VECTOR INTEGRATION

The concept of a line integral is a natural generalization of the concept of a definite integral $\int_a^b f(x)\,dx$ where the integral $f(x)$ exists for all x in the interval $[a,b]$.

The concept of a surface integral is a natural generalization of the concept of a double integral, which occur in many applications for example, in connection with centre of gravity of a curved lamina, the potential due to charges distributed on the surfaces, etc: In a double integral, the integrand is a function $f(x,y)$ defined for all (x,y) in a closed bounded region R of the xy-plane.

The concept of a volume integral is a generalization of the double integral. In this case the integrand is a function $f(x,y,z)$ defined in a double closed region R in space ("closed" means that the boundary is part of the region and "bounded" means that the region can be enclosed in a sphere of sufficiently large radius.)

Definitions

Indefinite integral

Integration is the inverse operation of differentiation. Let $\bar{F}(t)$ be a differentiable vector fn of a scalar variable t and let $\frac{d}{dt}\bar{F}(t) = \bar{f}(t)$. Then $\int \bar{f}(t)\,dt = \bar{F}(t)$ and $\bar{F}(t)$ is called the primitive of $\bar{f}(t)$. The set of all primitives of $\bar{f}(t)$, that is $\int \bar{f}(t)\,dt = \bar{F}(t) + \bar{c}$ where $\bar{c}$ is any arbitrary constant vector is called indefinite integral of $\bar{f}(t)$. Hence the indefinite integral of $\bar{f}(t)$ is not unique.

PROPERTIES:

1) $\int K\bar{f}(t)\,dt = K\int \bar{f}(t)\,dt$, K is a real constant

2) $\int [\bar{f}(t) \pm \bar{g}(t)]\,dt = \int \bar{f}(t)\,dt \pm \int \bar{g}(t)\,dt$

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③ If $\bar{f}(t) = f_1(t)\bar{i} + f_2(t)\bar{j} + f_3(t)\bar{k}$, then

$$\int \bar{f}(t)\,dt = \bar{i}\int f_1(t)\,dt + \bar{j}\int f_2(t)\,dt + \bar{k}\int f_3(t)\,dt + \bar{c}$$

i.e the integration of a vector function reduces to the evaluation of three ordinary real integrals.

② <u>Definite integral</u>:

Let $\int \bar{f}(t)\,dt = \bar{F}(t) + \bar{c}$

Then $\int_a^b \bar{f}(t)\,dt = \bar{F}(b) - \bar{F}(a)$

This is called the definite integral of $\bar{f}(t)$ between the limits $t=a$ and $t=b$.

<u>NOTE</u>: If $\bar{f}(t) = f_1(t)\bar{i} + f_2(t)\bar{j} + f_3(t)\bar{k}$ then

$$\int_a^b \bar{f}(t)\,dt = \bar{i}\int_a^b f_1(t)\,dt + \bar{j}\int_a^b f_2(t)\,dt + \bar{k}\int_a^b f_3(t)\,dt$$

<u>Standard results</u>:

(i) $\int\left(\frac{d\bar{r}}{dt}\cdot\bar{s} + \bar{r}\cdot\frac{d\bar{s}}{dt}\right) = \bar{r}\cdot\bar{s} + \bar{c}$ Here c is a scalar constant, since $\bar{r}\cdot\bar{s}$ is a scalar.

(ii) $\int\left(2\bar{r}\cdot\frac{d\bar{r}}{dt}\right)dt = \bar{r}^2 + \bar{c}$

(iii) $\int 2\left(\frac{d\bar{r}}{dt}\cdot\frac{d^2\bar{r}}{dt^2}\right)dt = \left(\frac{d\bar{r}}{dt}\right)^2 + \bar{c}$

(iv) $\int\left(\frac{d\bar{r}}{dt}\times\bar{s} + \bar{r}\times\frac{d\bar{s}}{dt}\right) = \bar{r}\times\bar{s} + \bar{c}$

(v) If $\bar{a}$ is a constant vector, then

$\int\left(\bar{a}\times\frac{d\bar{r}}{dt}\right)dt = \bar{a}\times\bar{r} + \bar{c}$

(vi) $\int\left(\bar{r}\times\frac{d^2\bar{r}}{dt^2}\right)dt = \bar{r}\times\frac{d\bar{r}}{dt} + \bar{c}$

$\frac{d}{dt}\left(\bar{r}\times\frac{d\bar{r}}{dt}\right)$
$\bar{r}\times\frac{d^2\bar{r}}{dt^2} + \left(\frac{d\bar{r}}{dt}\right)^2$

(vii) If $\bar{c}$ is a scalar constant, then $\int \bar{c}\,\bar{r}\,dt = \bar{c}\int \bar{r}\,dt$

(viii) $\int(\bar{r} \pm \bar{s})\,dt = \int \bar{r}\,dt \pm \int \bar{s}\,dt$

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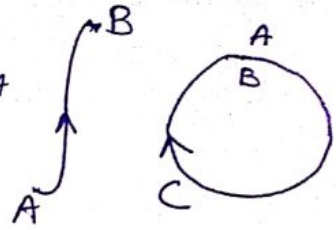
Closed Curve

Let C be a curve in space. Let A be the initial point and B be the terminal point of the curve C.

When the direction along C oriented from A to B is positive then the direction from B to A is called negative direction. If the two points A and B coincide the curve C is called the closed curve.



Smooth curve: A curve $\bar{r} = \bar{f}(t)$ is called a smooth curve if $\bar{f}(t)$ is continuously differentiable.

A curve C is said to be piece-wise smooth if it is the union of finite number of smooth curves.

Line integrals:

Let $\bar{r} = \bar{f}(t)$ define a smooth curve C joining the points A and B. Let ds be the differential of arc length at $P \in C$. Then $\frac{d\bar{r}}{ds} = \bar{t}$ is the unit vector along the tangent to the curve C at P. Let $\bar{F}(r)$ be a vector point fn defined and continuous along C. The component of $\bar{F}(r)$ along the tangent at P is $\bar{F}(r) \cdot \bar{t}$. The integral $\int \bar{F} \cdot \bar{t}\, ds$ taken along the curve C is called the line integral of $\bar{F}$ along C. This is written as

$$\int_A^B \bar{F} \cdot \bar{t}\, ds = \int_C \left(\bar{F} \cdot \frac{d\bar{r}}{ds}\right) ds = \boxed{\int_C \left(\bar{F} \cdot d\bar{r}\right)}$$

NOTE: other types of line integrals are $\int_C \bar{F} \times d\bar{r}$ and $\int_C \phi\, d\bar{r}$

In general any integral which is to be evaluated along a curve is called a line integral.

CIRCULATION:

If $\bar{v}$ represents the velocity of a fluid particle and C is a closed curve, then the integral $\oint_C \bar{v} \cdot d\bar{r}$ is called the circulation of $\bar{v}$ around the curve C.

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If $\int_C \bar{V}\cdot d\bar{r} = 0$, then the field $\bar{V}$ is called conservative i.e no work is done and the energy is conserved.

If the circulation of $\bar{V}$ round every closed curve in a region D vanishes, then $\bar{V}$ is said to be irrotational in D.

WORK DONE BY A FORCE:

If $\bar{F}$ represents the force vector acting on a particle moving along an arc AB, then the work done during a small displacement $\delta\bar{r}$ is $\bar{F}.\delta\bar{r}$. Hence the total work done by $\bar{F}$ during displacement from A to B is given by the line integral $\int_A^B \bar{F}\cdot d\bar{r} = \int_C \bar{F}\cdot d\bar{r}$

If the force $\bar{F}$ is conservative (i.e) $\bar{F} = \nabla\phi$, then the work done is independent of the path and vice-versa. In this case curl $\bar{F}$ = curl (grad ϕ) $= \bar{0}$ and ϕ is called scalar potential.

NOTE: ① $\bar{F}$ is conservative force field if $\nabla\times\bar{F} = \bar{0}$

② A conservative force field is also irrotational. ($\because \nabla\times\bar{F} = \bar{0}$)

PROBLEMS

① If $\bar{F} = 3xy\,\bar{i} - y^2\,\bar{j}$ evaluate $\int \bar{F}\cdot d\bar{r}$ where C is the curve $y = 2x^2$ in the xy plane from (0,0) to (1,2).

Sol: The eqn of the curve C is $y = 2x^2$

$dy = 4x\,dx$.

Given $\bar{F} = 3xy\,\bar{i} - y^2\,\bar{j}$

Since integration is performed in the XY-plane ($z=0$) therefore $\bar{r} = x\bar{i} + y\bar{j}$ and $\bar{F}\cdot d\bar{r} = F_1 dx + F_2 dy$ and x varies from 0 to 1.

$$\int_C \bar{F}\cdot d\bar{r} = \int_C F_1 dx + F_2 dy = \int_C 3xy\,dx - y^2 dy$$

$$= \int_{x=0}^{1} 3x(2x^2)\,dx - 4x^4 \times 4x\,dx$$

$$= \int_{x=0}^{1} (6x^3 - 16x^5)\,dx = \left[\frac{6x^4}{4} - \frac{16x^6}{6}\right]_{x=0}^{1}$$

$$= \frac{6}{4} - \frac{16}{6} = \frac{3}{2} - \frac{8}{3} = \frac{9-16}{6} = \frac{-7}{6}$$

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(2) Prove that force field given by $\bar{F} = 2xyz^3\,\bar{i} + x^2z^3\,\bar{j} + 3x^2yz^2\,\bar{k}$ is conservative. Find the work done by moving a particle from $(1,-1,2)$ to $(3,2,-1)$ in this force field. (9)

<u>Sol:</u> If integral $\int_{P_1}^{P_2} \bar{F}\cdot d\bar{r}$ is independent of the path joining P_1 and P_2 then $\bar{F}$ is called a conservative field.

Alternatively, $\bar{F}$ is called a conservative field if $\text{curl}\,\bar{F} = \bar{0} \Leftrightarrow$ there exists a scalar point fn such that $\bar{F} = \nabla\phi$

$$\text{curl}\,\bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xyz^3 & x^2z^3 & 3x^2yz^2 \end{vmatrix}$$

$$= \bar{i}\left[\frac{\partial}{\partial y}(3x^2yz^2) - \frac{\partial}{\partial z}(x^2z^3)\right] - \bar{j}\left[\frac{\partial}{\partial x}(3x^2yz^2) - \frac{\partial}{\partial z}(2xyz^3)\right]$$

$$+\bar{k}\left[\frac{\partial}{\partial x}(x^2z^3) - \frac{\partial}{\partial y}(2xyz^3)\right]$$

$$= \bar{i}\left[3x^2z^2 - 3x^2z^2\right] - \bar{j}\left[6xyz^2 - 6xyz^2\right] + \bar{k}\left[2xz^3 - 2xz^3\right] = \bar{0}$$

$\text{curl}\,\bar{F} = 0$

$\Rightarrow \bar{F}$ is conservative.

To find $\phi \ni \bar{F} = \text{grad}\,\phi$

$$2xyz^3\,\bar{i} + x^2z^3\,\bar{j} + 3x^2yz^2\,\bar{k} = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$$

$$\frac{\partial\phi}{\partial x} = 2xyz^3, \quad \frac{\partial\phi}{\partial y} = x^2z^3, \quad \frac{\partial\phi}{\partial z} = 3x^2yz^2$$

We have $d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz$

$$d\phi = 2xyz^3dx + x^2z^3dy + 3x^2yz^2dz$$

$$d\phi = d(x^2yz^3)$$

$$\Rightarrow \phi = x^2yz^3 + \text{constant}$$

Work done $= \int_{P_1}^{P_2} \bar{F}\cdot d\bar{r}$

We have $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k} \Rightarrow d\bar{r} = dx\,\bar{i} + dy\,\bar{j} + dz\,\bar{k}$

$$\bar{F}\cdot d\bar{r} = 2xyz^3dx + x^2z^3dy + 3x^2yz^2dz$$

$$= d(x^2yz^3)$$

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Hence work done:

$$\int_{P_1}^{P_2} \bar{F}\cdot d\bar{r} = \int_{(1,-1,2)}^{(3,2,-1)} d(x^2yz^3)$$

$$= \left[x^2yz^3\right]_{(1,-1,2)}^{(3,2,-1)}$$

$$= 9\times 2\times(-1)^3 - (1)^2(-1)(2)^3$$

$$= -18+8 = -10$$

③ Find the work done by $\bar{F}=(2x-y-z)\bar{i}+(x+y-z)\bar{j}+(3x-2y-5z)\bar{k}$ along a curve C in the XY-plane given by

(i) $x^2+y^2=9,\ z=0$

(ii) $x^2+y^2=4,\ z=0$

parametric eqns of circle
$x = r\cos\theta = 3\cos\theta$
$y = r\sin\theta = 3\sin\theta$

Sol: $\bar{F} = (2x-y-z)\bar{i}+(x+y-z)\bar{j}+(3x-2y-5z)\bar{k}$

$\bar{r} = x\bar{i}+y\bar{j}+z\bar{k} \Rightarrow d\bar{r} = dx\bar{i}+dy\bar{j}+dz\bar{k}$

In the XY-plane, $z=0 \Rightarrow dz=0$

$$\bar{F}\cdot d\bar{r} = (2x-y)dx+(x+y)dy$$

(i) Now work done $= \int_C \bar{F}\cdot d\bar{r}$, where C is the circle $x^2+y^2=9$

Take $x=3\cos\theta,\ y=3\sin\theta$,
$dx=-3\sin\theta\, d\theta,\ dy=3\cos\theta\, d\theta$. θ varies from 0 to 2π

Work done $= \int_C \bar{F}\cdot d\bar{r}$

$$= \int_C (2x-y)dx+(x+y)dy$$

$$= \int_{\theta=0}^{2\pi} \left[6\cos\theta-3\sin\theta\right]\times -3\sin\theta\, d\theta + (3\cos\theta+3\sin\theta)\times 3\cos\theta\, d\theta$$

$$= \int_{\theta=0}^{2\pi} \left[-18\sin\theta\cdot\cos\theta + 9\sin^2\theta + 9\cos^2\theta + 9\sin\theta\cdot\cos\theta\right] d\theta$$

$$= \int_{\theta=0}^{2\pi} \left[9 - 9\sin\theta\cdot\cos\theta\right] d\theta.$$

$$= 9\int_{\theta=0}^{2\pi} 1\, d\theta - \frac{9}{2}\int_{\theta=0}^{2\pi} \sin 2\theta\, d\theta.$$

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$$\therefore = 9\left[\theta\right]_0^{2\pi} - \frac{9}{2}\left[\frac{\cos 2\theta}{2}\right]_{\theta=0}^{2\pi}$$

$$= 9(2\pi) - \frac{9}{4}\left[\cos 4\pi - \cos 0\right]$$

$$= 18\pi - \frac{9}{4}\left[1-1\right] = 18\pi$$

(ii) $x^2+y^2=4,\ z=0$

Work done $\int_C \bar{F}\cdot d\bar{r}$, where C is the circle $x^2+y^2=4$

$$\bar{F} = (2x-y-z)\,\bar{i} + (x+y-z)\,\bar{j} + (3x-2y-5z)\,\bar{k}$$

$$d\bar{r} = dx\,\bar{i} + dy\,\bar{j} + dz\,\bar{k}$$

In xy-plane, $z=0 \Rightarrow dz=0$

$$\bar{F}\cdot d\bar{r} = (2x-y)\,dx + (x+y)\,dy$$

$x^2+y^2=4$

$x = 2\cos\theta,\ y = 2\sin\theta$

$dx = -2\sin\theta\, d\theta,\ dy = 2\cos\theta\, d\theta$

$$\text{work done} = \int_C \bar{F}\cdot d\bar{r}$$

$$= \int_{\theta=0}^{2\pi} (4\cos\theta - 2\sin\theta)\times -2\sin\theta\, d\theta + (2\cos\theta + 2\sin\theta)\times 2\cos\theta\, d\theta$$

$$= \int_{\theta=0}^{2\pi} \left[-8\sin\theta\cdot\cos\theta + 4\sin^2\theta + 4\cos^2\theta + 4\sin\theta\cdot\cos\theta\right] d\theta$$

$$= \int_{\theta=0}^{2\pi} \left[-4\sin\theta\cdot\cos\theta + 4\right] d\theta$$

$$= \int_{\theta=0}^{2\pi} \left[-2\sin 2\theta + 4\right] d\theta$$

$$= \left[\frac{+\cancel{2}\cos 2\theta}{\cancel{2}} + 4\theta\right]_{\theta=0}^{2\pi}$$

$$= (\cos 4\pi + 8\pi) - (\cos 0 + 0)$$

$$= \cancel{1} + 8\pi - \cancel{1} = 8\pi$$

(4) Evaluate $\int_C \bar{F}\cdot d\bar{r}$, where $\bar{F} = (x-3y)\,\bar{i} + (y-2x)\,\bar{j}$ and C is the closed curve in the xy-plane, $x = 2\cos t$, $y = 3\sin t$ from $t=0$ to $t=2\pi$

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$\bar{F} = (x-3y)\bar{i} + (y-2x)\bar{j}$

$\bar{r} = x\bar{i} + y\bar{j}$ in xy plane

$d\bar{r} = dx\,\bar{i} + dy\,\bar{j}$

$\bar{F}\cdot d\bar{r} = (x-3y)dx + (y-2x)dy$

$x = 2\cos t,\quad y = 3\sin t$ from $t=0$ to $t=2\pi$

$$\int_C \bar{F}\cdot d\bar{r} = \int_{t=0}^{2\pi} (x-3y)dx + (y-2x)dy$$

$$= \int_{t=0}^{2\pi} [2\cos t - 9\sin t] \times (-2\sin t)\,dt + [3\sin t - 4\cos t] \times 3\cos t\,d$$

$$= \int_{t=0}^{2\pi} [-4\sin t\cdot\cos t + 18\sin^2 t + 9\sin t\cdot\cos t - 12\cos^2 t]\,dt$$

$$= \int_{t=0}^{2\pi} [18\sin^2 t + 5\sin t\cos t - 12\cos^2 t]\,dt$$

$\cos 2t = 1 - 2\sin^2 t$

$2\sin^2 t = 1 - \cos 2t$

$$= \int_{t=0}^{2\pi} \left[9(1-\cos 2t) - 6(1+\cos 2t) + \frac{5}{2}\sin 2t\right] dt$$

$$= \int_{t=0}^{2\pi} \left[3 - 15\cos 2t + \frac{5}{2}\sin 2t\right] dt$$

$$= \left[3t + 15\frac{\sin 2t}{2} - \frac{5\cos 2t}{4}\right]_{t=0}^{2\pi}$$

$$= \left(6\pi - \frac{15\sin 4\pi}{2} - \frac{5\cos 4\pi}{4}\right) - \left(0 - 0 - \frac{5}{4}\right)$$

$$= 6\pi - \cancel{\frac{5}{4}} + \cancel{\frac{5}{4}} = 6\pi$$

(5) Find the work done in moving a particle in the force field

(a) $\bar{F} = 3x^2\bar{i} + \bar{j} + z\bar{k}$

(b) $\bar{F} = 3x^2\bar{i} + (2xz - y)\bar{j} + z\bar{k}$ along the straight line from $(0,0,0)$ to $(2,1,3)$

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Sol: Work done $= \int_{P_1}^{P_2} \bar{F}\cdot d\bar{r}$

$\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$, Then $d\bar{r} = dx\bar{i} + dy\bar{j} + dz\bar{k}$

$\bar{F} = 3x^2\bar{i} + \bar{j} + z\bar{k}$

$\bar{F}\cdot d\bar{r} = 3x^2dx + dy + z\,dz$

$$\int_{(0,0,0)}^{(2,1,3)} \bar{F}\cdot d\bar{r} = \int_{(0,0,0)}^{(2,1,3)} (3x^2dx + dy + z\,dz)$$

$$= \left[\frac{3x^3}{3} + y + \frac{z^2}{2}\right]_{(0,0,0)}^{(2,1,3)}$$

$$= 8 + 1 + \frac{9}{2} = 9 + \frac{9}{2} = \frac{27}{2}$$

$$= \int_{t=0}^{1} \left[3\times 4t^2\times 2dt + dt + 3t\times 3dt\right]$$

$$= \int_{t=0}^{1} \left[24t^2 + 1 + 9t\right]dt$$

$$= \left[24\times\frac{t^3}{3} + t + 9\times\frac{t^2}{2}\right]_{t=0}^{1}$$

$$= 8 + 1 + \frac{9}{2} = 9 + \frac{9}{2}$$

$$= \frac{27}{2} \text{ units}$$

(b) $\bar{F} = 3x^2\bar{i} + (2xz - y)\bar{j} + z\bar{k}$

Equation of OA is $\frac{x-0}{2} = \frac{y-0}{1-0} = \frac{z-0}{3-0} = t$ (say)

$x = 2t,\ y = t,\ z = 3t$ are its parametric equations

The points $(0,0,0)$ and $(2,1,3)$

$x = 2t$, $0 = t$, $t = 0$, $z = 0$ | $2 = 2t \Rightarrow t = 1$, $1 = t \Rightarrow t = 1$, $3 = 3t \Rightarrow t = 1$

$\therefore$ Work done in moving along the line from $O(0,0,0)$ to $A(2,1,3)$

$$\int_C \bar{F}\cdot d\bar{r} = \int_{OA} 3x^2dx + (2xz - y)dy + z\,dz$$

$$= \int_{t=0}^{1} 3(4t^2)\times 2dt + \left[2\times 2t\times 3t - t\right]dt + 3t\,3dt$$

$24t^2 + 12t^2 - t + 9t$

$$= \int_{t=0}^{1} (36t^2 + 8t)\,dt = \left[\frac{36t^3}{3} + \frac{8t^2}{2}\right]_{t=0}^{1}$$

$$= \left[12t^3 + 4t^2\right]_{t=0}^{1} = 12 + 4 = 16 \text{ units}$$

$3x^2dx + (2xz - y)dy + z\,dz$

$$\left[x^3 + 2xzy - \frac{y^2}{2} + \frac{z^2}{2}\right]_{(0,0,0)}^{(2,1,3)}$$

$$= 8 + 2\times 2\times 3\times 1 - \frac{1}{2} + \frac{9}{2}$$

$$= 8 + 12 + \frac{8}{2} = 20.$$

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(6) If $\bar{F} = xy\bar{i} - z\bar{j} + x^2\bar{k}$ and c is the curve $x=t^2$, $y=2t$, $z=t^3$ from $t=0$ to $t=1$. Evaluate $\int_c \bar{F}.d\bar{r}$

$\bar{F} = xy\bar{i} - z\bar{j} + x^2\bar{k}$, $x=t^2$, $y=2t$, $z=t^3$

$dx = 2t\,dt$, $dy = 2dt$, $dz = 3t^2 dt$

$$\bar{F} = xy\bar{i} - z\bar{j} + x^2\bar{k}$$
$$= 2t^3\bar{i} - t^3\bar{j} + t^4\bar{k}$$

$$\int_c \bar{F}.d\bar{r} = \int_c (2t^3\bar{i} - t^3\bar{j} + t^4\bar{k}).(dx\bar{i} + dy\bar{j} + dz\bar{k})$$

$$= \int_{t=0}^{1} \left[2t^3 dx - t^3 dy + t^4 dz\right]$$

$$= \int_{t=0}^{1} \left[2t^3 \times 2t\,dt - t^3 \times 2dt + t^4 \times 3t^2 dt\right]$$

$$= \int_0^1 \left[4t^4 - 2t^3 + 3t^6\right] dt$$

$$= \left[\frac{4 \times t^5}{5} - \frac{2 \times t^4}{4} + \frac{3t^7}{7}\right]_{t=0}^{1}$$

$$= \frac{4}{5} - \frac{\cancel{2}^{1}}{\cancel{4}_{2}} + \frac{3}{7} = \frac{4\times14 - 1\times35 + 3\times10}{70} = \frac{56-35+30}{70}$$
$$= \frac{51}{70}$$

(7) If $\bar{F} = (x^2-27)\bar{i} - 6yz\bar{j} + 8xz^2\bar{k}$, evaluate $\int_c \bar{F}.d\bar{r}$ from the point $(0,0,0)$ to the point $(1,1,1)$ along the straight line from $(0,0,0)$ to $(1,0,0)$, $(1,0,0)$ to $(1,1,0)$ and $(1,1,0)$ to $(1,1,1)$

<u>Sol:</u> $\bar{F} = (x^2-27)\bar{i} - 6yz\bar{j} + 8xz^2\bar{k}$

$\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$d\bar{r} = dx\bar{i} + dy\bar{j} + dz\bar{k}$

$\bar{F}.d\bar{r} = (x^2-27)dx - 6yz\,dy + 8xz^2 dz$

(i) Along the straight line from $O(0,0,0)$ to $A(1,0,0)$

Here $y=0=z \Rightarrow dy=dz=0$. Also x changes from 0 to 1.

$$\int_{OA} \bar{F}.d\bar{r} = \int_{x=0}^{1} (x^2-27)dx = \left[\frac{x^3}{3} - 27x\right]_{x=0}^{1}$$
$$= \frac{1}{3} - 27 = -\frac{80}{3}$$

$\frac{27}{\cancel{13}}$ 81

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(ii) Along the straight line from $A=(1,0,0)$ to $B(1,1,0)$.

Here $x=1$, $z=0$ $\quad$ y changes from 0 to 1.

$dx=0 \Rightarrow dz=0$

$$\int_{AB} \bar{F}\cdot d\bar{r} = \int_{y=0}^{1} -6yz\,dy = \int_{y=0}^{1} -6y(0)\,dy = 0$$

(iii) Along the straight line from $B=(1,1,0)$ to $C(1,1,1)$

$x=1$, $y=1$, z changes from 0 to 1

$dx=0$, $dy=0$

$$\int_{BC} \bar{F}\cdot d\bar{r} = \int_{z=0}^{1} (8xz^2)\,dz$$

$$= \int_{z=0}^{1} (8z^2)\,dz = \left[\frac{8xz^3}{3}\right]_{z=0}^{1} = \frac{8}{3}$$

(i)+(ii)+(iii) $\quad = \cancel{\frac{80}{3}}$ $\quad \int_C \bar{F}\cdot d\bar{r} = \int_{OA} \bar{F}\cdot d\bar{r} + \int_{AB} \bar{F}\cdot d\bar{r} + \int_{BC} \bar{F}\cdot d\bar{r}$

$$= -\frac{80}{3} + 0 + \frac{8}{3} = \frac{-72}{3} = -24$$

(8) If $\bar{F} = (5xy-6x^2)\bar{i} + (2y-4x)\bar{j}$, evaluate $\int_C \bar{F}\cdot d\bar{r}$ along the curve C in xy-plane $y=x^3$ from $(1,1)$ to $(2,8)$.

Given $\bar{F} = (5xy-6x^2)\bar{i} + (2y-4x)\bar{j}$ — ①

Along the curve $y=x^3 \Rightarrow dy = 3x^2dx$

$$\bar{F} = (5x\cdot x^3 - 6x^2)\bar{i} + (2x^3-4x)\bar{j}$$

$$= (5x^4-6x^2)\bar{i} + (2x^3-4x)\bar{j}$$

$$d\bar{r} = dx\bar{i} + dy\bar{j} + dz\bar{k}$$

$$= dx\bar{i} + dy\bar{j}$$

$$= dx\bar{i} + 3x^2dx\,\bar{j}$$

$$\bar{F}\cdot d\bar{r} = (5x^4-6x^2)dx + (2x^3-4x)\cdot 3x^2\,dx$$

$$= [5x^4-6x^2+6x^5-12x^3]\,dx$$

$$= (6x^5+5x^4-12x^3-6x^2)\,dx$$

$$\int_C \bar{F}\cdot d\bar{r} = \int_{x=1}^{2} (6x^5+5x^4-12x^3-6x^2)\,dx$$

$$= \left[\frac{6x^6}{6} + \frac{5x^5}{5} - \frac{12x^4}{4} - \frac{6x^3}{3}\right]_{x=1}^{2}$$

$$= \left[x^6+x^5-3x^4-2x^3\right]_{x=1}^{2} = [64+32-48-16] - [1+1-3-2] = 32+3 = 35$$

$$\begin{array}{r} 96 \\ 64 \\ \hline 32 \end{array}$$

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(9) Find the work done by the force $\bar{F} = z\bar{i} + x\bar{j} + y\bar{k}$, when it moves a particle along the arc of the curve $\bar{r} = \cos t\,\bar{i} + \sin t\,\bar{j} - t\bar{k}$ from $t=0$ to $t=2\pi$

$\bar{F} = z\bar{i} + x\bar{j} + y\bar{k}$, $\quad \bar{r} = \cos t\,\bar{i} + \sin t\,\bar{j} - t\bar{k}$

$x = \cos t,\ y = \sin t,\ z = -t$

$$d\bar{r} = (-\sin t\,\bar{i} + \cos t\,\bar{j} - \bar{k})\,dt$$

$$\bar{F}\cdot d\bar{r} = z\,dx + x\,dy + y\,dz$$

$$= (-t)(-\sin t)\,dt + (\cos t)(\cos t\,dt) + (\sin t)(-dt)$$

$$= (t\sin t + \cos^2 t - \sin t)\,dt$$

$$\text{Work done} = \int_{t=0}^{2\pi} \bar{F}\cdot d\bar{r}$$

$$= \int_{t=0}^{2\pi} (t\sin t + \cos^2 t - \sin t)\,dt$$

$\cos 2t = 2\cos^2 t - 1$

$\cos^2 t = \frac{1+\cos 2t}{2}$

$$= \left[t(-\cos t) - (1)(-\sin t) + \frac{1}{2}\left(t + \frac{\sin 2t}{2}\right) + \cos t\right]_{t=0}^{2\pi}$$

$$= \left[-t\cos t + \sin t + \frac{t}{2} + \frac{\sin 2t}{4} + \cos t\right]_{t=0}^{2\pi}$$

$$= \left(-2\pi\cos 2\pi + \sin 2\pi + \frac{2\pi}{2} + \frac{\sin 4\pi}{4} + \cos 2\pi\right)$$

$$-(0+0+0+0+1)$$

$$= -2\pi + \pi + \cancel{1} - \cancel{1} = -\pi$$

(10) Evaluate $\oint_C (yz\,dx + xz\,dy + xy\,dz)$ over arc of a helix $x = a\cos t,\ y = a\sin t,\ z = kt$, as t varies from 0 to 2π.

Tkr cet. ac.in

Given curve is $x = a\cos t,\ y = a\sin t,\ z = kt$

$$dx = -a\sin t\,dt,\quad dy = a\cos t\,dt,\quad dz = k\,dt$$

$$\oint_C (yz\,dx + xz\,dy + xy\,dz) = \int_{t=0}^{2\pi} \left[a\sin t\cdot kt\right](-a\sin t\,dt) + (a\cos t)(kt)(a\cos t\,dt) + (a\cos t)(a\sin t)(k\,dt)$$

$$= \int_{t=0}^{2\pi} \left[-a^2k\cdot t\sin^2 t + a^2k\cdot t\cos^2 t + a^2k\sin t\cos t\right]$$

$$= a^2k\int_{t=0}^{2\pi} t(\cos^2 t - \sin^2 t)\,dt + \frac{a^2k}{2}\int_{t=0}^{2\pi} \sin 2t\,dt$$

$$= a^2k\int_{t=0}^{2\pi} t\cos 2t\,dt + \frac{a^2k}{2}\left[-\frac{\cos 2t}{2}\right]_{t=0}^{2\pi}$$

$$= a^2k\left[t\cdot\frac{\sin 2t}{2} + 1\times\frac{-\cos 2t}{4}\right]_0^{2\pi}$$

$$a^2k\left[\left(\pi\sin 4\pi + \frac{\cos 4\pi}{4}\right) - \frac{1}{4}\right]$$

$$+\frac{a^2k}{2}\left[-\frac{\cos 4\pi}{2} + \frac{1}{2}\right]$$

$$= 0$$

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(11) Find the work done in moving in a particle in the force field (13)
$\bar{F} = (3x^2)\bar{i} + (2zx - y)\bar{j} + z\bar{k}$, along the curve defined by $x^2 = 4y$, $3x^3 = 8z$
from $x=0$ to $x=2$

Given, $\bar{F} = (3x^2)\bar{i} + (2zx - y)\bar{j} + z\bar{k}$
and curve is $x^2 = 4y$, $3x^3 = 8z$ — (1)

$$\bar{F} = 3x^2\bar{i} + \left(2 \times \frac{3}{8}x^3 \cdot x - \frac{x^2}{4}\right)\bar{j} + \frac{3}{8}x^3\bar{k}$$

$$= 3x^2\bar{i} + \left(\frac{3}{4}x^4 - \frac{x^2}{4}\right)\bar{j} + \frac{3}{8}x^3\bar{k}$$

$\cancel{\bar{F}\cdot d\bar{r}} =$ $\cancel{x^2 = 4y}$, $\cancel{3x^3 = 8z}$

$\cancel{2x} = \bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$$= x\bar{i} + \frac{x^2}{4}\bar{j} + \frac{3}{8}x^3\bar{k}$$

$$d\bar{r} = \left(\bar{i} + \frac{x}{2}\bar{j} + \frac{9}{8}x^2\bar{k}\right)dx$$

$$\bar{F}\cdot d\bar{r} = 3x^2dx + \left(\frac{3}{4}x^4 - \frac{x^2}{4}\right)\frac{x}{2}\,dx + \frac{3}{8}x^3 \cdot \frac{9}{8}x^2\,dx$$

$$= \left(3x^2 + \frac{3}{8}x^5 - \frac{x^3}{8} + \frac{27}{64}x^5\right)dx$$

Hence work done $= \int_{x=0}^{2} \bar{F}\cdot d\bar{r}$

$$= \int_{x=0}^{2}\left(3x^2 + \frac{3}{8}x^5 - \frac{x^3}{8} + \frac{27}{64}x^5\right)dx$$

$$= \left[3 \times \frac{x^3}{3} + \frac{3}{8} \times \frac{x^6}{6} - \frac{x^4}{8} \times \frac{1}{4} + \frac{27}{64} \times \frac{x^6}{6}\right]_{x=0}^{2}$$

$$= 8 + \frac{\cancel{3}}{\cancel{8}} \times \frac{1}{\cancel{6}_{\cancel{3}}} \times \cancel{2}\times\cancel{2}\times\cancel{2}\times\cancel{2}\times\cancel{2}\times\cancel{2} - \frac{\cancel{2}\times\cancel{2}\times\cancel{2}\times\cancel{2}}{\cancel{8}} \times \frac{1}{\cancel{4}_{2}} + \frac{27}{64} \times \frac{\cancel{8}\times\cancel{8}}{\cancel{6}_{2}}$$

$$= 8 + 4 - \frac{1}{2} + \frac{9}{2} = 12 + 4 = 16$$

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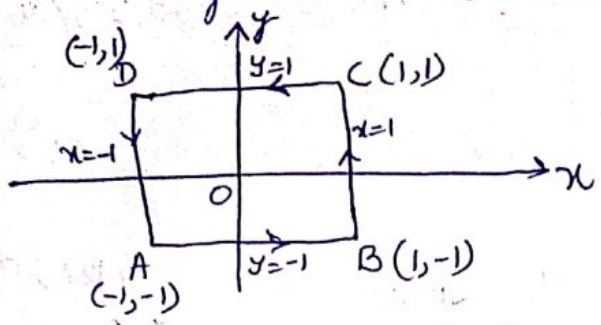
Ex) Evaluate the line integral $\int_C [(x^2+xy)dx+(x^2+y^2)dy]$ where C is the square formed by the lines $x=\pm1$ and $y=\pm1$

Sol:



$$\int_C \bar{F}\cdot d\bar{r} = \int_C [(x^2+xy)dx+(x^2+y^2)dy]$$

In the counter (or anti) clockwise direction,

$$\int_C \bar{F}\cdot d\bar{r} = \int_{AB} \bar{F}\cdot d\bar{r} + \int_{BC} \bar{F}\cdot d\bar{r} + \int_{CD} \bar{F}\cdot d\bar{r} + \int_{DA} \bar{F}\cdot d\bar{r} \quad -(1)$$

Along AB: Here $y=-1 \Rightarrow dy=0$

$$\int_{AB} \bar{F}\cdot d\bar{r} = \int_{x=-1}^{1}(x^2-x)dx = \int_{-1}^{1}x^2dx - \int_{-1}^{1}x\,dx$$

$$= 2\int_0^1 x^2dx = 2\left[\frac{x^3}{3}\right]_{x=0}^{1} = \frac{2}{3}$$

Along BC: Here $x=1 \Rightarrow dx=0$, y varies from -1 to 1

$$\int_{BC} \bar{F}\cdot d\bar{r} = \int_{y=-1}^{1}(1+y^2)dy = \left[y+\frac{y^3}{3}\right]_{y=-1}^{1} = \left(1+\frac{1}{3}\right)-\left(-1-\frac{1}{3}\right) = 1+1+\frac{1}{3}+\frac{1}{3} = \frac{8}{3}$$

Along CD: Here $y=1 \Rightarrow dy=0$, x varies from 1 to -1

$$\int_{CD} \bar{F}\cdot d\bar{r} = \int_{x=1}^{-1}(x^2+x)dx = \int_{x=1}^{-1}x^2dx + \int_{x=1}^{-1}x\,dx = -\int_{x=-1}^{1}x^2dx + -\int_{x=-1}^{1}x\,dx$$

$$= -2\int_{x=0}^{1}x^2dx = -2\times\left[\frac{x^3}{3}\right]_{x=0}^{1}$$

$$= -\frac{2}{3}$$

Along DA: Here $x=-1 \Rightarrow dx=0$, y varies from 1 to -1

$$\int_{DA} \bar{F}\cdot d\bar{r} = \int_{y=1}^{-1}((-1)^2+y^2)dy = \int_{y=1}^{-1}(1+y^2)dy = \left[y+\frac{y^3}{3}\right]_{y=1}^{-1} = \left[-1-\frac{1}{3}-1-\frac{1}{3}\right]$$

$$= -2-\frac{2}{3} = -\frac{8}{3}$$

$$\int_C \bar{F}\cdot d\bar{r} = \frac{2}{3}+\frac{8}{3}-\frac{2}{3}-\frac{8}{3} = 0$$

Ex) Find the work done by the force $\bar{F}=(2y+3)\bar{i}+xz\bar{j}+(yz-x)\bar{k}$ when it moves a particle from the point $(0,0,0)$ to $(2,1,1)$ along the curve $x=2t^2$, $y=t$ and $z=t^3$.

$x=2t^2$ $\quad y=t$ $\quad z=t^3$

$dx=4t\,dt$ $\quad dy=dt$ $\quad dz=3t^2dt$

L.L. $x=0, y=0, z=0$

$2t^2=0, t=0, t^3=0$

$t=0, t=0, t=0$

U.L: $x=2, y=1, z=1$

$2t^2=2, t=1, t^3=1$

$t=1, t=1, t=1$

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t is from 0 to 1.

Work done $= \int_C \bar{F}\cdot d\bar{r} = \int (2y+3)\,dx + xz\,dy + (yz-x)\,dz$

$$= \int_{t=0}^{1} (2t+3)(4t\,dt) + (2t^2)(t^3)(dt) + (t^4-2t^2)(3t^2\,dt)$$

$$= \int_{t=0}^{1} (8t^2+12t+2t^5+3t^6-6t^4)\,dt$$

$$= \int_{t=0}^{1} (3t^6+2t^5-6t^4+8t^2+12t)\,dt$$

$$= \left[\frac{3t^7}{7}+\frac{2t^6}{6}-\frac{6t^5}{5}+\frac{8t^3}{3}+6t^2\right]_{t=0}^{1}$$

$$= \frac{3}{7}+\frac{1}{3}-\frac{6}{5}+\frac{8}{3}+\frac{6}{1} = \frac{45+35-126+280+630}{105}$$

$$= \frac{\cancel{864}^{\,288}}{\cancel{105}_{\,35}} = \frac{288}{35}$$

180
280
560

360
630
990

990
−126
864

$\frac{4}{12}$ $\frac{35}{+8}$ 280

105
6
630

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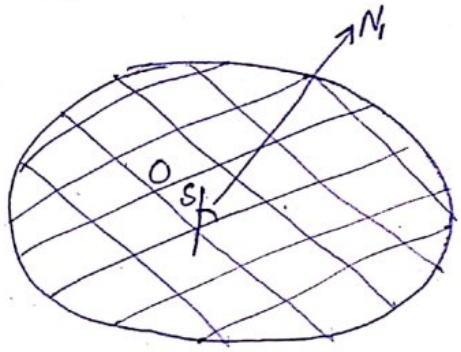
SURFACE INTEGRALS

A surface $\bar{r} = \bar{f}(u,v)$ is called a smooth surface if $\bar{f}(u,v)$ is continuous and possesses partial derivatives. Let $\bar{F}(\bar{r})$ be a continuous vector point function, defined over the smooth surface $\bar{r} = \bar{f}(u,v)$. Let S be the region of the surface. Divide the region into m subregions of areas $\delta S_1, \delta S_2, \delta S_3 \ldots \delta S_i \ldots \delta S_m$. Let P_i be a point of δS_i and $\bar{n}_i$ be the unit normal to δS_i at P_i. Let δA_i be the vector area of δS_i.

Then $\delta \bar{A}_i = \bar{n}_i \, \delta S_i$



Form the sum

$$I_m = \sum_{i=1}^{m} \bar{F}(\bar{r}_i) \cdot \delta A_i$$

$$= \sum \bar{F}(\bar{r}_i) \cdot \bar{n}_i \, \delta S_i$$

Let m tend to infinity in such a way that each δS_i shrinks to a point. The limit of I_m if it exists is called the normal surface integral of $\bar{F}(\bar{r})$ over the region S of the surface $\bar{r} = \bar{f}(u,v)$ and is denoted by $\boxed{\int_S \bar{F}(\bar{r}) \cdot d\bar{A} \text{ (or) } \int_S \bar{F} \cdot \bar{n} \, ds}$

NOTE: Other types of surface integrals are $\int \bar{F} \times d\bar{A}$ (or) $\int \phi \, d\bar{A}$. Any integral which is to be evaluated over a surface is called a surface integral.

Let S be a closed surface
$\bar{n} \rightarrow$ outward drawn unit normal vector to surface S.

1) Physical interpretation:

The surface integral of a vector point function $\bar{F}$ expresses the normal flux through a surface. If $\bar{F}$ represents the velocity vector of a fluid then the surface integral $\int \bar{F} \cdot \bar{n} \, ds$ over a closed surface S represents the rate of flow of fluid through the surface. If the flux of $\bar{F}$ across every closed surface S in the region R is zero, then $\bar{F}$ is said to be a solenoidal vector function.

Cartesian form: $\bar{F} = F_1 \bar{i} + F_2 \bar{j} + F_3 \bar{k}$, where F_1, F_2, F_3 are cont & differentiable fns of x, y, z.

$$\int \bar{F} \cdot \bar{n} \, ds = \iint_S F_1 \, dy \, dz + F_2 \, dx \, dz + F_3 \, dx \, dy$$

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2) <u>Cartesian form</u>: Let $\bar{F}(\bar{r}) = F_1\bar{i} + F_2\bar{j} + F_3\bar{k}$ where F_1, F_2, F_3 are continuous and differentiable functions of x, y, z.

Then $\int \bar{F}\cdot\bar{n}\, ds = \iint_S F_1\, dy\, dz + F_2\, dz\, dx + F_3\, \cancel{dy\, dz}\; dx\, dy$

<u>NOTE</u>: Let R_1 be the projection of S on xy plane

Then

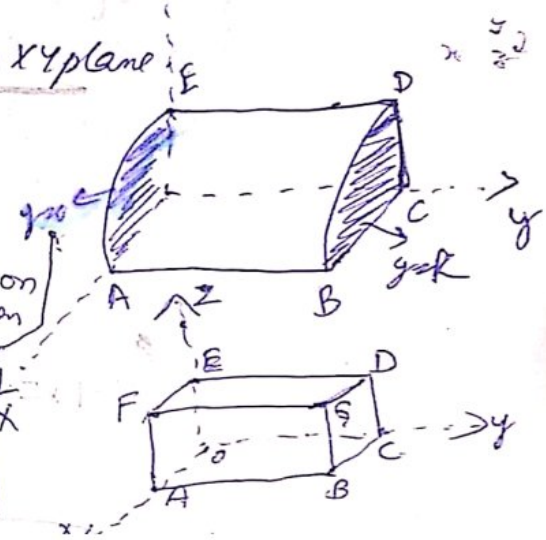
$$\int_S \bar{F}\cdot\bar{n}\, ds = \iint_{R_1} \frac{\bar{F}\cdot\bar{n}\, dx\, dy}{|\bar{n}\cdot\bar{k}|}$$

Let R_2 be the projection of S on YZ plane then

$$\int_S \bar{F}\cdot\bar{n}\, ds = \iint_{R_2} \frac{\bar{F}\cdot\bar{n}}{|\bar{n}\cdot\bar{i}|}\, dy\, dz$$

Let R_3 be the projection of S on ZX plane, then

$$\int_S \bar{F}\cdot\bar{n}\, ds = \iint_{R_3} \frac{\bar{F}\cdot\bar{n}}{|\bar{n}\cdot\bar{j}|}\, dz\, dx$$



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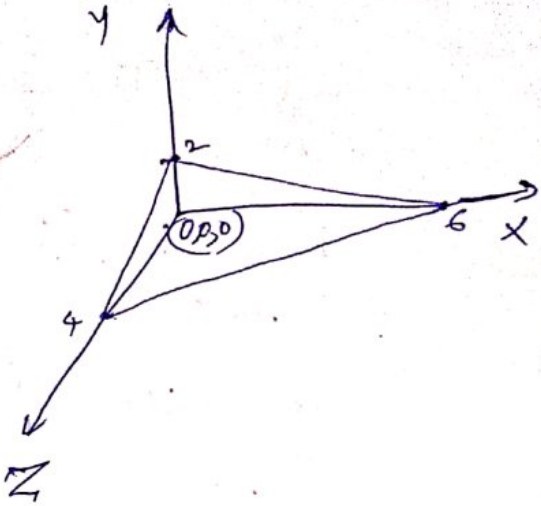
① Evaluate $\int_S \bar{F}\cdot\bar{n}\,ds$ where $\bar{F} = 18z\bar{i} - 12\bar{j} + 3y\bar{k}$ and 'S' is the part of the surface of the plane $2x+3y+6z=12$ located in the first octant. (4)



$$\frac{x}{6}+\frac{y}{4}+\frac{z}{2}=1$$

Let $\phi = 2x+3y+6z-12$.

Normal to the plane is $\nabla\phi$

$$\nabla\phi = \bar{i}(2)+\bar{j}(3)+\bar{k}(6)$$

$$= 2\bar{i}+3\bar{j}+6\bar{k}$$

$$\frac{\nabla\phi}{|\nabla\phi|} = \frac{2\bar{i}+3\bar{j}+6\bar{k}}{\sqrt{4+9+36}} = \frac{2\bar{i}+3\bar{j}+6\bar{k}}{7} = \bar{n}$$

Let 'R' be the projection of 'S' on to XY-plane. Then

$$\int_S \bar{F}\cdot\bar{n}\,ds = \iint_R \bar{F}\cdot\bar{n}\,\frac{dx\,dy}{|\bar{n}\cdot\bar{k}|}$$

$$= \iint \left[18z\times\frac{2}{7} - 12\times\frac{3}{7} + 3y\times\frac{6}{7}\right]\times\frac{dx\,dy}{\frac{6}{7}}$$

$$= \iint \left[6z - 6 + 3y\right] dx\,dy$$

$$= \iint (12-2x-\cancel{3y}-6+\cancel{3y})\,dx\,dy$$

$$= \iint (6-2x)\,dx\,dy$$

Given surface is $2x+3y+6z=12$

Let 'R' be the projection of S onto XY-plane; i.e. $z=0$

$\therefore$ R of XY plane is $2x+3y=12$

$$y = \frac{12-2x}{3}$$

$\Rightarrow$ y varies from 0 to $\frac{12-2x}{3}$

x varies from 0 to 6

$$\text{Surface integral} = \iint \frac{\bar{F}\cdot\bar{n}}{|\bar{n}\cdot\bar{k}|}\,dx\,dy$$

$$= \iint (6-2x)\,dx\,dy$$

$$= 2\int_{x=0}^{6}\int_{y=0}^{\frac{12-2x}{3}} (3-x)\,dy\,dx$$

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$$= 2\int_{x=0}^{6} (3-x)\left[y\right]_0^{\frac{12-2x}{3}} dx$$

$$= 2\int_{x=0}^{6} (3-x)\frac{(12-2x)}{3} dx$$

$$= \frac{4}{3}\int_{x=0}^{6} \left[(3-x)(6-x)\right] dx$$

$$= \frac{4}{3}\int_{x=0}^{6} \left[18-9x+x^2\right] dx$$

$$= \frac{4}{3}\left[18x - \frac{9x^2}{2} + \frac{x^3}{3}\right]_{x=0}^{6}$$

$$= \frac{4}{3}\left[18\times 6 - \frac{9\times 6\times \cancel{6}^{3}}{\cancel{2}} + \frac{6\times 6\times \cancel{6}^{2}}{\cancel{3}}\right]$$

$$= \frac{4}{3}\left[108 - 162 + 72\right] = \frac{4\times \cancel{18}^{6}}{\cancel{3}} = 24$$

$$\begin{array}{r} 18 \\ 9 \\ \hline 162 \end{array} \qquad \begin{array}{r} 36 \\ 2 \\ \hline 72 \end{array} \qquad \begin{array}{r} 108 \\ 72 \\ \hline 180 \\ 252 \\ \hline 18 \end{array}$$

yz plane

$$= \iint_R (\bar{F}\cdot\bar{n}) \frac{dy\,dz}{|\bar{n}\cdot\bar{i}|}$$

$$= \iint_R \frac{6}{\cancel{2}}(6z-6+3y) \times \frac{dy\,dz}{\frac{2}{\cancel{2}}}$$

$$= 3\iint (6z-6+3y)\,dy\,dz$$

$$= 3\int_{y=0}^{4}\int_{z=0}^{\frac{4-y}{2}} (6z-6+3y)\,dz\,dy$$

$$= 3\int_{y=0}^{4} \left[3z^2 - 6z + 3yz\right]_{z=0}^{\frac{4-y}{2}} dy$$

$$= 3\int_{y=0}^{4} \left[\frac{3(4-y)^2}{4} - 6\left(\frac{4-y}{2}\right) + 3y\left(\frac{4-y}{2}\right)\right] dy$$

$2x+3y+6z=12$

yz plane, $x=0$

$3y+6z=12$

$z=0$ to $z=\frac{12-3y}{6}$

$\Rightarrow z=\frac{4-y}{2}$

$y=0$ to 4

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$$= 3\int_{y=0}^{4}\left[\frac{3}{4}(16-8y+y^2) - 3(4-y) + \frac{12y}{2} - \frac{3y^2}{2}\right]dy$$

$$= 3\int_{y=0}^{4}\left[\cancel{12} - \cancel{6y} + \frac{3}{4}y^2 - \cancel{12} + 3y + \cancel{6y} - \frac{3}{2}y^2\right]dy$$

$$= 3\int_{y=0}^{4}\left[-\frac{3}{4}y^2 + 3y\right]dy$$

$$= 3\left[-\frac{3}{4}\times\frac{y^3}{3} + \frac{3y^2}{2}\right]_{y=0}^{4}$$

$$= 3\left[-\frac{\cancel{3}}{\cancel{4}}\times\frac{\cancel{4}\times4\times4}{\cancel{3}} + \frac{3}{\cancel{2}}\times\cancel{4}^{2}\times4\right]$$

$$= 3\left[-16+24\right] = 3\times8 = 24$$

$$= \frac{2}{3}\int_{x=0}^{6}\left[36-18x+2x^2\right]dx$$

$$= \frac{2}{3}\left[36x - 18\frac{x^2}{2} + 2\frac{x^3}{3}\right]_{x=0}^{6}$$

$$= \frac{2}{3}\left[\frac{36\times6}{6} - 18\times\frac{6\times\cancel{6}^{3}}{\cancel{2}} + \frac{2\times\cancel{6}^{2}\times6\times6}{\cancel{3}}\right]$$

$$= \frac{2}{3}\left[216 - 324 + 144\right]$$

$$= \frac{2}{\cancel{3}}\times\cancel{36}^{12} = 24$$

<u>ZX-plane</u> $= \iint_R \frac{\bar{F}\cdot\bar{n}}{|\bar{n}\cdot\bar{j}|}\,dz\,dx$

$$= \iint_R \frac{6}{7}(6z-6+3y)\times\frac{dz\,dx}{\frac{3}{7}}$$

$$= 2\iint_R (6z-6+3y)\,dz\,dx$$

$$= 2\iint_R \left[\cancel{6z}-6+12-2x-\cancel{6z}\right]dz\,dx$$

$$= 2\iint_R \left[6-2x\right]dz\,dx$$

$$= 2\int_{x=0}^{6}\int_{z=0}^{\frac{6-x}{3}}(6-2x)\,dz\,dx$$

$$= 2\int_{x=0}^{6}(6-2x)\left(\frac{6-x}{3}\right)dx$$

$2x+3y+6z=12$

ZX plane

$y=0$

$2x+6z=12$

$6z = 12-2x$

$z = \frac{12-2x}{6}$

$z = \frac{6-x}{3}$

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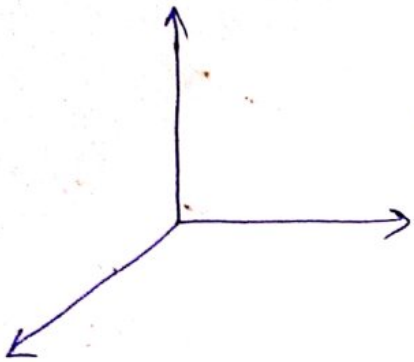
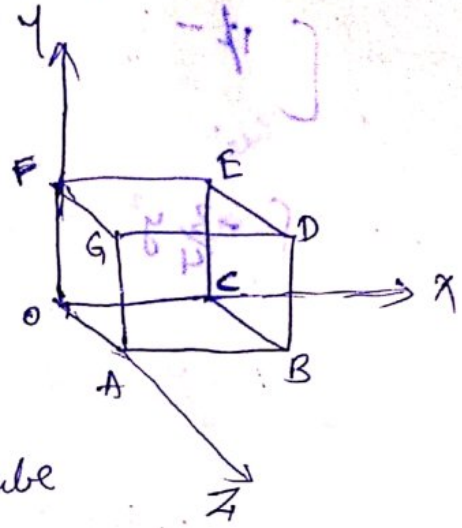
② If $\bar{F} = 4xz\bar{i} - y^2\bar{j} + yz\bar{k}$, evaluate $\int_S \bar{F}\cdot\bar{n}\,ds$ where S is the surface of the cube bounded by $x=0, x=a, y=0, y=a, z=0, z=a$





Sol: Consider the volume within the cube OABCDEFG bounded by
$x=0, x=a, y=0, y=a, z=0, z=a$

Here $\bar{F} = 4xz\bar{i} - y^2\bar{j} + yz\bar{k}$

Let us calculate $\iint_S \bar{F}\cdot\bar{n}\,ds$ for each face of the cube.

(i) For $R_1 = OAGF$, $x=0$, $ds = dz\,dy$, $\bar{n} = -\bar{i}$

$$\iint_{R_1} \bar{F}\cdot\bar{n}\,ds = \int_{z=0}^{a}\int_{y=0}^{a} (-4xz)\,dz\,dy = 0$$

(ii) For $R_2 = BCED$, $x=a$, $ds = dy\,dz$, $\bar{n} = \bar{i}$

$$\iint_{R_2} \bar{F}\cdot\bar{n}\,ds = \int_{z=0}^{a}\int_{y=0}^{a} 4xz\,dy\,dz$$

$$= 4a\int_{z=0}^{a}\int_{y=0}^{a} z\,dy\,dz$$

$$= 4a \times \int_{z=0}^{a} z\,[y]_0^a\,dz$$

$$= 4a \times a\int_{z=0}^{a} z\,dz$$

$$= 4a^2 \times \left[\frac{z^2}{2}\right]_{z=0}^{a} = \frac{4a^4}{2} = 2a^4$$

YZ

$\int_S \bar{F}\cdot\bar{n}\,ds$

$= \iint \bar{F}\cdot\bar{n}\,\frac{dy\,dz}{|\bar{n}\cdot\bar{i}|}$

$= \iint (\bar{F}\cdot\bar{i})\,dy\,dz$

$= \int_{y=0}^{a}\int_{z=0}^{a} 4xz\,dy\,dz$

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(iii) For $R_3 = OCEF$, $z=0$, $ds = dx\,dy$, $\bar{n} = -\bar{k}$

$$\int_{R_3} \bar{F}\cdot\bar{n}\,ds = \int_{x=0}^{a}\int_{y=0}^{a} (-yz)\,dx\,dy = 0$$

(iv) For $R_4 = ABDG$, $z=a$, $ds = dx\,dy$, $\bar{n} = \bar{k}$

$$\int_{R_4} \bar{F}\cdot\bar{n}\,ds = \int_{x=0}^{a}\int_{y=0}^{a} (yz)\,dx\,dy$$

$$= \int_{x=0}^{a}\int_{y=0}^{a} (ay)\,dx\,dy$$

$$= a\int_{x=0}^{a} \left(\frac{y^2}{2}\right)_{y=0}^{a} dx$$

$$= \frac{a^3}{2}\int_{x=0}^{a} 1\,dx = \frac{a^3}{2}\times a = \frac{a^4}{2}$$

xy plane

$\int_S \bar{F}\cdot\bar{n}\,ds$ $\quad \bar{n} = \bar{k}$

$= \iint (\bar{F}\cdot\bar{k})\,\frac{dx\,dy}{(n\cdot k)}$

$= \iint (yz)\,dx\,dy$

$= \int_{x=0}^{a}\int_{y=0}^{a} ay\,dx\,dy$

(v) For $R_5 = OABC$, $\bar{n} = -\bar{j}$, $ds = dx\,dz$, $y=0$

$$\int_{R_5} \bar{F}\cdot\bar{n}\,ds = \int_{x=0}^{a}\int_{z=0}^{a} (+y^2)\,dx\,dz = 0$$

(vi) For $R_6 = GDEF$, $\bar{n} = \bar{j}$, $ds = dx\,dz$, $y=a$

$$\int_{R_6} \bar{F}\cdot\bar{n}\,ds = \int_{x=0}^{a}\int_{z=0}^{a} (-y^2)\,dx\,dz = -a^2\int_{x=0}^{a}\int_{z=0}^{a} 1\,dz\,dx$$

$$= -a^2\int_{x=0}^{a} [z]_{z=0}^{a}\,dx$$

$$= -a^2\times a\int_{x=0}^{a} 1\,dx = -a^3\times a = -a^4$$

Hence $\int_S \bar{F}\cdot\bar{n}\,ds = \iint_{R_1} + \iint_{R_2} + \iint_{R_3} + \iint_{R_4} + \iint_{R_5} + \iint_{R_6}$

$$= 0 + 2a^4 + 0 + \frac{a^4}{2} + 0 - a^4 = a^4 + \frac{a^4}{2} = \frac{3a^4}{2}$$

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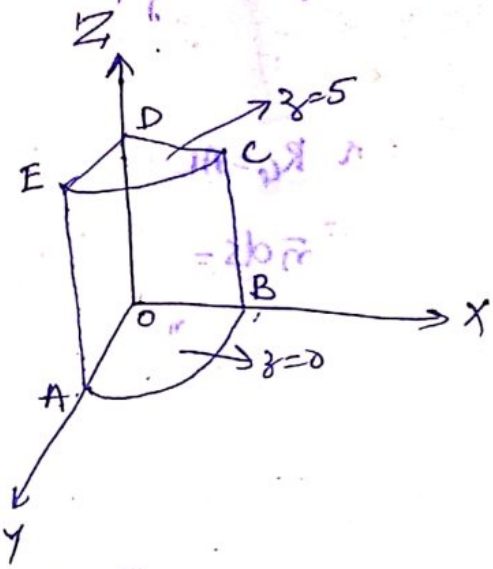
③ Evaluate $\int \bar{F}.\bar{n}\, ds$ where $\bar{F} = z\bar{i} + x\bar{j} - 3y^2z\bar{k}$ and S is the surface $x^2+y^2=16$ included in the first octant between $z=0$ and $z=5$.

Surface S is the curved surface ABCEA.



$\phi = x^2+y^2-16$.

normal to the surface S is gradϕ

$\text{grad}\,\phi = \bar{i}(2x) + \bar{j}(2y) + \bar{k}(0)$

$= 2x\bar{i} + 2y\bar{j}$

unit normal $=$

$$\bar{n} = \frac{2x\bar{i}+2y\bar{j}}{\sqrt{4x^2+4y^2}} = \frac{x\bar{i}+y\bar{j}}{\sqrt{16}} = \frac{x\bar{i}+y\bar{j}}{4}$$

Let 'R' be the projection of 'S' on YZ plane. Then 'R' is the rectangle OAED.

$$\iint_S \bar{F}.\bar{n}\, ds = \iint_R (z\bar{i}+x\bar{j}-3y^2z\bar{k}).\left(\frac{x\bar{i}+y\bar{j}}{4}\right)\frac{dy\,dz}{\frac{x}{4}}$$

$$= \iint_R \bar{F}.\bar{n}\,\frac{dy\,dz}{|\bar{n}.\bar{i}|}$$

$$= \iint_R \left(\frac{zx}{4}+\frac{xy}{4}\right)\times\frac{4}{x}\,dy\,dz$$

$x^2+y^2=16$ projected onto YZ plane, so $x=0$

$y^2=16$

$y=\pm 4$

$y=4$

$$= \int_{z=0}^{5}\int_{y=0}^{4}(z+y)\,dy\,dz$$

$$= \int_{z=0}^{5}\left[zy+\frac{y^2}{2}\right]_{y=0}^{4} dz$$

$$= \int_{z=0}^{5}\left(4z+\frac{4\times 4}{2}\right)dz$$

$$= \int_{z=0}^{5}(4z+8)\,dz$$

$$= \left[\frac{4z^2}{2}+8z\right]_{z=0}^{5}$$

$$= 4\times\frac{25}{2}+40$$

$$= 50+40 = 90$$

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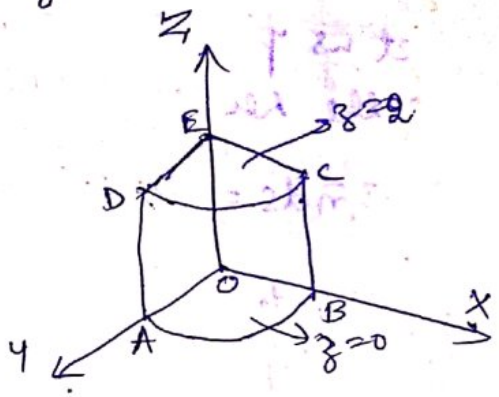
(4) Evaluate $\iint_S \bar{F}\cdot d\bar{s}$ if $\vec{F} = yz\bar{i} + 2y^2\bar{j} + xz^2\bar{k}$ and S is the surface of the cylinder $x^2+y^2=9$ contained in the first octant between the planes $z=0$ and $z=2$.



$\phi = x^2+y^2-9=0$

$\nabla\phi = \bar{i}(2x) + \bar{j}(2y)$

$= 2x\bar{i} + 2y\bar{j}$

$\bar{n} = \dfrac{\nabla\phi}{|\nabla\phi|} = \dfrac{2x\bar{i}+2y\bar{j}}{\sqrt{4x^2+4y^2}}$

$= \dfrac{x\bar{i}+y\bar{j}}{\sqrt{x^2+y^2}} = \dfrac{x\bar{i}+y\bar{j}}{3}$

$\bar{F}\cdot\bar{n} = yz\times\dfrac{x}{3} + 2y^2\times\dfrac{y}{3} = \dfrac{xyz}{3} + \dfrac{2y^3}{3}$

Let R_1 be the projection of 'S' onto YZ plane, we get a rectangle OADE.

YZ plane
$\Rightarrow x=0$
$y^2=9$
$y=\pm 3$
$y=3$

$\iint_S \bar{F}\cdot\bar{n}\,ds = \iint_{R_1} \bar{F}\cdot\bar{n}\,\dfrac{dy\,dz}{|\bar{n}\cdot\bar{i}|}$

$= \int_{z=0}^{2}\int_{y=0}^{3} \left(\dfrac{xyz+2y^3}{3}\right) \times \dfrac{1}{\frac{x}{3}}\,dy\,dz$

$= \int_{z=0}^{2}\int_{y=0}^{3} \left(yz + \dfrac{2y^3}{x}\right) dy\,dz$

$= \int_{z=0}^{2}\int_{y=0}^{3} \left(yz + \dfrac{2y^3}{\sqrt{9-y^2}}\right) dy\,dz$

$= \int_{z=0}^{2}\int_{y=0}^{3} yz\,dy\,dz + 2\int_{z=0}^{2}\int_{y=0}^{3} \dfrac{y^3}{\sqrt{9-y^2}}\,dy\,dz$

$= \int_{z=0}^{2} z\left[\dfrac{y^2}{2}\right]_0^3 dz + \dfrac{9}{2}\int_{z=0}^{2} z\,dz = \dfrac{9}{2}\left[\dfrac{z^2}{2}\right]_{z=0}^{2}$

$= \dfrac{9}{2}\int_{z=0}^{2} z\,dz = \dfrac{9}{2}\left[\dfrac{z^2}{2}\right]_{z=0}^{2}$

$= \dfrac{9}{2}\times 2 = 9$

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$$= 9 + 2\int_{z=0}^{2}\int_{y=0}^{3} \frac{y^3}{\sqrt{9-y^2}}\,dy\,dz$$

$$\int_{y=0}^{3} \frac{y^3}{\sqrt{9-y^2}}\,dy$$

Put $y = 3\sin\theta$

$dy = 3\cos\theta\, d\theta$

$$\int_{\theta=0}^{\frac{\pi}{2}} \frac{27\sin^3\theta}{\sqrt{9-9\sin^2\theta}} \times 3\cos\theta\, d\theta$$

$$= \frac{27\times\cancel{3}}{\cancel{3}}\int_{\theta=0}^{\frac{\pi}{2}} \frac{\sin^3\theta\cdot\cancel{\cos\theta}}{\cancel{\cos\theta}}\,d\theta$$

$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

$$= 27\int_{\theta=0}^{\frac{\pi}{2}} \sin^3\theta\cdot\cos^0\theta\, d\theta$$

$2m-1=3$

$m=2$

$$= 27\times\frac{1}{2}\beta\left(2,\frac{1}{2}\right)$$

$$= \frac{27}{2}\times\frac{\Gamma(2)\,\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{5}{2}\right)}$$

$$= \frac{27}{2}\times\frac{1\times\sqrt{\pi}}{\frac{3}{2}\times\frac{1}{2}\sqrt{\pi}} = \frac{\cancel{27}^{9}}{\cancel{2}}\times\frac{\cancel{2}}{\cancel{3}}\times 2 = 18$$

$=$

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(9) Evaluate $\iint_S \bar{F}\cdot\bar{n}\, ds$ where $\bar{F} = 12x^2y\,\bar{i} - 3yz\,\bar{j} + 2z\,\bar{k}$ and 'S' is the portion of the plane $x+y+z=1$ included in the first octant.

$$\bar{F} = 12x^2y\,\bar{i} - 3yz\,\bar{j} + 2z\,\bar{k}$$

$$\phi \equiv x+y+z-1=0$$

$$\nabla\phi = \bar{i}(1) + \bar{j}(1) + \bar{k}(1)$$

$$= \bar{i}+\bar{j}+\bar{k}$$

$$\bar{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{\bar{i}+\bar{j}+\bar{k}}{\sqrt{3}}$$

$$\bar{F}\cdot\bar{n} = \frac{12x^2y}{\sqrt{3}} - \frac{3yz}{\sqrt{3}} + \frac{2z}{\sqrt{3}}$$

$$= \frac{1}{\sqrt{3}}(12x^2y - 3yz + 2z)$$

Let us project onto XY plane. $z=0$, $x+y=1$, $y=1-x$

$$\iint_S \bar{F}\cdot\bar{n}\, ds = \iint_R \bar{F}\cdot\bar{n}\, \frac{dx\,dy}{|\bar{n}\cdot\bar{k}|}$$

where 'R' is the projection of 'S' on XY plane.

$$= \frac{1}{\sqrt{3}}\iint (12x^2y - 3yz + 2z) \times \frac{dx\,dy}{\frac{1}{\sqrt{3}}}$$

$$= \iint 12x^2y + (-3y+2)z\, dx\,dy$$

$$= \iint 12x^2y + (-3y+2)(1-x-y)\, dx\,dy$$

$$= \int_{x=0}^{1}\int_{y=0}^{1-x}\left[12x^2y - 3y + 3xy + 3y^2 + 2 - 2x - 2y\right] dx\,dy$$

$$= \int_{x=0}^{1}\left[12x^2\frac{y^2}{2} - 3\frac{y^2}{2} + 3x\frac{y^2}{2} + y^3 + 2y - 2xy - y^2\right]_{y=0}^{1-x} dx$$

$$= \int_{x=0}^{1}\left[12\frac{x^2y^2}{2} - \frac{5}{2}y^2 + 3\frac{xy^2}{2} + y^3 + 2y - 2xy\right]_{y=0}^{1-x}$$

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$$= \int_{x=0}^{1} \left[6x^2(1-x)^2 - \frac{5}{2}(1-x)^2 + \frac{3x}{2}(1-x)^2 + (1-x)^3 + 2(1-x) - 2x(1-x) \right] dx$$

$$= \int_{x=0}^{1} \left[\left(6x^2 + \frac{3x}{2} - \frac{5}{2}\right)(1-x)^2 + 1 - 3x + 3x^2 - x^3 + 2 - 2x - 2x + 2x^2 \right] dx$$

$$= \int_{x=0}^{1} \left[\left(6x^2 + \frac{3}{2}x - \frac{5}{2}\right)(1-2x+x^2) + 3 - 7x + 5x^2 - x^3 \right] dx$$

$$= \int_{x=0}^{1} \left[6x^2 + \frac{3}{2}x - \frac{5}{2} - 12x^3 - 3x^2 + 5x + 6x^4 + \frac{3}{2}x^3 - \frac{5}{2}x^2 + 3 - 7x + 5x^2 - x^3 \right] dx$$

$$= \int_{x=0}^{1} \left[-13x^3 + 6x^4 + \left(6x^2 - 3x^2 - \frac{5}{2}x^2 + 5x^2\right) + \left(\frac{3}{2}x + 5x - 7x\right) + \left(-\frac{5}{2} + 3\right) \right] dx$$

$+\frac{3}{2}x^3$ $\quad 8 - \frac{5}{2} = \frac{11}{2}$ $\quad \frac{16}{2}$ $\quad \frac{3}{2} - 2$

$$= \int_{x=0}^{1} \left[-\frac{23}{2}x^3 + 6x^4 + \frac{11}{2}x^2 - \frac{1}{2}x + \frac{1}{2} \right] dx$$

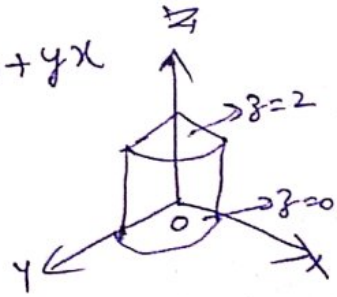
$$= \frac{1}{2} \int_{x=0}^{1} \left[12x^4 - 23x^3 + 11x^2 - x + 1 \right] dx$$

$$= \frac{1}{2} \left[12\frac{x^5}{5} - 23\frac{x^4}{4} + 11\frac{x^3}{3} - \frac{x^2}{2} + x \right]_{x=0}^{1}$$

$$= \frac{1}{2} \left[\frac{12}{5} - \frac{23}{4} + \frac{11}{3} - \frac{1}{2} + 1 \right] = \frac{49}{120}$$

→ Evaluate $\iint_S \bar{F} \cdot \bar{n}\, ds$ where $\bar{F} = z\bar{i} + x\bar{j} - 3y^2 z\bar{k}$ where 'S' is the surface of the cylinder $x^2 + y^2 = 1$ in the first octant between $z = 0$ and $z = 2$

$\bar{F} \cdot \bar{n} = zx + yx$



$\phi = x^2 + y^2 - 1 = 0$

$\nabla\phi = \bar{i}(2x) + \bar{j}(2y) + \bar{k}(0)$

$= 2x\bar{i} + 2y\bar{j}$

$$\bar{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2x\bar{i} + 2y\bar{j}}{\sqrt{4x^2 + 4y^2}} = \frac{x\bar{i} + y\bar{j}}{1}$$

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yz plane → project it
'R_1'

$$\iint_S \bar{F}\cdot\bar{n}\,ds = \iint_R \bar{F}\cdot\bar{n}\,\frac{dy\,dz}{|\bar{n}\cdot\bar{i}|}$$

$$= \iint_R x(y+z)\,\frac{dy\,dz}{x}$$

$$= \iint (y+z)\,dy\,dz$$

$x^2+y^2=1$ in yz plane
$x=0$

$y^2=1$
$y=1$

$$= \int_{y=0}^{1}\int_{z=0}^{2} (y+z)\,dy\,dz$$

$$= \int_{y=0}^{1} \left[yz+\frac{z^2}{2}\right]_{z=0}^{2} dy$$

$$= \int_{y=0}^{1} \left[2y+2\right] dy$$

$$= \left(y^2+2y\right)_{y=0}^{1} = 1+2=3$$

zx-plane

$$\iint_S \bar{F}\cdot\bar{n}\,ds = \iint_R \bar{F}\cdot\bar{n}\,\frac{dz\,dx}{|\bar{n}\cdot\bar{j}|}$$

$$= \iint x(y+z)\,\frac{dz\,dx}{y}$$

$$= \iint \left(x+\frac{xz}{y}\right) dz\,dx$$

$$= \iint \left(x+\frac{zx}{\sqrt{1-x^2}}\right) dz\,dx$$

$$= \int_{x=0}^{1}\int_{z=0}^{2} \left(x+\frac{zx}{\sqrt{1-x^2}}\right) dz\,dx$$

$x^2+y^2=1$ in zx-plane
$y=0$

$x=\pm 1$

$$= \cancel{\int_{z=0}^{2} x^2}$$

$$= \int_{x=0}^{1} \left[x\times 2 + \frac{x}{\sqrt{1-x^2}}\times\left(\frac{z^2}{2}\right)_{z=0}^{2}\right] dx$$

$$= \int_{x=0}^{1} \left[2x+\frac{2x}{\sqrt{1-x^2}}\right] dx$$

$$= \left[2\times\frac{x^2}{2} - 2\sqrt{1-x^2}\right]_{x=0}^{1}$$

$$= \left[x^2-2\sqrt{1-x^2}\right]_{x=0}^{1}$$

$$= 1-(0-2) = 3$$

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→ Construct the flux of water through the parabolic cycloid $y=x^2$, $0 \le x \le 2$; $0 \le z \le 3$ if the velocity vector $V = F = 3z^2\bar{i} + 6\bar{j} + 6xz\bar{k}$ m/sec.

Sol: Flux $= \iint \bar{F}\cdot\bar{n}\, ds$

$$\bar{F} = 3z^2\bar{i} + 6\bar{j} + 6xz\bar{k}$$

$$\bar{n} = \frac{\nabla\phi}{|\nabla\phi|}$$

given $\phi = y - x^2$ $[\because y = x^2]$

$$\nabla\phi = \bar{i}(-2x) + \bar{j}(1)$$

$$= -2x\bar{i} + \bar{j}$$

$$\bar{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{-2x\bar{i} + \bar{j}}{\sqrt{4x^2+1}}$$

Let R be the projection of the surface $y=x^2$ bounded by $0 \le x \le 2$; $0 \le z \le 3$ in the xz plane.

$$ds = \frac{dx\,dz}{|\bar{n}\cdot\bar{j}|} = \frac{dx\,dz}{\frac{1}{\sqrt{4x^2+1}}}$$

$\because 0 \le x \le 2$; $0 \le z \le 3$

$$\iint \bar{F}\cdot\bar{n}\, ds = \int_{x=0}^{2}\int_{z=0}^{3} (3z^2\bar{i} + 6\bar{j} + 6xz\bar{k})\cdot\frac{(-2x\bar{i}+\bar{j})}{\sqrt{4x^2+1}}\cdot\frac{dx\,dz}{\frac{1}{\sqrt{4x^2+1}}}$$

$$= \int_{x=0}^{2}\int_{z=0}^{3}\left[(-2x)(3z^2) + 6\right] dz\, dx$$

$$= \int_{x=0}^{2}\left[-\frac{6xz^3}{3} + 6z\right]_{z=0}^{3} dx = \int_{x=0}^{2}(-54x + 18)\,dx$$

$$= \left[-54\frac{x^2}{2} + 18x\right]_{x=0}^{2} = -72$$

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VOLUME INTEGRALS

Let V be the volume bounded by a surface $\bar{r} = \bar{f}(u,v)$. Let $\bar{F}(\bar{r})$ be a vector point fn defined over V. Divide V into m sub-regions of volumes $\delta V_1, \delta V_2, \ldots \delta V_i, \ldots, \delta V_m$.

Let $P_i(\bar{r}_i)$ be a point in δV_i. Then form the sum $I_m = \sum_{i=1}^{m} \bar{F}(\bar{r}_i)\,\delta V_i$. Let $m \to \infty$ in such a way that δV_i shrinks to a point. The limit of I_m if it exists, is called the volume integral of $\bar{F}(\bar{r})$ in the region V is denoted by $\int_V \bar{F}(\bar{r})\,dv$ or $\int_V \bar{F}\,dv$.

Cartesian form: Let $\bar{F}(r) = F_1\bar{i} + F_2\bar{j} + F_3\bar{k}$ where F_1, F_2, F_3 are functions of x, y, z. We know that $dv = dx\,dy\,dz$. The volume integral is given by

$$\int_V \bar{F}\,dv = \iiint (F_1\bar{i} + F_2\bar{j} + F_3\bar{k})\,dx\,dy\,dz$$

$$= \bar{i}\iiint F_1\,dx\,dy\,dz + \bar{j}\iiint F_2\,dx\,dy\,dz + \bar{k}\iiint F_3\,dx\,dy\,dz.$$

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Problems on Volume integrals

1) If $\bar{F} = 2xz\,\bar{i} - x\bar{j} + y^2\bar{k}$, evaluate $\int_V \bar{F}\,dv$ where V is the region bounded by the surfaces $x=0$, $x=2$, $y=0$, $y=6$, $z=x^2$ and $z=4$.

$$\bar{F} = 2xz\,\bar{i} - x\bar{j} + y^2\bar{k}$$

Volume integral is

$$\int_V \bar{F}\,dv = \iiint \left(2xz\,\bar{i} - x\bar{j} + y^2\bar{k}\right) dx\,dy\,dz.$$

$$= \bar{i}\int_{x=0}^{2}\int_{y=0}^{6}\int_{z=x^2}^{4} (2xz)\,dx\,dy\,dz$$

$$- \bar{j}\int_{x=0}^{2}\int_{y=0}^{6}\int_{z=x^2}^{4} (x)\,dx\,dy\,dz$$

$$+ \bar{k}\int_{x}\int_{y}\int_{z} (y^2)\,dx\,dy\,dz$$

$$= \bar{i}\int_{x=0}^{2}\int_{y=0}^{6} \left[xz^2\right]_{z=x^2}^{4} dx\,dy$$

$$- \bar{j}\int_{x=0}^{2}\int_{y=0}^{6} \left[xz\right]_{z=x^2}^{4}$$

$$+ \bar{k}\int_{x=0}^{2}\int_{y=0}^{6} \left[y^2 z\right]_{z=x^2}^{4} dx\,dy$$

$$= \bar{i}\int_{x=0}^{2}\int_{y=0}^{6} x(16-x^4)\,dx\,dy - \bar{j}\int_{x=0}^{2}\int_{y=0}^{6} x(4-x^2).$$

$$+ \bar{k}\int_{x=0}^{2}\int_{y=0}^{6} y^2(4-x^2)\,dx\,dy$$

$$= \bar{i}\int_{x=0}^{2} x(16-x^4)(6)\,dx - \bar{j}\int_{x=0}^{2} (4x-x^3)(6)$$

$$+ \bar{k}\int_{x=0}^{2} (4-x^2)\left[\frac{y^3}{3}\right]_0^6 dx \qquad \frac{6\times6\times6^2}{3}$$

$$= \bar{i}\int_{x=0}^{2} (96x - 6x^5)\,dx - \bar{j}\int_{x=0}^{2} (24x - 6x^3)\,dx$$

$$+ \bar{k}\int_{x=0}^{2} (288 - 72x^2)\,dx$$

$$= 96\times\frac{x^2}{2} - \frac{x^6}{6}$$

$\frac{36}{2}$ $\frac{72}{288}$

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$$= \left[48x^2 - x^6\right]_{x=0}^{2} - \bar{j}\left[12x^2 - \frac{6x^4}{4}\right]_{x=0}^{2} + \left[288x - \frac{72x^4}{3}\right]_{x=0}^{2} \bar{k}$$

$$= (48\times 4 - 64)\bar{i} - \bar{j}\left[12\times 4 - 6\times\frac{2\times2\times2\times2}{4}\right] + \left[288\times 2 - 24\times 8\right]\bar{k}$$

$$= \cancel{192-64}\ 128\bar{i} - 24\bar{j} + 384\bar{k}$$

(2) If $\bar{F} = (2x^2 - 3z)\bar{i} - 2xy\bar{j} - 4x\bar{k}$ then evaluate (i) $\int_V \nabla\cdot\bar{F}\, dv$ and (ii) $\int_V (\nabla\times\bar{F})\, dv$ where V is the closed region bounded by $x=0, y=0, z=0, 2x+2y+z=4$

$$\frac{\partial\bar{F}}{\partial x} = (4x)\bar{i} - 2y\bar{j} - 4\bar{k}$$

$$\frac{\partial\bar{F}}{\partial y} = -2x\bar{j}.$$

$$\frac{\partial\bar{F}}{\partial z} = -3\bar{i}.$$

Sol: (i) $\nabla\cdot\bar{F}$

$$= \bar{i}\cdot\frac{\partial\bar{F}}{\partial x} + \bar{j}\cdot\frac{\partial\bar{F}}{\partial y} + \bar{k}\cdot\frac{\partial\bar{F}}{\partial z}$$

$$= \cancel{\bar{i}\cdot(4x}$$

$$= 4x - 2x = 2x$$

Limits are $z=0$ to $z = 4-2x-2y$

$y=0$ to $\frac{4-2x}{2}$ i.e $2-x$

$x=0$ to $\cancel{x}=2$

$$\therefore \int \nabla\cdot\bar{F}\, dv = \int_{x=0}^{2}\int_{y=0}^{2-x}\int_{z=0}^{4-2x-2y} 2x\, dx\, dy\, dz$$

$$= \int_{x=0}^{2}\int_{y=0}^{2-x} 2x(4-2x-2y)\, dx\, dy$$

$2x(2)(2-x-y)$

$(2x-x^2-xy)$

$$= 4\int_{x=0}^{2}\int_{y=0}^{2-x}\left(2x - x^2 - xy\right) dx\, dy.$$

$$= 4\int_{x=0}^{2}\left(2xy - x^2y - \frac{xy^2}{2}\right)_{y=0}^{2-x}$$

$$= 4\int_{x=0}^{2} 2x(2-x) - x^2(2-x) - \frac{x}{2}(2-x)^2$$

$$= 4\int_{x=0}^{2}\frac{(2-x)}{2}\left[4x - 2x^2 - (2x - x^2)\right] dx$$

$$= 4\int_{x=0}^{2}\frac{2-x}{2}\left[4x - 2x^2 - 2x + x^2\right] dx$$

$$= 4\int_{x=0}^{2}\frac{2-x}{2}\left[2x - x^2\right] dx$$

$$= 2\int_{x=0}^{2}(2-x)(2x - x^2)$$

$$= 2\int_{x=0}^{2}\left[4x - 2x^2 - 2x^2 + x^3\right] dx$$

$$= 2\int_{x=0}^{2}\left[x^3 - 4x^2 + 4x\right] dx$$

$$= 2\left[\frac{x^4}{4} - \frac{4x^3}{3} + 2x^2\right]_{x=0}^{2}$$

$$= \left[\frac{x^4}{2} - \frac{8x^3}{3} + 4x^2\right]_{x=0}^{2}$$

$$= \frac{2\times2\times2\times2}{2} - \frac{8}{3}\times2\times2\times2 + 4\times2\times2$$

$$= 8 - \frac{64}{3} + 16 = \frac{24-64+48}{3}$$

$$\frac{72-64}{3} = \frac{8}{3}$$

$$2\left[4 - \frac{4\times8}{3} + 8\right]$$

$$2\times\left(\frac{12-32+24}{3}\right)$$

$$2\times\frac{4}{3} = \frac{8}{3}$$

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Volume integrals problems

$\nabla \times \bar{F} = \text{curl } \bar{F}$

$$= \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x^2-3z & -2xy & -4x \end{vmatrix}$$

$$= \bar{i}\left[0-0\right] + \bar{j}\left[-3+4\right] + \bar{k}\left[-2y-0\right]$$

$$= \bar{j} - 2y\bar{k}$$

$$\int_V (\nabla \times \bar{F})\, dv$$

$$= \iiint \bar{j}\, dx\,dy\,dz - \bar{k}\iiint 2y\, dx\,dy\,dz$$

$$= \bar{j}\int_{x=0}^{2}\int_{y=0}^{2-x}\int_{z=0}^{4-2x-2y} 1\, dx\,dy\,dz$$

$$= \bar{j}\int_{x=0}^{2}\int_{y=0}^{2-x}\left[z\right]_0^{4-2x-2y} dx\,dy$$

$$= \bar{j}\int_{x=0}^{2}\int_{y=0}^{2}(4-2x-2y)\, dx\,dy$$

$$= \bar{j}\int_{x=0}^{2}\left[4y-2xy-y^2\right]_0^{2-x} dx$$

$$= \bar{j}\int_{x=0}^{2}\left[8-4x-4\right] dx$$

$$= \bar{j}\int_{x=0}^{2}(4-4x)\, dx$$

$$= \bar{j}\left[4x-2x^2\right]_0^2$$

$$= \bar{j}\left[8-8\right]$$

$$= 0$$

$$= \bar{j}\int_{x=0}^{2}\left[4(2-x) - 2x(2-x) - (2-x)^2\right] dx$$

$$= \bar{j}\int_{x=0}^{2}\left[8-4x-4x+2x^2 - (4+x^2-4x)\right] dx$$

$$= \bar{j}\int_{x=0}^{2}\left[8-8x+2x^2-4-x^2+4x\right] dx$$

$$= \bar{j}\int_{x=0}^{2}\left[x^2-4x+4\right] dx$$

$$= \bar{j}\left[\frac{x^3}{3}-2x^2+4x\right]_0^2$$

$$= \frac{8}{3} - 8 + 8 = \frac{8}{3}\bar{j}$$

$$-2\bar{k}\iiint y\, dx\,dy\,dz$$

$$\int_{x=0}^{2}\int_{y=0}^{2-x}\int_{z=0}^{4-2x-2y} y\, dx\,dy\,dz$$

$$\int_{x=0}^{2}\int_{y=0}^{2-x} y(4-2x-2y)\, dx\,dy$$

$$\int_{x=0}^{2}\int_{y=0}^{2-x}(4y-2xy-2y^2)\, dx\,dy$$

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$$= \int_{x=0}^{2} \left[2y^2 - xy^2 - \frac{2y^3}{3} \right]_0^{2-x} dx$$

$$= \int_{x=0}^{2} \left[2(2-x)^2 - x(2-x)^2 - \frac{2}{3}(2-x)^3 \right] dx$$

$$= \int_{x=0}^{2} \left[(2-x)^3 - \frac{2}{3}(2-x)^3 \right] dx$$

$$= \frac{(2-x)^3}{-3} - \frac{2}{3}\frac{(2-x)^4}{(-4)}$$

$$= \left[\frac{1}{6}(2-x)^4 - \frac{(2-x)^3}{3} \right]_0^2$$

$$= \frac{2^4}{6} - 0 - \left[\frac{2^4}{6} - \frac{2^3}{3} \right]$$

$$= \frac{2^3}{3} - \frac{2^4}{6}$$

$$= \frac{8}{3} - \int_{x=0}^{2} \frac{(2-x)^3}{3} dx$$

$$= \frac{1}{3} \times \frac{(2-x)^4}{-4}$$

$$= \frac{-1}{12} \left[(2-x)^4 \right]_0^2$$

$$= \frac{-1}{12} \left[0 - 16 \right]$$

$$= \frac{\cancel{16}^{4}}{\cancel{12}_{3}} = \frac{4}{3}$$

$$-\frac{8}{3}\bar{k}$$

Ans $\frac{8}{3}(\bar{j} - \bar{k})$

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Green's theorem in a plane (Transformation between line integral and double integral)

If R is a closed region in xy plane bounded by a simple closed curve C and if M and N are continuous functions of x and y having continuous derivatives in R, then (1)

$$\boxed{\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy}$$

where C is traversed in the positive (anti-clockwise) direction

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Green's identities

If f and g are two continuous and differentiable scalar point functions over the region enclosed by the surface s, then

(i) $\int_V [f\nabla^2 g + \nabla f \cdot \nabla g]\, dv = \int_S (f\nabla g)\cdot \bar{n}\, ds$

(ii) $\int_V (f\nabla^2 g - g\nabla^2 f)\, dv = \int_S (f\nabla g - g\nabla f)\cdot \bar{n}\, ds$

Proof: Take $\bar{F} = f\nabla g$

Gauss divergence theorem states

$$\int_V \operatorname{div} \bar{F}\, dv = \int_S \bar{F}\cdot\bar{n}\, ds$$

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$$\Rightarrow \int_V \operatorname{div}(f\nabla g)\, dv = \int_S (f\nabla g)\cdot\bar{n}\, ds$$

$$\Rightarrow \int_V (f\nabla^2 g + \nabla f\cdot\nabla g)\, dv = \int (f\nabla g)\cdot\bar{n}\, ds \quad \text{——(1)}$$

This is called Green's first identity or theorem.

(ii) Interchanging f & g in (1), we get

$$\int_V (g\nabla^2 f + \nabla g\cdot\nabla f)\, dv = \int (g\nabla f)\cdot\bar{n}\, ds \quad \text{——(2)}$$

(1) − (2) gives,

$$\int_V (f\nabla^2 g - g\nabla^2 f)\, dv = \int [f\nabla g - g\nabla f]\cdot\bar{n}\, ds$$

This is called Green's second identity (or) Green's theorem.

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NOTE: ① Green's theorem converts a line integral around a closed curve into a double integral and is a special case of Stoke's theorem.

② Green's theorem in vector notation.

$\bar{F} = M\bar{i} + N\bar{j}$ and $\bar{r} = x\bar{i} + y\bar{j}$. Then

$$\bar{F}\cdot d\bar{r} = M\,dx + N\,dy.$$

$$\text{curl}\,\bar{F} = \nabla\times\bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & 0 \end{vmatrix} = \bar{i}(0) - \bar{j}(0-0) + \bar{k}\left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right]$$

By Stoke's theorem,

$$\oint_C \bar{F}\cdot d\bar{r} = \int_S \text{curl}\,\bar{F}\cdot\bar{n}\,ds \quad , \text{where } \bar{n} = \bar{k} \text{ for XY-plane.}$$

$$= \iint_R (\text{curl}\,\bar{F}\cdot\bar{n})\,dR \qquad dR = dx\,dy.$$

$$= \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy.$$

③ Area of the plane region 'R' bounded by a simple closed curve 'C'.

Let $N = x$, $M = -y$

Then $\oint_C x\,dy - y\,dx = \iint_R [1-(-1)]\,dx\,dy$

$$= 2\iint dx\,dy$$

$$\boxed{\frac{1}{2}\oint_C x\,dy - y\,dx = \int_A 1\,dA = A}$$

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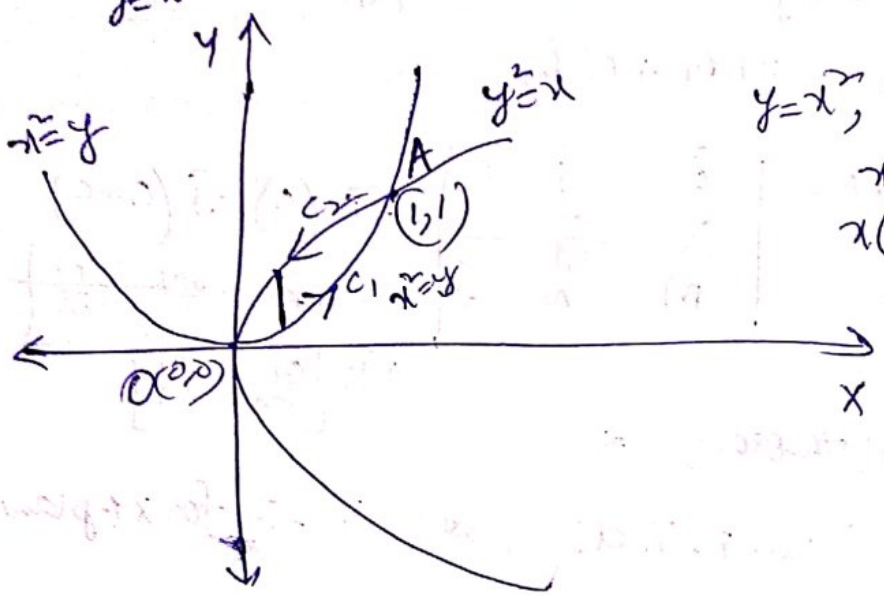
(1) Verify Green's theorem in plane for $\oint (3x^2-8y^2)\,dx + (4y-6xy)\,dy$ where 'C' is the region bounded by $y=\sqrt{x}$ and $y^2=x$

$y=x^2$



$y=x^2,\ y^2=x$

$x^4=x$

$x(x^3-1)=0$

$x=0$

$x=1$

(0,0)

(1,1)

$M=3x^2-8y^2, \qquad N=4y-6xy.$

$$\oint_C M\,dx+N\,dy = \iint_R \left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dx\,dy$$

$$R.H.S = \iint \left(-6y-(-16y)\right)dx\,dy$$

$$= \iint 10y\,dx\,dy$$

$$= 10\int_{x=0}^{1}\int_{y=x^2}^{\sqrt{x}} y\,dx\,dy$$

$$= 10\int_{x=0}^{1}\left[\frac{y^2}{2}\right]_{y=x^2}^{\sqrt{x}} dx$$

$$= 10\int_{x=0}^{1}\left[\frac{x}{2}-\frac{x^4}{2}\right]dx$$

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$$= \frac{10}{2} \int_{x=0}^{1} \left[x - x^4 \right] dx$$

$$= 5 \left[\frac{x^2}{2} - \frac{x^5}{5} \right]_{x=0}^{1}$$

$$= 5 \left[\frac{1}{2} - \frac{1}{5} \right] = \cancel{5} \left[\frac{5-2}{\cancel{10}_{2}} \right] = \frac{3}{2}$$

<u>LHS</u> $\oint_C (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy$

Line integral along 'C'.

= Line integral along OA + Line integral along AO.

Line integral along OA: $(0,0)$ to $(1,1)$.

along $x^2 = y$

$dy = 2x\,dx$

$$\oint_{OA} \left\{ \left[3x^2 - 8x^4 \right] dx + \left[4x^2 - 6x \cdot x^2 \right] \times 2x\,dx \right.$$

$$= \int_{x=0}^{1} \left[3x^2 - 8x^4 + 8x^3 - 12x^4 \right] dx$$

$$= \int_{x=0}^{1} \left[3x^2 + 8x^3 - 20x^4 \right] dx$$

$$= \left[\frac{3x^3}{3} + \frac{8x^4}{4} - 20\frac{x^5}{5} \right]_{x=0}^{1}$$

$$= 1 + 2 - 4 = -1$$

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Line integral along AO - i.e along $y^2 = x$

$y = \sqrt{x}$

$$\int_{AO} (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy$$

$$= \int (3x^2 - 8x)\,dx + (4\sqrt{x} - 6x\sqrt{x}) \times \frac{1}{2\sqrt{x}}\,dx$$

$$= \int \left[3x^2 - 8x + 2 - 3x\right] dx$$

$$= \int_{x=1}^{0} \left[3x^2 - 11x + 2\right] dx$$

$$= \left[\frac{3x^3}{3} - \frac{11x^2}{2} + 2x\right]_{x=1}^{0}$$

$$= 0 - \left[1 - \frac{11}{2} + 2\right] = -\left[3 - \frac{11}{2}\right]$$

$$= \frac{11}{2} - 3 = \frac{5}{2}$$

$$-1 + \frac{5}{2} = \frac{3}{2}$$

L.H.S = R.H.S

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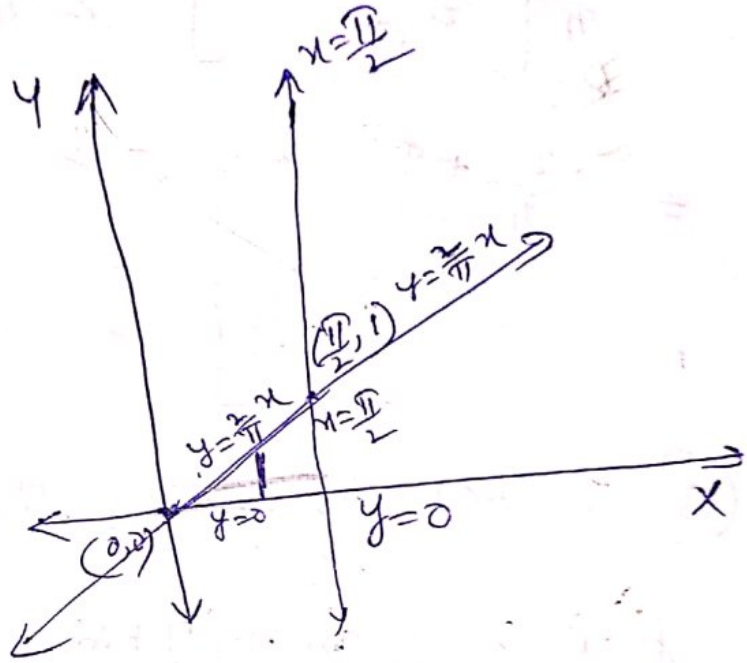
② Evaluate by Green's theorem

$\oint_C (y - \sin x)\,dx + \cos x\,dy$ where 'C' is the triangle enclosed by the lines $y=0$, $x=\frac{\pi}{2}$,

$\pi y = 2x$

$y = \frac{2}{\pi}x$



$y=\frac{2}{\pi}x$, $x=\frac{\pi}{2}$

$y=\frac{2}{\pi}\times\frac{\pi}{2}$

$y=1$, $x=\frac{\pi}{2}$

$$\oint_C \underbrace{(y-\sin x)}_{M}\,dx + \underbrace{\cos x}_{N}\,dy$$

$$\oint_C M\,dx + N\,dy = \iint_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$$= \iint \left[-\sin x - 1\right] dx\,dy$$

$$= -\int_{x=0}^{\frac{\pi}{2}} \int_{y=0}^{\frac{2}{\pi}x} (\sin x + 1)\,dy\,dx$$

$$= -\int_{x=0}^{\frac{\pi}{2}} (\sin x + 1)\,\frac{2x}{\pi}\,dx$$

$$= -\frac{2}{\pi}\left[\int_{x=0}^{\frac{\pi}{2}} x\sin x\,dx + \int_{x=0}^{\frac{\pi}{2}} x\,dx\right]$$

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$$= \frac{-2}{\pi}\left[x(-\cos x) - (1)(-\sin x) \right]_{x=0}^{\frac{\pi}{2}} + \left[\frac{x^2}{2}\right]_0^{\frac{\pi}{2}}$$

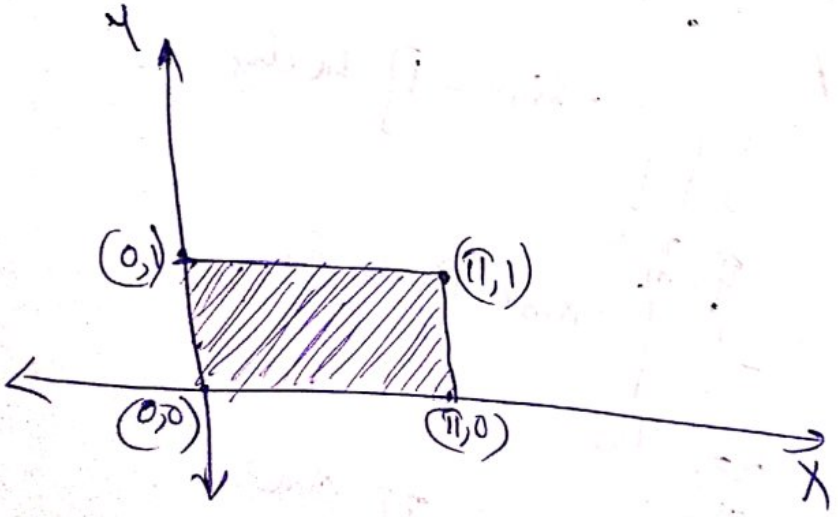
$$= \frac{-2}{\pi}\left[\cancel{\frac{\pi}{2} \times (0)}^{\,0} + 1 \right] + \frac{1}{\cancel{2}} \times \frac{\pi^2}{4} \times \frac{-\cancel{2}}{\pi}$$

$$= \cancel{\frac{-2}{\pi} + \frac{\pi^2}{8}} \quad \frac{-2}{\pi} - \frac{\pi}{4}$$

$$= -\left[\frac{\pi}{4} + \frac{2}{\pi}\right]$$

(3) Evaluate by Green's theorem

$\oint_C (x^2 - \cosh y)\,dx + (y + \sin x)\,dy$ where C is the rectangle with vertices $(0,0)$, $(\pi,0)$ $(\pi,1)$ $(0,1)$



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$$\oint_C \underbrace{(x^2-\cosh y)}_{M}dx + \underbrace{(y+\sin x)}_{N}dy$$

$$= \iint \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$$= \int_{x=0}^{\pi}\int_{y=0}^{1} \left[\cos x + \sinh y\right] dx\,dy$$

$$= \int_{x=0}^{\pi}\int_{y=0}^{1} \cos x\, dx\,dy + \int_{x=0}^{\pi}\int_{y=0}^{1} \sinh y\, dx\,dy$$

$$= \int_{x=0}^{\pi} \cos x \times 1\, dx + \int_{x=0}^{\pi} \left[\cosh y\right]_{y=0}^{1} dx \qquad \frac{e^0+e^0}{2}$$

$$= \left[\sin x\right]_{x=0}^{\pi} + \left[\cosh(1) - \cosh(0)\right]\pi$$

$$= 0 + \pi\left[\cosh(1) - 1\right]$$

$$= \pi\left[\cosh(1) - 1\right]$$

(4) A vector field is given by

$$\bar{F} = (\sin y)\bar{i} + x(1+\cos y)\bar{j}$$

Evaluate the line integral over the circular path $x^2+y^2=a^2$, $z=0$.

(i) directly (ii) By using Green's theorem.

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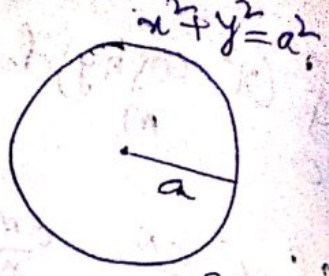
(i) Directly



$z=0$

$\Rightarrow$ In XY-plane

Line integral

$$= \oint_C F \cdot dr$$

$$= \oint_C \left[\sin y \, dx + x(1+\cos y)\, dy \right]$$

$$= \oint_C \left[\sin y \, dx + x \cos y \, dy \right] + x\, dy$$

$$= \oint d(x \sin y) + x\, dy \quad \cancel{= \oint \left[\sin(a\cos\theta)(-a\sin\theta) + a\cos\theta \cos(a\sin\theta) + a\cos\theta \, a\cos\theta \right.}$$

Given circle is $x^2+y^2=a^2$

Take $x = a\cos\theta$, $y = a\sin\theta$

$dx = -a\sin\theta \, d\theta$, $dy = a\cos\theta \, d\theta$

$\theta \to 0$ to 2π

$$= \int_{\theta=0}^{2\pi} d\left[a\cos\theta \sin(a\sin\theta) \right] + \int_{\theta=0}^{2\pi} a\cos\theta \times a\cos\theta \, d\theta$$

$$= \left[a\cos\theta \sin(a\sin\theta) \right]_{\theta=0}^{2\pi} + a^2 \int_{\theta=0}^{2\pi} \cos^2\theta \, d\theta$$

$$= \left[a\cos(2\pi) \sin(a\sin 2\pi) \right] - \left[a\cos 0 \times 0 \right]$$

$$= (0-0) + a^2 \int_{\theta=0}^{2\pi} \left(\frac{1+\cos 2\theta}{2} \right) d\theta$$

$$= \frac{a^2}{2} \left[\theta + \frac{\sin 2\theta}{2} \right]_{\theta=0}^{2\pi}$$

$$= \frac{a^2}{2} \left[(2\pi + 0) \right] = a^2\pi$$

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(ii) By using Green's theorem,

$$\oint_C M\,dx + N\,dy = \iint_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$$= \iint_S \left[1 + \cancel{\cos y} - \cancel{\cos y}\right] dx\,dy$$

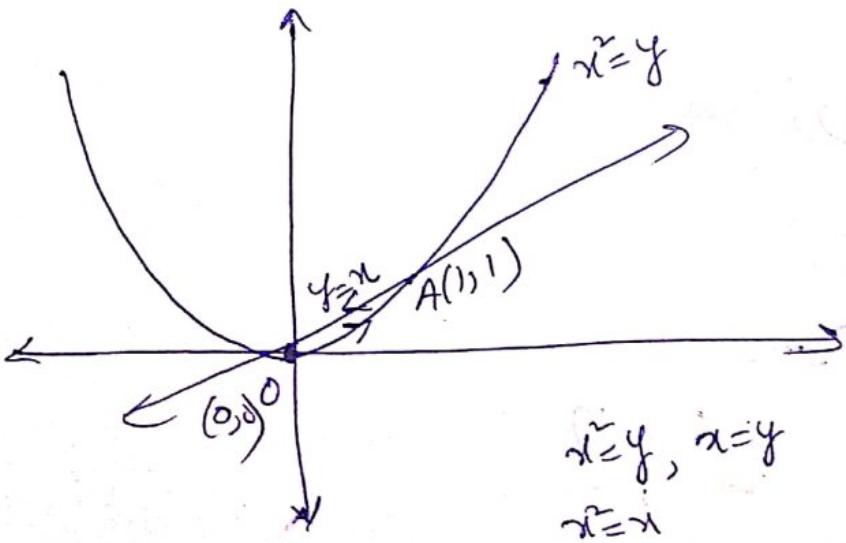
$$= \iint 1\,dx\,dy$$

$$= \int_{r=0}^{a}\int_{\theta=0}^{2\pi} 1\; r\,dr\,d\theta$$

$$= \int_{r=0}^{a} r\,(2\pi)\,dr$$

$$= 2\pi\left[\frac{r^2}{2}\right]_{r=0}^{a} = \cancel{2}\pi \times \frac{a^2}{\cancel{2}} = a^2\pi$$

(5) Verify Green's theorem for $\int_C \left[(xy+y^2)\,dx + x^2\,dy\right]$ where 'C' is bounded by $y=x$ and $y=x^2$.



$x^2=y,\ x=y$

$x^2=x$

$x^2-x=0$

$x(x-1)=0$

$x=0,1$

$y=0,1$

$(0,0)\ (1,1)$

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$$\oint_C (xy+y^2)\,dx + x^2\,dy =$$

$$\oint_C M\,dx + N\,dy = \iint_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

L.H.S $\oint_C M\,dx + N\,dy$

$$= \int_{OA} (M\,dx + N\,dy) + \int_{AO} (M\,dx + N\,dy)$$

<u>Along OA</u> <u>Along</u> <u>$y = x^2$</u> $dy = 2x\,dx$

$$= \int (xy+y^2)\,dx + x^2\,dy$$

$$= \int (x^3 + x^4)\,dx + x^2 \times 2x\,dx$$

$$= \int \left[x^3 + x^4 + 2x^3\right] dx$$

$$= \int_{x=0}^{1} \left[x^4 + 3x^3\right] dx$$

$$= \left[\frac{x^5}{5} + \frac{3x^4}{4}\right]_{x=0}^{1}$$

$$= \frac{1}{5} + \frac{3}{4} = \frac{4+15}{20} = \frac{19}{20}$$

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Along AO Along $y = x$

$dy = dx$

$$\int_{AO} (xy + y^2)\,dx + x^2\,dy$$

$$= \int (x^2 + x^2)\,dx + x^2\,dx$$

$$= \int_{x=1}^{0} (3x^2)\,dx = \left[x^3\right]_{x=1}^{0} = -1$$

$$-1 + \frac{19}{20} = \frac{-1}{20}$$

R.H.S

$$\iint_S \left[\frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial y}(xy + y^2)\right] dx\,dy$$

$$= \iint (2x - x - 2y)\,dx\,dy$$

$$= \int_{x=0}^{1} \int_{y=x^2}^{x} (x - 2y)\,dy\,dx$$

$$= \int_{x=0}^{1} \left[xy - y^2\right]_{y=x^2}^{x} dx$$

$$= \int_{x=0}^{1} \left[(x^2 - x^2) - (x^3 - x^4)\right] dx$$

$$= \int_{x=0}^{1} (x^4 - x^3)$$

$$= \left[\frac{x^5}{5} - \frac{x^4}{4}\right]_{x=0}^{1}$$

$$= \frac{1}{5} - \frac{1}{4} = \frac{4-5}{20}$$

$$= \frac{-1}{20}$$

$$= \int_{x=0}^{1} \int_{y=x^2}^{x} x\,dx\,dy$$

$$= \int_{x=0}^{1} x\left[x - x^2\right] dx$$

$$= \int_{x=0}^{1} \left[x^2 - x^3\right] dx$$

$$= \frac{x^3}{3} - \frac{x^4}{4} = \frac{1}{3} - \frac{1}{4} = \frac{4-3}{12} = \frac{1}{12}$$

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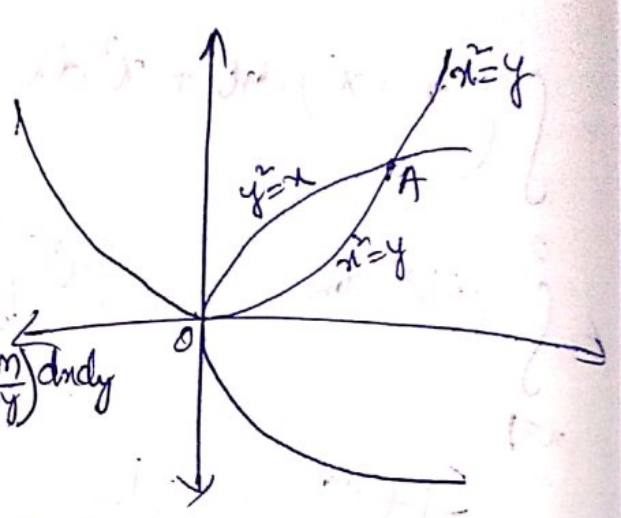
(6) Using Green's theorem evaluate $\int_C (2xy - x^2)\,dx + (x^2+y^2)\,dy$ where 'c' is the closed curve of the region bounded by $y = x^2$ and $y^2 = x$



$\cancel{\oint_C \bar{F}\,d\bar{r} = \iint_S}$

$$\oint_C M\,dx + N\,dy = \iint_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

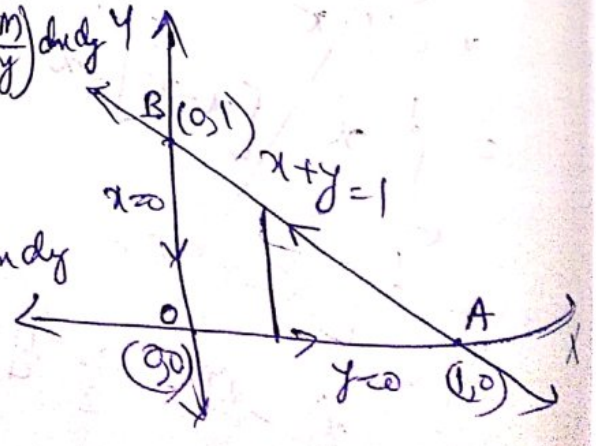
$$= \iint_S \left[2x - 2x\right] dx\,dy$$

$$= 0$$

(7) Verify Green's theorem for

$\int_C \left[(3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy\right]$ where 'c' is the region bounded by $x=0$, $y=0$, $x+y=1$

$$\oint_C M\,dx + N\,dy = \iint \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$



R.H.S

$$= \iint \left[-6y - (-16y)\right] dx\,dy$$

$$= 10 \int_{x=0}^{1} \int_{y=0}^{1-x} y\,dy\,dx$$

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$$= 10 \int_{x=0}^{1} \left[\frac{y^2}{2}\right]_{y=0}^{1-x} dx$$

$$= \frac{10}{2} \int_{x=0}^{1} (1-x)^2 \, dx$$

$$= 5 \left[\frac{(1-x)^3}{-3}\right]_{x=0}^{1}$$

$$= 5\left[0 + \frac{1}{3}\right] = \frac{5}{3}$$

$$L.H.S = \oint_C M\,dx + N\,dy$$

<u>Along OA</u>, $y=0$, $dy=0$

$$= \oint_C \underbrace{\cancel{(2xy - x^2)}}_{(3x^2-8y^2)} dx + \underbrace{\cancel{(x^2+y^2)}}_{(4y-6xy)} dy$$

$$= \int_{OA} \cancel{\left(\frac{x^2}{x}\right)} dx$$

$$3x^2\,dx$$

$$= -\int_{x=0}^{1} x^2 \, dx$$

$$\int_{x=0}^{1} 3x^2 \, dx = \left[x^3\right]_{x=0}^{1} = 1$$

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Along AB

$$\int_{AB} (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy)$$

$x + y = 1$

$y = 1 - x$

$dy = -dx$

$$= \int \left[3x^2 - 8(1-x)^2\right] dx + \left[4(1-x) - 6x(1-x)\right] - dx$$

$$= \int \left[3x^2 - 8\{1 + x^2 - 2x\} + \{4 - 4x - 6x + 6x^2\}(-1)\right] dx$$

$$= \int \left[3x^2 - 8 - 8x^2 + 16x - 4 + 4x + 6x - 6x^2\right] dx$$

$$= \int_{x=1}^{0} \left[-11x^2 + 26x - 12\right] dx$$

$$= \left[-11\frac{x^3}{3} + 26\frac{x^2}{2} - 12x\right]_{x=1}^{0} = -\left[-\frac{11}{3} + 13 - 12\right]$$

$$= -\left[-\frac{11}{3} + 1\right] = \frac{11}{3} - 1 = \frac{8}{3}$$

Along BO

Along $x = 0$

$dx = 0$

$$\int_{BO} (4y - 6xy)\,dy$$

$$= \int_{y=1}^{0} (4y)\,dy = 4\left[\frac{y^2}{2}\right]_{y=1}^{0}$$

$$= 4\left[0 - \frac{1}{2}\right] = -2$$

Total line integral

$$= 1 + \frac{8}{3} - 2$$

$$= \frac{8}{3} - 1 = \frac{5}{3}$$

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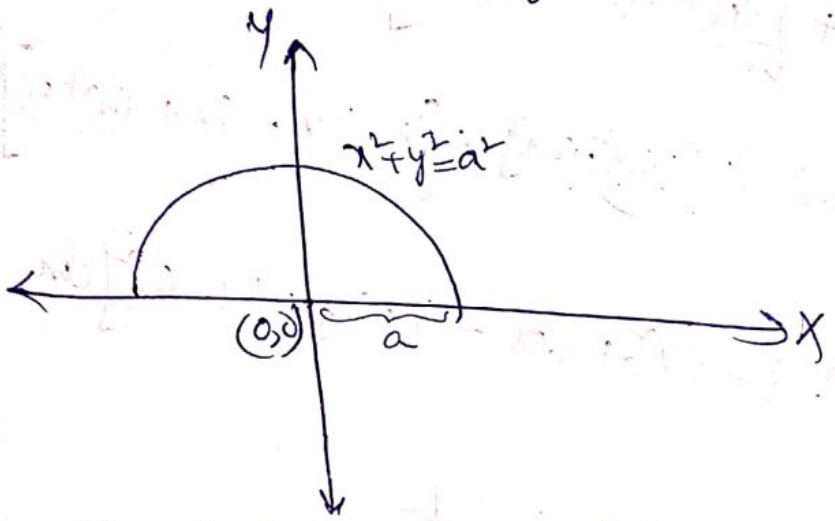
(8) Apply Green's theorem to evaluate $\oint_C (2x^2-y^2)\,dx + (x^2+y^2)\,dy$, where 'c' is the boundary of the area enclosed by the X-axis and upper half of the circle $x^2+y^2=a^2$



$M = 2x^2 - y^2$, $N = x^2 + y^2$

$\frac{\partial N}{\partial x} = 2x$, $\frac{\partial M}{\partial y} = -2y$

$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 2x + 2y = 2(x+y)$

$\oint_C M\,dx + N\,dy = \iint \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

$= 2\iint (x+y)\,dx\,dy$

$x = r\cos\theta$, $y = r\sin\theta$ $\quad \theta \to 0 \text{ to } \pi$

$= 2\int_{r=0}^{a}\int_{\theta=0}^{\pi} r(\cos\theta + \sin\theta)\, r\,dr\,d\theta$

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$$= 2\int_{r=0}^{a} r^2 \int_{\theta=0}^{\pi} (\cos\theta + \sin\theta)\, d\theta\, dr$$

$$= 2\int_{r=0}^{a} r^2 \left[\sin\theta - \cos\theta\right]_{\theta=0}^{\pi} dr$$

$$= 2\int_{r=0}^{a} r^2 \left[\{0-(-1)\} - \{0-1\}\right] dr$$

$$= 2\int_{r=0}^{a} r^2 \left[2\right] dr$$

$$= 4\left[\frac{r^3}{3}\right]_{r=0}^{a} = \frac{4a^3}{3}$$

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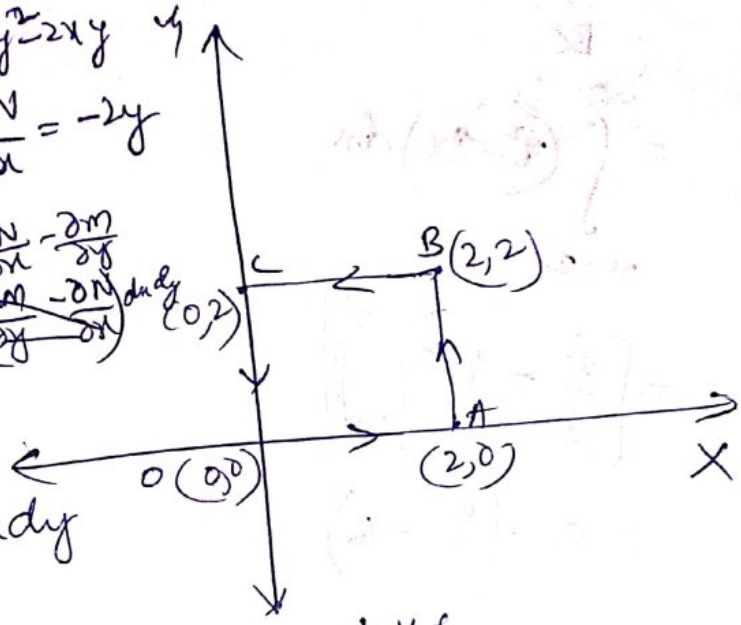
(10) Verify Green's theorem in the plane $\int_C (x^2-xy^3)dx + (y^2-2xy)dy$ where 'C' is the square with vertices $(0,0)$ $(2,0)$ $(2,2)$ $(0,2)$

$M = x^2-xy^3$, $N = y^2-2xy$

$\frac{\partial M}{\partial y} = -3xy^2$, $\frac{\partial N}{\partial x} = -2y$



$$\oint_C M\,dx + N\,dy = \iint \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

RHS

$$\iint (+3xy^2 - 2y)\,dx\,dy$$

$$= \int_{x=0}^{2}\int_{y=0}^{2} (3xy^2-2y)\,dy\,dx$$

$$= \int_{x=0}^{2} \left[xy^3 - y^2\right]_{y=0}^{2} dx$$

$$= \int_{x=0}^{2} \left[8x - 4\right] dx$$

$$= \left[4x^2 - 4x\right]_{x=0}^{2}$$

$$= 16 - 8 = 8$$

LHS

Along OA $\quad y=0$, $dy=0$

$$\int_{OA} (x^2-xy^3)\,dx$$

$$\int_{x=0}^{2} x^2\,dx = \left[\frac{x^3}{3}\right]_{x=0}^{2} = \frac{8}{3}$$

Along AB $\quad x=2$, $dx=0$

$$\int (y^2-2xy)\,dy$$

$$= \int_{y=0}^{2} (y^2-4y)\,dy$$

$$= \left[\frac{y^3}{3} - 2y^2\right]_{y=0}^{2}$$

$$= \frac{8}{3} - 8$$

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Along BC $\quad y=2$

$dy=0$

$$\int_{BC} (x^2-xy^3)\,dx$$

$$= \int_{x=2}^{0} (x^2-8x)\,dx$$

$$= \left[\frac{x^3}{3} - 4x^2\right]_{x=2}^{0}$$

$$= 0 - \left(\frac{8}{3} - 16\right)$$

$$= 16 - \frac{8}{3} = \frac{40}{3}$$

Along CO $\quad x=0$

$dx=0$

$$\int (y^2-2xy)\,dy$$

$$= \int_{y=2}^{0} y^2\,dy$$

$$= \left[\frac{y^3}{3}\right]_{y=2}^{0} = -\frac{8}{3}$$

Total L.H.S

$$= \frac{8}{3} + \cancel{\frac{8}{3}} \cancel{-8} + \frac{40}{3} - \cancel{\frac{8}{3}}$$

$$= \frac{48}{3} - 8 = \frac{48-24}{3} = \frac{24}{3}$$

$$= 8$$

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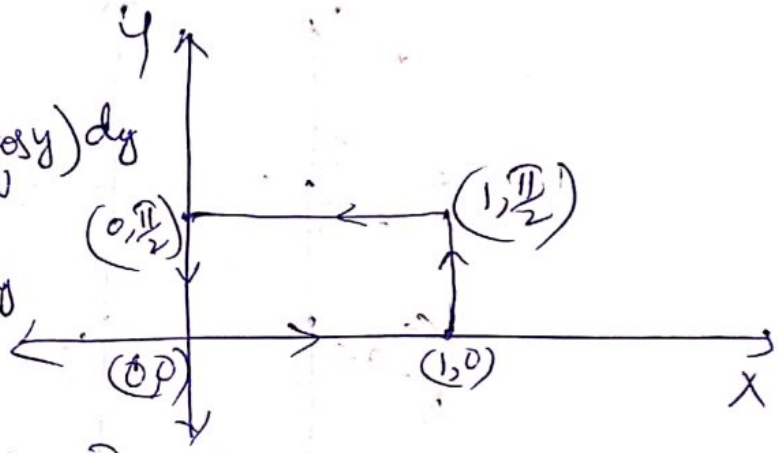
(11) Find the circulation of $\bar{F}$ round the curve C where $\bar{F} = (e^x \sin y)\,\bar{i} + (e^x \cos y)\,\bar{j}$ and 'C' is the rectangle whose vertices are $(0,0)$ $(1,0)$ $(1,\frac{\pi}{2})$, $(0,\frac{\pi}{2})$

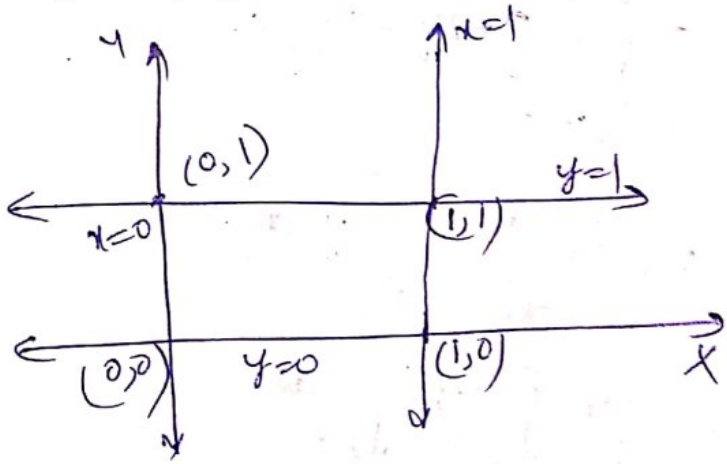


$$\oint_C \underbrace{(e^x \sin y)}_{M} dx + \underbrace{(e^x \cos y)}_{N} dy$$

$$= \iint_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$$= \iint_S \left[e^x \cos y - e^x \cos y\right] dx\,dy = 0$$

(12) Evaluate $\int_C \left[(x^2+xy)\,dx + (x^2+y^2)\,dy\right]$ where 'C' is the boundary of the region bounded by the lines $x=0$, $x=1$, $y=0$, $y=1$



$$\iint_S (2x - x)\,dx\,dy$$

$$= \int_{x=0}^{1}\int_{y=0}^{1} x\,dx\,dy$$

$$= \int_{x=0}^{1} x \times 1\,dx$$

$$= \left[\frac{x^2}{2}\right]_{x=0}^{1} = \frac{1}{2}$$

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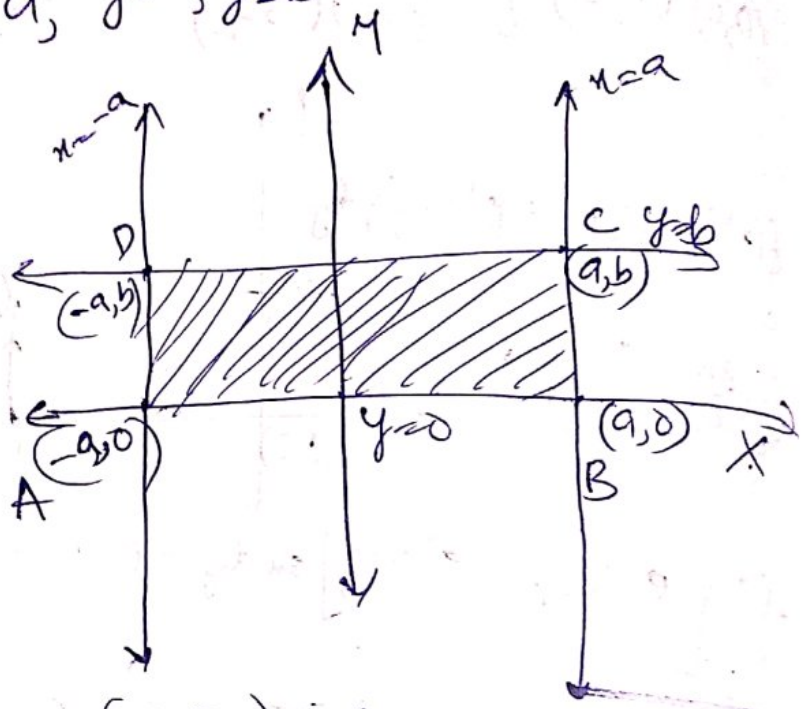
(13) Verify Green's theorem for $\bar{F} = (x^2+y^2)\,\bar{i} - 2xy\,\bar{j}$ taken around the rectangle bounded by the lines $x=a,\ x=-a,\ y=0,\ y=b$



$$\oint M\,dx + N\,dy = \iint_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

R.H.S

$$= \int_{x=-a}^{a}\int_{y=0}^{b} \left[-2y - 2y\right] dy\,dx$$

$$= -4 \int_{x=-a}^{a} \int_{y=0}^{b} y\,dy\,dx$$

$$= -4 \int_{x=-a}^{a} \left[\frac{y^2}{2}\right]_{y=0}^{b} dx$$

$$= -4 \times \frac{b^2}{2} \int_{x=-a}^{a} 1\,dx$$

$$= -2b^2 \left[x\right]_{x=-a}^{a} \qquad = -2b^2[2a]$$

$$= -4ab^2$$

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Along AB $\qquad \bar{F}\cdot d\bar{r} = (x^2+y^2)dx$

$y=0 \qquad -2xy\,dy$

$dy=0$

$$\int_{AB} (x^2+y^2)\,dx$$

$$\int_{x=-a}^{a} x^2\,dx = \left[\frac{x^3}{3}\right]_{x=-a}^{a} = \frac{a^3}{3}+\frac{a^3}{3} = \frac{2a^3}{3}$$

Along BC

$x=a,\ dx=0 \qquad y \to 0 \text{ to } b$

$$\int_{y=0}^{b} -2ay\,dy = -\cancel{2}a\left[\frac{y^2}{\cancel{2}}\right]_0^b = -ab^2$$

Along CD

$y=b \qquad x \to a \text{ to } -a$

$\Rightarrow dy=0$

$$\int_{x=a}^{-a} (x^2+y^2)\,dx = \int_{x=a}^{-a} (b^2+x^2)\,dx$$

$$= \left[b^2x+\frac{x^3}{3}\right]_{x=a}^{-a}$$

$$= b^2(-a-a) + \left[-\frac{a^3}{3}-\frac{a^3}{3}\right]$$

$$= -2ab^2 - \frac{2}{3}a^3$$

Along DA $\qquad x=-a \Rightarrow dx=0, \quad y \to b \text{ to } 0$

$$\int_{y=b}^{0} 2ay\,dy = 2a\left(\frac{y^2}{2}\right)_{y=b}^{0}$$

$$= \cancel{2}a\left[-\frac{b^2}{\cancel{2}}\right] = -ab^2$$

Total $= -4ab^2$

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Gauss's divergence theorem

(Transformation between surface integral and volume integral)

Let S be a closed surface enclosing a volume V. If $\bar{F}$ is a continuously differentiable vector point function, then

$$\int_V \operatorname{div} \bar{F}\, dv = \int_S \bar{F}\cdot\bar{n}\, ds$$

where $\bar{n}$ is the outward drawn unit normal vector at any point of S.

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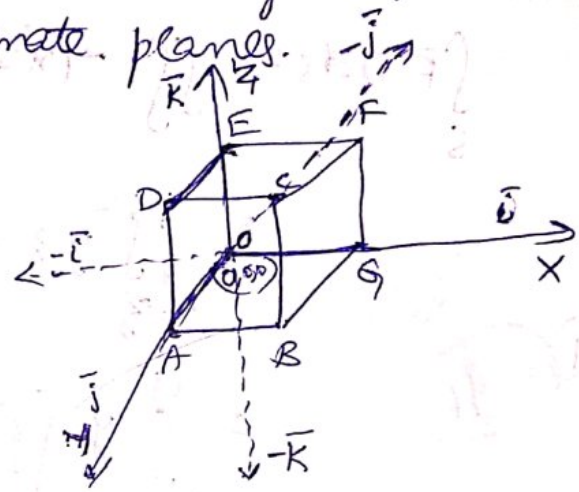
① Verify Gauss divergence theorem for

$\bar{F} = (x^3 - yz)\bar{i} - 2x^2y\bar{j} + z\bar{k}$ taken over the surface of the cube bounded by the planes $x = y = z = a$ and coordinate planes.



OABG

$ds = dx\,dy$,

$\bar{n} = -\bar{k}$

$z = 0 \Rightarrow dz = 0$

$$\int_{OABG} \bar{F}\cdot\bar{n}\,ds = \int_{OABG} (-z)\,dx\,dy = 0$$

DCFE

$ds = dx\,dy$, $\bar{n} = \bar{k}$, $z = a$, $dz = 0$

$$\int_{DCFE} (z)\,dx\,dy = \int_{y=0}^{a}\int_{x=0}^{a} a\,dx\,dy = a\int_{y=0}^{a} [x]_0^a\,dy$$

$$= a \times a \int_{y=0}^{a} 1\,dy = a^3$$

EFGO

$ds = dx\,dz$, XZ-plane $\Rightarrow y = 0$, $dy = 0$. $\bar{n} = -\bar{j}$

$$\int_{EFGO} \bar{F}\cdot\bar{n}\,ds = \iint_{EFGO} (2x^2y)\,dx\,dz = 0$$

ABCD

$ds = dx\,dz$, $y = a \Rightarrow dy = 0$, $\bar{n} = \bar{j}$

$$\int_{ABCD} \bar{F}\cdot\bar{n}\,ds = \iint_{ABCD} (-2x^2y)\,dx\,dz = -2a\int_{z=0}^{a}\int_{x=0}^{a} x^2\,dx\,dz$$

$$= -2a \times \frac{a^3}{3} \times a$$

$$= -\frac{2a^5}{3}$$

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OEDA

$ds = dy\,dz, \quad x=0, \quad \bar{n} = -\bar{i}$

$dx = 0$

$$\int_S \bar{F}\cdot\bar{n}\,ds = \iint -(x^3 - yz)\,dy\,dz$$

$$= \int_{y=0}^{a}\int_{z=0}^{a} yz\,dy\,dz$$

$$= \int_{y=0}^{a} y\left[\frac{z^2}{2}\right]_0^a dy$$

$$= \frac{a^2}{2}\left[\frac{y^2}{2}\right]_{y=0}^{a} = \frac{a^2}{2}\times\frac{a^2}{2} = \frac{a^4}{4}$$

CBGF

$ds = dy\,dz, \quad x=a, \quad \bar{n} = \bar{i}$

$dx = 0$

$$\int_S \bar{F}\cdot\bar{n}\,ds = \int_{z=0}^{a}\int_{y=0}^{a} (x^3 - yz)\,dy\,dz$$

$$= \int_{z=0}^{a}\int_{y=0}^{a} (a^3 - yz)\,dy\,dz$$

$$= \int_{z=0}^{a}\left[a^3 y - \frac{y^2 z}{2}\right]_{y=0}^{a} dz$$

$$= \int_{z=0}^{a}\left[a^4 - \frac{a^2 z}{2}\right] dz$$

$$= \left[a^4 z - \frac{a^2}{2}\times\frac{z^2}{2}\right]_{z=0}^{a}$$

$$= a^5 - \frac{a^4}{4}$$

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Surface integral

$$= 0 + a^3 + 0 - \frac{2a^5}{3} + \cancel{\frac{a^4}{4}} + a^5 - \cancel{\frac{a^4}{4}}$$

$$R.H.S = a^3 + \frac{1}{3}a^5$$

$$\int_V \operatorname{div} \bar{F}\, dv = \int_S \bar{F}\cdot\bar{n}\, ds$$

$$\operatorname{div} \bar{F} = \frac{\partial}{\partial x}(x^3 - yz) + \frac{\partial}{\partial y}(-2x^2y) + \frac{\partial}{\partial z}(z)$$

$$= 3x^2 - 2x^2 + 1 = x^2 + 1$$

$$\int_V \operatorname{div} \bar{F}\, dv = \int_{x=0}^{a}\int_{y=0}^{a}\int_{z=0}^{a} (x^2+1)\, dx\, dy\, dz$$

$$= \int_{x=0}^{a}\int_{y=0}^{a} (x^2+1)\, a\, dy\, dz$$

$$= \int_{x=0}^{a} a(x^2+1)\, a\, dz$$

$$= a^2 \left[\frac{x^3}{3} + x\right]_{x=0}^{a} = a^2\left[\frac{a^3}{3} + a\right] = \frac{a^5}{3} + a^3$$

L.H.S = R.H.S

$\Rightarrow$ Gauss divergence theorem is verified.

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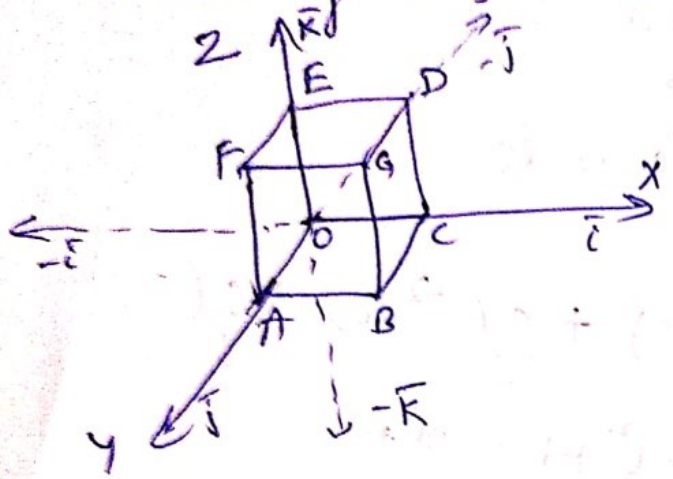
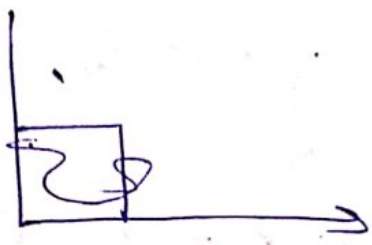
(2) Verify Gauss divergence theorem for $\vec{F} = x^3\bar{i} + y^3\bar{j} + z^3\bar{k}$ taken over the cube bounded by $x=0,\ x=a,\ y=0,\ y=a,\ z=0,\ z=a$.





$$\int_V \operatorname{div}\bar{F}\, dv = \int_S \bar{F}\cdot\bar{n}\, ds$$

$$\bar{F} = x^3\bar{i} + y^3\bar{j} + z^3\bar{k}$$

L.H.S

$$\operatorname{div}\bar{F} = 3x^2 + 3y^2 + 3z^2 = 3(x^2+y^2+z^2)$$

$$3\iiint_{x\,y\,z} (x^2+y^2+z^2)\, dx\,dy\,dz$$

$$= 3\iint_{x\,y} \left[x^2 z + y^2 z + \frac{z^3}{3}\right]_{z=0}^{a} dx\,dy$$

$$= 3\iint_{x\,y} \left[x^2 a + y^2 a + \frac{a^3}{3}\right] dx\,dy$$

$$= 3\int_x x^2 a \times a + \frac{y^3}{3} a + \frac{a^3}{3} \times a$$

$$= 3\int_x \left[a^2 x^2 + \frac{a^4}{3} + \frac{a^4}{3}\right] dx$$

$$= 3\left[a^2 \frac{x^3}{3} + \frac{a^4}{3} x + \frac{a^4}{3} x\right]$$

$$= 3\left[\frac{a^5}{3} + \frac{a^5}{3} + \frac{a^5}{3}\right]$$

$$= 3a^5$$

R.H.S

OABC

$\bar{n} = -\bar{k}$, $ds = dx\,dy$, $z = 0$

$$\int_S \bar{F}\cdot\bar{n}\, ds = \int_{x=0}^{a}\int_{y=0}^{a} (-z^3)\, dx\,dy = 0$$

EFGD

$\bar{n} = \bar{k}$, $ds = dx\,dy$, $z = a$

$$\int_S \bar{F}\cdot\bar{n}\, ds = \int_{x=0}^{a}\int_{y=0}^{a} (z^3)\, dx\,dy = a^3 \int_{x=0}^{a}\int_{y=0}^{a} 1\, dx\,dy$$

$$= a^3 \times a^2 = a^5$$

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OAFE

$\bar{n} = -\bar{i}$, $ds = dy\,dz$, $x = 0$

$$\int_S \bar{F}\cdot\bar{n}\,ds = \iint (-x^3)\,dy\,dz = 0$$

CDGB

$\bar{n} = \bar{i}$, $ds = dy\,dz$, $x = a$

$$\int_S \bar{F}\cdot\bar{n}\,ds = \int_{y=0}^{a}\int_{z=0}^{a} (-x^3)\,dy\,dz$$

$$= a^3 \times a^2 = a^5$$

OCDE

$\bar{n} = -\bar{j}$, $ds = dx\,dz$, $y = 0$

$$\int_S \bar{F}\cdot\bar{n}\,ds = \iint (-y^3)\,dx\,dz = 0$$

FABG

$\bar{n} = \bar{j}$, $ds = dx\,dz$, $y = a$

$$\int_S \bar{F}\cdot\bar{n}\,ds = \iint (y^3)\,dx\,dz$$

$$= \int_{x=0}^{a}\int_{z=0}^{a} a^3\,dx\,dz = a^5$$

$$0 + a^5 + 0 + a^5 + 0 + a^5 = 3a^5$$

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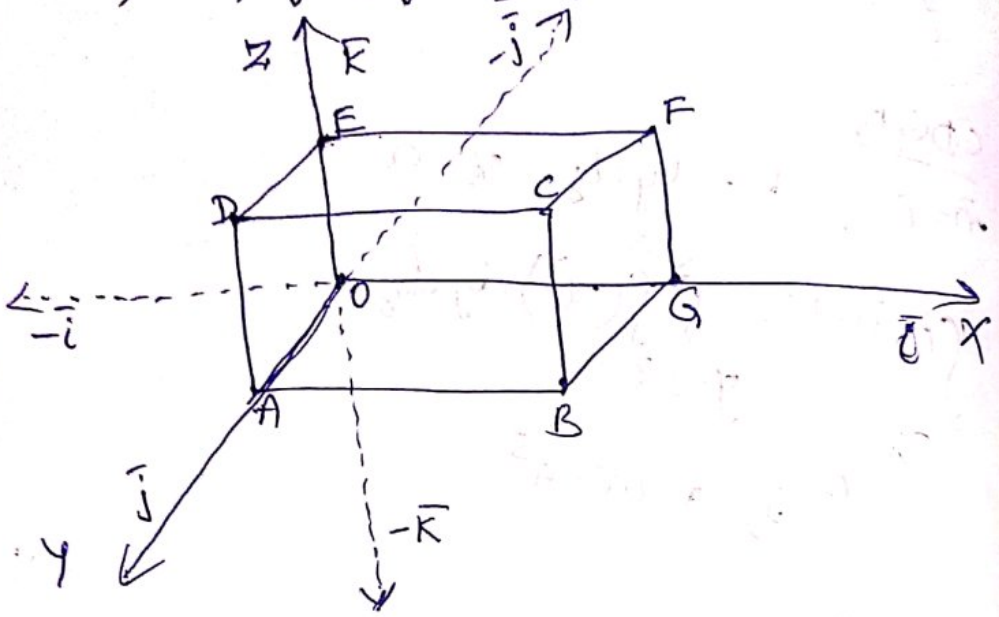
(3) Verify G.D.T for $\bar{F} = x^2\bar{i} + y^2\bar{j} + z^2\bar{k}$ over the cuboid formed by the planes

$x=0, x=a, y=0, y=b, z=0, z=c$



$$\int_V \operatorname{div}\bar{F}\, dv = \int_S \bar{F}\cdot\bar{n}\, ds$$

$$\operatorname{div}\bar{F} = \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial z}(z^2)$$

$$= 2x + 2y + 2z$$

L.H.S

$$\int_V \operatorname{div}\bar{F}\, dv = 2\iiint (x+y+z)\, dx\, dy\, dz$$

$$= 2\int_{x=0}^{a}\int_{y=0}^{b}\int_{z=0}^{c} (x+y+z)\, dz\, dy\, dx$$

$$= 2\int_{x=0}^{a}\int_{y=0}^{b} \left[xz + yz + \frac{z^2}{2}\right]_{z=0}^{c} dy\, dx$$

$$= 2\int_{x=0}^{a}\int_{y=0}^{b} \left[xc + yc + \frac{c^2}{2}\right] dy\, dx$$

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$$= \cancel{2\int_{x=0}^{a}\int_{y=0}^{b}}$$

$$= 2\int_{x=0}^{a}\left[xcy + \frac{y^2}{2}c + \frac{c^2}{2}y\right]_{y=0}^{b} dx$$

$$= 2\int_{x=0}^{a}\left[xcb + \frac{b^2}{2}c + \frac{c^2}{2}b\right] dx$$

$$= 2\left[\frac{x^2}{2}cb + \frac{b^2c}{2}x + \frac{c^2b}{2}x\right]_{x=0}^{a}$$

$$= \cancel{2}\left[\frac{a^2bc + ab^2c + abc^2}{\cancel{2}}\right]$$

$$= abc(a+b+c)$$

<u>Along OABG</u>

$ds = dx\,dy$, $\bar{n} = -\bar{k}$, $z = 0$

$$\int_S \bar{F}\cdot\bar{n}\,ds = \iint (-z^2)\,dx\,dy = 0$$

<u>DEFC</u>

$ds = dx\,dy$, $\bar{n} = \bar{k}$, $z = c$

$$\int_S \bar{F}\cdot\bar{n}\,ds = \iint (+z^2)\,dx\,dy$$

$$= \int_{x=0}^{a}\int_{y=0}^{b} c^2\,dx\,dy$$

$$= c^2\,ba$$

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OADE

$\bar{n} = -\bar{i}$, $ds = dy\,dz$, $x = 0$

$$\int_S \bar{F}\cdot\bar{n}\,ds = \iint (-x^2)\,dy\,dz = 0$$

CFGB

$\bar{n} = \bar{i}$, $ds = dy\,dz$, $x = a$

$$\int_S \bar{F}\cdot\bar{n}\,ds = \iint (x^2)\,dy\,dz$$

$$= \int_{y=0}^{b}\int_{z=0}^{c} a^2\,dy\,dz$$

$$= a^2bc$$

OEFG

$\bar{n} = -\bar{j}$, $ds = dx\,dz$, $y = 0$

$$\int_S \bar{F}\cdot\bar{n}\,ds = \iint (-y^2)\,dx\,dz = 0$$

ABCD

$\bar{n} = \bar{j}$, $ds = dx\,dz$, $y = b$

$$\int_S \bar{F}\cdot\bar{n}\,ds = \iint (y^2)\,dx\,dz = b^2\int_{x=0}^{a}\int_{z=0}^{c} 1\,dx\,dz$$

$$= b^2ac$$

$$= 0 + c^2ba + 0 + a^2bc + 0 + b^2ac$$

$$= abc(c+a+b)$$

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(4) Compute $\int (ax^2+by^2+cz^2)\,ds$ over the surface of the sphere $x^2+y^2+z^2=1$

$$\phi = x^2+y^2+z^2-1=0$$

$$\nabla\phi = \bar{i}(2x)+\bar{j}(2y)+\bar{k}(2z)$$

$$\frac{\nabla\phi}{|\nabla\phi|} = \frac{2x\bar{i}+2y\bar{j}+2z\bar{k}}{\sqrt{4x^2+4y^2+4z^2}} = \frac{x\bar{i}+y\bar{j}+z\bar{k}}{1} = \bar{n}$$

$$\bar{F}\cdot\bar{n} = ax^2+by^2+cz^2$$

$$\bar{F}\cdot(x\bar{i}+y\bar{j}+z\bar{k}) = ax^2+by^2+cz^2$$

$$\bar{F} = ax\bar{i}+by\bar{j}+cz\bar{k}$$

$$\int (ax^2+by^2+cz^2)\,ds = \int_S \bar{F}\cdot\bar{n}\,ds = \int_V \operatorname{div}\bar{F}\,dv$$

$$= \int_V [a+b+c]\,dv$$

$$= (a+b+c)\int_V 1\,dv$$

$$= (a+b+c)\times V$$

$$= (a+b+c)\times \frac{4}{3}\pi\times 1^3 = \frac{4\pi}{3}(a+b+c)$$

(5) Show that $\int (ax\bar{i}+by\bar{j}+cz\bar{k})\cdot\bar{n}\,ds = \frac{4\pi}{3}(a+b+c)$, where 's' is the surface of the sphere $x^2+y^2+z^2=1$

$$\bar{F} = ax\bar{i}+by\bar{j}+cz\bar{k}$$

$$\operatorname{div}\bar{F} = a+b+c$$

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$$\int_S \bar{F}\cdot\bar{n}\,ds = \int_V (\nabla\cdot\bar{F})\,dv$$

$$= \int_V (a+b+c)\,dv$$

$$= (a+b+c)\int_V 1\,dv$$

$$= (a+b+c)\,V$$

$$= (a+b+c)\,\frac{4}{3}\pi r^3$$

$$= \frac{4\pi}{3}(a+b+c)$$

(6) Using Divergence theorem, evaluate

$$\iint_S (x\,dy\,dz + y\,dz\,dx + z\,dx\,dy)$$

where $S: x^2+y^2+z^2=a^2$

$F_1 = x$, $F_2 = y$, $F_3 = z$

$\bar{F} = x\bar{i} + y\bar{j} + z\bar{k}$

$$\iint_S (x\,dy\,dz + y\,dz\,dx + z\,dx\,dy)$$

$$= \int_V \operatorname{div}\bar{F}\,dv$$

$$= \int_V \left[\frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z)\right] dv$$

$$= 3\int_V dv$$

V = volume of the sphere with radius a.

$$= 3V =$$

$$= 3\times\frac{4}{3}\pi a^3 = 4\pi a^3$$

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(7) Apply divergence theorem to evaluate

$$\iint_S (x+z)\,dy\,dz + (y+z)\,dz\,dx + (x+y)\,dx\,dy.$$

where 'S' is the surface of the sphere

$x^2+y^2+z^2=4$

$F_1 = x+z, \quad F_2 = y+z, \quad F_3 = x+y$

$\bar{F} = (x+z)\bar{i} + (y+z)\bar{j} + (x+y)\bar{k}$

$$\iint_S F_1\,dy\,dz + F_2\,dz\,dx + F_3\,dx\,dy = \int_V \operatorname{div}\bar{F}\,dv$$

$$= \int_V [1+1+0]\,dv$$

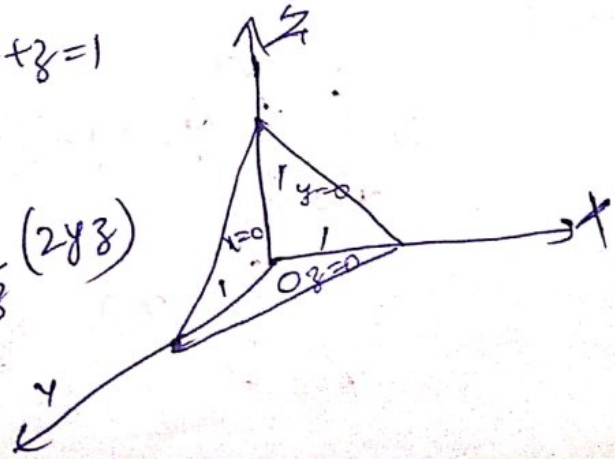
$$= 2\int_V 1\,dv$$

$$= 2V$$

$$= 2\times\frac{4}{3}\pi\times(2)^3 = 2\times\frac{4}{3}\times 8\pi$$

$$= \frac{64}{3}\pi$$

(8) Evaluate $\int_S \bar{F}\cdot\bar{n}\,ds$, if $\bar{F} = xy\bar{i} + z^2\bar{j} + 2yz\bar{k}$ over the tetrahedron bounded by $x=0$, $y=0$, $z=0$ and the plane $x+y+z=1$



$\bar{F} = xy\bar{i} + z^2\bar{j} + 2yz\bar{k}$

$\operatorname{div}\bar{F} = \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}(z^2) + \frac{\partial}{\partial z}(2yz)$

$= y + 0 + 2y = 3y$

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$$\int_S \bar{F}\cdot\bar{n}\,ds = \int_V \operatorname{div}\bar{F}\,dv$$

$$= \iiint (3y)\,dx\,dy\,dz$$

$$= 3\iiint y\,dx\,dy\,dz$$

$$= 3\int_{x=0}^{1}\int_{y=0}^{1-x}\int_{z=0}^{1-x-y} y\,dz\,dy\,dx$$

$x+y+z=1$

<u>$z=0$ to $1-x-y$</u>

<u>y limits</u>, put $z=0$

$y=0$ to $1-x$

<u>x limits</u>, put <u>$y=0, z=0$</u>

$x=0$ to $x=1$

$$= 3\int_{x=0}^{1}\int_{y=0}^{1-x} y\,(1-x-y)\,dy\,dx$$

$$= 3\int_{x=0}^{1}\int_{y=0}^{1-x} (y-xy-y^2)\,dy\,dx$$

$$= 3\int_{x=0}^{1}\left[\frac{y^2}{2}-x\frac{y^2}{2}-\frac{y^3}{3}\right]_{y=0}^{(1-x)} dx$$

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$$= 3\int_{x=0}^{1}\left[\left(\frac{1-x}{2}\right)y^2 - \frac{y^3}{3}\right]_{y=0}^{1-x} dx$$

$$= 3\int_{x=0}^{1}\left[\left(\frac{1-x}{2}\right)(1-x)^2 - \frac{(1-x)^3}{3}\right] dx$$

$$= 3\int_{x=0}^{1}\left[(1-x)^3\left(\frac{1}{2}-\frac{1}{3}\right)\right] dx$$

$$= 3\times\frac{1}{6}\left[\frac{(1-x)^4}{-4}\right]_{x=0}^{1}$$

$$= \frac{1}{2}\left[0+\frac{1}{4}\right] = \frac{1}{8}$$

(9) Use divergence theorem to evaluate $\iint_S \bar{F}\cdot d\bar{S}$ where $\bar{F} = x^3\bar{i} + y^3\bar{j} + z^3\bar{k}$ and 'S' is the surface of the sphere $x^2+y^2+z^2=a^2$

$$\iint_S \bar{F}\cdot d\bar{S} = \int_V \operatorname{div}\bar{F}\, dv$$

$$= \int_V \left[3x^2+3y^2+3z^2\right] dv$$

$$= 3\int_V (x^2+y^2+z^2)\, dv$$

$$= 3\iiint (x^2+y^2+z^2)\, dx\, dy\, dz$$

'S' is the surface of the sphere $x^2+y^2+z^2=r^2$

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$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi,$

$z = r\cos\theta$

$dx\,dy\,dz = r^2\sin\theta\, dr\, d\theta\, d\phi$

$r \to 0$ to a

$\theta \to 0$ to π

$\phi \to 0$ to 2π

$$= 3\iiint \left[r^2\sin^2\theta\cos^2\phi + r^2\sin^2\theta\sin^2\phi + r^2\cos^2\theta\right] r^2\sin\theta\, dr\, d\theta\, d\phi$$

$$= 3\iiint \left(r^2\sin^2\theta + r^2\cos^2\theta\right) r^2\sin\theta\, dr\, d\theta\, d\phi$$

$$= 3\int_{r=0}^{a}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} r^4\sin\theta\, dr\, d\theta\, d\phi$$

$$= 3\times 2\pi \int_{r=0}^{a}\int_{\theta=0}^{\pi} r^4\sin\theta\, d\theta\, dr$$

$$= 6\pi \int_{r=0}^{a} r^4 \left[-\cos\theta\right]_{\theta=0}^{\pi} dr$$

$$= 6\pi \int_{r=0}^{a} r^4 \left[-\cos\pi + \cos 0\right] dr$$

$$= 12\pi \left(\frac{r^5}{5}\right)_{r=0}^{a} = \frac{12\pi a^5}{5}$$

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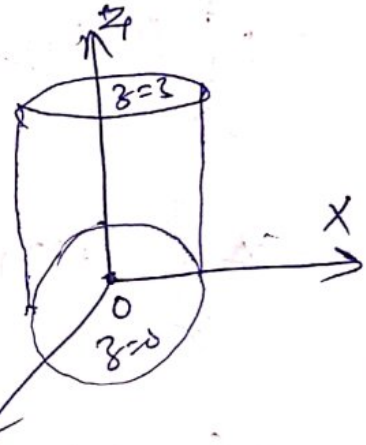
(10) Use divergence theorem to evaluate $\iint \bar{F}.ds$ where $\bar{F} = 4x\bar{i} - 2y^2\bar{j} + z^2\bar{k}$ and 'S' is the surface bounded by the region $x^2+y^2=4$, $z=0$ and $z=3$.



$$\operatorname{div}\bar{F} = 4 + (-4y) + 2z$$
$$= 4 - 4y + 2z$$

$$\iint \bar{F}.ds = \int_V \operatorname{div}\bar{F}\, dv$$

$$= \int_{z=0}^{3}\int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{x=-2}^{2} (4-4y+2z)\, dx\, dy\, dz$$

$$= \int_{z=0}^{3}\int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4-4y+2z)(4)\, dy\, dz$$

$$= \int_{z=0}^{3}\left[\int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4+2z)\, dy - 4\left[\frac{y^2}{2}\right]_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\right] dz$$

$$= \int_{z=0}^{3}\left[(4+2z)\times 2\sqrt{4-x^2} - 0\right] dz$$

$$= \int_{z=0}^{3}\int_{x=-2}^{2}\int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4+2z-4y)\, dy\, dx\, dz$$

$$= \int_{z=0}^{3}\int_{x=-2}^{2}\left[(4+2z)\, 2\sqrt{4-x^2} - 0\right] dx\, dz$$

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$$= \int_{z=0}^{3} \int_{x=-2}^{2} 2\sqrt{4-x^2}\,(4+2z)\,dx\,dz$$

$$= 2\int_{z=0}^{3} (4+2z) \int_{x=-2}^{2} \sqrt{4-x^2}\,dx\,dz$$

$$= 4\int_{z=0}^{3} (4+2z) \int_{x=0}^{2} \sqrt{4-x^2}\,dx\,dz$$

$$= 4\int_{z=0}^{3} (4+2z) \left[\frac{x}{2}\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\left(\frac{x}{2}\right) \right]_{x=0}^{2} dz$$

$$= 4\int_{z=0}^{3} (4+2z) \left[\cancel{2} \times \frac{\pi}{\cancel{2}} \right] dz$$

$$= 4\pi \left[4z + 2\frac{z^2}{2} \right]_{z=0}^{3}$$

$$= 4\pi\left[12+9\right] = 84\pi$$

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(11) By transforming into triple integral, evaluate $\iint x^3\,dy\,dz + x^2y\,dz\,dx + x^2z\,dx\,dy$ where 'S' is the closed surface consisting of the cylinder $x^2+y^2=a^2$ and the circular discs $z=0$, $z=b$.

$F_1 = x^3$, $F_2 = x^2y$, $F_3 = x^2z$

$\bar{F} = x^3\bar{i} + x^2y\,\bar{j} + x^2z\,\bar{k}$

$\operatorname{div}\bar{F} = 3x^2 + x^2 + x^2 = 5x^2$

$x^2+y^2=a^2$, $z=0$, $z=b$

$y^2 = a^2-x^2$

$y = -\sqrt{a^2-x^2}$ to $\sqrt{a^2-x^2}$

Put $y=0$

$\cancel{y^2=a^2}$ $\quad x^2=a^2$

$\cancel{y=\pm a}$ $\quad x=\pm a$

$\cancel{y=-a \text{ to } a}$ $\quad x=-a$ to a

$$\int_V \operatorname{div}\bar{F}\,dv$$

$$= \iiint (5x^2)\,dx\,dy\,dz$$

$$= 5\int_{z=0}^{b}\int_{x=-a}^{a}\int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} x^2\,dy\,dx\,dz$$

$$= 5\int_{z=0}^{b}\int_{x=-a}^{a} x^2\left[2\sqrt{a^2-x^2}\right]dx\,dz$$

$$= 10\int_{z=0}^{b}\int_{x=-a}^{a} x^2\sqrt{a^2-x^2}\,dx\,dz$$

$$= 10\int_{z=0}^{b} 2\int_{x=0}^{a} x^2\sqrt{a^2-x^2}\,dx\,dz$$

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$$= 20 \int_{z=0}^{b} \int_{x=0}^{a} x^2 \sqrt{a^2-x^2}\, dx\, dz$$

$$= \cancel{20 \int_{z=0}^{b} [\quad]}$$

consider,

$$\int_{x=0}^{a} x^2 \sqrt{a^2-x^2}\, dx$$

$x = a\cos\theta$

$L.L \Rightarrow x=0 \Rightarrow \cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}$

$U.L \Rightarrow x=a \Rightarrow \cos\theta = 1 \Rightarrow \theta = 0$

$$= \int_{\theta=\frac{\pi}{2}}^{0} a^2\cos^2\theta \sqrt{a^2-a^2\cos^2\theta} \times -a\sin\theta\, d\theta$$

$$= a^4 \int_{\theta=0}^{\frac{\pi}{2}} \cos^2\theta \sin^2\theta\, d\theta$$

$2m-1=2$

$m=\frac{3}{2}$

$$= a^4 \times \frac{1}{2} B\left(\frac{3}{2}, \frac{3}{2}\right)$$

$$= a^4 \times \frac{1}{2} \times \frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma(3)}$$

$$= a^4 \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}\Gamma\left(\frac{1}{2}\right) \times \frac{1}{2}\Gamma\left(\frac{1}{2}\right)$$

$$= \frac{a^4\pi}{16}$$

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$$= 20 \int_{z=0}^{b} \frac{a^4\pi}{16}\, dz$$

$$= 20 \times \frac{a^4\pi}{16} \left[z \right]_0^b$$

$$= \frac{\cancel{20}^{5}}{\cancel{16}_{4}} \times a^4 \pi b$$

$$= \frac{5a^4 b\pi}{4}$$

(12) Applying Gauss Divergence theorem, prove that

$$\int \bar{r} \cdot \bar{n}\, ds = 3V$$

or $\int \bar{r} \cdot d\bar{S} = 3V$

$$\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$$

$$\int \bar{r} \cdot \bar{n}\, ds = \int_V \operatorname{div} \bar{r}\, dv$$

$$= \int_V (1+1+1)\, dv$$

$$= 3V$$

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Stoke's theorem (Transformation between line integral and surface integral)

Let 'S' be a open surface bounded by a closed, non intersecting curve 'c'. If $\bar{F}$ is any differential vector point function then

$$\oint_C \bar{F}\cdot d\bar{r} = \int_S \text{curl}\,\bar{F}\cdot\bar{n}\,ds$$

where c is traversed in the positive direction and $\bar{n}$ is unit outward drawn normal at any point of the surface.

Cartesian form: Let $\bar{F} = F_1\bar{i} + F_2\bar{j} + F_3\bar{k}$. Let the unit normal vector $\bar{n}$ drawn outward make angles α, β, γ with the +ve direction of x, y, z axes. Then

$$\bar{n} = \cos\alpha\,\bar{i} + \cos\beta\,\bar{j} + \cos\gamma\,\bar{k}$$

$$\bar{F}\cdot d\bar{r} = F_1\,dx + F_2\,dy + F_3\,dz$$

$$\nabla\times\bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \bar{i}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) + \bar{j}\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) + \bar{k}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

$$(\nabla\times\bar{F})\cdot\bar{n} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\cos\alpha + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\cos\beta + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\cos\gamma$$

$\therefore$ Stoke's theorem is equivalent to

$$\oint F_1\,dx + F_2\,dy + F_3\,dz = \iint\left[\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\cos\alpha + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\cos\beta + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\cos\gamma\right]ds$$

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① Evaluate by Stoke's theorem $\int_C (e^x\,dx + 2y\,dy - dz)$ where 'c' is the curve $x^2+y^2=9$ and $z=2$.

$\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ $\qquad$ $\bar{F} = e^x\bar{i} + 2y\bar{j} - \bar{k}$

$d\bar{r} = dx\,\bar{i} + dy\,\bar{j} + dz\,\bar{k}$

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By Stokes's theorem,

$$\int_C \bar{F}\cdot d\bar{r} = \iint_S (\text{curl}\,\bar{F})\cdot \bar{n}\, ds$$

$$\text{curl}\,\bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^x & 2y & -1 \end{vmatrix}$$

$$= \bar{i}\,[0-0] - \bar{j}\,[0-0] + \bar{k}\,[0-0] = \bar{0}$$

$$\therefore \iint_S (\bar{0}\cdot\bar{n})\, ds = 0$$

$$\therefore \int_C (e^x\, dx + 2y\, dy - dz) = 0$$

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⇒ Verify Stokes theorem for

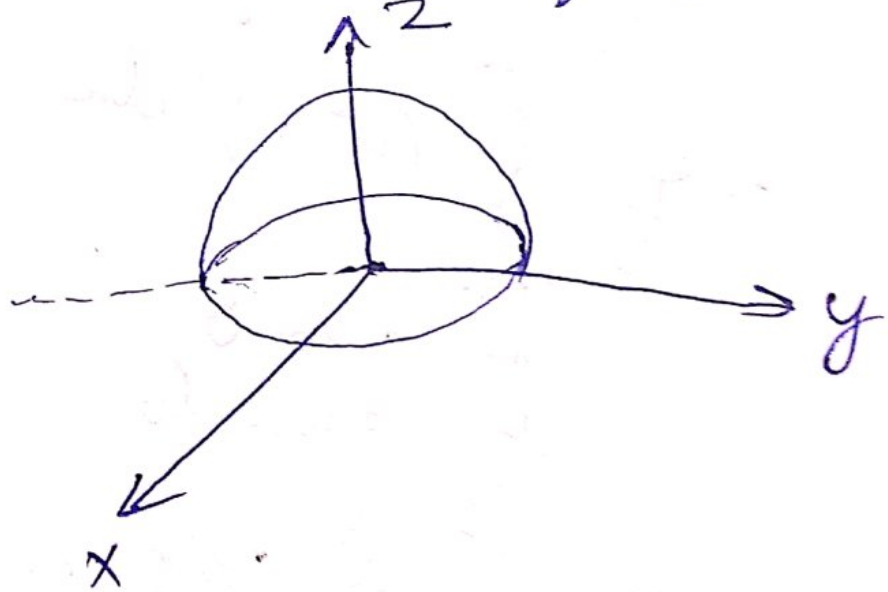
$\bar{F} = y^2\bar{i} + y\bar{j} - zx\bar{k}$ and 'S' is the upper half of the sphere $x^2+y^2+z^2=a^2$ and $z \geq 0$

$$\oint_c \bar{F}\cdot d\bar{r} = \iint (\text{curl}\,\bar{F})\cdot\bar{n}\, ds$$

'C' is the boundary of the given hemisphere

→ 'C' is $x^2+y^2=a^2$

On the curve 'C' : $z=0$, $x^2+y^2=a^2$



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By Stroke's theorem,

$$\oint_C \bar{F}\cdot d\bar{r} = \iint_S \overset{(\text{curl}\,\bar{F})}{\bar{F}}\cdot\bar{n}\,ds$$

$$LHS = \oint_C \bar{F}\cdot d\bar{r}$$

parametric equations of circle $x^2+y^2=a^2$

$x = a\cos\theta,\ y = a\sin\theta,\quad \theta \to 0 \text{ to } 2\pi$

$$= \int \left[y^2 dx + y\,dy - zx\,dz \right]_{z=0}$$

$$= \int (y^2 dx + y\,dy)$$

$$= \int_{2\pi} \left[a^2\sin^2\theta \times -a\sin\theta\,d\theta + a\sin\theta \times a\cos\theta \right] d\theta$$

$$= \int_{\theta=0} \left[-a^3\sin^3\theta + a^2\sin\theta\cos\theta \right] d\theta$$

$\sin 3\theta$
$= 3\sin\theta - 4\sin^3\theta$

$$= 0$$

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$$\text{Curl}\,\bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & y & -zx \end{vmatrix}$$

$$= \bar{i}\left[0-0\right] - \bar{j}\left[-z-0\right] + \bar{k}\left[0-2y\right]$$

$$= z\bar{j} - 2y\bar{k}$$

$$\bar{n} = \frac{\bar{i}(2x) + \bar{j}(2y) + \bar{k}(2z)}{2\sqrt{x^2+y^2+z^2}}$$

$$= \frac{x\bar{i} + y\bar{j} + z\bar{k}}{a}$$

Project it onto XY-plane.

$x^2+y^2=a^2$ $\quad \iint_S \bar{F}\cdot\bar{n}\,ds = \iint_R \bar{F}\cdot\bar{n}\,\frac{dx\,dy}{|\bar{n}\cdot\bar{k}|}$

$$\iint_S (z\bar{j} - 2y\bar{k})\cdot\left(\frac{x\bar{i} + y\bar{j} + z\bar{k}}{a}\right)\frac{dx\,dy}{\frac{z}{a}}$$

$$= \iint \left[\frac{zy}{a} - \frac{2yz}{a}\right] \times \frac{a}{z}\,dx\,dy$$

$$= \iint (y - 2y)\,dx\,dy$$

$$= -\iint y\,dx\,dy$$

$x = r\cos\theta$, $y = r\sin\theta$, $\quad r \to 0$ to a, $\theta \to 0$ to 2π

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$$= -\int_{r=0}^{a}\int_{\theta=0}^{2\pi} r\sin\theta \; r\,dr\,d\theta$$

$$= -\int_{r=0}^{a} r^2 \left[-\cos\theta\right]_{\theta=0}^{2\pi} dr$$

$$= -\int_{r=0}^{a} r^2 \left[-\cos 2\pi + \cos 0\right] dr$$

$$= 0$$

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$$= \bar{j}\int_0^2 (2-x)^2\,dx - \bar{k}\int_0^2 \frac{2}{3}(2-x)^3\,dx$$

$$= \bar{j}\left[\frac{(2-x)^3}{3}\right]_0^2 - \frac{2\bar{k}}{3}\left[\frac{(2-x)^4}{-4}\right]_0^2$$

$$\int_V (\nabla\times F)\,dv = \frac{8}{3}(\bar{j}-\bar{k})$$

→ Make use of stokes Theorem for the verification of $F = y\bar{i}+z\bar{j}+x\bar{k}$ and surface is the part of sphere $x^2+y^2+z^2=1$ above the xy plane.

Sol: stokes Theorem is $\int \bar{F}\cdot dr = \int \text{curl}\,\bar{F}\cdot n\,ds$

given $\bar{F} = y\bar{i}+z\bar{j}+x\bar{k}$ over the surface $x^2+y^2+z^2=1$ is xy plane.

$$\bar{F}\cdot dr = (y\bar{i}+z\bar{j}+x\bar{k})\cdot(dx\bar{i}+dy\bar{j}+dz\bar{k})$$

$$= y\,dx + z\,dy + x\,dz$$

$$\int_C \bar{F}\cdot dr = \int_C y\,dx + z\,dy + x\,dz = \int_C y\,dx$$

since in xy plane. $z=0$ $\therefore$ $dz=0$.

let $x=\cos\theta$; $y=\sin\theta$ $dx = -\sin\theta\,d\theta$

$dy = \cos\theta\,d\theta$. वा

$$\int_C \bar{F}\cdot dr = \int_C y\cdot dx = \int_0^{\pi} \sin\theta(-\sin\theta)\,d\theta.$$

$\because$ $x^2+y^2=1$; $z=0$. is a circle when we change it into polar coordinate system

$$= -4\int_0^{\pi/2} \sin^2\theta\,d\theta.$$

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$$-4\int_0^{\pi/2} \frac{1-\cos 2\theta}{2}\, d\theta = -4\cdot\frac{1}{2}[\theta]_0^{\pi/2} - \frac{4}{2}\frac{\sin 2\theta}{2}\Big|_0^{\pi/2}$$

$$= -2\cdot\frac{\pi}{2} - \sin 2\cdot\frac{\pi}{2} = -\pi$$

$$\int_C \bar{F}\cdot dr = -\pi$$

Now $\text{curl}\,\bar{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & z & x \end{vmatrix} = -(i+j+k)$

unit vector $\bar{n} = \dfrac{\nabla f}{|\nabla f|}$

$\because f = x^2+y^2+z^2-1$

$\nabla f = \dfrac{\partial f}{\partial x} i + \dfrac{\partial f}{\partial y} j + k\dfrac{\partial f}{\partial z} = 2xi + 2yj + 2zk$

$$\bar{n} = \frac{2xi+2yj+2zk}{\sqrt{4x^2+4y^2+4z^2}} = \frac{2(xi+yj+zk)}{2\sqrt{x^2+y^2+z^2}} = xi+yj+zk$$

$\because x^2+y^2+z^2=1$

$\text{curl}\,F\cdot n = -(i+j+k)\cdot(xi+yj+zk)$; $ds = dx\,dy$

$$\iint \text{curl}\,F\cdot n\, ds = -\iint (x+y+z)\, dx\, dy$$

Considering spherical polar coordinates

$x = r\sin\theta\cos\phi$; $y = r\sin\theta\sin\phi$; $z = r\cos\theta$

$\because r = 1$ here.

$x = \sin\theta\cos\phi$; $y = \sin\theta\sin\phi$; $z = \cos\theta$.

$$-\int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi} (\sin\theta\cos\phi + \sin\theta\sin\phi + \cos\theta)\sin\theta\, d\theta\, d\phi$$

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$$= -\int_0^{\pi/2} \left[\sin\theta \sin\phi - \sin\theta \cos\phi + \phi \cos\theta\right]_0^{2\pi} \sin\theta \, d\theta$$

$$= -2\pi \int_0^{\pi/2} \cos\theta \sin\theta \, d\theta$$

$$= -\pi \int_0^{\pi/2} \sin 2\theta \, d\theta = -\pi \left[-\frac{\cos 2\theta}{2}\right]_0^{\pi/2}$$

$$\iint \text{curl}\,\bar{F}\cdot n\,ds = -\frac{\pi}{2}\left[\cos 2\cdot\frac{\pi}{2} - \cos 0\right] = \frac{\pi}{2}\left[-1-1\right] = -\pi$$

Hence $\int_C \bar{F}\cdot dr = \iint \text{curl}\,\bar{F}\cdot n\,ds$

Hence Stokes theorem varified.

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verify stoke's theorem for

$\bar{F} = (y-z+2)\,\bar{i} + (yz+4)\,\bar{j} - xz\,\bar{k}$

over the region

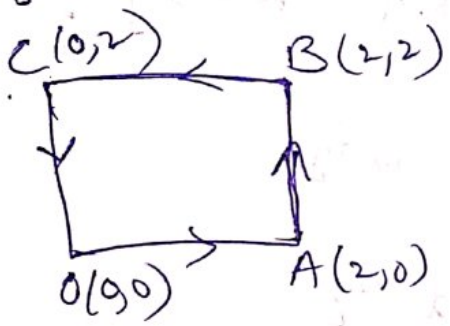
$x=0, y=0, z=0, \; x=2, y=2, z=2$

$$\text{curl}\,\bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y-z+2 & yz+4 & -xz \end{vmatrix}$$

$$= \bar{i}\left[0-y\right] - \bar{j}\left[\underbrace{-z-(-1)}_{-z+1}\right] + \bar{k}\left[0-1\right]$$

$$= -y\,\bar{i} + (z-1)\,\bar{j} - \bar{k}$$

$$\int_S (\text{curl}\,\bar{F})\cdot\bar{n}\,ds = \int(-1)\,ds = (-1)\times 2\times 2$$
$$= -4$$



<u>Along OA</u> $\qquad z=0$

$$\int (y-z+2)\,dx + (yz+4)\,dy - xz\,dz$$

$$= \int (y+2)\,dx + 4\,dy \qquad y=0 \quad x \to 0 \text{ to } 2 \quad dy=0$$

$$= \int_{x=0}^{2} 2\,dx = 2\times 2 = 4$$

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Along AB

$x=2$ $\quad y \to 0$ to 2

$dx=0$

$$\int (y+2)\,dx + 4\,dy$$

$$= \int_{y=0}^{2} 4\,dy = 4\times 2 = 8$$

Along BC

$y=2$ $\quad x \to 2$ to 0

$dy=0$

$$\int_{x=2}^{0} (y+2)\,dx$$

$$= 4\int_{x=2}^{0} 1\,dx$$

$$= -8$$

~~$x=2$ $\quad y \to 2$ to 0~~

~~$dx=0$~~

~~0~~

~~$\int_{y=2} 4\,dy = 4\times 2 = 8$~~

Along CO

$x=0$ $\quad y \to 2$ to 0

$dx=0$

$$\int_{y=2}^{0} 4\,dy = -8$$

total line integral

$$= 4 + \cancel{8} - \cancel{8} - 8 = \underline{-4}$$

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(9) if $\bar{F} = (4xy - 3x^2z^2)\bar{i} + 2x^2\bar{j} - 2x^3z\bar{k}$, prove that $\int_C \bar{F}\cdot dr$ ie workdone is independent of the curve joining the two points.

Sol) w.k.that,

if the force is conservative ie $\bar{F} = \nabla\phi$, then the workdone is independent of the path, and conversely

Now, we have to show that $\text{curl}\,\bar{F} = 0$

where $\bar{F} = \nabla\phi$

$$\text{curl}\,\bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 4xy - 3x^2z^2 & 2x^2 & -2x^3z \end{vmatrix}$$

$$= \bar{i}\left[0 - 0\right] - \bar{j}\left[-6x^2z + 6x^2z\right] + \bar{k}\left[4x - 4x\right]$$

$$= 0.$$

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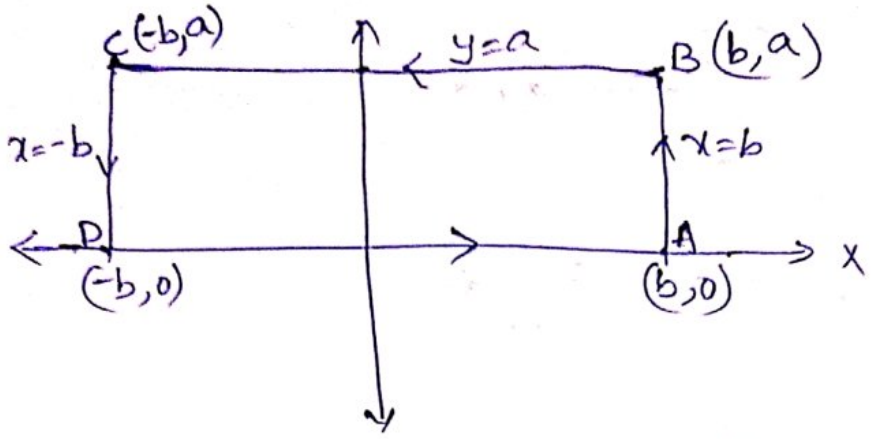
Q) Verify Stoke's theorem for $\bar{F} = y^2\bar{i} - 2xy\bar{j}$, taken round the rectangle bounded by $x = \pm b$, $y=0$, $y=a$



Stoke's theorem states that

$$\int_C \bar{F}\cdot d\bar{r} = \iint_S (\text{curl}\,\bar{F})\cdot\bar{n}\,ds$$

Given $\bar{F} = y^2\bar{i} - 2xy\,\bar{j}$

$$\text{curl}\,\bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & -2xy & 0 \end{vmatrix}$$

$$= \bar{i}\,[0-0] - \bar{j}\,[0-0] + \bar{k}\,[-2y-2y]$$

$$= -4y\bar{k}$$

For the given surface S, $\bar{n} = \bar{k}$

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$$(\text{curl } \bar{F}) \cdot \bar{n} = -4y$$

<u>R.H.S</u>

$$\iint_S (\text{curl } \bar{F}) \cdot \bar{n} \, ds$$

$$= \int_{y=0}^{a} \int_{x=-b}^{b} (-4y) \, dx \, dy$$

$$= \int_{y=0}^{a} -4y \left[x\right]_{-b}^{b} dy$$

$$= \int_{y=0}^{a} (-4y)(2b) \, dy$$

$$= -8b \int_{y=0}^{a} y \, dy$$

$$= -8b \times \left[\frac{y^2}{2}\right]_{y=0}^{a}$$

$$= -8b \times \frac{a^2}{2} = -4a^2b$$

<u>L.H.S</u>

$$= \int_C \bar{F} \cdot d\bar{r} = \int_{DA} \bar{F} \cdot d\bar{r} + \int_{AB} \bar{F} \cdot d\bar{r} + \int_{BC} \bar{F} \cdot d\bar{r} + \int_{CD} \bar{F} \cdot d\bar{r}$$

$$\bar{F} \cdot d\bar{r} = (y^2 \bar{i} - 2xy \bar{j}) (dx \bar{i} + dy \bar{j} + dz \bar{k})$$

$$= y^2 dx - 2xy \, dy$$

<u>Along DA</u> $\Rightarrow$ $y = 0 \Rightarrow dy = 0$

$$\int_{DA} \bar{F} \cdot d\bar{r} = 0$$

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Along AB $\Rightarrow$ $x = b$; $dx = 0$

$$\int_{AB} \bar{F} \cdot d\bar{r} = \int_{y=0}^{a} \left[y^2 dx - 2xy\, dy \right]$$

$$= \int_{y=0}^{a} -2by\, dy$$

$$= -2b \int_{y=0}^{a} y\, dy$$

$$= (-2b) \times \left[\frac{y^2}{2} \right]_{y=0}^{a} = (-2b) \times \frac{a^2}{2} = -a^2 b$$

Along BC $\Rightarrow$ $y = a$; $dy = 0$

x varies from b to $-b$

$$\int_{BC} \bar{F} \cdot d\bar{r} = \int_{x=b}^{-b} \left[y^2 dx - 2xy\, dy \right]$$

$$= \int_{x=b}^{-b} a^2\, dx = a^2 \left[x \right]_{b}^{-b}$$

$$= a^2 \left[-b - b \right] = -2a^2 b$$

Along CD $\Rightarrow$ $x = -b \Rightarrow dx = 0$

y varies from a to 0

$$\int_{CD} \bar{F} \cdot d\bar{r} = \int_{y=a}^{0} \left[y^2 dx - 2xy\, dy \right]$$

$$= \int_{y=a}^{0} -2(-b)y\, dy$$

$$= 2b \left[\frac{y^2}{2} \right]_{y=a}^{0} = 2b \left[0 - \frac{a^2}{2} \right]$$

$$= -a^2 b$$

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$$\therefore \oint_C \bar{F} \cdot d\bar{r} = 0 + (-a^2b) + (-2a^2b) + (-a^2b)$$

$$= -4a^2b$$

$$\Rightarrow L.H.S = R.H.S$$

$$\therefore \oint \bar{F} \cdot d\bar{r} = \iint_S (\text{curl } \bar{F}) \cdot \bar{n} \, ds$$

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