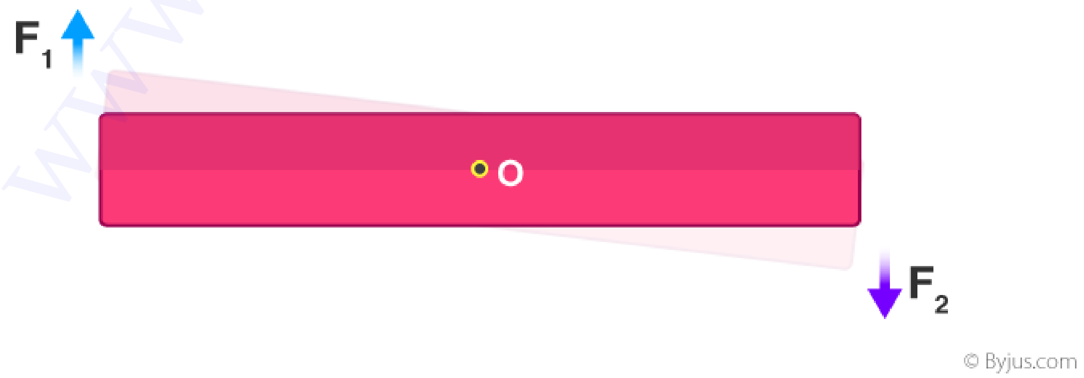
Torque can be either static or dynamic. Static torque is a torque that does not produce an angular acceleration. A few examples of static torque are as follows:

• A person pushing a closed-door is applying a static torque because the door isn't rotating despite the force applied.

• Pedalling a cycle at a constant speed is also an example of static torque as there is no acceleration.

The drive shaft in a racing car accelerating from the start line exhibits dynamic torque because it must be producing an angular acceleration of the wheels, given that the car is accelerating along the track.

To explain torque in detail let us consider the figure.



- We can see that the net force on the body is zero if the magnitude of forces F1 and F2 are equal.

- Hence, the body is in translational equilibrium.
- But it tends to rotate, thus the turning effect produced by force is known as moment of force or torque.

Now we will consider the example of a door and try to formulate the equation for torque.

- If we apply force closer to the hinge, then a larger force is required to rotate the door.
- Also, it depends on the direction in which the force is being applied.
- If it is perpendicular to the line joining the hinge and the point of application of force then a smaller force is required.

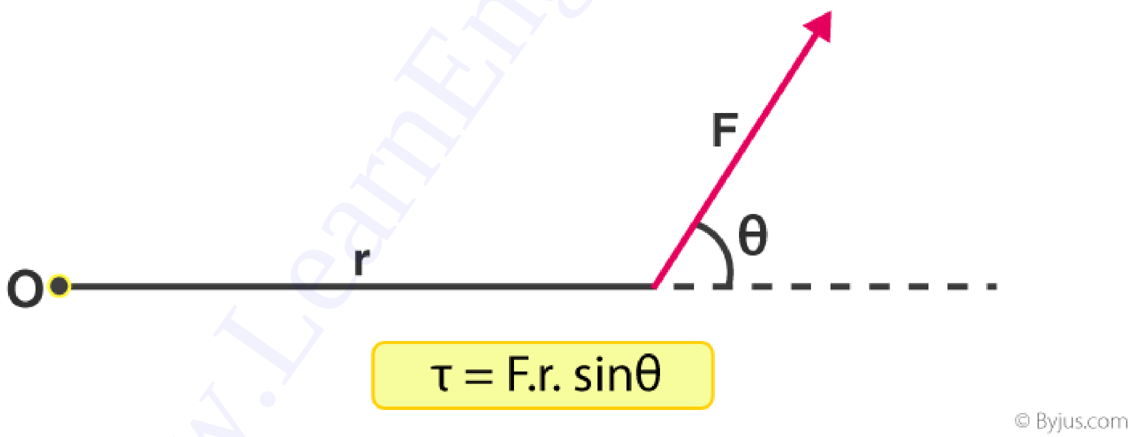
How is Torque Calculated?

A simple way to calculate the magnitude of the torque is to first determine the lever arm and then multiply it times the applied force.

Now, from the above observation, we conclude that the torque produced depends on the magnitude of the force and the perpendicular distance between the point about which torque is calculated and the point of application of force. So, mathematically torque is represented as:

$\tau = Fr\sin\Theta$

where r is the length of the lever arm and θ is the angle between the force vector and the lever arm.



Measurement of Torque

The unit of torque is Newton–meter (N-m). The above equation can be represented as the vector product of force and position vector.

So, as it is a vector product hence torque also must be a vector. Using vector product notations we can find the direction of torque.

Applications of Torque

In any object experiencing torque, there is a pivot point. Some applications are provided below:

Seesaws and Wrenches

Gyroscopes

A pendulum or a parachute is applying torque when swinging

A person riding a bicycle

Flag flying on a mast.

Rotational dynamics of the rigid body

Rigid body— Translational and Rotational motion

A rigid body is defined as that body which does not undergo any change in shape or volume when external forces are applied on it. When forces are applied on a rigid body, the distance between any two particles of the body will remain unchanged, however, large the forces may be.

Actually, no body is perfectly rigid. Every body can be deformed more or less by the application of the external force. The solids, in which the changes produced by external forces are negligibly small, are usually considered as rigid body.

Rotational motion

When a body rotates about a fixed axis, its motion is known as rotatory motion. A rigid body is said to have pure rotational motion, if every particle of the body moves in a circle, the centre of which lies on a straight line called the axis of rotation (Fig-1 shown below). The axis of rotation may lie inside the body or even outside the body. The particles lying on the axis of rotation remains stationary.

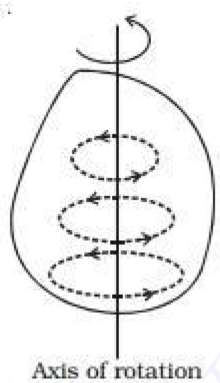


Fig Rotational motion

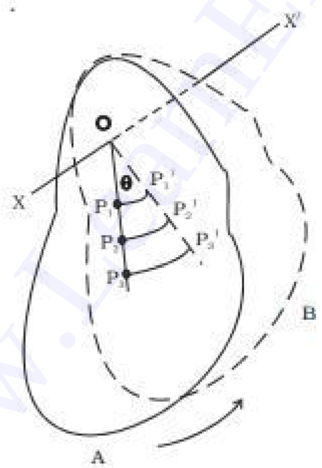


Fig Rotational motion of a rigid body

The position of particles moving in a circular path is conveniently described in terms of a radius vector r and its angular displacementθ .

Let us consider a rigid body that rotates about a fixed axis XOX′ passing through O and perpendicular to the plane of the paper as shown in Fig. Let the body rotate from the position A to the position B. The different particles at $P1$,$P2$,$P3$. ,... in the rigid body covers unequal istances $P1P1'$, $P2P2'$, $P3P3'$,... in the same interval of time. Thus their linear velocities are different. But in the same time interval, they all rotate through the same angle θ and hence the angular velocity is the same for the all the particles of the rigid body.

Thus, in the case of rotational motion, different constituent particles have different linear velocities but all of them have the same angular velocity.

$$\omega = \omega_o + \alpha t$$

$$\theta = \omega_o t + \frac{1}{2}\alpha t^2$$

$$\omega^2 = \omega_0^2 + 2\alpha\,\theta$$

The above are the equations of rotational motion.

Conservation of angular momentum

The principle of conservation of angular momentum in the context of rotation about a fixed axis.

$$\frac{d}{dt}(I\omega) = \tau$$

From the above Eq. if the external torque is zero,

$L_z = I\omega$ = constant

For symmetric bodies, from Eq. (below), L_z may be replaced by L.(L and L_z are respectively the magnitudes of L and L_z.)

$$\mathbf{L} = \mathbf{L}_z = I\omega\hat{\mathbf{k}}$$

This then is the required form, for fixed axisrotation, of below equation which expresses thegeneral law of conservation of angularmomentum of a system of particles.

If $\boldsymbol{\tau}_{ext} = \mathbf{0}$,

Eq. (7.28b) reduces to

$$\frac{d\mathbf{L}}{dt} = 0$$

or $\mathbf{L}$ = constant.

Thus, if the total external torque on a systemof particles is zero, then the total angularmomentum of the system is conserved, i.e. remains constant.

(Fig-A demonstration of conservation of angular momentum. A girl sits on a swivel chair and stretches her arms/ brings her arms closer to the body.)

Consider this experiment withyour friend. Sit on a swivel chair (a chair with aseat, free to rotate about a pivot) with your armsfolded and feet not resting on, i.e., away from,the ground. Ask your friend to rotate the chairrapidly. While the chair is rotating with considerable angular speed stretch your armshorizontally. What happens? Your angularspeed is reduced. If you bring back your arms closer to your body, the angular speed increasesagain. This is a situation where the principle of conservation of angular momentum isapplicable. If friction in the rotationalmechanism is neglected, there is no externaltorque about the axis of rotation of the chairand hence Iω is constant. Stretching the armsincreases I about the axis of rotation, resultingin decreasing the angular speed ω. Bringing the arms closer to the body has the oppositeeffect.

Rotational energy state of a rigid diatomic molecule

The M.I. of a diatomic molecule about an axis through its centre of mass and perpendicular to the distance r_0 (the bond length) between the two atoms is $\mu r_0{}^2$where μ is its reduced mass. If, therefore, ω be the angular velocity of the molecule about its axis ofrotation, its angular momentum j = Iω, and the angular momentum vector $\vec{j}$ lies in the plane perpendicular to the plane of rotation or along the axis of rotation.

Thus, $$j^2 = I^2\omega^2. \quad \text{Or,} \quad \omega^2 = j^2/I^2.$$

Now, as we know, the kinetic energy of rotation of a rigid body about an axis, about which its M.I. is I, is given by $E = 1/2\, I\omega^2$, where ω is its angular velocity about the axis.So that, substituting the value of ω^2 as obtained above, we have

$$E = \frac{1}{2} I \frac{j^2}{I^2} = \frac{j^2}{2I} = \frac{j^2}{2\mu r_0^2}$$

This is the relation for the kinetic energy of rotation of a diatomic molecule on the basis of classical mechanics, where j and, therefore, E may have any possible values and not only certain specified ones.However, the angular momentum in the case of small particles is quantised and, as deduced on the basis of wave mechanics, may have only discrete values

$\sqrt{j(j+1)}$ in terms of ħ or h/2π units, where j is the total angular momentum quantum number, equal to 0, 1, 2, 3 etc., i.e., the magnitude of the angular momentum of the molecule =

$$= \sqrt{j(j+1)}\frac{h}{2\pi}.$$

Substituting this in place of j, therefore, in the expression for rotational kinetic energy of the diatomic molecule, we have

$$E = \frac{j(j+1)h^2}{8\pi^2 I},$$

an expression which is also borne out by actual experiment.

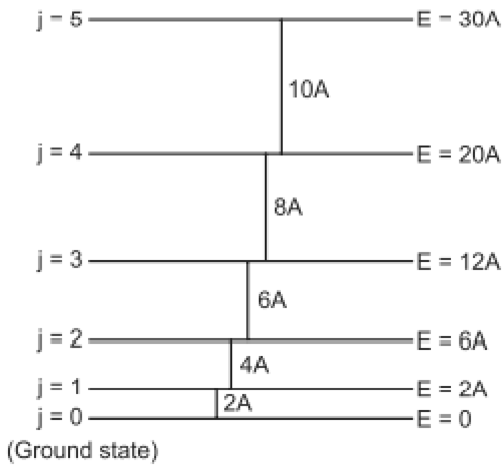
This means, then, that the rotational kinetic energy of a diatomic molecule is also quantised and may have only discrete values corresponding to j = 0, 1, 2, 3 etc., and no other intermediate values

Putting $h^2/8\pi^2 I = A$, we have $E = Aj(j+1)$. So that, for $j = 0$, $E = 0$; for $j = 1$, $E = 2A$; for $j = 2$, $E = 6A$ and so on.

These **are called the rotational energy states, (or rotational energy levels) of the diatomic molecule,** indicating that it can have no rotational energy between 0 and 2A, between 2A and 6A and so on.

For convenience, these are usually shown in what is called an energy level diagram, as indicated in Fig. below, which consists of a set of parallel lines, one above the other, representing the various energy levels, the lowest or the base line representing the ground state of the molecule, withj = 0 and hence also E = 0. The upper lines correspond to j = 1, j = 2 etc. and therefore, E = 2A, 6A etc., the distances of the lines (or energy levels) from the base line (or the ground state) beingproportional to the values of energy represented by them.

At or near the absolute zero, the molecule is in the ground state, with E = 0 corresponding to j = 0. As the temperature rises, the energy state or the energy level of the molecule also rises as shown, so that E attains the values 2A, 6A, 12A, 20A, 30A etc., corresponding to j = 1, 2, 3, 4 and 5 etc. respectively, i.e., there is a transition of the molecule from a lower to a higher energy state as it absorbs more and more energy.



On the other hand, if the transition occurs from a higher to a lower energy state, the molecule gives out its surplus energy which appears in the form of an electromagnetic radiation. These transitions are indicated by vertical lines in the figure along with the surplus energy emitted as electromagnetic radiation in each successive transition.

In either case, however, only those transitions are possible for which the value of j changes by $\Delta J = \pm 1$, i.e., from j to (j + 1) or from j to (j – 1).

The change in energy corresponding to a change $\Delta j = \pm 1$ in the value of j can be easily obtained from the difference in energy at energy levels corresponding to j and (j – 1)

Thus, the change in energy,

$$\Delta E = E_{(j)} - E_{(j-1)} = j(j+1)A - (j-1)jA = 2Aj.$$

Or,
$$\Delta E = 2\frac{h^2}{8\pi^2 I}j = \frac{h^2}{4\pi^2 I}j \qquad ...(i)$$

So that, $hv = \frac{h^2}{4\pi^2 I}j$ and , therefore, $v = \frac{h}{4\pi^2 I}j = \frac{2Aj}{h}$ ...(ii)

This shows that the frequencies (v) of the transition radiations emitted differ by the same amount 2A/h for successive pairs of energy levels.

Thus, the frequency of radiation due to transition between energy levels corresponding to j = 1 and j = 0 is 2A/h; that of the radiation due to transition between energy levels corresponding to j = 2 and j = 0 is 4A/h and so on, the common difference between the frequencies emitted between successive pairs of energy levels being 2A/h all along. This means, clearly, that the spectral lines emitted due to transition between successive pairs of energy levels must all be equally spaced. The frequencies of these spectral lines being quite small (and hence their wavelengths large) because of the small amount of the energy involved in the transition, they are found in the infra-red or the micro-radio wave region of the electromagnetic spectrum.

It will be readily seen that this common difference (2A/h) in the frequencies of the successive spectral lines enables us to obtain the value of the M.I. of a diatomic molecule about its own axis of rotation.

For, $2A/h = h/4\pi^2 I$, whence, $I = h^2/8\pi^2 A$.

And, since $I = \mu r_0^2$, where μ is the reduced mass of the molecule and r_0, the internuclear distance or bond length.

we can also obtain the value of r_0.Now, defining wave number v as the number of wavelengths per cm,,

we have $v = 1/\lambda = v/c$ *per cm*,

where c is the velocity of light or electromagnetic radiation, in general.If, therefore, E be the energy of radiation, we have E = hv or, v = E/h

Or, $\bar{v} = 1/\lambda = E/hc = E$, where $E/hc = \epsilon$ is referred to as *energy in wave numbers.*

We can thus put our equation I above in terms of ∈, when it becomes

$$v = \frac{E_{(j)}}{hc} - \frac{E_{(j-1)}}{hc} = E_{(j)} - E_{(j-1)} = \frac{2A}{hc} j = \frac{hj}{4\pi^2 Ic} = \frac{hj}{4\pi^2 \mu r_0^2 c} \quad ...(iii)$$

This shows at once that the common spacing between the spectral lines emitted is $h/4\pi^2 Ic = h/4\pi^2 \mu r_0^2$ c. So that, knowing the spacing between the spectral lines we can again obtain the M.I. of the diatomic molecule and its intranuclear distance or bond length.

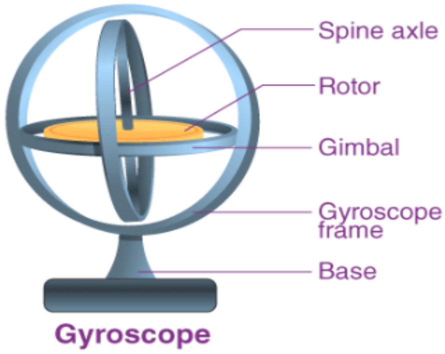
Gyroscope

'Gyre' is a Greek word, meaning 'circular motion' and Gyration means the whirling motion. A gyroscope is a spatial mechanism which is generally employed for the study of precessional motion of a rotary body. Gyroscope is generally employed for the control of angular motion of a body.

If we attempt to move some of its parts, it does not only resist this motion but even evades it. This resistance to change in the direction of rotational axis is called the gyroscopic effect. Gyroscope finds applications in gyrocompass, used inaircraft, naval ship, control system of missiles and space shuttle.

A gyroscope is defined as the device has a spinning disc that is mounted on the base such that it can move freely in more than one direction so that the orientation is maintained irrespective of the movement in the base.

Gyroscope Diagram



Parts of Gyroscope

A gyroscope consists of the following parts:

- Spin axis
- Gimbal
- Rotor
- Gyroscope frame

Design of Gyroscope

A gyroscope can be considered as a massive rotor that is fixed on the supporting rings known as the gimbals. The central rotor is isolated from the external torques with the help of frictionless bearings that are present in the gimbals. The spin axis is defined by the axle of the spinning wheel.

The rotor has exceptional stability at high speeds as it maintains the high-speed rotation axis at the central rotor. The rotor has three degrees of rotational freedom.

Gyroscope Working Principle

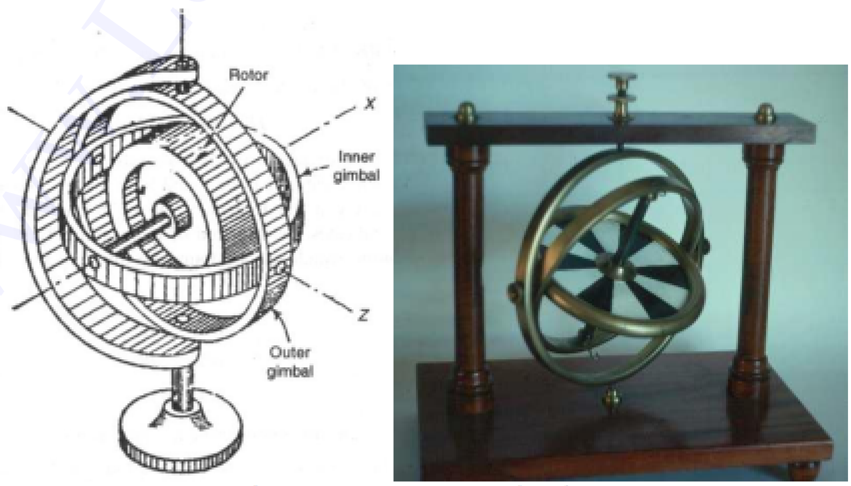


Fig. 1 Gyroscope mechanism

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A gyroscope consists of a rotor mounted in the inner gimbal. The inner gimbal ismounted in the outer gimbal which itself is mounted on a fixed frame as shown in Fig.1.

When the rotor spins about X-axis with angular velocity ω rad/s and the inner gimbal precesses (rotates) about Y-axis, the spatial mechanism is forced to turn about Z-axis other than its own axis of rotation, and the gyroscopic effect is thus setup. The resistance to this motion is called gyroscopic effect.

The working principle of gyroscope is based on gravity and is explained as the product of angular momentum which is experienced by the torque on a disc to produce a gyroscopic precession in the spinning wheel.

This process is termed gyroscopic motion or gyroscopic force and is defined as the tendency of a rotating object to maintain the orientation of its rotation.

We know that the rotating object possesses angular momentum and this needs to be conserved. This is done because when there is any change in the axis of rotation, there will be a change in the orientation which changes the angular momentum. **Therefore, it can be said the working principle of gyroscope is based on the conservation of angular momentum.**

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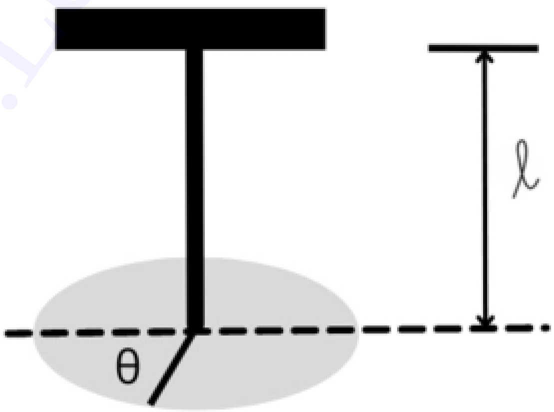
Torsional pendulum

The torsional pendulum is the disc suspended to the thin bar which creates twisting oscillations around the axis of the bar. The restoring force developed by twisting or torsional action creates the oscillations in the disc.If the initial angular displacement θ is given to the disc by applying twisting torque, the thin rod generates the restoring torque, which causes the disc to revolve in the opposite direction.



This mechanism creates simple harmonic motion in the torsional pendulum.

Torsional pendulum formula The required equations for the analysis of the torsional pendulum are listed below,

1) Restoring torque (T):

$$T = -C\theta$$

Here θ is angular twist and C is torque per unit twist of the pendulum that is given by,

$$C = \frac{\pi \times \eta \times r^4}{2l}$$

Where,

η = Modulus of rigidity

r = Wire radius

l = Length of wire

2) Torsional pendulum period equation:

The equation for the period of the original pendulum is given by,

$$T = 2\pi\sqrt{\frac{I}{C}}$$

3) Equation of modulus of rigidity for torsional pendulum:

The equation for the torsional rigidity of the torsional pendulum is given by,

$$\eta = \frac{8\pi I}{r^4}\left(\frac{L}{T^2}\right)$$

Torsional pendulum period derivation

The restoring torque is directly proportional to the angle of twist in the wire that is given by,

T= – Cθ ——(1)

Where C = Torsion constant

In angular motion, the equation for torque is,

$$T = I \times \alpha$$

$$\text{As, } \alpha = \frac{d^2\theta}{dt^2}$$

$$\text{Therefore, } T = I\frac{d^2\theta}{dt^2} \text{ ——(2)}$$

Where, I = Moment of inertia of the disc

Equating the equations 1 and 2 we get,

$$I\frac{d^2\theta}{dt^2} = -C\theta$$

$$\frac{d^2\theta}{dt^2} + \left(\frac{C}{I}\right)\theta = 0 \text{ ——(3)}$$

The equation of angular simple harmonic motion is given by,

$$\frac{d^2\theta}{dt^2} + \omega^2\theta = 0 \text{ ——(4)}$$

Now by comparing equations 3 and 4 we get,

$$\omega^2 = \frac{C}{I}$$

$$\therefore \omega = \sqrt{\frac{C}{I}} \text{ ——(5)}$$

This is the equation for the angular speed of torsional pendulum.

Now the frequency of oscillation is given by,

$$\omega = 2\pi f$$

$$f = \frac{\omega}{2\pi}$$

$$f = \frac{1}{2\pi}\sqrt{\frac{C}{I}} \text{ ——(6)}$$

This is the equation for frequency of oscillation of the torsional pendulum.

Now the period of torsional pendulum is given by,

Period of torsional pendulum $= t = 2\pi\sqrt{\frac{I}{C}}$ ——(7)

This is the required equation for the period of oscillation of the torsional pendulum.

Derivation for the torsional rigidity of torsional pendulum:

From equation 7, the period of oscillation is,

$$t = 2\pi\sqrt{\frac{I}{C}}$$

Put the value of ,

$$C = \frac{\pi\eta r^4}{2l}$$

$$\therefore t = 2\pi\sqrt{\frac{I2l}{\pi\eta r^4}}$$

$$\therefore \eta = \frac{8\pi I}{r^4}\left(\frac{l}{t^2}\right)$$

This is the equation to determine torsional rigidity of pendulum.

The double pendulum

Consider the double pendulum shown on figure 1. A double pendulum is formed by attaching a pendulum directly to another one. Each pendulum consists of a bob connected to a massless rigid rod which is only allowed to move along a vertical plane. The pivot of the first pendulum is fixed to a point O. All motion is friction**less.**

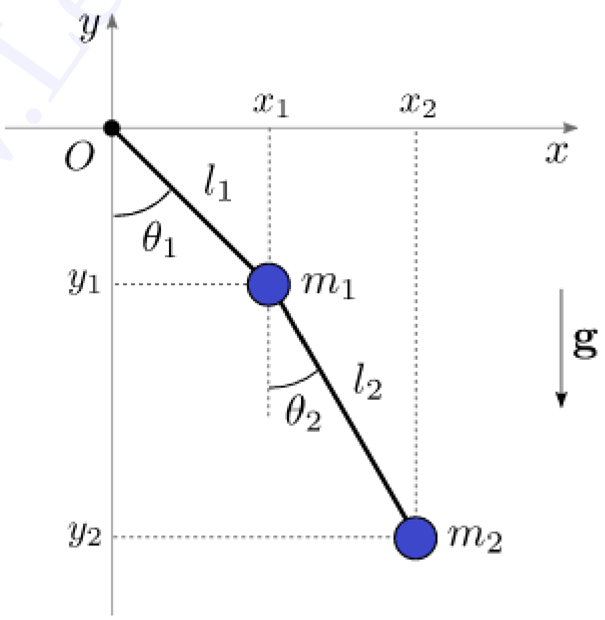


Fig. 1.A double pendulum.

In our discussion, the fixed point O will be taken as the origin of the Cartesian coordinate system with the x axis pointing along the horizontal direction and the y axis pointing vertically upwards. Let θ1 and θ2 be the angles which the first and second rods make with the vertical direction respectively. As can be seen on figure 1.A, the positions of the bobs are given by:

$$
\begin{aligned}
x_1 &= l_1 \sin\theta_1 & y_1 &= -l_1 \cos\theta_1 \\
x_2 &= l_1 \sin\theta_1 + l_2 \sin\theta_2 & y_2 &= -l_1 \cos\theta_1 - l_2 \cos\theta_2
\end{aligned}
$$

Differentiating the quantities above with respect to time, we obtain the velocities of the bobs**:**

$$
\begin{aligned}
\dot{x}_1 &= l_1 \dot{\theta}_1 \cos\theta_1 & \dot{y}_1 &= l_1 \dot{\theta}_1 \sin\theta_1 \\
\dot{x}_2 &= l_1 \dot{\theta}_1 \cos\theta_1 + l_2 \dot{\theta}_2 \cos\theta_2 & \dot{y}_2 &= l_1 \dot{\theta}_1 \sin\theta_1 + l_2 \dot{\theta}_2 \sin\theta_2
\end{aligned}
$$

The double pendulum is a very interesting system as it is very simple but can show chaotic behavior for certain initial conditions. In this regime, slightly changing the initial values of the angles (θ_1,θ_2) and angular velocities ($\dot{\theta}_1$,$\dot{\theta}_2$) makes the trajectoriesof the bobs become very different from the original ones.

The Lagrangian for the double pendulum is given by L=T−V, where T and V are the kinetic and potential energies of the system respectively. The kinetic energy T is given by:

$$
\begin{aligned}
T &= \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 \\
&= \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2) \\
&= \frac{1}{2} m_1 l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 \left[l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2 + 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2) \right]
\end{aligned}
$$

where above we used the fact that $\cos\theta_1\cos\theta_2+\sin\theta_1\sin\theta_2= \cos(\theta_1-\theta_2)$. The potential energy V is given by:

$$
\begin{aligned}
V &= m_1 g y_1 + m_2 g y_2 \\
&= -m_1 g l_1 \cos\theta_1 - m_2 g (l_1 \cos\theta_1 + l_2 \cos\theta_2) \\
&= -(m_1 + m_2) g l_1 \cos\theta_1 - m_2 g l_2 \cos\theta_2
\end{aligned}
$$

The Lagrangian equation for the double pendulum of the system is then:

$$L = \frac{1}{2}(m_1 + m_2)l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2 l_2^2\dot{\theta}_2^2 + m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2)$$
$$+ (m_1 + m_2) g l_1 \cos\theta_1 + m_2 g l_2 \cos\theta_2$$

Introduction to nonlinear oscillations

A linear oscillator can oscillate with only one frequency, its motion is sinusoidal and periodic.
If the return force in the spring shown in Fig. 1 is not linear, the motion will still repeatitself, but it will no longer have only a single frequency in its motion. The oscillations willrepeat over and over, always with the same period, but the position as a function of timewill not be given by
$y = A\cos(2\pi f_1 t)$, where $f_1 = 1/P$.

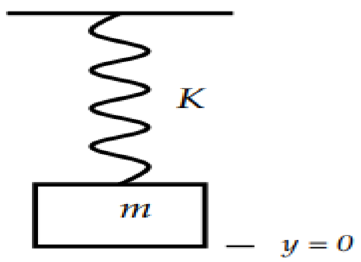
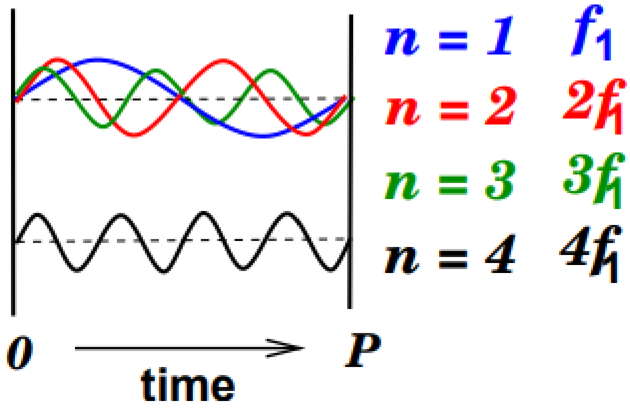


Fig. 1: The Simple Oscillator

The reason for giving (1/P) thename f_1 will soon be apparent.So what are the building blocks of this complex motion? What property do we need forthese building blocks? We need building blocks that depend on time, since we are tryingto describe the motion of the nonlinear oscillator. The simplest building block would be asine wave, but with a different frequency, but the immediate question is "What frequency?"
If we want to describe motion that exactly repeats itself cycle after cycle, we need building blocks that oscillate with exactly the same period P, or at least integer fractions of the period, namely P, $P/_2$, $P/_3$, etc., which we can write symbolically as $P_n = P/n$ where n is an integer, n = 1, 2, 3, ·……   Since the frequency f_n is the inverse of the period,

$$f_n = \frac{n}{P} = n\frac{1}{P} = nf_1.$$



(Fig-2: The first four harmonics. The fourth harmonic is shown below the others for clarity. The dashed vertical line shows half the period, so t = P/2. Note the differencebetween the behavior of the odd (1 and 3) harmonics and the even harmonics (2 and4) just after t = P/2.)

These building blocks, called harmonics, are simple sine waves with frequencies that areinteger multiples of the lowest frequency f1 = 1/P. They go through exactly 1, 2, etccomplete oscillations in the period P. The frequency f1 is called the fundamental of theharmonic series. The first four harmonics are shown in Fig-2, which are given by

$$y_n = A\cos\{n(2\pi f_1)t)\} \quad n = 1, 2, 3, 4.$$

Notice that at the midpoint, all of the harmonics are zero, but the even harmonics have gonethrough an integer number of cycles, and are going positive again while the odd harmonics.have gone through 1/2, 3/2, 5/2, etc cycles, and are going negative. This will becomeimportant when we talk about periodic waves,.If the period of the oscillation is P, then the frequencies present in the motion are

$$f_1 = \frac{1}{P},\ f_2 = 2f_1 = \frac{2}{P},\ f_3 = 3f_1,\ f_4 = 4f_1 4f,\ \text{etc.}$$

To summarize, the motion contains the frequency f_1 which is the inverse of the period,plus harmonics (integer multiples) of this frequency. This is very different from thesimple oscillator. In the simple oscillator we had one frequency which only depended onthe stiffness and inertia of the system. Now, with a nonlinear return force, we get somethingquite new. The motion of the nonlinear oscillator consists of a complex motion made up ofharmonics of f1. The participation of each harmonic in a complex oscillation depends on thedetails of the nonlinearity.

There are two important characteristics of the nonlinear oscillator.

1. The effects of the nonlinearity become much more important as the amplitude is increased.

2. For some types of nonlinearity, the frequency of the oscillator will change with amplitude.Thus when we drive a nonlinear system, the larger the amplitude the more important thehigher harmonics are. The second property can make for very interesting response when thesystem is driven at different amplitudes producing a very curious shape to the resonancecurve, but we will not discuss it here. We note that the pendulum is not a simple oscillator,but rather a nonlinear one, with the frequency decreasing with increased amplitude.

UNIT II - ELECTROMAGNETIC WAVES

Introduction

The greatest theoretical achievement of physics in the 19th century was the discovery of electromagnetic waves. The history of electromagnetic theory begins with ancient measures to understand atmospheric electricity, in particular lightning, but were unable to explain the phenomena. People then had little understanding of electricity except these facts. Electric forces in nature come in two kinds. First, there is the electric attraction between unlike (+) and ( ) charges or repulsion between like (+) and (+) or ( ) and ( ) electric charges. It is possible to use this to define a unit of electric charge, as the charge which repels a similar charge at a distance of, say, 1 meter, with a force of unit strength.

Then Faraday showed that a magnetic field which varied in time like the one produced by an alternating current(AC) could drive electric currents, if (say) copper wires were placed in it in the appropriate way. That was “magnetic induction,” the phenomenon on which electric transformers are based. So, magnetic fields could produce electric currents, and we already know that electric currents produce magnetic fields. In the 19th century it had become clear that electricity and magnetism were related, and their theories were unified: wherever charges are in motion electric current results, and magnetism is due to electric current. The source for electric field is electric charge, whereas that for magnetic field is electric current (charges in motion).

In 1864 Maxwell theoretically proposed that electromagnetic disturbance travels in free space with the speed of light. Although the idea was remained hidden in his set of equations but virtually never said anything about the waves nor he said anything about the generation of such waves. Later on

Hertz in 1888 succeeded in producing and observing electromagnetic waves of wavelength of the order of 6m in thelaboratory.

J. C. Bose in 1895 succeeded in producing and observing electromagnetic waves of much shorter wavelength 25 mm- 5 mm.

- G. Marconi in the same year succeeded in transmitting electromagnetic waves over distances of many kilometers.

Basics & Terminologies

The Lorentz force law characterizes the observable effects of electric and magnetic fields on charges, Maxwellsequations characterize the origins of those fields and their relationships to each other. The simplest representation of Maxwells equations is in differential form, which leads directly to waves. But before going to the Maxwell's equations let us refresh ourself with few terminologies which will have regular appearances in the equations. The four Maxwell

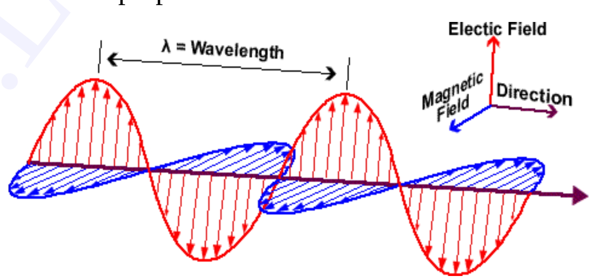
Field variables	Names	Unit
$\vec{E}$	Electric Field	volts/meter; Vm^{-1}
$\vec{H}$	Magnetic Intensity	amperes/meter; Am^{-1}
$\vec{B}$	Magnetic Flux Density	Tesla, T
$\vec{D}$	Electric Displacement	coulombs/m^2; Cm^{-2}
$\vec{J}$	Electric current density	amperes/m^2; Am^{-2}
ρ	Electric charge density	coulombs/m^3; Cm^{-3}

equations which will be derived in the next section invoke one scalar and five vector quantities comprising 16 variables. Some variables only characterize how matter alters field behavior. In vacuum we can eliminate three vectors (9variables) by noting $\vec{D} = \varrho_0 \vec{E}$, $\vec{B} = \mu_0 \vec{H}$ *and* $\vec{J} = \rho \vec{v} = \sigma \vec{E}$. where $\varrho_0 = 8.8542 \times 10^{-12}$ [farads m^{-1}] is the absolute permittivity of vacuum, $\mu_0 = 4\pi$ 10^{-7} [henries m^{-1}] is the absolute permeability of vacuum, $\vec{v}$ is the drift velocity of the local net charge density ρ, and σ is the conductivity of a medium [Siemens m^{-1}]. If we regard the electrical sources ρ and $\vec{J}$ as given, then the equations can be solved for all remaining unknowns. Specifically, we can then find $\vec{E}$ and $\vec{B}$, and thus compute the forces on all charges present.

E -M Waves

Definition:

Electromagnetic waves or EM waves are oscillating magnetic and electric fields at right angles to each other, self-propagating in direction perpendicular to both the electric and magnetic fields.



Properties of E -M Waves

Listed below are some of the important characteristics of electromagnetic waves.

Property 1: In electromagnetic waves the electric field vector $\vec{E}$ and magnetic field vector $\vec{B}$ and propagation vector $\vec{K}$ are mutually perpendicular for a right-handed system. Hence electromagnetic waves are transverse in nature.

Property 2: Electromagnetic waves travel with speed of light.

Property 3: Electromagnetic waves are self-propagating. They keep on moving even without the source thatcreated them.

Property 4: Electromagnetic waves doesn't need any medium to propagate.

Property 5: Electromagnetic waves are not deflected by electric or magnetic field.

Property 6: Electromagnetic waves can show interference or diffraction and can be polarized.

Property 7: Electromagnetic waves carry energy with them and exerts pressure on the medium they incident upon.

Basic Definitions-

Electric field intensity (E) - It isd efined as the ratio of electrostatic force to the electric charge. Electric Field = F/q. Unit of E is NC^{-1} or Vm^{-1}.

Electric displacement vector (D) - It is defined as electric flux (Q) per unit area. It is also known as Electric flux density.

Electrical permittivity (ε): It is defined as the ratio of displacement current 9D) to the electric field intensity(E). It is given by

ε = D /E and its unit is Unit: $C^2N^{-1}m^{-2}$

Dielectric constant (or) relative permittivity (ε_r)**:** The dielectric constant of a substance is the ratio of the permittivity of the substance to the permittivity of the free space. It shows the extent to which a material can hold electric flux within it. It is given by $\epsilon_r = \epsilon_0 \epsilon$

here,

ϵ_r is the dielectric constant

ϵ is the permittivity of the substance

$\epsilon 0$ is the permittivity of the free space

Magnetic flux density ($\vec{B}$): It is defined as the number of magnetic lines of force (φ_m) passing normally through unit area of cross section.

Unit: 'weber/m^2' or Tesla

Magnetic field intensity ($\vec{H}$): It is defined as the force experienced by a unit north pole placed at the point in a magnetic field. It is given by H = F/M

Unit: Ampere / meter.

Magnetic permeability (μ): It is the ratio of magnetic flux density ($\vec{B}$) to the magnetic field intensity ($\vec{H}$). It is given by

$$\mu = \mu_0 \mu_r = \frac{B}{\vec{H}}$$

μ_0 → permeability of free space μr → relative permeability

Unit: Ns^2C^{-2}

Relative permeability (μ_r):

It is the ratio of absolute permeability (μ) of the medium to the permeability of free space(μ_0). It is a number. For air and non – magnetic material, its value is '1'.

$$\mu_r = \mu / \mu_0$$

$\mu_0 = 4\pi \times 10^{-7} Ns^2C^{-2}$

Maxwell's Field Equations

From a long view of the history of mankind seen from, say, ten thousand years from now, there can be little doubt that the most significant event of the 19th century will be judged as Maxwell's discovery of the laws of electrodynamics.

The Feynman Lectures on Physics (1964), Richard Feynman

Maxwell's equations are a set of four differential equations that, together with the Lorentz force law, form the foundation of classical electromagnetism, classical optics, and electric circuits. These equations describe how electric and magnetic fields propagate, interact, and how they are influenced by objects. He was an Einstein/Newton-level genius who took a set of known experimental laws (Faraday's Law, Ampere's Law) and unified them into a symmetric coherent set of Equations known as Maxwell's Equations. Maxwell was one of the first to determine the speed of propagation of electromagnetic (EM) waves was the same as the speed of light - and hence to conclude that EM waves and visible light were really the same thing.

The four Maxwell's equations can be divided into two major subsets. The first two, Gauss's law for electrostatics and one people used to say as Gauss's law for magnetism, however it is not exactly so, describe how fields emanate from charges and magnets respectively. The other two, Faradays law and Ampere's law with Maxwell's correction, describe how induced electric and magnetic fields circulate around their respective sources.

Each of Maxwell's equations can be looked at from the "microscopic" perspective, which deals with total charge and total current, and the "macroscopic" set, which defines two new auxiliary fields that allow one to perform calculations without knowing microscopic data like charges at the atomic level.

Let us now discuss them one by one.

Maxwell's First equation, Gauss Law for electrostatics

The integral of the outgoing electric field $\vec{E}$ over an area enclosing a volume V equals the total charge Q enclosed by the volume divided by ε_0 in vacuum. It represents completely covering the surface with a large number of tiny patches having areas $d\vec{S}$. We represent these small areas as vectors pointing outwards, because we can then take the dot product with the electric field to select the component of that field pointing perpendicularly outwards (it would count negatively if the field were pointing inwards) - this is the only component of the field that contributes to actual flow across the surface. (Just as a river flowing parallel to its banks has no flow across the banks).

Physical Significance:

The net quantity of the electric flux leaving a volume is proportional to the charge inside the volume.

Maxwell's Second equation

The second law states that there are no "magnetic charges (or monopoles)" analogous to electric charges, and that magnetic fields are instead generated by magnetic dipoles. Such dipoles can be represented as loops of current, but in many ways are similar in appearance to positive and negative "magnetic charges" that are inseparable and thus haveno formal net "magnetic charge." This can be derived from Biot-Savart Law. Magnetic field lines form loops such that all field lines that go into an object leave it at some point. Thus, the total magnetic flux through a surface surrounding a magnetic dipole is always zero.

Physical Significance:

Magnetic monopole does not exist.

Maxwell's Third equation, Faraday's law of EM induction

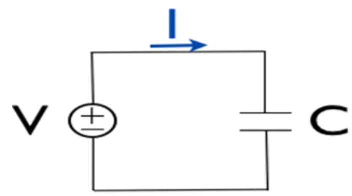
Faraday demonstrated the fact that whenever the magnetic flux associated with any closed loop changes an induced emf develops in the circuit and that sends current through the circuit which last so long as the change of flux lasts. He also showed that the induced emf produced is directly proportional to magnetic flux linked with the coil. In mathematical language Faradays law states that the closed integral of the induced electric field is minus the time rate of change of the magnetic flux through the loop. Or simply saying a time-varying magnetic field (or flux) induces an electric field. In fact, the straight forward outcome of this equation says that work is needed to take a charge around a closed curve in an electric field. Magnetic field lines form loops such that all field lines that go into an object leave it at some point. Thus, the total magnetic flux through a surface surrounding a magnetic dipole is always zero.

Physical Significance:

Magnetic monopole does not exist.

Maxwell's Fourth equation, Modification of Ampere's law

Amperes law states that the line integral of the magnetic field $\vec{B}$ around any closed path or circuit is equal to the current enclosed by the path. In a simple note magnetic field could be created by electrical current. However, in the presence of time dependent charge densities it cannot be correct. Because here electric field which grows continuously since there has been an accumulation of charge in the capacitor plates. Thus, there is a time varying electric field present between the plates. That implies there must also a magnetic field present inside the capacitor plate. And then if you place a compass needle between the capacitor plates the needle gets displaced ie there is some deflection. So, the point here is that between the plates no conductor is there ie no conduction current should be there but still the needle is showing deflection and the circuit shows a current reading. Maxwell resolved this contradiction by creating something called a displacement current. This was an analogy with a dielectric material.If a dielectric material is placed in an electric field, the molecules are distorted, their positive charges moving slightly to the right, say, the negative charges slightly to the left. Now consider what happens to a dielectric in an increasing electric field. The positive charges will be displaced to the right by a continuously increasing distance, so, as long as the electric field is increasing in strength, these charges are moving: there is actually a displacement current. This electric field that produces the current and makes the circuit continuous. Maxwell added this displacement term in Ampere's law and he showed that it is equal to the permittivity of free space times the rate of change of electric flux with respect to time. Let us now see how this was achieved.



Therefore, this is the way to generalize Ampere's law from the magnetostatic situation to the case where charge densities are varying with time.

Physical Significance:

Magnetic field B around any closed path or circuit is equal to the conductions current plus the time derivative ofelectric displacement through any surface bounded by the path.

2.1 MAXWELL'S EQUATIONS

There are four Maxwell's equations in electromagnetic theory.

- Gauss law for electricity

$$\vec{\nabla}.\vec{D} = \rho$$

- Gauss law for magnetism

$$\vec{\nabla}.\vec{B} = 0$$

- Faraday's law of electromagnetic induction

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

- Ampere's law

$$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

1. Gauss law for electricity

Gauss law for electricity states that the total electrical flux through any closed surface in an electric field is equal to the total charge enclosed by the surface.

Let us consider a dielectric medium of surface 'S' and volume 'V'. Let 'Q' is the total charge in the dielectric material and 'ρ' be the charge density, then

According to Gauss law, we can write

$$\oint_S \vec{E}.ds = \frac{Q}{\epsilon_0}$$

$$\oint_S \epsilon_0 \vec{E}.ds = Q \qquad ------(1)$$

We know $\vec{D} = \epsilon \vec{E}$ $\qquad ------(2)$

but $\epsilon = \epsilon_0 \epsilon_r$

for air medium $\epsilon_r = 1$

$$so \quad \epsilon = \epsilon_0 \qquad ------(3)$$

Substituting (3) in (2) we get,

$$\vec{D} = \epsilon_0 \vec{E} \qquad ------(4)$$

Substituting (4) in (1) we get,

$$\oint_S \vec{D}.ds = Q \qquad ------(5)$$

Now the charge density ρ is the charge per unit volume

$$Charge\ density, \quad \rho = \frac{Q}{V}$$

$$so \qquad Q = \rho V \qquad ------(6)$$

Substituting (6) in (5) we get.

$$\oint_S \vec{D}.ds = \oint_V \rho.dv \qquad -----(7)$$

This is the Maxwell's first equation in integral form.

Differential form

Take the integral form of Maxwell's equation

$$\oint_S \vec{D}.ds = \oint_V \rho.dv \qquad -----(8)$$

Apply Gauss divergence theorem on the left hand side, we get

$$\oint_S \vec{D}.ds = \oint_V \vec{\nabla}.\vec{D}dv \qquad -----(9)$$

Substituting (9) in (8) we get.

$$\oint_V \vec{\nabla}.\vec{D}dv = \oint_V \rho.dv$$

Two volume integrals are equal if their integrands are equal.

$$\vec{\nabla}.\vec{D} = \rho \qquad -----(10)$$

This is Maxwell's first equation in differential form.

2. Gauss law for magnetism:

Gauss law for magnetism states that the total magnetic flux through any closed surface in an magnetic field is equal to zero.

$$i.e \quad \varphi = 0 \qquad -----(1)$$

The magnetic flux in terms of magnetic induction 'B' is given as

$$\varphi = \oint_S \vec{B}.ds \qquad -----(2)$$

Substituting (2) in (1) we get

$$\oint_S \vec{B}.ds = 0 \qquad -----(3)$$

This is the Maxwell's second equation in integral form.

Differential form

Take the integral form of Maxwell's equation

$$\oint_S \vec{B}.ds = 0 \qquad -----(4)$$

Apply Gauss divergence theorem on the left hand side, we get

$$\oint_S \vec{B}.ds = \oint_V \vec{\nabla}.\vec{B}dv \qquad ------(5)$$

Substituting (5) in (4) we get

$$\oint_V \vec{\nabla}.\vec{B}dv = 0$$

$$\vec{\nabla}.\vec{B} = 0$$

This is the Maxwell's second equation in differential form.

3. Faraday's law of electromagnetic induction:

Faraday's law of electromagnetic induction states that the emf induced across the coil is equal to the rate of change of flux in the coil.

$$\epsilon = -\frac{d\varphi}{dt} \qquad -----(1)$$

Where 'ϵ' is the electromotive force and 'φ' is the magnetic flux.

Electromotive force in terms of electric field is written as

$$\epsilon = \oint_l \vec{E}.dl \qquad ----(2)$$

Magnetic Flux 'φ'in terms of the magnetic induction 'B' is written as

$$\varphi = \oint_S \vec{B}.ds \qquad ----(3)$$

Substituting (2) and (3) in (1) we get

$$\oint_l \vec{E}.dl = -\frac{d}{dt}\left(\oint_S \vec{B}.ds\right)$$

$$\oint_l \vec{E}.dl = -\oint_S \frac{d\vec{B}}{dt}.ds \qquad -----(4)$$

This is the Maxwell's third equation in integral form.

Differential form

Take the integral form of Maxwell's equation

$$\oint_l \vec{E}.dl = -\oint_S \frac{d\vec{B}}{dt}.ds \qquad ------(5)$$

Apply stokes theorem on L.H.S. w get,

$$\oint_l \vec{E}.dl = \oint_S \vec{\nabla}\times\vec{E}.dl \qquad ------(6)$$

Substituting (6) in (5) we get

$$\oint_S \vec{\nabla}\times\vec{E}.dl = -\oint_S \frac{d\vec{B}}{dt}.ds \qquad -----(7)$$

Two surface integrals are equal if their integrands are equal.

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \qquad ------(8)$$

This is the Maxwell's third equation in differential form.

4. Ampere's law (Maxwell's Fourth Equation)

Ampere's law states that the line integral of the magnetic field intensity 'H' on any closed path is equal to the current 'I' enclosed in that path.

$$\oint_l \vec{H}.\vec{dl} = I \qquad -----(1)$$

Relation between current and current density

$$I = \oint_S \vec{J}.\vec{ds} \qquad -----(2)$$

Substituting (2) in (1) we get

$$\oint_l \vec{H}.dl = \oint_S \vec{J}.\vec{ds} \qquad -----(3)$$

From Stokes's theorem, Line integral can be converted into surface integral

$$\oint_l \vec{H}.dl = \oint_S \vec{\nabla}\times\vec{H}.\vec{ds} \qquad ------(4)$$

It can be written as

$$\oint_l \vec{H}.dl = \oint_S \left(\vec{J}+\frac{\partial\vec{D}}{\partial t}\right).\vec{ds}$$

This is the Maxwell's fourth equation in integral form (Refer Maxwell's fourth differential equation)

Comparing (3) and (4)

$$\oint_S \vec{\nabla}\times\vec{H}.\vec{ds} = \oint_S \vec{J}.\vec{ds} \qquad -----(5)$$

$$\vec{\nabla}\times\vec{H} = \vec{J} \qquad -----(6)$$

Applying Gauss divergence theorem

$$\vec{\nabla}.(\vec{\nabla} \times \vec{H}) = \vec{\nabla}.\vec{J} \qquad ------(7)$$

From Vector identity

$$\vec{\nabla}.(\vec{\nabla} \times \vec{H}) = 0 \qquad ------(8)$$

Substituting (8) in (7) we get

$$\vec{\nabla}.\vec{J} = 0 \qquad ------(9)$$

According to equation of continuity, we can write,

$$\vec{\nabla}.\vec{J} + \frac{\partial \rho}{\partial t} = 0$$

$$\vec{\nabla}.\vec{J} = -\frac{\partial \rho}{\partial t}$$

$$\text{If } \vec{\nabla}.\vec{J} = 0 \quad \text{then} \quad \frac{\partial \rho}{\partial t} = 0 \qquad ------(10)$$

Maxwell modified (6) by adding current density 'J_d'

$$\vec{\nabla} \times \vec{H} = \vec{J} + \vec{J_d} \qquad ------(11)$$

Taking divergence,

$$\vec{\nabla}.(\vec{\nabla} \times \vec{H}) = \vec{\nabla}.(\vec{J} + \vec{J_d}) \qquad ------(12)$$

From Vector identity

$$\nabla.(\vec{\nabla} \times \vec{H}) = 0 \qquad ------(13)$$

Substituting (13) in (12) we get

$$\vec{\nabla}.\vec{J} + \vec{\nabla}.\vec{J_d} = 0 \qquad ------(14)$$

We know

$$\vec{\nabla}.\vec{J} = -\frac{\partial \rho}{\partial t} \qquad ------(15)$$

Substituting (15) in (14) we get

$$-\frac{\partial \rho}{\partial t} + \vec{\nabla}.\vec{J_d} = 0$$

$$\vec{\nabla}.\vec{J_d} = \frac{\partial \rho}{\partial t} \qquad ------(16)$$

From Maxwell's first equation, $\vec{\nabla}.\vec{D} = \rho \qquad ------(17)$

Substituting (17) in (16) we get

$$\vec{\nabla}.\vec{J_d} = \frac{\partial(\vec{\nabla}.\vec{D})}{\partial t}$$

$$\vec{\nabla}.\vec{J_d} = \vec{\nabla}.\frac{\partial \vec{D}}{\partial t}$$

$$\vec{J_d} = \frac{\partial \vec{D}}{\partial t} \qquad ------(18)$$

Substituting (18) in (11) we get

$$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \qquad ------(19)$$

This is the Maxwell's fourth equation in differential form

2.2. ELECTROMAGNETIC WAVE EQUATIONS

The general electromagnetic wave equations can be derived from Maxwell's equations.

Maxwell's equations are

$$\vec{\nabla}.\vec{D} = \rho \qquad ------(1)$$

$$\vec{\nabla}.\vec{B} = 0 \qquad ------(2)$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad ------(3)$$

$$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \qquad ------(4)$$

Let us consider a homogeneous linear medium, of permittivity ϵ, permeability μ, conductivity σ and charge density $\rho = 0$

We know

$$\vec{J} = \sigma\vec{E}$$

$$\vec{D} = \epsilon\vec{E}$$

$$\vec{B} = \mu\vec{H}\ ;\ \ \vec{H} = \frac{\vec{B}}{\mu}$$

$$\rho = 0$$

substituting these values in the above equations, we get

$$\vec{\nabla}.\epsilon\vec{E} = 0$$

$$\epsilon\vec{\nabla}.\vec{E} = 0$$

$$\epsilon \neq 0$$

$$so \qquad \vec{\nabla}.\vec{E} = 0 \qquad ------(5)$$

$$\vec{\nabla}.\vec{B} = 0 \qquad ------(6)$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad ------(7)$$

$$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

$$\vec{\nabla} \times \frac{\vec{B}}{\mu} = \sigma\vec{E} + \frac{\partial \epsilon\vec{E}}{\partial t}$$

$$\vec{\nabla} \times \vec{B} = \mu\sigma\vec{E} + \mu\epsilon\frac{\partial \vec{E}}{\partial t} \qquad ------(8)$$

Wave equations in terms of electric field

Consider equation (7)

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

Take curlon both sides, we get

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = \vec{\nabla} \times \left(-\frac{\partial \vec{B}}{\partial t}\right) \qquad ------(9)$$

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = -\frac{\partial}{\partial t}(\vec{\nabla} \times \vec{B}) \qquad -------(10)$$

Substituting (8) in (10) we get

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = -\frac{\partial}{\partial t}\left(\sigma\mu\vec{E} + \epsilon\mu\frac{\partial \vec{E}}{\partial t}\right)$$

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = -\sigma\mu\frac{\partial \vec{E}}{\partial t} - \epsilon\mu\frac{\partial^2 \vec{E}}{\partial t^2} \qquad -------(11)$$

Using vector identity on the L.H.S

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = \vec{\nabla}.(\vec{\nabla}.\vec{E}) - \vec{\nabla}^2\vec{E} \qquad -----(12)$$

Substituting (12) in (11) we can write

$$\vec{\nabla}.(\vec{\nabla}.\vec{E}) - \vec{\nabla}^2\vec{E} = -\sigma\mu\frac{\partial \vec{E}}{\partial t} - \epsilon\mu\frac{\partial^2 \vec{E}}{\partial t^2} \qquad -----(13)$$

From (5) $\quad \vec{\nabla}.\vec{E} = 0$

So the above equation becomes

$$-\vec{\nabla}^2\vec{E} = -\sigma\mu\frac{\partial \vec{E}}{\partial t} - \epsilon\mu\frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\vec{\nabla}^2\vec{E} = \sigma\mu\frac{\partial \vec{E}}{\partial t} + \epsilon\mu\frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\vec{\nabla}^2\vec{E} - \sigma\mu\frac{\partial \vec{E}}{\partial t} - \epsilon\mu\frac{\partial^2 \vec{E}}{\partial t^2} = 0 \qquad -----(14)$$

This is the wave equation in terms of electric field.

Wave equations in terms of magnetic field

Consider equation (8)

$$\vec{\nabla} \times \vec{B} = \mu\sigma\vec{E} + \mu\epsilon\frac{\partial \vec{E}}{\partial t}$$

Take curl on both sides, we get

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{B}) = \vec{\nabla} \times \left(\mu\sigma\vec{E} + \mu\epsilon\frac{\partial \vec{E}}{\partial t}\right)$$

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{B}) = \vec{\nabla} \times \mu\sigma\vec{E} + \vec{\nabla} \times \left(\mu\epsilon\frac{\partial \vec{E}}{\partial t}\right)$$

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{B}) = \mu\sigma(\vec{\nabla} \times \vec{E}) + \mu\epsilon\,\frac{\partial(\vec{\nabla} \times \vec{E})}{\partial t} \qquad ------(15)$$

Substituting (7) in (15)

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{B}) = \mu\sigma\left(-\frac{\partial \vec{B}}{\partial t}\right) + \mu\epsilon\frac{\partial}{\partial t}\left(-\frac{\partial \vec{B}}{\partial t}\right)$$

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{B}) = -\mu\sigma \frac{\partial \vec{B}}{\partial t} - \mu\epsilon \frac{\partial^2 \vec{B}}{\partial t^2} \qquad ------(16)$$

Using vector identity on the L.H.S

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{B}) = \vec{\nabla}.(\vec{\nabla}.\vec{B}) - \vec{\nabla}^2 \vec{B} \qquad ------(17)$$

Substituting (17) in (16) we can write

$$\vec{\nabla}.(\vec{\nabla}.\vec{B}) - \vec{\nabla}^2 \vec{B} = -\mu\sigma \frac{\partial \vec{B}}{\partial t} - \mu\epsilon \frac{\partial^2 \vec{B}}{\partial t^2}$$

From (6) $\quad \vec{\nabla}.\vec{B} = 0$

So the above equation becomes

$$-\vec{\nabla}^2 \vec{B} = -\mu\sigma \frac{\partial \vec{B}}{\partial t} - \mu\epsilon \frac{\partial^2 \vec{B}}{\partial t^2}$$

$$\vec{\nabla}^2 \vec{B} = \mu\sigma \frac{\partial \vec{B}}{\partial t} + \mu\epsilon \frac{\partial^2 \vec{B}}{\partial t^2}$$

$$\vec{\nabla}^2 \vec{B} - \mu\sigma \frac{\partial \vec{B}}{\partial t} - \mu\epsilon \frac{\partial^2 \vec{B}}{\partial t^2} = 0$$

This is the wave equation in terms of magnetic field.

2.3 PLANE PROGRESSIVE WAVE EQUATIONS IN VACUUM OR FREE SPACE:

If the electromagnetic wave field vector is constant over any plane perpendicular to the directio of wave propagation at any instant then this wave is called plane progressive wave.

Consider a plane electromagnetic wave vacuum or free space. Let permittivity ϵ_0, permeabilit μ_0 are constants. And Conductivity, $\sigma = 0$

Maxwell's equations for free space can be written as

$$\vec{\nabla}.\vec{E} = 0 \qquad ------(1)$$

$$\vec{\nabla}.\vec{B} = 0 \qquad ------(2)$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad ------(3)$$

$$\vec{\nabla} \times \vec{B} = \epsilon_0\mu_0 \frac{\partial \vec{E}}{\partial t} \qquad ------(4)$$

Plane electromagnetic wave equations in terms of electric field

Consider equation (3)

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

Take curl on both sides, we get

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = \vec{\nabla} \times \left(-\frac{\partial \vec{B}}{\partial t}\right)$$

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = -\frac{\partial (\vec{\nabla} \times \vec{B})}{\partial t} \qquad ------(5)$$

Substituting (4) in (5)

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = -\frac{\partial}{\partial t}\left(\epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}\right)$$

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = -\epsilon_0 \mu_0 \frac{\partial^2 \vec{E}}{\partial t^2} \qquad ------(6)$$

Using vector idednity on the L.H.S

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = \vec{\nabla}.(\vec{\nabla}.\vec{E}) - \vec{\nabla}^2 \vec{E} \qquad ------(7)$$

Substituting (7) in (6) we can write

$$\vec{\nabla}.(\vec{\nabla}.\vec{E}) - \vec{\nabla}^2 \vec{E} = -\epsilon_0 \mu_0 \frac{\partial^2 \vec{E}}{\partial t^2} \qquad ------(8)$$

From (5) $\quad \vec{\nabla}.\vec{E} = 0$

So the above equation becomes

$$-\vec{\nabla}^2 \vec{E} = -\epsilon_0 \mu_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\vec{\nabla}^2 \vec{E} = \epsilon_0 \mu_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\vec{\nabla}^2 \vec{E} - \epsilon_0 \mu_0 \frac{\partial^2 \vec{E}}{\partial t^2} = 0$$

$$\vec{\nabla}^2 \vec{E} - \mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} = 0 \qquad ------(9)$$

This is the plane electromagnetic wave equation in terms of electric field.

Plane electromagnetic wave equations in terms of magnetic field

Consider equation (4)

$$\vec{\nabla} \times \vec{B} = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

Take curl on both sides, we get

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{B}) = \vec{\nabla} \times \left(\mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}\right)$$

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{B}) = \mu_0 \epsilon_0 \frac{\partial}{\partial t}(\vec{\nabla} \times \vec{E}) \qquad ------(10)$$

Substituting (3) in (10) we can write

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{B}) = \mu_0 \epsilon_0 \frac{\partial}{\partial t}\left(-\frac{\partial \vec{B}}{\partial t}\right)$$

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{B}) = -\mu_0 \epsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2}$$

Using vector idednity on the L.H.S

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{B}) = \vec{\nabla}.(\vec{\nabla}.\vec{B}) - \vec{\nabla}^2\vec{B} \qquad ------(11)$$

Substituting (11) in (10) we can write

$$\vec{\nabla}.(\vec{\nabla}.\vec{B}) - \vec{\nabla}^2\vec{B} = -\mu_0\epsilon_0\frac{\partial^2\vec{B}}{\partial t^2}$$

From (6) $\quad \vec{\nabla}.\vec{B} = 0$

So the above equation becomes

$$-\vec{\nabla}^2\vec{B} = -\mu_0\epsilon_0\frac{\partial^2\vec{B}}{\partial t^2}$$

$$\vec{\nabla}^2\vec{B} = \mu_0\epsilon_0\frac{\partial^2\vec{B}}{\partial t^2}$$

$$\vec{\nabla}^2\vec{B} - \mu_0\epsilon_0\frac{\partial^2\vec{B}}{\partial t^2} = 0 \qquad -------(12)$$

This is the plane electromagnetic wave equation in terms of magnetic field.

2.4 PROPERTIES OF ELECTROMAGNETIC WAVES - SPEED, ORIENTATION, PHASE AND AMPLITUDE IN MATTER (dielectric medium)

(I)Speed of electromagnetic waves in matter (dielectric medium):

The general equations of electromagnetic waves in terms of electric field and magnetic field vector is given as

for electric field $\quad \vec{\nabla}^2\vec{E} - \mu\epsilon\frac{\partial^2\vec{E}}{\partial t^2} = 0 \qquad -----(1)$

for magnetic field $\quad \vec{\nabla}^2\vec{B} - \mu\epsilon\frac{\partial^2\vec{B}}{\partial t^2} = 0 \qquad ------(2)$

In common we can write the above equations as

$$\vec{\nabla}^2\vec{f} - \mu\epsilon\frac{\partial^2\vec{f}}{\partial t^2} = 0 \qquad ------(3)$$

Where $\vec{f}$ is the scalar wave function which represents the electric and magnetic field components (i.e E_x, E_y, E_z, and B_x, B_y, B_z)

The standard form of the wave equation is

$$\vec{\nabla}^2\vec{\psi} - \frac{1}{v^2}\frac{\partial^2\vec{\psi}}{\partial t^2} = 0 \qquad -----(4)$$

Comparing equations (3) and (4) we can write

$$\frac{1}{v^2} = \mu\epsilon$$

$$\frac{1}{\mu\epsilon} = v^2$$

$$\frac{1}{\sqrt{\mu\epsilon}} = v$$

Substituting the values of μ and ϵ as

$$\mu = \mu_0\mu_r$$
$$\epsilon = \epsilon_0\epsilon_r$$
$$\frac{1}{\sqrt{\mu_0\mu_r\epsilon_0\epsilon_r}} = v$$

But we know

$$\frac{1}{\sqrt{\mu_0\epsilon_0}} = c$$
$$\frac{c}{\sqrt{\mu_r\epsilon_r}} = v \qquad ------(5)$$

From the above equation we can see that the speed of electromagnetic waves is less than the speed of light.

Substituting the values of μ and ϵ in equation (1) and (2)

for electric field $$\vec{\nabla}^2\vec{E} - \mu_r\epsilon_r\mu_0\epsilon_0\frac{\partial^2\vec{E}}{\partial t^2} = 0 \qquad ------(6)$$

for magnetic field $$\vec{\nabla}^2\vec{B} - \mu_r\epsilon_r\mu_0\epsilon_0\frac{\partial^2\vec{B}}{\partial t^2} = 0 \qquad ------(7)$$

But we know

$$\frac{1}{\sqrt{\mu_0\epsilon_0}} = c$$

squaring we get $$\frac{1}{\mu_0\epsilon_0} = c^2 \qquad ------(8)$$

Substituting (8) in (6) and (7) we get

for electric field $$\vec{\nabla}^2\vec{E} - \frac{\mu_r\epsilon_r}{c^2}\frac{\partial^2\vec{E}}{\partial t^2} = 0 \qquad ------(9)$$

for magnetic field $$\vec{\nabla}^2\vec{B} - \frac{\mu_r\epsilon_r}{c^2}\frac{\partial^2\vec{B}}{\partial t^2} = 0 \qquad ------(10)$$

The above equations represent the wave equations of electric and magnetic field vectors in terms of the speed.

(ii)Orientation of electromagnnetic waves in matter (dielectric medium):

The solution of equations (9) and (10) can be written as

$$\vec{E}(r,t) = \vec{E_0}e^{i(K.r-\omega t)}$$
$$\vec{B}(r,t) = \vec{B_0}e^{i(K.r-\omega t)}$$

Where E_0 and B_0 are the amplitude of electric and magnetic fields.

K is the wave vector

Maxwell's equations in a dielectric medium are

$$\vec{\nabla}.\vec{E} = 0 \qquad ------(11)$$
$$\vec{\nabla}.\vec{B} = 0 \qquad ------(12)$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad ------(13)$$

$$\vec{\nabla} \times \vec{B} = \mu\epsilon \frac{\partial \vec{E}}{\partial t} \qquad ------(14)$$

The del operator $\vec{\nabla}$ is equivalent to $i\vec{k}$ and $\frac{\partial}{\partial t}$ is equivalent to $-i\omega$

i.e $\qquad \vec{\nabla} = i\vec{k} \qquad and \qquad \frac{\partial \vec{E}}{\partial t} = -i\omega$

so equation (11) becomes

$$\vec{\nabla}.\vec{E} = 0$$

$$i\vec{k}.\vec{E} = 0$$

$$i \neq 0$$

$$so \qquad \vec{k}.\vec{E} = 0 \qquad ------(15)$$

Similarly equation (12) becomes $\quad \vec{\nabla}.\vec{B} = 0$

$$i\vec{k}.\vec{B} = 0$$

$$i \neq 0$$

$$so \qquad \vec{k}.\vec{B} = 0 \qquad ------(16)$$

equation (13) becomes

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$i\vec{k} \times \vec{E} = -(-i\omega)\vec{B}$$

$$i\vec{k} \times \vec{E} = i\omega\vec{B}$$

$$\vec{k} \times \vec{E} = \omega\vec{B} \qquad ------(17)$$

equation (14) becomes

$$\vec{\nabla} \times \vec{B} = \mu\epsilon \frac{\partial \vec{E}}{\partial t}$$

$$i\vec{k} \times \vec{B} = \mu\epsilon(-i\omega)\vec{E}$$

$$i\vec{k} \times \vec{B} = -i(\omega\mu\epsilon)\vec{E}$$

$$\vec{k} \times \vec{B} = -(\omega\mu\epsilon)\vec{E} \qquad ------(18)$$

From equation (15) and (16) we can say, the wave vector $\vec{k}$ is perpendicular to electric and magnetic fields respectively.

From (17) we can say the magnetic field vector $\vec{B}$ is perpendicular to wave vector $\vec{k}$ and electric field vector $\vec{E}$.

From (18) we can say the electric field vector $\vec{E}$ is perpendicular to wave vector $\vec{k}$ and magnetic field vector $\vec{B}$

So $\vec{E}$, $\vec{E}$and $\vec{E}$ are mutually perpendicular to each other.

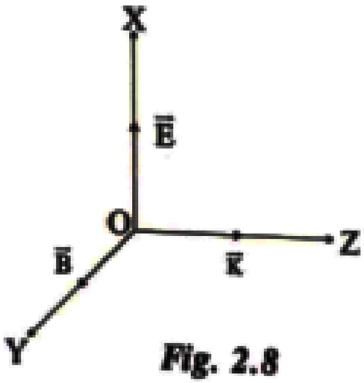


Fig. 2.8

(iii)Amplitude and phase of electromagnetic waves in matter (dielectric medium):

From equation (17)

$$\vec{k} \times \vec{E} = \omega \vec{B}$$

$$\vec{k} = k\hat{n}$$

$$k\hat{n} \times \vec{E} = \omega \vec{B}$$

$$\hat{n} \times \vec{E} = \frac{\omega}{k} \vec{B}$$

For dielectrics, velocity

$$v = \frac{\omega}{k}$$

So the above equation becomes

$$\hat{n} \times \vec{E} = v\vec{B}$$

$$\vec{B} = \frac{1}{v} \hat{n} \times \vec{E}$$

The magnitude of the magnetic field vector can be written as

$$|\vec{B}| = \frac{1}{v} |\vec{E}|$$

or $$B = \frac{1}{v} E$$

since $B = \mu H$, the above equation becomes

$$\mu H = \frac{1}{v} E$$

$$\mu v = \frac{E}{H} \qquad ------(19)$$

But the velocity

$$v = \frac{1}{\sqrt{\varepsilon \mu}}$$

$$\frac{E}{H} = \frac{\mu}{\sqrt{\varepsilon\mu}}$$

$$\frac{E}{H} = \sqrt{\frac{\mu}{\varepsilon}} \qquad ------(20)$$

Substituting the values of μ and ε in (20)

$$\mu = \mu_0\mu_r$$

$$\epsilon = \epsilon_0\epsilon_r$$

We get

$$\frac{E}{H} = \eta = \sqrt{\frac{\mu_0\mu_r}{\epsilon_0\epsilon_r}} \qquad ------(21)$$

Where η is the intrinsic impedence in vacuum.

We know

$$\sqrt{\frac{\mu_0}{\epsilon_0}} = \eta_0 \text{ where } \eta_0 \text{ is the intrinsic impedence in matter.}$$

Substituting η_0 in (21) we get

$$\frac{E}{H} = \eta = \eta_0\sqrt{\frac{\mu_r}{\epsilon_r}}$$

We have

$$\frac{E}{H} = \eta \qquad ------(22)$$

$$E = \eta H$$

i.e, the electric field vector is η times the magnetic field vector

or $\quad H = \dfrac{E}{\eta}$

i.e, the magnetic field vector is $\frac{1}{\eta}$ times the electric field vector.

From equation (22) we can say that electric and magnetic field vectors are in the same phase.

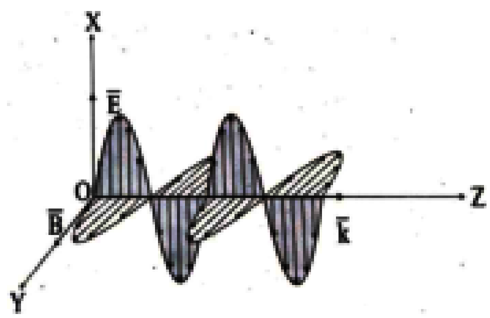


Fig. 2.9 Electromagnetic wave in dielectric

Summary:

- EM waves in dielectric medium travel with a speed less than the speed of light.
- EM wave field vectors $\vec{E}$ and $\vec{B}$ are perpendicular to each other and is also perpendicular to the direction of propagation of the wave. Hence it is confirmed that EM waves are transverse in nature.

The electric and magnetic field vectors are in the same phase.

2.5 POLARIZATION:

Polarization refers to the relationship between the orientation of electric and magnetic field vectors and the direction of propagation of the EM waves.

Polarization vectors:

Consider an EM wave propagating along the Z-axis. This means the electric and magnetic field vectors does not move along X and Y axes.

The plane wave solution of electric and magnetic field vectors are

$$\vec{E}(z,t) = \vec{E}_0 e^{i(K_z z - \omega t)} \quad ------ (1)$$

$$\vec{B}(z,t) = \vec{B}_0 e^{i(K_z z - \omega t)} \quad ------ (2)$$

Where $\vec{E_0}$ is the polarization vector of electric field and $\vec{B_0}$ is the polarization vector of magnetic field.

Since EM wave is propagating along z-axis, the component of electric and magnetic field polarization vectors are zero.

So we can write

$$\vec{E}_0 = \hat{i}\vec{E}_{ox} + \hat{j}\vec{E}_{oy} \quad ------ (3)$$

$$\vec{B}_0 = \hat{i}\vec{B}_{ox} + \hat{j}\vec{B}_{oy} \quad ------ (4)$$

Now the plane wave solution of electric and magnetic field vectors becomes

$$\vec{E}(z,t) = [\hat{i}\vec{E}_{ox} + \hat{j}\vec{E}_{oy}]e^{i(K_z z - \omega t)}$$

$$\vec{E}(z,t) = \hat{i}\vec{E}_{ox} e^{i(K_z z - \omega t)} + \hat{j}\vec{E}_{oy} e^{i(K_z z - \omega t)} \quad ------ (5)$$

similarly

$$\vec{B}(z,t) = [\hat{i}\vec{B}_{ox} + \hat{j}\vec{B}_{oy}]e^{i(K_z z - \omega t)}$$

$$\vec{B}(z,t) = \hat{i}\vec{B}_{ox} e^{i(K_z z - \omega t)} + \hat{j}\vec{B}_{oy} e^{i(K_z z - \omega t)} \quad ------ (6)$$

From equations (5) and (6) we say that the plane YZ is the plane of polarisation and the plane XZ is the plane of vibration.

Based on the values of $\vec{E}_{ox}, \vec{E}_{oy}, \vec{B}_{ox}$ *and* $\vec{B}_{oy}$ and based on the phases of electric and magnetic field vectors, we can predict whether the wave is plane polarised or circularly polarised or elliptically polarised.

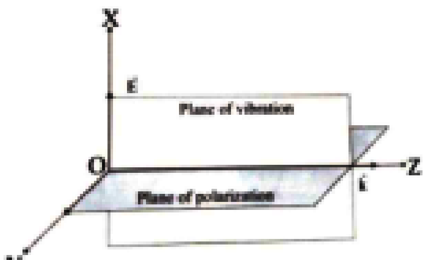


Fig. 2.10

Plane polarised wave:

If $\vec{E}_{ox}$ *and* $\vec{E}_{oy}$ are real numbers and if the electric and magnetic field vectors are in the same phase, then it is plane polarized wave.

Case (i)

If of $\vec{E}_{ox} \neq 0$ *and* , $\vec{E}_{oy} = 0$ then equation (5) becomes

$$\vec{E}(z,t) = \hat{i}\vec{E}_{ox} e^{i(K_z z - \omega t)} + 0$$

$$\vec{E}(z,t) = \hat{i}\vec{E}_{ox} e^{i(K_z z - \omega t)}$$

Now the wave is said to be polarized in the x direction

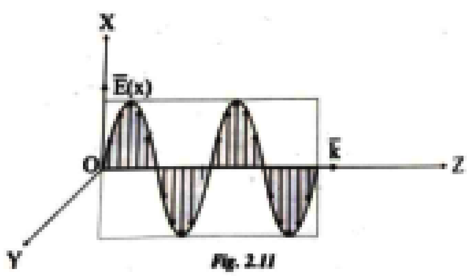


Fig. 2.11

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Case (ii)

If of $\vec{E}_{oy} \neq 0$ *and* , $\vec{E}_{ox} = 0$ then equation (5) becomes

$$\vec{E}(z,t) = 0 + \hat{j}\vec{E}_{oy}e^{i(K_z.z-\omega t)}$$

$$\vec{E}(z,t) = \hat{j}\vec{E}_{oy}e^{i(K_z.z-\omega t)}$$

Now the wave is said to be polarized in the y direction

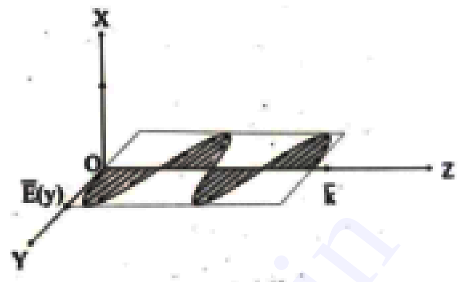

Fig. 2.12

Case (iii)

If of $\vec{E}_{ox} \neq 0$ *and* , $\vec{E}_{oy} \neq 0$ then equation (5) becomes

$$\vec{E}(z,t) = \hat{i}\vec{E}_{ox}e^{i(K_z.z-\omega t)} + \hat{j}\vec{E}_{oy}e^{i(K_z.z-\omega t)}$$

$$\vec{E}(z,t) = \hat{i}\vec{E}_{ox}e^{i(K_z.z-\omega t)} + \hat{j}\vec{E}_{oy}e^{i(K_z.z-\omega t)}$$

Now the wave is said to be linearly polarized.

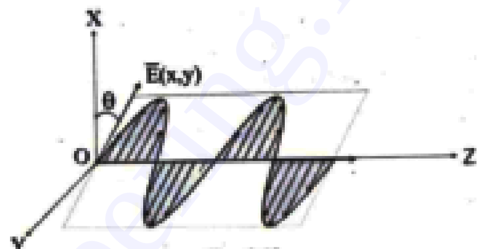

Fig. 2.13

Circularly and elliptically polarised wave:

If $\vec{E}_{ox}$ *and* $\vec{E}_{oy}$ are equal and if the electric and magnetic field vectors are not in the same phas and if the phase difference is $\pm\frac{\pi}{2}$, then it is circularly or elliptically polarized wave.

If $\vec{E}_{ox} = \vec{E}_{oy} = \vec{E}_0^1$ then

$$\vec{E}(z,t) = \hat{i}\vec{E}_0^1 e^{i(K_z.z-\omega t)} + \hat{j}\vec{E}_0^1 e^{i\left(K_z.z-\omega t \pm \frac{\pi}{2}\right)}$$

$$\vec{E}(z,t) = \hat{i}\vec{E}_0^1 e^{i(K_z.z-\omega t)} + \hat{j}\vec{E}_0^1 e^{i(K_z.z-\omega t)}e^{i\left(\pm\frac{\pi}{2}\right)}$$

$$e^{i\left(\pm\frac{\pi}{2}\right)} = \pm i$$

So the above equation becomes

$$\vec{E}(z,t) = \hat{i}\vec{E}_0^1 e^{i(K_z.z-\omega t)} + \hat{j}\vec{E}_0^1 e^{i(K_z.z-\omega t)}(\pm i)$$

$$\vec{E}(z,t) = \hat{i}\vec{E}_0^1 e^{i(K_z.z-\omega t)} \pm i\hat{j}\vec{E}_0^1 e^{i(K_z.z-\omega t)}$$

$$\vec{E}(z,t) = (\hat{i} \pm i\hat{j})\vec{E}_0^1 e^{i(K_z.z-\omega t)}$$

Now the wave is said to be circularly polarized.

If $\vec{E}_{ox} \neq \vec{E}_{oy}$ then

$$\vec{E}(z,t) = \hat{i}\vec{E}_{ox}e^{i(K_z.z-\omega t)} + \hat{j}\vec{E}_{oy}e^{i\left(K_z.z-\omega t \pm \frac{\pi}{2}\right)}$$

$$\vec{E}(z,t) = \hat{i}\vec{E}_{ox}e^{i(K_z.z-\omega t)} + \hat{j}\vec{E}_{oy}e^{i(K_z.z-\omega t)}e^{i\left(\pm\frac{\pi}{2}\right)}$$

$$e^{i\left(\pm\frac{\pi}{2}\right)} = \pm i$$

So the above equation becomes

$$\vec{E}(z,t) = \hat{i}\vec{E}_{ox}e^{i(K_z.z-\omega t)} + \hat{j}\vec{E}_{oy}e^{i(K_z.z-\omega t)}(\pm i)$$

$$\vec{E}(z,t) = \hat{i}\vec{E}_{ox}e^{i(K_z.z-\omega t)} \pm i\hat{j}\vec{E}_{oy}e^{i(K_z.z-\omega t)}$$

$$\vec{E}(z,t) = \left(\hat{i}\vec{E}_{ox} \pm i\hat{j}\vec{E}_{oy}\right)e^{i(K_z.z-\omega t)}$$

Now the wave is said to be elliptically polarized

2.6 PRODUCING ELECTRO MAGNETIC WAVES:

We know a stationary charged particle produces an electric field and exerts force on nearby charged particles.

If a charged particle is moving, it can produce magnetic field and exerts force on nearby moving charged particle.

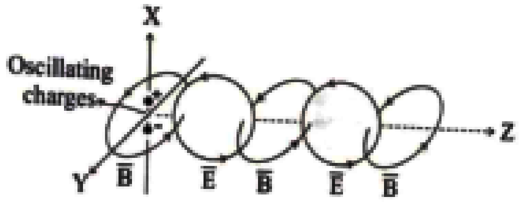

Fig.2.14

If a charged particle is oscillating w.r.t an equilibrium position, then it is known as accelerating charged particle.

An accelerating charged particle produce EM waves of its own frequency 'f'.

The wavelength λ of EM wave is given by

$$\lambda = \frac{c}{f}$$

2.7 ENERGY AND MOMENTUM OF EM WAVES:

Total energy of EM waves is the sum of time average of energy content due to electric field and magnetic field.

Energy content due to electric field U_E:

Energy density due to electric field

$$U_E = \frac{1}{2}\epsilon_0 \vec{E}^2$$

Substitute $\vec{E} = \vec{E}_0 \sin(kz - \omega t)$

$$U_E = \frac{1}{2}\epsilon_0 \vec{E}_0^{\,2} \sin^2(kz - \omega t)$$

Time average of energy density

$$U_E = \frac{1}{T}\int_0^T \frac{1}{2}\epsilon_0 \vec{E}_0^{\,2} \sin^2(kz - \omega t)dt$$

$$U_E = \frac{1}{2}\epsilon_0 \vec{E}_0^{\,2} \frac{1}{T}\int_0^T \sin^2(kz - \omega t)dt$$

Since the value of $\frac{1}{T}\int_0^T \sin^2(kz - \omega t)dt = \frac{1}{2}$, the above equation becomes

$$U_E = \frac{1}{2}\epsilon_0 \vec{E}_0^{\,2} \frac{1}{2}$$

i.e $$U_E = \frac{1}{4}\epsilon_0 \vec{E}_0^{\,2}$$

The above equation represents the energy content in electromagnetic waves due to electric field.

Energy content due to magnetic field U_B:

Energy density due to magnetic field

$$U_B = \frac{1}{2\mu_0}\vec{B}^2$$

Substitute $\vec{B} = \vec{B}_0 \sin(kz - \omega t)$

$$U_B = \frac{1}{2\mu_0}\vec{B_0}^2 \sin^2(kz - \omega t)$$

Time average of energy density

$$U_B = \frac{1}{T}\int_0^T \frac{1}{2\mu_0}\vec{B_0}^2 \sin^2(kz - \omega t)dt$$

$$U_B = \frac{1}{2\mu_0}\vec{B_0}^2 \frac{1}{T}\int_0^T \sin^2(kz - \omega t)dt$$

Since the value of $\frac{1}{T}\int_0^T \sin^2(kz - \omega t)dt = \frac{1}{2}$, the above equation becomes

$$U_B = \frac{1}{2\mu_0}\vec{B_0}^2 \frac{1}{2}$$

i.e $$U_B = \frac{1}{4\mu_0}\vec{B_0}^2$$

but $$\vec{B_o} = \frac{\vec{E_o}}{c}$$

$$U_B = \frac{1}{4\mu_0}\left(\frac{\vec{E_o}}{c}\right)^2$$

$$U_B = \frac{1}{4\mu_0}\frac{\vec{E_o}^2}{c^2}$$

But we know

$$\frac{1}{c^2} = \mu_0\epsilon_0$$

So the above equation becomes

$$U_B = \frac{1}{4\mu_0}\vec{E_o}^2 \mu_0\epsilon_0$$

$$U_B = \frac{1}{4}\epsilon_0\vec{E_o}^2$$

The above equation represents the energy content in electromagnetic waves due to magnetic field.

Total energy $$U = U_E + U_B$$

$$U = \frac{1}{4}\epsilon_0\vec{E_o}^2 + \frac{1}{4}\epsilon_0\vec{E_o}^2$$

$$U = \frac{1}{2}\epsilon_0\vec{E_o}^2$$

Intensity of electromagnetic waves:

The magnitude of the time average of pointing vector is called intensity of electromagnetic waves.

The pointing vector

$$\vec{S} = (\vec{E} \times \vec{H})$$

$$(\vec{E} \times \vec{H}) = |\vec{E}||\vec{H}|sin\theta\hat{n}$$

$$\vec{S} = |\vec{E}||\vec{H}|sin\theta\hat{n}$$

$\vec{E}$ *and* $\vec{H}$ are normal to each other. Therefore $\theta = 90^0$,and since $sin90^0 = 1$, we can write

$$\vec{S} = |\vec{E}||\vec{H}|\hat{n} \qquad ------(1)$$

We know $\quad \vec{B} = \mu_0\vec{H}$

or $\quad \vec{H} = \frac{\vec{B}}{\mu_0}$

Substituting in equation (1) we get

$$\vec{S} = |\vec{E}|\frac{\vec{B}}{\mu_0}\hat{n} \qquad -----(2)$$

We can write the solution of the EM wave in the sine form as

$$\vec{E} = \vec{E}_0 sin\,(k.r - \omega t) \qquad -----(3)$$

$$\vec{B} = \vec{B}_0 sin\,(k.r - \omega t) \qquad -----(4)$$

Substituting (3) and (4) in (2) we get

$$\vec{S} = \vec{E}_0 sin\,(k.r - \omega t)\frac{1}{\mu_0}\vec{B}_0 sin\,(k.r - \omega t)\hat{n}$$

but $\quad \vec{B}_0 = \frac{\vec{E}_0}{c}$

Substituting in the above equation we get

$$\vec{S} = \frac{1}{\mu_0}\frac{\vec{E}_0^{\,2}}{c}\hat{n}sin^2\,(k.r - \omega t)$$

The time average of the pointing vector can be written as

$$\vec{S}_{time\ avg} = \frac{1}{T}\int_0^T \frac{1}{\mu_0}\frac{\vec{E}_0^{\,2}}{c}\hat{n}sin^2\,(k.r - \omega t)\,dt$$

since $\quad \frac{1}{T}\int_0^T sin^2\,(k.r - \omega t)\,dt = \frac{1}{2}$

$$\vec{S}_{time\ avg} = \frac{1}{\mu_0}\frac{\vec{E}_0^{\,2}}{c}\hat{n}\frac{1}{2}$$

$$\vec{S}_{time\ avg} = \frac{\vec{E}_0^{\,2}}{2\mu_0 c}\hat{n}$$

Substituting for

$$\frac{1}{c} = \sqrt{\epsilon_0 \mu_0}$$

$$\vec{S}_{time\ avg} = \frac{\vec{E}_0^{\ 2}\sqrt{\epsilon_0 \mu_0}}{2\mu_0}\hat{n}$$

$$\vec{S}_{time\ avg} = \frac{\vec{E}_0^{\ 2}\sqrt{\epsilon_0}}{2\sqrt{\mu_0}}\hat{n}$$

$Intensity\ I\ or\ magnitude\ of\ \vec{S}_{time\ avg} = |\vec{S}_{time\ avg}|$

$$I = \frac{\vec{E}_0^{\ 2}}{2}\sqrt{\frac{\epsilon_0}{\mu_0}}$$

$since \qquad \eta_0 = \sqrt{\frac{\mu_0}{\epsilon_0}}$

$$I = \frac{\vec{E}_0^{\ 2}}{2\eta_0}$$

This equation represents the intensity of EM waves.

Momentum of EM waves:

Momentum of EM waves can be found in terms of

- Energy
- Pointing vector

Momentum in terms of energy

We know

$$E = mc^2 \qquad ------(1)$$

Similarly the energy 'u' in terms of the effective mass of electromagnetic radiation can be written as

$$u = mc^2$$

$$m = \frac{u}{c^2} \qquad ------(2)$$

The momentum of the particle of mass 'm' and velocity 'v' is

$$\vec{p} = mv \qquad ------(3)$$

Substituting (2) in (3) we get

$$\vec{p} = \frac{u}{c^2}v \qquad ------(4)$$

If the EM wave is travelling along z-axis with velocity 'c' it is represented as

$$c = c\hat{k}$$

Substituting for 'v' in (4) we get

$$\vec{p} = \frac{u}{c^2}c\hat{k}$$

$$\vec{p} = \frac{u}{c^2} c\hat{k}$$

$$\vec{p} = \frac{u}{c}\hat{k} \qquad ------(5)$$

Magnitude of the momentum is

$$\vec{p} = \frac{u}{c}$$

This represents the momentum of EM waves in terms of energy 'u'.

Momentum in terms of pointing vector:

The pointing vector

$$\vec{S} = (\vec{E} \times \vec{H}) = uc\hat{k}$$

$$u\hat{k} = \frac{\vec{S}}{c} \qquad ------(6)$$

Substituting (6) in (5) we get

$$\vec{p} = \frac{1}{c}\frac{\vec{S}}{c}$$

$$\vec{p} = \frac{\vec{S}}{c^2}$$

$$since \qquad c^2 = \frac{1}{\epsilon_0 \mu_0}$$

We can write the above equation as

$$\vec{p} = \epsilon_0 \mu_0 \vec{S}$$

$$\vec{p} = \epsilon_0 \mu_0 (\vec{E} \times \vec{H})$$

The above equation represents the momentum of EM waves in terms of pointing vector.

Radiation pressure of electromagnetic waves:

When an EM waves strikes the surface, it exerts a force on the surface due to the change in momentum. The amount of force exerted per unit area on the surface due to the force is called radiation pressure.

We know the momentum

$$\vec{p} = \frac{u}{c}\hat{k}$$

The magnitude of momentum

$$\vec{p} = \frac{u}{c} \qquad ------(1)$$

Similarly we know the pointing vector

$$\vec{S} = uc\hat{k}$$

The magnitude of the pointing vector is

$$\vec{S} = uc \qquad ------(2)$$

According to pointing theorem, the electromagnetic energy passing normal to the surface per unit area and unit time is given by

$$\vec{S} = \frac{u}{At} \qquad ------(3)$$

Where 'A' is the area and 't' is the time.

Comparing equation (2) and (3) we get

$$uc = \frac{u}{At}$$

$$c = \frac{1}{At}$$

$$At = \frac{1}{c} \qquad ------(4)$$

Substituting (4) in (1) we get

$$\vec{p} = uAt \qquad ------(5)$$

According to Newton's law, the force 'F' acting on the surface is given by

$$F = \frac{p}{t} \qquad ------(6)$$

Substituting (5) in (6) we get,

$$F = \frac{uAt}{t}$$

$$F = uA \qquad ------(7)$$

We know the radiation pressure P_{rad} exerted on the surface is given by

$$P_{rad} = \frac{F}{A} \qquad ------(8)$$

Substituting (7) in (8) we get,

$$P_{rad} = \frac{uA}{A}$$

$$i.e \quad P_{rad} = u$$

Thus the radiation pressure of EM wave is equal to the energy of the striking EM wave.

2.8 CELL PHONE RECEPTION:

One of the important applications of electromagnetic spectrum is cell phone communication. The communication from one cell to another phone is done through radio waves.

Transmission and reception unit:

Cell phone is a two way communicating radio, consisting of a radio wave transmitter and a radio wave receiver. Cell phones contain at least one radio antenna in order to transmit and receive radio signals.

When an antenna converts an electric signal into radio waves, it acts as a transmitter and when it converts the radio waves into an electric signal, it then acts as a receiver.

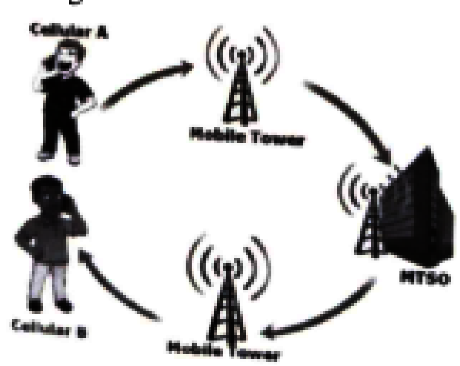

Fig. 2.15

Reception mechanism:

When a call is made on the cell phone, it converts input voice into an electrical signal, which is transmitted through radio waves to the nearest mobile tower.

Radio waves transport digitized voice or data in the form of oscillating electric and magnetic fields. Cell phones transmit radio waves in all directions and it carries the information and travel with the speed of light in air.

The network of mobile towers then relays the radio waves to destination tower through Mobile Telephone Switching Office (MTSO) and in turn to the other cell phone, there in the electrical signal is converted back to sound again.

2.9 REFLECTION AND TRANSMISSION OF ELECTROMAGNETIC WAVES FROM A NON-CONDUCTING MEDIUM-VACUUM INTERFACE FOR NORMAL INCIDENCE:

Let us consider an electromagnetic wave which travels from a non-conducting medium to vacuum.

At the interface of the two medium, one part of the incident wave is reflected into the same medium and the part is transmitted into the next medium.

We know that the non-conducting medium and vacuum will have different electrical permittivity $\epsilon_0 and\ \epsilon_r$ and magnetic permeability $\mu_0 and\ \mu_r$.

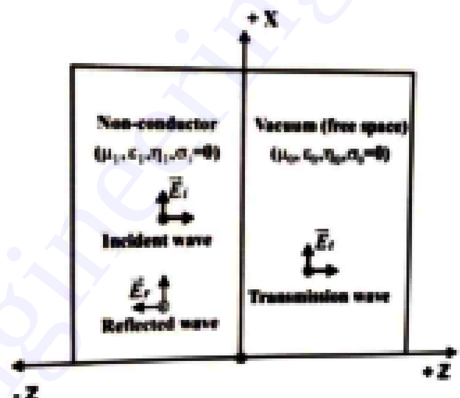

Fig.2.16

Let $E_i\ and\ E_r$ corresponds to the electric field vectors of the incident and reflected waves respectively.

Let $H_i\ and\ H_r$ corresponds to the magnetic field vectors of the incident and reflected waves respectively.

We can write

$$E_i + E_r = E_t \qquad ------(1)$$

$$H_i + H_r = H_t \qquad ------(2)$$

E_t represent the transmitted electric field vector

H_t represent transmitted magnetic field vector.

Equation (1) and (2) relate the electric and magnetic fields respectively, at both the medium and the interface between two media. These equations are used to deduce the laws of reflection and transmission of electromagnetic waves at normal incidence.

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UNIT-3

OSCILLATIONS, OPTICS AND LASERS

Introduction

Oscillatory motion is a repeating motion, which occurs extensively in nature. Oscillations (vibrations) and waves pervade all sciences. We are familiar with several general examples of oscillatory motion such as fluttering of tree leaves in a gentle breeze, swinging of a swing, the beating of the heart etc. Vibrations of atoms in a solid, electric and magnetic fields in light waves and radio waves, large structures swaying due to earth quake etc have an oscillatory nature. Electrical and mechanical oscillations as well as vibrations in structures are every day topics in the world of engineering. A repeating and periodic disturbance (oscillation) moving through a medium from one location to another gives rise to a wave. Waves are also encountered extensively in sciences and technology. Water waves, waves on a string, sound waves, radio waves, microwaves, light waves, and earthquake waves are a few of the examples. Therefore, the study of oscillations and waves constitutes the core topic in engineering and technology.

Oscillations

We all know that when a constant force acts on a body, the body moves with a constantacceleration. Rectilinear motion and uniform circular motion are examples of such a motion. However, when the force acting on the body varies in time, the acceleration of the body will also change with time. Oscillatory motion is one type of motion that can occur when a body is subjected to a force that varies in time. The first systematic observations of oscillations were made by Galileo, who had observed certain rhythm in the swinging of chandeliers in the cathedral at Pisa. He determined the time taken for the oscillations with the help of his pulse. He found to his surprise that the time taken by each oscillation was constant. Later, this property of constant time period for oscillations was exploited in making pendulum clocks.

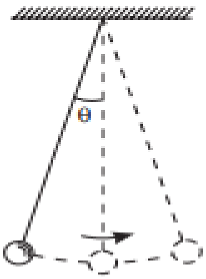


Fig. 1.1

A pendulum serves as the simplest example of an oscillating body (Fig. 1.1). When a pendulum bob is at the lowest position, the pendulum is in a state of rest or it is said to be in its equilibrium position. The forces that act on the pendulum bob are the force of gravity and the force of tension in the string. When the bob is displaced from this equilibrium position, the gravitational force gives rise to a vertical component, directed along the string and a tangential component, directed perpendicular to the string . The tangential component pulls the bob continually back to the equilibrium position and therefore it is known as the restoring force. The bob will not come to the state of rest immediately. When the bob reaches the midpoint, the restoring force vanishes. However, the inertia property causes the bob to overshoot the equilibrium position and the motion continues. Once again, the restoring force comes into action but with a change in its direction. The restoring force, which is maxima at the extreme positions of the bob, stops the bob and pulls it back toward the equilibrium position. The result is a continuing oscillatory motion of the bob back and forth along an arc. Thus, the constant play between the restoring force and inertia property is responsible for the oscillatory motion. The oscillatory motion is periodic and repeats itself in equal intervals of time. In general, an oscillation is a periodic fluctuation in the value of a physical quantity above and below some equilibrium value. In mechanical oscillations, the body undergoes linear or angular displacement whereas non-mechanical oscillations involve the variation of quantities such as voltage or current in electrical circuits or the electric and magnetic fields in TV signals, light waves, UV-rays and X-rays.

Simple harmonic motion

Any motion, which repeats itself at regular intervals according to a sinusoidal law, is called a harmonic motion. The oscillations of a simple pendulum or the motion of a mass m under a restoring force is an idealized model of harmonic motion. In these cases, the force is directly proportional to displacement. The oscillatory motion in which the force is directly proportional to the displacement is called simple harmonic motion (S.H.M.). Since force is proportional to the displacement, the acceleration is not constant but varies with time. S.H.M. is thus a non-uniformly accelerated motion. Hence the equations of motion with constant acceleration are not applicable to simple harmonic motion..

Equation of Simple Harmonic Motion

We now obtain the expressions for displacement, velocity, and acceleration of a body moving with simple harmonic motion.

For studying simple harmonic motion, we considera block of mass m attached to a spring (see Fig. 1.2). When the mass is pulled and left to it, it oscillates about its equilibrium position. The directed distance of the mass from its equilibrium position is called its displacement. The restoring force F acting on the body is due to the stiffness of the spring and is given by Hooke's law.

$F = -kx$ --------------------- (1.1)

where x is the displacement from the equilibrium position. k is called the elastic constant which represents the force required to displace the mass one unit of distance.The negative sign in the expression indicates that the force F is opposite to the displacement. When the mass is pulled to right, the spring gets stretched. Then, x is positive and the force is negative and is directed to the left. When x is negative, the spring is compressed and F is positive and directed to the right.

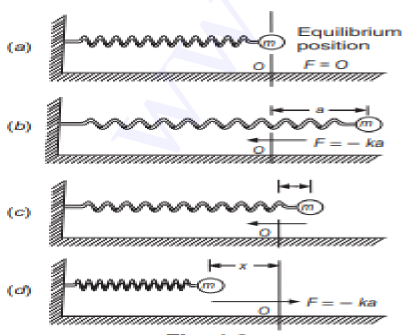


Fig. 1.2

According to Newton's second law, the restoring force produces acceleration. Thus, F = – kx = ma

(restoring force) (inertial force)

$$\therefore \quad m\frac{d^2x}{dt^2} = -kx$$

$$\text{or} \quad \frac{d^2x}{dt^2} + \frac{k}{m}x = 0 \qquad (1.2)$$

This equation is merely another way of writing Newton's second law and it is known as the differential equation of simple harmonic motion. Putting k/m = ω^2 into the above equation, we get

$$\frac{d^2x}{dt^2} + \omega^2 x = 0 \qquad \text{-------------------------------------(1.3)}$$

where w is called the angular frequency and is given by

$$\omega = \sqrt{\frac{k}{m}}.$$

. The time period of oscillations is given by

$$T = \sqrt{\frac{m}{k}} \qquad (1.4)$$

It is seen from equ. (1.4) that the time period of the mass is independent of the amplitude. Secondly, for a given elastic constant, the period increases with the increase in the mass of the block; a heavier mass oscillates more slowly. For a given mass, the period decreases as kincreases; a stiffer spring causes quicker oscillations.All simple harmonic oscillators obey a differential equation of the form (1.3).

Three Conditions for the Occurrence of Simple Harmonic Oscillations

In case of mechanical oscillators, three conditions must be satisfied for the occurrence of simple harmonic oscillations.

(i) There must be a position of stable equilibrium.

(ii) There must be no dissipation of energy.

(iii) The acceleration should be proportional to the displacement and opposite in direction.

Resonance

Resonance describes the phenomenon of increased amplitude that occurs when the frequency of an applied periodic force is equal or close to a natural frequency of the system on which it acts. When an oscillating force is applied at a resonant frequency of a dynamic system, the system will oscillate at a higher amplitude than when the same force is applied at other, non-resonant frequencies.

Frequencies at which the response amplitude is a relative maximum are also known as resonant frequencies or resonance frequencies of the system. Small periodic forces that are near a resonant frequency of the system have the ability to produce large amplitude oscillations in the system due to the storage of vibrational energy.

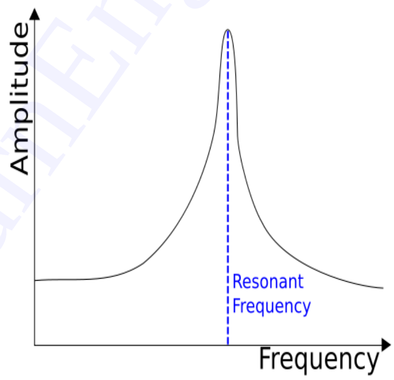
Resonance occurs when an oscillating system is driven (made to oscillate from an outside source) at a frequency which is the same as its own natural frequency. All oscillating systems require some form of an elastic force and a mass e.g. a mass at the end of a spring. All oscillators have a natural frequency. If you have a mass on a spring, and give it an amplitude, it will resonate at a frequency:

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

This frequency is independent of the amplitude you give the oscillator to start with. It is the natural frequency of the oscillator. If you keep giving the oscillator amplitude at this frequency, it will not change the frequency of the oscillation. But, you are still doing work. This energy must go somewhere. The only place it can go is into additional kinetic and gravitational potential energy in the oscillation. If you force an oscillation at its resonant frequency, you add significantly to its amplitude.

In simple terms, resonance occurs when the driving frequency of an oscillation matches the natural frequency, giving rise to large amplitudes.

If you were to force an oscillation at a range of frequencies, and measure the amplitude at each, the graph would look something like the following:



Resonance phenomena occur with all types of vibrations or waves: there is mechanical resonance, Orbital resonance, acoustic resonance, electromagnetic resonance, nuclear magnetic resonance (NMR), electron spin resonance (ESR) and resonance of quantum wave functions. Resonant systems can be used to generate vibrations of a specific frequency (e.g., musical instruments), or pick out specific frequencies from a complex vibration containing many frequencies (e.g., filters).

The term resonance (from Latin resonantia, 'echo', from resonare, 'resound') originated from the field of acoustics, particularly the sympathetic resonance observed in musical instruments, e.g., when one string starts to vibrate and produce sound after a different one is struck.

Analogy between mechanical and electrical oscillations

There is a similarity between mechanical and electrical oscillations as discussed below:

$$\frac{d^2x}{dt^2} + \frac{s}{m}x = 0 \qquad ...(i)$$

where x is the displacement, s the force constant of proportionally or stiffness and m the mass of oscillator. The angular frequency is given by

$$\omega^2 = \frac{s}{m}$$

$$\text{or} \quad \omega = \sqrt{\frac{s}{m}}$$

or frequency

$$n = \frac{1}{2\pi}\sqrt{\frac{s}{m}}$$

The equation for displacement is given by

$$x = a \sin(\omega t + \phi)$$

where a and φ are constants, known as amplitude and initial phase angle.
The total energy of the mechanical oscillator

$$E = \frac{1}{2}mv^2 + \frac{1}{2}sx^2 = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}sx^2$$

where $= \frac{1}{2}m\dot{x}^2$ is the $K.E.$ and $\frac{1}{2}sx^2$ the $P.E.$

The equation of motion of a simple harmonic electrical oscillator is given by

$$\frac{d^2Q}{dt^2} + \frac{1}{LC}Q = 0$$

where Q is the change, L the inductance and C the capacitance of the electrical circuit.
The angular frequency is given by

$$\omega = \sqrt{\frac{1}{LC}}$$

$$\text{or frequency} \qquad n = \frac{1}{2\pi\sqrt{LC}}$$

The charge on the capacitance varies harmonically and is represented by an equation similar to displacement equation i.e

$$Q = Q_0 \sin(\omega t + \phi)$$

where Q_0 is the amplitude of charge and φ phase difference.

The current I = dQ/dt corresponds to velocity v = dx/dt and is given by

$$I = \omega Q_0 \cos(\omega t + \phi).$$

The voltage across the capacitor V =

$$\frac{Q}{C} = \frac{Q_0}{C} \sin(\omega t + \phi)$$

Both I and V, therefore vary harmonically with the same angular velocity ω.
The total energy of an electrical oscillator is the sum of the magnetic energy and electric energy.
The magnetic energy can be calculated from the current I and potential inductance and is given by

$$V = L\frac{dI}{dt}$$

Across the inductance and is given by

$$\int VI\,dt = \int L\frac{dI}{dt} I\,dt = \int LIdI\,\frac{1}{2}LI^2 = \frac{1}{2}L\dot{Q}^2$$

Compare it with kinetic energy in a mechanical oscillator given by

$$\frac{1}{2}m\dot{x}^2$$

Thus mass in a mechanical circuit corresponds to inductance in an electrical circuit and velocity to electric current. The electrostatic energy can be calculated from the voltage across the capacitor and is given by

$$\frac{1}{2}CV^2 = \frac{1}{2}C\left(\frac{Q}{C}\right)^2 = \frac{1}{2}\frac{Q^2}{C}$$

Compare it with potential energy in a mechanical oscillator given $1/2sx^2$. Thus stiffness s in a mechanical circuit corresponds to 1/ C in an electrical circuit.

Waves and it types

A wave is a disturbance in a medium that transports energy without causing net particle movement. Waves come in two kinds, longitudinal and transverse. Transverse waves are like those on water, with the surface going up and down, and longitudinal waves are like of those of sound, consisting of alternating compressions and rarefactions in a medium.

Characteristics of Waves

Waves include the following characteristics:

- The particles of the medium traversed by a wave vibrate only slightly about their mean positions, but they are not permanently displaced in the wave's propagation direction.
- Along with or perpendicular to the wave's line of travel, each succeeding particle of the medium performs a motion quite identical to its predecessors.
- During wave motion, only energy is transferred, but not a piece of the medium.

Types of Waves

The several forms of waves are listed here:

1. Transverse Waves:
Transverse waves have particles vibrating perpendicular to the direction of wave propagation. These waves can be produced in solids and liquids.

Transverse waves have particles vibrating up and down in the direction in which the wave is moving.

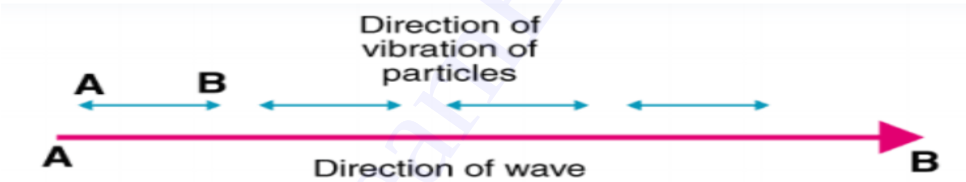
Examples of transverse waves:

- Water waves (ripples of gravity waves, not sound through water)
- Light waves
- S-wave earthquake waves
- Stringed instruments
- Torsion wave

A crest is the highest point of a transverse wave. It's a trough at the bottom.

2. Longitudinal Wave:

The movement of the particles in the medium in a longitudinal wave is in the same dimension as the wave's movement direction. The particles vibrate in the back and forth in the same direction. These waves can be produced in solids, liquids, and gases.



Examples of longitudinal waves:

- Sound waves
- Compression wave

Parts of longitudinal waves:

- Compression-The particles are close together in this case.
- Rarefaction-Where the particles are dispersed

Properties of Waves

The following are the primary characteristics of waves:

- Amplitude – A wave is a form of energy transmission. The amplitude of a wave is its height, which is commonly measured in meters. It is proportional to the quantity of energy transported by a wave.

- Wavelength – A wavelength is a distance between identical locations in adjacent cycles of crests of a wave. In addition, it is measured in meters.
- Period – A wave's period is the amount of time it takes a particle on a medium to complete one complete vibrational cycle. Because the period is a unit of time, it is measured in seconds or minutes.
- Frequency – The number of waves passing a spot in a certain amount of time is referred to as the frequency of a wave. The hertz (Hz) unit of frequency measures one wave every second.

The frequency's reciprocal is the period, and vice versa.

Period=1 / Frequency

OR

Frequency = 1 / Period

- Speed – The speed of an object refers to how quickly it moves and is usually stated as the distance travelled divided by the time it takes to travel. The distance travelled by a specific point on the wave (crest) in a given amount of time is referred to as the wave's speed. A wave's speed is thus measured in meters per second or m/s.

Waves on string

- The type of wave that occurs in a string is called a transverse wave. In a transverse wave, the wave direction is perpendicular the the direction that the string oscillates in.

- The period of a wave is indirectly proportional to the frequency of the wave $T=1/f$.

- The speed of a wave is proportional to the wavelength and indirectly proportional to the period of the wave$v= \lambda / T$
- This equation can be simplified by using the relationship between frequency and period $v = \lambda f$.

When studying waves, it is helpful to use a string to observe the physical properties of waves visually. Imagine you are holding one end of a string, and the other end is secured and the string is pulled tight. Now, if you were to flick the string either up and down. **The wave that occurs due to this motion is called a transverse wave**. A transverse wave is defined as a wave where the movement of the particles of the medium is perpendicular to the direction of the propagation of the wave. Figure 1 shows this in a diagram. In this case, the medium through which the waves propagate is the rope. The wave traveled from one end to the other, while the rope moved up and down.

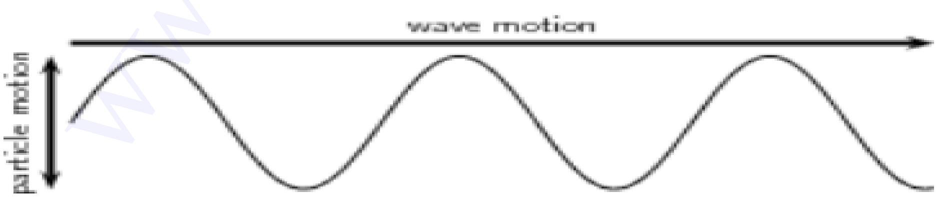


Figure 1: In transverse waves, the media the wave is traveling in moves perpendicular to the direction of the wave.

Wave Properties

Transverse waves have what are called peaks and troughs. The peak is the crest, or top point of the wave and the trough is the valley or bottom point of the wave. Refer to Figure 2 for a visual representation of these terms. The amplitude is the maximum displacement of a particle from its equilibrium position. Wavelength, usually denoted with a lambda (λ) and measured in meters, is the distance from either one peak to the next peak, or one trough to the next trough. Period, usually denoted as T and measured in seconds, is the time it takes for two successive peaks, or one wavelength, to pass through a fixed point. Frequency, f, is the number of wavelengths that pass through a given point in 1 second. Frequency is measured by taking the reciprocal of a period:

$$f = 1/T.$$

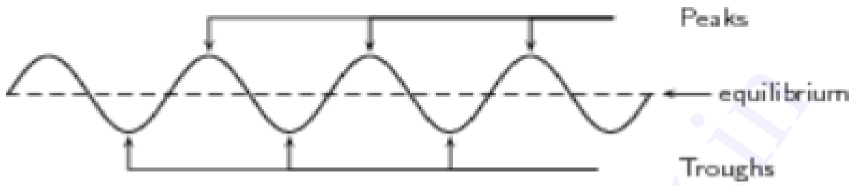


Figure 2: Peaks are the top most points of the waves and troughs are the bottom, or valleys of the waves.

Speed of a Wave on a String

Velocity is found by dividing the distance traveled by the time it took to travel that distance. In waves, this is found by dividing the wavelength by

$V = \lambda / T$

$= \lambda / f$

Speed of a Wave on a Vibrating String

Another example of waves on strings are of the waves on vibrating strings, such as in musical instruments. Pianos and guitars both use vibrating strings to produce music. In these cases, the frequency is what characterizes the pitch and therefore the note. The speed of a wave on this kind of string is proportional to the square root of the tension in the string and inversely proportional to the square root of the linear density of the string:

$$v = \sqrt{\frac{T}{\mu}}$$

Reflections of Transverse Waves

When transverse waves in strings meet one end, they are reflected, and when the incident wave meets the reflected wave, interference occurs.The way in which a transverse wave reflects depends on whether or not it is fixed at both ends. First we will look at waves that are fixed at both ends. Below figure shows an image of a transverse wave that is reflected from a fixed end. When a transverse wave meets a fixed end, the wave is reflected, but inverted. This swaps the peaks with the troughs and the troughs with the peaks.

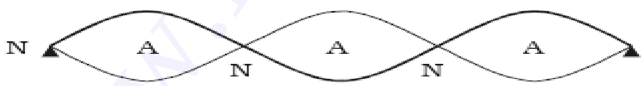
Transverse Wave With a Fixed End Point: A transverse wave that is fixed at the end point. The reflected wave is inverted.

If an image of a transverse wave on a string that meets a free end. The wave is reflected, but unlike a transverse wave with a fixed end, it is not inverted.

Transverse Wave With a Free End: When a transverse wave meets a free end, it is reflected.

Standing Waves

- When two progressive waves of same amplitude and wavelength travelling along a straight line in opposite directions superimpose on each other, stationary waves are formed.
- When either of the two scenarios of above diagram wave reflection occurs, the incident wave meets the reflected wave. These waves move fast each other in opposite directions, causing interference. When these two waves have the same frequency, the product of this is called the standing waves. Standing waves appear to be standing still, hence the name. To understand how standing waves occur, we can analyze them further.When the incident wave and reflected wave first meet, both waves have an amplitude is zero. As the waves continue to move fast each other, they continue to interfere with each other either constructively or destructively.
- when waves are completely in phase and interfere with each other constructively, they are amplified, and when they are completely out of phase and interfere destructively they cancel out. As the waves continue to move fast each other, and are reflected from the opposite end, they continue to interfere both ways, and a standing wave is produced.
- Every point in the medium containing a standing wave oscillates up and down and the amplitude of the oscillations depends on the location of the point. When we observe standing waves on strings, it looks like the wave is not moving and standing still. The principle of standing waves is the basis of resonance and how many musical instruments get their sound. The points in a standing wave that appear to remain flat and do not move are called nodes. The points which reach the maximum oscillation height are called antinodes.



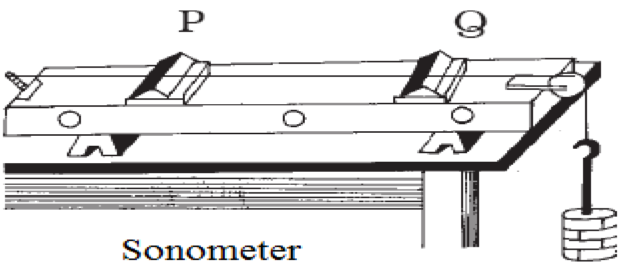
Stationary waves

Standing waves in strings

In musical instruments like sitar, violin, etc. sound is produced due to the vibrations of the stretched strings. Here, we shall discuss the different modes of vibrations of a string which is rigidly fixed at both ends.

When a string under tension is set into vibration, a transverse progressive wave moves towards the end of the wire and gets reflected. Thus stationary waves are formed.

Sonometer



The sonometer consists of a hollow sounding box about a metre long. One end of a thin metallic wire of uniform cross-section is fixed to a hook and the other end is passed over a pulley and attached to a weight hanger as shown in figure. The wire is stretched over two knife edges P and Q by adding sufficient weights on the hanger. The distance between the two knife edges can be adjusted to change the vibrating length of the wire.A transverse stationary wave is set up in the wire. Since the ends are fixed, nodes are formed at P and Q and antinode is formed in the middle.

The length of the vibrating $l = \lambda/2$
Thus $\lambda = 2l$.
If n is the frequency of the vibrating segment, then,
$n = v/\lambda = v/2l$ …... (1)

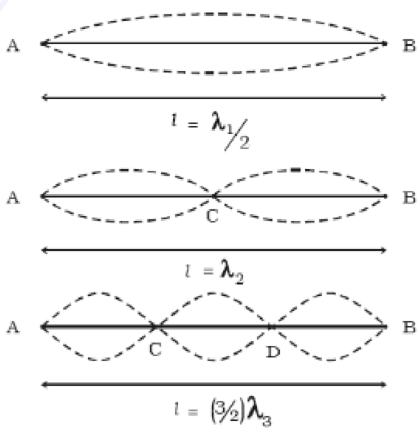
We know that, $v = \sqrt{T/m}$ where T is the tension and m is the mass per unit length of the wire.
Thus,

$$n = \frac{1}{2l}\sqrt{\frac{T}{m}} \quad \text{--(2)}$$

Modes of vibration of stretched string

(i)Fundamental frequency

If a wire is stretched between two points, a transverse wave travels along the wire and is reflected at the fixed end. A transverse stationary wave is thus formed as shown in figure.



Fundamental and overtones in stretched string

When a wire AB of length l is made to vibrate in one segment then, $l = \lambda_1/2$
$\lambda_1 = 2l$.

This gives the lowest frequency called fundamental frequency, $n_1 = v/\lambda_1$
So, $n_1 = (1/2l)\sqrt{T/m}$ ---------------(3)

(ii) Overtones in stretched string

If the wire AB is made to vibrate in two segments then $l = \lambda_2/2 + \lambda_2/2$
So, $\lambda_2 = l$
But, $n_2 = v/\lambda_2$
So, $n_2 = 1/l\sqrt{T/m} = 2n1$ …... (4)
n_2 is the frequency of the first overtone.

Since the frequency is equal to twice the fundamental, it is also known as second harmonic.

Similarly, higher overtones are produced, if the wire vibrates with more segments. If there are P segments, the length of each segment is
$l/p = \lambda p/2$
Or, $\lambda p = 2l/P$
So, frequency, $n_p = (P/2l)\sqrt{T/m} = Pn_1$ …... (5)
(i.e) P^{th} harmonic corresponds to $(P–1)^{th}$ overtone.

Laws of transverse vibrations of stretched strings

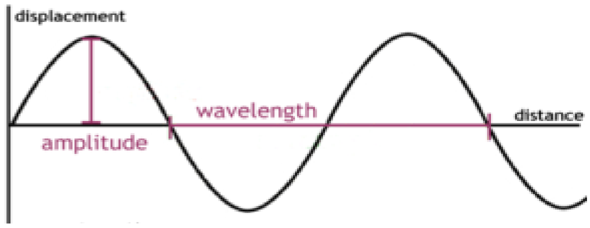
The laws of transverse vibrations of stretched strings are (i) the law of length (ii) law of tension and (iii) the law of mass.
(i) For a given wire (m is constant), when T is constant, the fundamental frequency of vibration is inversely proportional to the vibrating length (i.e)
$n \propto 1/l$
Or, nl = constant

(ii) For constant l and m, the fundamental frequency is directly proportional to the square root of the tension (i.e)
$n \propto \sqrt{T}$.

(iii) For constant l and T, the fundamental frequency varies inversely as the square root of the mass per unit length of the wire
(i.e) $n \propto 1/\sqrt{m}$

Travelling waves

All the waves we have seen so far have not moved. We call them "stationary", or "standing" waves. They oscillate in place, but they don't move. Imagine stretching a string and fixing both its ends on two points. Now grab the midpoint of the string and pull it down, before letting it go. Chances are that you'll see the midpoint of the string oscillating with an amplitude, with the end points fixed at their respective positions. These kind of waves are what we call the standing waves.
Now, for the next experiment, get into a hall with your friend and call out to him. If you shout loud enough and your friend hears well, chances are that he'll hear your call. Your voice reached him by the motion of sound waves, i.e. travelling waves. Had the sound waves been stationary, your voice would have never reached him.
If a wave that propagates in a medium is continuous then it is known as progressive wave or travelling wave.



Characteristics of progressive waves

1. Particles in the medium vibrate about their mean positions with the same amplitude.

2. The phase of every particle ranges from 0 to 2π.

3. No particle remains at rest permanently. During wave propagation, particles come to the rest position only twice at the extreme points.

4. Transverse progressive waves are characterized by crests and troughs whereas longitudinal progressive waves are characterized by compressions and rarefactions.

5. When the particles pass through the mean position they always move with the same maximum velocity.

6. The displacement, velocity and acceleration of particles separated from each other by $n\lambda$ are the same, where n is an integer, and λ is the wavelength.

Analytical Equation and Energy transfer in stationary wave

When two progressive waves of same amplitude and wavelength travelling along a straight line in opposite directions, they superimpose on each other which results in formation of stationary waves.

Consider a progressive wave of amplitude a and wavelength λ travelling in the x-axis direction.

$$y_1 = a\sin 2\pi\left(\frac{t}{T} - \frac{x}{\lambda}\right)$$

This wave is reflected from a free end and it travels in the negative x-axis direction. It will have same characteristics expect x changes to –x.

$$y_2 = a\sin 2\pi\left(\frac{t}{T} + \frac{x}{\lambda}\right)$$

Now, according to principle of superposition, the resultant displacement will be

$$y = y_1 + y_2$$

$$y = a\sin 2\pi\left(\frac{t}{T} - \frac{x}{\lambda}\right) + a\sin 2\pi\left(\frac{t}{T} + \frac{x}{\lambda}\right)$$

Using $\sin A + \sin B = 2\sin[(A+B)/2]\cos[(A-B)/2]$,

$$y = a\left[2\sin\left(\frac{2\pi t}{T}\right)\cos\left(\frac{2\pi x}{\lambda}\right)\right]$$

$$y = A\sin\left(\frac{2\pi t}{T}\right) \text{ where } A = 2a\cos\left(\frac{2\pi x}{\lambda}\right)$$

The above equation of y is an equation of stationary wave and it's amplitude is

$A = 2a\cos(\frac{2\pi x}{\lambda})$.

This represents that at some values of x the resultant amplitude is maximum known as antinodes and for some values of x it will be minimum (zero) known as nodes.

The energy transfer in a stationary wave

A pure standing wave does not transfer energy from the source to the destination.
$E_{transferr} = 0$

Since there is no movement takes place one particle to other.

The standing wave is actually the superposition of two travelling waves which were moving in opposite directions, so the energy transferred by a single wave in one direction is completely compensated by energy transfer in the opposite direction. This keeps the total energy (sum of potential and kinetic energies) of every particle in the standing wave constant. A further insight might be brought by observing that propagating waves are 'propagating' because each segment of the string propagates its disturbances (momentum, energy) to the neighbouring particle. So if we stop furnishing energy, displacement at the initial position stops and the wave moves further as energy keeps getting transferred in the direction of wave velocity. However, in the case of standing waves (ideally speaking) you do not need energy to sustain the motion at any point. The energy is kept within the already oscillating string segment, exactly why a standing wave does not travel.

Consider for example a transverse standing wave on a string, with wavefunction

$$\psi(z,t) = A\sin kz \sin\omega t.$$

The power transmitted by the wave (in the +z direction) is

$$P(z,t) = F(z,t)v(z,t) = 4\mu v\omega^2 A^2 \cos kz \sin kz \cos\omega t \sin\omega t$$

where μ is the linear mass density of the string. Taking the time average of the power,

$$\langle P\rangle = \frac{2\pi}{\omega} P_0(z) \int_0^{2\pi/\omega} dt \cos\omega t \sin\omega t$$

we find ⟨P⟩=0, since the integral is 0 by exactness.

For waves of equal amplitude travelling in opposing directions, there is on average no net propagation of energy.

The energy in a progressive wave

When a progressive wave travels through a medium, the displacement of a particle of the medium at any instant of time

$$y = a sin\frac{2\pi}{\lambda}\left(vt - x\right)$$

So, the velocity of the particle

$$U = \frac{dy}{dt} = \frac{2\pi v}{\lambda} a cos\frac{2\pi}{\lambda}\left(vt - x\right)$$

So, acceleration of the particle

$$f = \frac{d^2y}{dt^2} = -\frac{4\pi^2v^2}{\lambda^2}asin\frac{2\pi}{\lambda}\left(vt - x\right)$$

$$f = \frac{d^2y}{dt^2} = -\frac{4\pi^2v^2}{\lambda^2}y$$

Let ρ be the density of the medium. So, kinetic energy per unit volume at any instant of time

$$E_{K.E.} = \frac{1}{2}\rho\left(\frac{dy}{dt}\right)^2$$

$$E_{K.E.} = \frac{1}{2}\rho\left(\frac{2\pi v}{\lambda}acos\frac{2\pi}{\lambda}\left(vt - x\right)\right)^2$$

$$E_{K.E.} = \frac{1}{2}\rho\frac{4\pi^2v^2}{\lambda^2}a^2cos^2\frac{2\pi}{\lambda}\left(vt - x\right) \quad \text{............(1)}$$

Now, potential energy $dE_{P.E.}$= workdone for the displacement dy = dy × force

$$dE_{P.E.} = dy \times \rho\frac{d^2y}{dt^2}$$

$$dE_{P.E.} = \rho\frac{4\pi^2v^2}{\lambda^2}ydy$$

Total potential energy for the displacement y

$$E_{P.E.} = \rho\frac{4\pi^2v^2}{\lambda^2}\int_0^y ydy$$

$$E_{P.E.} = \rho\frac{4\pi^2v^2}{\lambda^2}\frac{y^2}{2}$$

$$E_{P.E.} = \frac{1}{2}\rho\frac{4\pi^2v^2}{\lambda^2}a^2sin^2\frac{2\pi}{\lambda}\left(vt - x\right) \quad \text{................(2)}$$

Total energy per volume at any instant of time

$$E = E_{K.E.} + E_{P.E.}$$

$$E = \frac{1}{2}\rho\frac{4\pi^2v^2}{\lambda^2}a^2cos^2\frac{2\pi}{\lambda}\left(vt - x\right) + \frac{1}{2}\rho\frac{4\pi^2v^2}{\lambda^2}a^2sin^2\frac{2\pi}{\lambda}\left(vt - x\right)$$

$$E = \frac{1}{2}\rho\frac{4\pi^2v^2}{\lambda^2}a^2\left(cos^2\frac{2\pi}{\lambda}\left(vt - x\right) + sin^2\frac{2\pi}{\lambda}\left(vt - x\right)\right)$$

$$E = \frac{1}{2}\rho\frac{4\pi^2v^2}{\lambda^2}a^2 = constant$$

The above equation represents the energy transfer in a progressive wave.

Sound Wave

Waves are disturbances that propagate through a material medium or empty space. Light waves can travel through a vacuum and some forms of matter, and are transverse in nature, which means that the

oscillations are perpendicular to the direction of propagation. However, sound waves are pressure waves that travel through an elastic medium like air, and are longitudinal in nature, which means the oscillations are parallel to the direction of propagation. When sound is introduced to a medium by a vibrating object, like the vocal chords of a person or strings in a piano, the particles in the air experience forward and backward motion as the vibrating object moves forward and backward. This results in regions in the air where the air particles are compressed together, called compressions, and other regions where they are spread apart, called rarefactions. The energy created by a sound wave oscillates between the potential energy created by the compressions and the kinetic energy of the small movements and speeds of the particles of the medium.
Compressions and rarefactions can be used to define the relationship between sound wave velocity and frequency.

Nature Of Sound

The sound produced by a guitar is different from the sound produced by a drum. This is because the sound produced by different sources have different characteristics. Sound can be characterized by its frequency, wavelength, and amplitude.

Frequency of sound

The number of rarefactions and compressions that occur per unit time is known as the frequency of a sound wave. The formula of the frequency of a wave is given as

$$f = \frac{1}{T}$$

Where,
f is the frequency of a sound wave and
T is the time period.

Wavelength of sound

The distance between the successive compression and rarefaction is known as the wavelength of a sound wave. The wavelength of the sound formula is given as follows

$$\lambda = \frac{v}{F}$$

Where, f is the frequency of the sound wave and v is the velocity of the sound wave.

Amplitude of sound

The amplitude of the sound is the magnitude of the maximum disturbance in a sound wave. The amplitude is also a measure of energy. Higher the amplitude higher the energy in a sound wave. Humans can hear a limited range of frequencies of sound. Physicists have identified the audio frequency spectrum of the human ear to be between 20 Hz and 20,000 Hz. Under ideal laboratory conditions, the human ear can detect frequencies that are as low as 12 Hz and as high as 20,000 Hz.

Speed of Sound

The speed at which sound waves propagate through a medium is known as the speed of sound. The speed of sound is different in different media. The speed of sound is highest in solids because the atoms in solid are highly compressed. The interaction between atoms in a particle is highly dependent on the distance between them. Higher the interaction between the atoms, the quicker the energy is transferred. As the interaction of the particles in solids is high, the speed of sound is faster than liquids and gases. The table below lists the speed of sound in different media. The formula used to calculate the speed of sound is given as

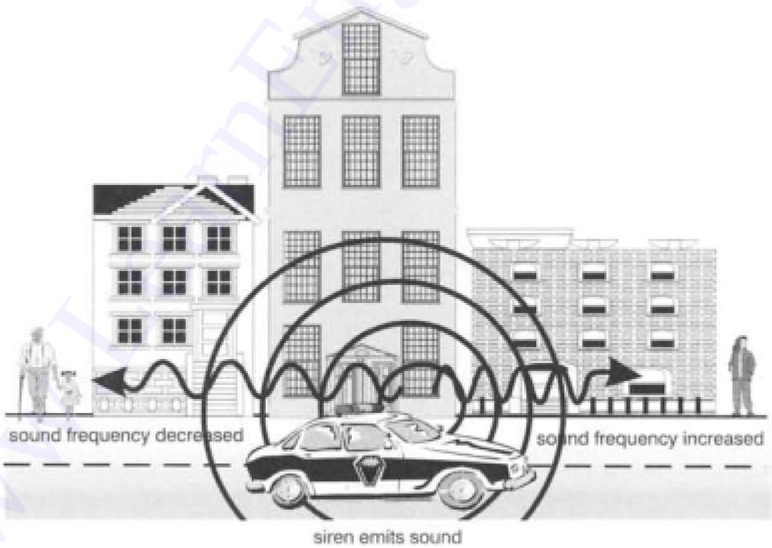
$$c = \frac{d}{t}$$

Where,
d is the distance traveled by sound
t is the time taken to cover the distance.

Medium	Speed of sound
Water	1481 m/s
Air	343.2 m/s
Copper	4600 m/s
Hydrogen	1270 m/s
Glass	4540 m/s

Doppler effect

Doppler effect. As a sound source approaches an observer, the sound's frequency increases while it decreases as it moves away from an observer.



when the observer and wave source are moving away from each other. The Doppler effectresults in greater wave frequency when the relative motion of the source and observer is toward each other, and lower frequency when the relative motion is away from eachother.The Doppler effect results in the change in pitch of a sound from higher to lower as a vehicle travels past a person. It is also used toestimate the speed and distance to distant galaxies.If the universe is expanding, all objects in the universe tend to be moving away fromeach other. The light emitted from these galaxies would, therefore, be shifted toward alonger wavelength or toward the red end ofthe visible light spectrum. The amount of redshift is proportional to how fast the galaxies are receding from observers on Earth. Astronomical measurements have demonstrated that the most distant galaxies show the greatest red shift and, hence, are moving awayfrom Earth at the greatest rate.

Another area where the Doppler effect has practical applications is in radar. The word "radar" is an acronym for radio detection and ranging. Police use radar andthe Doppler effect to measure the speed of vehicles. Coaches monitor pitches with radar guns to determine if pitchers are losing velocity and becoming fatigued. Another widespread use of radar is for weather forecasting. Doppler radar images are standard on the nightly weather forecast. Doppler radar works by emitting waves with at aspecific frequency from a rotating emitter.As the waves encounter precipitation in the atmosphere (in the form of rain or snow) carried by winds, the signal is reflected back to a receiver that produces color-enhanced radar images. The standard color pattern displays movement toward the radar as shades of green and movement away from the radar as shades of red. The intensity of the signal represents the strength of the weather pattern. Thunderstorms and severe weather such as tornadoes can be identified by trained weather personnel by examining the color patterns and the strength of the signal. Currently, approximately 150 Doppler radar sites are located throughout the United States.

Formula

Doppler effect is the apparent change in the frequency of waves due to the relative motion between the source of the sound and the observer. We can deduce the apparent frequency in the Doppler effect using the following equation:

$$f' = \frac{(V \pm V_0)}{(V \pm V_s)} f$$

f' = observed frequency
f = actual frequency
V = velocity of sound waves
V_0 = velocity of observer
V_s = velocity of the source

While there is only one Doppler effect equation, the above equation changes in different situations depending on the velocities of the observer or the source of the sound. Let us see below how we can use the equation of the Doppler effect in different situations.

(a) Source Moving Towards the Observer at Rest

In this case, the observer's velocity is zero, so V_0 is equal to zero. Substituting this into the Doppler effect equation above, we get the equation of the Doppler effect when a source is moving towards an observer at rest as:

$$f' = \frac{V}{(V - V_s)} f$$

f' = observed frequency
f = actual frequency
V = velocity of sound waves
V_s = velocity of the source

(b) Source Moving Away from the Observer at Rest

Since the velocity of the observer is zero, we can eliminate V_0 from the equation. But this time, the source moves away from the observer, so its velocity is negative to indicate the direction. Hence, the equation now becomes as follows:

$$f' = \frac{V}{(V - (-V_s))} f$$

f' = observed frequency
f = actual frequency
V = velocity of sound waves
V_s = velocity of the source

(c) Observer Moving Towards a Stationary Source

In this case, v_s will equal to zero, hence we get the following equation:

$$f' = \frac{(V + V_0)}{V} f$$

f' = observed frequency
f = actual frequency
V = velocity of sound waves
V_0 = velocity of observer

(d)Observer Moving Away from a Stationary Source

Since the observer is moving away, the velocity of the observer becomes negative. So, instead of adding V_0, we now subtract, since V_0 is negative.

$$f' = \frac{(V - V_0)}{V} f$$

f' = observed frequency
f = actual frequency
V = velocity of sound waves
V_0 = velocity of observer

Uses of Doppler Effect

Many people mistake the Doppler effect to be applicable only for sound waves. It works with all types of waves including light. Below, we have listed a few applications of the doppler effect:

- Sirens
- Radar
- Astronomy
- Medical Imaging
- Blood Flow Measurement

- Satellite Communication
- Vibration Measurement

Doppler Effect Limitations

- Doppler Effect is applicable only when the velocities of the source of the sound and the observer are much less than the velocity of sound.
- The motion of both source and the observer should be along the same straight line.

OPTICS - Reflection and Refraction of light waves

Reflection

Light is a form of energy due to which we are able to see the objects which emits light for example objects like sun, lamp, candle emits light of their own and thus they are known as luminous objects. There are objects like table , chair etc. which are not luminous objects and still we are able to see them and this happens because they reflect lights which fall on them from a luminous object like sun, lamp etc. and when this reflected light reaches our eyes we are able to see such non luminous objects.

Light rays basically consist of electromagnetic waves which do not require any material medium (like solid, liquid or gas) for their propagation. Wavelength of visible light waves is very small and is of the order of 4×10^{-7}m to 8×10^{-7}m.

Speed of light waves depends on the medium through which they pass as speed of light in air is slightly less than the speed of light in vacuum 8×108m/s, the same way the speed of light in water and glass is much less than that in air. When light falls on the surface of an object it can either be

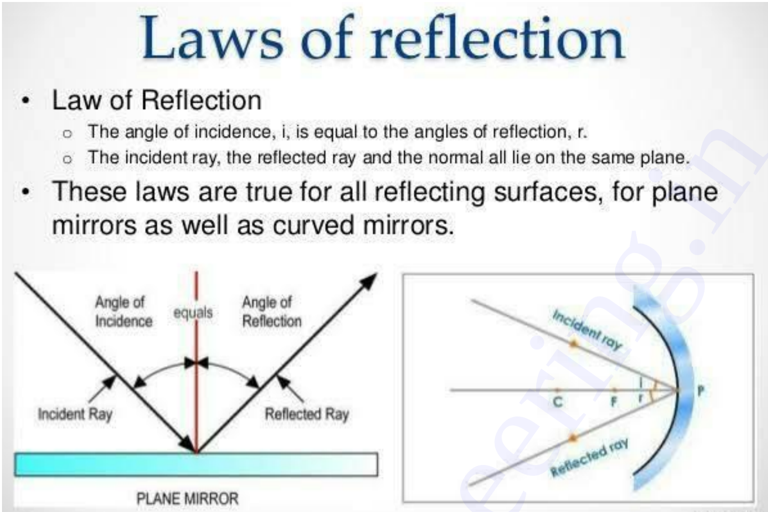
1. Absorbed:- If an object absorbs all the light falling on it , then it will appear perfectly black for example a blackboard
2. Transmitted: - An object is said to transmit light if it allows light to pass through itself and such objects are transparent.
3. Reflected:- If an object sends back light rays falling on its surface then it is said to have reflected the light

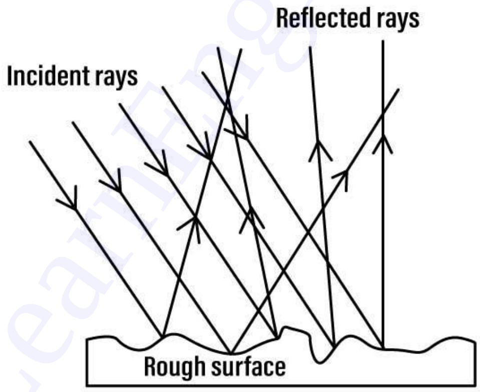
Reflection of Light

The process of sending back light rays which falls on the surface of an object is called reflection of light. Silver metal is one of the best reflectors of light. Mirrors we use on our dressing tables in our home are plane mirrors. A ray of light is the straight line along which the light traveled and a bundle of light rays is called a beam of light.

Laws of Reflection of light

1. The angle of incidence is equal to the angle of reflection, and 2. The incident ray, the reflected ray and the normal to the mirror at the point of incidence all lie in the same plane. These laws of reflection are applicable to all types of reflecting surfaces including spherical surfaces.



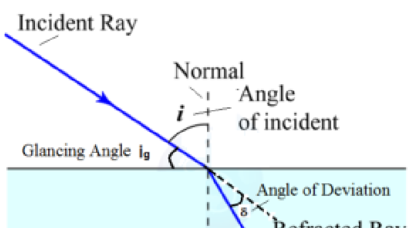


Irregular or diffused reflection

At each point on the surface, the laws of reflection are obeyed, but the angle of incidence, and so the angle of reflection, varies from point to point. The reflected rays are scattered randomly. Most objects, being rough, and seen by diffuse reflection. Because of diffuse reflection in all directions, an ordinary object can be seen at many different angles by the light reflected from it.

REFRACTION

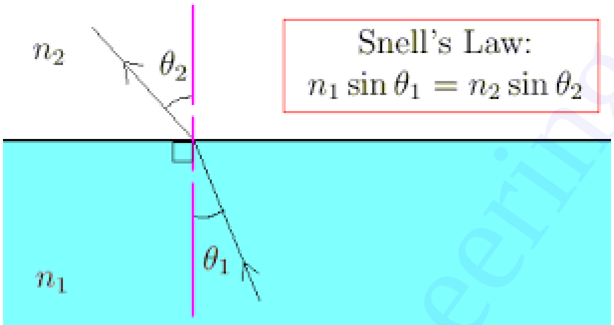
When a beam of light ray travels from one transparent medium into another medium, it bends while crossing the interface, separating the media. This phenomenon is called refrection.



LAWS OF REFRACTION

First law: The incident ray and the normal to the refracting surface at the point of incidence are in the same plane.

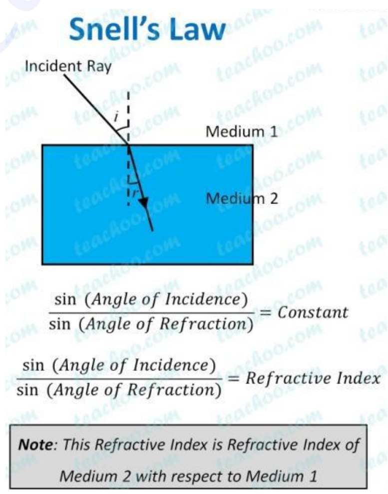
Second law: The ratio of the sine of the angle of incidence to the sine of the angle of refraction is a constant for any two given medium for one particular wavelength.



n_1 and n_2 are the refractive indices of the denser and rarer medium and θ_1 , θ_2 corresponds to the angle of incidence and refraction respectively and measured from the normal. Snell's law of refraction gives the amount of bending of light.

Refractive index of medium

The ratio between the sine of the angle of incidence and the sine of the angle of refraction in cllaed refractive index of the given medium.



Critical angle

The phenomenon of total internal reflection was introduced. Total internal reflection (TIR) is the phenomenon that involves the reflection of all the incident light off the boundary. TIR only takes place when both of the following two conditions are met:

- a light ray is in the more dense medium and approaching the less dense medium.
- the angle of incidence for the light ray is greater than the so-called critical angle.

Expression for critical angle

Let's consider two different media - creatively named medium i (incident medium) and medium r (refractive medium). The critical angle is the θ_i that gives a θ_r value of 90° If this information is substituted into Snell's Law equation, a generic equation for predicting the critical angle can be derived. The derivation is shown below.

n_i *• sine(θ_i) = n_r • sine (Θ_r)

n_i • sine(θ_{crit}) = n_r • sine(90 degrees)

n_i • sine(θ_{crit}) = n_r

sine(θ_{crit}) = n_r/n_i

θ_{crit}= sine^{-1} (n_r/n_i) = sin^{-1} (n_r/n_i)

The above equation gives the expression for critical angle.

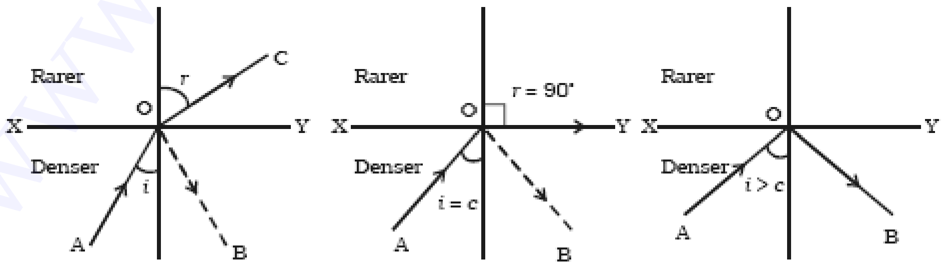
Total Internal Reflection

When a ray of light *AO* passes from an optically denser medium to a rarer medium, at the interface *XY*, it is partly reflected back into the same medium along *OB* and partly refracted into the rarer medium along *OC* as shown in figure.

If the angle of incidence is gradually increased, the angle of refraction *r* will also gradually increase and at a certain stage *r* becomes 90°. Now the refracted ray *OC* is bent so much away from the normal and it grazes the surface of separation of two media.

The angle of incidence in the denser medium at which the refracted ray just grazes the surface of separation is called the critical angle c of the denser medium.

If *i* is increased further, refraction is not possible and the incident ray is totally reflected into the same medium itself. This is called *total internal reflection.*

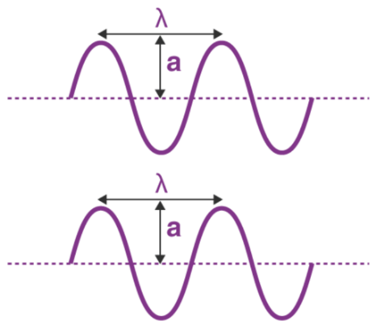


i,e $\theta_i > \theta_c$

Interference Of Light

Interference is a natural phenomenon that happens at every place and at every moment. Yet we don't see interference patterns everywhere. Interference is the phenomenon in which two waves superpose to form the resultant wave of the lower, higher or same amplitude.

Coherence

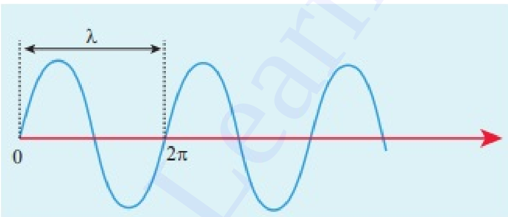


Two waves are said to be coherent when they maintain a constant phase difference between them. For this waves must have same amplitude and wavelength and travel almost in the same direction. Coherence is a necessary condition for producing stable interference patterns. Suppose phase difference between two waves keep changing, the positions of maximum and minimum amplitudes vary with time.

Phase difference and path difference

Phase is the angular position of a vibration. As a wave is progressing, there is a relation between the phase of the vibration and the path travelled by the wave. One can express the phase in terms of path and vice versa. In the path of the wave, one wavelength λ corresponds to a phase of 2π as shown in Figure. A path difference δ corresponds to a phase difference ϕ as given by the equation,

$$\delta = \frac{\lambda}{2\pi} \times \phi \text{ (or) } \phi = \frac{2\pi}{\lambda} \times \delta$$



For constructive interference, the phase difference should be, $\phi = 0, 2\pi, 4\pi \ldots$ Hence, the path difference must be, $\delta = 0, \lambda, 2\lambda \ldots$ In general, the integral multiples of λ.

$$\delta = n\lambda \text{ where, } n = 0, 1, 2, 3 \ldots$$

For destructive interference, phase difference should be, $\phi = \pi, 3\pi, 5\pi \ldots$ Hence, the path difference must be, $\delta = \lambda/2 , 3\lambda/2$ .....

In general, the half integral multiples of λ.

$$\delta = (2n-1)\frac{\lambda}{2} \text{ where, } n = 1, 2, 3 \ldots$$

Superposition principle

Considering two waves, travelling simultaneously along the same stretched string in opposite directions, as shown in the figure above. We can see images of waveforms in the string at each instant of time. It is observed that the net displacement of any element of the string at a given time is the algebraic sum of the displacements due to each wave.

$y_1 = a_1 \sin \omega t$ $y_2 = a_2 \sin(\omega t + \phi)$

$_2(\omega t+\phi)$

If the phase difference, ϕ = 0, ±2π, ±4π. . . , it corresponds to the condition for maximum intensity of light called as constructive interference.

If the phase difference, ϕ = ±π, ±3π, ±5π. . . , it corresponds to the condition for minimum intensity of light called destructive interference.

Types of Interference

Interference of light waves can either be constructive interference or destructive interference.

- **Constructive interference:** Constructive interference takes place when the crest of one wave falls on the crest of another wave such that the amplitude is maximum. These waves will have the same displacement and are in the same phase.
- **Destructive interference:** In destructive interference, the crest of one wave falls on the trough of another wave such that the amplitude is minimum. The displacement and phase of these waves are not the same.

MICHELSON INTERFEROMETER

An interferometer is an instrument that uses interference phenomenon in the measurement of the wavelength of light in terms of standard of length or the measurement of distance in terms of the known wavelength of the light.

The Michelson interferometer produces interference fringes by splitting a beam of light so that one beam strikes a fixed mirror and the other a movable mirror. When the reflected beams are brought back together, an interference pattern results.

MICHELSON INTERFEROMETER PRINCIPLE

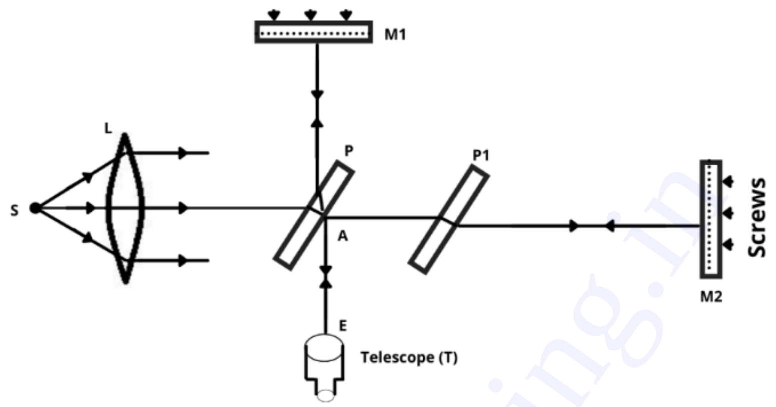
In this instrument, light from an extended source is divided into two parts by partial reflection and transmission. These two beams are sent at right angles to each other in the two directions. They get reflected from the mirror and form interference fringes which are observed and investigated.

Construction of Michelson Interferometer

It consists of a semi silver plate P placed at an angle of 45 degrees to the horizontal. M1 and M2 are two highly polished silver mirrors. These Mirrors are provided with a labeling screw at the back.

Mirror M1 can be moved towards or away from plate P with the help of micrometer screws. P1 is another glass plate whose thickness is exactly equal to the thickness of plate P.

It is transparent and parallel to P. S is a monochromatic source of light, say sodium lamp. Lens L is a Converging lens whose function is made to the source of light as an extended one.



As these two waves entering the telescope are derived from the same source S, hence these waves are coherent waves. Coherent waves interfere with each other and interference fringes formed and seen through the telescope. The ray which gets reflected from M1 crosses the plate P twice while the ray which from M2 Travels only in the air.This means that the ray which travels along the mirror M1 covers an additional part 2 (n-1) t. Where t is the thickness of the plate P and n is its refractive index.Which does not get any difficulty if we are using monochromatic light but if white light is used. Then it does create a problem because of the variation of n with wavelength.

To overcome this difficulty another plate P1 is introduced between P and mirror M2. This plate P1 is called a compensating plate.The plate P and P1 are of equal thickness being cut from a single optical plane parallel plate to ensure the equality of thickness and the nature of the material.The function of the compensating plate is that the Ray of light traveling towards M1 and M2 must travel equally **pass through the glass plate.** The phase changes on reflection at mirror M1 and M2 are smaller, the phase changes due to reflection in air and class are also similar, is equal to **π.** The two rays reaching the telescope interfere constructively or destructively depending upon the path difference.The path difference between the two rays reaching the telescope is mλ**. Where M is the integer and λ is the wavelength of light used.** Then constructive interference takes place. On the other hand, if the path difference is (2m+1) **λ/2,** Then destructive interference occurs. The path difference between the two beams AM1A and AM2A can be altered by moving mirror M1.

https://www.youtube.com/watch?v=lzBKlY4f1XA

TYPES OF FRINGES IN MICHELSON INTERFEROMETER

The shape of the fringes formed in the michelson interferometer depends on the inclination between mirror M1 and M2. Let M2 be the virtual image of mirror M1 as S' be the image of source S. Let D be the separation between mirror M1 and the virtual image M2' of the mirror M2. Therefore, the interference pattern formed will be due to air film enclosed between M1 and M2'. Let S1' and S2' be the images of the Saras M1 and M2 respectively. It means, we have two coherent sources S1' and S2' to obtain due to the process of the division of amplitude.

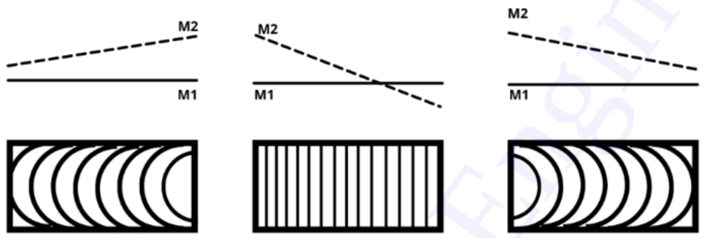
CIRCULAR FRINGES

If the mirror M1 and M2 are exactly at right angles, then the mirror ember and the image M2' two of the mirror M2 are parallel to each other.

Then circular fringes are formed. Let D be the distance of mirror M1 and mirror M2 from the plate P then the distance between M1 and M2' who is also d. It means, the thickness of air film enclosed between M1 and M2' is d as shown in the figure.

However, the distance between two coherent sources S1' and S2' is 2d. The path difference between the rays coming from S1' and S2'=2d cos**θ.** Where **θ** is the angle of inclination of the ray Falling On The Mirror.

Localised Fringes



When mirrors M1 and M2' are not exactly parallel or not equally distant, a wedge-shaped film is formed between them. The path of two reflected rays, originating from the same incident ray why reflection from M1 and M2' you are no longer parallel.

They intersect near M1 and hence the fringes are formed near M1. The fringes are called localized fringes and to see them the eye must be focused on the vicinity of M1.

These fringes are curved with their convex side toward the thin edge of the wedge as shown in the figure. The thin edge of the wedge is to the left and therefore the observer fringes are convex toward the left.

As we go on decreasing the separation between M1 and M2', the fringes move across the field of view away from the thin as of the wedge and at the same time gradually become straight. When M1 and M2' intersect, the lines are perfectly straight as shown in the figure.

We have to wedges opposing each other so the line should appear curved on both sides of the intersection but for a small field of view, they appear straight. When M1 is still moved such that the mirror M1 and the virtual image M2' of mirror M2 get a position as shown in the figure.

The fringes are again curved but with their convex side towards the right. Localized fringes become invisible for large path differences of the order of several millimeters.

White Light Fringes

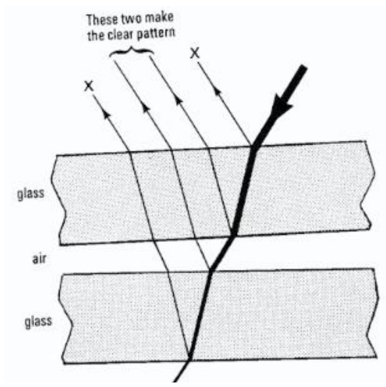
If monochromatic light is replaced by white light, its constituent wavelength gives rise to its own set of fringes two different widths. The zero-order fringes corresponding to each wavelength will coincide and hence we get a dark central fringe.
But if the path difference between the interfering rays is considerable. The central fringe will be surrounded by a few colored fringes. And then there will be so much overlapping that the pattern appears to be white.
The importance of these white light fringes is that the position of zero-order fringe which is dark can be located very easily in the Michelson interferometer.

Applications of Michelson Interferometer

Michelson Interferometer is used to determine:

- Wavelength of monochromatic light.
- The refractive index of a thin film.
- Resolution of spectral lines.
- The evolution of meters in terms of the wavelength of light.
- The angular diameter of stars.
- Presence of ether.
- The accuracy of the surface of the prism and lens.

THEORY OF AIR WEDGE

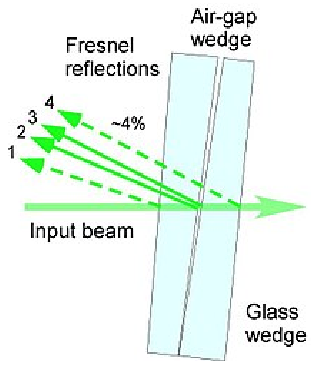


There are four streams of reflected light: two from the inner faces of the sandwich, where the glass meets the thin wedge of air; and two from the outer surfaces of the glass plates. The two streams from the inner surfaces have a small path difference (about twice the thickness of the air wedge at each place}; and you will see the interference bands of bright black and yellow. The streams reflected from the outer surfaces of the glass plates have too great a path difference to show an interference pattern noticeably.

More advanced students should understand that light reflected from top surface of the lower slide undergoes a phase change of pi (180 degrees). This means that a bright fringe is formed whenever the path difference between the two waves is twice the thickness of the air gap plus a half wavelength.

Suppose you counted the stripes all the way from one end of the sandwich to the other. Knowing the wavelength of yellow light (about 600 nm), you can estimate the thickness of very thin materials. Newton discovered that, when a thin lens is placed on a flat piece of glass, the circular air film between lens and plate will produce circular fringes. These are known as Newton's rings.

The air-wedge shearing interferometer is probably the simplest type of interferometer designed to visualize the disturbance of the wavefront after propagation through a test object. This interferometer is based on utilizing a thin wedged air-gap between two optical glass surfaces and can be used with virtually any light source even with non-coherent white light.



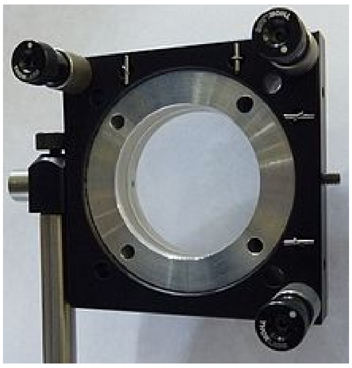
An air-wedge shearing interferometer is described in[1] and was employed in set of experiments described in. This interferometer consists of two optical glass wedges (~2-5deg), pushed together and then slightly separated from one side to create a thin air-gap wedge. This air-gap wedge has a unique property: it is very thin (micrometer scale) and it has perfect flatness (~λ/10).
There are four nearly equal intensity Fresnel reflections (~4% for refraction coefficient 1.5) from the air-wedge interferometer (Fig.)

1. from the exterior surface of the first glass block
2. from the interior surface of the first glass block
3. from the interior surface of the second glass block
4. from the exterior surface of the second glass block

The angle between beams 1-2 and 3-4 is non adjustable and depends only on the shape of the glass wedge. The angle between beams 2-3 is easily adjusted by varying the air-wedge angle. The distance between the air-wedge and an image plane should be long enough to spatially separate reflections 1 from 2 and 3 from 4The overlap of beams 2 and 3 in the image plane creates an interferogram.

ALIGNMENT

To minimize image aberrations the angle plane of the glass wedges has to be placed orthogonal to the angle plane of the air-wedge. Because intensity of Fresnel reflections from a glass surface are polarization and angle dependent, it is necessary to keep the air-wedge plane nearly perpendicular to the incident beam (± 5deg) to minimize instrumentally induced intensity variation. This is very important when coupling the air-wedge interferometer to imaging optics. The air-wedge interferometer has a very simple design and requiring only 2 standard BK7 glass wedges and 1 mirror holder.



APPLICATIONS

Because of its extremely thin air-gap, the air-wedge interferometer was successfully applied in experiments with femto-second high-power lasers. Figure 4 shows an interferogram of laser interactions with a He jet in a vacuum chamber. The probing beam has ~500-fs duration, and ~1-μm wavelength. The air-wedge interferogram from even this very short coherence length laser beam exhibits clear, high-contrast interference lines.

ADVANTAGES

The air-wedge shearing interferometer is similar to the classical shearing interferometer but is micrometers thick, can operate with virtually any light source even with non-coherent white light, has an adjustable angular beam split, and uses standard inexpensive optical elements. Replacement of the second glass wedge by a plane-concave lens, will turn the lateral-shearing air-wedge interferometer to a radial-shearing interferometer, which is important for some specific applications.

The principle of interference from the air-wedge between two plane-parallel glass plates is described in a number of elementary optics textbooks. But this "classical" air-wedge arrangement has never been used for interferometry with field visualization owing to the overlap of all four reflected beams in the image plane. Design described in this article eliminates this obstruction and makes the air-wedge interferometer effective for practical applications with a visualization field interferometry.

LASERS - Introduction

LASER stands for **L**ight **A**mplification by **S**timulated **E**mission of **R**adiation. Laser is a device which emits a powerful, monochromatic collimated beam of light. The emitted light waves are coherent in nature.

The first laser, ruby laser was invented by Dr.T.H. Maiman in the year 1960. Since then, the development of lasers is extremely rapid. The laser action is being demonstrated in many solids, liquids, gases and semiconductor.

CHARACTERISTICS OF LASER

Laser is basically a light source. Laser light has the following important characteristics

- High Directionality
- High Intensity
- Highly Monochromatic
- Highly Coherence

Directionality

Ordinary light spreads in all directions and its angular spread is 1m/m.

Fig. Directionality property of Laser

But it is found that laser is highly directional and is angular spread is 1mm/m. For example, the laser beam can be focused to very long distance with a few divergences or angular spread shown in Fig. 1.1.

Intensity

Since an ordinary light spread in all directions, the intensity reaching the target is very less.

But in the case of laser, due to high directionality, the intensity of laser beam reaching the target is of high intense beam. For example, 1 mill watt power of He-Ne laser appears to be brighter than the sunlight (Fig. 1.2).

Fig. Intensity variation

Monochromatic

Laser beam is highly monochromatic; the wavelength is single, whereas in ordinary light like mercury vapour lamp, many wavelengths of light are emitted Fig..

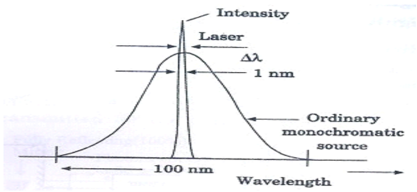


Fig. Monochromaticity nature of Laser

Coherence

It is an important characteristic of laser beam. In lasers the wave trains of same frequency are in phase, the radiation given out is in mutual agreement not only in phase but also in the direction of emission and polarization. Thus, it is a coherent beam. Due to high coherence, it results in an extremely high power. Fig. shows the coherence nature of Laser

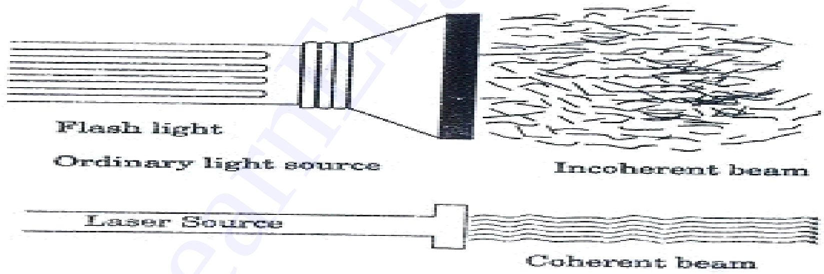


Fig. shows the coherence property of Laser

Differences between ordinary light and Laser beam.

S.No.	**Ordinary light**	**Laser beams**
1	In ordinary light the angular spread is more	In laser beam the angular spread is less.
2	They are not directional.	They are highly directional.
3	It is less intense	It is highly intense
4	It is not a coherent beam and is not in phase.	It is a coherent beam and is in phase
5	The radiation are polychromatic	The radiations are monochromatic
6	Example: Sun light, Mercury vapor lamp	He- Ne Laser, Co2 laser

STIMULATED ABSORPTION, SPONTANEOUS EMISSION AND STIMULATED EMISSION

Process 1 - Stimulated absorption

$h\nu$ An atom in the lower energy level or ground state energy level E_1 absorbs the incident photon radiation of energy and goes to the higher energy level or excited level E_2 as shown in figure. This process is called absorption.

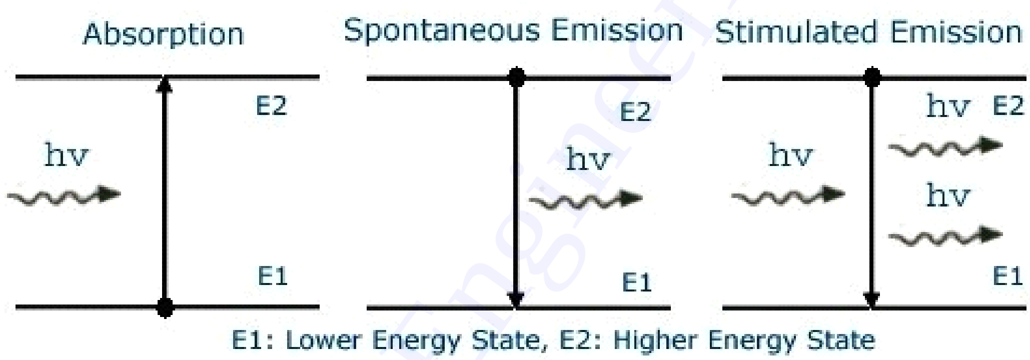


Fig. Absorption and emission process in Laser

Process 2- Spontaneous Emission

The atom in the excited state returns to the ground state by emitting a photon of energy

$E = (E_2 - E_1) = h\nu$ spontaneously without any external triggering as shown in the figure. This process is known as spontaneous emission. Such an emission is random and is independent of incident radiation.

Process 3 - Stimulated Emission

The atom in the excited state can also return to the ground state by external triggering or inducement of photon thereby emitting a photon of energy equal to the energy of the incident photon, known as stimulated emission. Thus, results in two photons of same energy, phase difference and of same directionality as shown.

Differences between Stimulated and spontaneous emission of radiation

S. No.	Stimulated Emission	Spontaneous emission
1.	An atom in the excited state is induced to return to the ground state , thereby resulting in two photons of same	The atom in the excited state returns to the ground state thereby emitting a photon,

	frequency and energy is called Stimulated emission	without any external inducement is called Spontaneous emission.
2	The emitted photons move in the same direction and is highly directional	The emitted photons move in all directions and are random
3	The radiation is highly intense, monochromatic and coherent	The radiation is less intense and is incoherent
4	The photons are in phase, there is a constant phase difference	The photons are not in phase (i.e.) there is no phase relationship between them.
5	The rate of transition is given by $R_{12}(Sp) = A_{21} \quad N_2$	The rate of transition is given by $R_{12}(Sp) = A_{21} \quad N_2$

POPULATION INVERSION

Population Inversion creates a situation in which the number of atoms in higher energy state is more than that in the lower energy state. Usually at thermal equilibrium, the number of atoms N_2 i.e., the population of atoms at higher energy state is much lesser than the population of the atoms at lower energy state N_1 that is $N_1 > N_2$. The Phenomenon of making $N_2 > N_1$ is known as Population Inversion (Fig. 1.6).

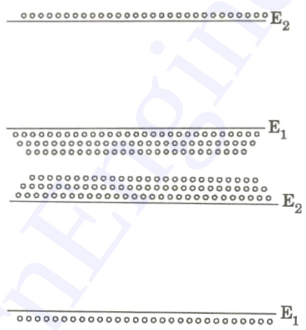


Fig. Population Inversion

Condition for Population inversion

- There must be at least two energy levels $E_2 > E_1$.
- There must be a source to supply the energy to the medium.
- The atoms must be continuously raised to the excited state.

META STABLE STATES

An atom can be excited to a higher level by supplying energy to it. Normally, excited atoms have short life times and release their energy in a matter of nano seconds (10^{-9}) through spontaneous emission. It means atoms do not stay long to be stimulated. As a result, they undergo spontaneous emission and rapidly return to the ground level; thereby population inversion could not be established. In order to do so, the excited atoms are required to 'wait' at the upper energy level till a large number of atoms accumulate at that level. In other words, it is necessary that excited state have a longer lifetime.

A Meta stable state is such a state. Metastable can be readily obtained in a crystal system containing impurity atoms. These levels lie in the forbidden gap of the host crystal. There could be no population inversion and hence no laser action, if metastable states don't exist.

EINSTEIN'S "A & B" COEFFICIENTS - DERIVATION

We know that, when light is absorbed by the atoms or molecules, then it goes from the lower energy level (E_1) to the higher energy level (E_2) and during the transition from higher energy level (E_2) to lower energy level (E_1) the light is emitted from the atoms or molecules. Fig.process involved in Laser.

Let us consider an atom exposed to light photons of energy, three distinct processes take place. ***(a). Absorption (b). Spontaneous emission (c). Stimulated Emission***

- ***Absorption***

An atom in the lower energy level or ground state energy level E_1 absorbs the incident photon radiation of energy $h\nu$ and goes to the higher energy level or excited level E_2 as shown in figure. This process is called absorption.

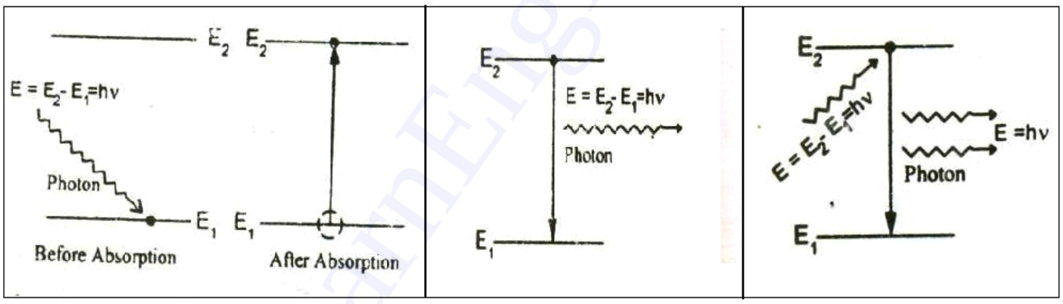


Fig. Various process involved in Laser

If there are many numbers of atoms in the ground state then each atom will absorb the energy from the incident photon and goes to the excited state then,
The rate of absorption (R_{12}) is proportional to the following

$R_{12} \propto \rho_\nu N_1 \quad \text{---}\rightarrow \quad (1) \quad = \quad B_{12}\rho_\nu N_1 \quad \text{---}\rightarrow \quad (2)$ = Energy density of incident radiation,

N_1 = no. of atoms in the ground state and B_{12}- is a constant which gives the probability of absorption transition per unit time. E Normally, the atoms in the excited state will not stay there for a long period of time, rather it comes to ground state by emitting a photon of energy $= h\nu$. Such an emission takes place by one of the following two methods.

- ***Spontaneous emission:***

$h\nu$ The atom in the excited state returns to the ground state by emitting a photon of energy

E = ($E_2 - E_1$) = spontaneously without any external triggering as shown in the figure. This process is known as spontaneous emission. Such an emission is random and is independent of incident radiation. If N_1 and N_2 are the numbers of atoms in the ground state (E_1) and excited state (E_2) respectively, then

$$R_{21}(Sp) \quad = A_{21} \quad \dashrightarrow \quad (4) \quad \dashrightarrow$$

The rate of spontaneous emission is $R_{21}(Sp)$ (3)

Where A_{21} is a constant which gives the probability of spontaneous emission transitions per unit time.

- ***Stimulated Emission:***

The atom in the excited state can also return to the ground state by external triggering or inducement of photon thereby emitting a photon of energy equal to the energy of the incident photon, known as stimulated emission. Thus results in two photons of same energy, phase difference and of same directionality as shown.

$$R_{21}(St) \propto \rho_\nu N_2 \quad \dashrightarrow \quad (5)$$

Therefore, the rate of stimulated emission is given by

$$R_{21}(St) = B_{21}\rho_\nu N_2 \quad \dashrightarrow \quad (6)$$

Where B_{21} is a constant which gives the probability of stimulated emission transitions per unit time.

Einstein's theory

Einstein's theory of absorption and emission of light by an atom is based on Planck's theory of radiation. Also under thermal equilibrium, the population of energy levels obeys the Maxwell Boltzmann distribution law

$= \quad B_{12}\rho_\nu N_1 = A_{21}$ ***Under thermal equilibrium***

(or) $\quad B_{21}N_2) = A_{21} \quad N_2 \quad B_{12}\rho_\nu N_1 -$

$\rho_\nu \quad \dashrightarrow \quad (7) \quad \rho_\nu$

We know from the Boltzmann distribution law

$$N_2 = N_0\, e^{-E_1/K_B T}$$

Where K_B is the Boltzmann Constant, T is the absolute temperature and N_0 is the number of atoms at absolute zero.

At equilibrium, we can write the ratio of population levels as follows

Substituting equation (8) in equation (9)

$$\frac{N_1}{N_2} = e^{(E_2 - E_1)/KBT}$$

This equation has a very good agreement with Planck's energy distribution radiation law.

Therefore comparing equations (6) and (7) , we can write

$8\pi h\nu^3 \quad B_{12}$

$1 = = B \quad \dashrightarrow \quad (12) \quad$ and $\quad B_{21}$ Taking $A_{21} = A$

The constants ***A*** and ***B*** are called as Einstein Coefficients, which accounts for spontaneous and stimulated emission probabilities.

Generally Spontaneous emission is more predominant in the optical region (Ordinary light). To increase the number of coherent photons stimulated emission should dominate over spontaneous emission. To achieve this, an artificial condition called Population Inversion is necessary.

PRINCIPLE OF LASER ACTION

Let as consider many number atoms in the excited state. We know the photons emitted during stimulated emission have same frequency, energy and are in phase as the incident photon. Thus result (fig. 1.7) in 2 photons of similar properties.

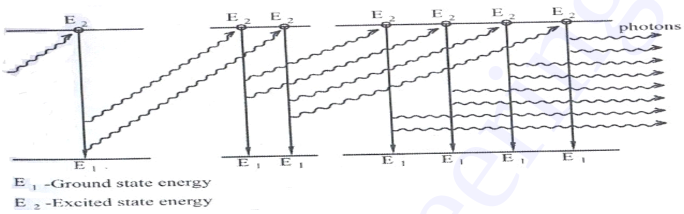


Fig. Amplification in Laser process

These two photons induce stimulated emission of 2 atoms in excited state there by resulting in 4 photons. These 4 photons induce 4 more atoms and give rise to 8 photons etc., as shown in figure.

Principle:

Due to stimulated emission the photons multiply in each step-giving rise to an intense beam of photons that are coherent and moving in the same direction. Hence the light is amplified by Stimulated Emission of the Radiation termed LASER.

ACTIVE MEDIUM

A medium in which population inversion can be achieved is known as active medium.

ACTIVE CENTER

The material in which the atoms are raised to the excited state to achieve Population Inversion is called Active Center.

PUMPING ACTION

The process to achieve the population inversion in the medium is called Pumping action. It is essential requirement for producing a laser beam.

Methods of pumping action

The methods commonly used for pumping action are:

- Optical pumping (Excitation by Photons)
- Electrical discharge method (Excitation by electrons)
- Direct conversion
- In elastic atom – atom collision between atoms

- **Optical pumping**

When the atoms are exposed to light radiations energy, atoms in the lower energy state absorb these radiations and they go to the excited state. This method is called Optical pumping. It is used in solid state lasers like ruby laser and Nd-YAG laser. In ruby laser, xenon flash lamp is used as pumping source.

- **Electrical discharge method (Excitation by electrons)**

In this method, the electrons are produced in an electrical discharge tube. These electrons are accelerated to high velocities by a strong electrical field. These accelerated electrons collide with the gas atoms.

By the process, energy from the electrons is transferred to gas atoms. Some atoms gain energy and they go to the excited state. This results in population inversion. This method is called Electrical discharge method. It is represented by the equation

$$A + e^* = A^* + e$$

Where A – gas atom in the ground state A* = same gas atom in the excited state e* = Electrons with higher Kinetic energy e – Same electron with lesser energy.

- This method of pumping is used in gas lasers like argon and CO_2 Laser.
- **Direct Conversion**

In this method, due to electrical energy applied in direct band gap semiconductor like Ga As, recombination of electrons and holes takes place. During the recombination process, the electrical energy is directly is converted into light energy.

- **In elastic atom – atom collision**

In this method, a combination of two gases (Say A and B are used). The excited states of A and B nearly coincides in energy.

In the first step during the electrical discharge atoms of gas A are excited to their higher energy state A* (metastable state) due to collision with the electrons . $A + e^* = A^* + e$

Now A* atoms at higher energy state collide with b atoms in the lower state. Due to inelastic atom - atom collision B atoms gain energy and they are excited to a higher state B*. Hence, A atoms lose energy and return to lower state. $A^* + B = A + B^*$

OPTICAL RESONATOR

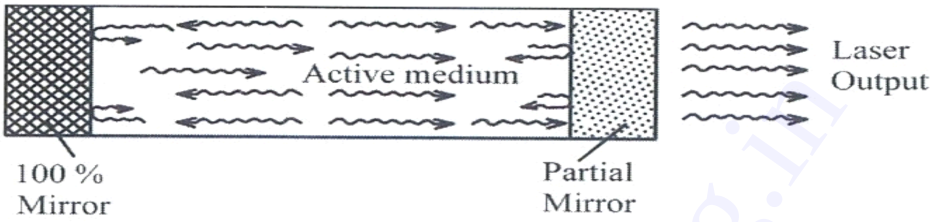


Fig. 1.8 View of optical resonator

An optical resonator consists of a pair of reflecting surfaces in which one is fully reflecting (R_1) and the other is partially reflecting (R_2). The active material is placed in between these two reflecting surfaces. The photons generated due to transitions between the energy states of active material are bounced back and forth between two reflecting surfaces. This will induce more and more stimulated transition leading to laser action.

- Interaction of radiation with matter is better explained using concept of photon rather than by the wave concept.
- Energy exchange can take place only at certain discrete values for which the photon energy is the minimum energy unit that light can give or accept.
- Wave picture of light is Classical and Photon picture is Quantum Mechanical.
- Laser- inherently a Quantum Mechanical device its operation depends on the existence of photons.
- Maxwell: Light belongs to group of EM waves; propagate with speed "c" in vacuum.

Frequency and wavelength related through Light incident on a substance, may undergo reflection, transmission, absorption and scattering to varying degrees depending on nature of substance.

- Results in loss of energy and hence decrease in light intensity with distance
- Absorption or Attenuation
- Attenuation Coefficient (a) - A measure of absorption of light in an optical medium. Is different for different medium and is a function of incident energy.
- At temperature above 0K, Atoms always have some thermal energy;
- Distributed among available energy levels according to their energy.
- At Thermal Equilibrium; Population at each energy level decreases with increase of energy level, For energy levels E_1 and E_2,
- Populations can be computed with Boltzmann¢s equation Ratio of populations, N_2/N_1 is called Relative Population.

- Relative Population (N_2/N_1); dependent on two factors Energy difference (E_2-E_1)

Temperature, T - At Lower Temperature; All atoms are in the ground states. At higher Temperature; Atoms move to higher states

Important Conclusions

As long as the material is in thermal equilibrium, the population of the higher state cannot exceed the population of lower states

- Excitation: Electron in the ground state receives an amount of energy equal to the difference of energy of ground state and one of the excited states, absorbs energy and jumps to the excited state. Electron cannot stay in the excited state for a longer time.
- Has to get rid of the excess energy in order to come to the lower energy level
- Only mechanism is through emission of a photon.
- De-excitation: The excited electron emits a photon of energy, hn= (E_2 - E_1) and jumps from excited state to the ground state ⇒ Spontaneous Emission
- METASTABLE STATE
- An atom can be excited to a higher level by supplying energy to it. Normally, excited states have short lifetimes ⇒ nanoseconds (10^{-9} s) and release their excess energy by spontaneous emission.
- Atoms do not stay at such excited states long enough to be stimulated to emit their energy. Though, the pumping agent continuously raises the atoms to the excited level, many of them rapidly undergo spontaneous transitions to the lower energy level Population inversion cannot be established.
- For establishing population inversion, the excited atoms are required to "wait" at the upper lasing level till a large number of atoms accumulate at that level.

longer-lived upper levels from where an excited atom does not return to lower level at once, but remains excited for an appreciable time, are known as Metastable States.

- Atoms stay in metastable states for about 10^{-6} to 10^{-3}s. This is 10^3 to 10^6 times longer than the time of stay at excited levels.
- Possible for a large number of atoms to accumulate at a metastable level.
- The metastable state population can exceed the population of a lower level and lead to the state of population inversion.
- If the metastable states do not exist, there could be no population inversion, no stimulated emission and hence no laser operation.

THRESHOLD CONDITION

Light bouncing back and forth in the optical resonator Undergoes amplification as well as suffers various losses Losses occur mainly due to

- Transmission at the output mirror
- Scattering & Diffraction of light within the active medium. For the proper build up of oscillations
- Essential is that the amplification between two consecutive reflections of light from

reflecting end mirror can balance losses.

- Determination of threshold gain by considering the change in intensity of a beam of light undergoing a round trip within the resonator ?
- Consider the laser medium fills the space between the mirrors M_1 & M_2, of reflectivity R_1 & R_2 respectively and mirrors separated by a distance L.
- Let I_0 - the intensity of the light beam at M_1
- Traveling from mirror M_1 to mirror M_2 ⇒ beam intensity increases from I_0 to I(L), After reflection at M_2, the beam Intensity will be;
- After a complete round trip (Reflection from M_1), the final Intensity will be

$$I(2L) = R_1 R_2 I_0 e^{(\gamma - \alpha_s)2L}$$

Growth of output Power Through Cavity

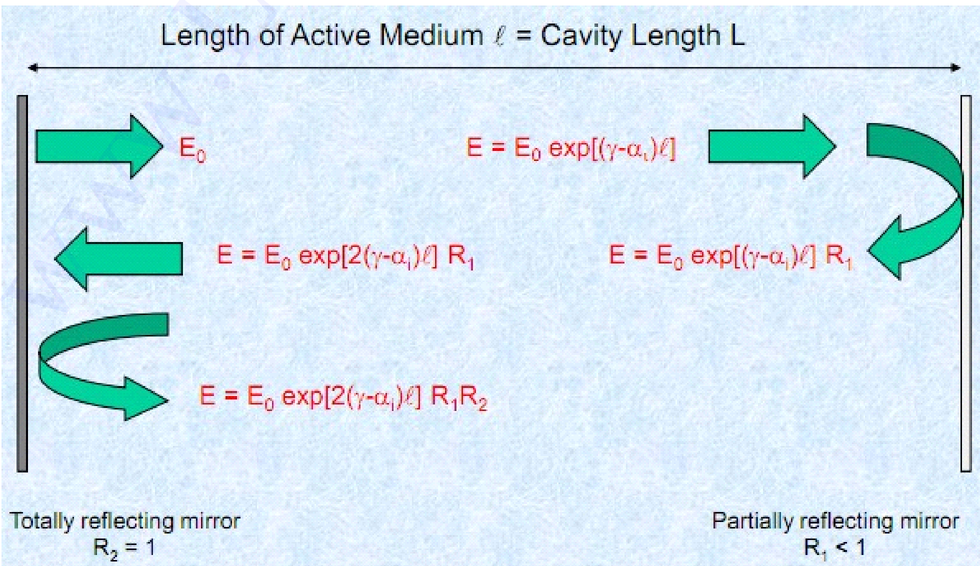


Fig. setup of optical resonator

Consider the laser medium fills the space between the mirrors M_1 & M_2, of reflectivity R_1 & R_2 respectively and mirrors separated by a distance L

Let I_0 - the intensity of the light beam at M_1 Let E_0 – the Energy of the light beam at M_1

Product R_1 R_2 represents the losses at the mirrors, whereas a_s includes all the distributed losses such as scattering, diffraction and absorption occurring in the medium.

$$2L(\gamma-\alpha_s) \geq - \ln(R_1 R_2)$$

$$(\gamma-\alpha_s) \geq -\frac{1}{2L}\ln(R_1 R_2)$$

$$\gamma \geq \alpha_s - \frac{1}{2L}\ln(R_1 R_2)$$

$$\gamma \geq \alpha_s + \frac{1}{2L}\ln\left(\frac{1}{R_1 R_2}\right)$$

Condition for Lasing

Shows that the initial gain must exceed the sum of losses in the cavity. The condition is used to determine the threshold value of pumping energy necessary for lasing action.

'g - Amplification of the laser, dependent on how hard the laser medium is pumped. As the pump power is slowly increased, a value of 'g_{th}' called threshold value will be reached and the laser starts oscillating.

Threshold value 'g_{th}' is given by

$$\gamma_{th} = \alpha_s + \frac{1}{2L}\ln\left(\frac{1}{R_1 R_2}\right)$$

For the laser to oscillate, $\gamma > \gamma_{th}$ $\Rightarrow$ **Threshold condition for lasing**

- This states the criterion when the net gain would be able to counteract the effect of losses in the cavity

Value of ‘g must be atleast ‘g_{th}’ for laser oscillations to commence If $g > g_{th}$ the waves grow and the amplifier reaches saturation.

It lowers the value of g in turn and eventually an equilibrium value is attained at g_{th}

Febry-Perot resonator

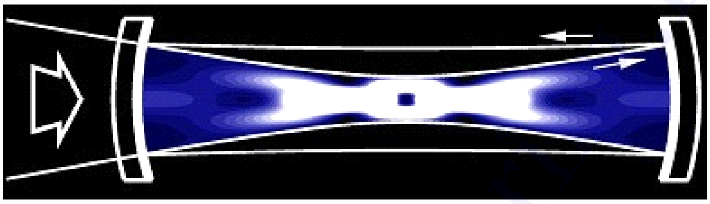


Fig. View of Febry-Perot resonator

Laser Modes

A wave of frequency n, that travel along the axis of cavity forms a series of standing waves within the cavity. They are discrete resonant conditions determined by the physical dimensions of the cavity. Modes governed by the cross-sectional dimension of the optical cavity - Transverse modes. Modes governed by the axial dimension of the resonant cavity - Longitudinal or Axial modes. In a cavity flanked by two plane parallel mirrors, the standing waves in the cavity satisfy the condition. The axial modes contribute to a single spot of light in the laser spot.

Pumping Schemes

- Atoms characterized by a large number of energy levels.
- Only two, three or four levels are pertinent to the pumping process.
- Types are - Two-level, Three-level and Four –level schemes.

Two-level Pumping system: Appears to be most simple and straight-forward method to establish population inversion; Pumping an excess of atoms into the higher energy state by applying intense radiation.

A two-level pumping scheme is not suitable for attaining P.I.

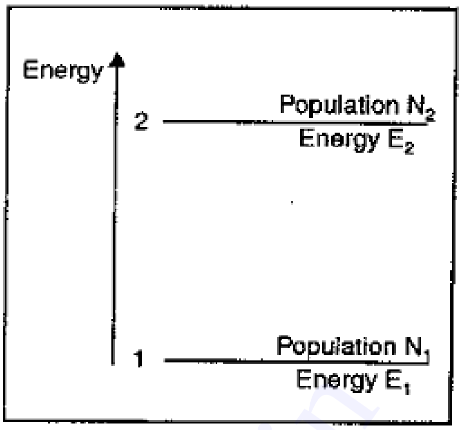


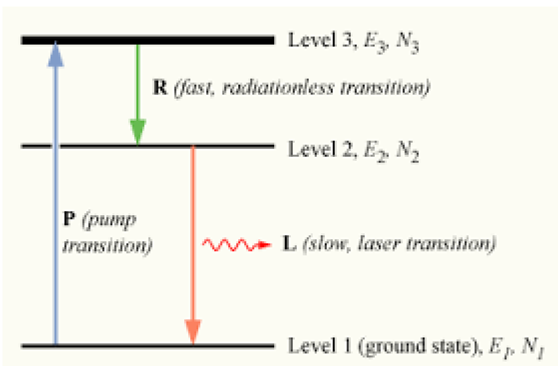
Fig. Two level Laser system

P.I. requires the lifetime Dt of upper level E_2 must be longer.

- Achieving population inversion in a two-level atom is not very practical.
- Such a task would require a very strong pumping transition that would send any decaying atom back into its excited state.
- This would be similar to reversing the flow of water in a water fall. It can be done, but is very energy costly and inefficient.
- In a sense, the pumping transition would have to work against the lasing transition.
- That is to say, once the population inversion is achieved the laser would lase.
- But immediately it would end up with more atoms in the lower level.
- Such two-level lasers involve a more complicated process.
- inversion is a familiar prevalent physical system this is not the usual case. Because the probalities for raising an electron to the upper level and inducing the decay of electrons to lower level known as population inversion are exactly the same, so optical pumping will at most only achieve equal population of a two-level system.
- In simple words, when both levels are equally populated,the no of electrons going up and down will be same and so the most important ingredient population inversion cannot be achieved in case of two levels.
- The only way out is to use a third METASTABLE STATE to solve the problem.

Three Level Pumping Scheme

A three level scheme; Lower level is either the ground state or a level whose separation from the ground state is small compared to kT. A photon of hn ($=E_2-E_1$) can induce



stimulated emission and laser action. Major disadvantage of a three level scheme ⇒ it requires very high pump powers. Terminal level of the laser transition is the ground state.

As the ground state is heavily populated, large pumping power is to be used to depopulate the ground level to the required extent ($N_2 > N_1$)
<u>Three level scheme can produce light only in Pulses</u>.

- Once stimulated emission commences, the metastable state E_2 gets depopulated very rapidly and the population of the ground state increases quickly.

- As a result the population inversion ends. One has to wait till the population inversion is again established.

- Three level lasers operate in Pulsed Mode.

Four Level Pumping Scheme

E_3 - a metastable level

❖ Laser transition takes the atoms to the level E2

❖ Atoms lose the rest of their excess energy & finally reach the ground state E1.

❖ Atoms are once again available for excitation.

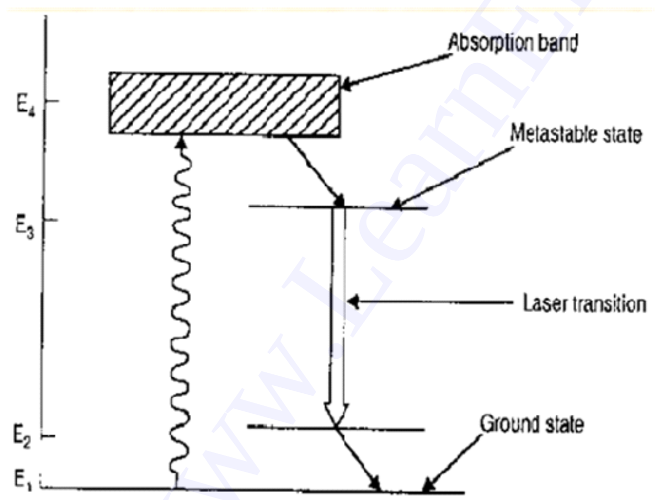


Fig. Four level Laser System

In Four level scheme, the terminal laser level E_2 is well above the ground level such that $(E_2-E_1) >> kT$.

- It guarantees that the thermal equilibrium population of E_2 level is negligible.
- In contrast to three level scheme, the lower laser transition level in four level scheme is not the ground state and is virtually vacant.
- It requires less pumping energy than does a three level laser.

- Further, the lifetime of the lower laser transition level E_2 is much shorter, hence atoms in level E_2 quickly drop to the ground state.
- This steady depletion of E_2 level helps sustain the population inversion by avoiding an accumulation of atoms in the lower lasing level fig.
- Four level lasers can operate in Continuous Wave mode
- Most of the working lasers are based on Four Level Scheme

Comparison of Three level and Four level Systems Three level laser,

$N_{th} = (N_2\text{-}N_1)$ and $N_0 = N_2+N_1$

For the laser to begin lasing ; $N_2 > \frac{N_0}{2} + \frac{N_{th}}{2}$

As $N_0 >> N_{th}$, therefore, $N_2 > \frac{N_0}{2}$

Four level laser

$$N_2 = N_1 \exp\left(\frac{(E_2 - E_1)}{kT}\right)$$

For the laser to begin lasing; $(N_3\text{-}N_2) > N_{th}$ i.e. $N_3 > N_{th}$

$$\therefore \quad \frac{(N_{th})_{3-level}}{(N_{th})_{4-level}} = \frac{N_0}{2N_{th}} \Rightarrow \textbf{a very large quantity}$$

Implies that it is much easier to pump a four level laser than a three level laser. This is the reason why most of the lasers are of four-level.

TYPES OF LASERS

Solid State Lasers – Ruby and Nd-YAG Laser – Gas Lasers – He-Ne and CO_2 lasers – semiconductor lasers – Heterojunction Lasers – Liquid Dye Lasers

Several ways to classify the different types of lasers What material or element is used as active medium Mode of operation : CW or Pulsed

- Classification may be done on basis of other parameters Gain of the laser medium
- Power delivered by laser Efficiency or Applications

Preference to classify the lasers on the basis of material used as Active Medium. Broadly divided into four categories

Solid lasers Gas lasers Liquid lasers

Semiconductor lasers

TYPES OF LASERS

Based on the type of active medium, Laser systems are broadly classified into the following categories.

S.No	TYPES OF LASER	EXAMPLES
1.	Solid State laser Ruby Laser	Nd:YAG laser
2.	Gas laser He-Ne Laser,	CO_2 Laser, Argon – ion laser
3.	Liquid Laser	$SeOCL_2$ Laser, Europium Chelate Laser
4.	Dye laser	Rhodamine 6G laser, Coumarin dye laser
5	Semiconductor Laser	GaAs laser, GaAsP laser

Solid state Laser - NdYAG LASER

It is a solid state and 4 level system as it consists of 4 energy levels. Nd ion is rare earth metal and it is doped with solid state host.Due to doping, yttrium ions get replaced by the Nd^{3+} ions. Also, the doping concentration is around 0.725% by weight.

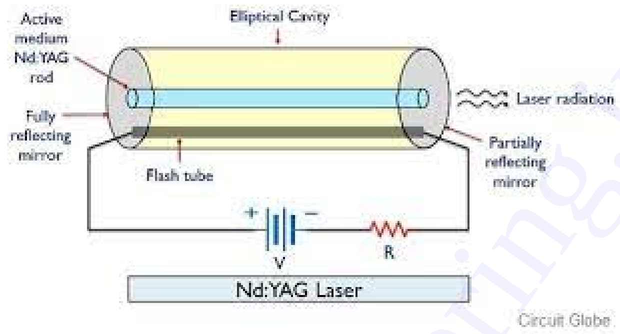
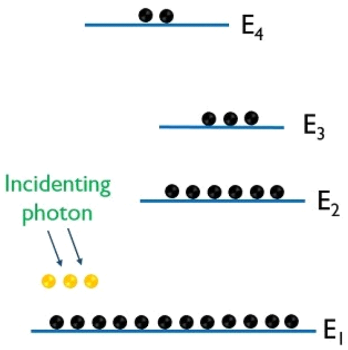


Fig. Construction of Nd:YAG laser

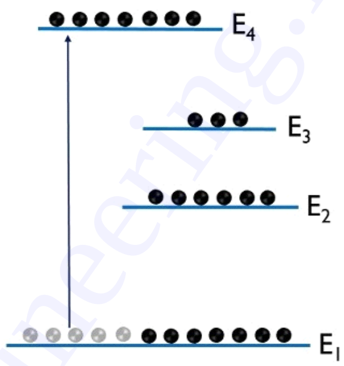
Active medium: when the external energy source is provided then the electrons from lower energy state moves to higher energy state thereby causing lasing action

External Energy source: optical pumping, xenon or krypton flash tube is taken Nd:YAG rod and the flash tube are placed inside an elliptical cavity

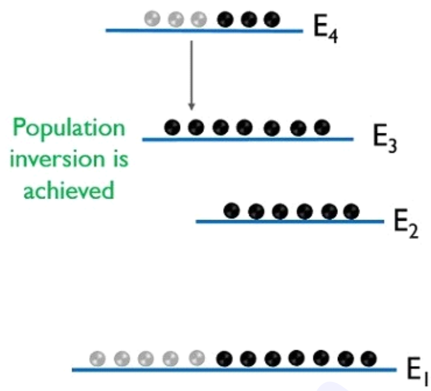
Optical resonator: two ends of the Nd:YAG rod is coated with silver. - to achieve maximum light reflection. other end is partially coated in order to provide a path for the light ray from an external source to reach the active medium. E_1 is the lowest energy state while E_4 is the highest energy level, electrons present in the energy state E_1 gains energy and moves to energy state E_4.E_4 is an unstable state. electrons that were excited to this state by the application external pumping will not stay at this state for much longer duration and comes to lower energy state E_3 very fastly but without radiating any photon. E_3 is the metastable state and exhibits longer lifespan. Thereby attaining population inversion. lifetime of the electrons at the metastable state gets exhausted then these electrons by releasing photons come to lower energy state E_2. E_2 also exhibit shorter lifespan like E_4. Thus, electrons present in E_2 state will come to E_1



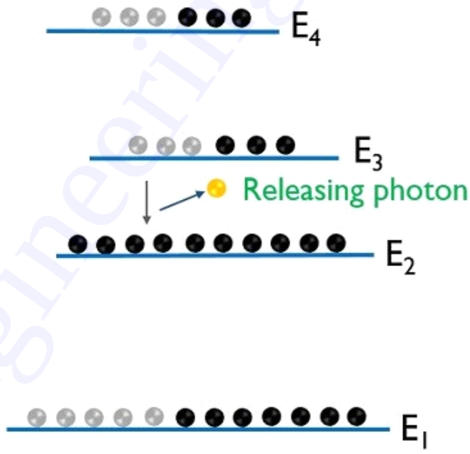
When external pumping is provided to the active medium of the Nd:YAG laser.



Due to external pumping electron in lowest energy state gains energy and move to highest unstable energy state.

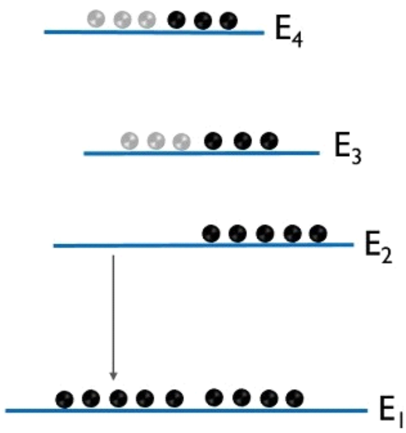


Once carrier lifespan is exhausted at unstable state, electrons come to metastable state and population inversion is achieved.



The electrons on coming to E_2 state emits energy in the form of photon.

Electrons by gaining single photon of energy releases the energy of 2 photons. Also, as the system is equipped with optical resonators so, more number of photons will get generated as the pumped energy will get reflected inside the active medium.



Due to shorter lifespan of E_2, electrons comes back to E_1 state.

several electrons on stimulation produce photons thereby generating a coherent laser beam of 1.064 µm.

Applications of Nd:YAG Laser

Military applications to find the desired target. Application in medical field for the surgical purpose.

Used in welding and cutting of steel and

Used in communication system

MOLECULAR GAS LASER -CO_2

In a molecular gas laser, laser action is achieved by transitions between vibrational and rotational levels of molecules. Its construction is simple and the output of this laser is continuous.

Molecular Gas laser

In a molecular gas laser, laser action is achieved by transitions between vibrational and rotational levels of molecules. Its construction is simple and the output of this laser is continuous. In CO2 molecular gas laser, transition takes place between the vibrational states of Carbon dioxide molecules.

CO2 **Molecular gas laser**

It was the first molecular gas laser developed by Indian born American scientist Prof.C.K.N.Pillai. It is a four level laser and it operates at 10.6 µm in the far IR region. It is a very efficient laser.

Energy states of CO2 molecules.

A carbon dioxide molecule has a carbon atom at the center with two oxygen atoms attached, one at both sides. Such a molecule exhibits three independent modes of vibrations.
They are

- Symmetric stretching mode.
- Bending mode
- Asymmetric stretching mode.
- **Symmetric stretching mode**

In this mode of vibration, carbon atoms are at rest and both oxygen atoms vibrate simultaneously along the axis of the molecule departing or approaching the fixed carbon atoms.

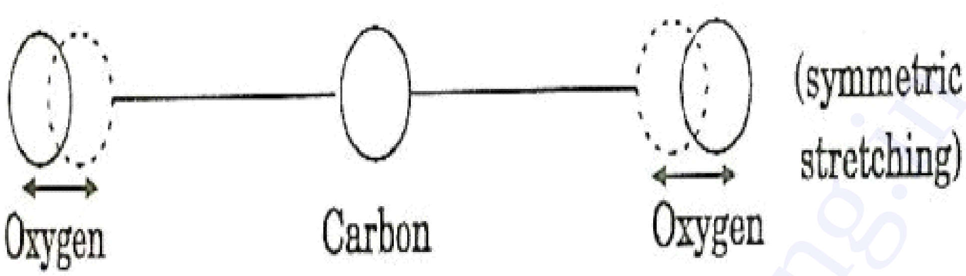


Fig. 2.9

- **Bending mode:**

In this mode of vibration, oxygen atoms and carbon atoms vibrate perpendicular to molecular axis.

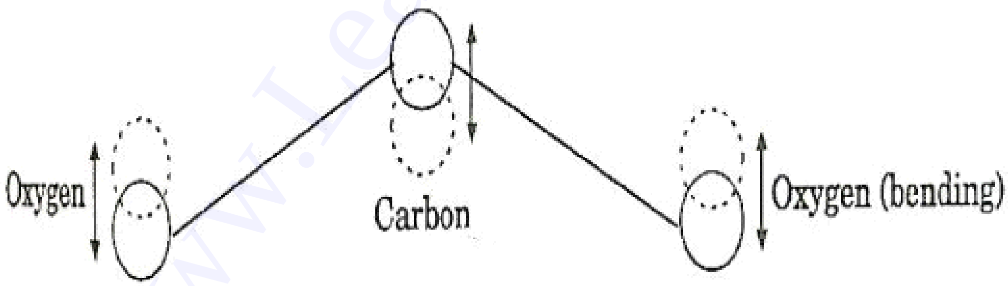
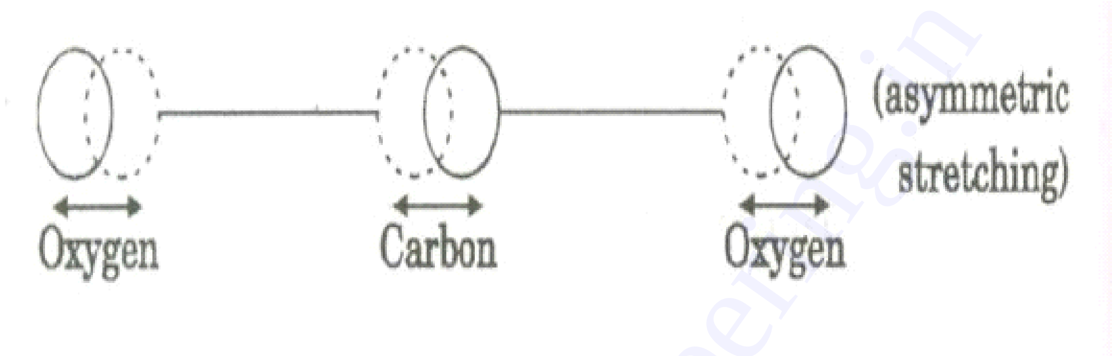


Fig.2.10 2.1

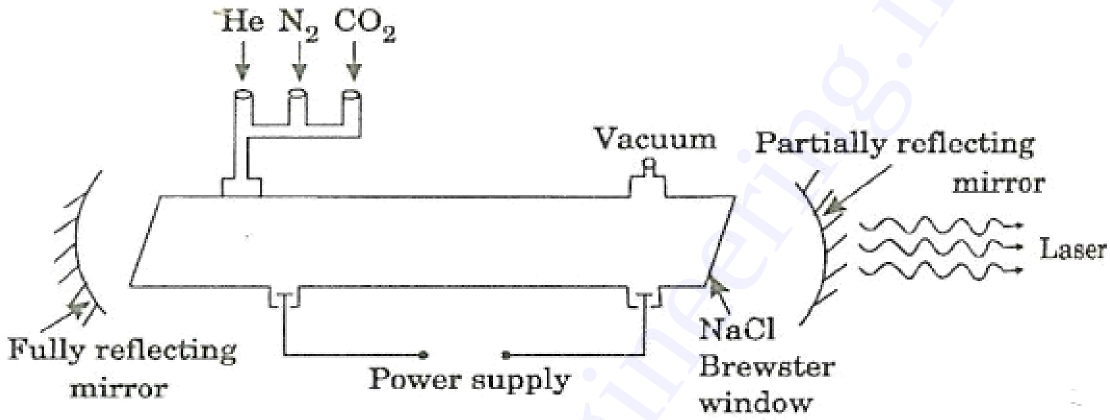
- **Asymmetric stretching mode**:



In this mode of vibration, oxygen atoms and carbon atoms vibrate asymmetrically, i.e., oxygen atoms move in one direction while carbon atoms in the other direction.

Principle:

The active medium is a gas mixture of CO2, N2 and He. The laser transition takes place between the vibrational states of CO2molecules.



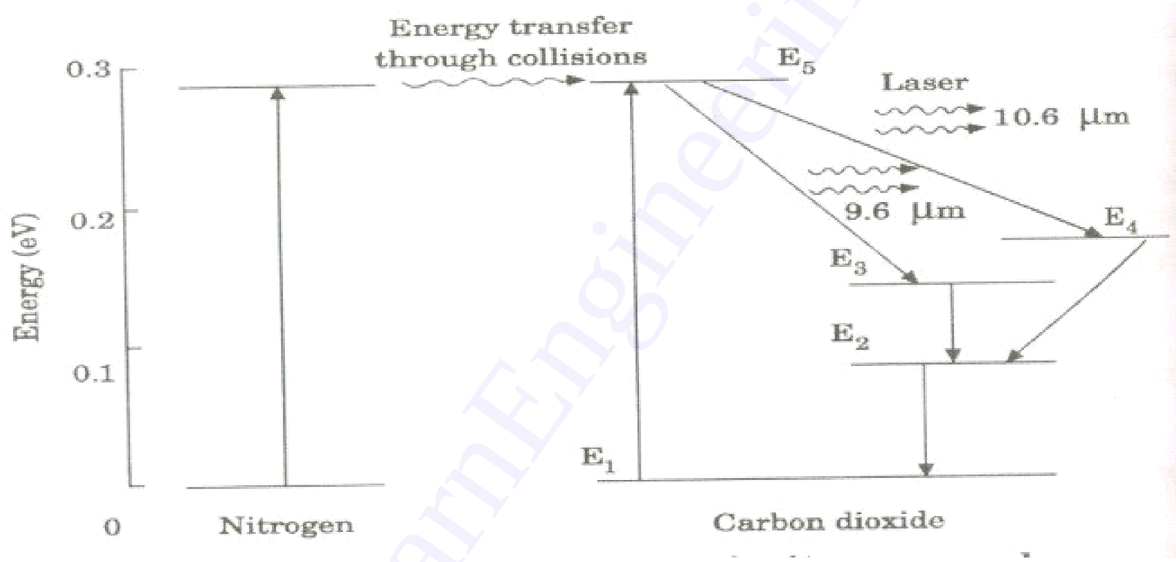
Construction:

It consists of a quartz tube 5 m long and 2.5 cm in the diameter. This discharge tube is filled with gaseous mixture of CO2(active medium), helium and nitrogen with suitable partial pressures.

The terminals of the discharge tubes are connected to a D.C power supply. The ends of the discharge tube are fitted with NaCl Brewster windows so that the laser light generated will be polarized.
Two concave mirrors one fully reflecting and the other partially form an optical resonator.

Working:

Figure shows energy levels of nitrogen and carbon dioxide molecules.



When an electric discharge occurs in the gas, the electrons collide with nitrogen molecules and they are raised to excited states. This process is represented by the equation

$$N_2 + e^* = N_2^* + e$$

N_2 = Nitrogen molecule in ground state e^* = electron with kinetic energy N_2^* = nitrogen molecule in excited state e= same electron with lesser energy

Now N_2 molecules in the excited state collide with CO2 atoms in ground state and excite to higher electronic, vibrational and rotational levels.
This process is represented by the equation

$$N2^* + CO2 = CO2^* + N2$$

N2* = Nitrogen molecule in excited state. CO2 = Carbon dioxide atoms in ground state CO2* = Carbon dioxide atoms in excited state N2 = Nitrogen molecule in ground state. Since the excited level of nitrogen is very close to the E5 level of CO2 atom, population in E5 level increases. As soon as population inversion is reached, any of the spontaneously emitted photon will trigger laser action in the tube. There are two types of laser transition possible.

- **Transition E5 to E4** : This will produce a laser beam of wavelength 10.6μm

- **Transition E5 to E3** This transition will produce a laser beam of wavelength 9.6µm. Normally 10.6µm transition is more intense than 9.6µm transition. The power output from this laser is 10kW
- **Characteristics**

- Type: It is a molecular gas laser.
- Active medium: A mixture of CO_2 , N_2 and helium or water vapour is used as active medium
- Pumping method: Electrical discharge method is used for Pumping action
- Optical resonator: Two concave mirrors form a resonant cavity
- Power output: The power output from this laser is about 10kW.
- Nature of output: The nature of output may be continuous wave or pulsed wave.

Wavelength of output: The wavelength of output is 0.6µm and 10.6µm.

Advantages

1.The output of this laser is continuous.
2. It has high efficiency.
3.It has very high output power.
4. The output power can be increased by extending the length of the gas tube.

Disadvantages

1. The contaminationof oxygen by carbon monoxide will have some effect on laser action.
2. The operating temperature plays an important role in determining the output power of laser.
3. The corrosion may occur at the reflecting plates.
4. Accidental exposure may damage our eyes, since it is invisible (infra red region) to our eyes.

Applications

1. High power CO2 laser finds applications in material processing, welding, drilling, cutting soldering etc.
2. The low atmospheric attenuation (10.6µm makes CO2 laser suitable for open air communication.
3. It is used for remote sensing
4. It is used for treatment of liver and lung diseases.
5.It is mostly used in neuro surgery and general surgery.
6. It is used to perform microsurgery and bloodless operations.

SEMICONDUCTOR LASER

Laser action can also be produced semiconductors. The most compact of all the lasers in semiconductor diode laser. It is also called injection laser. There are two types of semiconductor diode lasers (*i.) Homo - junction laser (ii.) Hetero- Junction laser.*

HOMO – JUNCTION SEMICONDUCTOR DIODE LASER

Definition

It is specifically fabricated p-n junction diode. This diode emits laser light when it is forward biased.

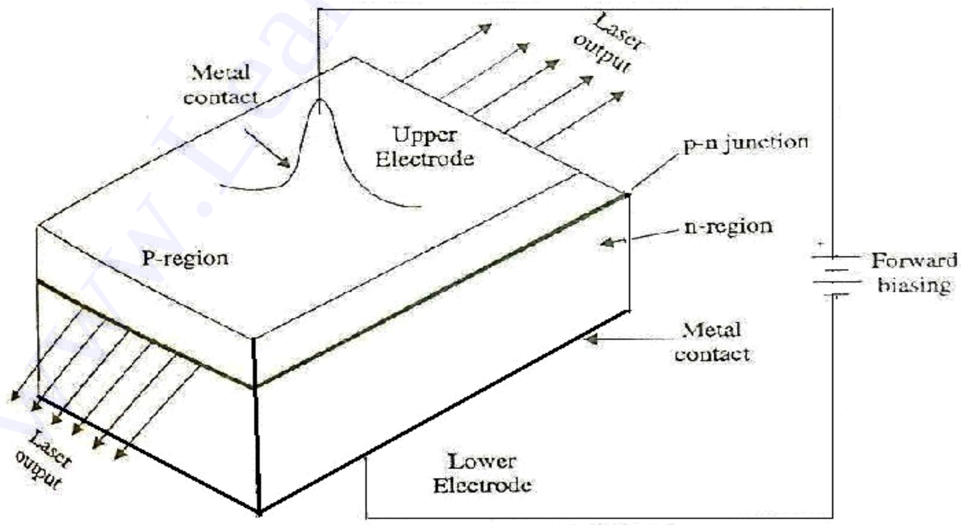
Principle

When a p-n junction diode is forward biased, the electrons from n – region and the holes from the p-region cross the junction and recombine with each other. During the recombination process, the light radiation (photons) is released from a certain specified direct band gap semiconductors like Ga-As. This light radiation is known as recombination radiation.

The photon emitted during recombination stimulates other electrons and holes to recombine. As a result, stimulated emission takes place which produces laser.

Construction

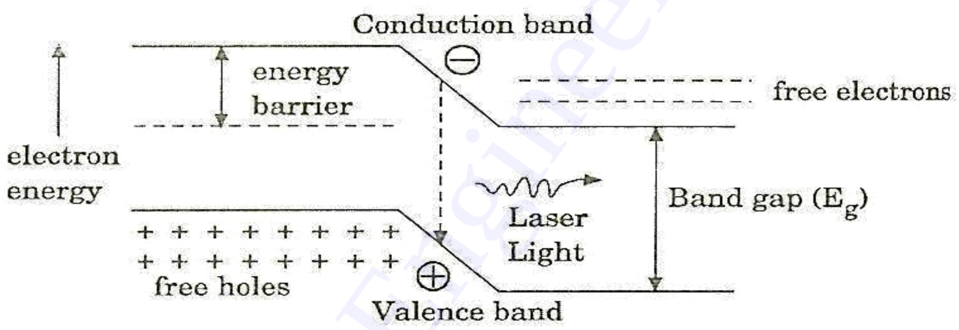
Figure shows the basic construction of semiconductor laser. The active medium is a p-n junction diode made from the single crystal of gallium arsenide. This crystal is cut in the form of a platter having thickness of 0.5μm. The platelet consists of two parts having an electron conductivity (n-type) and hole conductivity (p-type).



The photon emission is stimulated in a very thin layer of PN junction (in order of few microns). The electrical voltage is applied to the crystal through the electrode fixed on the upper surface. The end faces of the junction diode are well polished and parallel to each other. They act as an optical resonator through which the emitted light comes out.

Working

Figure shows the energy level diagram of semiconductor laser. When the PN junction is forward biased with large applied voltage, the electrons and holes are injected into junction region in considerable concentration. The region around the junction contains a large number of electrons in the conduction band and a large number of holes in the valence band.



If the population density is high, a condition of population inversion is achieved. The electrons and holes recombine with each other and this recombination's produce radiation in the form of light. When the forward – biased voltage is increased, more and more light photons are emitted and the light production instantly becomes stronger.These photons will trigger a chain of stimulated recombination resulting in the release of photons in phase. The photons moving at the plane of the junction travels back and forth by reflection between two sides placed parallel and opposite to each other and grow in strength.

After gaining enough strength, it gives out the laser beam of wavelength 8400A^0 The wavelength of laser light is given by $=$ $h\nu = E_g$

where Eg is the band gap energy in joule.

Characteristics

- ***Type:*** It is a solid state semiconductor laser.
- ***Active medium:*** A PN junction diode made from single crystal of gallium arsenide is used as an active medium.
- ***Pumping method:*** The direct conversion method is used for pumping action.
- ***Power output:*** The power output from this laser is a few mW.
- ***Nature of output:*** The nature of output is continuous wave or pulsed output.
- ***Wavelength of Output***: gallium arsenide laser gives infrared radiation in the wavelength 8300 to 8500A^0.

Advantages:

- It is very small in dimension. The arrangement is simple and compact.
- It exhibits high efficiency.
- The laser output can be easily increased by controlling the junction current
- It is operated with lesser power than ruby and CO_2 laser.
- It requires very little auxiliary equipment
- It can have a continuous wave output or pulsed output.

Disadvantages

- It is difficult to control the mode pattern and mode structure of laser.
- The output is usually from 5° to 15° i.e., laser beam has large divergence.
- The purity and monochromacity are poor than other types of laser
- Threshold current density is very large ($400A/mm^2$).
- It has poor coherence and poor stability.

Applications

- It is widely used in fiber optic communication
- It is used to heal the wounds by infrared radiation
- It is also used as a pain killer
- It is used in laser printers and CD writing and reading.
-

HETERO - JUNCTION SEMICONDUCTOR DIODE LASER

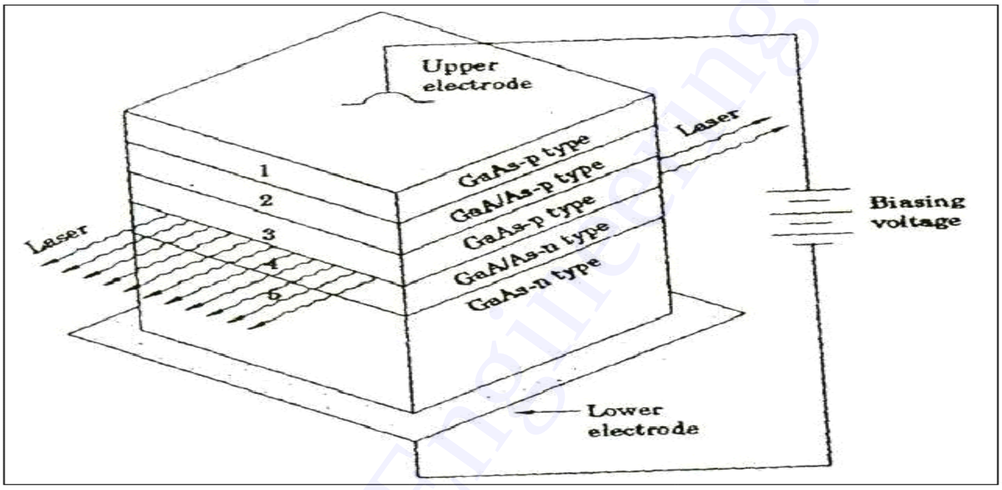
A p-n junction made up of the different materials in two regions ie., n type and p type is known as Hetero junction.

Principle:

When a PN junction diode is forward biased, the electrons from the n region and holes from the p region recombine with each other at the junction. During recombination process, light is released from certain specified direct band gap semiconductors.

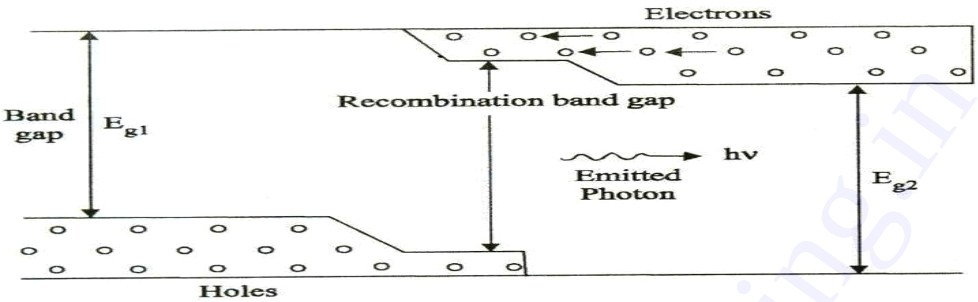
Construction:

This laser consists of five layers as shown in the figure. A layer of Ga-As p – type (3rd layer) will act as the active region. This layer is sand witched between two layers having wider band gap viz. Ga Al As-p – type (2nd layer) and Ga Al As-n- type (4th layer). The end faces of the junctions of 3rd and 4th layer are well polished and parallel to each other. They act as an optical resonator.



Working:

When the PN junction is forward biased, the electrons and holes are injected into the junction region. The region around the junction contains large number of electrons in the conduction band and holes in the valence band. Thus, the population inversion is achieved. At this stage, some of the injected charge carriers recombines and produce radiation in the form of light.When the forward biased voltage is increased, more and more light photons are emitted and the light intensity is more. These photons can trigger a chain of stimulated recombination's resulting in the release of photons in phase.



The photons moving at the plane of the junction travels back and forth by reflection between two sides and grow its strength. A coherent beam of laser having wavelength nearly 8000A^0 emerge out from the junction region.

Characteristics:

- ***Type:*** It is a Hetero junction semiconductor laser
- ***Active medium:*** PN junctions made from different layers.
- ***Pumping method:*** Direct conversion method
- ***Power output:*** The power output of laser beam is 1 mW
- ***Nature of the Output:*** Continuous wave form
- ***Wavelength of the output:*** Nearly 8000 A^0

Advantages:

- It produces continuous wave output.
- The power output is very high.

Disadvantages:

- It is very difficult to grow different layers of PN junction.
- The cost is very high.

Applications:

- This type of laser is mostly used in optical applications
- It is widely used in computers, especially on CD-ROMs.
-

APPLICATIONS OF LASER

Application of laser in industry – cutting and welding – Drilling – Surface Hardening – Medical applications – Laser as diagnostic and therapeutic tool – Holography – Theory of recording and reconstruction – application of Holography.

Introduction

Lasers deliver coherent, monochromatic, well-controlled, and precisely directed light beams. A priori, therefore, lasers would seem to be poor choices for general-purpose illumination, however, they are ideal for concentrating light in space, time, or particular wavelengths. Lasers have been regularly used to measure, cut, drill, weld, read, write, send messages, solve crimes, burn plaque out of arteries, and perform delicate eye operations.

Laser	Wavelength (μm)	Peak power (watts)	Pulse repetition rate (pulses per second)	Typical use
Nd:YAG (repetitively pulsed)	1.06	10^6	10	Drilling metals, scribing silicon wafers
Nd:YAG (repetitively Q-switched)	1.06	10^5	5000	Trimming resistors
Nd:YAG (continuous)	1.06	up to 5400	—	Cutting metals
CO_2 (repetitively pulsed)	10.6	10^5	100	Hole drilling in alumina circuit boards
CO_2 (TEA)	10.6	10^6	100	Marking components
CO_2 (continuous)	10.6	up to 20,000	—	Cutting metals, plastics, cloth
Copper vapor	0.512, 0.578	3×10^5	6500	Drilling metals
Excimer	0.193, 0.248	10^6–10^7	100	Drilling plastics, ceramics
Ruby	0.694	10^6	low (<1)	Drilling gemstones
Nd:glass	1.06	10^6	low (<1)	Drilling hard metals

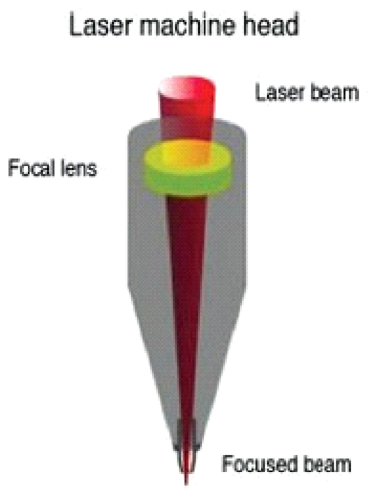
Over and over again the laser has proved to be an extremely practical tool. Nevertheless, lasers have also proved their usefulness in non-practical applications, especially in the realm of art and

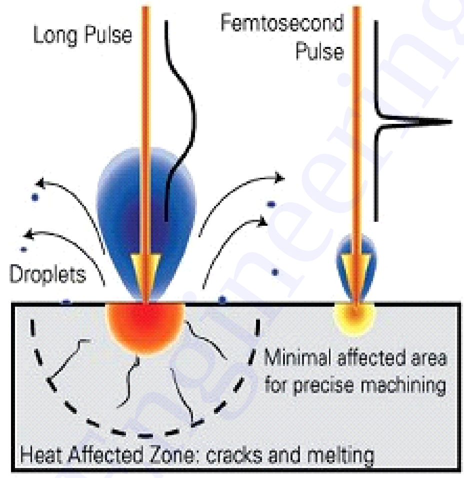
entertainment. Lasers are involved in almost all aspects of these fields, from "light shows" to Compact Discs (CDs) and Digital Video Discs (DVDs), to special effects in the movies. Some other commonplace applications of lasers are as Laser pointers, barcode scanners, laser printers, etc.

Still, much of the important modern day celebrated applications lie in the fiber-optic communication, laser machining and fabrication, trace element detection, laser metrology and medical imaging.

Laser Machining and cutting

Laser energy can be focused in space and concentrated in time so that it heats, burns away, or vaporizes many materials. Although the total energy in a laser beam may be small, the concentrated power on small spots or during short intervals can be enormous. Although lasers cost much more than mechanical drills or blades, their different properties allow them to perform otherwise difficult tasks.

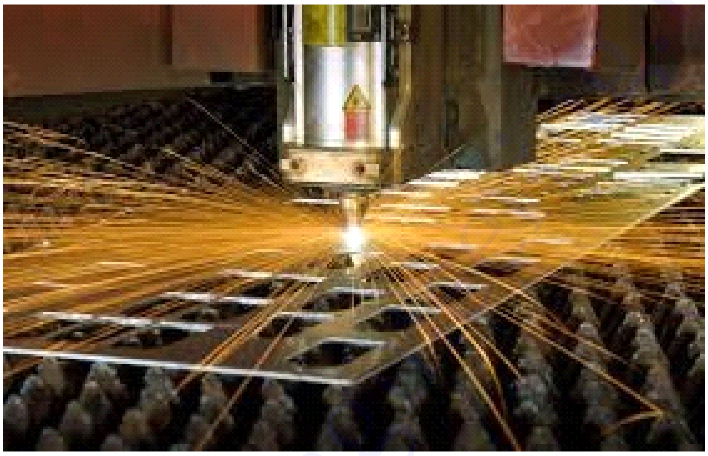




A laser beam does not deform flexible materials as a mechanical drill would, so it can drill holes in materials such as soft rubber nipples for baby bottles. Likewise, laser beams can drill or cut into extremely hard materials without dulling bits or blades. Laser machining is not dependent on the material hardness but on the optical properties of the laser and the optical and thermo- physical properties of the material. For example, lasers have drilled holes in diamond dies used for drawing wire.Several recent research have shown that laser cutting is best achieved with ultrafast lasers as the material only ablates and does not get a chance to melt under such ultrafast time scale interactions.

Laser cutting

In the simplest terms, a CNC laser cutter uses a coherent beam of light to cut material, most often sheet metal, but also wood, diamond, glass, plastics and silicon. In the beginning, the beam was directed through a lens via mirrors, but these days fiber optics are much more common. The lens focuses the beam at the work zone to burn, melt or vaporize the material. Exactly which process(es) the material undergoes depends on the type of laser cutting involved. Broadly speaking, laser cutting can be divided into two types: laser fusion cutting and ablative laser cutting. Laser fusion cutting involves melting material in a column and using a high- pressure stream of gas to shear the molten material away, leaving an open cut kerf. In contrast, ablative laser cutting removes material layer by layer using a pulsed laser—it’s like chiseling, only with light and on a microscopic scale. This generally means evaporating the material, rather than melting it. Two other key factors distinguish laser fusion cutting from ablative laser cutting.



First, ablative laser cutting can be used to make partial cuts in a material, whereas laser fusion cutting can only be used to cut all the way through it. This is due to fusion cutting operating with lasers either in continuous waves or with significantly longer pulses than ablative cutting (micro- or milliseconds vs. nanoseconds), which causes a molten pool to penetrate the entire depth of the metal. This molten material must be sheared away via gas stream, otherwise it can stay in the kerf and weld back cut edges upon cooling.

The second and more significant factor that distinguishes these two types of laser cutting is speed. "With sheet metal cutting—which makes up the bulk of the cutting industry. At the current state of laser technology, laser fusion cutting is much faster for those setups. Ablative cutting takes more time, for now.

Fiber Lasers vs CO_2

The two most common types of laser cutting machines are fiber laser and CO_2.

CO_2 lasers use an electromagnetically stimulated gas—typically, a mixture of carbon dioxide, nitrogen and sometimes hydrogen, xenon or helium—as their active laser medium. In contrast,

fiber lasers—which are a type of solid-state laser—use an optical fiber doped with rare-earth elements, such as erbium, ytterbium, neodymium or dysprosium. As indicated by Houldcroft's experiments, the industry began with CO_2, and that technology dominated until only recently. "Potentially, CO_2 lasers will be replaced completely. If so, this would happen mid-term while the fiber laser technology further evolves. Currently, CO_2 lasers still have some specific advantages, e.g., better edge quality in thick material and smaller burrs.

CNC laser cutters are used on a wide range of materials in a variety of industries. Since cutting sheet metal is the most common application, it's worth focusing on the particularities involved. For instance, reflectance and surface thickness are two of the most important factors to consider.

Laser cutting uses high-pressure gas—5-25 bars for nitrogen cutting—so you need the parts to either be supported by their own weight, which works if they're thicker than 2-3 mm and relatively large in size, but for the parts that are thin and small, to resist the force of the gas stream, small sections need to remain uncut," Sarrafi said. "These micro-joints are very small, 0.2-0.4 mm wide, so they're easy to break in post processing, but sometimes they're necessary to connect the parts to the frame so the parts don't fly away

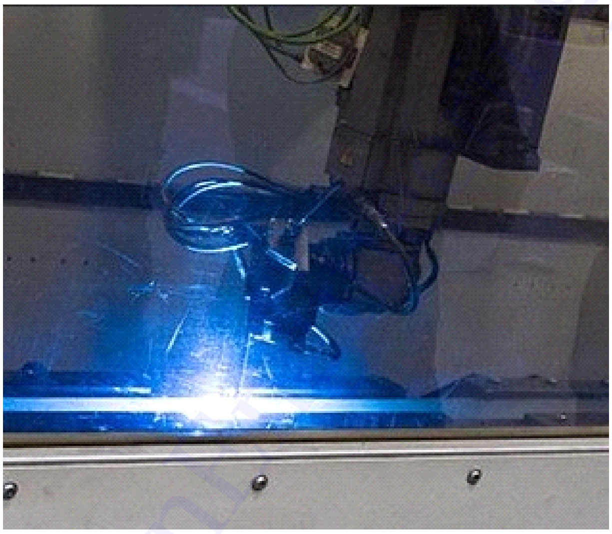
Laser welding

Laser welding is used more frequently in industrial processes because it has wider application than traditional welding as less heat is created because the beam is so focused. This means that heat transfer to the workpiece is much less and the metallurgical structure is less affected and the quality of the weld is much higher than with traditional forms of welding.

Laser welding is a much more accurate manufactoring process and welds can be as small as one hundredths of a millimetre. Small pulses of heat are used to create the weld which leads to a higher quality finish which is stronger providing a better depth to width ratio. Depending on the power of the laser, welding penetration up to 15 millimetre of steel or stainless steel can be achieved.

Another distinct advantage of laser welding over other methods is that lasers can weld a greater variety of metals such as high strength stainless steel, titanium, aluminium, carbon steel as well as precious metals like gold and silver.

With laser welding, welds are much more accurate and finish is superior as is strength. The manufactoring process is therefore excellent for fine components and it can be used in areas where there is limited access. Lasers enable precision and quality where required for fine components.



Summary of Laser Welding Advantages

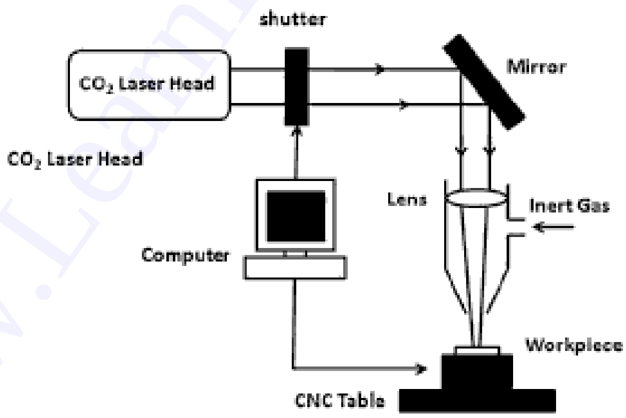
- Aesthetically better weld finishes
- More suited to high value items such as jewelry
- Great for inaccessible places
- Ideal for solenoids and machined components
- Perfect for medical devices where weld quality is vital for hygiene and precision
- Better weld quality for a variety of metals and metal depths
- No concerns for weld weaknesses due to minimal distortion
- Workpieces can be handled almost immediately because heat transference is low
- Overall improved productivity

The benefits of laser welding for modern processes over traditional welding are many. Laser welding overall has a much wider application and an ability to weld a greater number of metals to a much higher quality which is vital where precision engineering is required.

Laser - Hole drilling

We will describe the physical processes that occur in the interaction of high-power laser radiation with surfaces. An understanding of these processes is important for understanding the capabilities and limitations of laser vaporization. We will emphasize metallic targets, but much of what is said applies to other absorbing surfaces as well.

Lasers used—The Nd:YAG laser has often been used for drilling holes in metals. It can deliver an irradiance of 106–109 watts/cm2 to a target surface. For most metals, it offers lower reflectivity than the CO_2 laser, so that less light energy is lost by reflection. It also offers high processing speed. The CO_2 laser, with a wavelength 10 times larger than the Nd:YAG laser, has less importance in drilling of metals, because the beam cannot be focused to as small a spot, and because the absorption is not so high as for the Nd:YAG laser. But for many nonmetals, like alumina, the absorption is much higher for the CO_2 laser than for the Nd:YAG laser. Thus, CO_2 lasers have an important role in the drilling of materials like ceramics and plastic. The copper vapor laser, with a high pulse repetition rate, has also found a role in the drilling of metals. Excimer lasers offer material removal with relatively little heating of the surrounding material, because the chemical bonds in the target can be broken by shorter, ultraviolet wavelengths of the excimer laser. The material is removed without significant thermal conduction of heat into the interior of the workpiece. Thus, excimer lasers may be used for hole drilling in materials that are sensitive to heat, like plastics.



Depth of holes—When high-power laser radiation is absorbed by a target, the surface is heated by the incoming laser light. The surface temperature goes quickly through the melting point and reaches the vaporization temperature (boiling point). Material begins to vaporize and a hole is produced in the surface. When a pulsed laser beam with duration around 1 millisecond interacts with a surface, the process of material involves conventional heating, melting, and vaporization. The time scale is 10 Optics and Photonics Series, Photonics-Enabled Technologies: Manufacturing long enough so that vaporized material can flow away from the point of the interaction. Vaporization occurs at a continually retreating surface.

	Absorbed laser irradiance (watts/cm^2)		
Metal	**10^5**	**10^6**	**10^7**
Lead	118 μs	1.18 μs	12 ns
Zinc	128 μs	1.28 μs	13 ns
Magnesium	245 μs	2.45 μs	24.5 ns
Nickel	1.84 ms	184 μs	184 ns
Iron	1.86 ms	186 μs	186 ns
Aluminum	2.67 ms	26.7 μs	267 ns
Molybedenum	5.56 ms	55.6 μs	556 ns
Copper	8.26 ms	82.6 μs	826 ns
Tungsten	10.46 ms	104.6 μs	1.05 μs

Advantages

Hole drilling with lasers offers many advantages over competing techniques.

- There is no contact of external materials with the workpiece, and hence, no contamination.
- Hard, brittle materials that are difficult to drill with conventional techniques are often easily drilled with lasers.
- The heat-affected zones around the holes can be very small.
- It is possible to produce very small holes in thin materials.
- Laser drilling is compatible with automation, so that it is possible to produce large numbers of holes and complex patterns of holes in a completely automated process.
- There is no wear of expensive tool bits, so that in some cases, laser drilling offers an economic advantage.
- Holes can be drilled with high throughput rate, so that the cost is low. Limitations

Laser hole drilling, of course, will not completely replace conventional hole drilling. There are a number of limitations for laser hole drilling.

- Laser energy is relatively expensive and may not compete economically with other processes for specific applications.

• The holes drilled by lasers tend to have limited depth. One might think that one could use a CO_2 laser and allow it to dwell on a spot for a long time. But the heat then spreads over a 1arger volume and much of the advantage in using lasers is lost.

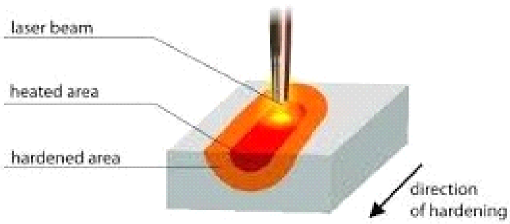
• There may be a recondensation of vaporized material around the entrance to the hole, which forms a crater-like lip. The lip can be removed fairly easily, but this adds one more step to the laser-drilling process.

Surface hardening

Laser beams are invisible electromagnetic radiations in the infra-red portion of the spectrum, and are increasingly being used for surface-hardening of ferrous materials to improve mechanical properties like wear resistance and even fatigue resistance. There are two main type of Lasers used- YAG Solid-state type and the carbon-dioxide gas type. The output of YAG laser has much shorter wavelength, 1.064 µm, whereas the carbon dioxide laser emits radiations with
10.8 µm wavelength. Carbondioxide laser is more commonly used and is suitable for surface hardening, particularly when the process requires more than 500 W of power.

The power density of laser beam is usually expressed as watts per square centimeter. The power densities used in laser surface hardening are in the range of 500 to 5000 W/cm2 with dwell times in the range of 0.1 s to 10 s. For carbon steels, power densities used are from 1000 to 1500W/cm2 with dwell time of 1 to 2 s.

During Laser surface hardening, a laser can generate very intense energy fluxes at the surface of the component, when the Laser radiations impinge on it, and are absorbed to generate heat energy. This heat is then conducted inside the component. When the power density of the laser beam is high, the rate of heat generation is much higher than the rate of heat conduction. The temperature of the surface layer increases rapidly to soon attain the austenitising temperature.



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A moderate power density of 500 W/m2, results in temperature gradient of 500°C/mm. The laser beam may be moved over the surface of the component as illustrated in Fig. 8.78. The surface which meets the laser beam gets heated up. Once the beam passes over, the heated volume gets subsequently 'self-quenched'. Thus, by selecting power density and the speed of the laser spot (i.e., the dwell time), a desired case depth can be hardened.

Laser-surface-hardening is similar to any other surface-hardening method such as induction, or flame, except that the laser beam is used to generate heat here. The heating time to the austenitising temperature, particularly in laser heating, is very short-fractions of seconds to few seconds. The dwell time cannot be made very large as surface melting may occur which is undesirable.

Alloy steels intended to have higher hardenabilities should have very fine carbides particles even then their dissolution is difficult. Diffusion of carbon though faster than alloying elements requires longer dwell time (low speed of motion of laser spot) to obtain homogeneous structure.

$$Y = -0.11 + 3.02\, P/\sqrt{D_b\, V}$$

where,

Y = depth of hardening (mm), P = Laser power (W),

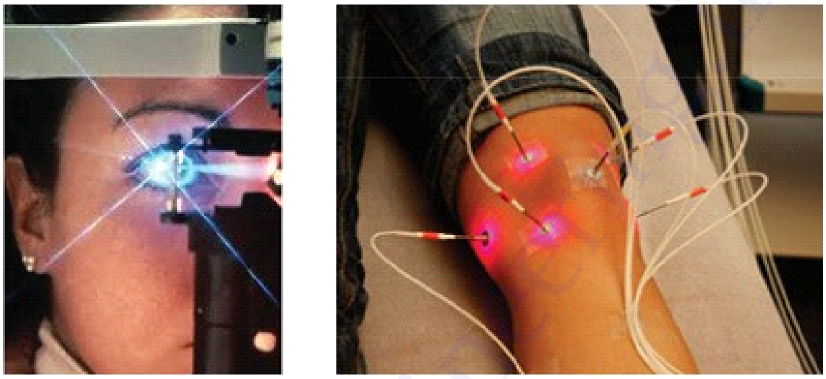
Db = incident beam diameter (mm) V = travel speed (mm/s)

but with a considerable scatter of experimental data. At a constant value of P/√DhV, the depth of hardening can vary by a factor of 2.

Advantages and Disadvantages of Laser Hardening:

- Non-hardenable steels like mild steels can be surface hardened.
- Hardness obtained is slightly higher than conventional hardening.
- Closer control over power inputs helps in eliminating dimensional distortion.
- Beam (with the help of optical parts) can easily reach the inaccessible areas of components, and re-entrant surfaces.
- No vacuum or protective atmosphere is required.
- The last optical element of the Laser and the component to be surface hardened may be far-placed.
- Very long and irregular-shaped components can be hardened easily. Disadvantages:
- High initial cost particularly of large lasers.
- Lasers use 10% of the input energy, i.e., there are inefficient.
- The depth of case is very limited.
- Working cost is high.
- **Medical Applications**

Surgical removal of tissue with a laser is a physical process similar to industrial laser drilling. Carbon-dioxide lasers operating at 10.6 micrometers can burn away tissue as the infrared beams are strongly absorbed by the water that makes up the bulk of living cells. A laser beam cauterizes the cuts, stopping bleeding in blood-rich tissues such as gums. Similarly, laser wavelengths near one micrometer (Neodymium-YAG Laser) can penetrate the eye, welding a detached retina back into place, or cutting internal membranes that often grow cloudy after cataract surgery Less-intense laser pulses can destroy abnormal blood vessels that spread across the retina in patients suffering from diabetes, delaying the blindness often associated with the disease. Ophthalmologists surgically correct visual defects by removing tissue from the cornea, reshaping the transparent outer layer of the eye with intense ultraviolet pulses from Excimer Lasers.



(a) (b)

Fig. (a) Schematic of Laser Eye Surgery. (b) Laser energy delivery to precise spots in joints for arthroscopic surgery.

Laser light can be delivered to places within the body that the beams could not otherwise reach through optical fibers similar to the tiny strands of glass that carry information in telephone systems. One important example involves threading a fiber through the urethra and into the kidney so that the end of the fiber can deliver intense laser pulses to kidney stones. The laser energy splits the stones into fragments small enough to pass through the urethra without requiring surgical incisions. Fibers also can be inserted through small incisions to deliver laser energy to precise spots in the knee joint during arthroscopic surgery Another medical application for lasers is in the treatment of skin conditions. Pulsed lasers can bleach certain types of tattoos as well as dark-red birthmarks called port-wine stains. Cosmetic laser treatments include removing unwanted body hair and wrin-kles.

Biomedical Imaging and superresolution

Confocal microscopy is a ubiquitous imaging tool for imaging thick specimen in a wide range of investigations in biological, medical and material sciences. It uses UV or visible light for the single photon excitation of fluorophore from ground state to the excited state followed by deactivation through fluorescence emission which is detected through high quantum efficiency photomultiplier tube (PMT) in the range of near ultraviolet, visible and near infrared spectral region. The basic difference of confocal Light Scanning Microscope with the conventional optical microscope is the confocal aperture arranged in a plane conjugate to the intermediate image plane and thus, to the object plane of the microscope. The PMT can only detect the light that passed the pinhole. As the laser beam is focused to a diffraction limited spot, which illuminates only a point of the object at a time, the point illuminated and the point observed are situated in conjugate planes, i.e. they are focused onto each other. The perfection of focused beam which is connected to the resolution has always been a matter of concern in the far-field fluorescence microscopy. Still, optical microscopy remains the best choice for monitoring live specimens despite the resolution advantage of, say electron microscopes, since the energy deposited in electron microscopy adversely affects the viability of live specimens. This practical compromise implicitly sets resolution enhancement as one of the most important developments in optical microscopy. Finally, all these images are combined into one super-resolved image with complete structural information. They demonstrated this method first in 2006 and called it Photo Activated Localization Microscopy (PALM)

Laser Imaging and Holography

Holography is a much broader field than most people have perceived. Recording and displaying truly three-dimensional images are only small parts of it. Holographic optical elements (HOE) can perform the functions of mirrors, lenses, gratings, or combinations of them, and they are used in myriad technical devices. Holographic interferometry measures microscopic displacements on the surface of an object and small changes in index of refraction of transparent objects like plasma and heat waves.

The coherence of laser light is crucial for interferometry and holography, which depend on interactions between light waves to make extremely precise measurements and to record three- dimensional images. The result of adding light waves together depends on their relative phases. If the peaks of one align with the valleys of the other, they will interfere destructively to cancel each other out; if their peaks align, they will interfere constructively to produce a bright spot. This effect can be used for measurement by splitting a beam into two identical halves that follow different paths. Changing one path just half a wavelength from the other will shift the two out of phase, producing a dark spot. This technique has proved invaluable for precise measurements of very small distances. Holograms are made by splitting a laser beam into two identical halves, using one beam to illuminate an object. This object beam then is combined with the other half— the reference beam—in the plane of a photographic plate, producing a random-looking pattern of light and dark zones that record the wave front of light from the object. Later, when laser light illuminates that pattern from the same angle as the reference beam, it is scattered to reconstruct an identical wave front of light, which appears to the viewer as a three-dimensional image of the object. Holograms now can be mass-produced by an embossing process, as used on credit cards, and do not have to be viewed in laser light.

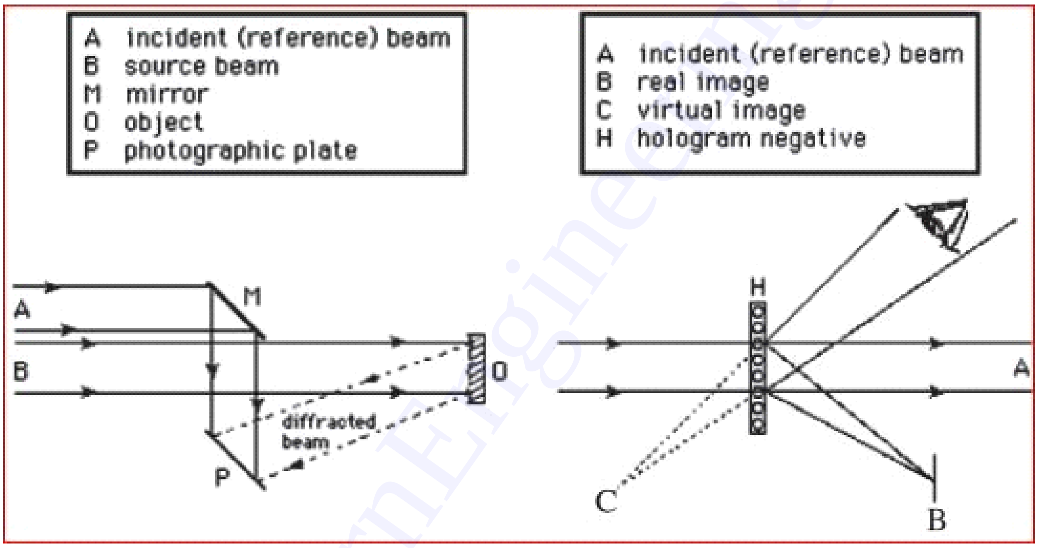


Fig. Schematic of Holography process where the laser beam is split into three components. First two beams are needed to create the hologram which is viewed with the help of the third.

TYPES OF HOLOGRAMS

A hologram is a recording in a two- or three-dimensional medium of the interference pattern formed when a point source of light (the reference beam) of fixed wavelength encounters light of the same fixed wavelength arriving from an object (the object beam). When the hologram is illuminated by the reference beam alone, the diffraction pattern recreates the wave fronts of light from the original object. Thus, the viewer sees an image indistinguishable from the original object. The reflection hologram Transmission holograms Hybrid holograms

Recording and reconstruction of holograms

Recording of hologram. The recording of hologram is based on the phenomenon of interference. It requires a laser source, a plane mirror or beam splitter, an object and a photographic plate. A laser beam from the laser source is incident on a plane mirror or beam splitter. As the name suggests, the function of the beam splitter is to split the laser beam. One part of splitted beam, after reflection from the beam splitter, strikes on the photographic plate. This beam is called reference beam. While other part of splitted beam (transmitted from beam splitter) strikes on the photographic plate after suffering reflection from the various points of object. This beam is called object beam. The object beam reflected from the object interferes with the reference beam when both the beams reach the photographic plate. The superposition of these two beams produces an interference pattern (in the form of dark and bright fringes) and this pattern is recorded on the photographic plate. The photographic plate with recorded interference pattern is called hologram. Photographic plate is also known as Gabor zone plate in honour of Denis Gabor who developed the phenomenon of holography. Each and every part of the hologram receives light from various points of the object. Thus, even if hologram is broken into parts, each part is capable of reconstructing the whole object.

Reconstruction of image.

In the reconstruction process, the hologram is illuminated by laser beam and this beam is called reconstruction beam. This beam is identical to reference beam used in construction of hologram.The hologram acts a diffraction grating. This reconstruction beam will undergo phenomenon of diffraction during passage through the hologram. The reconstruction beam after passing through the hologram produces a real as well as virtual image of the object.One of the diffracted beams emerging from the hologram appears to diverge from an apparent object when project back. Thus, virtual image is formed behind the hologram at the original site of the object and real image in front of the hologram. Thus an observer sees light waves diverging from the virtual image and the image is identical to the object. If the observer moves round the virtual image then other sides of the object which were not noticed earlier would be observed. Therefore, the virtual image exhibits all the true three dimensional characteristics. The real image can be recorded on a photographic plate.

Applications of holography

The three-dimensional images produced by holograms have been used in various fields, such as technical, educational also in advertising, artistic display etc.

Holographic diffraction gratings: The interference of two plane wavefronts of laser beams on the surface of holographic plate produces holographic diffraction grating. The lines in this grating are more uniform than in case of conventional grating.

___ Hologram is a reliable object for data storage, because even a small broken piece of hologram contains complete data or information about the object with reduced clarity.

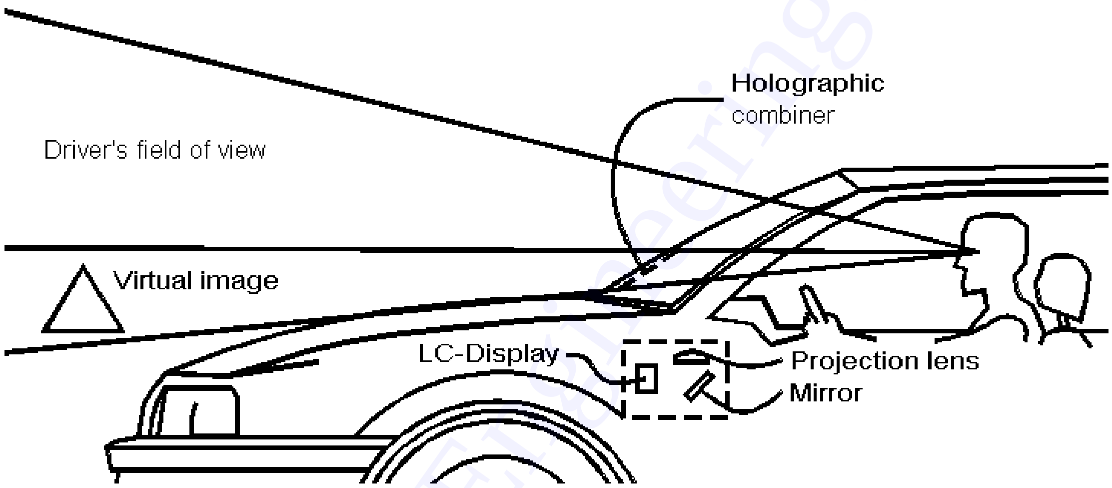
The information-holding capacity of a hologram is very high because many objects can be recorded in a single hologram, by slightly changing ...

Holography –Future Applications

Holography is a very useful tool in many areas, such as in commerce, scientific research, medicine, and industry.

Some current applications that use holographic technology are:

- Holographic interferometry is used by researchers and industry designers to test and design many things, from tires and engines to prosthetic limbs and artificial bones and joints.
- Supermarket and department store scanners use a holographic lens system that directs laser light onto the bar codes of the merchandise.
- Holographic optical elements (HOE's) are used for navigation by airplane pilots. A holographic image of the cockpit instruments appears to float in front of the windshield. This allows the pilot to keep his eyes on the runway or the sky while reading the instruments. This feature is available on some models of automobiles.



• Medical doctors can use three-dimensional holographic CAT scans to make measurements without invasive surgery. This technique is also used in medical education.

• Holograms are used in advertisements and consumer packaging of products to attract potential buyers.

• Holograms have been used on covers of magazine publications. One of the most memorable Sports Illustrated covers was the December 23, 1992 issue featuring Michael Jordan. Holograms have also been used on sports trading cards.

• The use of holograms on credit cards and debit cards provide added security to minimize counterfeiting.

• Holography has been used to make archival recordings of valuable and/or fragile museum artifacts.

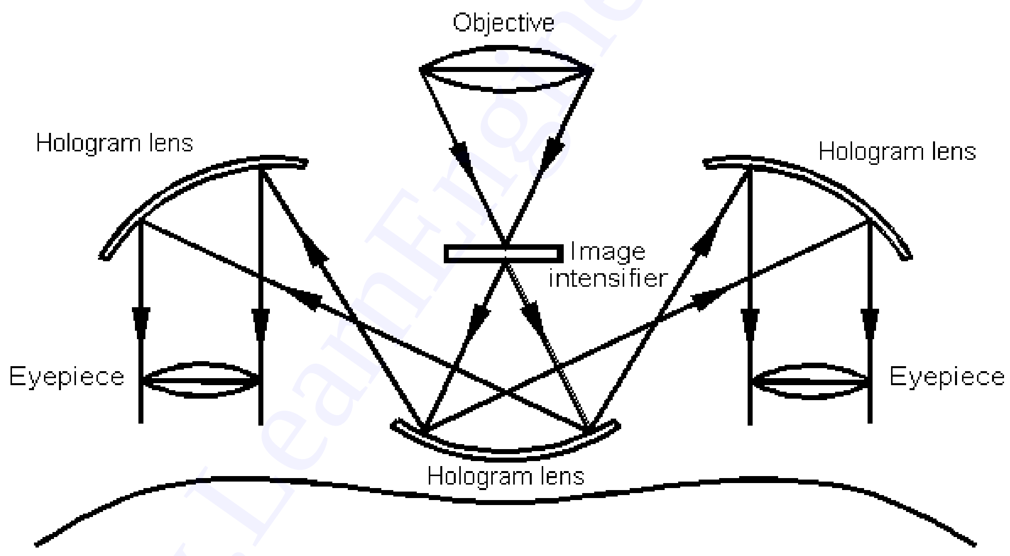
• Sony Electronics uses holographic technology in their digital cameras. A holographic crystal is used to allow the camera to detect the edge of the subject and differentiate between it and the background. As a result, the camera is able to focus accurately in dark conditions.

• Holography has been use by artists to create pulsed holographic portraits as well as other works of art.

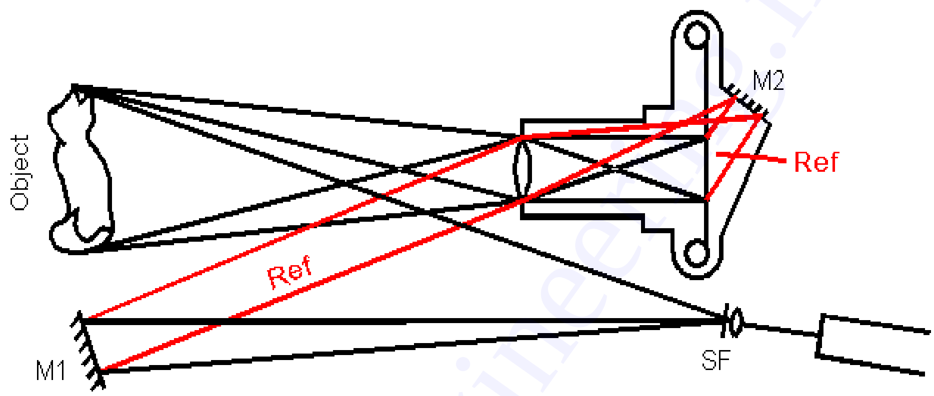
Future applications of holography include:

• Future colour liquid crystal displays (LCD's) will be brighter and whiter as a result of holographic technology. Scientists at Polaroid Corp. have developed a holographic reflector that will reflect ambient light to produce a whiter background.

• Holographic night vision goggles



• Many researchers believe that holographic televisions will become available within 10 years at a cost of approximately \$5000. Holographic motion picture technology has been previously attempted and was successful in the 1970s. The future of holographic motion pictures may become a reality within the next few years.



• Holographic memory is a new optical storage method that can store 1 terabyte (= 1000 GB) of data in a crystal approximately the size of a sugar cube. In comparison, current methods of storage include CD's that hold 650 to 700 MB, DVD's that store 4.7 GB, and computer hard drives that hold up to 120 GB.

• Optical computers will be capable of delivering trillions of bits of information faster than the latest computers.

• **MEDICAL ENDOSCOPE - FIBER OPTIC: CONSTRUCTION AND WORKING**

Optical fibers are very much useful in medical field. Using low quality, large diameter and short length silica fibers we can design a fiber optic endoscope or fibroscope.

MEDICAL ENDOSCOPE

Optical fibers are very much useful in medical field. Using low quality, large diameter and short length silica fibers we can design a fiber optic endoscope or fibroscope. A medical endoscope is a tubular optical instrument, used to inspect or view the internal parts of human body which are not visible to the naked eye. The photograph of the internal parts can also be taken using this endoscope.

Construction

Figure shows the structure of endoscope. It has two fibers viz., 1. Outer fiber(f0)
2. The inner fiber (fi).

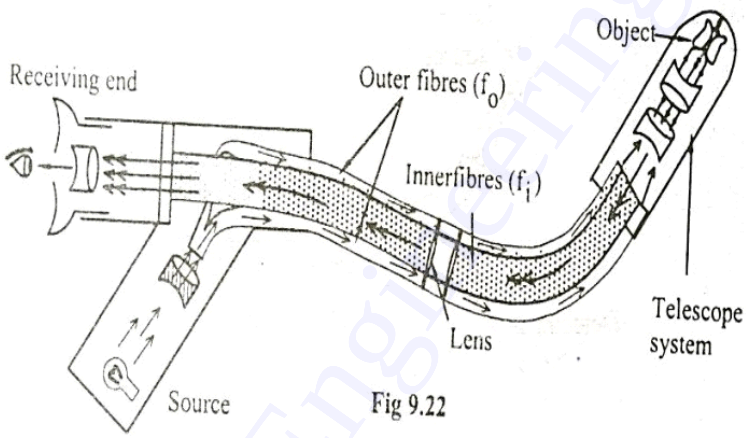


Fig 9.22

Outer fiber

The outer fiber consists of many fibers bundled together without any particular order of arrangement and is called incoherent bundle. These fiber bundles as a whole are enclosed in a thin sleeve for protection. The outer fiber is used to illuminate or focus the light onto the inner parts of the body.

Inner fiber

The inner fiber also consists of a bundle of fibers, but in perfect order. Therefore, this arrangement is called coherent bundle. This fiber is used to collect the reflected light from the object. A tiny lens is fixed to one end of the bundle in order to effectively focus the light, reflected from the object. For a wider field of view and better image quality, a telescope system is added in the internal part of the telescope.

Working

Light from the source is passed through the outer fiber (f0). The light is illuminated on the internal part of the body. The reflected light from the object is brought to focus using the telescope to the inner fiber (fi). Here each fiber picks up a part of the picture from the body. Hence the picture will be collected bit by bit and is transmitted in an order by the array of fibers. As a result, the whole picture is reproduced at the other end of the receiving fiber as shown in the figure. The output is properly amplified and can be viewed through the eye piece at the receiving end. The cross sectional view is as shown in the figure.In figure, we can see that along with input and output fibers, we have two more channels namely, (i) Instrumental Channel (C1) and (ii) Irrigation channel (C2) used for the following purposes.

Instrumental Channel (C1): It is used to insert or take the surgical instruments needed for operation.

Irrigational Channel (C2)

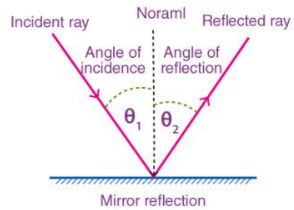
It is used to blow air or this is used to clear the blood in the operation region, so that the affectedparts of the body can be clearly viewed.

Chapter 02 – Optics

PART A

1. State laws of reflection

1. The law of reflection defines that upon reflection from a smooth surface, the angle of the reflected ray is equal to the angle of the incident ray, with respect to the normal to the surface that is to a line perpendicular to the surface at the point of contact.
2. The reflected ray is always in the plane defined by the incident ray and the normal to the surface at the point of contact of the incident ray.



2. State laws of refraction

1. The incident ray, the refracted ray and the normal at the point of incidence, all lie in the same plane.
2. The ratio of the sine of the angle of incidence i to the sine of the angle of refraction is constant for the pair of given media. This constant is called the refractive index of the second medium w.r.t. the first medium.

$_1\mu_2=\sin i / \sin r$

Note: When light ray is incident normally, only speed changes and direction of light remains the same.When light ray passes from rarer medium to denser medium, it bends towards the normal. When light ray passes from denser medium to rarer medium, it bends away from the normal.

3. Define refractive index of the medium

Refractive index of a medium is defined as ratio of speed of light in vacuum to that in the medium. It is a dimensionless quantity that describes how the light travels in the medium.

μ=vc

4. What is total internal reflection?

The phenomenon which occurs when the light rays travel from a more optically denser medium to a less optically denser medium. Consider the following situation. A ray of light passes from a medium of water to that of air. Light ray will be refracted at the junction separating the two media. Since it passes from a medium of a higher refractive index to that having a lower refractive index, the refracted light ray bends away from the normal. At a specific angle of incidence, the incident ray of light is refracted in such a way that it passes along the surface of the water. This particular angle of incidence is called the critical angle. Here the angle of refraction is 90 degrees. When the angle of incidence is greater than the critical angle, the incident ray is reflected back to the medium. We call this phenomenon total internal reflection.

5. Define critical angle.

Total internal reflection is a complete reflection of a ray of light within a medium such as water or glass from the surrounding surfaces back into the medium. It only occurs when both of the following two conditions are met: A light ray is in the more dense medium and approaching the less dense medium. The angle of incidence for the light ray is greater than the so-called critical angle.

The critical angle is the angle of incidence, for which the angle of refraction is 90°. If light enters a denser medium from a comparatively rarer medium, then the direction of light changes and the light ray bends towards the normal.

6. Give conditions of total internal reflection.

- The light ray moves from a more dense medium to a less dense medium.
- The angle of incidence must be greater than the critical angle.

7. Write expression for critical angle.

$n_1 \sin \theta_1 = n_2 \sin \theta_2$. Thus we have the following expression for the critical angle: $\mathbf{\sin \theta_c = n_2/n_1}$.

8. Mention a few applications of total internal reflection.

- Total internal reflection is also used in optical fibres. An optical fibre consists of an inner core of high refractive index glass and surrounded by an outer cladding of lower refractive index.
- When light is introduced into the inner core at one end, it will propagate along the fibre in a zigzag path undergoing a series of total internal reflections.
- Optical fibres are useful for getting light to inaccessible places. They are used in many important practical applications. This includes fibre optic diagnostic tools in medicine and fibre optic cables in telecommunications.
- An endoscope is an instrument made of a fibre optic cable. It is used by doctors to see the inside of the human body such as the stomach and the duodenum.
- In telecommunications, copper cables are now replaced by fibre optic cables in the telephone system.
- Multiple signals can be sent at high speeds through bundles of fibres by using flashes of light from a laser.

9. What is interference?

Interference is what happens when two or more waves meet each other. Depending on the overlapping waves' alignment of peaks and troughs, they might add up, or they can partially or entirely cancel each other. As per the interference definition, it is defined as ,the phenomenon in which two or more waves superpose to form a resultant wave of greater, lower or the same amplitude.

10. What is air wedge?

An air wedge is a simple interferometer used to visualize the disturbance of the wave front after propagation through a test object.

11. What is the expression for the fringe width in air wedge experiment?

Light rays coming from slits S_1 and S_2 interfere at point P

From figure, $S_1P = \sqrt{(y - \frac{d}{2})^2 + D^2}$

Or $S_1P = \sqrt{y^2 + \frac{d^2}{4} - yd + D^2}$

Or $S_1P = D\sqrt{\frac{y^2}{D^2} + \frac{d^2}{4D} - \frac{yd}{D^2} + 1}$

As $D >> d$ and $y << D$, we neglect $d^2/4D$ and y^2/D^2.

We get

Or $S_1P = D\sqrt{1 - \frac{yd}{D^2}}$

Or $S_1P = D[1 - \frac{yd}{2D^2}]$

Or $S_1P = D - \frac{yd}{2D}$

Similarly we get $S_2P = D + \frac{yd}{2D}$

Path difference between two light rays $\Delta x = S_2P - S_1P$

Or $\Delta x = (D + \frac{yd}{2D}) - (D - \frac{yd}{2D}) = \frac{yd}{D}$

For constructive interference, $\Delta x = n\lambda$

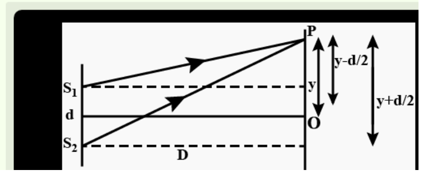
$\therefore \frac{yd}{D} = n\lambda$

$\Rightarrow y = \frac{n\lambda D}{d}$

Fringe width $\beta = y_n - y_{n-1}$

$\therefore \beta = \frac{n\lambda D}{d} - (n-1)\frac{\lambda D}{d}$

$\Rightarrow \beta = \frac{\lambda D}{d}$



12. What is the expression for the thickness of the wire in air wedge experiment?
$d/l = \lambda / 2\mu\beta$ i.e $d = l\lambda / 2\beta$
for an air film $\mu = 1$, βwhich is distance between two dark or bright/dark bands can be measured experimentally. Therefore, diameter can be determined.

13. What is Michelson interferometer?

The Michelson interferometer produces interference fringes by splitting a beam of light so that one beam strikes a fixed mirror and the other a movable mirror. When the reflected beams are brought back together, an interference pattern results.

14. What are the applications of Michelson interferometer?

- The detection of gravitational waves – LIGO is a massive interferometer with two lasers positioned thousands of kilometres apart
- In astronomical interferometry.
- In optical coherence tomography.
- In fibre optics.

PART B

1. Explain the formation of interference fringes in an air-wedge shaped film.
   How is the thickness of the wire determined by this method?
2. How will you use Michelson's interferometer to determine the thickness of a thin transparent film or plate?
3. Explain the construction, types of fringes and applications of Michelson's interferometer.
4. (i) Describe the construction of a Michelson's interferometer and discuss the different types of interference fringes formed in it. (ii) How will you use it to determine the wavelength of a monochromatic source?

BASIC QUANTUM MECHANICS

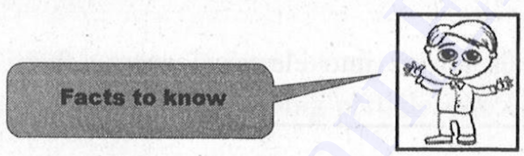
4.1 INTRODUCTION

Classical concepts like Newton's laws of motion can be applied to explain the motion of macro particles which are either observable directly or through microscope. When the classical mechanics concepts were applied to the particles of atomic dimensions like electrons, protons etc., they failed to describe the actual behaviour of the particles.

Quantum mechanics is a physical science which predicts the motion and interaction of subatomic particles and their wave like behaviour. In general, *physics at the atomic level is termed as quantum mechanics*, also called *wave mechanics*.

In the case of classical mechanics, the position, mass, velocity, acceleration etc. of a particle can be measured accurately.

In contrast, quantum mechanics deals with principles which are purely probabilistic in nature.



The realization that matter, on its finest scale is not continuous but is made up of lumps of material called atoms was the beginning of understanding for quantum mechanics. We describe, the lumpiness of matter, by saying that, matter is ***quantized***.

(In Latin, '*quanta*' means '*how much*'). The discrete (*not continuous*), indivisible units (of matter or energy) called **quanta** is one of the main feature of quantum mechanics.

4.2 PHOTONS AND LIGHT WAVES

Since all light ultimately comes from a radiating source, this suggests that perhaps *light is transmitted as tiny particles,* or **photons.**

A photon is a bundle of electromagnetic energy. It is the basic unit that makes up all light and is referred to as a '***quantum***' of electromagnetic energy. The photon is not a wave by itself, but, actually creates a wave that propagates through space.

The photon does not decide how to behave - whether to act as a wave or as a particle. How the photon appears depends on the circumstances of the phenomenon under observation.

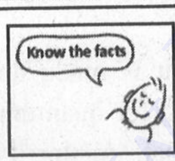
Max Planck in 1900 first proposed the quantized nature of the energy of light to successfully explain the black body radiation.

Einstein believed light is a particle (photon) and the flow of photons is a wave. The photoelectric effect where radiation interacts with matter (i.e. the absorption of light by electrons in solids) indicates radiant energy behaving like particles. It emphasizes the quantized nature of light, which indicates the light has a particle nature leading to the concept of photons.

Photons behave like particles as they can interact with matter.

When the energy of the photon is absorbed by the matter, in some case, the extra energy may be emitted as heat. Black top of the road getting hot in the sun is an example for this case.

When a photon strikes our eye, it is turned into electrical energy that is then transmitted to brain to form an image.



Light radiation behaves as waves in experiments based on interference, diffraction etc. These phenomena require the presence of two waves at the same position at the same time. Obviously it is difficult for the two particles to occupy the same position at the same time. Hence, we conclude that radiation behave like waves.

4.3 ELECTRONS AND MATTER WAVES

In 1923, Louis de Broglie extended the idea of symmetry in nature and claimed that if light sometimes behaves like a wave and sometimes like a particle, then particles

> **NOTE**
>
> Electrons are very delicate and they behave like either particles or waves, depending on the kind of experiment performed on them. In any case, it is impossible to measure both the wave and particle properties simultaneously.

> **Interesting fact**
>
> It is interesting to recall that G.P. Thompson, who shared the 1937 Nobel Prize with Davisson for the experiments which proved that electrons are waves, is the son of J.J. Thompson who received the Nobel Prize in 1906 for proving that cathode rays were actually particles - electrons! And the amazing thing is that they were both right.

4.4 DE BROGLIE HYPOTHESIS AND EXPRESSION FOR DE BROGLIE WAVELENGTH

According to de Broglie, every particle of matter is associated with a wave to guide it as it travels. This means, all material particles, when moving, possess wave nature.

The wave-particle dualism of material particles is known as **de Broglie hypothesis.**

The connection between the particle and wave properties is deduced by de Broglie based on quantum mechanical duality described by the Planck and Einstein's equations.

In 1901, Max Planck showed that atoms can absorb or emit radiation (electromagnetic waves) in discrete energy quanta called photons. The photon energy (E) is given by

$$\boxed{E = h\nu} \quad \text{...(1)}$$

where, 'ν' is the frequency of the radiation and 'h' is Planck's constant and its value is, $h = 6.62 \times 10^{-34}$ Js.

Albert Einstein was the first to show that each photon has a momentum 'p' given by

$$\boxed{p = \frac{E}{C}} \qquad ...(2)$$

where 'C' is the velocity of light. Hence, Planck and Einstein demonstrated that light has not only wavelike properties but also have particle like properties.

Combining equ.(1) and equ.(2), we have

$$p = \frac{h\nu}{c} \qquad ... (3)$$

But, we know that,

Velocity $[c]$ = frequency $[\nu] \times$ wavelength $[\lambda]$

i.e, $c = \nu\lambda$

$$\therefore \frac{\nu}{c} = \frac{1}{\lambda}$$

Therefore equation (3) becomes,

$$p = \frac{h}{\lambda}$$

$$\therefore \quad \boxed{\lambda = \frac{h}{p}} \qquad ...(4)$$

The momentum, of the particle is given by

$$p = mv$$

Therefore equation (4) can be written as

$$\boxed{\lambda = \frac{h}{mv}} \qquad ...(5)$$

The above relation gives the **de Broglie wavelength** (λ) and is applicable to each body, whether an electron, a proton, an atom, a butterfly, a person, a car, a planet, a star etc.

The relationship between wavelength and momentum is fundamental for all moving objects. An object of large mass and less speed has such a small wavelength that interference and diffraction are negligible. But, for smaller particles, like electrons, diffraction can be appreciable.

The concept of both particles and waves possess dual nature of both waves and particles is known as the **principle of Wave-Particle Duality.**

(i) When the particle is in motion, wave nature plays the dominant role and hence will behave like a wave.

(ii) When the particle is at rest, particle nature plays the dominant role and hence will behave like a particle.

4.4.1 de Broglie Wavelength in Various Forms

de Broglie wavelength		
In terms of energy	**In terms of voltage**	**In terms of temperature**
We know, kinetic energy, $E = \frac{1}{2} mv^2$ Multiplying by 'm' on both sides we get, $mE = \frac{1}{2} m^2 v^2$ $m^2 v^2 = 2mE$ (or) $mv = \sqrt{2mE}$ ∴ *de Broglie wavelength,* $\boxed{\lambda = \frac{h}{\sqrt{2mE}}}$	If an electron is accelerated from rest through a potential difference V, it gains kinetic energy, $\frac{1}{2} mv^2 = eV$ (or) $v = \sqrt{\frac{2eV}{m}}$ where 'e' is the electron charge and 'm' is its mass. Substituting this value for 'v' in the de Broglie expression for wavelength gives, *de Broglie wavelength,* $\lambda = \frac{h}{\sqrt{2meV}}$ **(or)** $\boxed{\lambda = \frac{12.25}{\sqrt{V}} \text{ Å}}$	An electron moving inside a system having temperature T possess energy given by $\frac{1}{2} mv^2 = \frac{3}{2} k_B T$ where v is the velocity of the particle and k_B is Boltzmann constant. $m^2 v^2 = 3mk_B T$ $mv = \sqrt{3mk_B T}$ Substituting this value for 'mv' in the de Broglie expression for wavelength gives, *de Broglie wavelength,* $\boxed{\lambda = \frac{h}{\sqrt{3mk_B T}}}$

The **electron volt** (eV) is the kinetic energy gained by an electron passing through a potential difference of one volt (1 V).

A **volt** is not a measure of energy, but the **electron volt** is a unit of energy.

4.4.2 Properties of Matter Waves

(i) Matter waves are produced whether the particles are charged or not.

(ii) Lighter the particle, greater is the wavelength associated with it.

(iii) Smaller the velocity of the particle, greater is the wavelength.

$$\text{(i.e)} \quad \lambda \propto \frac{1}{v}$$

(iv) The de Broglie wavelength of the matter wave is given by

$$\lambda = \frac{h}{p} = \frac{h}{mv}$$

(v) Matter waves show similar properties as other waves such as interference and diffraction.

(vi) Matter waves can travel faster than light.

NOTE

(i) Wave properties are only dominant in very small (micro) particles.

(ii) The wave and particle aspects of a moving particle can never obtain together.

Table 4.1 Distinction between matter waves and electromagnetic waves

S.No.	Matter Waves	Electromagnetic Waves
1.	Matter waves are associated with moving particles (charged or uncharged)	Electromagnetic waves are produced only by accelerated charged particles.
2.	Wavelength depends on the mass of the particle and its velocity, $\lambda = \frac{h}{mv}$	Wavelength depends on the energy of photon.
3.	Matter wave is not electromagnetic wave.	Electric field and magnetic field oscillate perpendicular to each other.
4.	Matter waves can travel with a velocity greater than the velocity of light.	Travel with velocity of light $c = 3 \times 10^8$ m/s
5.	Matter wave require medium for propagation. i.e., they cannot travel through vacuum.	Electromagnetic waves do not require any medium for propagation i.e., they can pass through vacuum.

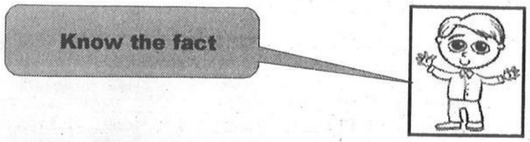
4.5 COMPTON EFFECT

Compton effect is a phenomenon of scattering of light by weakly bound electrons. It is essentially *the increase in wavelength of X-rays* and other energetic electromagnetic radiations *when they collide with and are scattered by weakly bound electrons in matter*.

In 1923, Arthur Compton discovered that when a photon is scattered by a particle like electron, the scattered radiation contains two components in which one having a lower frequency or longer wavelength called ***modified radiation*** and the other one having the same frequency or wavelength as that of the incident radiation called ***unmodified radiation*** **(or)** ***Thomson component.***

This is an example for ***inelastic scattering*** as the scattered photon has less energy (longer wavelength) than the incident photon.

This effect has proved to be one of the cornerstones of quantum mechanics, which accounts for both wave and particle properties of radiation as well as of matter.



Light is made up of many number of photons. "**Photon**" is the name given to the particle-like aspect of light's behavior (in addition to its wave-like behavior).

The Compton effect is based on treating light as consisting of particles of a given energy related to the frequency of the light wave. In this context, the particle of light is given the name "photon".

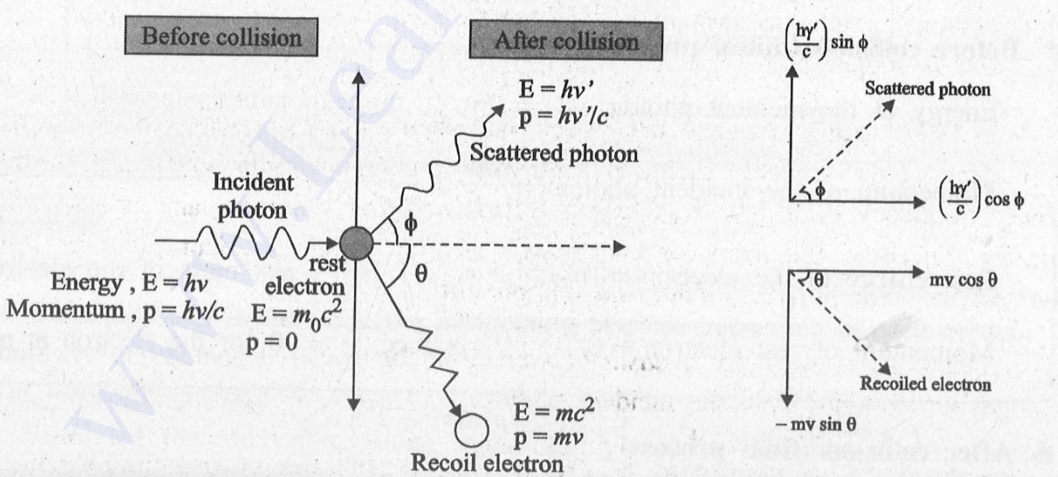
Statement:

The interaction of a high energy *X*-ray photon with a material of low atomic number results in a scattered photon of longer wavelength.

The phenomenon of change in wavelength of an *X*-ray photon after scattering by a target material is known as **Compton effect**.

4.5.1 Theory of Compton Effect

Consider a collision between an *X*-ray photon and an electron. Compton assumed that the electron is free and is at rest before collision with the *X*-ray photon. After collision, the relativistic mass of the electron is considered.



(a) Compton Effect **(b) Momentum components after colllision**

Fig. 4.1

Let $h\nu$ be the energy of the incident X- ray photon in which ν is the frequency associated with it. Moreover the momentum (p) of a massless particle can be related to its energy by the formula,

$$E = pc \qquad ...(1)$$

Since the energy of the incident photon is $h\nu$, its momentum is

$$p = \frac{E}{c} = \frac{h\nu}{c} \qquad ...(2)$$

During collision with the electron, the incident photon gives a fraction of its energy to the free electron and is scattered away from its original direction at an angle ϕ. The electron gains kinetic energy and recoils at an angle θ as shown in fig.4.1.

Here ϕ and θ are the scattering angle and recoil angle respectively.

Also, the scattered photon will have a lower frequency, ν'.

4.5.2 Derivation of Compton Wavelength

Applying the laws of conservation of energy and momentum, the expression for Compton wavelength can be derived.

Let us now find the energy and momentum components before and after collision.

✶ Before collision (initial process)

Energy of the incident photon $= h\nu$

Momentum of the incident photon $= \dfrac{h\nu}{c}$

Rest energy of the electron $= m_0c^2$ [m_0 – rest mass of the electron]

Momentum of rest electron $= 0$ [since, $v = 0$ for the electron at rest]

✶ After collision (final process)

Energy of the scattered photon $= h\nu'$

Momentum of the scattered photon $= \frac{h\nu'}{c}$

Energy of the recoiled electron $= mc^2$

Momentum of the recoiled electron $= mv$

where 'v' is the velocity of the electron after collision.

From the law of conservation of energy, we have

Total energy of the system before collision = Total energy of the system after collision

$$\therefore \quad \begin{bmatrix} \text{Incident photon energy +} \\ \text{Rest electron energy} \end{bmatrix} = \begin{bmatrix} \text{Scattered photon energy +} \\ \text{Recoiled electron energy} \end{bmatrix}$$

Hence, we get

$$h\nu + m_0c^2 = h\nu' + mc^2 \qquad \text{... (1)}$$

From the law of conservation of momentum, we have,

$$\begin{bmatrix} \text{Total initial momentum of} \\ \text{the system before collision} \end{bmatrix} = \begin{bmatrix} \text{Total final momentum of} \\ \text{the system after collision} \end{bmatrix}$$

$$\therefore \begin{bmatrix} \text{Incident photon momentum} \\ \text{before collision} \\ + \\ \text{Rest electron momentum} \\ \text{before collision} \end{bmatrix} = \begin{bmatrix} \text{Scattered photon momentum} \\ \text{after collision} \\ + \\ \text{Rest electron momentum} \\ \text{after collision} \end{bmatrix}$$

NOTE

Momentum, unlike energy, is a vector quantity that incorporates direction as well as magnitude and in the collision, momentum must be conserved in each of two mutually perpendicular directions.

Resolving the momentum along x-axis and y-axis for the photon and electron fore and after collision, we get

Table 4.2

Direction	Momentum before collision		Momentum after collision	
	Incident Photon	Rest Electron	Scattered Photon	Recoiled Electron
x-component momentum	$\frac{h\nu}{c}$	0	$\frac{h\nu'}{c}\cos\phi$	$m\nu\cos\theta$
y-component momentum	0	0	$\frac{h\nu'}{c}\sin\phi$	$-m\nu\sin\theta$

Along incident photon direction, we have

✶ ***x*-component momentum conservation**

$$\frac{h\nu}{c}+0 = \frac{h\nu'}{c}\cos\phi + m\nu\cos\theta \qquad \text{... (2)}$$

Along perpendicular to incident photon direction, we have

✶ ***y*-component momentum conservation**

$$0+0 = \frac{h\nu'}{c}\sin\phi - m\nu\sin\theta$$ [–ve sign indicates negative y direction]

i.e. $$0 = \frac{h\nu'}{c}\sin\phi - m\nu\sin\theta \qquad \text{... (3)}$$

Let us now eliminate θ by combining the momentum equations (2) and (3).

The first step is to multiply equation (2) and (3) by 'c' and the result is,

$$h\nu = h\nu'\cos\phi + m\nu c\cos\theta \qquad \text{... (4)}$$

and, $$0 = h\nu'\sin\phi - m\nu c\sin\theta \qquad \text{... (5)}$$

The second step is to rewrite the above equation (4) and (5) as

$$m\nu c\cos\theta = h(\nu - \nu'\cos\phi) \qquad \text{... (6)}$$

$$m\nu c\sin\theta = h\nu'\sin\phi \qquad \text{... (7)}$$

$$\therefore\ m^2 (c^2 - v^2) = m_0^2 c^2$$

Multiplying both sides by c^2, we get

$$m^2 c^2 (c^2 - v^2) = m_0^2 c^4 \qquad \text{... (12)}$$

Substituting equation (12) in equation (10), we obtain

$$m_0^2 c^4 = -2h^2 vv' (1 - \cos\phi) + 2h (v - v') m_0 c^2 + m_0^2 c^4$$

$$\therefore \qquad 2h (v - v') m_0 c^2 = 2h^2 vv' (1 - \cos\phi)$$

$$\text{(or)} \qquad \frac{v - v'}{vv'} = \frac{h}{m_0 c^2} (1 - \cos\phi)$$

$$\boxed{\frac{1}{v'} - \frac{1}{v} = \frac{h}{m_0 c^2} (1 - \cos\phi)} \qquad \text{...(13)}$$

Eqn.(13) shows that $v' < v$ as h, m_0, c are constants with positive values. This shows that the scattered frequency is always smaller than the incident frequency.

1. Change in wavelength:

Multiplying equation (13) by 'c' on both sides, we get

$$c\left[\frac{1}{v'} - \frac{1}{v}\right] = \frac{h}{m_0 c} (1 - \cos\phi) \qquad \text{...(14)}$$

But, we know that, $\dfrac{\text{velocity}}{\text{frequency}} = \text{wavelength}$

$$\therefore \qquad \frac{c}{v} = \lambda \text{ and } \frac{c}{v'} = \lambda'$$

where, λ is the wavelength of the incident photon and λ' is the wavelength of the scattered photon.

Equation (14) becomes,

$$\boxed{\lambda' - \lambda = \frac{h}{m_0 c} (1 - \cos\phi)} \qquad \text{...(15)}$$

Squaring and adding the above two equations, we get

$$\begin{bmatrix} m^2 v^2 c^2 \cos^2\theta \\ + \\ m^2 v^2 c^2 \sin^2\theta \end{bmatrix} = \begin{bmatrix} h^2 ((\nu - \nu' \cos\phi)^2) \\ + \\ h^2 \nu'^2 \sin^2\phi \end{bmatrix}$$

(i.e) $$m^2 v^2 c^2 (\cos^2\theta + \sin^2\theta) = h^2 (\nu^2 + \nu'^2 \cos^2\phi - 2\nu\nu' \cos\phi) + h^2 \nu'^2 \sin^2\phi$$

$$m^2 v^2 c^2 = h^2 (\nu^2 + \nu'^2 \cos^2\phi - 2\nu\nu' \cos\phi + \nu'^2 \sin^2\phi)$$

$$[\because \cos^2\theta + \sin^2\theta = 1]$$

$$m^2 v^2 c^2 = h^2 (\nu^2 + \nu'^2 - 2\nu\nu' \cos\phi) \qquad ...(8)$$

$$[\because \nu'^2 \cos^2\phi + \nu'^2 \sin^2\phi = \nu'^2 (\cos^2\phi + \sin^2\phi) = \nu'^2]$$

Rewriting eqn. (1), we get,

$$mc^2 = h(\nu - \nu') + m_0 c^2$$

Squaring the above equation, we get

$$[mc^2]^2 = [h(\nu - \nu') + m_0 c^2]^2$$

$$m^2 c^4 = [h(\nu - \nu')]^2 + m_0^2 c^4 + 2h(\nu - \nu') m_0 c^2$$

$$m^2 c^4 = h^2 (\nu^2 + \nu'^2 - 2\nu\nu') + 2h(\nu - \nu') m_0 c^2 + m_0^2 c^4$$

(or) $$m^2 c^2 c^2 = h^2 (\nu^2 + \nu'^2 - 2\nu\nu') + 2h(\nu - \nu') m_0 c^2 + m_0^2 c^4 \qquad ...(9)$$

Subtracting equation (8) from equation (9), we get

$$m^2 c^2 (c^2 - v^2) = -2h^2 \nu\nu' (1 - \cos\phi) + 2h(\nu - \nu') m_0 c^2 + m_0^2 c^4 \qquad ...(10)$$

But from Einstein's relation, we have

$$m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}} \qquad ...(11)$$

Squaring the above equation,

$$m^2 \left(1 - \frac{v^2}{c^2}\right) = m_0^2$$

$$\therefore\ m^2 (c^2 - v^2) = m_0^2 c^2$$

Multiplying both sides by c^2, we get

$$m^2 c^2 (c^2 - v^2) = m_0^2 c^4 \qquad \text{... (12)}$$

Substituting equation (12) in equation (10), we obtain

$$m_0^2 c^4 = -2h^2 \nu\nu' (1 - \cos\phi) + 2h(\nu - \nu') m_0 c^2 + m_0^2 c^4$$

$$\therefore \quad 2h(\nu - \nu') m_0 c^2 = 2h^2 \nu\nu' (1 - \cos\phi)$$

(or) $$\frac{\nu - \nu'}{\nu\nu'} = \frac{h}{m_0 c^2}(1 - \cos\phi)$$

$$\boxed{\frac{1}{\nu'} - \frac{1}{\nu} = \frac{h}{m_0 c^2}(1 - \cos\phi)} \qquad \text{...(13)}$$

Eqn.(13) shows that $\nu' < \nu$ as h, m_0, c are constants with positive values. This shows that the scattered frequency is always smaller than the incident frequency.

1. Change in wavelength:

Multiplying equation (13) by 'c' on both sides, we get

$$c\left[\frac{1}{\nu'} - \frac{1}{\nu}\right] = \frac{h}{m_0 c}(1 - \cos\phi) \qquad \text{...(14)}$$

But, we know that, $\frac{\text{velocity}}{\text{frequency}} = \text{wavelength}$

$$\therefore \quad \frac{c}{\nu} = \lambda \text{ and } \frac{c}{\nu'} = \lambda'$$

where, λ is the wavelength of the incident photon and λ' is the wavelength of the scattered photon.

Equation (14) becomes,

$$\boxed{\lambda' - \lambda = \frac{h}{m_0 c}(1 - \cos\phi)} \qquad \text{...(15)}$$

(or) Change in wavelength ($\Delta\lambda = \lambda' - \lambda$) is given by

$$\boxed{\Delta\lambda = \frac{h}{m_0 c}(1 - \cos\phi)} \qquad ...(16)$$

*Equation (16) is known as **Compton equation** which gives the change in wavelength of a photon that is scattered through an angle ϕ by a particle of rest mass m_0.*

Conclusion:

The Compton equation shows that the Compton shift ($\Delta\lambda$)

(i) Is independent of the incident photon wavelength (λ)

(ii) Depends only on the scattering angle, ϕ.

(iii) Is larger if the rest mass of the scatterer is small.

(iv) Is independent of the nature of the scattering material.

The above conclusions agree with the experimental results.

Significance of compton effect:

The effect is important because it demonstrates that light cannot be explained purely as a wave phenomenon.

Thus, Compton effect provides direct support that light consists of point like quanta of energy called *photons* and gives direct evidence in support of the particle nature of electromagnetic radiation.

3. Special Cases:

From equation (16), it is clear that

(i) When $\phi = 0$:

$\lambda' - \lambda = 0$, which indicates that there is no scattering along the direction of incident angle.

(ii) When $\phi = \frac{\pi}{2}$ (or) 90°:

$$\lambda' - \lambda = \frac{h}{m_0 C}$$

$$= \frac{6.626 \times 10^{-34}\ \text{Js}}{9.1 \times 10^{-31}\ \text{kg} \times 3 \times 10^{8}\ \text{m/s}}$$

$$\lambda' - \lambda = \mathbf{0.02426\ \text{Å}}$$

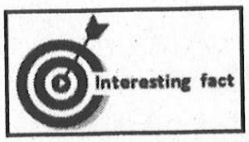
This difference in wavelength is known as **Compton wavelength** (λ). It is a constant quantity.

Here, the term $\Delta\lambda = \lambda' - \lambda$ is known as ***Compton shift*** and it gives the scale of wavelength change of the incident photon.

(iii) When $\phi = \pi$ (or) 180°:

$$\lambda' - \lambda = \frac{2h}{m_0 C} = 0.04852 \text{ Å}$$

Thus, the change in wavelength ranges from 0 at $\phi = 0°$ to twice the Compton wavelength at $\phi = 180°$, which indicates that *the greatest wavelength change corresponds to $\phi = 180°$.*



The Compton shift actually increases as $\lambda \to 0$ and hence the effect is more easily observable for shorter wavelength radiations.

Indeed, this is the reason why the sky is blue, because all of the atoms/molecules in the atmosphere scatter blue light more.

4.5.3 Experimental Verification of Compton Effect

The experimental demonstration of the Compton effect is shown in fig. 4.2.

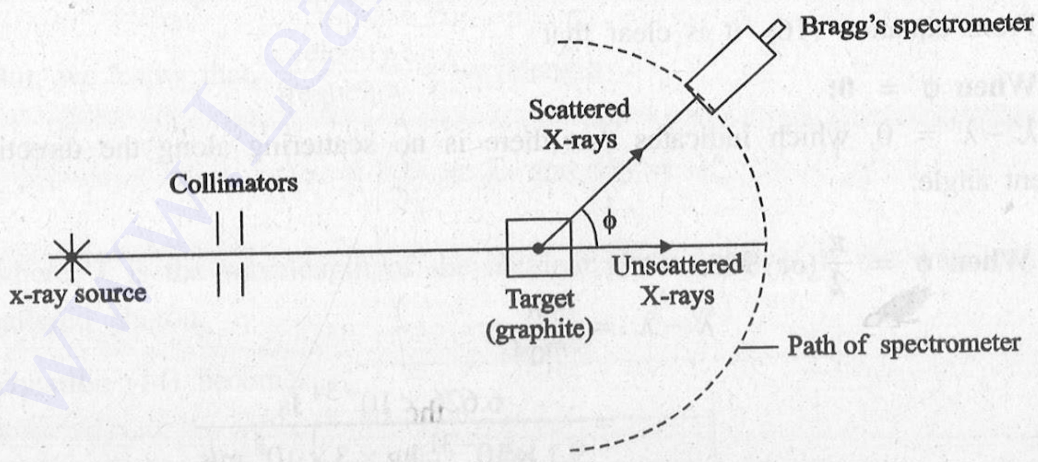


Fig. 4.2 Experimental verification of Compton effect

- A beam of X-ray is incident on a scattering target, for which Compton used carbon.
- A rotating detector measures the wavelengths of the scattered X-rays at various angles ϕ.
- Compton's results are illustrated in fig 4.3.

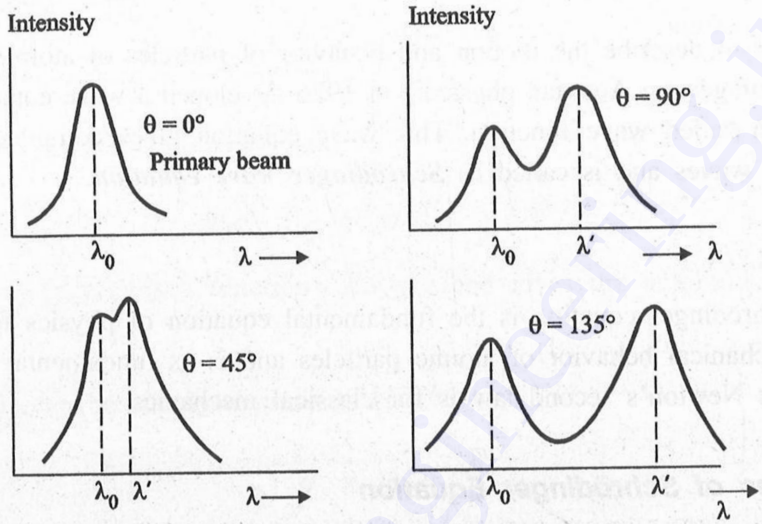


Fig 4.3 Compton's results for X-ray scattering

Inference:

- At each angle, two peaks appear, corresponding to scattered X-ray photons with two different energies or wavelengths.
- The wavelength of one peak does not change as the angle is varied. This peak corresponds to scattering from the tightly *bound inner electrons* of the target atom. These electrons are so tightly attached to the atoms that the photon scatters and loses no energy.
- The wavelength of the other peak varies strongly with angle as can be seen from fig. 4.3. This change in wavelength corresponds to scattering from the loosely bound outer most electrons in the target atom which absorbs certain amount of energy from the incident photon.

The direction of recoil electron is given by, $\tan\theta = \dfrac{\cot \phi/2}{\left[1+\left(\dfrac{hv}{m_0 c^2}\right)\right]}$

4.6 THE SCHRÖDINGER EQUATION

In order to describe the motion and behavior of particles of atomic dimensions, Erwin Schrödinger, an Austrian physicist, in 1926 developed a wave equation in terms of a function called wave function. This wave equation gives complete information about matter waves and is called as ***Schrödinger wave equation***.

Significance:

The Schrödinger equation is the fundamental equation of physics for describing quantum mechanical behavior of atomic particles and is as fundamental for quantum mechanics as Newton's second law is for classical mechanics.

4.6.1 Types of Schrödinger Equation

The two forms of Schrödinger wave equation are

1. **Time independent Schrödinger wave equation**
2. **Time dependent Schrödinger wave equation**

1. Schrödinger Time Independent Wave Equation

Consider a system of stationary waves associated with a moving particle of mass 'm' and velocity 'v'.

Let x, y, z be the coordinates of the particle and ψ, the wave displacement for the de Broglie waves at any instant of time t.

The wave motion of the particle in three dimensions is given by the classical differential equation as,

$$\frac{\partial^2 \psi}{\partial t^2} = v^2\left[\frac{\partial^2 \psi}{\partial x^2}+\frac{\partial^2 \psi}{\partial y^2}+\frac{\partial^2 \psi}{\partial z^2}\right] \quad ...(1)$$

(or) $$\frac{\partial^2 \psi}{\partial t^2} = v^2 \nabla^2 \psi \qquad ...(2)$$

Where $\nabla^2 = \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right]$ is called the Laplacian operator and 'v' is the wave velocity.

Here ψ is called the wave function which represents the complex displacement of the waves associated with the particle.

On solving equation (2), we get,

$$\psi_{(x, y, z, t)} = \psi_{0\,(x, y, z)}\, e^{-i\omega t} \qquad ...(3)$$

where $\psi_0\,(x, y, z)$ is a function of x, y, z and gives the amplitude at the point considered.

Eqn. (3) can be simply expressed as,

$$\psi = \psi_0\, e^{-i\omega t} \qquad ...(4)$$

Differentiating equation (4) twice with respect to time t, we get,

$$\frac{\partial^2 \psi}{\partial t^2} = -\omega^2 \psi_0\, e^{-i\omega t} \qquad ...(5)$$

Substituting the value of ψ from equation (4) in equation (5), we have

$$\boxed{\frac{\partial^2 \psi}{\partial t^2} = -\omega^2 \psi} \qquad ...(6)$$

Substituting eqn. (1) in eqn. (6), we obtain,

$$v^2\left[\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2}\right] = -\omega^2 \psi$$

(or) $$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} = -\left(\frac{\omega^2}{v^2}\right)\psi$$

(i.e) $$\frac{\partial^2 \psi}{\partial x^2}+\frac{\partial^2 \psi}{\partial y^2}+\frac{\partial^2 \psi}{\partial z^2}+\frac{\omega^2}{v^2}\psi = 0 \qquad ...(7)$$

But, $\omega = 2\pi\nu = 2\pi\left(\frac{v}{\lambda}\right)$

where 'ν' is the frequency and is given by, $\nu = \frac{v}{\lambda}$

Hence, $$\frac{\omega}{v} = \frac{2\pi}{\lambda} \qquad ...(8)$$

Substituting eqn. (8) in eqn. (7), we get

$$\frac{\partial^2 \psi}{\partial x^2}+\frac{\partial^2 \psi}{\partial y^2}+\frac{\partial^2 \psi}{\partial z^2}+\left(\frac{4\pi^2}{\lambda^2}\right)\psi = 0$$

$$\text{(or)}\ \nabla^2\psi+\left(\frac{4\pi^2}{\lambda^2}\right)\psi = 0 \quad \left[\because \nabla^2 = \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right] \qquad ...(9)$$

According to de Broglie, the wavelength of matter wave is given by

$$\lambda = \frac{h}{mv}$$

Hence, equation (9) becomes, $$\nabla^2\psi+\left(\frac{4\pi^2}{h^2}m^2 v^2\right)\psi = 0 \qquad ...(10)$$

The total energy of the particle is given by,

Total energy = Kinetic energy + Potential energy

If E is the total energy, V is the potential energy and $\frac{1}{2}mv^2$ is the kinetic energy of the particle, then we can write

$$E = \frac{1}{2}mv^2 + V$$

(or) $\frac{1}{2}mv^2 = E - V$, which gives

$$m^2 v^2 = 2m(E-V) \qquad ...(11)$$

Using eqn. (10) and (11), we obtain

$$\nabla^2 \psi + \left[\frac{4\pi^2}{h^2} \times 2m\,(E - V) \right] \psi = 0$$

(or) $$\boxed{\nabla^2 \psi + \frac{8\pi^2 m}{h^2}(E - V)\,\psi = 0} \quad ...(12)$$

The above equation is known as ***Schrödinger time independent wave equation*** or ***steady state form of Schrödinger equation.***

Substituting $\hbar = \frac{h}{2\pi}$ in equation (12), we arrive at

$$\boxed{\nabla^2 \psi + \frac{2m}{\hbar^2}(E - V)\,\psi = 0} \quad ...(13)$$

where, $\hbar = \frac{h}{2\pi} = 1.055 \times 10^{-34}$ Js is called **reduced Planck's constant.**

Special Case

For a free particle, the potential energy, $V = 0$, write

$$\boxed{\nabla^2 \psi + \frac{2mE}{\hbar^2}\psi = 0} \quad ...(14)$$

The above equation represents the ***Schrödinger wave equation for a free particle.***

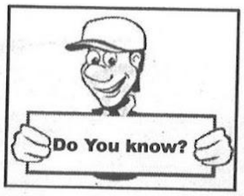
Eigen Function and Eigen Value

Schrödinger's time - independent equation is an example of a type of differential equation called an **Eigen value equation.**

In general, we can write an Eigen value equation as

$$\boxed{H\psi = E\psi}$$

The function ψ is then called an *Eigen function* of the operator H and the corresponding value for E is called the *Eigen value.*



The concept of an operator is of great importance in quantum mechanics.

An operator is an instruction to carry out a mathematical operation upon a mathematical function.

If the application of an operator to a function returns the original function multiplied by some constant, then this constant is termed as the **Eigen value** of that function.

If the application of an operator to a function gives back the original function multiplied by some constant, then the function is termed as an **Eigen function** of that operator.

2. Schrödinger Time Dependent Wave Equation

Let us assume that the wave function ψ for a particle moving freely is specified by

$$\psi = \psi_0 e^{-i\omega t} \quad \text{...(1)}$$

Differentiating eqn. (1) with respect to time, we get

$$\frac{\partial \psi}{\partial t} = -i\omega \psi_0 e^{-i\omega t}$$

(or)

$$\frac{\partial \psi}{\partial t} = -i(2\pi\nu)\psi_0 e^{-i\omega t} \qquad [\because \omega = 2\pi\nu]$$

i.e

$$\frac{\partial \psi}{\partial t} = -i(2\pi\nu)\psi \qquad [\because \psi = \psi_0 e^{-i\omega t}]$$

...(2)

Substituting, $E = h\nu$ in eqn. (2), we get

$$\frac{\partial \psi}{\partial t} = -i\left[2\pi\left(\frac{E}{h}\right)\right]\psi \quad ...(3)$$

Substituting, $\hbar = \frac{h}{2\pi}$, we have

$$\frac{\partial \psi}{\partial t} = -i\left(\frac{E}{\hbar}\right)\psi \quad ...(4)$$

Since, $i^2 = -1$, we can rewrite eqn.(4) as

$$E\psi = i\hbar\left(\frac{\partial \psi}{\partial t}\right) \quad ...(5)$$

The Schrödinger time independent wave equation is given by

$$\nabla^2 \psi + \frac{2m}{\hbar^2}(E - V)\psi = 0$$

(or) $$\nabla^2 \psi + \frac{2m}{\hbar^2}(E\psi - V\psi) = 0$$

(i.e) $$\nabla^2 \psi = -\left[\frac{2m}{\hbar^2}(E\psi - V\psi)\right] \quad ...(6)$$

Substituting eqn. (5) in eqn. (6), we obtain

$$\nabla^2 \psi = -\frac{2m}{\hbar^2}\left(i\hbar\frac{\partial \psi}{\partial t} - V\psi\right)$$

(or) $$-\left(\frac{\hbar^2}{2m}\right)\nabla^2 \psi = i\hbar\frac{\partial \psi}{\partial t} - V\psi$$

(or) $$-\left(\frac{\hbar^2}{2m}\right)\nabla^2 \psi + V\psi = i\hbar\frac{\partial \psi}{\partial t}$$

$$\boxed{\left[-\left(\frac{\hbar^2}{2m}\right)\nabla^2 + V\right]\psi = i\hbar\frac{\partial \psi}{\partial t}} \quad ...(7)$$

Since equation (7) involves time, it is known as ***Schrödinger's time dependent wave equation.***

4.6.2 Wave Function

A mathematical function which represent both a particle's wave like characteristics and its uncertainty in location is known as **wave function.**

Physical significance of wave function, ψ

(i) It is a complex quantity representing the variation of matter wave.

(ii) The wave function ψ gives the probability of finding a particle at a given place at a particular instant of time.

(iii) $|\psi|^2$ represents the probability density of the particle, which is real, positive and can be measured.

(iv) The wave function relates the particle and the wave statistically.

(v) When the particle is certainly to be found somewhere in space, the total probability of finding the particle is given by $\int_{-\infty}^{\infty} |\psi|^2 \, dx\, dy\, dz = 1$.

The wave function satisfying the above condition is said to be **normalized.**

Thus, a moving particle's motion and its behavior can be analyzed from the wave function.

Know the fact:

Normalization is the process by which the probability of finding the particle inside a potential box can be done.

NOTE

1. If we define $P(x)$ as the probability density (probability per unit length) in one dimension, then we have

$$\boxed{P(x)\, dx = |\psi(x)|^2\, dx}$$

The quantity $|\psi(x)|^2$ gives the probability to find the particle in the interval dx at x (that is, between x and $x + dx$)

2. A large value of $|\psi|^2$ means the strong possibility of the particle's presence, while small value of $|\psi|^2$ means the slight possibility of its presence.
3. The absolute magnitude is needed to make the probability everywhere positive.

4.7 MEANING OF WAVE FUNCTION

A mathematical function which represent the wave properties of a moving particle is known as wave **function.**

The **wave function** (ψ) gives the probability amplitude which describes the quantum state of a moving particle and how it behaves. It can be either *real or complex*.

The wave function ψ depends on the position coordinates x, y, z and time (t) and mathematically describes the motion of a particle. Thus, the wave properties of the particle are described by the wave function, ψ, since this function represents a wave.

In general, it is not possible to locate a particle precisely at a position at a given time (x, y, z, t), there is only a probability of the particle being at the specific point (x, y, z) at the given instant of time (t).

4.7.1 Physical Significance of Wave Function

- The wave function ψ is the probability amplitude and gives the probability of finding a particle at a given place at a particular instant of time.
- The square of the magnitude of the wave function $|\psi|^2$ represents the probability density of the particle which is a measure of the energy density or energy per unit volume.
- A large value of $|\psi|^2$ means the strong possibility of the particle's presence, while small value of $|\psi|^2$ means the slight possibility of its presence.
- When the particle is certainly to be found somewhere in space, the total probability of finding the particle is given by $\iiint |\psi|^2 \, dx\, dy\, dz = 1$.

4.7.1.1 Normalization

Let us consider a particle to be present within an element volume $d\tau = dx\, dy\, dz$.

Then, the probability of the particle to be present within that element volume d is $|\psi|^2 \, dx\, dy\, dz$.

The total probability of finding the particle somewhere in the element volume is $d\tau$ ***unity*** and is given by

$$\boxed{|\psi|^2\, dx\, dy\, dz = 1}$$

Now, the particle certainly to be found somewhere in space is given by

$$\iiint |\psi|^2\, dx\, dy\, dz = 1$$

A wave function that satisfies the above equation is said to be normalized.

Besides being normalizable, the wave function (ψ) must fulfill the following requirements.

(i) **It must be finite everywhere:** If ψ is infinite at a particular point, then it would mean an infinitely large probability of finding the particle at that point. This is not possible. Hence, ψ must have finite or zero value at any point.

(ii) **It must be single valued:** If ψ has more than one value at any point, it means more than one value of probability of finding the particle at that point which is not possible.

(iii) **It must be continuous:** For Schrödinger equation to be finite everywhere, $d\psi/dx$ should have no discontinuities at any boundary where potential changes. This implies that ψ too must be continuous across a boundary.

4.8 FREE PARTICLE

A **free particle** is one that is not subjected to any force and we can therefore ıke its potential energy to be zero.

In quantum mechanics, a **free particle** is described as a particle free to move a small space surrounded by impenetrable barriers.

For example, a particle in an infinite potential well is a simplified system presenting a free particle moving horizontally within an infinitely deep well from ıich it cannot escape.

4.8.1 Particle in an Infinite Potential Well

Consider a particle like electron of mass 'm' which is bound to move in a one dimensional crystal of length 'L'. This situation can be considered as an electron restricted to move within a deep potential well.

A particle under the influence of such a potential is free (no forces) between $x = 0$ and $x = L$, and is completely excluded (infinite potential) outside that region as illustrated in fig. 4.4.

The electron can move inside the potential well along the x - direction and its motion can be pictured as a particle bouncing back and forth between the walls of the box.

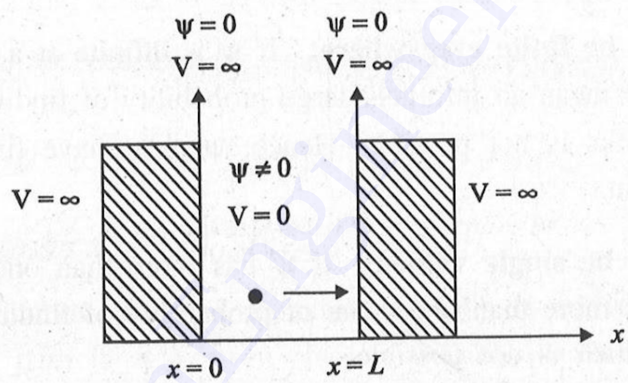


Fig. 4.4 Particle in a one dimensional box

Within the well, the potential energy is zero (i.e), $V = 0$. As the walls are located at $x = 0$ and $x = L$, the potential rises abruptly to a large value, say infinity at the edges of the box.

$$\text{i.e.,}\quad \left.\begin{aligned} &V = \infty \text{ at } x = 0, \\ &V = \infty \text{ at } x = L, \text{ and} \\ &V = 0 \text{ for } 0 < x < L \end{aligned}\right\} \quad \dots (1)$$

Therefore, the probability of finding the electron outside the well must be zero, that is,

$$\left.\begin{aligned} \psi &= 0 \text{ at } x \le 0, \text{ and} \\ \psi &= 0 \text{ at } x \ge L, \end{aligned}\right\} \quad \dots (2)$$

Equations (1) and (2) are known as ***the boundary conditions*** for the free electron.

$$\nabla^2 \psi + \frac{8\pi^2 m}{h^2} E \psi = 0 \quad ...(3)$$

Substituting $\frac{8\pi^2 mE}{h^2} = k^2$ in the above equation, we have

$$\nabla^2 \psi + k^2 \psi = 0 \quad ...(4)$$

The general solution of this equation is

$$\psi = A \sin kx + B \cos kx \quad ...(5)$$

Now, the boundary conditions can be applied to evaluate the constants A and B.

Since, $\psi = 0$ at $x = 0$, we have

$$0 = A \sin k(0) + B \cos k(0)$$

which gives, $B = 0$

Moreover, $\psi = 0$ at $x = L$, hence, we have

$$0 = A \sin kL \qquad [\because B = 0]$$

which means $A \neq 0$ and $\sin kL = 0$

But, $\sin kL = 0$, only when

$$kL = n\pi$$

Where n is an integer

$$\therefore k = \frac{n\pi}{L} \quad ...(6)$$

Substituting the above value of 'k' in equ (5), we have

$$\boxed{\psi_n(x) = A \sin\left(\frac{n\pi x}{L}\right)} \quad ...(8)$$

The above equation is known as the wave function of the free electron.

We know that, $k^2 = \frac{8\pi^2 mE}{h^2}$ and therefore we can write the energy of the free

on as

$$E_n = \frac{k^2 h^2}{8\pi^2 m}$$

But, we have, $k = \frac{n\pi}{L}$ and hence the energy of the free electron can be written as,

$$E_n = \frac{h^2 n^2 \pi^2}{L^2 8\pi^2 m}$$

Thus, the energy of the free electron is

$$\boxed{E_n = \frac{n^2 h^2}{8mL^2}} \qquad ...(9)$$

where the *principal quantum number*, $n = 1, 2, 3 \ldots$ represent various quantum states.

Salient features of the result:,

- The energy of a particle is **quantized** and can have *only discrete values called energy levels* given by
  $E_1 = \frac{h^2}{8mL^2}$ $(n = 1)$, $E_2 = \frac{4h^2}{8mL^2}$ $(n = 2)$, $E_3 = \frac{9h^2}{8mL^2}$ $(n = 3)$ and so on.........
- The lowest possible energy of a particle is not *zero*. This is called the **zero-point energy** and means the particle can never be at rest because it always has some kinetic energy.

4.8.2 Normalization of Wave Function

Normalization is used to find the amplitude of the function, ψ.

It is sure that the electron is somewhere inside the box and hence the probability of finding an electron somewhere inside the box is ***unity***. Hence for a normalized wave function, we have

$$\int_0^L \psi \psi^* \, dx = 1 \qquad ...(1)$$

Therefore, $$A^2 \int_0^L \sin^2\left(\frac{n\pi x}{L}\right) dx = 1 \qquad \left[\because \psi = A \sin\left(\frac{n\pi x}{L}\right)\right]$$

$$\text{(or) } A^2 \int_0^L \left[\frac{1-\cos\left(\frac{2\pi nx}{L}\right)}{2}\right] dx = 1 \qquad \left[\because \sin^2(\theta) = \frac{1-\cos 2\theta}{2}\right]$$

$$A^2\left[\int_0^L \frac{dx}{2} - \frac{1}{2}\int_0^L \cos\left(\frac{2\pi nx}{L}\right)\right] = 1$$

$$\left(\frac{A^2}{2}\right)\int_0^L dx = 1 \qquad \left[\because \int_0^L \cos\left(\frac{2\pi nx}{L}\right) = 0\right]$$

$$\left(\frac{A^2}{2}\right)[x]_0^L = 1$$

$$\left(\frac{A^2}{2}\right)L = 1$$

(or) $$A^2 = \frac{2}{L}$$

(i.e) $$\boxed{A = \sqrt{\frac{2}{L}}} \qquad ...(2)$$

Hence, the normalized wave function of the electron can be written as,

$$\boxed{\psi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)} \qquad ...(3)$$

Also,

$$|\psi_n(x)|^2 = \left(\frac{2}{L}\right)\sin^2\left(\frac{n\pi x}{L}\right) \qquad ...(4)$$

Where $|\psi_n|^2$ represents the probability density of finding an electron inside the ential box.

The wave functions $\psi_n(x)$ and their probabilities $|\psi_n|^2$ for the values of $n = 1, 2, 3$ are shown in fig. 4.5.

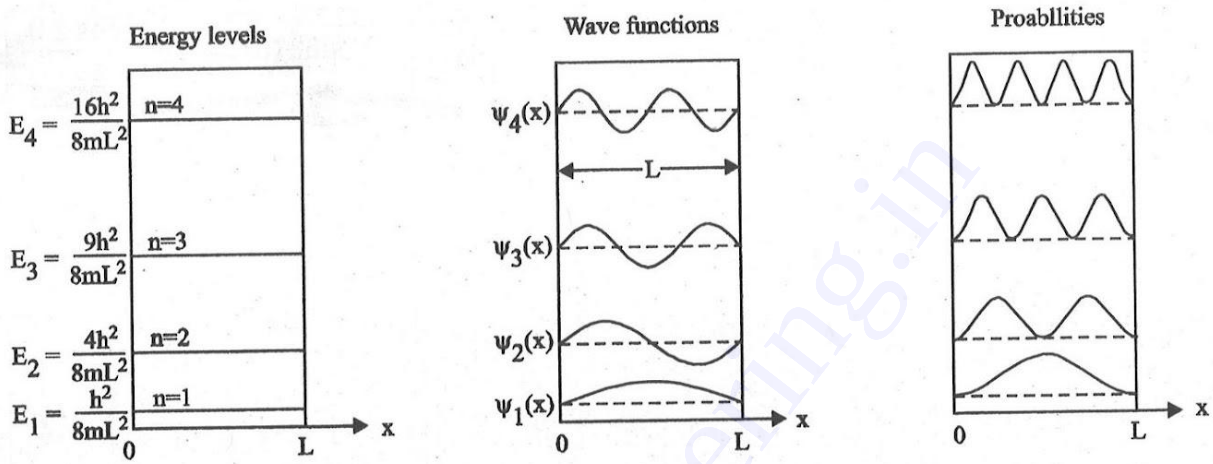


Fig 4.5 The allowed energy levels, wave functions and the corresponding probability distributions for an electron in a one dimensional potential box.

Interpretation and significance of the results:

From the fig. (4.2), it is evident that for the energy state $n = 1$, the probability density $|\psi_n|^2$ is largest in the middle while it decreases towards the boundaries. Notice that the number of **nodes** (places where the particle has zero probabily of being located) increases with increasing energy, n.

As the energy of the particle becomes greater, the quantum mechanical model breaks down because the energy levels get closer together and overlap, forming a continuum. This continuum means the particle is free and can have any energy value.

At such high energies, the classical mechanical model is applied as the particle behaves more like a continuous wave. Therefore, the particle in a box problem is an example of *wave-particle duality*.

4.9 PARTICLE IN AN INFINITE POTENTIAL WELL - 2D BOX

The quantum particle in the 1D box problem can be expanded to consider a particle within higher dimensions.

Let us now consider the Schrödinger equation for an electron confined to a two dimensional box, $0 < x < a$ and $0 < y < b$.

Hence, within this rectangle, the electron wave function behaves as a free particle, where

$$V(x, y) = 0$$

Also the walls are impenetrable, so the wave function becomes,

$$\psi(x, y, t) = 0 \text{ at the walls.}$$

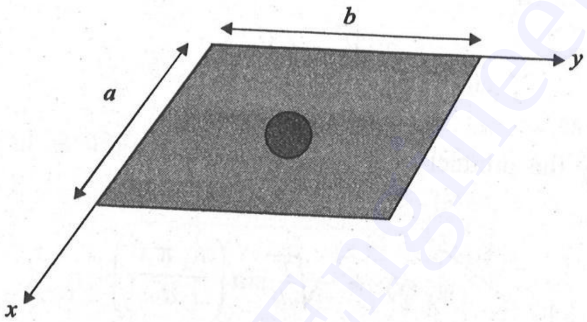


Fig. 4.6 A particle in a 2-dimensional box

Outside the box, we have the Schrödinger equation for 1 D box as,

$$\psi(x, y) = 0 \qquad \text{... (1)}$$

Inside the box, we have the Schrödinger equation for 1 D box as,

$$-\frac{\hbar}{2m}\frac{d^2\psi(x)}{dx^2} + V(x)\psi(x) = E\psi(x) \qquad \text{... (2)}$$

Extending it to two dimension, we have

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2\psi(x, y)}{\partial x^2} + \frac{\partial^2\psi(x, y)}{\partial y^2}\right) + V(x, y)\psi(x, y) = E\psi(x, y) \qquad \text{... (3)}$$

The boundary condition implies that,

$$\begin{pmatrix} V(x, y) = 0 \text{ within the box, and} \\ V(x, y) = \infty \text{ outside the box} \end{pmatrix} \quad \dots (4)$$

Hence, equ (2) can be simplified for the particle in a 2D box as

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2 \psi(x, y)}{\partial x^2} + \frac{\partial^2 \psi(x, y)}{\partial y^2}\right) = E\,\psi(x, y) \quad \dots (5)$$

The wave function inside the box can be solved by separation of variables $\psi(x, y) = \psi_x(x)\,\psi_y(y)$ which can be shown to lead to the equations

$$-\frac{\hbar^2}{2m}\frac{d^2 \psi_x(x)}{dx^2} = E_x\,\psi_x(x) \quad \dots (6)$$

$$\text{and } -\frac{\hbar^2}{2m}\frac{d^2 \psi_y(y)}{dy^2} = E_y\,\psi_y(y) \quad \dots (7)$$

This is simply the product of a two independent particles in 1-D boxes. So, we have

$$\psi_x(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n_x \pi x}{a}\right) \quad \dots (8)$$

$$\text{and } \psi_y(y) = \sqrt{\frac{2}{b}}\sin\left(\frac{n_y \pi_y}{b}\right) \quad \dots (9)$$

$$\text{where } E_{nx} = \frac{n_x^2 h^2}{8ma^2} \quad \dots (10)$$

$$E_{ny} = \frac{n_y^2 h^2}{8mb^2} \quad \dots (11)$$

The total energy E is now quantized by two numbers, n_x and n_y and is given by

$$\boxed{E_{n_x, n_y} = \frac{h^2}{8m}\left(\frac{n_x^2}{a^2} + \frac{n_y^2}{b^2}\right)} \quad \dots (12)$$

The lowest energy state is $n_x = n_y = 1$.

For the ground state of the particle in a 2D box, there is only one wave function with the specific energy given by equ (12). Hence, the ground state and the eigen value are said to be **non-degenerate.**

However, in the 2-D box potential, the energy of a state depends upon the sum of the squares of the two quantum numbers. The particle having a particular value of energy in the excited state may has several different stationary states or wave functions. If so, these states and energy eigen values are said to be **degenerate.**

4.10 PARTICLE IN A 3-DIMENSIONAL BOX

Consider a particle which can move freely with in a box of dimensions $a \times b \times c$ with impenetrable walls as shown in the fig. 4.7.

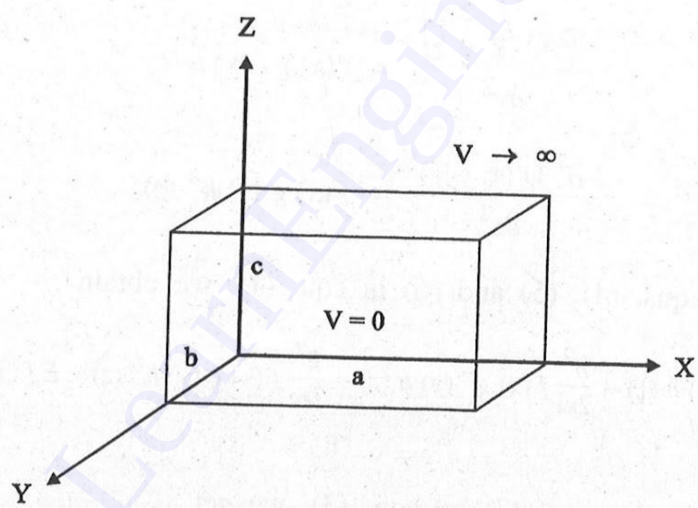


Fig. 4.7 Particle in a 3-dimensional box

In terms of potential energy, we can write

$$V(x, y, z) = \begin{cases} 0 & \text{inside the box} \begin{bmatrix} 0 < x < a \\ 0 < y < b \\ 0 < z < c \end{bmatrix} \\ \infty & \text{outside the box and on the walls} \end{cases} \quad \dots (1)$$

We know that the particle outside the box must be zero and the wavefunction must vanish everywhere outside the box.

Inside the box, where the potential energy is everywhere zero, the Schrodinger equation reads

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\frac{\partial^2}{\partial z^2}\right)\psi(x, y, z) = E\,\psi(x, y, z) \quad \text{... (2)}$$

To solve this equation we assume that the wavefunction can be written as

$$\psi(x, y, z) = f(x)\, g(y)\, h(z) \quad \text{... (3)}$$

Let us differentiate this function twice with respect to x, y, z to obtain,

$$\frac{\partial^2 \psi(x, y, z)}{\partial x^2} = f''(x)\, g(y)\, h(z) \quad \text{... (4)}$$

$$\frac{\partial^2 \psi(x, y, z)}{\partial y^2} = f(x)\, g''(y)\, h(z) \quad \text{... (5)}$$

$$\frac{\partial^2 \psi(x, y, z)}{\partial z^2} = f(x)\, g(y)\, h''(z) \quad \text{... (6)}$$

Substituting equs. (4), (5) and (6) in equ. (1), we obtain

$$-\frac{\hbar^2}{2m}f''(x)\, g(y)\, h(z)-\frac{\hbar^2}{2m}f(x)\, g''(y)\, h(z)-\frac{\hbar^2}{2m}f(x)\, g(y)\, h''(z)-E f(x)\, g(y)\, h(z)=0$$

... (7)

Dividing the entire equ. (7) by equ. (3), we get

$$-\frac{\hbar^2 f''(x)}{2m\, f(x)} = \frac{\hbar^2 g''(x)}{2m\, h(z)}-\frac{\hbar^2 h''(z)}{2m\, g(y)}-E = 0$$

$$\text{or} \quad -\frac{\hbar^2 f(x)}{2m\, f(x)} = \frac{\hbar^2 g''(y)}{2m\, g(y)}+\frac{\hbar^2 h''(z)}{2m\, h(z)}+E \quad \text{... (8)}$$

$$\text{or} \quad E_x = \frac{\hbar^2 g''(y)}{2m\, g(y)}+\frac{\hbar^2 h''(z)}{2m\, h(z)}+E \quad \text{... (9)}$$

where,

$$E_x = -\frac{\hbar^2 f''(x)}{2m f(x)} \quad \text{... (10)}$$

Similarly, we have

$$E_y = -\frac{\hbar^2 g''(y)}{2m\, g(y)} \quad \text{... (11)}$$

$$E_z = -\frac{\hbar^2 h''(z)}{2m\, h(z)} \quad \text{... (12)}$$

The total energy is then given by

$$E = E_z + E_y + E_z \quad \text{... (13)}$$

Using separation of variable to transform the 3D partial differential equation into 1 D differential equations, we have

$$\frac{d^2 f(x)}{dx^2} + \frac{2m}{\hbar^2} E_x f(x) = 0 \quad \text{... (14)}$$

$$\frac{d^2 g(y)}{dy^2} + \frac{2m}{\hbar^2} E_y\, g(y) = 0 \quad \text{... (15)}$$

$$\frac{d^2 h(z)}{dz^2} + \frac{2m}{\hbar^2} E_z\, h(z) = 0 \quad \text{... (16)}$$

We know the solution to these equation of the 1 D box, and the wavefunction in the x direction is

$$f(x) = \sqrt{\frac{2}{a}}\left[\sin\left(\frac{n_x \pi x}{a}\right)\right] \quad \text{... (17)}$$

and the energy

$$E_x = \frac{n_x^2 h^2}{8ma^2}, \quad \text{where, } n_x = 1, 2, 3, \ldots \quad \text{... (18)}$$

Similarly, the solution in the y direction is

$$g(y) = \sqrt{\frac{2}{b}}\left[\sin\left(\frac{n_y \pi y}{y}\right)\right] \qquad \text{... (19)}$$

$$E_y = \frac{n_y^2 h^2}{8mb^2}, \quad \text{where, } n_y = 1, 2, 3, \ldots \qquad \text{... (20)}$$

and z direction

$$h(z) = \sqrt{\frac{2}{z}}\left[\sin\left(\frac{n_z \pi z}{z}\right)\right] \qquad \text{... (21)}$$

$$E_z = \frac{n_z^2 h^2}{8mz^2}, \quad \text{where, } n_z = 1, 2, 3, \ldots \qquad \text{... (22)}$$

Thus the energy of the particle in a 3D box is

$$\boxed{E = \frac{h^2}{8m}\left(\frac{n_x^2}{a^2} + \frac{n_y^2}{b^2} + \frac{n_z^2}{z^2}\right)} \qquad \text{... (23)}$$

and the wavefunction

$$\psi(x, y, z) = \sqrt{\frac{8}{abc}}\sin\left(\frac{n_z \pi x}{a}\right)\sin\left(\frac{n_y \pi y}{b}\right)\sin\left(\frac{n_z \pi z}{c}\right)$$

or
$$\boxed{\psi(x, y, z) = \sqrt{\frac{8}{V}}\sin\left(\frac{n_z \pi x}{a}\right)\sin\left(\frac{n_y \pi y}{b}\right)\sin\left(\frac{n_z \pi z}{c}\right)} \qquad \text{... (24)}$$

where $V = abc$ is the volume of the box.

The normalized wavefunction is

$$\int_{-\infty}^{\infty} |\psi(x, y, z)|^2 \, d\tau = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\psi(x, y, z)|^2 \, dx\, dy\, dz \qquad \text{... (25)}$$

$$= \int_0^a |f(x)|^2 \, dx \int_0^b |g(y)|^2 \, dy \int_0^c |h(z)|^2 \, dz = 1 \qquad \text{... (26)}$$

Now lets consider a cube, i.e. $a = b = c$. The energy of a particle in cubi box becomes.

$$\boxed{E = \frac{h^2}{8m}\left(\frac{n_x^2}{a^2}+\frac{n_y^2}{a^2}+\frac{n_z^2}{a^2}\right) = \frac{h^2}{8ma^2}(n_x^2+n_y^2+n_z^2)} \quad \text{... (27}$$

Conclusion

Quantum systems with symmetry generally exhibit *degeneracy* in their energy levels.

4.11 DEGENERATE ENERGY STATES

Energy states having distinct eigen functions which share the same eigen value are called **degenerate energy states.**

Explanation

Consider three different combinations of three integers n_x, n_y and n_z such as 2 1 1, 1 2 1 and 1 1 2. Now we can write the energy values for those three different combinations as E_{211}, E_{121} and E_{112} respectively.

Now from the energy equation, we have

$$E_{211} = \frac{h^2}{8mL^2}(2^2+1^2+1^2) = \frac{6h^2}{8mL^2}$$

$$E_{121} = \frac{h^2}{8mL^2}(1^2+2^2+1^2) = \frac{6h^2}{8mL^2}$$

$$E_{112} = \frac{h^2}{8mL^2}(1^2+1^2+2^2) = \frac{6h^2}{8mL^2}$$

Here E_{211}, E_{121} and E_{112} represents three independent stationary states having quantum numbers (2,1,1), (1,2,1) and (1,1,2) for n_x, n_y, n_z.

Note that all the states have the ***same energy value*** $\frac{\mathbf{6h^2}}{\mathbf{8mL^2}}$. But the states will have different wave functions ψ_{211}, ψ_{121} and ψ_{112}.

Such energy states having different wave functions but having the same energy value or energy level are said to be ***degenerate energy states***.

Non-degenerate states

If there is only one wave function for a particular energy state, then the state is said to be a ***non-degenerate energy state***.

NOTE

With the application of a magnetic field or electric field to a system, the degeneracy breaks down.

The degree of degeneracy is the number of states with the same energy. An energy state which belongs to 'n' different eigen functions is termed as n-fold degenerate.

4.12 CORRESPONDENCE PRINCIPLE

Quantum mechanics is highly successful in describing microscopic entities like atoms and elementary particles. But, *macroscopic systems,* like tennis ball, automobile etc, are accurately described by classical mechanics.

The quantum world and the laws that govern it are universal, but as we move from atoms (microscopic world) to tennis balls and automobiles (macroscopic world), the fact of quantization becomes less noticeable and finally totally undetectable. The graininess effectively disappears and the laws of classical mechanics that govern the motions of large objects emerge as special limiting forms of the more general laws of quantum mechanics.

The agreement of a new scientific theory to an earlier scientific theory under appropriate circumstances is known as **correspondence principle.**

New theory and old theory must correspond; that is, they must overlap and agree in the region where the results of the old theory have been fully verified.

When the techniques of quantum mechanics are applied to macroscopic systems rather than atomic systems, there must be some limit in which quantum mechanics reduces to classical mechanics.

The conditions under which quantum and classical physics agree are referred to as the **correspondence limit**, or the **classical limit**.

Example:

Einstein's special relativity satisfies the correspondence principle, because it reduces to classical mechanics in the limit of velocities small compared to the speed of light.

Significance:

The correspondence principle indicates today's scientific truth can be outdated tomorrow by another truth that is more accurate and more broadly applicable. However, it defines that today's truth will not lose any value when that happens. This *requirement of* ***backwards-compatibility*** is known as the ***correspondence principle.***

4.13 SOLVED PROBLEMS

EXAMPLE 1

An electron is accelerated by a potential difference of 150 V. What is the wavelength of that electron wave?

Given data

Accelerating voltage applied to the electron $V = 150$ V

Solution:

We know that the de-Broglie wavelength

$$\lambda = \frac{12.25 \times 10^{-10}}{\sqrt{V}} \text{ m}$$

Substituting the given values, we have

$$\lambda = \frac{12.25 \times 10^{-10}}{\sqrt{150}} \text{ m}$$

$$\lambda = \frac{22.25 \times 10^{-10}}{12.24} \text{ m}$$

$$\lambda = 1.001 \times 10^{-10} \text{ m}$$

$$\boxed{\lambda = 1.001 \text{ Å}}$$

APPLIED QUANTUM MECHANICS

5.1 THE HARMONIC OSCILLATOR

Harmonic motion occurs when a system vibrates about an equilibrium position.

A pendulum, a particle attached to a spring, or many vibrations in atoms and molecules can be described as a harmonic oscillator.

A simple realization of the harmonic oscillator is a mass, attached to the end of a simple spring as shown in fig. 5.1.

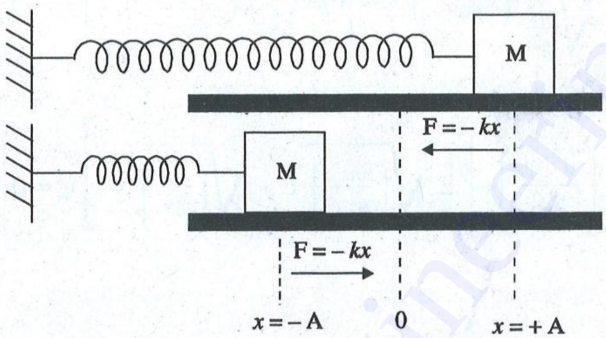


Fig. 5.1

Let us consider a paritcle of mass, m executes simple harmonic motion along x-axis.

Let x ***is the position of the mass as a function of time***, t. The constant k is known as the spring constant.

The study of quantum mechanical harmonic motion begins with the specification of Schrodinger equation.

Now, the potential energy of the particle (i) given by

$$\int_0^x kxdx = \frac{1}{2}kx^2$$

The Schrodinger equation for harmonic oscillator is

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{2m}{\hbar^2}\left(E - \frac{1}{2}kx^2\right)\psi = 0 \qquad \text{... (1)}$$

Multiply the above equation by $\frac{1}{\sqrt{k}}$ and rearranging, we have

$$\frac{\partial^2 \psi}{\partial x^2} + 2\sqrt{\left(\frac{m}{\hbar^2}\right)}\sqrt{\frac{mk}{\hbar^2}}\left[\frac{E}{\sqrt{k}} - \frac{1}{2}\sqrt{k}\,x^2\right]\psi = 0$$

(or)
$$\frac{1}{\sqrt{\frac{mk}{\hbar^2}}}\left(\frac{\partial^2 \psi}{\partial x^2}\right) + 2\sqrt{\frac{m}{\hbar^2}}\left[\frac{E}{\sqrt{k}} - \frac{1}{2}\sqrt{k}\,x^2\right]\psi = 0$$

$$\frac{1}{\sqrt{\frac{mk}{\hbar^2}}}\left(\frac{\partial^2 \psi}{\partial x^2}\right) + \left[2\sqrt{\left(\frac{m}{\hbar^2 k}\right)}E - \sqrt{\left(\frac{mk}{\hbar^2}\right)}\,x^2\right]\psi = 0 \qquad \text{... (2)}$$

Let,
$$\sqrt{\left(\frac{mk}{\hbar^2}\right)} = \alpha^2 \qquad \text{... (3)}$$

$$2\beta\sqrt{\left(\frac{m}{\hbar^2 k}\right)} = \lambda \qquad \text{... (4)}$$

Applying equ (3) and (4) in equ (2), we get

$$\frac{1}{\alpha^2}\left(\frac{\partial^2 \psi}{\partial x^2}\right) + (\lambda - \alpha^2 x^2)\,\psi = 0 \qquad \text{... (5)}$$

Let us introduce a new variable q related to x, such that

$$q = \alpha x$$

$$\therefore \frac{\partial q}{\partial x} = \alpha \qquad \text{... (6)}$$

But,
$$\frac{\partial \psi}{\partial x} = \frac{\partial \psi}{\partial q} \cdot \frac{\partial q}{\partial x}$$

$$\frac{\partial \psi}{\partial x} = \alpha \frac{\partial \psi}{\partial q} \qquad \text{... (7)}$$

And,
$$\frac{\partial^2 \psi}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial x}\right)$$

$$= \frac{\partial}{\partial q}\left(\frac{\partial \psi}{\partial x}\right)\frac{\partial q}{\partial x}$$

(or)
$$\frac{\partial^2 \psi}{\partial x^2} = \frac{\partial}{\partial q}\left(\alpha \frac{\partial \psi}{\partial q}\right)\alpha$$

$$= \alpha^2 \frac{\partial^2 \psi}{\partial q^2} \qquad \text{... (8)}$$

Using equations (7) and (8) in equation (5), we get

$$\frac{\partial^2 \psi}{\partial q^2} + (\lambda - q^2)\,\psi = 0 \qquad \text{... (9)}$$

To solve above differential equation, a solution of the form given below can be used

$$\psi = f(q)\, e^{-q^2/2} \qquad \text{... (10)}$$

where $f(q)$ is a function of q.

Now, equ (9) can take the form,

$$\frac{\partial^2 f}{\partial q^2} - 2q\frac{\partial f}{\partial q} + (\lambda - 1)\, f = 0 \qquad \text{... (11)}$$

Writing $(\lambda - 1) = 2n$; equation (11) becomes

$$\boxed{\frac{\partial^2 f}{\partial q^2} - 2q\frac{\partial f}{\partial q} + 2nf = 0} \quad \dots (12)$$

This is a standard mathematical equation known as ***Hermite equation***. The solution of equation is known as the *Hermite's polynomial given by*

$$H_n(q) = f(q) = (-1)^n (e^{q^2}) \frac{\partial^n}{\partial q^n} [e^{(-q^2)}] \quad \dots (13)$$

Therefore, the eigen functions of harmonic oscillator are as follows:

$$\boxed{\psi_n(q) = NH_n(q)\, e^{-q^2/2}} \quad \dots (14)$$

where N is the normalization constant.

The eigen values (permitted values of total energy) are given by

$$(\lambda - 1) = 2n \quad \dots (15)$$

Now, from equation (15), we have

$$\lambda = 2n + 1$$

or
$$2E\sqrt{\frac{m}{\hbar^2 k}} = 2n \pm 1$$

where, angular frequency, $\omega = \sqrt{\frac{k}{m}}$ (or) $k = \omega^2 m$

Since harmonic motion has a characteristic angular frequency, it makes sense to measure energy in terms of ω.

Hence, the allowed energies are

$$\boxed{E_n = \left(n + \frac{1}{2}\right)\hbar\omega} \quad \text{for } n = 0, 1, 2, 3, \ldots$$

where the ground state is usually designated with the quantum number, $n = 0$

Therefore, we have

$$E_0 = \left(\frac{1}{2}\right)\hbar\omega$$

$$E_1 = \left(\frac{3}{2}\right)\hbar\omega$$

$$E_2 = \left(\frac{5}{2}\right)\hbar\omega$$

$$E_3 = \left(\frac{7}{2}\right)\hbar\omega \qquad \text{and so on.}$$

It is clear that the difference between successive energy eigen values has a constant value given by,

$$\Delta E = E_{n+1} - E_n = \hbar\omega$$

The potential energy function and first few energy levels for the harmonic oscillator are shown in fig. 5.2

As the quantum number n increases, the energy of the oscillator and therefore the amplitude of oscillation increases.

A packet of energy $\hbar\omega$ is needed to make the quantum harmonic oscillator to move from a lower energy state to a higher energy state.

Here, the ground-state energy, $E_0 = \left(\frac{1}{2}\right)\hbar\omega$ is greater than the classical value of zero, which is a consequence of the uncertainty principle. This means that the oscillator is always oscillating.

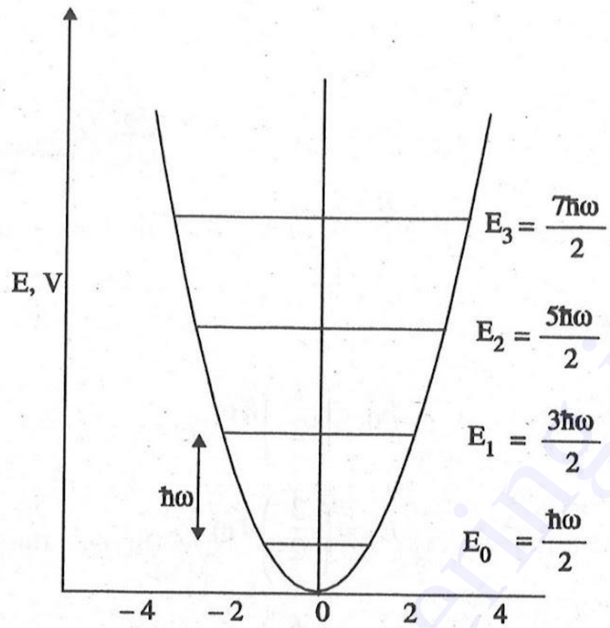


Fig. 5.2 Energy states and potential well of a quantum harmonic oscillator

The salient features of harmonic oscillator are,

- The energies are quantized and the energy levels are evenly spaced.
- There is a non-zero ground state energy.
- $E = 0$ is not allowed by Heisenberg uncertainty principle.

Significance

(i) It serves as a prototype in the mathematical treatment of phenomena like elasticity, acoustics, AC circuits, molecular and crystal vibrations, electromagnetic fields and optical properties of matter.

(ii) The physics of quantized electromagnetic oscillations (photons) and quantized mechanical oscillations (phonons) is intimately related to the quantum harmonic oscillator.

5.2 BARRIER PENETRATION AND QUANTUM TUNNELLING

According to quantum mechanics, a particle such as an electron can penetrate a barrier into a region forbidden by classical mechanics.

This phenomenon is known as **barrier penetration** and can happen only when the particle exhibits wave nature.

Tunnelling is a quantum phenomenon where particles with less energy than that of a potential barrier can still cross the energy barrier, by penetrating through it.

Explanation

Let us consider a particle of mass m traveling to the right along the x axis. The particle encounters a narrow potential barrier whose height V_0 is greater than E and whose thickness is L.

- Classically all the particles.

(i) will be reflected back (at $x = 0$) if $E < V_0$, and

(ii) will be transmitted to $x > L$ if $E > V_0$

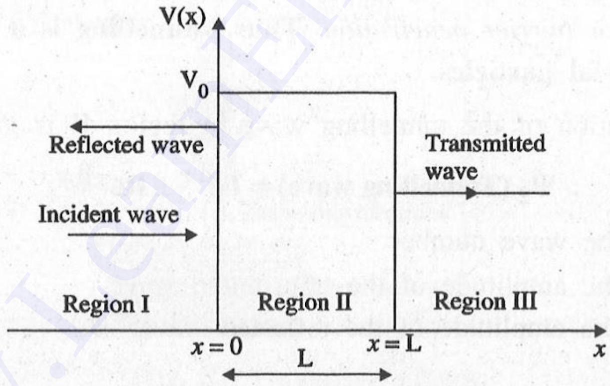


Fig. 5.3 Barrier penetration

But, quantum mechanics predicts a non-zero probability for finding the particle on the other side of the barrier even when $E < V_0$.

This can happen as the approaching particle has a sinusoidal wave function as shown in fig. 5.4.

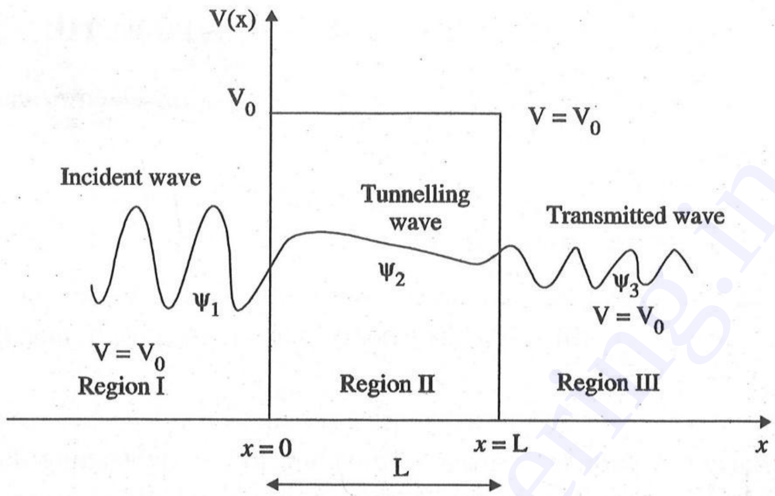


Fig. 5.4 Tunnelling

Within the barrier the wave is decaying and before it dies away to zero, there is again a sinusoidal wave function, at $x = L$. But it is a sine wave of greatly reduced amplitude.

Since, $[\psi]^2$ is non-zero beyond the barrier, it is evident that there is a non-zero probability that the particle penetrates the barrier. This process is called *tunnelling* through the barrier or *barrier penetration*. **Thus, tunnelling is a result of the wave properties of material particles.**

The wave function of the tunnelling wave at region II is given by

$$\Psi_2 \text{ (Tunnelling wave)} = Te^{\beta x} + Re^{-\beta x}$$

where, β is the wave number
T is the amplitude of the transmitted wave
R is the amplitude of the reflected wave.

Tunnelling probability

The tunneling probability can be described with a *transmission coefficient, T,* and a *reflection coefficient, R.*

Since an incident particle must either reflect or tunnel through, we have, $T + R = 1$.

Transmission coefficient

The probability that the particle gets through the barrier is called **transmission coefficient** (*T*).

$$T = \frac{\text{Probability density of transmitted wave}}{\text{Probability density of incident wave}}$$

The **transmission coefficient is given by,**

$$\boxed{T = e^{-2\beta L}}$$

where the wave number β is given by

$$\beta = \sqrt{\frac{2m(V_0 - E)}{\hbar}}$$

The transmission probability increases with decrease in height and width of the barrier.

5.3 TUNNELLING MICROSCOPE

An electron microscope that works by quantum tunnelling phenomenon and creates atomic scale imaging of surfaces is known as **tunnelling microscope.**

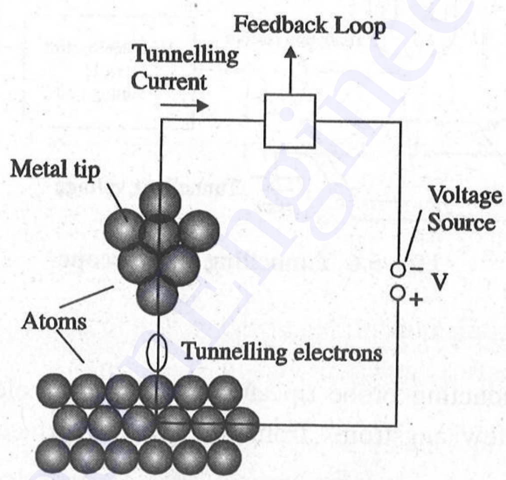


Fig. 5.5 Tunnelling process

For the electrons in the sample and in the metal tip, it is forbidden to stop in the gap between sample and tip. However, this gap is so small that the electrons are able to tunnel and flow through the gap.

Principle

When a voltage is applied between a conducting tip and a surface close to it, electrons can tunnel through the vacuum between the atoms of the tip and the surface.

The tunnelling current that results depends upon the distance between probe tip and sample surface.

Construction

The basic components are,

- Piezoelectric tube
- Tunnelling current amplifier
- Distance control unit and scanning unit
- Data processing and display

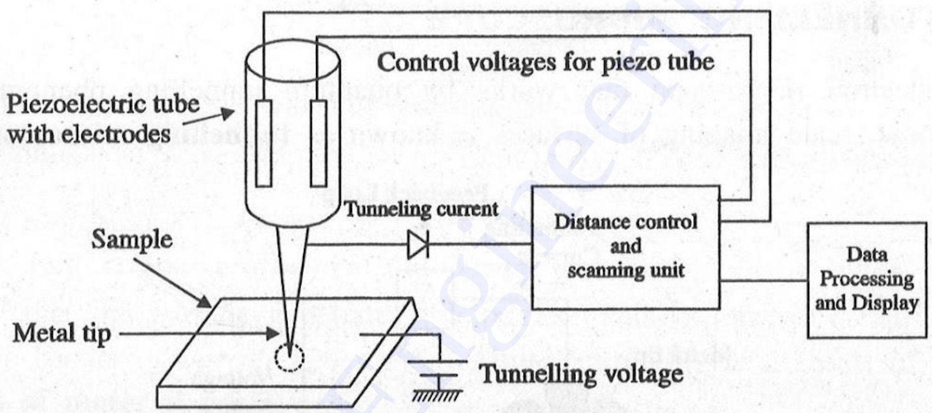


Fig. 5.6 Tunnelling microscope

Working

- The sharp conducting probe tip attached to a piezoelectric tube is positioned at a distance few angstroms from the sample surface.
- A small voltage applied between the probe tip and the surface causes electrons to tunnel from the tip to the sample surface.
- As the probe is scanned over the surface, the voltage applied to the piezotube is altered to maintain a constant tip-surface distance.
- Changes in this voltage registers variations in the tunnelling current.
- The changes in the tunnelling current are recorded and then used to generate a map of the sample surface on the display unit.

Merits

- The high resolution of STMs enable researchers to examine surfaces at an atomic level.
- Gives three dimensional profile of a surface.
- Versatile and can be used in ultrahigh vacuum, air, water and other liquids and gases.
- Operate in temperatures as low as zero Kelvin up to a few hundred degrees Celsius.

Demerits

- Require very stable, clean surfaces and conducting surface.
- Difficult to use effectively.
- The electronics required are extremely sophisticated as well as very expensive.

Applications

- The tunnelling microscope is widely used in both industrial and fundamental research to obtain atomic-scale images of metal surfaces.
- Used as diagnostic tool in the fields like solid state physics, electrochemistry, biology, organic chemistry, nano machining etc.,
- Defects and physical structure of synthetic chemical compounds can be studied.
- To study charge transport mechanisms in molecules.
- Used in research surrounding semiconductors and microelectronics.

5.4 RESONANT DIODE

A diode with a resonant tunnelling structure that allows electrons to tunnel through various resonant states at certain energy levels is known as a **resonant diode**.

Principle

Tunnelling of electrons through a finite-height potential well that occurs only when electron energies match an energy level in the well.

Construction

The structure of resonant diode is shown below.

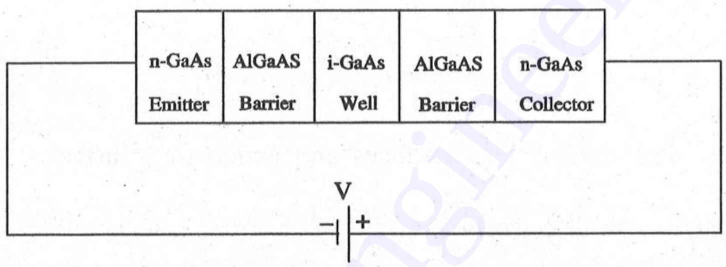


Fig. 5.7 Structure of resonant diode

- It consists of an intrinsic GaAs quantum well region sandwiched between two thin barrier regions made of AlGaAs.
- The regions at the extreme ends on both sides are made of heavily doped n-GaAs and they serve the purpose of emitter and collector.
- By applying a bias voltage, the tunnelling current across the diode can be controlled.

Working

According to quantum mechanics, electrons can tunnel from outside into the well through barrier under suitable conditions.

The energy band diagram of the resonant diode in shown in fig. 5.8.

Without any voltage bias, the electron energy levels in the well is higher than the incident electron energy (E).

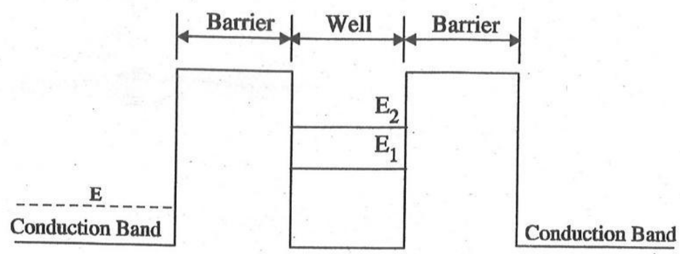


Fig. 5.8 Energy Band Diagram of resonantdiode (with no voltage bias)

So, no electron in the conduction band can tunnel to the well, and there is no current.

On increasing the bias voltage, the incident electron energy level, E on the left becomes higher and matches an energy level in the potential well. Now some electrons can tunnel into the well and the current from left to right increases due to tunnelling.

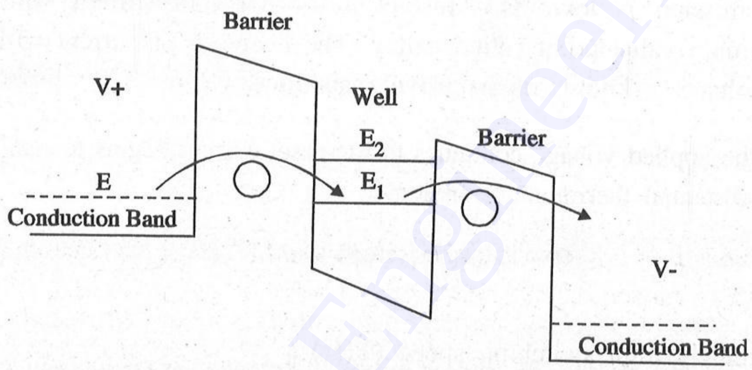


Fig. 5.9 Energy Band Diagram of resonant diode (with applied voltage bias)

Thus, when the electric field increases to the point where the energy level of the electrons in the emitter coincides with the energy level of the quasi-bound state of the well, the current reaches a maximum. This type of tunnelling is known as **resonant tunnelling**.

I-V characteristics of resonant diode:

- The current–voltage (V) characteristic for resonant diode is shown in fig. 5.10.
- As voltage increases, E also increases and hence the tunnelling current increases and reaches a peak point.

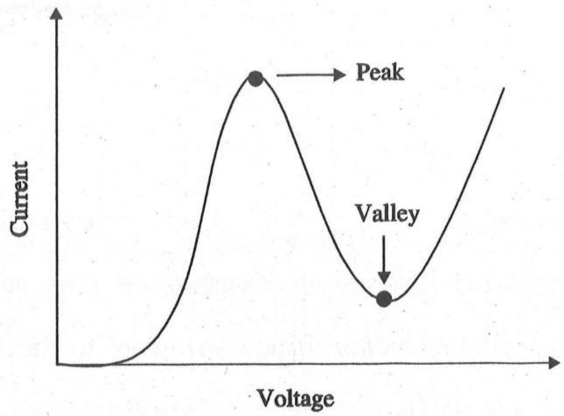


Fig. 5.10 I-V characteristics of resonant diode

- Further increase in voltage alters the energy value E and hence the transmission is low. This results in decrease in current which reaches a minimum value point called valley. The decrease of current with an increase in voltage is known as **negative resistance.**
- As the applied voltage continues to increase, current begins to rise again because of substantial thermionic emission.

Advantages

- Extremely high switching speed (THz)
- Low power consumption
- Room temperature operation

Applications of resonant tunnelling

Resonant tunneling has numerous applications in semiconductor devices such as electronic circuit components or integrated circuits that are designed at nanoscales.

- High speed electronic devices
- Optical communications
- Terahertz generation

5.5 FINITE POTENTIAL WELL

A box with finite potential walls in which a particle is confined to it is known as a **finite potential well**.

Unlike the infinite potential well, there is a probability associated with the particle being found outside the box.

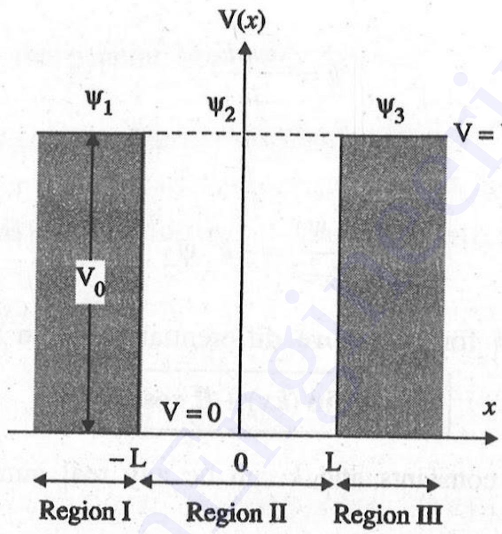


Fig. 5.11 A finite potential well

Let us consider a particle with total energy $E < V_0$ confined within a well bouncing back and forth between the turning points at $x = -L$ and $x = L$.

The potential well is divided into three regions (1, 2, 3) with associated wave functions as follows.

$$\psi = \begin{cases} \psi_1, \text{ if } x < -L & \text{(the region outside the box)} \\ \psi_2, \text{ if } -L < x < L & \text{(the region inside the box)} \\ \psi_3, \text{ if } x > L & \text{(the region outside the box)} \end{cases}$$

The time independent Schroedinger's equation can be written as

$$\frac{-\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V\psi = E\psi \qquad \text{... (1)}$$

Wavefunction at Region-II (Inside the well)

For this region, inside the box $V(x) = 0$ and equ.(1) reduces to

$$-\frac{\hbar^2}{2m}\frac{d^2\psi_2}{dx^2} = E\,\psi_2 \quad \text{... (2)}$$

Letting

$$k = \frac{\sqrt{2mE}}{\hbar},$$

eqn (2) becomes

$$\frac{d^2\psi_2}{dx^2} = -k^2\,\psi_2$$

The general solution for the above differential equation is

$$\boxed{\psi_2 = A\sin(kx) + B\cos(kx)} \quad \text{... (3)}$$

Here, A and B are constants and k can be any real number.

Hence,

$$\boxed{E = \frac{k^2\hbar^2}{2m}}$$

Wave function at Region I and III (Outside the well)

In the regions, $x < -L$ and $x > L$, we have, $V = V_0$, and now the Schrodinger equation is

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V_0\,\psi = E\,\psi$$

We rewrite this as

$$\frac{d^2\psi}{dx^2} - \left[\frac{2m(V_0 - E)}{\hbar^2}\right]\psi = 0 \quad \text{... (4)}$$

Let us assume that E is less than V_0, so the particle is "trappped" in the well. There might be only one such **bound state** or more.

We define a constant G by

$$G^2 = \frac{2m\,(V_0 - E)}{\hbar^2} \quad \text{... (5)}$$

and rewrite the Schrodinger equation as

$$\frac{d^2\psi}{dx^2} - G^2\,\psi = 0 \quad \text{... (6)}$$

This equation has the general solution

$$\psi_{1,\,3} = Ce^{Gx} + De^{-\,Gx} \quad \text{... (7)}$$

In the region, $x < -L$, x is always negative, so D must be zero. Similarly in region, $x > L$, where x is always positive, C must be zero.

Hence,

$$\psi_1 = Ce^{Gx} \quad \text{... (8)}$$

$$\psi_3 = De^{-\,Gx} \quad \text{... (9)}$$

Conclusion

(i) In a finite potential well, the wave function extends into these classically forbidden regions (Region I and III), where the total energy is less than the potential energy. This situation is possible and is consistent with the uncertainty principle.

(ii) However, in both these regions, the wave function decreases exponentially with distance from the well.

(iii) If the particle manages to acquire energy, $E > V_0$, then, it will escape from the well.

NOTE

An electron confined within a semiconductor by an electric force has a potential energy that can be modelled as a finite potential well. Similarly, a proton confined within the nucleus by the nuclear force has a potential energy that can be modelled as a finite potential well. Hence any situation in which a particle is confined can be modelled as a finite potential well.

Characteristics of finite potential well

- The number of bound state energies is finite.
- The number of bound states increases with the width and depth of the well.
- Tunnelling into the barrier (wall) is possible.
- Higher energy states are less tightly bound than lower ones.
- A particle provided with enough energy can escape the well (unbound state).

5.6 BLOCH THEOREM

One of the characteristic features of many solids is the regular arrangement of their atoms forming a crystal. The potential energy of electrons in such a crystal is the result of the positively charged ions producing a columbic attraction.

In a crystal, electrons move in a potential $V(x)$ which is produced by regularly-spaced ion cores as shown in fig. 5.12 (a).

The potential of the electron at the site of positive ions is zero and is maximum in between the sites of two positive ions as shown in fig. 5.12 (b).

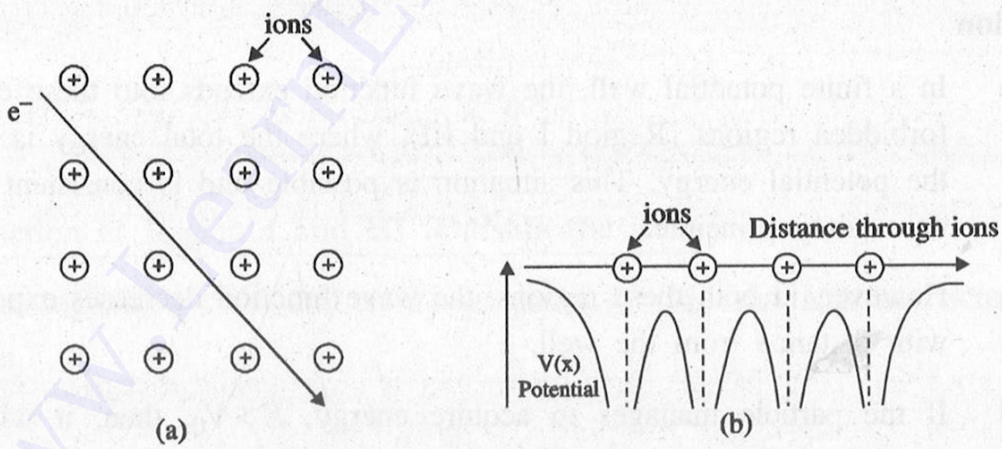


Fig. 5.12 Electrons in a periodic potential

Let a one-dimensional lattice (ie), only an array of ionic cores along x-axis is considered.

Since the potential energy of any particle bound in a field of attraction is negative and since the conduction electron is bound to the solid, its potential energy V is negative.

Further, as it approaches the site of an ionic core, $V \rightarrow -\infty$.

Since this occurs symmetrically on either side of the core, it is referred to as potential well.

The width of the potential well 'b' is not uniform, but has a tapering shape.

If V_0 is the potential at a given depth of the well, then the variation is such that

$$b \rightarrow 0, \text{ as } V_0 \rightarrow \infty, \text{ and hence,}$$

$$bV_0 = \text{constant.}$$

Now, since the lattice is a repetitive structure of the ion arrangement in a crystal, the type of variation of also repeats itself.

If 'α' is the inter-ionic distance, then, as we move in x-direction, the value of V will be same at all points which are separated by a distance equal to 'a'.

$$\text{ie, } V(x) = V(x + a)$$

where, x is distance of the electron from the core.

Such a potential is said to be a periodic potential.

The equation which gives the eigen function for an electron moving in a perfectly periodic potential is known as **Bloch theorem.**

The **Bloch theorem** states that, for a particle moving in a periodic potential, the eigen functions for a conduction electron are of the form,

$$\psi(x) = e^{ikx}\, U(x)$$

where, e^{ikx} represents a plane wave, and $U(x) = U(x + a)$

The function $U(x)$ has the same periodicity as the potential energy of the electron, and is called the *modulating function.*

Significance

- A large number of materials are well described by regular atomic spacing and a periodic potential for a crystal lattice which is like a string of finite wells.
- The presence of periodic potential in a crystal leads to energy bands, which are essentially energy intervals between which energy levels are nearly continuous.

5.7 THE KRONIG PENNEY MODEL

The study of essential behaviour of electrons by approximating the potentials inside a crystal to the shape of rectangular steps is called **Kronig-Penney model of potentials.**

The Kronig-Penney model describes the one dimensional representation of electron potential in a periodic lattice. (Fig. 5.13).

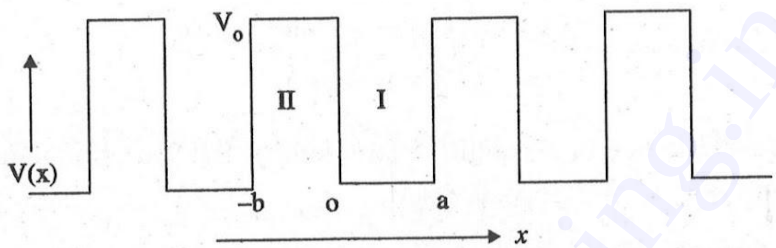


Fig. 5.13 Periodic Potential - Kronig-Penney Model

This model consist of an infinite row of rectangular potential wells separated by barriers of width 'b". Each well has a width 'a" and a depth V_0.

It is assumed that when an electron is near the positive ion site, potential energy is taken as zero. Whereas, outside the well, that is, in between two positive ions, potential energy is assumed to be V_0.

Hence, we have

$$V(x) = V_0 \quad \text{for} \quad -b < x < 0$$

and

$$V(x) = 0 \quad \text{for} \quad 0 < x < a$$

The period of the potential is, $(a+b)$.

The possible states that the electron can occupy are determined by the Schrödinger equation,

$$\frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2}(E - V)\psi = 0$$

The Schrodinger equation for the two regions can be written as

Region I $$\frac{\partial^2 \psi}{\partial x^2} + \frac{2m}{\hbar^2}(E)\,\psi = 0 \qquad \text{for } 0 < x < a \qquad [\text{since, } V = 0] \qquad (1)$$

Region II $$\frac{\partial^2 \psi}{\partial x^2} + \frac{2m}{\hbar^2}(E - V_0)\,\psi = 0 \qquad \text{for } -b < x < 0 \qquad [\text{since, } V = 0] \qquad (2)$$

We rewrite the above equation as

$$\frac{d^2 \psi}{dx^2} + \alpha^2 \psi = 0 \qquad \text{for } 0 < x < a \qquad \ldots (3)$$

and $$\frac{d^2 \psi}{dx^2} - \beta^2 \psi = 0 \qquad \text{for } -b < x < 0 \qquad \ldots (4)$$

where,

$$\alpha^2 = \frac{2mE}{\hbar^2}$$

$$\beta^2 = \frac{2m}{\hbar^2}(V_0 - E)$$

Bloch has given the solution for Schrodinger equation as

$$\psi(x) = U(x) \cos kx \qquad \ldots (5)$$

where, $U(x) = U(x + a)$ ... (6)

Solving the above equation (3) and (4) by applying boundary conditions, we get

$$\frac{8\pi^2 mV_0}{2\hbar^2 a} b \cdot \sin(\alpha a) + \cos(\alpha a) = \cos(ka) \qquad \ldots (7)$$

$$\frac{P}{\alpha a} \sin(\alpha a) + \cos(\alpha a) = \cos(ka) \qquad \ldots (8)$$

where, $P = \dfrac{8\pi^2 mV_0 \alpha b}{2\hbar^2}$ = potential barrier strength

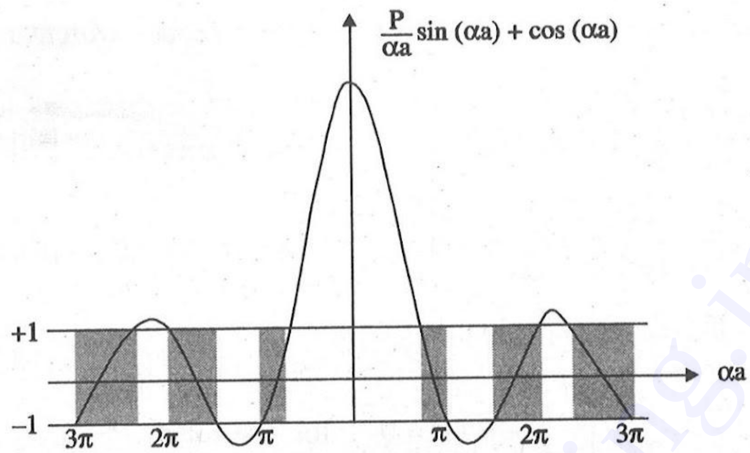


Fig. 5.14

By varying $\alpha\, a$, a wave mechanical nature could be plotted as shown in fig. 5.14.

The shaded portion of the wave shows the bands of allowed energy with the forbidden region as unshaded portion.

Thus, the energies of an electron moving under a periodic lattice potential lie only in certain allowed zones; other energies are forbidden.

Results from Kronig – Penney Model

1. The Kronig-Penney model demonstrates that a simple one-dimensional periodic potential yields energy bands as well as energy band gaps.
2. If potential barrier between wells is strong, energy bands are narrowed and spaced far apart. This corresponds to crystals in which electrons are tightly bond to ion cores, and wave functions do not overlap much with adjacent cores.
3. If potential barrier between wells is weak, energy bands are wide and spaced close together.
4. The energy spectrum of electrons consist of an infinite number of allowed energy bands separated by intervals in which, there are no allowed energy levels. These are known as forbidden regions.
5. When 'α' increases, the width of the allowed energy bands also increases and forbidden energy regions become narrow.