UNIT — III

INITIAL VALUE PROBLEMS

AND

APPLICATIONS

Unit-IV.

Differential equations and its applications

Differential eqn :- An eqn which involves the derivates of one or more dependent variables with respect to one or more independent variables. is called differential eqn.

* The differential eqns can be classified into two types.

i) ordinary diffential eqns (O.D.E)

ii) partial differential eqns (P.D.E)

i) **ordinary differential eqn :-** A D.E is said to be an ordinary if the derivatives w.r.t only one independent variable.

ii) **Partial differential eqn :-** A D.E is said to be a partial, if the derivates w.r.t two or more independent variables.

Order of D.E :- The highest derivative in a D.E is known as a order of D.E.

Degree :- The power of the highest derivative in a D.E is known as Degree. (which is free from fractions & radicals)

eg :- $\frac{d^2y}{dx^2} + \left[1+\left(\frac{dy}{dx}\right)^3\right]^{3/2} = 0$

$$\left(\frac{d^2y}{dx^2}\right) = -\left[1+\left(\frac{dy}{dx}\right)^3\right]$$

S.O.B.S

$$\left(\frac{d^2y}{dx^2}\right)^2 = \left[1+\left(\frac{dy}{dx}\right)^3\right]^3$$ — order -2, degree -2.

Solution of a D.E :- The relation between dependent and independent variables, do not contain any derivates, which satisfying the given D.E is known as the solution of a D.E.

General solution :- A solution of a D.E is said to be a general sol, if the order of D.E is equal to the no. of arbitrary constants.

Ex: i) $y = c_1 e^x + c_2 e^{-x}$, order -2 is the general solution for

$$\frac{d^2y}{dx^2} - y = 0$$

(ii) $y = A\cos x + B\sin x$ is the general sol^n for $\frac{d^2y}{dx^2} + y = 0$.

Particular sol^n :- A particular sol^n which is obtained from the general sol^n to giving some particular values to the arbitary constants.

Ex :- $y = 2e^x + 3e^{-x}$.

Formation of a D.E :-

~~Af~~formed the D.E $y = e^x(A\cos x + B\sin x)$ where A, B are the arbitary constants.

Given eqn $y = e^x(A\cos x + B\sin x) \rightarrow (1)$

diff. both sides w.r.t "x".

$$\frac{dy}{dx} = y' = e^x[-A\sin x + B\cos x] + [A\cos x + B\sin x]e^x.$$

$$y' = e^x[-A\sin x + B\cos x] + y$$

$$y' - y = e^x[-A\sin x + B\cos x]$$

Diff. w.r.t "x"

$$y'' - y' = e^x[-A\cos x - B\sin x] + [-A\sin x + B\cos x]e^x$$

$$\cancel{y'' - 2y' = e^x[-A\cos x}$$

$$y'' - y' = e^x(-A\cos x - B\sin x] + y' - y$$

$$y'' - 2y' + y = e^{\cdot} - y$$

$$\underline{y'' - 2y' + 2y = 0}$$

** Form the D.E from $y = ae^x + be^{2x} + ce^{3x}$ where a, b, c are arbitary constant.

Given eqn $y = ae^x + be^{2x} + ce^{3x}$.

diff. both sides w.r.t "x".

$$y' = ae^x + 2be^{2x} + 3ce^{3x}$$

$$y' = (ae^x + be^{2x} + ce^{3x}) + be^{2x} + 2ce^{3x}.$$

$$y' = y + be^{2x} + 2ce^{3x}$$

$$y' - y = be^{2x} + 2ce^{3x}.$$

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diff. w.r.t 'x'

$$y'' - y' = 2be^{2x} + 6ce^{3x}.$$

$$y'' - y' = 2[be^{2x} + 2ce^{3x}] + 2ce^{3x}$$

$$y'' - y' = 2[y' - y] + 2ce^{3x}$$

$$y'' - 3y' + 2y = 2ce^{3x}.$$

Diff. w.r.t 'x'.

$$y''' - 3y'' + 2y' = 6ce^{3x}.$$

$$y''' - 3y'' + 2y' = 3[y'' - 3y' + 2y]$$

$$y''' - 3y'' + 2y' = 3y'' - 9y' + 6y$$

$$y''' - 6y'' + 11y' - 6y = 0$$

~~Varia~~ 1st order & 1°degree differential eqns :-

* The D.E $\frac{dy}{dx} = f(x,y)$ is known as the first order. → ①

and 1st degree D.E.

The D.E ① can be solved by using the following methods.

i) variable seperable

ii) Homogeneous D.E

iii) Exact D.E

iv) Linear D.E

v) Bernoulli's D.E.

1) Variable - Seperable Method :-

Problems :-

* Solve the D.E $(y-yx)dx+(x+xy)dy=0$

Sol :-

$$y(1-x)dx+x(1+y)dy=0$$

$$y(1-x)dx=-x(1+y)dy.$$

$$\left(\frac{1-x}{x}\right)dx=-\frac{(1+y)}{y}dy$$

$$\frac{(1+y)}{y}dy=-\frac{(1-x)}{x}dx.$$

Now Integration

$$\int\left(\frac{1}{y}+1\right)dy=\int\left(1-\frac{1}{x}\right)dx+c$$

$$\log y+y=(x-\log x)+c$$

* Solve the D.E $\frac{y}{x}\frac{dy}{dx}=\sqrt{1+x^2+x^2y^2+y^2}$

Sol :-

$$\frac{y}{x}\frac{dy}{dx}=\sqrt{(1+x^2)+y^2(1+x^2)}$$

$$\frac{y}{x}\frac{dy}{dx}=\sqrt{(1+x^2)(1+y^2)}$$

$$\frac{y}{\sqrt{1+y^2}}dy=x(\sqrt{1+x^2})dx.$$

$$\int\frac{y}{\sqrt{1+y^2}}dy=\int x\cdot\sqrt{1+x^2}\,dx.$$

$1+y^2=t$ and $1+x^2=k$

$2y\,dy=dt$ $\quad$ $2x\,dx=dk$

$y\,dy=\frac{1}{2}dt$, $\quad$ $x\,dx=\frac{1}{2}dk$.

$$\Rightarrow \int \frac{\frac{1}{2}dt}{\sqrt{t}} = \int \frac{1}{2}\cdot\sqrt{k}\,dk.$$

$$= \int t^{-1/2}dt = \int k^{1/2}dk$$

$$= \frac{t^{1/2}}{1/2} = \frac{k^{3/2}}{3/2}.$$

$$2\cancel{2}.(1+y^2)^{1/2} = \frac{\cancel{2}^{2}}{3}(1+x^2)^{3/2}.+c$$

~~$(1+y^2)^{1/2} =$~~

ii) **Homogeneous Differential eqns :-**

Homogeneous funⁿ :- A function $f(x,y)$ is said to be a homogeneous if $f(kx,ky)=k^n f(x,y)$ where $n=$ constant.

Ex :- $f(x,y)=x^2+y^2+2xy$.

$f(x,y)=x^3+y^3+x^2y+xy^2$

Homogeneous D.E :- A D.E $\frac{dy}{dx}=f(x,y)$ is said to be a homogeneous, if $f(x,y)$ must be a homogeneous function.

Problems :-

1. Solve $(y^2-2xy)dx = (x^2-2xy)dy$.

Sol :- Given $\frac{dy}{dx} = \frac{(y^2-2xy)}{(x^2-2xy)}$ be a D.E.

Let $y = vx$.

Diff. w.r.t

$$\frac{dy}{dx} = v + x \cdot \frac{dv}{dx}$$

$$v + x \cdot \frac{dv}{dx} = \frac{(v^2x^2 - 2x \cdot vx)}{(x^2 - 2x \cdot vx)}$$

$$v + x \cdot \frac{dv}{dx} = \frac{\cancel{x^2}(v^2 - 2v)}{\cancel{x^2}(1 - 2v)}$$

$$v + x \cdot \frac{dv}{dx} = \frac{v^2 - 2v}{(1 - 2v)}$$

$$x \cdot \frac{dv}{dx} = \frac{(v^2 - 2v)}{(1 - 2v)} - v.$$

$$x \cdot \frac{dv}{dx} = \frac{v^2 - 2v - v + 2v^2}{(1 - 2v)}$$

$$x \cdot \frac{dv}{dx} = \frac{3v^2 - 3v}{(1 - 2v)}$$

$$\frac{(1 - 2v)}{3(v^2 - v)} \cdot dv = \frac{1}{x} \text{ to } dx.$$

$$\Rightarrow -\frac{(2v - 1)}{3(v^2 - v)} dv = \frac{1}{x} dx.$$

Integrating on both sides.

$$\Rightarrow -\frac{1}{3} \log(v^2 - v) = \log x + \log c.$$

$$= \log(v^2 - v)^{-1/3} = \log xc$$

$$(v^2 - v)^{-1/3} = xc.$$

$$\Rightarrow \frac{1}{(v^2-v)^{1/3}} = xc.$$

$$= \frac{1}{\left(\frac{y^2}{x^2} - \frac{y}{x}\right)^{1/3}} = xc$$

2. Solve $\frac{dy}{dx} = \frac{y^3+3x^2y}{x^3+3xy^2}$

Sol :- Given $\frac{dy}{dx} = \frac{y^3+3x^2y}{x^3+3xy^2}$ be a D.E.

$$y = vx$$

$$\frac{dy}{dx} = v + x\cdot\frac{dv}{dx}.$$

$$v + x\cdot\frac{dv}{dx} = \frac{(v^3x^3 + 3x^2\cdot vx)}{(x^3 + 3x\cdot v^2x^2)}$$

$$v + x\cdot\frac{dv}{dx} = \frac{\cancel{x^3}(v^3+3v)}{\cancel{x^3}(1+3v^2)}$$

$$x\cdot\frac{dv}{dx} = \frac{(v^3+3v)}{(1+3v^2)} - v.$$

$$x\cdot\frac{dv}{dx} = \frac{v^3+3v-\cancel{v^4+3v^3}-v-3v^3}{(1+3v^2)}$$

$$x\cdot\frac{dv}{dx} = \frac{-2v^3+2v}{(1+3v^2)}$$

$$\frac{(1+3v^2)}{(2v-2v^3)}\,dv = \frac{1}{x}\,dx.$$

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$$\frac{1+3v^2}{v(1+v)(1-v)} = \frac{A}{v} + \frac{B}{(1+v)} + \frac{C}{(1-v)}$$

$$1+3v^2 = A(1+v)(1-v) + B(1-v)v + C(1+v)v$$

put $v=0$

$1 = A \Rightarrow A=1$

put $v=1$

$4 = 2C \Rightarrow C=2$

put $v=-1$

$4 = -2B \Rightarrow B=-2$

$$\Rightarrow \frac{1}{2}\left[\frac{1}{v} - \frac{2}{1+v} + \frac{2}{1-v}\right] dv = \frac{1}{x}\,dx.$$

Integrating on b.s.

$$\Rightarrow \frac{1}{2}\left[\log v - 2\log(1+v) - 2\log(1-v)\right] = \log x + \log c.$$

$$\Rightarrow \frac{1}{2}\left[\log\left(\frac{y}{x}\right) - 2\log\left(1+\frac{y}{x}\right) - 2\log\left(1-\frac{y}{x}\right)\right] = \log x + \log c.$$

Linear Diff. eqn :- A D.E is of the form $\frac{dy}{dx} + Py = Q$ is called first order first degree linear differential eqn in y where P, Q are the constants (or) functions in terms of 'x'.

Step-1 :- Find I.F $= e^{\int P\,dx}$.

Step-2 :- General soln $y \cdot I.F = \int Q \cdot I.F\,dx + C$.

Note :- suppose the D.E $\frac{dx}{dy} + P(y) \cdot x = Q(y)$ .. is L.D.E in "x".

Step-1 :- Find $I.F = e^{\int P\,dy}$

step-2 :- General solⁿ. $x \cdot I.f = \int Q(y) \cdot I.f \, dy + c$

Problems :-

* Solve $(1-x^2)\frac{dy}{dx} + 2xy = x\sqrt{1-x^2}$.

Sol :- Dividing both side with $1-x^2$,

$$\frac{dy}{dx} + \frac{2xy}{(1-x^2)} = \frac{x}{\sqrt{1-x^2}}$$

$$P(x) = \frac{2x}{1-x^2}, \quad Q(x) = \frac{x}{\sqrt{1-x^2}}$$

$$\text{Integrating factor} = e^{\int \frac{2x}{1-x^2} dx}$$

$$= e^{-\int \frac{-2x}{1-x^2} dx}$$

$$= e^{-\log(1-x^2)} = e^{\log(1-x^2)^{-1}}$$

$$I.f = (1-x^2)^{-1} = \frac{1}{(1-x^2)}$$

General solution $= y \cdot I.f = \int Q \cdot I.f \, dx + c$.

$$y \cdot \frac{1}{(1-x^2)} = \int \frac{x}{\sqrt{1-x^2}} \cdot \frac{1}{(1-x^2)} dx + c$$

$$= \int \frac{x}{(1-x^2)^{3/2}} dx + c$$

Put $(1-x^2) = t$

$$-2x\,dx = dt$$

$$x\,dx = -\tfrac{1}{2}\,dt$$

$$= \int \frac{-\frac{1}{2}dt}{t^{3/2}} \cancel{dt} + c$$

$-3/2+1$

$$= \frac{-1}{\cancel{2}} \left[\frac{t^{-1/2}}{-1/2}\right] + c$$

$$= \frac{1}{\sqrt{t}} + c.$$

$$y \cdot \frac{1}{1-x^2} = \frac{1}{\sqrt{1-x^2}} + c.$$

$$\underline{\underline{y = \sqrt{1-x^2} + c(1-x^2)}}$$

* Solve the D.E $(x+1)\frac{dy}{dx} - y = e^{3x}(x+1)^2$

Sol :- Dividing both sides with $(x+1)$

$$\frac{dy}{dx} - \frac{1}{(1+x)}y = e^{3x}(x+1).$$

is linear in y.

Integrating factor $= e^{\int P dx}$.

$$p(x) = \frac{-1}{(1+x)}, \quad q(x) = e^{3x}\frac{\cancel{(x+1)}}{(x+1)}$$

$$If = e^{\int \frac{-1}{(1+x)} dx}$$

$$= e^{-\log(x+1)}$$

$$= e^{\log(x+1)^{-1}}$$

$$If = \frac{1}{1+x}$$

General sol^n $y \cdot If = \int Q(x)\, If\, dx + c$

$$y \cdot \frac{1}{1+x} = \int e^{3x}\cancel{(x+1)} \cdot \frac{1}{\cancel{(1+x)}} dx + c$$

$$\frac{y}{(1+x)} = \frac{e^{3x}}{3} + c.$$

$$y = (1+x).\frac{e^{3x}}{3} + c(1+x)$$

* Solve the D.E $(1+x^2)\frac{dy}{dx} + 2xy = 4x^2.$

Sol:- Dividing both sides with $(1+x^2)$.

$$\frac{dy}{dx} + \frac{2x}{(1+x^2)} y = \frac{4x^2}{(1+x^2)}$$

$$p(x) = \frac{2x}{1+x^2}, \quad Q(x) = \frac{4x^2}{(1+x^2)}$$

$$I.f = e^{\int p dx} = e^{\int \frac{2x}{1+x^2} dx}$$

$$= e^{\log(1+x^2)} = (1+x^2)$$

General sol^n $y.I.f = \int Q(x) . I.f \, dx + c$

$$y.(1+x^2) = \int \frac{4x^2}{\cancel{(1+x^2)}} . \cancel{(1+x^2)} \, dx + c$$

$$y(1+x^2) = 4\frac{x^3}{3} \cancel{dx} + c$$

$$y = \frac{4x^3}{3(1+x^2)} + c$$

* Solve the D.E $\frac{dy}{dx} + \frac{y}{x\log x} = \frac{\sin 2x}{\log x}.$

Sol:- $p(x) = \frac{1}{x\log x}, \quad Q(x) = \frac{\sin 2x}{\log x}$

$I.f = e^{\int p dx} = e^{\int \frac{1}{x \log x} dx}$ $\quad \log x = t$

$\frac{1}{x} dx = dt$

$= e^{\int \frac{1}{t} dt}$

$I.f = e^{\log t} = t = \cancel{x} \log x.$

General soln $y . If = \int Q(x) . I.f \, dx + c.$

$y . \log x = \int \frac{\sin 2x}{\log x} . \log x \, dx + c$

$y . \log x = -\frac{\cos 2x}{2} + c$

$$y = \frac{-\cos 2x}{2 \log x} + \frac{c}{\log x}.$$

* Solve the D.E $\frac{dy}{dx} + (y-1) \cos x = e^{-\sin x} \cos^2 x.$

Sol:- given ~~the~~ D.E is linear in "y".

$\frac{dy}{dx} + \cos x \, y - \cos x = e^{-\sin x} \cos^2 x.$

$\frac{dy}{dx} + y . \cos x = e^{-\sin x} \cos^2 x + \cos x.$

$p(x) = \cancel{e^{\int p}} \cos x, \quad Q(x) = e^{-\sin x} \cos^2 x + \cos x$

$I.F = e^{\int p dx} = e^{\int \cos x \, dx}$

$= e^{\sin x}$

General soln $\cancel{y . If = e^{-\sin x} . \cos^2 x}$

$y . If = \int Q(x) . If \, dx + c.$

$$y \cdot e^{\sin x} = \int e^{-\sin x} \cos^2 x + \cos x \cdot e^{\sin x} \, dx + c$$

$$= \int \cos^2 x \, dx + \int \cos x \cdot e^{\sin x} \, dx + c$$

$\sin x = t$
$\cos x \, dx = d$

$$= \int \left(\frac{1 + \cos 2x}{2} \right) dx + \int e^t \, dt + c$$

$$y \cdot e^{\sin x} = \frac{1}{2}\left[x + \frac{\sin 2x}{2}\right] + e^{\cos x} + c$$

* Solve the D.E. $(1+y^2)dx = (\tan^{-1}y - x)\, dy$.

Sol:-
$$\frac{dx}{dy} = \frac{\tan^{-1}y}{(1+y^2)} - \frac{x}{1+y^2}$$

$$\frac{dx}{dy} + \frac{x}{1+y^2} = \frac{\tan^{-1}y}{(1+y^2)}$$

$$P(y) = \frac{1}{1+y^2}, \quad Q(y) = \frac{\tan^{-1}y}{(1+y^2)}$$

$$I.F = e^{\int P dy} = e^{\int \frac{1}{1+y^2} dy}$$

$$= e^{\tan^{-1}y}$$

Put $y^2 = t$
$2y \cdot dy = dt$
$dy = \frac{1}{2y} \cdot dt$

General sol^n $x \cdot If = \int Q(y) \cdot If \, dy + C$.

$$x \cdot e^{\tan^{-1}y} = \int \frac{\tan^{-1}y}{(1+y^2)} \cdot e^{\tan^{-1}y} \, dy + C.$$

put $\tan^{-1}y = t$

$$\frac{1}{1+y^2} dx = dt$$

$$= \int t \cdot e^t \, dt$$

$$= t \cdot e^t - 1 \cdot e^t + c$$

$y \cdot e^{\tan^{-1}y} = e^{\tan^{-1}y}(\tan^{-1}y - 1) + c$

$y = (\tan^{-1}y - 1) + c(e^{\tan^{-1}y})$.

* solve the D.E $(1+y^2) + (x - e^{\tan^{-1}y})\frac{dy}{dx} = 0$.

Sol :-
$$(x - e^{\tan^{-1}y})\frac{dy}{dx} = -(1+y^2)$$

$$x - e^{\tan^{-1}y} = -(1+y^2) \cdot \frac{dx}{dy}.$$

$$\frac{dx}{dy} = \frac{-(x - e^{\tan^{-1}y})}{(1+y^2)}$$

$$\frac{dx}{dy} = \frac{e^{\tan^{-1}y}}{1+y^2} \mp \frac{x}{1+y^2}$$

$$\frac{dx}{dy} + \frac{x}{(1+y^2)} = \frac{e^{\tan^{-1}y}}{1+y^2}.$$

$$P(y) = \frac{1}{1+y^2}\ ;\ Q(y) = \frac{e^{\tan^{-1}y}}{1+y^2}.$$

$$I.F = e^{\int P dy} = e^{\int \frac{1}{1+y^2} dy}$$
$$= e^{\tan^{-1}y}.$$

General soln is $x \cdot If = \int Q(y) \cdot If\, dy + c$

$$x \cdot e^{\tan^{-1}y} = \int \frac{e^{\tan^{-1}y}}{(1+y^2)} \cdot e^{\tan^{-1}y} + c$$

put $\tan^{-1}y = t$

$\frac{1}{1+y^2} dy = dt$

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$$x \cdot e^{\tan^{-1}y} = \int e^{2t}\, dt + c$$

$$= \frac{e^{2t}}{2} + c$$

$$x \cdot e^{\tan^{-1}y} = \frac{e^{2\tan^{-1}y}}{2} + c$$

$$x = \frac{e^{\tan^{-1}y}}{2} + c(e^{-\tan^{-1}y}).$$

* Solve the D.E $\sin 2x \cdot \frac{dy}{dx} - y = \tan x$.

* $(x + \tan y)\, dy = \sin 2y\, dx$.

* $(y - e^{\sin^{-1}x}) \frac{dx}{dy} + \sqrt{1-x^2} = 0$.

* Solve the D.E $\sin 2x \cdot \frac{dy}{dx} - y = \tan x$.

Sol:- diving both sides $\sin 2x$.

$$\frac{dy}{dx} - \frac{y}{\sin 2x} = \frac{\tan x}{\sin 2x}.$$

$$P(x) = \frac{-1}{\sin 2x}, \quad Q(x) = \frac{\cancel{\sin x}}{2\,\cancel{\sin x} \cos^2 x} = \frac{1}{2}\sec^2 x.$$

$$I.F = e^{\int P\, dx} = e^{-\int \frac{1}{\sin 2x} dx} = e^{-\int \operatorname{cosec} 2x\, dx}$$

$$= e^{-\log\left(\frac{\operatorname{cosec} 2x - \cot 2x}{2}\right)}$$

$$= \left(\frac{\operatorname{cosec} 2x - \cot 2x}{2}\right)^{-1}$$

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Bernouli's differential eqn :-

A D.E is of the form $\frac{dy}{dx} + P(x)y = Q(x) \cdot y^n$ is called a Bernouli's eqn where n = real constant.

If $n=0$ then Bernouli's D.E is known as linear Differential eqn.

Problems :-

* solve D.E ~~show that~~ $\frac{dy}{dx} + \tan x\, y = \sec x\, y^2$.

sol :- Given D.E $\frac{dy}{dx} + \tan x\, y = \sec x\, y^2$.

dividing both sides with "y^2."

$$\frac{1}{y^2}\frac{dy}{dx} + \frac{\tan x}{y} = \sec x.$$

let $\frac{1}{y} = u$

$$\frac{-1}{y^2}\frac{dy}{dx} = \frac{du}{dx}$$

$$-\frac{du}{dx} + \tan x \cdot u = \sec x.$$

$\frac{du}{dx} - \tan x \cdot u = -\sec x$ is linear in 'u'

$P(x) = -\tan x$, $Q(x) = -\sec x$.

$$If = e^{\int P dx} = e^{-\int \tan x\, dx} = e^{-\log \sec x} = \frac{1}{\sec x}$$

General soln is $u \cdot If = \int Q(x) \cdot If\, dx + c.$

$$\Rightarrow \frac{1}{y} \cdot \frac{1}{\sec x} = \int -\cancel{\sec x} \cdot \frac{1}{\cancel{\sec x}}\, dx + c$$

$\frac{1}{y \cdot \sec x} = -x + c.$

$\Rightarrow -xy\sec x + cy\sec x = 1$

* Solve the de $\frac{dy}{dx} - y\tan x = \frac{\sin x \cos^2 x}{y^2}$

Sol: Given D.E. $\frac{dy}{dx} - y\tan x = \frac{\sin x\cos^2 x}{y^2}$

Multiplying with 'y^2' on both sides

$y^2 \cdot \frac{dy}{dx} - y^3\tan x = \sin x\cos^2 x.$

let $y^3 = u$

$3y^2 \cdot \frac{dy}{dx} = \frac{du}{dx}.$

$y^2 \cdot \frac{dy}{dx} = \frac{1}{3}\frac{du}{dx}$

$\frac{1}{3}\frac{du}{dx} - u\cdot\tan x = \sin x\cos^2 x.$

$\frac{du}{dx} - 3\cdot u\tan x = 3\sin x\cos^2 x.$

$P(x) = -3\tan x,\ Q(x) = 3\sin x\cos^2 x$

$I.f = e^{\int P dx} = e^{-3\int \tan x\, dx} = e^{-3\log \sec x} = (\sec x)^{-3}$

$I.f = \underline{\cos^3 x}$

General soln is $u \cdot I.f = \int Q(x) \cdot I.f\, dx + c$

$\Rightarrow y^3 \cdot \cos^3 x = \int 3\sin x\cos^2 x \cdot \cos^3 x\, dx + c$

$= 3\int \sin x\cos^5 x\, dx + c.$

$\cos x = t \Rightarrow -\sin x\, dx = dt.$

$= -3\int t^5\, dt + c$

$= -\cancel{3}\left[\frac{t^6}{\cancel{6}_2}\right] + c$

$y^3 \cos^3 x = \frac{-t^6}{2} + c$

$y^3 \cos^3 x = \frac{-\cos^6 x}{2} + c$

* Solve the D.E $3x(1-x^2) \cdot y^2 \frac{dy}{dx} + (2x^2-1) y^3 = ax^3$.

Sol:- Given D.E is $3x(1-x^2) \cdot y^2 \frac{dy}{dx} + (2x^2-1) y^3 = ax^3$.

Dividing b.s with $x(1-x^2)$

$$3y^2 \frac{dy}{dx} + \frac{(2x^2-1)}{x(1-x^2)} \cdot y^3 = a \cdot \frac{x^3}{x(1-x^2)}.$$

Let $y^3 = u$.

$$\Rightarrow 3y^2 \cdot \frac{dy}{dx} = \frac{du}{dx}$$

$$\frac{du}{dx} + \frac{(2x^2-1)}{x(1-x^3)} \cdot u = a \cdot \frac{x^2}{(1-x^2)}$$ & is linear in 'u'

$$P(x) = \frac{(2x^2-1)}{x(1-x^3)}, \quad Q(x) = \frac{ax^2}{(1-x^2)}$$

$$I.f = e^{\int P dx}$$

$$= e^{\int \frac{(2x^2-1)}{x(1-x^3)} dx}$$

$$= e^{-\int \frac{x^2 + (x^2-1)}{x(1-x^3)} dx}$$

$$= e^{-\int \left[\frac{x^2}{x(x^2-1)} + \frac{(x^2-1)}{x(x^2-1)} \right] dx}$$

$$= e^{-\int \left(\left(\frac{x}{x^2-1} \right) + \frac{1}{x} \right) dx}$$

$$= e^{-\left[\frac{1}{2} \log(x^2-1) + \log x \right]}$$

$$= e^{-[\log (x^2-1)^{1/2} + \log x]}$$

$$= e^{-[\log x\sqrt{x^2-1}]}$$

$$= e^{\log [x\sqrt{x^2-1}]^{-1}}$$

$$I.F = \frac{1}{x\sqrt{x^2-1}}$$

General Soln $\quad u \cdot If = \int Q(x) \cdot If \, dx + c.$

$$y^3 \cdot \frac{1}{x\sqrt{x^2-1}} = \int \frac{a x^2}{(1-x^2)} \cdot \frac{1}{x\sqrt{x^2-1}} \, dx + c$$

$$= -a \int \frac{a}{(x^2-1)^{3/2}} \, dx + c.$$

let $x^2-1 = t$

$2x \, dx = dt$

$\Rightarrow x \, dx = \frac{1}{2} dt.$

$$= -\frac{1}{2} a \int \frac{1}{t^{3/2}} \, dt + c$$

$$= -\frac{1}{2} a \left[\frac{t^{-3/2+1}}{-3/2+1} \right] + c \Rightarrow \frac{a}{\sqrt{t}} + c$$

$$y^3 \cdot \frac{1}{x\sqrt{x^2-1}} = \frac{a}{\sqrt{x^2-1}} + c$$

* Solve the D.E $x\frac{dy}{dx} + y = y^2 x^3 \cos x.$

Sol:- Given the D.E is $x\frac{dy}{dx} + y = y^2 \cdot x^3 \cos x$

Dividing both sides with y^2

$$\frac{x}{y^2} \frac{dy}{dx} + \frac{1}{y} = x^3 \cos x.$$

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put $\frac{1}{y} = u$

$$-\frac{1}{y^2}\frac{dy}{dx} = \frac{du}{dx}.$$

$$-x\frac{du}{dx} + u = x^3 \cos x.$$

$$x.\frac{du}{dx} - u = -x^3\cos x \Rightarrow \frac{du}{dx} - \frac{1}{x}u = -x^2\cos x.$$

$P(x) = -1,\ Q(x) = -x^3\cos x$

$I.F = e^{\int P dx} = \cancel{e^{-\log}}\ e^{-\int 1 dx} \Rightarrow e^{-x}$

General soln $\quad u.If = \int Q(x).If + c.$

$$\frac{1}{y}.e^{-x} = \int x^3\cos x.e^{-x} + c$$

$P(x) = \frac{-1}{x},\ Q(x) = -x^2\cos x$

$I.F = e^{\int P dx} = e^{-\log x} = \frac{1}{x}.$

General soln $\quad u.If = \int Q(x).If + c$

$$\frac{1}{y}.\frac{1}{x} = \int -x^2\cos x.\frac{1}{x} + c$$

$$\frac{1}{xy} = \int -x\cos x + c$$

$$\Rightarrow \frac{1}{xy} = \left[x.(\sin x) - 1.(-\cos x)\right]$$

$$\frac{1}{xy} = -x\sin x - \cos x + c$$

* Solve the D.E $x^2y\,dx - (x^3+y^3)\,dy = 0$

Sol :- Given D.E is $x^2y\,dx - (x^3+y^3)\,dy = 0$.

$$x^2y \cdot \frac{dx}{dy} - x^3 - y^3 = 0$$

$$x^2y\frac{dx}{dy} - x^3 = y^3$$

$$x^2\frac{dx}{dy} - \frac{1}{y}x^3 = y^2$$

$$\text{let } x^3 = u$$

$$\Rightarrow 3x^2 \cdot \frac{dx}{dy} = \frac{du}{dy}.$$

$$\Rightarrow x^2 \cdot \frac{dx}{dy} = \frac{1}{3}\frac{du}{dy}.$$

$$\frac{1}{3}\cdot\frac{du}{dy} - \frac{1}{y}\cdot u = y^2.$$

$$\frac{du}{dy} - \frac{3}{y}\cdot u = 3y^2.$$

L.E. $P(y) = \frac{-3}{y}$, $Q(y) = 3y^2$.

$$IF = e^{\int P\,dy} = e^{-3\int\frac{1}{y}dy} = e^{\log y^3} \Rightarrow \frac{1}{y^3}.$$

General Soln $u \cdot IF = \int Q(y) \cdot IF + C$

$$x^3 \cdot \frac{1}{y^3} = \int 3\cancel{y^2}\cdot\frac{1}{y^{\cancel{3}}} + C$$

$$\frac{x^3}{y^3} = 3\log y + C$$

* Solve the D.E $3.\frac{dy}{dx} - y\cos x = y^4(\sin 2x - \cos x)$

Sol:- Given the D.E $3.\frac{dy}{dx} - y\cos x = y^4(\sin 2x - \cos x)$

$$\frac{3}{y^4}\frac{dy}{dx} - \frac{1}{y^3}\cos x = (\sin 2x - \cos x).$$

put $\frac{-1}{y^3} = u.$

$$\frac{3}{y^4}\frac{dy}{dx} = \frac{du}{dx}.$$

$$\frac{du}{dx} + u\cos x = (\sin 2x - \cos x)$$

$$P(x) = +\cos x, \quad Q(x) = (\sin 2x - \cos x).$$

$$I.F = e^{\int P dx} = e^{+\int \cos x\, dx} = e^{\sin x} + c.$$

General soln $u. If = \int Q(x). If + C.$

$$\frac{-1}{y^3}. e^{\sin x} = \int (\sin 2x - \cos x). e^{\sin x} + c$$

$$= \int e^{\sin x}(2\sin x\cos x - \cos x)\, dx + c$$

$$= \int e^{\sin x}\cos x(2\sin x - 1)\, \underline{dx} + c$$

put $\sin x = t$

$+\cos x\, dx = dt.$

$$= +\int e^t.(2t-1)\, dt$$

$$= \int e^t . 2t - \int e^t dt$$

$$= \cancel{e^t . t^2} - \cancel{e^t}$$

$$= e^t \cancel{(t^2-1)}\, dt$$

$= (2t-1)e^t - 2e^t + c.$

$\frac{-1}{y^3} \cdot e^{\sin x} = (2\sin x - 1)e^{\sin x} - 2e^{\sin x} + c$

* Solve the D.E $\frac{dy}{dx} + yx = y^2 \cdot e^{x^2/2} \sin x.$

Sol:- Given the D.E $\frac{dy}{dx} + yx = y^2 \cdot e^{x^2/2} \sin x.$

$$\frac{1}{y^2}\frac{dy}{dx} + \frac{1}{y}x = e^{x^2/2} \sin x.$$

put $\frac{1}{y} = u$

$$-\frac{1}{y^2} \cdot \frac{dy}{dx} = \frac{du}{dx}.$$

$$-\frac{du}{dx} + u \cdot x = e^{x^2/2} \sin x.$$

$$\frac{du}{dx} - ux = -e^{x^2/2} \sin x.$$

$P(x) = -x, \quad Q(x) = -e^{x^2/2} \sin x.$

$I.F = e^{\int P dx} = e^{-\int x dx} = e^{-x^2/2}$ ~~dx+c~~

General sol^n $u \cdot I.f = \int Q(x) \cdot I.f + c$

$$\frac{1}{y} \cdot e^{-x^2/2} = \int \cancel{e^{+x^2/2}} \sin x \cdot \cancel{e^{-x^2/2}} + c$$

$$\frac{1}{y} \cdot e^{-x^2/2} = \cos x + c$$

* $(1-x^2)\dfrac{dy}{dx} + xy = y^3 \sin^{-1}x.$

* $\cos x\, dy = y(\sin x - y)\,dx$

* $x^3 \cdot \dfrac{dy}{dx} - x^3 y + y^4 \cos x = 0.$

* Solve the D.E $2y \cdot \cos y^2 \dfrac{dy}{dx} - \dfrac{2}{x+1} \sin y^2 = (x+1)^2.$

Sol:- Given D.E $2y \cdot \cos y^2 \dfrac{dy}{dx} - \dfrac{2}{(x+1)} \sin y^2 = (x+1)^2.$

let $\sin y^2 = u.$

$$\cos y^2 \cdot 2y \cdot \frac{dy}{dx} = \frac{du}{dx}.$$

$$\frac{du}{dx} - \frac{2}{(x+1)} u = (x+1)^3.$$

$P(x) = \dfrac{-2}{(x+1)}$, $Q(x) = (x+1)^3.$

$$I.F = e^{\int P dx} = e^{-\int \frac{2}{(x+1)} dx} = e^{-2\log(x+1)}$$

$$= \frac{1}{(x+1)^2}$$

General sol^n $u \cdot If = \int Q(x) \cdot If + c$

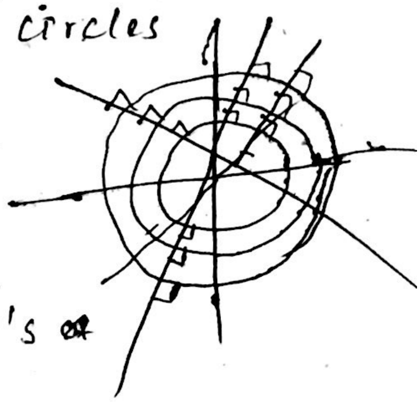
$$\sin y^2 \cdot \frac{1}{(x+1)^2} = \int (x+1)^3 \cdot \frac{1}{(x+1)^2} dx + c$$

$$\sin y^2 \cdot \frac{1}{(x+1)^2} = \cancel{\log(x+1)} \frac{x^2}{2} + x + c$$

***

Orthogonal Trajectories :- The family curves in which intersects the given family of curves at right angles is known as orthogonal Trajectories to the given family of curves.

Ex :- $x^2+y^2=c^2$, be a given family of circles



The family of st. lines $y=cx$ intersects the family of circles at right angles.

$\therefore$ the family of st lines are the O.T's to the family of circles.

Working Rule :-

Suppose $f(x,y,c)=0$ be a given family of curves. $\rightarrow (1)$

Step-1 :- Eliminating arbitary constants "c" by differentiating w.r.t "x". i.e, $f(x,y,\frac{dy}{dx})=0 \rightarrow (2)$.

Step-2 :- For an orthogonal trajectory replace $\frac{dy}{dx}$ with $\frac{-dx}{dy}$

i.e., $f(x,y,-\frac{dx}{dy})=0 \rightarrow (3)$

Step-3 :- Solve the eqn (3) by known methods, to attain the O.T. to the given family of curves.

* Find the O.T's for the family of circles $x^2+y^2=c^2$.

Sol :- given family of curves $x^2+y^2=c^2$

diff. w.r.t 'x'

$$2x+2y.\frac{dy}{dx}=0$$

* Replace $\frac{dy}{dx}$ with $-\frac{dx}{dy}$

$$2x-2y.\frac{dx}{dy}=0$$

$$2x=2y.\frac{dx}{dy}$$

$$x=y.\frac{dx}{dy}.$$

$$\frac{1}{y}dy=\frac{dx}{x}.$$

$$\Rightarrow \log y=\log x+\log c.$$

$$\Rightarrow y=cx$$

* Find the O.T for the family of circles $x^2+(y-c)^2=c^2$,

Sol :- Given the family of circles $x^2+(y-c)^2=0$.

diff. w.r.t "x"

$$2x+2(y-c)\frac{dy}{dx}=0$$

$$x=-(y-c)\frac{dy}{dx}.$$

$$(y-c)=-x\frac{dx}{dy}$$

$$c=y+x\frac{dx}{dy} \rightarrow (1)$$

$$\therefore x^2+\left[-x\frac{dx}{dy}\right]^2=\left[y+x\cdot\frac{dx}{dy}\right]^2.$$

$$\Rightarrow x^2+\cancel{x^2\left(\frac{dx}{dy}\right)^2}=y^2+\cancel{x^2\left(\frac{dx}{dy}\right)^2}+2\cdot y\cdot x\frac{dx}{dy}$$

$$x^2=y^2+2xy\cdot\frac{dx}{dy}\cdot\rightarrow (2)$$

FOr O.T's Replace $\frac{dx}{dy}$ with $\frac{-dy}{dx}$.

$$x^2=y^2+2xy\cdot\left(\frac{-dy}{dx}\right)$$

$$x^2=y^2-2xy\frac{dy}{dx}.$$

$$\Rightarrow 2xy\frac{dy}{dx}-y^2=-x^2.$$

$$2y\frac{dy}{dx}-\frac{1}{x}y^2=-x.$$

Put $y^2=u$

$$2y.\frac{dy}{dx}=\frac{du}{dx}.$$

$$\frac{du}{dx}-\frac{1}{x}u=-x.$$ is linear in u

$$P(x)=-\frac{1}{x},\ Q(x)=-x.$$

$$I.F=e^{\int Pdx}=e^{-\int\frac{1}{x}dx}=e^{-\log x}=1/x.$$

General sol^n $\Rightarrow$ $u.IF=\int Q(x).If\,dx+C.$

$$y^2\cdot\frac{1}{x}=\int -x\cdot\frac{1}{x}+C$$

$$\frac{y^2}{x}=-x+C$$

$$y^2=-x^2+cx$$

$$x^2+y^2=cx$$

* Find the O.T's for family of circles $x^2+y^2+2gx+c=0$. where "g" is a parameter.

Sol :- Given family of circles $x^2+y^2+2gx+c=0 \rightarrow (1)$

diff w.r.t "x"

$$2x+2y\cdot\frac{dy}{dx}+2g=0$$

$$2g=-2x-2y\cdot\frac{dy}{dx}$$

Substitute $2g$ in eq (1)

$$x^2+y^2+\left[-2x-2y\cdot\frac{dy}{dx}\right]x+c=0$$

$$x^2+y^2-2x^2-2xy\cdot\frac{dy}{dx}+c=0$$

$$\Rightarrow -x^2+y^2-2xy\cdot\frac{dy}{dx}+c=0 \rightarrow (2)$$

For O.T Replace $\frac{dy}{dx}$ with $-\frac{dx}{dy}$.

$$\Rightarrow -x^2+y^2+2xy\cdot\frac{dx}{dy}+c=0 \rightarrow (3)$$

$$2xy\cdot\frac{dx}{dy}=x^2-y^2-c$$

$$2xy\frac{dx}{dy}-x^2=-y^2-c$$

$$2x\cdot\frac{dx}{dy}-\frac{1}{y}\cdot x^2=\frac{-y^2-c}{y}$$

Let $x^2=u$

$$2x\cdot\frac{dx}{dy}=\frac{du}{dx}$$

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$$\frac{du}{dx} - \frac{1}{y}u = -y - c/y.$$

$$P(y) = \frac{-1}{y}, \quad Q(y) = -y - c/y.$$

$$I.F = e^{\int P dy} = e^{-\int \frac{1}{y} dy} = e^{-\log y} = \frac{1}{y}.$$

General sol^n $u \cdot If = \int Q \cdot If \, dy + c_1$

$$\frac{x^2}{y} = \int \left(-y - c/y\right) \cdot 1/y \, dy + c_1$$

$$\frac{x^2}{y} = \int \left(-1 - c/y^2\right) dy + c_1$$

$$\frac{y^{-2+1}}{-2+1}$$

$$\frac{x^2}{y} = -y + \frac{c}{y} + c_1$$

$$\frac{x^2}{y} = -y^2 + c + c_1 y$$

$$\underline{\underline{x^2 + y^2 - c_1 y = c}}$$

* Find the O.T's of the family of parabolas $y^2 = 4ax$.

$\underline{\underline{Sol}}$:- Given system of parabolas $y^2 = 4ax \to (1)$

diff. w.r.t 'x'.

$$2y \cdot \frac{dy}{dx} = 4a.$$

Substitute $4a$ in eq. (1)

$$y^2 = 2y \cdot \frac{dy}{dx} \cdot x$$

$$y = 2x \frac{dy}{dx}$$

For O.T replace $\frac{dy}{dx}$ with $-\frac{dx}{dy}$.

$$y = -2x\frac{dx}{dy}.$$

$$\int y\,dy = -2x\,dx.$$

$$\int y\,dy = \int -2x\,dx$$

$$\frac{y^2}{2} = -\cancel{2}\cdot\frac{x^2}{\cancel{2}}\,dx + c$$

$$y^2 = -2x^2 \cancel{d} + c$$

$$\underline{\underline{y^2 + 2x^2 = c^2}}$$

* Prove that the system of rectangular hyperbolas $x^2 - y^2 = a^2$ and $xy = c^2$ are mutually orthogonal trajectories.

__Sol:__ Given the family of rectangular hyperbolas $x^2 - y^2 = a^2$

and $xy = c^2$

consider $x^2 - y^2 = a^2$.

$$2x - 2y\cdot\frac{dy}{dx} = 0.$$

$$x = y\cdot\frac{dy}{dx}.$$

Replace $\frac{dy}{dx}$ with $-\frac{dx}{dy}$.

$$x = -y\cdot\frac{dx}{dy}$$

$$\int\frac{1}{x}\,dx = \int -\frac{1}{y}\,dy$$

$$\log x = -\log y + c$$

$\log x + \log y = c$

$c = xy$

consider $xy = c^2$.

diff w.r.t "x"

$$x \cdot \frac{dy}{dx} + y = 0$$

replace $\frac{dy}{dx}$ with $-\frac{dx}{dy}$.

$$-x \cdot \frac{dx}{dy} + y = 0$$

$$+\int x \, dx = \int y \, dy.$$

$$\frac{x^2}{2} = \frac{y^2}{2} + c$$

$$\frac{x^2}{2} - \frac{y^2}{2} = c$$

$$x^2 - y^2 = 2c$$

* Prove that the system of parabolas $y^2 = 4a(x+a)$ are self orthogonal.

Sol Given system of parabolas $y^2 = 4a(x+a)$

$= 4ax + 4a^2$

diff with r. to "x"

$$2y \cdot \frac{dy}{dx} = 4a$$

$$a = \frac{1}{2} y \frac{dy}{dx}$$

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$$y^2 = 2y\cdot\frac{dy}{dx}\left(x+\frac{1}{2}y\frac{dy}{dx}\right)$$

$$= 2yy'\left(x+\frac{1}{2}yy'\right)$$

$$2\times\tfrac{1}{2}\, y = 2y'x + \tfrac{1}{2}y'$$

$$y = 2y'x + y(y')^2 \rightarrow (2).$$

For O.T, Replace $\frac{dy}{dx}$ with $-\frac{dx}{dy} \Rightarrow \frac{-1}{dy/dx}$

y' with $\frac{-1}{y'}$.

$$\Rightarrow y = \frac{-2x}{y'} + y\left(\frac{-1}{y'}\right)^2$$

$$y = \frac{-2x}{y'} + \frac{y}{(y')^2}$$

$$y(y')^2 = -2xy' + y.$$

$$y = 2xy' + y(y')^2.$$

By observing that the given system of parabolas are self orthogonal.

** Show that the system of [non-homogeneous] confocal conics $\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1$ is self orthogonal. where λ is a parameter

Sol:- Given the system of confocal conics is $\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1 \rightarrow (1)$

diff. (1) w.r.t 'x'

$$\frac{2x}{a^2+\lambda} + \frac{2y\cdot y'}{b^2+\lambda} = 0$$

$$x(b^2+\lambda) + y \cdot y'(a^2+\lambda) = 0$$

$$b^2x + x\lambda + a^2y\cdot y' + \lambda y \cdot y' = 0$$

$$(b^2x + a^2y\cdot y') + \lambda(x + y\cdot y') = 0$$

$$\lambda(x+y\cdot y') = -(b^2x + a^2y\cdot y')$$

$$\boxed{\lambda = \frac{-(b^2x + a^2yy')}{(x+y\cdot y')}}$$

"λ" value sub in (1)

$$\Rightarrow \frac{x^2}{a^2 - \left[\frac{b^2x + a^2yy'}{x+y\cdot y'}\right]} + \frac{y^2}{b^2 - \left[\frac{b^2x + a^2yy'}{x+y\cdot y'}\right]} = 1$$

$$\Rightarrow \frac{x^2}{\frac{a^2(x+\cancel{y\cdot y'}) - [b^2x + a^2\cancel{yy'}]}{x+y\cdot y'}} + \frac{y^2}{\frac{\cancel{b^2}(x+y\cdot y') - [\cancel{b^2x}+a^2yy']}{x+y\cdot y'}}$$

$$\Rightarrow \frac{x^2(x+yy')}{a^2x - b^2x} + \frac{y^2(x+yy')}{b^2yy' - a^2yy'} = 1$$

$$\Rightarrow \frac{x^2(x+yy')}{x(a^2-b^2)} - \frac{y^2(x+yy')}{yy'(a^2-b^2)} = 1$$

$$\left(\frac{x+yy'}{a^2-b^2}\right)\left[x - \frac{y}{y'}\right] = 1$$

$$(x+yy')\cdot\left(x - \frac{y}{y'}\right) = a^2 - b^2 \longrightarrow (2)$$

For O.T; Replace y' with $-\frac{1}{y'}$

$$\Rightarrow \left(x - \frac{y}{y'}\right)(x + yy') = a^2 - b^2 \rightarrow (3)$$

By observing eqn (2) & (3) the given system of confocal conics are self orthogonal.

* Find the O.T of the family of curves $x^{2/3} + y^{2/3} = a^{2/3}$.

Sol :- Given the system of family of curves $x^{2/3} + y^{2/3} = a^{2/3}$

diff. w.r.t "x".

$\frac{2}{3} - 1$

$$\frac{\cancel{2}}{3} x^{-1/3} + \frac{\cancel{2}}{3} y^{-1/3} \frac{dy}{dx} = 0.$$

$$x^{-1/3} + y^{-1/3} \frac{dy}{dx} = 0 \rightarrow (1)$$

Replacing $\frac{dy}{dx}$ with $\frac{-dx}{dy}$.

$$x^{-1/3} - y^{-1/3} \frac{dx}{dy} = 0$$

$$\int y^{1/3}\, dy = \int x^{1/3}\, dx$$

$\frac{1}{3} + 1$

$$\frac{y^{4/3}}{4/3} = \frac{x^{4/3}}{4/3}$$

$$x^{4/3} - y^{4/3} = c$$

Orthogonal Trajectories in polar form :-

Suppose the system of ~~eqns~~ curves is in the form of $f(r, \theta, c) = 0 \rightarrow (1)$,

where "c" is a arbitary const or parameter.

Step-1 :- Differentiating the eqn (1) w.r.t "θ" to eliminate the arbitrary constant "c".

i.e., $f\left(r, \theta, \frac{dr}{d\theta}\right) = 0 \rightarrow (2)$.

Step-2 :- For orthogonal trajectory replace $\frac{dr}{d\theta}$ with $-r^2 \frac{d\theta}{dr}$

i.e., $f\left(r, \theta, -r^2 \frac{dr}{d\theta}\right) \rightarrow (3)$.

Step-3 :- Solve the D.E (3) by known methods to obtain the required O.T for the given system of curves.

Problems :-

* Find the O.T's ~~of~~ the family of circles $r = a\cos\theta$ where a is a parameter.

Sol :- Given system of circles $r = a\cos\theta \rightarrow (1)$

diff. (1) w.r.t "θ"

$$\frac{dr}{d\theta} = -a\sin\theta$$

$$a = \frac{-1}{\sin\theta}\frac{dr}{d\theta}.$$

Substitute a in eqn ①.

$$r = \frac{-1}{\sin\theta} \cdot \frac{dr}{d\theta} \cos\theta .$$

$$r = -\cot\theta \cdot \frac{dr}{d\theta} .$$

For O.T, Replace $\frac{dr}{d\theta}$ with $\cancel{\frac{-1}{r^2}/dr}$ $-r^2\frac{d\theta}{dr}$

$$\cancel{r} = +\cot\theta \cdot r^{\cancel{2}} \frac{d\theta}{dr}$$

$\cancel{-\cot\theta d\theta = r dr}$ $\frac{1}{r} dr = \cot\theta d\theta$

Integrating on both sides

$$\int \frac{1}{r} dr = \int \cot\theta \, d\theta$$

$$\log r = \log\sin\theta + \log c .$$

$$\underline{\underline{r = c\sin\theta}}$$

*Find the O.T's for the given family of curves $r = a.(1+\cos\theta)$.

Sol: Given family of curves is $r = a(1+\cos\theta) \rightarrow ①$

diff eq ① w.r.t θ

$$\frac{dr}{d\theta} = a(-\sin\theta)$$

$$a = \frac{-1}{\sin\theta} \frac{dr}{d\theta}$$

Sub a in eq ②

$$r = -\frac{1}{\sin\theta}(1+\cos\theta)\frac{dr}{d\theta}$$

$$\cancel{r} \cancel{-\frac{1}{\sin\theta}} + \cancel{\cot\theta}.$$

$$r = -\frac{2\cos^2\theta/2}{2\sin\theta/2\cos\theta/2}\frac{dr}{d\theta}$$

$$r = -\cot\theta/2\,\frac{dr}{d\theta}.$$

For O.T, Replace $\frac{dr}{d\theta}$ with $-r^2\frac{d\theta}{dr}$.

$$r = -\cot\theta/2\left[-r^2\frac{d\theta}{dr}\right]$$

$$1 = \cot\theta/2\,\frac{d\theta}{dr}\cdot r$$

$$\int\frac{1}{r}dr = \int\cot\theta/2\cdot d\theta$$

$$\log r = 2\log\sin\theta/2 + \log c$$

$$\log r = \log\sin^2\theta/2 + \log c.$$

$$r = (\sin^2\theta/2\; c)$$

* Find the O.T's of system of curves is $r^n = a^n\cos n\theta$.

Sol:- Given system of curves is $r^n = a^n\cos n\theta$. $\rightarrow$ (1)

~~diff w.r.t 'θ'~~ Taking log on both sides

$$n\log r = \log a^n + \log\cos n\theta$$

$$n\log r = n\log a + \log\cos n\theta. \rightarrow (2)$$

diff. w.r.to "θ"

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$$\phi \cdot \frac{1}{r}\frac{dr}{d\theta} = \frac{1}{\cos n\theta}(-\sin n\theta)\cdot n$$

$$\frac{1}{r}\frac{dr}{d\theta} = -\tan n\theta.$$

For O.T, Replace $\frac{dr}{d\theta}$ with $-r^2\frac{d\theta}{dr}$.

$$\frac{1}{r}\cdot -r^2\frac{d\theta}{dr} = +\tan n\theta$$

$$\frac{1}{r}dr = \frac{1}{\tan n\theta}d\theta$$

$$\int \frac{1}{r}dr = \int \cot n\theta \, d\theta.$$

By integrating,

$$\log r = \frac{\log \sin n\theta}{n} + \log c$$

$$\log \sin n\theta = n\log r + \log c$$

$$\log \sin n\theta = \log[c^n r^n]$$

$$\sin n\theta = r^n c^n$$

$$\boxed{r^n = c_1^n . \sin n\theta}$$

** $r^n \sin n\theta = a^n$,

Sol :- Given the system of curves is $r^n \sin n\theta = a^n$

Taking log on both sides.

$$\log r^n + \log \sin n\theta = \log a^n.$$

$$n\log r + \log \sin n\theta = n\log a.$$

diff w.r.t "θ".

$$n \cdot \frac{1}{r}\frac{dr}{d\theta} + \frac{1}{\sin n\theta}(\cos n\theta) \cdot n = 0$$

$$\frac{1}{r}\frac{dr}{d\theta} + \tan n\theta = 0.$$

for O.T's

Replacing $\frac{dr}{d\theta}$ with $-r^2\frac{d\theta}{dr}$

$$\frac{1}{r} \cdot -r^2\frac{d\theta}{dr} + \tan n\theta = 0 \quad -\tan n\theta \cdot \frac{1}{d\theta}$$

$$\int \cot n\theta \, d\theta = \int \frac{1}{r} dr.$$

$$\frac{\log \sin n\theta}{n} = \log r + \log c.$$

$$\underline{\underline{\sin n\theta = r^n c^n}}$$

* Find the O.T of family of curves $r = \frac{2a}{(1+\cos\theta)}$

Sol :- Given $r = \frac{2a}{(1+\cos\theta)}$

diff w.r.t "θ".

$$r(-\sin\theta) + (1+\cos\theta)\frac{dr}{d\theta} = 0$$

$$(1+\cos\theta)\frac{dr}{d\theta} = r\sin\theta$$

$$\frac{dr}{d\theta} = \frac{r\sin\theta}{(1+\cos\theta)}$$

for OT,

*Newton's law of cooling :- The rate of change of temperature of a body is proportional to the diff. b/w the temperature of a body and temp. of surrounding medium

i.e., where 'θ' be temp. of a body and

'θ_0' be a temp. of a surrounding medium

Then By newton's law of cooling, $\frac{d\theta}{dt} \propto (\theta-\theta_0)$.

$\Rightarrow \frac{d\theta}{dt} = -K(\theta-\theta_0)$.

where K = proportional constant.

Problems :-

i) A body kept in air with temperature $25^\circ C$ ~~whos~~ cools from $140^\circ C$ to $80^\circ C$ in 20 min. Find the when the body cools down to $35^\circ C$.

Sol :- Let "θ" be the temp. of a body.

By Newton's law of cooling, $\frac{d\theta}{dt} = -K(\theta-\theta_0)$

Given $\theta_0 = 25^\circ C$.

$$\frac{d\theta}{dt} = -K(\theta-25)$$

$$\frac{d\theta}{(\theta-25)} = -Kdt$$

By integrating on both sides.

$$\log(\theta-25) = -Kt + C$$

when $t=0$ (initially) $\theta = 140^\circ C$

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from ①.

$$\log(140-25) = k(0) + c$$

$$\Rightarrow c = \log 115$$

"c" value in eq ①.

$$\log(\theta-25) = -kt + \log 115$$

$$\log(\theta-25) - \log 115 = -kt \rightarrow ②$$

$t = 20, \ \theta = 80°c$

from ② $\Rightarrow \log(80-25) - \log 115 = -20k \rightarrow ③$

$$\frac{eq ②}{eq ③} \Rightarrow \frac{\log(\theta-25) - \log 115}{\log 55 - \log 115} = \frac{-kt}{-20k}$$

$$\Rightarrow \frac{\log\left(\frac{\theta-25}{115}\right)}{\log\left(\frac{55}{115}\right)} = \frac{t}{20}$$

Now $\theta = 35°c$.

$$\Rightarrow \frac{\log\left(\frac{35-25}{115}\right)}{\log\left(\frac{55}{115}\right)} = \frac{t}{20}$$

$$\Rightarrow t = 20\left[\frac{\log\left(\frac{10}{115}\right)}{\log\left(\frac{55}{115}\right)}\right]$$

~~t = 66.224 min.~~ $t = 66.224$

~~t = 85.614 min~~

* The temp. of a body drops from $100°c$ to $75°c$ in 10 min in the surrounding air is at $20°c$. What will be its temp. after half an hour and when the temp will be $25°c$.

Sol:- Let "θ" be the temp. of a body.

By Newton's law of cooling,

$$\frac{d\theta}{dt} = -k(\theta - \theta_0)$$

Given $\theta_0 = 20°c$

$$\frac{d\theta}{dt} = -k(\theta - 20)$$

$$\frac{d\theta}{(\theta-20)} = -k\,dt$$

on integrating

$$\log(\theta - 20) = -kt + c \rightarrow (1)$$

when $t=0$ (initially) i.e. $\theta = 100°c$.

$$\log(100-20) = -kt + c.$$

$$\log 80 = -k(0) + c.$$

$$c = \log 80.$$

"c" value in eq (1)

$$\log(\theta - 20) = -kt + \log 80$$

$$\log(\theta-20) - \log 80 = -kt \rightarrow (2).$$

$t = 10$ min, $\theta = 75°$

$$\log(75-20) - \log 80 = -10k$$

$$\log 55 - \log 80 = -10k \rightarrow (3)$$

$$\frac{eq\,①}{eq\,③} \Rightarrow \frac{\log(\theta-25)-\log 80}{\log 55-\log 80} = \frac{-kt}{-10k}$$

$$\Rightarrow \frac{\log\left(\frac{\theta-20}{80}\right)}{\log\left(\frac{55}{80}\right)} = \frac{t}{10} \quad \longrightarrow ④$$

ii) After ~~d~~ 30 min = t

$$\frac{\log\left(\frac{\theta-20}{80}\right)}{\log\left(\frac{55}{80}\right)} = \frac{3\cancel{0}}{1\cancel{0}}$$

$$\log\left(\frac{\theta-20}{80}\right) = \cancel{3}\log\left(\frac{55}{80}\right)^3$$

$$\frac{\theta-20}{80} = \left(\frac{55}{80}\right)^3$$

$$\cancel{\theta=\theta} \cdot \theta-20 = 25.99$$

$$\cancel{\theta = 50.9960} \quad \theta = 45.99 \simeq \underline{46^\circ C}$$

From ④,

$$\frac{\log\left(\frac{25-20}{80}\right)}{\log\frac{55}{80}} = \frac{t}{10}$$

$$t = 10\left[\frac{\log\left(\frac{5}{80}\right)}{\log\left(\frac{55}{80}\right)}\right]$$

$$t = 73.9$$

* the air temp is 30°c and the water at a temp. 100°c cools to 80°c in 10min. Find when the temp. of water will become 40°c.

Sol:- let "θ" be the temp. of a body.

By Newton's law of cooling,

$$\frac{d\theta}{dt} = -K(\theta - \theta_0)$$

Given $\theta_0 = 30°c$

$$\frac{d\theta}{dt} = -k(\theta - 30)$$

$$\frac{d\theta}{(\theta - 30)} = -k\,dt$$

on Integrating

$$\log(\theta - 30) = -kt + c \rightarrow ①$$

when $t=0$ (~~partially~~, $\theta = 100°c$

$$\log(100-30) = -k(0) + c$$

$$c = \log 70°$$

"c" value in eq. ①

$$\log(\theta - 30) - \log 70 = \cancel{0} - kt \rightarrow ②$$

when $t = 10$ min, $\theta = 80°c$

$$\log(50) - \log 70 = -10k \rightarrow ③$$

$$\frac{②}{③} \Rightarrow \frac{\log\left(\frac{\theta-30}{70}\right)}{\log\left(\frac{50}{70}\right)} = \frac{\cancel{-}kt}{\cancel{+}10k}$$

At $\theta = 40^\circ$,

$$\frac{\log\left(\frac{40-30}{70}\right)}{\log\left(\frac{50}{70}\right)} = \frac{t}{10}$$

$$t = 10 \cdot \left(\frac{\log\left(\frac{10}{70}\right)}{\log\left(\frac{50}{70}\right)}\right).$$

$$t = \underline{\underline{57.8327}}$$

* The air is maintained at 30°, $80^\circ C$ to $60^\circ C$ in 12 min.

Find the temp of body after i) 36 min, ii) 24 min.

Sol:-

$$\frac{d\theta}{dt} = -k(\theta - \theta_0)$$

$$\theta_0 = 30^\circ$$

$$\frac{d\theta}{dt} = -k(\theta - 30)$$

$$\int \frac{d\theta}{\theta - 30} = \int -kt$$

$$\log(\theta - 30^\circ) = -kt + c$$

when $t = 0$, $\theta = 80^\circ$

law of Natural Growth & decay:-

If the rate of change of a substance "N" at any time is directly proportional to the availability substance at that time.

$$\frac{dN}{dt} \propto N.$$

$$\frac{dN}{dt} = -KN.$$

Note-1:- The D.E $\frac{dN}{dt} = KN$ as time "t" is increases and "N" increases indicates the law of natural growth.

Note-2:- The D.E $\frac{dN}{dt} = -KN$ as time "t" increases and the availability of substance "N" decreases indicates the law of natural decay.

Problems:-

1) A bacterial culture ingrowing exponentially increases from 100 to 400 gms in 10 hrs. How much was present after 3 hours from the initial instant.

Sol:- let 'N' be the no. of bacteria at any time t, we know that by the law of natural growth, $\frac{dN}{dt} = KN$

$$\frac{dN}{dt} = KN \Rightarrow \frac{dN}{N} = kdt$$

$$\int \frac{dN}{N} = \int kdt.$$

$$\log N = kt + c \rightarrow (1)$$

* The rate at which bacteria multiply is proportional to the instantaneous "N" numbers present if the original number doubles in 2 hrs. when it will be triple.

Sol:- Let "N" be the no. of bacteria at any time t,

we know that, law of Natural growth,

$$\frac{dN}{dt} = kN$$

$$\frac{dN}{N} = k\,dt$$

$$\log N = kt + c \rightarrow (1)$$

Given at $t=0, N=N_0$ (say)

from (1). $\log N_0 = c$.

$$\Rightarrow \log N = kt + \log N_0$$

$$\Rightarrow \log N - \log N_0 = kt \rightarrow (2)$$

and also given that when $t = 2\text{hrs}, N = 2N_0$

from (2)

$$\log 2N_0 - \log N_0 = 2k \rightarrow (3)$$

$$(2) \div (3) \Rightarrow \frac{\log \frac{N}{N_0}}{\log \frac{2N_0}{N_0}} = \frac{kt}{2k}$$

$$\frac{\log \frac{3N_0}{N_0}}{\log 2} = \frac{t}{2}$$

$$t = 2\left(\frac{\log 3}{\log 2}\right)$$

$$t = 3.1699$$

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2014,
* The bacteria in a culture grows exponentially so that the initial number has doubled in 3 hrs. How many times the initial number will be present after 9 hrs.

Sol:- let "N" be the no. of bacteria at time t.

we k.T, Law of natural growth.

$$\frac{dN}{dt} = kN$$

$$\frac{dN}{N} = k\,dt$$

$$\log N = kt + c \rightarrow ①$$

Given at time $t=0$, $N = N_0$ (say)

from ① $\Rightarrow$ $\log N_0 = c$.

$$\Rightarrow \log N - \log N_0 = kt \rightarrow ②$$

And also given that when $t = 3$ hrs, $N = 2N_0$

$$\Rightarrow \log 2N_0 - \log N_0 = 3k \rightarrow ③$$

$2 \div 3$, $\Rightarrow$ $$\frac{\log \frac{N}{N_0}}{\log \frac{2N_0}{N_0}} = \frac{\cancel{k}t}{3\cancel{k}}$$

$$\frac{\log \frac{N}{N_0}}{\log 2} = \frac{\cancel{9}^{3}}{\cancel{3}_{1}}$$

$$\log \frac{N}{N_0} = 3 \log 2 ..$$

$$\frac{N}{N_0} = 2^3 = 8$$

$$N = 8N_0.$$

At $t=0$, $N=100$ gms.

from ① $\log 100 = c$

$\Rightarrow \log N = kt + \log 100$

$\Rightarrow \log N - \log 100 = kt \rightarrow ②$.

and also given that when $t = 10$ hrs, $N = 400$ gms.

from ②.

$\log 400 - \log 100 = 10k \rightarrow ③$.

② ÷ ③

$$\frac{\log N - \log 100}{\log 400 - \log 100} = \frac{kt}{10k}$$

$$\frac{\log \frac{N}{100}}{\log 4} = \frac{t}{10}$$

$$\log_{10} \frac{N}{100} = \frac{3}{10} \log 4.$$

$$\log_{10} \frac{N}{100} = 0.18061$$

$$\frac{N}{100} = 10^{0.18061}$$

$N = \cancel{142.794}$ $\underline{151.568}$

* The number "N" of bacteria in a culture grew at a proportional to "N". The value of "N" was initially 100 and increased to 332 in 1hr. What was the value of "N" after $1\frac{1}{2}$ hour.

let "N" be the no. of bacteria at any time t,

Sol:

we know that law of Natural growth,

$$\frac{dN}{dt} = kN$$

$$\frac{dN}{N} = k\,dt$$

$$\Rightarrow \log N = kt + c.$$

Given at $t=0$, $N=100$

from ① $\Rightarrow \log 100 = c.$

$$\Rightarrow \log N = \log 100 + kt$$

$$\log N - \log 100 = kt \rightarrow ②.$$

and $t = 1$ hr, $N = 332$

$$\log 332 - \log 100 = k \rightarrow ③.$$

② ÷ ③ $\Rightarrow$ $$\frac{\log \frac{N}{100}}{\log \frac{332}{100}} = \frac{\cancel{k}\,\cancel{t}\,3/2}{\cancel{k}}$$

$$\log_{10} \frac{N}{100} = \frac{3}{2} \log \frac{332}{100}.$$

$$\log \frac{N}{100} = \cancel{0.78170}\ 0.78170$$

$$\cancel{N} = \cancel{100 \cdot e^{0.78170}}.$$

$$\cancel{N = 218.518}$$

$$N = 100 \times 10^{0.78170}$$

$$N = \underline{604.922}$$

Higher order Diff. eqn :-

8

Auxilary eqn $f(m)=0 \rightarrow (3)$.

By solving (3),

let $m_1, m_2, m_3 ---- m_n$ are the roots of the auxiliary eqn.

case-i :- Suppose the roots $m_1, m_2, m_3 --- m_n$ are all real and distinct. Then

$$y_c = c_1e^{m_1x} + c_2e^{m_2x} + c_3e^{m_3x} + --- + c_ne^{m_nx}.$$

case-ii :- Suppose $m_1=m_2$ (real and equal) and then $m_3, m_4, --- m_n$ are all real and distinct, then

$$y_c = (c_1+c_2x)e^{m_1x} + c_3e^{m_3x} + c_4e^{m_4x} + --- + c_ne^{m_nx},$$

case-iii :- Suppose $m_1=m_2=m_3$ (equal) and rest of roots are real and distinct, then

$$y_c = (c_1+c_2x+c_3x^2)e^{m_1x} + c_4e^{m_4x} + c_5e^{m_5x} + --- + c_ne^{m_nx}.$$

case-iv :- Suppose m_1, m_2 $(\alpha \pm i\beta)$ are complex & rest of roots are real and distinct.

$$y_c = e^{\alpha x}[c_1\cos\beta + c_2\sin\beta x] + c_3e^{m_3x} + c_4e^{m_4x} + ---- + c_ne^{m_nx}.$$

Case-V:- Suppose $\alpha \pm i\beta$ repeated i.e. (m_1, m_2, m_3, m_4) and $m_5, m_6, m_7, \ldots$ are all real and distinct $(m_1, m_2, \ldots, m_n)$

$$y_c = e^{\alpha x}\left[(c_1 + c_2 x)\cos\beta x + (c_3 + c_4 x)\sin\beta x\right] + c_5 e^{m_5 x} + \ldots + c_n e^{m_n x}.$$

Case-VI:- Suppose $m_1, m_2\ (\alpha \pm \sqrt{\beta})$ are irrational and $m_3, m_4, m_5, \ldots m_n$ are all real and distinct.

$$y_c = e^{\alpha x}\left[c_1\cosh\sqrt{\beta}\,x + c_2\sinh\sqrt{\beta}\,x\right] + c_3 e^{m_3 x} + c_4 e^{m_4 x} + \ldots + c_n e^{m_n x}.$$

Problems :-

* $$\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + P_2 \frac{d^{n-2} y}{dx^{n-2}} + \ldots + P_n y = Q(x).$$

Sol:- General sol^n $y = C.F + P.I$

$= y_c + y_p$

$$\frac{d}{dx} = D$$

$$D^n y + P_1 D^{n-1} y + \ldots + P_n y = Q(x)$$

$$\Rightarrow \left[D^n + P_1 D^{n-1} + P_2 D^{n-2} + \ldots + P_n\right] y = Q(x).$$

$$\Rightarrow f(D)\,y = Q(x)$$

where $f(D) = D^n + P_1 D^{n-1} + P_2 D^{n-2} + \ldots + P_n$

Problems :-

* Solve the D.E $\frac{d^3y}{dx^3} - 9\frac{d^2y}{dx^2} + 23\frac{dy}{dx} - 15y = 0$.

Sol :- Given D.E $\frac{d^3y}{dx^3} - 9\frac{d^2y}{dx^2} + 23\frac{dy}{dx} - 15y = 0$.

$$\boxed{D^3y - 9D^2y + 23Dy - 15y = 0}$$

$$[D^3 - 9D^2 + 23D - 15]y = 0.$$

let $f(D) = D^3 - 9D^2 + 23D - 15$.

Now the auxiliary eqn $f(m) = 0$

$$f(m) = m^3 - 9m^2 + 23m - 15 = 0$$

$$\begin{array}{c|cccc} & 1 & -9 & 23 & -15 \\ m=1 & 0 & 1 & -8 & 15 \\ \hline & 1 & -8 & 15 & 0 \end{array}$$

$$(m-1)(m^2 - 8m + 15) = 0$$

$m = 1, 3, 5$ (real & Distinct)

$$y = C_1e^{x} + C_2e^{3x} + C_3e^{5x}$$

* Solve the D.E $(D^4 - 2D^3 - 3D^2 + 4D + 4)y = 0$.

Sol :- Given that $f(D) = D^4 - 2D^3 - 3D^2 + 4D + 4$.

Now the auxiliary eqn $f(m) = 0$

$$f(m) = \cancel{m^3 - 9m^2 + 23m - 15} = 0$$

$$= m^4 - 2m^3 - 3m^2 + 4m + 4$$

$$
\begin{array}{c|ccccc}
m=-1 & 1 & -2 & -3 & 4 & 4 \\
 & 0 & -1 & 3 & 0 & -4 \\
\hline
 & 1 & -3 & 0 & 4 & \boxed{0}
\end{array}
$$

$$(m+1)(m^3-3m^2+4)=0$$

$$m=-1,-1,2,2$$

$$y=(c_1+c_2x)e^{-x}+(c_3+c_4x)e^{2x}$$

* Solve the DE $y''+y'+y=0$

Sol:- Given the eqn is $y''+y'+y=0$

$$\frac{d^2y}{dx^3}+\frac{d^{\cancel{2}}y}{dx}+y=0$$

$$D^2y+Dy+y=0$$

$$\text{let } f(D)=(D^2+D+1)$$

Now the auxillary eqn is $f(m)=m^2+m+1$

$$f(m)=m^2+m+1=0$$

$$m=-\frac{1}{2}\pm\frac{\sqrt{3}}{2}i \quad \text{(complex)}$$

$$y=e^{-\frac{1}{2}x}\left[c_1\cos\frac{\sqrt{3}}{2}x+c_2\sin\frac{\sqrt{3}}{2}x\right]$$

To find y_p :-

Suppose the given differential eqn $f(D)y_i = Q(x)$

then $y_p = \frac{1}{f(D)} \cdot Q(x)$.

1

case-I :- If $Q(x) = e^{ax}$.

$$y_p = \frac{1}{f(D)} \cdot Q(x)$$

$$y_p = \frac{1}{f(D)} e^{ax}.$$

put $D=a$

$$y_p = \frac{1}{f(a)} \cdot e^{ax}$$

NOTE :- If $f(a)=0$, put x in the numerator, simultaneously diff. the denominator i.e., $y_p = \frac{x}{f'(D)} \cdot e^{ax}$.

put $D=a$

$$= \frac{x}{f'(a)} \cdot e^{ax}$$

$\Rightarrow$ if $f'(a)=0$

then $y_p = \frac{x^2}{f''(D)} \cdot e^{ax}$.

put $D=a$

$$y_p = \frac{x^2}{f''(a)} \cdot e^{ax}$$

~~This process, will continue~~

continue this process, till the denominator not equal to zero.

Problems :-

1) Solve the D.E $(D^3-5D^2+8D-4)y = e^{3x}$.

$y_p = \frac{1}{3^3-5\cdot3^2+\ldots}$

Sol :- Given the D.E $(D^3-5D^2+8D-4)y = e^{3x}$.

let $f(D) = D^3-5D^2+8D-4$

Now the auxillary eqn $f(m)=0$.

$$\Rightarrow m^3-5m^2+8m-4=0$$

$$m=1,2,2.$$

$$y_c = c_1 e^{x} + (c_2+c_3 x)e^{2x}.$$

$$y_p = \frac{1}{f(D)} \cdot Q(x)$$

$$= \frac{1}{D^3-5D^2+8D-4} \cdot e^{3x}. \qquad D=a.$$

put $D=3$

$$= \frac{1}{(3)^3-5(3)^2+8(3)-4} \cdot e^{3x}.$$

$$= \frac{1}{2} \cdot e^{3x}$$

$\therefore$ General soln $y = y_c + y_p$

$$= c_1 e^{x} + (c_2+c_3x)e^{2x} + \frac{1}{2}e^{3x}.$$

* Solve the D.E $(D^3-6D^2+11D-6)y = e^{-2x}+e^{-3x}$.

Sol:- let $f(D) = D^3-6D^2+11D-6$.

Now the auxiliary eqn $f(m)=0$

$$m^3-6m^2+11m-6=0$$

$m=1,2,3$. (real & distinct)

$$y_c = c_1 e^{x} + c_2 e^{2x} + c_3 e^{3x}.$$

$$y_p = \frac{1}{f(D)} \cdot Q(x)$$

$$= \frac{1}{D^3-6D^2+11D-6} \cdot e^{-2x} + e^{-3x}$$

$$= \frac{1}{D^3-6D^2+11D-6} \cdot e^{-2x} + \frac{1}{D^3-6D^2+11D-6} e^{-3x}.$$

put $D=-2$, put $D=-3$

$$= \frac{1}{-8-24-22-6} e^{-2x} + \frac{1}{-27-54-33-6} \cdot e^{-3x}$$

$$\begin{array}{r} 24 \\ 22 \\ \hline 48 \end{array}$$

$$= \frac{1}{-60} e^{-2x} + \frac{1}{-120} e^{-3x}.$$

G.S. $y = y_c + y_p$

$$= c_1 e^{x} + c_2 e^{2x} + c_3 e^{3x} - \left(\frac{1}{60} e^{-2x} + \frac{1}{120} e^{3x}\right)$$

* solve the D.E $(D^3-1) \cdot y = (e^{x}+1)^2$.

Sol:- let $f(D) = D^3 - 1$

Now the auxiliary eqn $f(m)=0$

$$m^3 - 1^3 = 0$$

$$(m-1)(m^2+m+1) = 0$$

$$m = 1, -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$$

$$y_c = c_1 e^{x} + e^{-\frac{1}{2}x}\left[c_2 \cos\frac{\sqrt{3}}{2}x + c_3 \sin\frac{\sqrt{3}}{2}x\right]$$

$$y_p = \frac{1}{f(D)} \cdot Q(x)$$

$$= \frac{1}{D^3-1} \cdot (e^{x}+1)^2.$$

$$= \frac{1}{(D^3-1)}(e^{2x}+2e^{x}+1)$$

$$= \frac{1}{(D^3-1)}e^{2x} + \frac{1}{(D^3-1)}\cdot 2e^{x} + \frac{1}{D^3-1}\,1$$

put $D=2$, put $D=1$,

$$= \frac{1}{2^3-1}e^{2x} + 2\cdot\frac{1}{1^3-1}e^{x} + \frac{1}{-1}$$

$$= \frac{1}{7}e^{2x} + 2\cdot\frac{x}{3D^2}\cdot e^{x} - 1$$

put $D=1$

$$y_p = \frac{1}{7}e^{2x} + 2\cdot\frac{x}{3}e^{x} - 1$$

General sol^n $\Rightarrow$ $y = y_c + y_p$

$$= c_1e^{x} + e^{-\frac{1}{2}x}\left[c_2\cos\frac{\sqrt{3}}{2}x + c_3\sin\frac{\sqrt{3}}{2}x\right]$$
$$+ \frac{1}{7}e^{2x} + \frac{2}{3}xe^{x} - 1$$

* Solve the D.E $(D^2+6D+9)y = 2e^{-3x}$

Sol:- let $f(D) = D^2+6D+9$

Now the auxiliary eq^n $f(m)=0$

$$f(m) = m^3+6m+9=0$$

$$(m+3)^2 = 0$$

$$m = -3, -3$$

$$y_c = (c_1+c_2x)e^{-3x}.$$

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$y_p = \frac{1}{f(D)} \cdot Q(x)$

$= \frac{1}{D^2+6D+9} \cdot 2e^{-3x}$.

$9 - \frac{18+9}{2D+6}$

$9-$ $=$ 2 put $D=-3$

$= \frac{1}{9-18+9} \; 2 \cdot e^{-3x}$

$= 2 \cdot \frac{x}{2D+6} e^{-3x}$.

$= \frac{x}{D+3} \cdot e^{-3x}$

$-3+3$ $D=$

put

$= \frac{x^2}{1} e^{-3x}$

$y_p = \underline{x^2 e^{-3x}}$

General $sol^n \Rightarrow y = y_c + y_p$

$= (C_1 + C_2 x) e^{-3x} + x^2 e^{-3x}$.

* Solve the D.E $(D^3 - 5D^2 + 7D - 3)y = e^{2x} \cosh x$.

Sol :- let $f(D) = D^3 - 5D^2 + 7D - 3$

Now the auxiliary eqn $f(m)=0$

$f(m) = m^3 - 5m^2 + 7m - 3 = 0$

$m = 1, 1, 3$.

$y_c = (C_1 + C_2 x) e^x + C_3 e^{3x}$.

$$y_p = \frac{1}{f(D)} \cdot Q(x)$$

$$= \frac{1}{D^3-5D^2+7D-3} \cdot e^{2x} \cosh x$$

$$= \frac{1}{D^3-5D^2+7D-3} \cdot e^{2x}\left[\frac{e^x+e^{-x}}{2}\right]$$

$$= \frac{1}{2}\left[\frac{1}{D^3-5D^2+7D-3} \cdot e^{3x}+e^x\right]$$

$$= \frac{1}{2}\left[\frac{1}{D^3-5D^2+7D-3} e^{3x} + \frac{1}{D^3-5D^2+7D-3} e^x\right]$$

put $D=3$, put $D=1$

$$= \frac{1}{2}\left[\frac{1}{27-45+21-3} e^{3x} + \frac{1}{1-5+7-3} e^x\right]$$

$$\frac{21}{48}$$

$$= \frac{1}{2}\left[\frac{x}{3D^2-10D+7} \cdot e^{3x} + \frac{x}{3D^2-10D+7} \cdot e^x\right]$$

$3(9)$

$$\frac{27}{30}$$

put $D=3$ put $D=1$

$$= \frac{1}{2}\left[\frac{x}{4} e^{3x} + \frac{x}{3-10+7} e^x\right]$$

$$= \frac{1}{2}\left[\frac{x}{4} e^{3x} + \frac{x^2}{6D-10} e^x\right]$$

$$= \frac{1}{2}\left[\frac{x}{4} e^{3x} - \frac{x^2}{4} e^x\right]$$

$\therefore$ General soln $\Rightarrow y = y_c + y_p$

$$= (c_1 + c_2 x)e^{x} + c_3 e^{3x} + \frac{1}{2}\left[\frac{x}{4} e^{3x} - \frac{x^2}{4} e^{x}\right]$$

* Solve the D.E $(D^3 + 2D^2 + D)y = e^{2x}$.

* $(D^3 - 3D^2 + 4)y = (1 + e^{-x})^3$. $1 + e^{-3x}$

* $(D^2 + 4D + 4)y = 18\cosh 2x$. $\cosh 2x = \frac{e^{2x} + e^{-2x}}{2}$

* Solve the D.E $(D^3 + 2D^2 + D)y = e^{2x}$

Sol:- let $f(D) = D^3 + 2D^2 + D$

Now A-eqn is $f(m) = 0$

$$m^3 + 2m^2 + m = 0$$

$$\cancel{m = -1,}$$

$$m(m^2 + 2m + 1) = 0$$

$$m = 0, -1, -1$$

$$y_c = c_1 + (c_2 + c_3 x)e^{-x}$$

$$y_p = \frac{1}{f(D)} \cdot Q(x)$$

$$= \frac{1}{f(D)} \cdot = \frac{1}{D^3 + 2D^2 + D} \cdot e^{2x}$$

put $D = 2$

$$= \frac{1}{8 + 8 + 2} \cdot e^{2x}$$

$$= \frac{1}{18} e^{2x}$$

General solution is $y = y_c + y_p$

$$y = c_1 e^{0} + (c_2 + c_3 x)\, e^{-x} + \frac{1}{18} \cdot e^{2x}$$

$-27+$

* solve the D.E $(D^3 - 3D^2 + 4)\, y = (1 + e^{-x})^3$.

<u>sol</u> :- let $f(D) = 0$

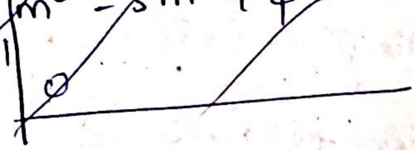
Now. A. eqn is $f(m) = 0$

$$m^3 - 3m^2 + 4 = 0$$

$m^3 -$

$\cancel{m = 1, 2}$ $\cancel{m^3 - 3m^2 + 4}$

$\cancel{m^3}$. $m = -1$



$m - 2$

$m^2 + 4 -$

$$\begin{array}{c|cccc} m = -1 & 1 & -3 & 0 & 4 \\ & 0 & -1 & 4 & -4 \\ \hline & +1 & -4 & 4 & \boxed{0} \end{array}$$

$$(m+1) \cdot (m^2 + 4m + 4) = 0$$

$$m = -1, -2, -2.$$

$$y_c = c_1 e^{-x} + (c_2 + c_3 x)\, e^{-2x}.$$

$$y_p = \frac{1}{D^3 - 3D^2 + 4} (1 + e^{-x})^3.$$

$3a^2b$

put $D = -1$ $\quad = \frac{1}{D^3 - 3D^2 + 4} (1 + e^{-3x} + 3e^{-x} + 3e^{\ldots}$

$$= \frac{1}{\cancel{-1 + 3 + 4}} \cancel{(1 + e^{-x})^3}$$

=

=

$$= \frac{1}{D^3-3D^2+4}\cdot 1 + \frac{1}{D^3-3D^2+4}\cdot e^{-3x} + \frac{1}{D^3-3D^2+4}\cdot 3e^{-x} + \frac{1}{D^3-3D^2+4}\cdot 3e^{-2x}$$

put $D=0$, $\quad D=-3$, $\quad D=-1$ $\quad D=-2$

$-1-3+4$

$-8-3(4$

$-3(9)$

$$= \frac{1}{4} + \frac{1}{-50}e^{-3x} + \frac{3x}{3D^2-6D}\cdot e^{-x} + \frac{1}{-16}\cdot 3e^{-2x}.$$

put $D=-1$ $\quad 3(D^2-2D)$

$1+2$

$$= \frac{1}{4} - \frac{1}{50}e^{-3x} + \frac{3x}{3}e^{-x} - \frac{1}{16}3e^{-2x}.$$

General solution $y = y_c + y_p$

$$= c_1 e^{-x} + (c_2+c_3x)e^{-2x} + \frac{1}{4} - \frac{1}{50}e^{-3x} + \frac{x}{3}e^{-x} - \frac{1}{16}3\cdot e^{-2x}.$$

* Solve the D.E $(D^2+4D+4)y = 18\cosh 2x$.

Sol:- let $f(D) = D^2+4D+4$.

Now, A. eqn is $f(m)=0$

$$m^2+4m+4=0$$

$$(m+2)^2=0$$

$$m=-2,-2$$

$$y_c = (c_1+c_2x)e^{-2x}.$$

$$y_p = \frac{1}{D^2+4D+4}\cdot 18\cosh 2x$$

$$= \frac{1}{D^2+4D+4}\cdot 18\left[\frac{e^{2x}+e^{-2x}}{2}\right]$$

$$= 9\left[\frac{1}{D^2+4D+4}\cdot e^{2x} + \frac{1}{D^2+4D+4}\cdot e^{-2x}\right]$$ $4-8+4$

$4+4(2)+4$ put $D=2$, put $D=-2$

$$= 9\left[\frac{1}{16}e^{2x} + \frac{x}{2D+4}e^{-2x}\right]$$

put $D=-2$

$$= 9\left[\frac{1}{16}e^{2x} + \frac{x^2}{2}e^{-2x}\right]$$

$\therefore$ General Solution $y = y_c + y_p$

$$= (c_1 + c_2 x)e^{-2x} + 9\left[\frac{1}{16}e^{2x} + \frac{x^2}{2}e^{-2x}\right]$$

Method-2 :- If $Q(x) = \sin ax$ (or) $\cos ax$ then

$$y_p = \frac{1}{f(D)} \cdot Q(x)$$

2

$$= \frac{1}{f(D)} \cdot \sin ax$$

put $D^2 = -a^2$

Problems :-

1) Solve the D.E $(D^2 - 2D + 2)y = \sin 9x$.

Sol :- ~~Let~~ Given the D.E is $(D^2 - 2D + 2) = \sin 9x$.

Now the ~~Qa~~ Auxiliary eqn $(\cancel{D^2 - 2D + 2})$ $f(m) = 0$

$$m^2 - 2m + 2 = 0$$

$$(m-1)^2 + 1 = 0$$

$$(m-1)^2 = -1$$

$$m - 1 = \pm i$$

$$m = 1 \pm i$$

$$y_c = e^x [c_1 \cos x + c_2 \sin x].$$

$$y_p = \frac{1}{D^2 - 2D + 2} \cdot \cancel{\sin} \, Q(x)$$

$$= \frac{1}{D^2 - 2D + 2} \sin 9x$$

$$= \cancel{\frac{1}{-81 - 2D + 2}}$$ put $D^2 = -9^2 = -81$

$$= \frac{1}{-81-2D+2} \cdot \sin 9x$$

$$= \frac{1}{-(2D+79)} \cdot \frac{\sin 9x}{2D-79}$$

$$= -\frac{2D-79}{4D^2-(79)^2} \cdot \sin 9x$$

put $D^2 = -9^2 = -81$

$$= -\frac{2D-79}{4(-81)-6241} \cdot \sin 9x.$$

$$= \frac{1}{6565} \left[18 \cos 9x - 79 \sin 9x\right]$$

$y = y_c + y_p$

$$= e^x \left[c_1 \cos x + c_2 \sin x\right] + \frac{1}{6565} \left[18 \cos 9x - 79 \sin 9x\right]$$

2) Solve the D.E $(D^2+3D+2)y = e^{-x} + \sin 2x.$

Sol:- Given the D.E is $(D^2+3D+2)y = e^{-x} + \sin 2x.$

Now the auxiliary eqn $f(m) = 0$

$$m^2+3m+2 = 0$$

$$m = -1, -2.$$

$$y_c = c_1 e^{-x} + c_2 e^{-2x}.$$

$$y_p = \frac{1}{D^2+3D+2} \cdot Q(x)$$

$$= \frac{1}{D^2+3D+2} \cdot \left[e^{-x} + \sin 2x\right]$$

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$$= \frac{1}{D^2+3D+2} \cdot e^{-x} + \frac{1}{D^2+3D+2} \sin 2x.$$

put $D = -1$, put $D^2 = -2^2 = -4$

$$= \frac{1}{1-3+2} \cdot e^{-x} + \frac{1}{-4+3D+2} \sin 2x$$

$$= \frac{x}{2D+3} \cdot e^{-x} + \frac{1}{3D-2} \sin 2x.$$

put $D = -1$

$$= \frac{x}{-2+3} e^{-x} + \frac{3D-2}{9D^2-4} \sin 2x$$

put $D^2 = -4$

$$= x e^{-x} + \frac{3D-2}{-40} \sin 2x$$

$$y_p = x e^{-x} - \frac{1}{40} \cdot 6\cos 2x + 2\sin 2x$$

General sol^n $y = y_c + y_p$

$$= c_1 e^{-x} + c_2 e^{-2x} + x e^{-x} - \frac{1}{40} 6\cos 2x + 2\sin 2x$$

* Solve the D.E $(D^2+16)y = e^{-3x} + \cos 4x.$

Sol:- Given D.E is $(D^2+16)y = e^{-3x} + \cos 4x.$

Now Auxiliary eqn is $\cancel{D^2+16} = f(m) = 0$

$$m^2 + 16 = 0$$

$$m^2 = -16$$

$$m = \pm 4i$$

$$y_c = \left[c_1 \cos 4x + c_2 \sin 4x\right]$$

$$y_p = \frac{1}{D^2 + \cancel{3D} \cancel{6} \cancel{2}} \cdot e^{-3x} + \cos 4x$$

$$= \frac{1}{D^2+3D+2} \cdot e^{-3x} + \frac{1}{D^2+3D+2} \cos 4x$$

put $D = -3$, $D^2 = -16$

$$= \frac{1}{9-6+2} e^{-3x} + \frac{1}{-16+3D+2} \cos 4x$$

$$= \frac{1}{5} e^{-3x}$$

$$= \frac{1}{D^2+16} \cdot e^{-3x} + \frac{1}{D^2+16} \cos 4x$$

put $D = -3$, put $D^2 = -16$

$$= \frac{1}{25} e^{-3x} + \frac{x}{2D} \cdot \cos 4x$$

$$y_p = \frac{1}{25} e^{-3x} + \frac{x}{2} \frac{\sin 4x}{4} .$$

//

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* Solve the DE $(D^4-2D^3+2D^2-2D+1)y = \cos x$.

Sol:- Given D.E is $(D^4-2D^3+2D^2-2D+1)y = \cos x$.

Now A-eqn is $f(m)=0$

$$m^4-2m^3+2m^2-2m+1=0$$

$$1-2+2-2+1$$

$$\begin{array}{c|ccccc} m=1 & 1 & -2 & 2 & -2 & 1 \\ & 0 & 1 & -1 & 1 & -1 \\ \hline & 1 & -1 & 1 & -1 & \boxed{0} \end{array}$$

$$(m-1)(m^3-m^2+m-1)=0$$

$$m=1,1,i,-i$$

$$y_c = (c_1+c_2x)e^{x} + (c_3\cos x + c_4\sin x)$$

$$y_p = \frac{1}{D^4-2D^3+2D^2-2D+1}\cdot\cos x$$

put $D^2=-1$

$$y_p = \frac{1}{1+2D-2-2D+1}\cos x.$$

$$= \frac{x}{4D^3-6D^2+4D-2}\cdot\cos x.$$

$$= \frac{x}{-4D+6+4D-2}\cos x$$

$$= \frac{x}{4}\cos x.$$

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$$

$1(45-48)-2(36-42)$
$+3(32-35)$

$-3-2(-6)+3(-3)$

$-3+12-9$

$-12+12$

$=0.$

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* Solve the D.E $(D^3-1)y = e^x + \sin 3x + 2$.

Sol:- Given D.E is $(D^3-1)y = e^x + \sin 3x + 2$.

Now A.eqn is $f(m) = 0$

$$m^3 - 1 = 0$$

$$m = 1, \frac{-1+\sqrt{3}i}{2}, \frac{-1-\sqrt{3}i}{2}.$$

$$y_c = c_1 e^x + e^{-\frac{1}{2}x}\left[c_1 \cos\frac{\sqrt{3}}{2}x + c_2 \sin\frac{\sqrt{3}}{2}x\right]$$

$$y_p = \frac{1}{(D^3-1)} e^x + \frac{1}{(D^3-1)} \sin 3x + \frac{1}{(D^3-1)} \cdot 2$$

put $D = 1$, put $D^2 = -9$

$$= \frac{x}{3D^2} e^x + \frac{1}{-9D-1} \cdot \sin 3x + \frac{2}{-1}$$

$$= \frac{x}{3} e^x - \frac{9D-1}{(81D^2-1)} \sin 3x - 2$$

$$= \frac{x}{3} e^x - \frac{9D-1}{-730} \sin 3x - 2$$

$$y_p = \frac{x}{3} e^x + \frac{27\cos 3x - \sin 3x}{730} - 2$$

General solution is $y = y_c + y_p$

$$y = c_1 e^x + e^{-\frac{1}{2}x}\left[c_1 \cos\frac{\sqrt{3}}{2}x + c_2 \sin\frac{\sqrt{3}}{2}x\right]$$

$$+ \frac{x}{3} e^x + \frac{27\cos 3x - \sin 3x}{730} - 2$$

* Solve the D.E $(D^2-3D+2)y = \cos 3x \cos 2x$.

Sol:- Given D.E is $(D^2-3D+2)y = \cos 3x \cdot \cos 2x$

Now A-eqn is $f(m)=0$

$$m^2-3m+2=0$$

$$m = 2, 1$$

$$y_c = C_1 e^{x} + C_2 e^{2x}$$

$$y_p = \frac{1}{(D^2-3D+2)} \cdot \cancel{\tfrac{1}{2}} \cos 3x \cos 2x$$

$$= \frac{1}{(D^2-3D+2)} \cdot \frac{1}{2}\left[\cos 5x + \cos x\right]$$

$$= \frac{1}{2}\left[\frac{1}{D^2-3D+2}\cos 5x + \frac{1}{D^2-3D+2}\cos x\right]$$

put $D^2=-25$, put $D^2=-1$ $-3D+1$

$$= \frac{-1}{2}\left[\frac{1}{23+3D}\cos 5x + \frac{1}{3D-1}\cos x\right]$$

$$= \frac{-1}{2}\left[\frac{3D-23}{9D^2-(23)^2}\cos 5x + \frac{3D+1}{9D^2-1}\cos x\right]$$

$$= \frac{-1}{2}\left[\frac{3D-23}{-754}\cos 5x + \frac{3D+1}{-10}\cos x\right]$$

$$= \frac{1}{2}\left[\frac{\cancel{15\cos}\ 15\sin 5x - 23\cos 5x}{754} + \frac{-3\sin x +}{10}\right.$$

$$y_p = \frac{1}{2}\left[\frac{-15\sin 5x - 23\cos 5x}{754} + \frac{-3\sin x + \cos x}{10}\right]$$

General solution is $y = y_c + y_p$

$$y = c_1 e^{x} + c_2 e^{2x} + \frac{1}{2}\left[\frac{-15\sin 5x - 23\cos 5x}{754} + \frac{-3\sin x + \cos x}{10}\right]$$

* Solve the D.E $(D^2+D+1)\cdot y = (1+\sin x)^2$

Sol:- Given D.E is $(D^2+D+1)y = (1+\sin x)^2$.

Now A.eqn is $f(m)=0$

$$m^2+m+1=0$$

$$m = \frac{-1}{2} \pm \frac{\sqrt{3}}{2} i$$

$$y_c = e^{-\frac{1}{2}x}\left[c_1 \cos \frac{\sqrt{3}}{2}x + c_2 \sin \frac{\sqrt{3}}{2}x\right]$$

$$y_p = \frac{1}{(D^2+D+1)}(1+\sin x)^2$$

$$= \frac{1}{(D^2+D+1)}\left\{1+\sin^2 x + 2\sin x\right]$$

$$= \frac{1}{D^2+D+1}\left\{1 + \frac{1-\cos 2x}{2} + 2\sin x\right]$$

$$= \frac{1}{D^2+D+1}\left[\frac{3}{2} - \frac{\cos 2x}{2} + 2\sin x\right]$$

$$= \left[\frac{1}{D^2+D+1}\cdot 3/2 - \frac{1}{2}\cdot\frac{1}{D^2+D+1}\cos 2x + 2\cdot\frac{1}{D^2+D+1}\sin x\right]$$

$$= \frac{3}{2} - \frac{1}{2}\left[\frac{1}{-1^2+D+1}\cos 2x\right] + 2\left[\frac{1}{-1^2+D+1}\sin x\right]$$

$$= \frac{3}{2} - \frac{1}{2}\left[\frac{1}{D}\cos 2x\right] + 2\left[\frac{1}{D}\sin x\right]$$

$$= \frac{3}{2} - \frac{1}{2}\frac{\sin 2x}{2} + 2(-\cos x)$$

$$\Rightarrow P.I = \frac{3}{2} - \frac{\sin 2x}{4} - 2\cos x$$

$$y = C.F + P.I \quad //$$

*Solve the D.E $(D^3+1)y = \sin(2x+1)$.

Sol:- Given D.E is $(D^3+1)y = \sin(2x+1)$

Now. A.eqn is $f(m)=0$

$$m^3+1=0.$$

$$m=-1, \frac{1}{2} \pm \frac{\sqrt{3}}{2}i$$

$$y_c = C_1 e^{-x} + e^{+\frac{1}{2}x}\left[C_2\cos\frac{\sqrt{3}}{2}x + C_3\sin\frac{\sqrt{3}}{2}x\right]$$

$$y_p = \frac{1}{(D^3+1)}\sin(2x+1)$$

put $D^2=-4$

$$= \frac{1}{-4D+1}\sin(2x+1)$$

$$= \frac{1+4D}{1-16D^2} \sin(2x+1)$$

put $D^2 = -4$

$1-16(-4)$

$$= \frac{1+4D}{65} \sin(2x+1)$$

$$= \frac{1}{65}\left(\sin(2x+1) + 8\cos(2x+1)\right)$$

General solution $y = y_c + y_p$

$$= c_1 e^{-x} + e^{1/2 x}\left[c_2 \cos\frac{\sqrt{3}}{2}x + c_3 \sin\frac{\sqrt{3}}{2}x\right]$$

$$+ \frac{1}{65}\left[\sin(2x+1) + 8\cos(2x+1)\right].$$

Method-3 :-

If $Q(x) = x^k$.

$$y_p = \frac{1}{f(D)} \cdot Q(x)$$

$$= \frac{1}{f(D)} \cdot x^k.$$

Now the denominator $f(D)$ can be written in the form of $1 \pm \phi(D)$ by taking out the lowest degree term in that i.e., $\frac{1}{1 \pm \phi(D)} \cdot x^k$

$$= \frac{1}{1 \pm \phi(D)} \cdot x^k$$

$$= \left[1 \pm \phi(D)\right]^{-1} \cdot x^k.$$

Note:-

i) $(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots$

ii) $(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots$

iii) $(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \cdots$

iv) $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \cdots$

Problems :-

i) Solve the D.E $(D^2 - 3D + 2)y = 2x^2$.

Sol:- Given the D.E $(D^2 - 3D + 2)y = 2x^2$.

Now A.eqn $\Rightarrow f(m) = 0$

$$m^2 - 3m + 2 = 0$$

$$m = 1, 2$$

$$y_c = c_1 e^x + c_2 e^{2x}.$$

$$y_p = \frac{1}{f(D)} \cdot Q(x)$$

$$= \frac{1}{D^2-3D+2} \cdot 2x^2.$$

$$= \frac{1}{\cancel{2}\left[1+\frac{D^2-3D}{2}\right]} \cdot \cancel{2}x^2$$

$$= \left[1+\frac{D^2-3D}{2}\right]^{-1} \cdot x^2$$

$$= \cancel{1+CD} = \left[1-\left(\frac{D^2-3D}{2}\right)+\left(\frac{D^2-3D}{2}\right)^2+ \cdots\right] \cdot x^2$$

$Dx^2 = 2x$

$D^2x^2 = 2.$

$$= \left[x^2 - \frac{2-3(2x)}{2} + \frac{9(2)}{4_2}\right]$$

$$= \left[x^2-1+3x+9/_2\right]$$

$\frac{9}{2}-1$

$$y_p = \underline{\underline{x^2+3x+7/_2}}$$

$\therefore$ General sol^n is $y = y_c + y_p$

$$\underline{\underline{y}} = c_1e^x + c_2e^{2x} + x^2 + 3x + 7/_2$$

*Solve the D.E $\quad \frac{d^2y}{dx^2} + \frac{dy}{dx} = x^2+2x+4$

Sol:- Given D.E $\quad (D^2+D)y = x^2+2x+4$

Now A eq^n $\Rightarrow f(m) = 0$

$m^2+m = 0$

$m(m+1) = 0$

$m=0, m=-1$

$y_c = c_1 + c_2 e^{-x}$

$$y_p = \frac{1}{f(D)} \cdot Q(x)$$

$$= \frac{1}{D^2+D}\left[x^2+2x+4\right]$$

$$= \cancel{\frac{1}{D^2+D} \cdot x^2 + \frac{1}{\ }} = \frac{1}{D(1+D)} x^2+2x+4$$

$$= \frac{(1+D)^{-1}}{D} \cdot (x^2+2x+4)$$

$D(x^2+2x+4) = 2x+2$

$D^2(x^2+2x+4) = 2$

$$= \frac{1}{D}(1-D+D^2)(x^2+2x+4)$$

$\cancel{Dx^2 = 2x+2}$

$\cancel{D^2x^2 = 2}$

$$= \frac{1}{D}\left[x^2+\cancel{2x}+4-\cancel{2x}-\cancel{2}+\cancel{2}\right]$$

$$= \frac{1}{D}\left[x^2+4\right]$$

$\frac{1}{D}$ = Integration.

$$= x^3+4x$$

\# Solve $\dfrac{d^3y}{dx^3} - \dfrac{d^2y}{dx^2} - 6\dfrac{dy}{dx} = 1+x^2$.

Sol:- Given D.E $(D^3-D^2-6D) = 1+x^2$.

A eqn $\Rightarrow$ $f(m)=0$

$m^3-m^2-6m=0$

$m=3,-2,0$

$y_c = c_1 + c_2 e^{-2x} + c_3 e^{3x}$.

$$y_p = \frac{1}{(D^3-D^2-6D)}(1+x^2)$$

$$= \frac{1}{-6D\left(1-\frac{D^3-D^2}{6D}\right)}(1+x^2)$$

$$= \frac{-1}{6D}\left(1-\frac{D^2-D}{6}\right)^{-1}(1+x^2)$$

$$= \frac{-1}{6D}\left(1+\frac{D^2-D}{6}+\frac{(D^2-D)^2}{36}\right)(1+x^2)$$

$D^4-2D^3+D^2$

$D(1+x^2) = 2x$

$D^2(1+x^2) = 2$

$$= \frac{-1}{6D}\left(1+x^2+\frac{2-2x}{6}+\frac{2}{36}\right)$$

$$= \frac{-1}{6D}\left(1+x^2+\frac{1-x}{3}+\frac{1}{18}\right)$$

$$= \frac{-1}{6D}\left(1+1/3+x^2-x/3+1/18\right)$$

$$= \frac{-1}{6D}\left(x^2-x/3+\frac{25}{18}\right)$$

$$= \frac{-1}{6}\left[\frac{x^3}{3}-\frac{x^2}{6}+\frac{25}{18}x\right]$$

* Solve the D.E $(D^3+2D^2+D)y = e^{2x}+\sin 2x+x^2+x$.

Sol:- Given D.E $(D^3+2D^2+D)y = e^{2x}+\sin 2x+x^2+x$.

Now. A. eqn $\Rightarrow f(m)=0$

$$m^3+2m^2+m=0$$

$$m=0,-1,-1$$

$$y_c = c_1+(c_2+c_3x)e^{-x}.$$

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$$y_p = \frac{1}{(D^3+2D^2+D)}(e^{2x} + \sin 2x + x^2 + x)$$

$$= \frac{1}{(D^3+2D^2+D)} e^{2x} + \frac{1}{D^3+2D^2+D} \sin 2x + \frac{1}{D^3+2D^2+D}(x^2+x)$$

let $y_{p_1} = \frac{1}{(D^3+2D^2+D)} e^{2x}.$

put $D = 2$

$$= \frac{1}{8+8+2} e^{2x} = \frac{1}{18} e^{2x}.$$

$$y_{p_2} = \frac{1}{(D^3+2D^2+D)} \sin 2x$$

put $D^2 = -4$

$$= \frac{1}{-4D-8+D} \cdot \sin 2x$$

$$= \frac{1}{-(3D+8)} \sin 2x.$$

$$= -\frac{(3D-8)}{9D^2-64} \sin 2x.$$

$$= -\frac{(3D-8)}{-36-64} \sin 2x$$

$$= \frac{3D-8}{100} \sin 2x$$

$$= \frac{3}{\cancel{100}_{50}} \cdot \cancel{2} \cos 2x - \frac{\cancel{8}^{4}}{\cancel{100}_{50}} \sin 2x.$$

$$= \frac{3}{50} \cos 2x - \frac{2}{25} \sin 2x.$$

$$y_{p_3} = \frac{1}{(D^3+2D^2+D)} \cdot (x^2+x)$$

$$= \frac{1}{D(1+D^2+2D)}(x^2+x)$$

$$= \frac{1}{D}(1+D^2+2D)^{-1}(x^2+x)$$

$$= \cancel{(1+D^2+2D} = \frac{1}{D}\left\{1-(D^2+2D)+(D^2+2D)^2\right\}(x^2+x)$$

D^4+4D^2

$d(x^2+x) = 2x+1$

$d^2(x^2+x) = 2$

$$= \frac{1}{D}\left(x^2+x-2-2(2x+1)+4(2)\right]$$

$$= \frac{1}{D}\left[x^2+x-2-4x-2+8\right]$$

$$= \frac{1}{D}\left[x^2-3x+4\right]$$

$$= \frac{x^3}{3} - \frac{3x^2}{2} + 4x$$

$$y_p = \frac{1}{18}e^{2x} + \frac{1}{100}\left[6\cos 2x - 8\sin 2x\right] + \frac{x^3}{3} - \frac{3x^2}{2} + 4x.$$

General soln is $y = y_c + y_p$

$$y = C_1 + (c_2 + c_3 x)e^{-x} + \frac{1}{18}e^{2x} + \frac{1}{100}\left[6\cos 2x - 8\sin\right.$$

$$+ \frac{x^3}{3} - \frac{3x^2}{2} + 4x.$$

* Solve the D.E $(D^2+2D+1)y = x^2+2x$

Sol: Now. A. eqn $\Rightarrow f(m)=0$

$$(m^2+2m+1)=0$$

$$(m+1)^2=0$$

$$m=-1,-1$$

$$y_c = (c_1+c_2x)e^{-x}.$$

$$y_p = \frac{1}{(D^2+2D+1)}\cdot(x^2+2x)$$

$$= \frac{1}{(1+D)^2}(x^2+2x)$$

$$= (1+D)^{-2}(x^2+2x)$$

$D(x^2+2x) = 2x+2$

$D^2(x^2+2x) = 2$

$$= (1-2D+3D^2)(x^2+2x)$$

$$= [x^2+2x-2(2x+2)+3(2)]$$

$$= [x^2+2x-4x-4+6]$$

$$y_p = [x^2-2x+2]$$

General soln is $y = y_c+y_p$

$$y = (c_1+c_2x)e^{-x} + x^2-2x+2.$$

* $(D^3-3D^2+3D-1)y = x^3+\sin x.$

* $(D^2-4D+4)y = e^{2x}+\sin 3x+x^2$

* $(D^4+D^2)y = 3x^2+4\sin x - 2\cos x.$

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Method-4 :- if $Q(x) = e^{ax} \cdot v(x)$

where $v(x) = \sin bx$ or $\cos bx$ or x^K. Then

$$y_p = \frac{1}{f(D)} \cdot Q(x)$$

$$= \frac{1}{f(D)} \cdot e^{ax} v(x)$$

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$$= e^{ax} \cdot \frac{1}{f(D+a)} \cdot v(x)$$

Note :- $\frac{1}{f(D)} \cdot e^{-ax} v(x)$

then $y_p = e^{-ax} \cdot \frac{1}{f(D-a)} \cdot v(x)$.

Problems :-

1) Solve the D.E $(D^3 - 7D^2 + 14D - 8)y = e^x \cos 2x$.

Sol :- Given D.E is $D^3 - 7D^2 + 14D - 8 = 0$

Now A.eqn $\Rightarrow f(m) = 0$

$$m^3 - 7m^2 + 14m - 8 = 0$$

$$m = 4, 2, 1.$$

$$y_c = c_1 e^x + c_2 e^{2x} + c_3 e^{4x}.$$

$$y_p = \frac{1}{f(D)} \cdot Q(x)$$

$$= \frac{1}{f(D)} \cdot e^x \cos 2x.$$

$$= \frac{1}{(D^3-7D^2+14D-8)} \cdot e^x \cos 2x.$$

$$= e^x \cdot \frac{1}{(D+1)^3 - 7(D+1)^2 + 14(D+1) - 8} \cdot \cos 2x$$

$$= e^x \cdot \frac{1}{D^3+1+3D^2+3D-7(D^2+1+2D)+14D+14-8} \cdot \cos 2x.$$

$$= e^x \frac{1}{D^3+1+3D^2+3D-7D^2-7-\cancel{14D}+\cancel{14D}+14-8} \cdot \cos 2x$$

$$= e^x \cdot \frac{1}{D^3-4D^2+3D} \cdot \cos 2x$$

$$= \quad \text{put } D^2 = -a^2$$
$$D^2 = -4.$$

$$= e^x \cdot \frac{1}{-4D+16+3D} \cdot \cos 2x$$

$$= e^x \cdot \frac{1}{16-D} \cdot \cos 2x.$$

$$= e^x \cdot \frac{16+D}{256-D^2} \cdot \cos 2x$$

$$= e^x \cdot \frac{16+D}{260} \cos 2x$$

$$= e^x \cdot \frac{1}{260}\left(16\cos 2x - 2\cdot \sin 2x\right)$$

General soln is $y = y_c + y_p$

$$= C_1 e^x + C_2 e^{2x} + C_3 e^{4x} + \frac{e^x}{260}\left(16\cos 2x - 2\sin 2x\right)$$

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* Solve the D.E $(D^2+2D-3)\,y = x^2 e^{-3x}$.

Sol:- Now. A-eqn is $f(D)=0$

$$(D^2+2D-3)=0$$

$$(D+1)^2-4=0$$

$$(D+1-2)(D+1+2)=0$$

$$(D-1)(D+3)=0$$

$$\underline{\underline{D=1,-3}}$$

$$\therefore\ y_c = c_1 e^{x} + c_2 e^{-3x}.$$

$$y_p = \frac{1}{f(D)} \cdot Q(x)$$

$$= \frac{1}{D^2+2D-3} \cdot e^{-3x} \cdot x^2$$

$$= e^{-3x} \cdot \frac{1}{(D-3)^2+2(D-3)-3} \cdot x^2$$

$$= e^{-3x} \cdot \frac{1}{D^2+9-6D+2D-6-3} \cdot x^2$$

$$= e^{-3x} \cdot \frac{1}{D^2-4D} \cdot x^2.$$

$$= e^{-3x} \cdot \frac{1}{-4D\left(1-\frac{D}{4}\right)} \cdot x^2$$

$$= e^{-3x} \cdot \frac{1}{-4D}\left[1-D/4\right]^{-1} \cdot x^2$$

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$$= e^{-3x} \cdot \frac{1}{-4D}\left(1+\frac{D}{4}+\frac{D^2}{16}\right) \cdot x^2$$

$$Dx^2 = 2x$$
$$D^2x^2 = 2$$

$$= \frac{-1}{4} \cdot e^{-3x} \cdot \frac{1}{D}\left(x^2 + \frac{2x}{4} + \frac{2}{16}\right)$$

$$= -\frac{1}{4} e^{-3x} \cdot \frac{1}{D}\left(x^2 + x/2 + 1/8\right)$$

$$y_p = \frac{-1}{4} e^{-3x} \cdot \left[\frac{x^3}{3} + \frac{x^2}{4} + 1/8\, x\right]$$

General soln is $y = y_c + y_p$

$$y = c_1 e^{x} + c_2 e^{-3x} - \frac{1}{4} e^{-3x}\left[\frac{x^3}{3} + \frac{x^2}{4} + 1/8\, x\right].$$

Solve the D.E $(D^2+4D+3)y = e^x \cos 2x - \cos 3x - 3x^3$.

Sol: Now A. eqn is $f(m) = 0$

$$m^2 + 4m + 3 = 0$$

$$m = -1, -3.$$

$$y_c = c_1 e^{-x} + c_2 e^{-3x}.$$

$$y_p = \frac{1}{f(D)} \cdot Q(x)$$

$$= \frac{1}{D^2+4D+3}\left(e^x \cos 2x - \cos 3x - 3x^3\right)$$

$$y_{p_1} = \frac{1}{D^2+4D+3} e^x \cos 2x$$

$$= e^x \cdot \frac{1}{D^2+1+2D+4D+4+3} \cdot \cos 2x$$

$= e^{x} \cdot \frac{1}{D^2+6D+8} \cos 2x$

put $D^2 = -a^2 = -4$

$= e^{x} \cdot \frac{1}{-4+6D+8} \cos 2x$

$= e^{x} \cdot \frac{1}{6D+4} \cos 2x$

$= e^{x} \cdot \frac{6D-4}{36D^2-16} \cos 2x$

$= e^{x} \cdot \frac{6D-4}{-160} \cos 2x$ $\qquad 6(-2$

$= \frac{e^{x}}{-160} \cancel{(-6\cdot 2 \sin 2x)} \qquad = \frac{e^{x}}{160}(12\sin 2x + 4\cos 2x)$

$y_{P_2} = \frac{1}{D^2+4D+3}(\cos 3x)$

put $D^2 = -9$

$= \frac{1}{-9+4D+3}(\cos 3x)$

$= \frac{1}{4D-6} \cos 3x$

$= \frac{4D+6}{16D^2-36} \cos 3x$

$= \frac{1}{-160}(\cancel{-36}\, 12\sin 3x + 6\cos 3x)$

$$= \frac{1}{180}\left(12\sin 3x \mp 6\cos 3x\right)$$

$$y_{P_3} = \frac{1}{D^2+4D+3}\cdot 3x^3$$

$$= \cancel{3}\cdot\frac{1}{\cancel{3}\left(1+\left(\frac{D^2+4D}{3}\right)\right)}\cdot x^3$$

$$= \left[1+\left(\frac{D^2+4D}{3}\right)\right]^{-1}\cdot x^3$$

$$= \left[1-\left(\frac{D^2+4D}{3}\right)+\left(\frac{D^2+4D}{3}\right)^2-\left(\frac{D^2+4D}{3}\right)^3\right]\cdot x^3$$

$$= \left[1-\frac{(D^2+4D)}{3}+\frac{(D^4+16D^2+8D^3)}{9}-\frac{(D^6+64D^3+\cancel{4}3D^4+12D^3)}{27}\right]x^3$$

put $Dx^3 = 3x^2$

$D^2x^3 = 6x$

$D^3x = 6$

$$= \left[x^3-\frac{(6x+12x^2)}{3}+\frac{(96x+\overset{48}{\cancel{24x^2}})}{9}-\frac{(0+384+0+72)}{27}\right]$$

$$= \left[x^3-2x-4x^2+\frac{32x+\overset{48}{\cancel{6x^2}}}{3}-\frac{152}{9}\right]$$

$$= \left[\cancel{x^3}+\frac{\cancel{2}6}{3}x-\frac{4}{3}\cancel{x^2}-\frac{152}{9}\right] = \left[x^3+\frac{26}{3}x-4x^2-\frac{8}{9}\right]$$

$y_p = y_{p_1} + y_{p_2} + y_{p_3}$

$$= \frac{e^x}{160}(12\sin 2x + 4\cos 2x) + \frac{1}{180}(12\sin 3x - 6\cos 3x)$$

$$- \left[x^3 + \frac{26}{3}x - 4x^2 - 8/9\right]$$

General sol^n is $y_e = y_c + y_p$

* $(D^2 - 2D + 1)y = x^2 e^{3x} - \sin 2x + 3$
* $(D^2 - 4D + 4)y = e^{2x} + \cos 2x + e^x \sin 2x$.
* $(D^3 - 3D^2 + 3D + 1)y = (x+1)e^x$.
* $(D^3 - 2D + 4)y = e^x \sin x$.

Method-5 :- If $Q(x) = x \cdot V(x)$ where $V(x) = \sin ax$ (or) $\cos ax$.

then $y_p = \frac{1}{f(D)} \cdot Q(x)$

$$= \frac{1}{f(D)} \cdot x \cdot V(x)$$

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$$= \left[x - \frac{f'(D)}{f(D)}\right] \frac{1}{f(D)} \cdot V(x)$$

Problems :-

1) Solve the D.E $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = xe^x \sin x$.

Sol :- Given D.E $(D^2+3D+2)y = e^x, x\sin x$

Now A-eqn is $f(m)=0$

$$m^2+3m+2=0$$

$$m=-1,-2.$$

$$y_c = c_1 e^{-x} + c_2 e^{-2x}.$$

$$y_p = \frac{1}{f(D)} \cdot Q(x)$$

$$= \frac{1}{(D^2+3D+2)} \cdot e^x (x\sin x)$$

$$= e^x \cdot \frac{1}{(D+1)^2+3(D+1)+2} \cancel{e^x}\, x\sin x$$

$$= e^x \cdot \frac{1}{D^2+5D+6} \cdot x\sin x.$$

$$= e^x \cdot \left[x - \frac{2D+5}{D^2+5D+6} \right] \cdot \frac{1}{D^2+5D+6} \cdot \sin x$$

put $D^2=-1$

$$= e^x \left[x - \frac{2D+5}{D^2+5D+6} \right] \cdot \frac{1}{5D+5} \cdot \sin x.$$

$$= e^x \left(x - \frac{2D+5}{D^2+5D+6} \right] \cdot \frac{5D-5}{25D^2-25} \sin x.$$

$$= e^x \left(x - \frac{2D+5}{D^2+5D+6} \right] \cdot \frac{5D-5}{-50} \sin x.$$

$$= \frac{e^x}{5}\left[x - \frac{2D+5}{D^2+5D+6}\right] \cdot \frac{D+1}{-2} \sin x$$

$$= \frac{e^x}{5}\left[x - \frac{2D+5}{D^2+5D+6}\right] \cdot \left[\frac{+\cos x - \sin x}{-2}\right]$$

$$= \cancel{\frac{-e^x}{10}\left[x - \frac{2D+5}{D^2}\right.}$$

$$= \frac{-e^x}{10}\left[x(\cos x - \sin x) - \frac{(2D+5)}{(D^2+5D+6)}(\cos x - \sin x)\right]$$

$$= \frac{-e^x}{10}\left[x(\cos x - \sin x) - \frac{2(-\sin x - \cos x) + 5(\cos x - \sin x)}{D^2+5D+6}\right]$$

$$= \frac{-e^x}{10}\left[x(\cos x - \sin x) - \frac{3\cos x - 7\sin x}{D^2+5D+6}\right]$$

$$= \frac{-e^x}{10}\left[x(\cos x - \sin x) - \frac{3\cos x - 7\sin x}{5D+5}\right]$$

$$= \frac{-e^x}{10}\left[x(\cos x - \sin x) - \frac{1}{5}\frac{(D-1)}{(D^2-1)}(3\cos x - 7\sin x)\right]$$

$$= \frac{-e^x}{10}\left[x(\cos x - \sin x) + \frac{1}{10}(-3\sin x - 7\cos x - 3\cos x + 7\sin x)\right]$$

$$= \frac{-e^x}{10}\left(x(\cos x - \sin x) + \frac{1}{10}(4\sin x - \cancel{8} \, 10\cos x)\right]$$

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General soln $y = y_c + y_p$

$$y = C_1 e^{-x} + C_2 e^{-2x} - \frac{e^x}{10}\left[x(\cos x - \sin x) + \frac{1}{10}(4\sin x - 10\cos x)\right]$$

* Solve the D.E $(D^2 - 2D + 1)y = x e^x \cos x$

Sol :- Now. A.eqn $f(m) = 0$

$$m^2 - 2m + 1 = 0$$

$$(m-1)^2 = 0$$

$$m = 1, 1$$

$$y_c = \cancel{C_1 e^x}\,(C_1 + C_2 x)e^x.$$

$$y_p = \frac{1}{f(D)} \cdot Q(x)$$

$$= \frac{1}{D^2 - 2D + 1} \cdot e^x \cdot x\cos x$$

$$= (D-1)^2$$

$$= e^x \cdot \frac{1}{[(D+1)-1]^2} \cdot x\cos x.$$

$$= e^x \cdot \frac{1}{D^2} \cdot x\cos x$$

$$= e^x\left(x - \frac{2D}{D^2}\right) \cdot \frac{1}{D^2}\cos x.$$

$$= e^x\left(x - \frac{2}{D}\right) \cdot \frac{1}{-1}\cos x.$$

$$= +e^x\left[-x\cos x + 2\sin x\right]$$

General soln is $y = y_c + y_p$

$$= (C_1 + C_2 x)e^x + e^x\left[-x\cos x + 2\sin x\right].$$

* Solve the D.E $(D^2-4D+4)y = 8x^2 e^{2x} \sin 2x$.

Sol :- Given $(D^2-4D+4)y=0$

Now A eqn is $f(m)=0$

$$m^2-4m+4=0$$

$$(m-2)^2=0$$

$$m=2,2.$$

$$y_c = (c_1+c_2x)e^{2x}.$$

$$y_p = \frac{1}{f(D)} \cdot 8x^2 e^{2x} \sin 2x.$$

$$= \frac{1}{D^2-4D+4} \cdot 8x^2 e^{2x} \sin 2x$$

$$= 8 \cdot e^{2x} \cdot \frac{1}{(D+2)^2-4(D+2)+4} \cdot x^2 \sin 2x$$

$$= 8e^{2x} \cdot \frac{1}{D^2} x^2 \sin 2x.$$

$$\boxed{e^{i2x} = \underset{R.P}{\cos 2x} + i \underset{I.P}{\sin 2x}}$$

$$= 8e^{2x} \cdot \frac{1}{D^2} \cdot x^2 \text{ Imaginary part of } e^{i2x}.$$

$$= 8e^{2x} \cdot \text{Imag.P of } e^{i2x} \cdot \frac{1}{D^2} x^2$$

$$= 8e^{2x} \cdot \text{Imag. part of } e^{i2x} \cdot \frac{1}{(D+2i)^2} \cdot x^2,$$

$$= 8e^{2x} \cdot \text{IP of } e^{i2x} \cdot \frac{1}{(2i)^2\left(1+\frac{D}{2i}\right)^2} x^2$$

$$= -2e^{2x} \text{ IP of } e^{i2x}\left(1+\frac{D}{2i}\right)^{-2} x^2.$$

$= -2e^{2x} \cdot \text{IP of } e^{i2x}\left[1 - \frac{2D}{2i} + 3\left(\frac{D}{2i}\right)^2 + \cdots\right]x^2$

$= -2e^{2x} \text{ IP of } e^{i2x}\left[1 + Di - \frac{3}{4}D^2 + \cdots\right]x^2.$

$Dx^2 = 2x$

$D^2x^2 = 2$

$= -2e^{2x} \cdot \text{IP of } e^{i2x}\left[x^2 + 2xi - \frac{3}{4}(2)\right]$

$= -2e^{2x} \text{ IP of } [\cos 2x + i\sin 2x][(x^2 - 3/2) + i\,2x] \to (1)$

$= -2e^{2x}\left[2x\cos 2x + \left(x^2 - \frac{3}{2}\right)\sin 2x\right]$

Note :- If $(D^2 - 4D + 4)y = 8x^2e^{2x}\cos 2x$ $\quad y_c = (c_1 + c_2x)e^{2x}$

then $y_p = -2e^{2x}\left[\left(x^2 - \frac{3}{2}\right)\cos 2x - 2x\sin 2x\right].$

from eqn (1) $= -2e^{2x}$ ~~IP~~ RP of $[\cos 2x + i\sin 2x][(x^2 - 3/2) + i2x]$

* Solve the D.E $(D^2 - 1)y = x^2\cos x.$

Sol :- Now A. eqn ~~eqn~~ is $f(m) = 0$

$(m^2 - 1^2) = 0$

$(m+1)(m-1) = 0$

$m = -1, 1$

$y_c = c_1e^{-x} + c_2e^{x}.$

$y_p = \frac{1}{f(D)} \cdot Q(x)$

$= \frac{1}{(D^2-1)} \cdot x^2\cos x$

$= \frac{1}{(D^2-1)} \cdot \text{RP of } e^{ix} \cdot x^2.$

Variation of Parameters :-

Suppose $\frac{d^2y}{dx^2} + P\cdot\frac{dy}{dx} + Qy = R(x)$ be a 2^{nd} order linear diff. eqn. → (1).

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Working Rule :-

Step-1 :- where P, Q are the functions / constants and "R" is a function of 'x'.

Step-1 :- write the standard form (operator form) if it is necessary.

Step-2 :- Find $y_c = c_1 u(x) + c_2 v(x)$

Step-3 :- where $u(x), v(x)$ are the solutions of the given D.E (1).

Step-3 :- Let $y_p = A\,u(x) + B\,v(x)$

where $A = -\int \frac{vR}{uv' - vu'}\, dx$

$$B = \int \frac{uR}{uv' - vu'}\, dx.$$

Step-4 :- General solⁿ $y = y_c + y_p$.

NOTE :- Here the Denominator $uv' - vu'$ is called the Wronskian of (u, v). and it is denoted by $W(u, v)$.

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Problems :-

1) Solve the D.E by the method of variation of parameters

$$\frac{d^2y}{dx^2} + y = \operatorname{cosec} x.$$

Sol :- Given D.E is $(D^2+1)y = \operatorname{cosec} x$

Now A-eqn $\Rightarrow f(m) = 0$

$$(m^2+1) = 0$$

$$m^2 = -1$$

$$m = \sqrt{-1} = i$$

$$m = -i, i$$

$$y_c = c_1 \cos x + c_2 \sin x.$$

$u(x) = \cos x, \quad v(x) = \sin x.$

let $y_p = A u(x) + B v(x)$

$$A = -\int \frac{vR}{uv' - vu'} \, dx$$

$$= -\int \frac{\sin x \cdot \operatorname{cosec} x}{\cos x (\cos x) - \sin x (-\sin x)} \, dx$$

$$= -\int dx = -x$$

$$B = \int \frac{uR}{uv' - vu'} \, dx$$

$$= \int \frac{\cos x \cdot \operatorname{cosec} x}{\cos^2 x + \sin^2 x} \, dx$$

$$= \int \cot x \, dx$$

$$= \log|\sin x|$$

then $y_p = -x\cos x + \sin x \log|\sin x|$.

General solⁿ ⇒ $y = y_c + y_p$

$$= c_1 \cos x + c_2 \sin x - x\cos x + \sin x \log|\sin x|.$$

4) Solve the D.E $\frac{d^2y}{dx^2} + 9y = \tan 3x$.

Sol:- Given D.E is $(D^2+9)y = \tan 3x$.

Now A-eqn ⇒ $f(m) = 0$

$$m^2 + 9 = 0$$

$$m^2 = -9$$

$$m = -3i, 3i$$

$$y_c = c_1 \cos 3x + c_2 \sin 3x;$$

where $u = \cos 3x$, $v(x) = \sin 3x$, $R = \tan 3x$.

$$w(u,v) = \begin{vmatrix} \cos 3x & \sin 3x \\ -3\sin 3x & 3\cos 3x \end{vmatrix}$$

$$= 3\cos^2 3x + 3\sin^2 3x = 3$$

let $y_p = A\,u(x) + B\,v(x)$.

where $A = -\int \frac{vR}{uv' - vu'}\,dx$

$$= -\int \frac{\sin 3x \cdot \tan 3x}{3}\,dx.$$

$$= \frac{-1}{3}\int \frac{\sin^2 3x}{\cos 3x}\,dx$$

$$= \frac{-1}{3}\int \frac{1-\cos^2 3x}{\cos 3x}\,dx.$$

$$= \frac{+1}{3}\int (\cos 3x - \sec 3x)\,dx$$

$$= \frac{1}{3}\left(\frac{\sin 3x}{3} - \frac{\log(\sec 3x - \tan 3x)}{3}\right)$$

$$A = \sin 3x - \log(\sec 3x - \tan 3x)$$

$$B = \int \frac{uR}{uv' - vu'}\,dx$$

$$= \int \frac{\cos 3x \cdot \tan 3x}{3}\,dx.$$

$$= \int \frac{\sin 3x}{3}\,dx$$

$$= \frac{+1}{3}\,\frac{\cos 3x}{3}$$

$$B = \frac{\cos 3x}{9}$$

Then $y_p = \sin 3x - \log|\sec 3x - \tan 3x) + \frac{\cos 3x}{9}$

General sol^n $\quad y = y_c + y_p$

$$= c_1 \cos 3x + c_2 \sin 3x + \sin 3x - \log(\sec 3x - \tan 3x) + \frac{\cos 3x}{9}.$$

Applications of higher order differential equations:-

Electrical circuits:-

LCR circuit:-

Suppose an electrical circuit consist inductance L, Resistence R, and capacitence C in series form.

The voltage drop across inductance $L = L\frac{di}{dt} = L\frac{d^2q}{dt^2}$

The voltage drop across resistence $R = Ri = R\frac{dq}{dt}$

The voltage drop across capacitence $C = \frac{q}{c}$

then the voltage law, voltage drop across at each part in a circuit is equal to the resultant electromotive force

$\therefore$ The differential equation in a LCR circuit is given by

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{c} = e$$

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{c} = 0 \text{ (without emf)}$$

L-R circuit:-

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} = e$$

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} = 0 \text{ (without emf)}$$

1. A circuit consist of an inductance 2H a resistence of 4 Ω and capacitence of 0.05 F if $q = i = 0$ at $t = 0$ then find $q(t)$ & $i(t)$ when constant emf of 100V. $L = 2H$

$$R = 4\Omega$$

$$C = 0.05F$$

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = E$$

The DE for LCR circuit states that

$$2\frac{d^2q}{dt^2} + 4\frac{dq}{dt} + \frac{q}{0.05} = 100.$$

$$2\frac{d^2q}{dt^2} + 4\frac{dq}{dt} + 20q = 100$$

$$(2D^2 + 4D + 20)q = 100 \longrightarrow (1)$$

AE of eqn-(1) is $f(m) = 0$

$$2m^2 + 4m + 20 = 0$$

$$m^2 + 2m + 10 = 0$$

$$m = \frac{-2 \pm \sqrt{4 - 4\cdot 10}}{2}$$

$$= \frac{-2 \pm \sqrt{-36}}{2}$$

$$= \frac{-2 \pm 6i}{2}$$

$$m = -1 \pm 3i$$

$$q_c = e^{-t}[c_1 \cos 3t + c_2 \sin 3t]$$

$$q_p = \frac{1}{2D^2+4D+20} 100.$$

put $D=0$

$$= \cancel{\frac{1}{20}} \cancel{\times 1000}^{5} \quad \frac{1}{20} \times 10\cancel{0}^{5}$$

$$q_p = 5$$

$$q = q_c + q_p$$

$$q(t) = e^{-t}[c_1 \cos 3t + c_2 \sin 3t] + 5$$

$$i = \frac{dq}{dt} = e^{-t}[-3c_1 \sin 3t + 3c_2 \cos 3t] - [c_1 \cos 3t + c_2 \sin 3t]e^{-t}$$

at $t=0$ $q=0$

$$\Rightarrow c_1 + 5 = 0 \quad c_1 = -5$$

and also at $t=0, i=0$

$$\Rightarrow 3c_2 - c_1 = 0$$

$$3c_2 = -5$$

$$c_2 = -5/3$$

Now sub c_1 and c_2 values in $q(t)$ & i

$$q = e^{-t}\left[-5\cos 3t - \frac{5}{3}\sin 3t\right] + 5$$

$$i = e^{-t}[15 \sin 3t - 5\cos 3t] - \left[-5\cos 3t - \frac{5}{3}\sin 3t\right]e^{-t}$$

$$= 15 e^{-t}\sin 3t - 5e^{-t}\cancel{\cos 3t} + 5e^{-t}\cancel{\cos 3t} + \frac{5}{3}e^{-t}\sin 3t$$

$$= \left(15 + \frac{5}{3}\right)e^{-t}\sin 3t$$

$$= \frac{50}{3}e^{-t}\sin 3t$$

Q. Determine charge q and current i in an RLC circuit with $L = 0.5H$, $R = 6\Omega$, $C = 0.02F$, $E = 24\sin 10t$ and initial conditions $q = i = 0$ at $t=0$.

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = E$$

$$0.5\frac{d^2q}{dt^2} + 6\frac{dq}{dt} + \frac{q}{0.02} = 24\sin 10t$$

$$\frac{1}{2}D^2q + 6Dq + 50q = 24\sin 10t$$

$$D^2q + 12Dq + 100q = 48\sin 10t$$

$$(D^2 + 12D + 100)q = 48\sin 10t \rightarrow (1)$$

AE of eqn-(1) is $f(m) = 0$

$$m^2 + 12m + 100 = 0$$

$$m = \frac{-12 \pm \sqrt{144 - 4.100}}{2}$$

$$= \frac{-12 \pm \sqrt{-256}}{2}$$

$$= \frac{-12 \pm 16i}{2}$$

$$= -6 \pm 8i$$

$$q_c = e^{-6t}\left[c_1\cos 8t + c_2\sin 8t\right]$$

$$q_p = \frac{1}{D^2 + 12D + 100} \cdot 24\sin 10t.$$

$$D^2 = -100.$$

$$= \frac{1}{-\cancel{100}+12D+\cancel{100}} 24\sin 10t$$

$$= -\frac{\cancel{24}^{2}}{\cancel{12}} \cdot \frac{\cos 10t}{\cancel{10}5}$$

$$q_p = -\frac{1}{5}\cos 10t$$

$$q = q_c + q_p$$

$$q = e^{-6t}\left[c_1\cos 8t + c_2\sin 8t\right] - \frac{1}{5}\cos 10t$$

$$i = \frac{dq}{dt} = e^{-6t}\left[-8c_1\sin 8t + 8c_2\cos 8t\right] - 6e^{-6t}\left[c_1\cos 8t + c_2\sin 8t\right] -$$

$$\frac{1}{\cancel{5}} \times \cancel{10}^{2} \times \sin 10t$$

$$i = e^{-6t}\left[-8c_1\sin 8t + 8c_2\cos 8t\right] - 6e^{-6t}\left[c_1\cos 8t + c_2\sin 8t\right] + 2\sin 10t$$

at $t=0$, $q=0$

$$c_1 - \frac{1}{5} = 0$$

$$c_1 = \frac{1}{5}$$

$t=0$, $i=0$

$$8c_2 - c_1 = 0$$

$$8c_2 = c_1$$

$$c_2 = \frac{c_1}{8}$$

$$c_2 = \frac{1}{40}$$

Now sub c_1, c_2 in $q(t)$ & i.

$$q = e^{-6t}\left[\frac{1}{5}\cos 8t + \frac{1}{40}\sin 8t\right] - \frac{1}{5}\cos 10t$$

$$i = e^{-6t}\left[-\frac{8}{5}\sin 8t + \frac{8}{40}\cos 8t\right] - 6e^{-6t}\left[\frac{1}{5}\cos 8t + \frac{1}{40}\sin 8t\right.$$

$$+ 2\sin 10t.$$