MATRICES

Definition:- A collection of numbers arranged in rows and columns is said to be an array, A matrix is a rectangular array of numbers closed in addition, substraction, multiplication, and division.

We represent a matrix by $A=[a_{ij}]_{m\times n}$, $i=1(1)m$, $j=1(1)n$.

i.e. $\therefore A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$

Diagonal matrix:-

$A=[a_{ij}]$ is such that $a_{ij}=0 \;\forall\; i\neq j$

i.e. $A = diag[d_1, d_2, \ldots, d_n]$

Eg. $A = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}$ is a 3x3 diagonal mtx.

Scalar matrix:- $A=[a_{ij}] \ni a_{ij}=0 \;\forall\; i\neq j$

$a_{ij}=k \;\forall\; i=j$, $k\in\mathbb{N}$.

Eg $A = \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}$

~~Upper~~ Triangular mtx:- If every element above or below the leading diagonal of a square matrix is zero, then the matrix is called a triangular matrix.

Upper triangular mtx:- $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$

Lower Triangular mtx:- $\begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

Equality of matrices:- Two matrices A and B of the same order are said to be equal, if and only if the corresponding elements are equal.

Multiplication of matrix by a scalar:-

Matrix multiplication is associative and distributive but not commutative.

$A = [a_{ij}]_{m\times n}$

$$kA = \begin{bmatrix} ka_{11} & ka_{12} \cdots & ka_{1n} \\ ka_{21} & ka_{22} \cdots & ka_{2n} \\ \vdots & & \vdots \\ ka_{m1} & ka_{m2} \cdots & ka_{mn} \end{bmatrix} = [ka_{ij}]_{m\times n}$$

Addition of Matrices:-

$A = [a_{ij}]_{m\times n} \quad B = [b_{ij}]_{m\times n}$

$A + B = [a_{ij} + b_{ij}]$

Matrix addition is commutative, associative and distributive.

Note:- Only a square matrix can have a determinant.

Transpose of a matrix:-

$A = [a_{ij}]_{m\times n}$

$A^T = [a_{ji}]_{n\times m}$

(1) $(A')' = A$.

(2) $(A+B)' = A' + B'$.

(3) $(AB)' = B'A'$.

Symmetric and Skew-symmetric matrices:-

For a symmetric mtx. A, $\quad a_{ij} = a_{ji} \ \forall\ i, j$.

For a skew-symmetric mtx A, $\quad a_{ij} = -a_{ji} \ \forall\ i \neq j$

$= 0 \quad \forall\ i = j$.

Ex.1. If A be any matrix, S.T. AA' and $A'A$ are symmetric.

Sol. $(AA')' = (A')'(A)' = AA'$

$(A'A)' = (A)'(A')' = A'A$

Hence AA' and $A'A$ both are symmetric.

Ex.2. If A and B are both symmetric, then AB is symmetric iff A and B commute.

Sol.

$A' = A, \quad B' = B$

$(AB)' = B'A'$

$= BA$

$= AB$ iff A and B commute.

This shows $(AB)'$ is symmetric.

Ex.3. S.T. A^2 is symmetric, if A is either symmetric or skew-symmetric.

Sol. A^2 exists only if A is square mtx.

Let $A = [a_{ij}]$, $i, j = 1, 2, \ldots, n$. Then

$A^2 = [c_{ij}]$, $i, j = 1, 2, \ldots, n$, where,

$c_{ij} = \sum_{k=1}^{n} a_{ik} a_{kj}$

Case I :- When A is symmetric, then $a_{kj} = a_{jk}$.

$$\therefore c_{ij} = \sum_{k=1}^{n} a_{ik} a_{jk}$$

$$\therefore c_{ji} = \sum a_{jk} a_{ik} \text{ ; on interchanging } i \text{ and } j.$$

$\therefore$ Clearly, $c_{ij} = c_{ji}$, $\therefore A^2$ is symmetric.

Case II :- When A is skew-symmetric, then $a_{kj} = -a_{jk}$.

$$\therefore c_{ij} = \sum_{k=1}^{n} a_{ik}(-a_{jk}) = -\sum_{k=1}^{n} a_{ik} a_{jk}$$

so that $c_{ji} = \sum a_{jk} a_{ik}$, on interchanging i and j.

Clearly, $c_{ij} = c_{ji}$

$\therefore$ Hence A^2 is again symmetric.

$\therefore A^2$ is symmetric if A is either symmetric or skew-symmetric.

Ex.4. S.T. all positive integral powers of a symmetric matrix are symmetric.

Sol. $A' = A$

$(A^n)' = (A\,A \ldots n \text{ times})'$, n be a positive integer

$= A'A' \ldots n$ times

$= A\,A \ldots n$ times as $A' = A$

$= A^n$; hence A^n is symmetric.

Ex.5. S.T. all positive odd (even) integral powers of a skew-symmetric matrix are skew-symmetric (symmetric).

Sol. $A' = -A$

$(A^n)' = (A\,A \ldots n \text{ times})'$

$= A'A' \ldots n$ times

$= (-A)(-A) \ldots n$ times as $A' = -A$.

$= (-1)^n A^n$

$$= \begin{cases} A^n, & n = \text{even} \quad \therefore A^n \text{ is symmetric.} \\ -A^n, & n = \text{odd} \quad \therefore A^n \text{ is skew-symmetric.} \end{cases}$$

Ex.6. If A is symmetric (skew-symmetric), show that $B'AB$ is symmetric (skew-symmetric).

Sol. Case I :- $A^T = A$ $\quad (B'AB)' = (B)'A'(B')'$

$= B'AB$

Case II :- $A^T = -A$, $\quad (B'AB)' = -B'AB$

Ex.7. If A and B are symmetric (skew-symmetric), S.T. A+B is symmetric (skew-symmetric).

Ex.8. If A be any sq. matrix, S.T. $A+A'$ is symmetric and $A-A'$ is skew-symmetric.

Sol.
$$(A+A')' = (A') + (A')' = A' + A = (A+A')$$
$$(A-A')' = (A') - (A')' = A' - A = -(A-A').$$

Ex.9. If A, B are symmetric, S.T. $AB+BA$ is symmetric and $AB-BA$ is skew-symmetric.

Ex.10. Show that every square matrix can be uniquely expressed as the sum of a symmetric and a skew-symmetric matrix.

Sol.
$$A = \frac{1}{2}(A+A') + \frac{1}{2}(A-A')$$
$$(A+A')' = A' + (A')' = A' + A$$
$$(A-A')' = A' - A = -(A-A')$$

$\therefore$ $A+A'$ is symmetric and $A-A'$ is skew-symmetric.

Conjugate and Triangulated of matrix:—

A matrix obtained by replacing each element of a mtx A by its complex conjugate is called the conjugate mtx of A and is denoted by $\bar{A}$.

A matrix is said to be real iff $\bar{A} = A$.

$(\bar{A})'$ is called triangulated matrix of A.

Hermitian and Skew-Hermitian matrices:—

$A^* = [a_{ij}]$ is Hermitian iff $a_{ij} = \bar{a}_{ij} \ \forall\ i, j$

$= \bar{a}_{ii} \ \forall\ i = j$

i.e. every diagonal element of a Hermitian mtx is real.

e.g. $\begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$, $\begin{bmatrix} 4 & 1-i \\ 1+i & 2 \end{bmatrix}$ are the examples of Hermitian mtx.

$A = [a_{ij}]$ is skew-Hermitian iff $a_{ij} = -\bar{a}_{ji} \ \forall\ i, j$.

i.e. Every diagonal element of a skew-hermitian mtx is either purely imaginary or zero.

e.g. $\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1-i \\ -1-i & 2i \end{bmatrix}$

Q. Show that $A = \begin{bmatrix} 3 & 1+2i \\ 1-2i & 2 \end{bmatrix}$ is Hermitian.

Sol. $\bar{A}$ = Conjugate of $A = \begin{bmatrix} 3 & 1-2i \\ 1+2i & 2 \end{bmatrix}$

and $A^* = (\bar{A})'$ transpose of $\bar{A} = \begin{bmatrix} 3 & 1+2i \\ 1-2i & 2 \end{bmatrix}$

Clearly, $A^* = A$; hence A is Hermitian.

Note:- The above all results of symmetric and skew-symmetric matrices are true if we replace symmetric mtx by

Hermitian matrix and skew-symmetric mtx by skew-Hermitian mtx.

Note:– 1. ~~A~~ Every square mtx. can be uniquely represented as the sum of a [redacted] hermitian and a skew-hermitian matrix.

2. Every square mtx. can be uniquely expressed as $P + iQ$, where P and Q are hermitian.

3. If A is Hermitian (skew-hermitian) mtx, then iA is a skew-hermitian (Hermitian) mtx.

Polynomials in square matrix with scalar coefficients:–

The algebra of polynomials in one square mtx A with scalar coefficients is the same as the algebra of ordinary polynomials. For example, two parallel identities are:

$$x^2 - (\alpha+\beta)x + \alpha\beta \equiv (x-\alpha)(x-\beta)$$

$$A^2 - (\alpha+\beta)A + \alpha\beta I \equiv (A-\alpha I)(A-\beta I)$$

More generally, if

$$x^n + p_1 x^{n-1} + \cdots + p_{n-1}x + p_n \equiv (x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n)$$

then $A^n + p_1 A^{n-1} + \cdots + p_{n-1}A + p_n I \equiv (A-\alpha_1 I)(A-\alpha_2 I)\cdots(A-\alpha_n I)$

Ex. 1. Find the scalar solutions of the mtx equation $A^2 - 5A + 7I = 0$, and show that $\begin{bmatrix} 3 & 1 \\ -2 & 2 \end{bmatrix}$ is a non-scalar solution.

Sol. Consider the algebric equation

$$x^2 - 5x + 7 = 0;$$

$$x = \frac{5 \pm i\sqrt{3}}{2}$$

Hence the scalar solution are $A = \frac{1}{2}(5 \pm i\sqrt{3})I$, where $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

i.e. $A = \begin{bmatrix} \frac{1}{2}(5+i\sqrt{3}) & 0 \\ 0 & \frac{1}{2}(5+i\sqrt{3}) \end{bmatrix}$

2nd part:– If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$, then $A^2 = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix}$

$$A^2 - 5A = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} - 5\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$$

$$= -7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= -7I$$

Hence $\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$ is a non-scalar solution.

Idempotent Matrix :-

Definition:- A square matrix A such that $A^2 = A$ is called idempotent. If $A^T = A$ and $A^2 = A$, then the square matrix A is called symmetric idempotent.

Theorem:- Every non-singular idempotent mtx is an identity mtx.

Proof:- A matrix A is said to be non-singular or regular if $|A| \neq 0$.

As A is idempotent so $A^2 = A$ or $AA = A$.

Now, a square mtx B of the same order as A, such that $AB = BA = I$, is called the inverse mtx of A and is denoted by A^{-1} and exists as A is non-singular.

$$A^{-1}(AA) = A^{-1}A$$
$$\Rightarrow IA = I$$
$$\Rightarrow A = I.$$

Theorem:- If A and B are idempotent matrices, then AB is idempotent if A and B commute.

Proof:-

$$A^2 = A \;,\; B^2 = B$$

$$(AB)^2 = (AB)(AB)$$
$$= A(BA)B$$
$$= A(AB)B$$
$$= (AA)(BB)$$
$$= A^2B^2 = AB$$ as A and B commute. i.e. $AB = BA$.

$\therefore$ AB is idempotent.

Theorem:- If A is idempotent and $A+B = I$, then B is idempotent and $AB = BA = 0$.

Proof:- since $A+B = I$, $\therefore B = (I-A)$

$$B^2 = (I-A)(I-A)$$
$$= I - A - A + A^2$$
$$= I - A \quad [\because A^2 = A]$$
$$= B.$$

$\therefore$ B is an idempotent matrix.

2nd Part :-

$$A+B = I$$
$$A(A+B) = AI$$
$$A^2 + AB = A$$
$$A + AB = A$$
$$\therefore AB = 0$$

Similarly $BA = 0$.

Theorem:- Show that the mtx A defined as $A = I_n - X(X'X)^{-1}X'$ is a symmetric and idempotent mtx.

Sol.

$$A = I_n - X\left(X^{-1}(X')^{-1}\right)X'$$
$$= I_n - \{(XX^{-1})\}\{(X')^{-1}X'\}$$
$$= I_n - (I_n)(I_n)$$
$$= I_n - I_n = 0$$

Therefore $A' = A$ and $A^2 = A$.

Nilpotent Matrix :-

Definition :- If A be a nilpotent mtx such that $A^m = 0$, where m is positive integer, then A is called a nilpotent matrix.

If m be the least positive integer for which $A^m = 0$, then A is said to be a nilpotent matrix of index m. Thus a square mtx $A \ni A^m = 0$, but $A^{m-1} \neq 0$ is a nilpotent mtx of order m, m being a positive integer.

Involuntary Matrix :-

Definition :- A square mtx A is such that

$$A^2 = I \quad \text{or,} \quad (I+A)(I-A) = 0$$

is called involutory. Clearly I is involutary.

Ex. Show that $A = \begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix}$ is nilpotent.

Sol.

$$A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Orthogonal matrix :-

Definition :- A square mtx A is said to be orthogonal if

$$A'A = I = AA'.$$

Note :- When A is orthogonal $|A|^2 = 1$, $|A| = \pm 1$. If $|A|$ is equal to 1, then A is called a proper mtx.

Theorem :- If A and B are n-square orthogonal matrices, then AB and BA are orthogonal matrices.

Proof :- Since A and B are orthogonal matrices, we have

$$AA' = I \quad \text{and} \quad BB' = I$$

$$\begin{aligned} (AB)(AB') &= (AB)(B'A') \\ &= A(BB')A' \\ &= AA' \\ &= I \end{aligned}$$

$\therefore AB$ is orthogonal, similarly BA is also orthogonal.

Ex.1. S.T. the mtx $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ is orthogonal.

Sol.

$$AA' = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$; hence A is orthogonal.

Ex.2. If A is real skew-symmetric matrix such that $A^2 + I = 0$. Show that A is orthogonal.

Sol.

$$\begin{aligned} A &= -A' \\ AA &= -AA' \\ A^2 &= -AA' \\ -I &= -AA' \\ \Rightarrow AA' &= I \end{aligned}$$

$\therefore A$ is orthogonal.

Unitary matrix:–

Definition:– A square matrix A is called unitary if

$$A^* A = I = AA^*.$$

If A is real then $A^* = A'$, so that A is unitary if

$$A'A = AA' = I.$$

Ex. Show that $A = \begin{bmatrix} \frac{1}{2}(1+i) & -\frac{1}{2}(1-i) \\ \frac{1}{2}(1+i) & \frac{1}{2}(1-i) \end{bmatrix}$ is unitary mtx.

Sol.

$$A = \frac{1}{2}\begin{bmatrix} 1+i & 1-i \\ 1+i & 1-i \end{bmatrix}$$

$$A^* = \frac{1}{2}\begin{bmatrix} 1+i & 1-i \\ -1-i & 1+i \end{bmatrix}$$

$$AA^* = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

∴ A is unitary.

Trace (or spur) of a square matrix:–

Definition:– The sum of the elements of the principal diagonal of a square matrix is called the trace (or, spur, a german word) of the matrix.

Thus if $A = [a_{ij}]$ be a square matrix of order n, then

$$\text{trace } A = a_{11} + a_{22} + a_{33} + \cdots + a_{nn} = \sum_{i=1}^{n} a_{ii}.$$

Remark:– For the identity mtx I_n, trace $I_n = n$.

Ex. 1. Show that for any sq. mtx A, B :–

(i) Trace (KA) = k trace (A), k being a scalar.

(ii) Trace (KA + B) = K trace (A) + trace (B).

Sol. (i) $A = [a_{ij}]_{n\times n}$. Then $KA = [Ka_{ij}]_{n\times n}$

$$\text{Trace}(KA) = \sum_{i=1}^{n} k a_{ii}$$

$$= k\,\text{Trace}(A).$$

(ii) $A = [a_{ij}]_{n\times n}$, $B = [b_{ij}]_{n\times n}$

$$KA + B = [Ka_{ij} + b_{ij}]_{n\times n}$$

$$\text{Trace}(KA+B) = \sum_{i=1}^{n} (ka_{ii} + b_{ii})$$

$$= k\sum_{i=1}^{n} a_{ii} + \sum_{i=1}^{n} b_{ii}$$

$$= k\,\text{trace}(A) + \text{trace}(B)$$

e.g. If trace(A) = 20, trace(B) = −8, then trace(A+B) = 12.

Ex. 2. Show that trace A' = trace A.

Sol. Let $A = [a_{ij}]_{n\times n}$, $A' = [a'_{ij}]_{n\times n} = [a_{ji}]_{n\times n}$

therefore when $i=j$, we have trace(A) = trace(A').

Ex. 3. Show that trace (AB) = trace (BA), if AB and BA co-exist.

Sol.

$A = [a_{ij}]_{m\times n}$ $B = [b_{ij}]_{n\times m}$

$AB = [c_{ij}]_{m\times m}$; $c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$

$$\therefore \text{trace}(AB) = \sum_{i=1}^{m} c_{ii}$$

$$= \sum_{i=1}^{m}\sum_{k=1}^{n} a_{ik} b_{ki}$$

$BA = [d_{ij}]_{n\times n}$

$$d_{ij} = \sum_{k=1}^{m} b_{ik} a_{kj}$$

$$\therefore \text{trace}(BA) = \sum_{i=1}^{n} d_{ii}$$

$$= \sum_{i=1}^{n}\sum_{k=1}^{m} b_{ik} a_{ki}$$

$$= \sum_{k=1}^{n}\sum_{i=1}^{m} b_{ki} a_{ik}$$, interchanging i and k.

$$= \sum_{i=1}^{n}\sum_{k=1}^{n} a_{ik} b_{ki}$$

$\therefore$ trace (AB) = trace (BA).

Ex. 4. S.T. trace$(AA') \geq 0$.

Sol. $A = [a_{ij}]_{m\times n}$, $A' = [a'_{ij}]_{n\times m}$

$AA' = [c_{ij}]_{m\times m}$, $c_{ij} = \sum_{k=1}^{n} a_{ik} a'_{kj}$

$$\therefore \text{trace}(AA') = \sum_{i=1}^{m} c_{ii}$$

$$= \sum_{i=1}^{m}\sum_{k=1}^{n} a_{ik} a'_{ki}$$

$$= \sum_{i=1}^{m}\sum_{k=1}^{n} (a_{ik})^2 \text{ as } a_{ik} = a'_{ki}$$

$$\geq 0$$

Ex. 5. S.T. trace $(C'AC)$ = trace (A), if C is an orthogonal mtx.

Sol. trace $(C'AC)$ = trace $\{C'(AC)\}$

= trace $\{(AC)C'\}$, as trace AB = trace BA.

= trace $\{A(CC')\}$, as C is orthogonal mtx.

= trace (A).

The adjoint of a square matrix:-

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, \quad \text{Adj } A = \begin{bmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & \vdots & & \vdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{bmatrix}$$

Theorem:- If A is a square matrix, then

$$A(\text{adj}A) = (\text{adj}A).A = |A|.I.$$

Sol. We have

The (i,j) the element of the product $A.(\text{Adj}A)$

= Product of the ith row of A and jth column of adj$A$

$= a_{i1}A_{j1} + a_{i2}A_{j2} + \cdots + a_{in}A_{jn}$

$$= \begin{cases} 0 & \text{if } i \neq j \\ |A| & \text{if } i = j \end{cases}$$

Thus in the product, only the diagonal elements exist and each is equal to $|A|$ while all other element are zero, so that

$$A.(\text{adj}A) = \begin{bmatrix} |A| & 0 & 0 & \cdots & 0 \\ 0 & |A| & 0 & \cdots & 0 \\ 0 & 0 & |A| & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & |A| \end{bmatrix}$$

$$= |A| \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

$$= |A|.I.$$

Similarly, $(\text{adj}A).A = |A|.I.$

Corollary 1:- If $|A| \neq 0$, we have

$$|A|.|\text{Adj}A| = \begin{vmatrix} |A| & 0 & \cdots & 0 \\ 0 & |A| & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & |A| \end{vmatrix} = |A|^n$$

$\therefore |\text{Adj}A| = |A|^{n-1}.$

Corollary 2:- If $|A| \neq 0$, then

$$A\left(\frac{1}{|A|}\text{Adj}A\right) = I = \left(\frac{1}{|A|}\text{adj}A\right)A.$$

Singular matrix:– A sq. matrix is said to be singular if its determinant is zero.

Theorem:- If A and B are n-square matrices, then

$$\text{Adj}AB = \text{Adj}A \cdot \text{Adj}B.$$

Sol. We have $AB.(\text{adj}AB) = |AB| \cdot I = (\text{adj}AB) \cdot AB$

Now, $AB(\text{adj}B)(\text{Adj}A) = A(B.\text{adj}B)(\text{Adj}A)$
$= A \cdot (|B| \cdot I)(\text{Adj}A)$
$= |B| \cdot (A \cdot \text{adj}A)$
$= |B|\,|A| \cdot I$
$= |AB| \cdot I$

Also, $(\text{adj}B)(\text{adj}A)\,AB = (\text{adj}B)[\text{adj}(A).A]B$
$= (\text{adj}B)|A| \cdot IB$
$= |A|[(\text{adj}B) \cdot B]$
$= |A| \cdot |B| \cdot I$
$= |AB| \cdot I.$

$\therefore \text{Adj}(AB) = \text{Adj}(A)\,\text{Adj}(B).$

Ex.1. Show that $\text{Adj}A' = (\text{Adj}A)'$

Sol. Obviously, the matrices $\text{adj}A'$ and $(\text{adj}A)'$ both will be of the same order as A. Now,

(i,j)th element of $\text{Adj}A'$ = the co-factor of (j,i)th element of A' in the determinant $|A'|$.
= the co-factor of (i,j)th element of A in the determinant $|A|$
= (j,i)th element of Adj A.

$\therefore \text{Adj}A' = (\text{adj}A)'.$

Ex.2. Show that every skew-symmetric mtx of odd order is singular.

Solution:– Let A be a skew-symmetric matrix of order n, where n is odd.

Since A is skew-symmetric, we have

$A' = -A.$

$|A'| = |-A|$
$= (-1)^n |A|$
$= -|A|$, where n is odd and $|A'| = |A|$

$\therefore |A| = -|A|$

$\therefore 2|A| = 0$

$\Rightarrow |A| = 0.$

Hence a skew-symmetric mtx of odd order is singular.

WORKED EXAMPLES :-

1. Let A be a non-singular sq. mtx of order 3. If B is the mtx obtained from A by adding 3-multiple of its first row to its second row, then the value of det $(2A^{-1}B)$ is
(A) 8 (B) 3 (C) 6 (D) 2

Sol. (D) $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$|A| \neq 0$,

$A \sim B = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 3a_{11}+a_{21} & 3a_{12}+a_{22} & 3a_{13}+a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

Since A and B are equivalent so $|A| = |B|$

$\therefore \text{Det}(2A^{-1}B) = 2|A^{-1}||B|$

$= 2 \cdot \frac{1}{|A|} \cdot |B|$

$= 2$.

2. Let $\underset{\sim}{u}$ be a unit column vector and $A = I - 2\underset{\sim}{u}\underset{\sim}{u}^T$. Then A^{-1} is
(A) $I - 2\underset{\sim}{u}\underset{\sim}{u}^T$ (B) $I + 2\underset{\sim}{u}\underset{\sim}{u}^T$ (C) $2\underset{\sim}{u}\underset{\sim}{u}^T - I$ (D) $4\underset{\sim}{u}\underset{\sim}{u}^T$.

Sol. $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 2 \cdot \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$= \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

$A^{-1} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = I - 2\underset{\sim}{u}\underset{\sim}{u}^T$.

3. Let P, Q, R be matrices of order 3×5, 5×7, 7×3 respectively. The number of scalar additions required to compute $P(QR)$ is
(A) 114 (B) 126 (C) 128 (D) 138.

Sol. We have to calculate $P(QR)$.
Q and R is of order 5×7 and 7×3 respectively.
When we multiply it we have to make 6 scalar addition for each entry so 5 column in Q and 3 rows in R when we multiply Q and R,
we do $6 \times 5 \times 3 = 90$ addition further QR is of 5×3 matrix.
Similarly when we multiply P to QR we perform $4 \times 3 \times 3$ addition
$\therefore$ Total no. of additions $= 90 + 36 = 126$.

4. If A is a square matrix and $A-\frac{1}{2}I$ and $A+\frac{1}{2}I$ are orthogonal, prove that A is skew-symmetric & $A^2 = -3/4\, I$.

Sol. Since $A-\frac{1}{2}I$ is orthogonal,

so, $(A-\frac{1}{2}I)(A-\frac{1}{2}I)^T = I$

or, $(A-\frac{1}{2}I)(A'-\frac{1}{2}I) = I$

or, $AA' - \frac{1}{2}IA' - \frac{1}{2}AI + \frac{1}{4}I^2 = I$

or, $AA' - \frac{1}{2}A' - \frac{1}{2}A = \frac{3}{4}I$ —— (1)

Similarly since $A+\frac{1}{2}I$ is orthogonal, we have

$AA' + \frac{1}{2}A' + \frac{1}{2}A = \frac{3}{4}I$. —— (2)

$\therefore$ (1) & (2) gives (1) − (2) $A' + A = 0$

$\Rightarrow A' = -A$

$\therefore$ A is skew-symmetric.

(1) + (2) gives $2AA' = \frac{3}{2}I$

$\therefore 2A(-A) = \frac{3}{2}I$

$\therefore A^2 = -\frac{3}{4}I$.

5. If $A = \begin{bmatrix} 1+i & 2-3i & 2 \\ 3-4i & 4+5i & 1 \\ 5 & 3 & 3-i \end{bmatrix}$. Find the conjugate of matrix A?

Sol. $\overline{A} = \begin{bmatrix} 1-i & 2+3i & 2 \\ 3+4i & 4-5i & 1 \\ 5 & 3 & 3+i \end{bmatrix}$

6. Let P and Q be two $n\times n$ nonzero matrices $\ni P+Q=0$, then show that, (i) P is non-singular

(ii) $P = Q^T$

(iii) $P = Q^{-1}$

(iv) $\text{Rank}(P) = \text{Rank}(Q)$.

Sol. Given that P and Q are two $n\times n$ matrices such that $P+Q=0$.

$\therefore P+Q=0 \Rightarrow P = -Q$

$\Rightarrow |P| = |Q|$

$\Rightarrow |P| \neq 0$ iff $|Q| \neq 0$

i.e. P is non-singular iff Q is non-singular.

If Q is non-singular then Q^{-1} exists. $|Q^{-1}| = |Q^T| = |Q|$

$\therefore P = Q^T = Q^{-1}$.

But if $P+Q=0$ then $\text{Rank}(P) = \text{Rank}(Q)$.

Example:- $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $Q = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

Then $P+Q=0$ and $\text{Rank}(P) = \text{Rank}(Q)$.

7. Show that all $n \times n$ symmetric matrices over F form a vector space of dimension $\frac{n(n+1)}{2}$ over F. What is the dimension of the space of skew-symmetric matrices?

Sol. Let $A = (a_{ij})$ be an $n \times n$ symmetric mtx.
Then $a_{ij} = a_{ji}$ $\forall i \neq j$. Thus the number of independent entries are a_{ij} $(i<j)$ and a_{ij} $(1 \le i \le n)$ and these are $\frac{n(n-1)}{2} + n = \frac{n(n+1)}{2}$ in number.

Hence the space has dimension $\frac{n(n+1)}{2}$.

For a skew-symmetric mtx, $a_{ij} = 0$, $1 \le i \le n$, $\forall i = j$ and $a_{ij} = -a_{ji}$ thus the number of LIN entries are $\frac{n(n-1)}{2}$ which is also the dimension of the space of skew-symmetric matrices.

8. (a) If A is a real skew-symmetric mtx and $A^2 + I = 0$, then S.T. A is orthogonal.
(b) If H is Hermitian mtx, what kind of mtx is e^{iH}?

Sol. (a) Let A be the real skew-symmetric mtx.
then $A^T = -A$.
Also we have $A^2 + I = 0$.
We have $A^T A = (-A)(A) = I$.
$\therefore$ A is orthogonal.

(b) Let H is Hermitian mtx. $H^* = H$.
For any mtx M, $(e^M)^* = e^{M^*}$

$$\text{Let } e^{iH} = A$$

$$\text{Then } A^* A = (e^{iH})^* . e^{iH}$$

$$= I.$$

Hence e^{iH} is unitary matrix.

Inverse or Reciprocal of a matrix:-

Let A be a square matrix of order n. Then the mtx B of order n, if it exists, such that

$$AB = BA = I_n,$$

is called the inverse or reciprocal of A and is denoted by A^{-1}.

$$A^{-1} = \left(\frac{Adj\,A}{|A|}\right) \cdot I, \text{ provided } |A| \neq 0.$$

Corollary:- We have $|A||A^{-1}| = |AA^{-1}| = |I| = 1$

Hence $|A^{-1}| = |A|^{-1}$.

Theorem:- The inverse of a matrix is unique.

Sol. If possible let B and C be two inverses of the same mtx A, then by definition

$$AB = BA = I$$
$$AC = CA = I$$

$\therefore$ $C(AB) = CI = C$

$\therefore$ $(CA)B = IB = B$

$\therefore$ $C = B$, $\therefore$ the inverse is unique.

Theorem:- A square matrix A has an inverse if and only if $|A| \neq 0$, i.e. only a non-singular matrix has an inverse.

Proof:- The condition is necessary. Let B be the inverse of the mtx A.

Then $AB = I$

so, that $|A||B| = |I| = 1$

$\therefore$ $|A| \neq 0$.

$\therefore$ The condition is sufficient.

Let $|A| \neq 0$, we assume that $B = \frac{Adj\,A}{|A|}$

$$\therefore AB = A\left(\frac{Adj\,A}{|A|}\right)$$
$$= \frac{1}{|A|} \cdot (A \cdot Adj\,A)$$
$$= \frac{|A| \cdot I}{|A|} = I$$

Similarly $BA = I$.

$\therefore$ $AB = BA = I$. $\therefore$ Hence A has an inverse.

Ex. 1 Find A^{-1} when $A = \begin{bmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \\ 0 & 6 & -7 \end{bmatrix}$.

Sol. $|A| = 20$; $\text{Adj}\,A = \begin{bmatrix} 2 & 6 & 4 \\ 21 & -7 & -8 \\ -18 & 6 & 4 \end{bmatrix}$

Hence $A^{-1} = \dfrac{\text{Adj}\,A}{|A|} = \begin{bmatrix} \frac{2}{20} & \frac{6}{20} & \frac{4}{20} \\ \frac{21}{20} & -\frac{7}{20} & -\frac{8}{20} \\ -\frac{18}{20} & \frac{6}{20} & \frac{4}{20} \end{bmatrix}$

Ex. 2. Show that the inverse of a regular symmetric (hermitian) matrix is symmetric (hermitian).

Sol. Let A be symmetric, so that $A' = A$

$\&\ AA^{-1} = I = A^{-1}A$

Since $I = I'$, so, $A^{-1}A = (A^{-1}A)'$

$AA^{-1} = (A^{-1}A)'$

$= A'.(A^{-1})'$

$= A(A^{-1})'$

$\therefore A^{-1} = (A^{-1})'$

$\therefore A^{-1}$ is symmetric.

2nd Part :- Let A be hermitian, so that

$A^{*} = A$

$A^{-1}A = AA^{-1} = I$

$\therefore (A^{-1}A)^{*} = I^{*} = I = (AA^{-1})$

$\& \ A^{*}(A^{-1})^{*} = AA^{-1}$

$\therefore A(A^{-1})^{*} = AA^{-1}$

$\therefore (A^{-1})^{*} = A^{-1}$

$\therefore A^{-1}$ is hermitian.

Ex. 3. Find the inverse of $A = \begin{bmatrix} 0 & 1 & 2 & 2 \\ 1 & 1 & 2 & 3 \\ 2 & 2 & 2 & 3 \\ 2 & 3 & 3 & 3 \end{bmatrix}$.

Sol.

$\boxed{A = I \times A}$

$$\begin{bmatrix} 0 & 1 & 2 & 2 \\ 1 & 1 & 2 & 3 \\ 2 & 2 & 2 & 3 \\ 2 & 3 & 3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A$$

$$\begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 2 & 1 & 3 \\ 1 & 3 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} A \quad \begin{bmatrix} C_1' \leftrightarrow C_2 - C_1 \\ C_3' \leftrightarrow C_4 - C_3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 3 \\ 1 & 1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A \quad \begin{bmatrix} R_2' \leftrightarrow R_3 - R_2 \\ R_4' \leftrightarrow R_4 - R_3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & -1 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & -1 & 1 & 0 \\ 2 & 2 & -1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} A \quad [R_4' \leftrightarrow R_4 - R_1] \quad [R_3' \leftrightarrow R_3 - 2R_2]$$

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & -1 & 0 \\ -1 & -1 & 1 & 0 \\ 2 & 2 & -1 & 0 \\ 1 & 2 & -1 & 1 \end{bmatrix} A \quad [R_4' = R_4 + R_3] \quad [R_1' = R_1 - R_2]$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -3 & 1 & -2 \\ -1 & -1 & 1 & 0 \\ -1 & -4 & 2 & -3 \\ 1 & 2 & -1 & 1 \end{bmatrix} A \quad [R_1' = R_1 - 2R_4] \quad [R_3' = R_3 - 3R_4]$$

$$I = A^{-1} A$$

$$\therefore A^{-1} = \begin{bmatrix} 0 & -3 & 1 & -2 \\ -1 & -1 & 1 & 0 \\ -1 & -4 & 2 & -3 \\ 1 & 2 & -1 & 1 \end{bmatrix}$$

Rank and Nullity of a matrix:—

The maximum order of the non-singular square sub-matrix of A is called the rank of A. The mtx A is said to be of rank r, if and only if it has at least one nonsingular square sub-mtx of order r and all square submatrices of $(r+1)$ and higher orders are singular. The rank of a mtx A is denoted by rank(A).

If A is a square mtx of order n, then $n - \text{rank}(A)$ is called the nullity of the mtx A and denoted by $N(A)$.

Remarks:—

1. If I is a unit mtx of order n, rank$(I) = n$.
2. If A is of order $m \times n$, then rank$(A) \le m$ and $\le n$.
3. If A' is the transpose of A, rank(A') = rank(A).

Ex.1. S.T. the rank of a skew-symmetric mtx can't be 1.

Sol. For a skew-symmetric mtx, the elements of the leading diagonal are all zero. If one element is non-zero, say x, then we have a minor of the form $\begin{vmatrix} 0 & x \\ -x & 0 \end{vmatrix} \neq 0$

Therefore rank of the mtx is ≥ 2. if all entries are zero then rank is 0.

Ex. 2. Find the rank of the mtx A, where $A = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 3 & -2 \\ 2 & 4 & -3 \end{bmatrix}$.

Sol.
$$|A| = 2(-9+8) + 2(-3+4) = 0$$
But $\begin{vmatrix} 2 & 1 \\ 0 & 3 \end{vmatrix} \neq 0$, so rank $(A) = 2$.

Normal form of a matrix :-

Theorem:- By means of elementary transformations every mtx A of order $m \times n$ and rank $r (>0)$ can be reduced to the following forms:

(i) $\left[\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right]$ (ii) $\begin{bmatrix} I_r \\ 0 \end{bmatrix}$ (iii) $[I_r | 0]$ (iv) $[I_r]$

Ex. Reduce the mtx A to its normal form, where $A = \begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 2 & 3 & 4 \\ 3 & 4 & 5 & 2 \end{bmatrix}$ and hence determine its rank.

Solution:-
$$\begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 2 & 3 & 4 \\ 3 & 4 & 5 & 2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & -1 \\ 0 & 1 & 2 & 5 \\ 0 & 1 & 2 & 5 \end{bmatrix} \qquad R_2' \leftrightarrow R_2 - R_1, \quad R_3' \leftrightarrow R_3 - 3R_1$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & -1 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad R_3' \leftrightarrow R_3 - R_2$$

$$\sim \begin{bmatrix} 1 & 0 & -1 & -6 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad R_1' \leftrightarrow R_1 - R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & -6 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad C_3' \leftrightarrow C_3 + C_1, \quad C_3' \leftrightarrow C_3 - 2C_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad C_4' \leftrightarrow C_4 + 6C_1, \quad C_4' \leftrightarrow C_4 - 5C_2$$

Hence the normal form of A is $\left[\begin{array}{c|c} I_2 & 0 \\ \hline 0 & 0 \end{array}\right]$

and rank$(A) = 2$.

WORKED EXAMPLES:-

1. If $r(A)$ denotes rank of a mtx A, then $r(AB)$ is
(a) $r(A)$ (b) $r(B)$ (c) $\leq \min[r(A), r(B)]$ (d) $> \min[r(A) r(B)]$

Sol. $rank(AB) \leq \min[r(A), r(B)]$

e.g. $A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$AB = \begin{bmatrix} 7 & 10 \\ 7 & 10 \end{bmatrix}$ $rank(A) = 1$, $r(B) = 2$.

$rank(AB) = 1 \leq \min[r(A), r(B)]$.

2. Show that $rank(A') = rank(A^*) = rank(A)$.

Sol. The transpose of a mtx is obtained by interchanging rows into columns into rows, and clearly this change does not alter the values of the determinants of minors. Hence

$rank(A') = rank(A)$.

Now assume that the value of a minor of A with complex elements be $a+ib$. Then the value of the corresponding minor of the conjugate mtx of A be $a-ib$.

incase $a+ib = 0$ then $a-ib = 0$.
in case $a+ib \neq 0$ then $a-ib \neq 0$.

Hence $rank(A^*) = rank(A) = rank(A')$.

3. If A is an $(n \times 1)$ non-zero mtx and B is a $(1 \times n)$ non-zero mtx, then show that $rank(AB) = 1$.

Sol. Let $A = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{bmatrix}$ and $B = [b_{11} \ b_{12} \cdots b_{1n}]$.

Then $AB = \begin{bmatrix} a_{11}b_{11} & a_{11}b_{12} \cdots\cdots & a_{11}b_{1n} \\ a_{21}b_{11} & a_{21}b_{12} \cdots\cdots & a_{21}b_{1n} \\ \vdots & \vdots & \vdots \\ a_{n1}b_{11} & a_{n2}b_{12} \cdots\cdots & a_{n1}b_{1n} \end{bmatrix}$

AB is non-zero and there will be at least one non-zero minor of order 1.

Hence, $rank(AB) = 1$ because all 2nd and higher order minors are all zero here.

4. Find the rank of the matrix $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & -2 & 1 \\ 2 & 0 & -3 & 2 \\ 3 & 3 & 0 & 3 \end{bmatrix}$

Solution:-

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & -2 & 1 \\ 2 & 0 & -3 & 2 \\ 3 & 3 & 0 & 3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 3 & -2 & 0 \\ 2 & 0 & -3 & 0 \\ 3 & 3 & 0 & 0 \end{bmatrix} \qquad C_4' \leftrightarrow C_4 - C_1$$

$\therefore$ rank (A) = No. of LIN columns in A

= 3.

5. Reduce the mtx A to its normal form, where $A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$ and hence find the rank of the mtx.

Solution:-

$$A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad R_4' \leftrightarrow R_4 - (R_1 + R_2)$$

$$\sim \begin{bmatrix} 0 & 1 & -3 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad C_1' \leftrightarrow C_1 - (C_2 + C_4)$$

$$\sim \begin{bmatrix} 1 & -1 & -3 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad C_2' = C_1 + C_2$$

5. Reduce the matrix A to its normal form, where

$$A = \begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 2 & 3 & 4 \\ 3 & 4 & 5 & 2 \end{bmatrix}$$

and hence determine its rank.

Sol. To reduce a mtx into its normal form, let us use congruent row and column operations.

$$A \sim \begin{bmatrix} 1 & 1 & 1 & -1 \\ 0 & 1 & 2 & 5 \\ 3 & 4 & 5 & 2 \end{bmatrix} \quad \text{By } R_{21}(-1)$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & -1 \\ 0 & 1 & 2 & 5 \\ 0 & 1 & 2 & 5 \end{bmatrix} \quad \text{By } R_{31}(-3)$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 2 & 5 \\ 0 & 1 & 2 & 5 \end{bmatrix} \quad \text{By } C_{21}(-1)$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 2 & 5 \\ 0 & 1 & 2 & 5 \end{bmatrix} \quad \text{By } C_{31}(-1)$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 5 \\ 0 & 1 & 2 & 5 \end{bmatrix} \quad \text{By } C_{41}(1)$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{By } R_{32}(-1)$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{By } C_{32}(-1)$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{By } C_{42}(-5)$$

Hence the normal form of A is $\left[\begin{array}{c|c} I_2 & 0 \\ \hline 0 & 0 \end{array}\right]$ and rank $(A) = 2$.

6. Reduce the matrix A to its normal form, where

$$A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$$

and hence find the rank of the matrix.

<u>Solution:-</u> By performing the operation R_{12}, we have

$$A \sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$$

By $R_{31}(-3)$, $R_{41}(-1)$
$$A \sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \end{bmatrix}$$

By $C_{31}(-1)$, $C_{41}(-1)$
$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \end{bmatrix}$$

By $R_{32}(-1)$, $R_{42}(-1)$
$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

By $C_{32}(3)$, $C_{42}(1)$, $A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

which is of the form $\left[\begin{array}{c|c} I_2 & 0 \\ \hline 0 & 0 \end{array}\right]$; hence rank(A) = 2.

System of Linear Equations

1. The equations $x - y + 2z = 4$
$3x + y + 4z = 6$
$x + y + z = 1$ have

(A) Unique solution (B) Infinite solutions (C) No solution (D) None.

Sol. → (B) $[A|b] = \left[\begin{array}{ccc|c} 1 & -1 & 2 & 4 \\ 3 & 1 & 4 & 6 \\ 1 & 1 & 1 & 1 \end{array}\right]$

$\sim \left[\begin{array}{ccc|c} 1 & -1 & 2 & 4 \\ 0 & 4 & -2 & -6 \\ 0 & 2 & -1 & -3 \end{array}\right]$

$\sim \left[\begin{array}{ccc|c} 1 & -1 & 2 & 4 \\ 0 & 4 & -2 & -6 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$\therefore$ rank $[A|b]$ = rank $(A) = 2 < 3$

$\therefore$ it has infinite solutions.

2. The system of linear equations $x + y + z = 2$, $2x + y - z = 3$, $3x + 2y + kz = 4$ has a unique solution if

(A) $k = 0$ (B) $k \neq 1$ (C) $k = 1$ (D) $k \neq 0$.

Sol. $[A|b] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 2 & 1 & -1 & 3 \\ 3 & 2 & k & 4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -1 & -3 & -1 \\ 0 & -1 & k-3 & -2 \end{array}\right]$

$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -1 & -3 & -1 \\ 0 & 0 & k & -1 \end{array}\right]$

$\therefore k \neq 0$

Alt. method:—

Given system has a unique solution if

$\begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 3 & 2 & k \end{vmatrix} \neq 0$

$\Rightarrow k \neq 0$

3. The system of equations

$$2x + y = 5$$
$$x - 3y = -1$$
$$3x + 4y = k$$

is consistent, when k is

(A) 1 (B) 2 (C) 5 (D) 10

Sol. (D) $[A|B] = \left[\begin{array}{cc|c} 2 & 1 & 5 \\ 1 & -3 & -1 \\ 3 & 4 & k \end{array}\right]$

$$A = \begin{bmatrix} 2 & 1 \\ 1 & -3 \\ 3 & 4 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 \\ 1 & -3 \\ 3 & 4 \end{bmatrix}$$

$$\text{rank}(A) = 2$$

$$\text{Det}[A|B] = -7k + 70$$

$\therefore$ The system is consistent if $\text{Det}[A|B] = 0$

$$\Rightarrow 7k = 70$$
$$\Rightarrow k = 10.$$

4. S.T. the S.O.E.s

$$\left.\begin{array}{l} x_1 - 2x_2 + x_3 - x_4 + 1 = 0 \\ 3x_1 - 2x_3 + 3x_4 + 4 = 0 \\ 5x_1 - 4x_2 + x_4 + 3 = 0 \end{array}\right\} \text{ is inconsistent.}$$

Sol. Here, we have

$$A = \begin{bmatrix} 1 & -2 & 1 & -1 \\ 3 & 0 & -2 & 3 \\ 5 & -4 & 0 & 1 \end{bmatrix}, \quad [A \vdots b] = \begin{bmatrix} 1 & -2 & 1 & -1 & -1 \\ 3 & 0 & -2 & 3 & -4 \\ 5 & -4 & 0 & 1 & -3 \end{bmatrix}$$

By elementary transformations, A and $[A \vdots b]$ can be reduced in the following forms:

$$A \sim \begin{bmatrix} 1 & 0 & \frac{1}{6} & 1 \\ 0 & 1 & -\frac{5}{6} & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad [A \vdots b] \sim \begin{bmatrix} 1 & 0 & 1/6 & 1 & -4/3 \\ 0 & 1 & -5/6 & 1 & -1/6 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$$

$\therefore$ rank$(A) = 2$ but rank$(A \vdots b) = 3$

So, the equations are inconsistent.

5. Find the rank of the matrix $A = \begin{bmatrix} 1 & a & b & 0 \\ 0 & c & d & 1 \\ 1 & a & b & 0 \\ 0 & c & d & 1 \end{bmatrix}$.

Sol.

$$A \sim \begin{bmatrix} 1 & a & b & 0 \\ 0 & c & d & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ rank$(A) = 2$.

6. Find the inverse of the matrix $\begin{bmatrix} 1 & 2 & 1 \\ 3 & 2 & 3 \\ 1 & 1 & 2 \end{bmatrix}$

Sol. $A = I_3 A$

Applying E-row transformation to the matrix applying $R_2 \to R_2 - 3R_1$, $R_3 \to R_3 - R_1$.

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 0 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} A$$

$R_2 \to -\frac{1}{4} R_2$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{3}{4} & \frac{1}{4} & 0 \\ -1 & 0 & 1 \end{bmatrix} A$$

$R_1 \to R_1 - 2R_2$, $R_3 \to R_3 + R_2$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1/2 & \frac{1}{2} & 0 \\ 3/4 & -1/4 & 0 \\ -1/4 & -1/4 & 1 \end{bmatrix} A$$

$R_1 \to R_1 - R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1/4 & 3/4 & -1 \\ 3/4 & -1/4 & 0 \\ -1/4 & -1/4 & 1 \end{bmatrix} A$$

$\therefore I_3 = BA$

$\therefore A^{-1} = B$.

$$\therefore A^{-1} = B = \begin{bmatrix} -1/4 & 3/4 & -1 \\ 3/4 & -1/4 & 0 \\ -1/4 & -1/4 & 1 \end{bmatrix}$$

7. Solve : $x_1 - x_2 + x_3 = 2$

$3x_1 - x_2 + 2x_3 = -6$

$3x_1 + x_2 + x_3 = -18$

Sol. Here, we have $A = \begin{bmatrix} 1 & -1 & 1 \\ 3 & -1 & 2 \\ 3 & 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 & 1 & 2 \\ 3 & -1 & 2 & -6 \\ 3 & 1 & 1 & -18 \end{bmatrix}$

Now, $|A| = 1(-1-2) + 1(3-6) + 1(3+3) = 0$

Also, $\begin{vmatrix} 1 & -1 \\ 3 & -1 \end{vmatrix} \neq 0$; hence rank $(A) = 2$.

We now proceed to reduce matrix B to its Echelon form

by $R_{21}(-3)$, $R_{31}(-3)$,

$$B \sim \begin{bmatrix} 1 & -1 & 1 & 2 \\ 0 & 2 & -1 & -12 \\ 0 & 4 & -2 & -24 \end{bmatrix}$$

By $R_{32}(-2)$,

$$B \sim \begin{bmatrix} 1 & -1 & 1 & 2 \\ 0 & 2 & -1 & -12 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

By $R_1(1/2)$

$$B \sim \begin{bmatrix} 1 & -1 & 1 & 2 \\ 0 & 1 & -1/2 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ rank$(B) = 2 =$ rank(A).

Hence the given equations are consistent.

Also the mtx equation becomes

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -6 \\ 0 \end{bmatrix}$$

$$\therefore \left.\begin{aligned} x_1 - x_2 + x_3 &= 2 \\ x_2 - \tfrac{1}{2}x_3 &= -6 \end{aligned}\right\} \quad \therefore x_1 = -\tfrac{1}{2}x_3 - 4, \quad x_2 = \tfrac{1}{2}x_3 - 6.$$

and x_1, x_2 canbe expressed in terms of x_3 which is arbitrary. Since the rank is 2, two unknowns x_1, x_2 are expressed in terms of x_3, the system has ~~finite~~ infinite number of solutions.

8. Find two non-singular matrices P and Q such that PAQ is in the normal form where

$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 3 & 1 & 1 \end{bmatrix}$. Also find the rank of the matrix A.

Solution:—

$$A = I_3 A I_3$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

E-operations on the matrix A (left hand equation) until it is reduced to the normal form.
Every E-row operation will also be applied to the pre factor I_3 of the product on the right hand member of the above equation and every column operation to the post-factor I_3

Applying $R_2 \to R_2 - R_1$, $R_3 \to R_3 - 3R_1$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & -2 \\ 0 & -2 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Performing $C_2 \to C_2 - C_1$, $C_3 \to C_3 - C_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & -2 \\ 0 & -2 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Performing $R_2 \to -\frac{1}{2} R_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -2 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & -1/2 & 0 \\ -3 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Performing $R_3 \to R_3 + 2R_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & -1/2 & 0 \\ -2 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Performing $C_3 \to C_3 - C_2$

$$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 0 \end{array}\right] = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & -1/2 & 0 \\ -2 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\left[\begin{array}{c|c} I_2 & 0 \\ \hline 0 & 0 \end{array}\right] = PAQ, \text{ where,}$$

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & -1/2 & 0 \\ -2 & -1 & 1 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$\therefore$ Rank(A) = 2.

EIGEN VALUES & VECTORS

Eigen Values :—

Let $A = [a_{ij}]_{n\times n}$ be any square matrix of order n and λ be an indeterminate.

$[A - \lambda I]$ is called characteristic matrix and $|A - \lambda I| = 0$ is called characteristic equation and roots of this equation is called the characteristic roots or characteristic values or eigen values or latent roots or proper values of the mtx A.

Note:– The set of the eigen values of A is called the spectrum of A.

Eigen Vectors :— If λ is a characteristic root of an $n\times n$ matrix A, then a non-zero vector $\underset{\sim}{x}$ such that

$$A\underset{\sim}{x} = \lambda \underset{\sim}{x}$$

is called a characteristic vector or eigen vector of A corresponding to the characteristic root λ.

Remark:— 1. λ is a characteristic root of a matrix A if and only if there exists a non-zero vector $\underset{\sim}{x}$ such that $A\underset{\sim}{x} = \lambda\underset{\sim}{x}$.

2. If $\underset{\sim}{x}$ is a characteristic vector of a matrix A corresponding to the characteristic value λ. Here k is any non-zero scalar, then $k\underset{\sim}{x}$ is also a characteristic vector of A corresponding to the same characteristic value λ.

3. The characteristic vectors corresponding to distinct characteristic roots of a matrix are linearly independent.

Nature of an eigen value of the special types of matrices :—

1. The eigen value of a Hermitian matrix are all real.
2. The eigen value of a real symmetric mtx are all real.
3. The eigen value of a skew-Hermitian mtx are either pure imaginary or zero.
4. The eigen value of a skew symmetric mtx are either pure imaginary or zero.
5. The eigen values of a unitary matrices & an orthogonal matrix are of unit modulus.

Ex. S.T. the eigen value of a triangular matrix are just the diagonal elements of the matrix.

Sol. Let, $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$ be a triangular mtx of order 3.

$$|A - \lambda I| = \begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ 0 & a_{22}-\lambda & a_{23} \\ 0 & 0 & a_{33}-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (a_{11}-\lambda)(a_{22}-\lambda)(a_{33}-\lambda) = 0$$

$$\Rightarrow \lambda = a_{11}, a_{22}, a_{33}.$$

Ex.2. Determine the eigen values and eigen vectors of the matrix

$$A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}.$$

Sol.

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (5-\lambda)(2-\lambda) - 4 = 0$$

$$\Rightarrow (\lambda - 6)(\lambda - 1) = 0$$

$$\therefore \lambda_1 = 6, \lambda_2 = 1$$

Eigen vectors $\underset{\sim}{X_1} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ of A corresponding to the eigen value 6.

$$(A - 6I)\underset{\sim}{x} = \underset{\sim}{0}$$

$$\Rightarrow \begin{bmatrix} 5-6 & 4 \\ 1 & 2-6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -1 & 4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ by } R_2 \leftrightarrow R_2 + R_1$$

rank of the coefficient matrix = 1, there are $(2-1) = 1$ L.I. solution.

$$4x_2 = x_1$$

$$\therefore x_2 = 4, \ x_1 = 1$$

$\therefore \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ is an eigen vector of A corresponding to the eigen value 6.

The eigen vectors $\underset{\sim}{X_2} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ corresponding to the eigen value 1.

$$(A - 1I)\underset{\sim}{x} = \underset{\sim}{0}$$

$$\Rightarrow \begin{bmatrix} 4 & 4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{cases} 4x_1 + 4x_2 = 0 \\ x_1 + x_2 = 0 \end{cases}$$

$$\therefore x_1 = -x_2$$

if $x_2 = 1, x_1 = -1$.

$\therefore \underset{\sim}{X_2} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is the eigen vector of A corresponding to the eigen value 1.

So, $\underset{\sim}{X_1} = k \begin{bmatrix} 4 \\ 1 \end{bmatrix}$, $\underset{\sim}{X_2} = h \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ are the eigenvectors corresponding to the eigen values 6, 1, respectively, where k, h are any non-zero scalars.

The cayley-Hamilton theorem:-

Every square matrix satisfies its characteristic equation.
i.e. if for a square matrix A of order n,

$$|A-\lambda I| = (-1)^n\left[\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots\cdots + a_n\right]$$

then the matrix equation

$$x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_nI = 0$$

is satisfied by $x = A$,

i.e. $A^n + a_1A^{n-1} + a_2A^{n-2} + \cdots + a_nI = 0$.

Corollary 1:- If A be a non-singular matrix, $|A| \neq 0$.

Premultiplying by A^{-1}

$$A^{n-1} + a_1A^{n-2} + a_2A^{n-3} + \cdots + a_{n-1}I + a_nA^{-1} = 0$$

or, $A^{-1} = -\left(\frac{1}{a_n}\right)\left(A^{n-1} + a_1A^{n-2} + \cdots + a_{n-1}I\right)$

Corollary 2:- If m be a positive integer such that $m \geq n$, then multiplying the results by A^{m-n}.

$$A^m + a_1A^{m-1} + \cdots + a_nA^{m-n} = 0.$$

Eigen values and Eigen vectors:-

If V is a vector space over the field F and T is a linear operator on V. An eigen value of T is a scalar c in F such that there is a non-zero vector $\alpha \in V$ with $T\alpha = c\alpha$.

If c is an eigen value of T, then

(a) Any α such that $T\alpha = c\alpha$ is called eigen vector of T associated with the eigen value c;

(b) The collection of all c such that $T\alpha = c\alpha$ is called the eigen space associated with c.

Eigen value of matrix A over F:- If A is an $n\times n$ matrix over the field F, an eigen value of A over F is a scalar c in F such that the matrix $(A - cI)$ is singular (not invertible).

Diagonalisable:- If T is a linear operator on the finite dimensional space V. Then T is diagonalizable if there is a basis for V each vector of which is an eigen vector of T.

Eigen polynomial:-

$$f(c) = |A - cI|.$$

Some Important Theorem:—

1. If T is a linear operation on a finite dimensional space V and c is any scalar. Then followings are equivalent:
   (a) c is an eigen value of T
   (b) The operator $(T-cI)$ is singular (not invertible)
   (c) $\det(T-cI)=0$.
2. Similar matrices have the same eigen polynomial.
3. If $T\alpha = c\alpha$ and F is any polynomial, then $F(T)\alpha) = F(c)\alpha$
4. Suppose T is a linear operator on the finite dimensional space V; $c_1, \ldots, c_k$ are k-distinct eigen values of T and W is the space of the eigen vectors associated with the eigen value c_i. If $W = W_1 + W_2 + \cdots + W_k$, then
$$\dim(W) = \dim(W_1) + \dim(W_2) + \cdots + \dim(W_k).$$
In fact, if B_i is an ordered basis for W_i, then $B=(B_1, \ldots, B_k)$ is an ordered basis for W.
5. If T is a linear operator on a finite dimensional values of T and W_i is a null space of $(T-c_iI)$. The followings are equivalent:
(i) T is diagonalizable
(ii) The eigen polynomial for T is $F=(x-c_1)^{d_1}(x-c_2)^{d_2}\cdots(x-c_k)^{d_k}$; with $\dim(W_i)=d_i$, $i=1(1)k$.
(iii) $\dim V = \dim(W_1) + \dim(W_2) + \cdots + \dim(W_k)$.

Theorem:— If $\underset{\sim}{\alpha}$ is a characteristic vector of T corresponding to the eigen value λ, then $k\underset{\sim}{\alpha}$ is also a ch. vector of T corresponding to the same ch. value λ. Here k is any non-zero scalar.

Proof:- Since $\underset{\sim}{\alpha}$ is a characteristic vector of T corresponding to the ch. value λ, therefore, $\underset{\sim}{\alpha} \neq \underset{\sim}{0}$ and
$$T(\underset{\sim}{\alpha}) = \lambda\underset{\sim}{\alpha} \quad ; \text{ let, } k \text{ be any non-zero scalar.}$$
$$T(k\underset{\sim}{\alpha}) = kT(\underset{\sim}{\alpha}) = k(\lambda\underset{\sim}{\alpha})$$
$$= \lambda(k\underset{\sim}{\alpha}).$$
$\therefore$ $k\underset{\sim}{\alpha}$ is a characteristic vector of T corresponding to the characteristic value λ.

Thus, corresponding to a ch. value λ, there may correspond more than one characteristic vectors.

Theorem:— If $\underset{\sim}{\alpha}$ is a ch. vector of T, then $\underset{\sim}{\alpha}$ can't correspond to more than one ch. values of T.

Proof:- $T\underset{\sim}{\alpha} = c_1\underset{\sim}{\alpha}$ and $T\underset{\sim}{\alpha} = c_2\underset{\sim}{\alpha}$; where c_1, c_2 are two distinct ch. values of T.
$$c_1\underset{\sim}{\alpha} = c_2\underset{\sim}{\alpha}$$
$$\Rightarrow (c_1-c_2)\underset{\sim}{\alpha} = \underset{\sim}{0} \quad [\because \underset{\sim}{\alpha} \neq \underset{\sim}{0}]$$
$$\Rightarrow c_1 - c_2 = 0$$
$$\Rightarrow c_1 = c_2.$$

So, a ch. vector of a matrix, can't correspond to more than one characteristic value of that mtx.

Theorem:— Let T be a linear operator on an n-dimensional vector space V and A be the matrix of T relative to any ordered basis B. Then a vector α in V is an eigenvector of T corresponding to its eigenvalue c if and only if its coordinate vector $\underset{\sim}{X}$ relative to the basis B is an eigen-vector of A corresponding to its eigenvalue c.

Proof:— We have $[T-cI]_B = [T]_B - c[I]_B$

$$= A - cI.$$

If $\alpha \neq 0$, then the coordinate vector $\underset{\sim}{X}$ of $\underset{\sim}{\alpha}$ is also non-zero.

Now, $[(T-cI)(\alpha)]_B = [T-cI]_B[\alpha]_B$

$$= (A-cI)\underset{\sim}{X}$$

$$\therefore (T-cI)(\alpha) = \underset{\sim}{0} \text{ iff } (A-cI)\underset{\sim}{X} = 0$$

or, $T(\alpha) = c\underset{\sim}{\alpha}$ iff $A\underset{\sim}{X} = c\underset{\sim}{X}$.

or, α is an eigenvector of T iff $\underset{\sim}{X}$ is an eigen vector of A.

Ex. 1. Let V be an n-dimensional vector space over F. What is the ch. polynomial of (i) the identity operator on V, (ii) the zero operator on V.

Sol. Let B be any ordered basis for V.

(i) If I is the identity operator on V, then $[I]_B = I$. The characteristic polynomial of I is $\det(I - xI)$

$$= \begin{vmatrix} 1-x & 0 & \cdots & 0 \\ 0 & 1-x & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1-x \end{vmatrix} = (1-x)^n.$$

(ii) If $\hat{0}$ is the zero operator on V, then $[\hat{0}]_B = \underset{\sim}{0}$, i.e. the null matrix of order n, then the ch. polynomial of $\hat{0}$ is

$$\det(\underset{\sim}{0} - xI) = \begin{vmatrix} -x & 0 & \cdots & 0 \\ 0 & -x & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & -x \end{vmatrix} = (-1)^n x^n.$$

Ex. 2. If $c \in F$ is a ch. value of a linear operator T on a vector space $V(F)$, then for any polynomial $p(x)$ over F, $p(c)$ is a ch. value of $p(T)$.

Sol. Since c is a ch. value of T, therefore $\exists$ a non-zero vector $\underset{\sim}{\alpha}$ in V $\ni$

$$T\underset{\sim}{\alpha} = c\underset{\sim}{\alpha}$$

$$\Rightarrow T(T\underset{\sim}{\alpha}) = T(c\underset{\sim}{\alpha})$$

$$\Rightarrow T^2\underset{\sim}{\alpha} = cT(\underset{\sim}{\alpha}) = c^2\underset{\sim}{\alpha}$$

$\therefore$ c^2 is a ch. value of T^2.

Repeating this process k times, we get

$$T^k\underset{\sim}{\alpha} = c^k\underset{\sim}{\alpha}.$$

$\therefore$ c^k is a ch. value of T^k, where k be any positive integer.

Let $p(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_m x^m$, where $a_i \in F$.
Then $p(T) = a_0 I + a_1 T + a_2 T^2 + \dots + a_m T^m$.

We have $[P(T)](\alpha) = (a_0 I + a_1 T + \dots + a_m T^m)(\alpha)$

$$= a_0 I \underset{\sim}{\alpha} + a_1 T(\underset{\sim}{\alpha}) + \dots + a_m T^m(\underset{\sim}{\alpha})$$

$$= a_0 \underset{\sim}{\alpha} + a_1 (c \underset{\sim}{\alpha}) + \dots + a_m (c^m \underset{\sim}{\alpha})$$

$$= (a_0 + c a_1 + \dots + c^m a_m) \underset{\sim}{\alpha}.$$

$\therefore P(c) = a_0 + c a_1 + \dots + a_m c^m$ is a ch. value of $P(T)$.

Ex. 3. Find all (complex) ch. values and ch. vectors of the following matrix $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

Sol.

$$|A - xI| = \begin{vmatrix} -x & 1 \\ 0 & -x \end{vmatrix} = 0$$

$$\Rightarrow x^2 = 0 \Rightarrow x = 0$$

$\therefore$ 0 is the only ch. value of A.

Let x_1, x_2 be the components of the ch. vector corresponding to this ch. value 0.

Let $\underset{\sim}{X} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

Now, $[A - 0.I] \underset{\sim}{X} = \underset{\sim}{0}$

$$\Rightarrow \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

i.e. $x_2 = 0$

let $x_1 = k$; where k is any non-zero complex number.

$\therefore \underset{\sim}{X} = \begin{bmatrix} k \\ 0 \end{bmatrix}$ is the ch. vector corresponding to the eigen value $\lambda = 0$, where k is any non-zero complex no.

WORKED EXAMPLES:—

1. The eigen values of the matrix $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ is

(A) 2, 3, 6 (B) 2, 6, 7 (C) −2, 3, 6 (D) None.

Sol. (C) $|A-\lambda I| = 0$

$$\Leftrightarrow \begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} 5-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} + 3\begin{vmatrix} 1 & 5-\lambda \\ 3 & 1 \end{vmatrix}$$

$$= (\lambda+2)(\lambda-3)(\lambda-6) = 0$$

$$\Rightarrow \lambda = -2, 3, 6.$$

2. Find the eigen values of $(A^4+3A-2I)$, where $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ are

(A) 2, 3, 20 (B) 2, 2, 2 (C) 2, 2, 20 (D) 20, 20, 2

Sol. (C)

$$A^2 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 3 \\ 0 & 4 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 1 & 3 & 3 \\ 0 & 4 & 3 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 & 3 \\ 0 & 4 & 3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 15 & 15 \\ 0 & 16 & 15 \\ 0 & 0 & 1 \end{bmatrix}$$

Now, $B = A^4+3A-2I$

$$= \begin{bmatrix} 1 & 15 & 15 \\ 0 & 16 & 15 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 3 & 3 \\ 0 & 6 & 3 \\ 0 & 0 & 3 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 18 & 18 \\ 0 & 20 & 18 \\ 0 & 0 & 2 \end{bmatrix}$$

Then the eigen values of $|B-\lambda I| = 0$

$$\Leftrightarrow \begin{vmatrix} 2-\lambda & 18 & 18 \\ 0 & 20-\lambda & 18 \\ 0 & 0 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda = 2, 20, 2.$$

3. Find the eigen values of A^4, where $A = \begin{pmatrix} 1 & 0 & -1 \\ 9 & 4 & 1 \\ 3 & 1 & 1 \end{pmatrix}$.

Sol.

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 9 & 4 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

Then the eigen values of the matrix can be determined from the ch. equation

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 0 & -1 \\ 9 & 4-\lambda & 1 \\ 0 & 1 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)(\lambda-2)(\lambda-3) = 0$$

$$\Rightarrow \lambda = 1, 2, 3.$$

$\therefore$ The eigen values of A are: 1, 2, 3.

So, the eigen values of A^4 are: $1^4, 2^4, 3^4$, i.e. 1, 16, 81.

4. Use Cayley Hamilton theorem, find the inverse of $A = \begin{bmatrix} -1 & 1 & 1 \\ 3 & 1 & -1 \\ 2 & 2 & 1 \end{bmatrix}$

Sol. $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} -1-\lambda & 1 & 1 \\ 3 & 1-\lambda & -1 \\ 2 & 2 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow -\lambda^3 + \lambda^2 - 4\lambda - 4 = 0$$

$$\Rightarrow A^3 - A^2 + 4A + 4 = 0$$

$$\Rightarrow A^{-1} = \frac{-A^2 + A + 4I}{4}$$

$$= \frac{1}{4}\left[- \begin{bmatrix} 6 & 2 & -1 \\ -2 & 2 & 1 \\ 6 & 6 & 1 \end{bmatrix} + \begin{bmatrix} -1 & 1 & 1 \\ 3 & 1 & -1 \\ 2 & 2 & 1 \end{bmatrix} + \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} \right]$$

$$= \frac{1}{4} \begin{bmatrix} -3 & -1 & 2 \\ 5 & 3 & -2 \\ 4 & -4 & 4 \end{bmatrix}$$

5. If λ_1 and λ_2 are the values of λ for which $\begin{vmatrix} 1 & \lambda & 0 \\ \lambda & 2 & 1 \\ 0 & 1 & 1 \end{vmatrix} = 0$, then $\lambda_1 + \lambda_2$ equals (A) -1 (B) 0 (C) 1 (D) 2.

Sol. (B) $\begin{vmatrix} 1 & \lambda & 0 \\ \lambda & 2 & 1 \\ 0 & 1 & 1 \end{vmatrix} = 0$

$$\Rightarrow 1(2-1) - \lambda(\lambda) = 0$$

$$\Rightarrow \lambda^2 = 1 \qquad \text{i.e. } \lambda_1 + \lambda_2 = 1 - 1 = 0.$$

$$\Rightarrow \lambda = \pm 1$$

6. Find the characteristic roots and corresponding characteristic vectors for each of the following matrix $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

Sol. The characteristic equation is

$$|A-\lambda I| = \begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = -\lambda^3 + 18\lambda^2 - 45\lambda = 0$$

$$\Rightarrow \lambda(\lambda-3)(\lambda-15) = 0$$

$\Rightarrow \lambda = 0, 3, 15$ are the 3 ch. roots of the mtx A.

If $\underset{\sim}{X}$ is a characteristic vector corresponding to the ch. root 0, then we have

$$A\underset{\sim}{X} = \lambda \underset{\sim}{X}$$

$$\Rightarrow [A - \lambda I]\underset{\sim}{X} = \underset{\sim}{0}$$

$$\Rightarrow \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow 8x_1 - 6x_2 + 2x_3 = 0, \quad -6x_1 + 7x_2 - 4x_3 = 0, \quad 2x_1 - 4x_2 + 3x_3 = 0$$

let $x_1 = 1$, then $x_2 = 2$, $x_3 = 2$.

$\therefore \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ is a ch. vector corresponding to the

It may similarly be shown by considering the equation

$(A - 3I)\underset{\sim}{X} = \underset{\sim}{0}$,

$(A - 15I)\underset{\sim}{X} = \underset{\sim}{0}$,

that the ch. vectors corresponding to the ch. roots 3 and 15 are arbitrary non-zero multiples of the vectors

$$\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}.$$

7. Find the eigen value of the mtx $\begin{bmatrix} 1 & 0 & 0 & -a/2 \\ 0 & 1 & 0 & -a/2 \\ 0 & 0 & 1 & -a/2 \\ 0 & 0 & 0 & a \end{bmatrix}$.

Sol.

$$\det[A - \lambda I] = 0$$

$$\Rightarrow (1-\lambda)(1-\lambda)(1-\lambda)(a-\lambda) = 0$$

$\Rightarrow \lambda = 1, 1, 1, a$ are the eigen values.

8. Let $M = \begin{pmatrix} 1 & 1+i & 2-i \\ 1-i & 2 & 3+i \\ 2+i & 3-i & 3 \end{pmatrix}$. If $B = \begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{pmatrix}$, where $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$ and $\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}$ are LIN eigen vectors of M, then the main diagonal of the matrix $B^{-1}MB$ has

(A) exactly one real entry (B) exactly two real entry
(C) exactly 3 real entry (C) no real entry.

Sol.

$M^T = M^*$

$\Rightarrow$ M is a Hermitian matrix

Since B is invertible mtx.

$\Rightarrow B^{-1}MB$ is a diagonal mtx whose diagonal entries are eigenvalues of M.

We know the eigen values of Hermitian mtx are real.

$\Rightarrow$ all ~~three~~ eigen values are real.

9. Let P be a 3X3 mtx $\ni$ for some c, the linear system $P = c$ has infinite number of solutions. Which one of the following is TRUE?

(A) The linear system $Px = b$ has infinite no. of solutions $\forall b$.
(B) Rank (P) = 3 (C) Rank (P) $\neq 1$ (D) Rank (P) ≤ 2.

Sol. (D) rank (P) < n $\Rightarrow$ $Px = b$ has infinite no. of solutions

$\therefore$ Rank (P) ≤ 2.

10. Let P be a 2X2 mtx $\ni P^{102} = 0$. Then

(A) $P^2 = 0$ (B) $(1-P)^2 = 0$ (C) $(1+P)^2 = 0$ (D) $P = 0$

Sol. (A) Since P is mtx of order 2 so its ch. equation is of order 2.

so P^{102} is equal to 0 iff $P^2 = 0$.

11. Let A be an nxn matrix $\ni P^{-1}AP > 0$ for every non-zero invertible mtx P where P is also an nxn mtx. Which of the following is TRUE?

(A) All eigen values of A are negative (B) All eigen values of A are positive.

Sol. (B) Since given that for mtx A, $P^{-1}AP > 0$.

Here $P^{-1}AP$ is a diagonal mtx whose diagonal elements are eigen values of matrix A. But $P^{-1}AP > 0$ shows the eigen values of A are all positive.

12) Let $P = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, The eigenvectors corresponding to the eigenvalues i and $-i$ are respectively

(A) $\begin{pmatrix} 1 \\ i \end{pmatrix}$ and $\begin{pmatrix} -1 \\ i \end{pmatrix}$ (B) $\begin{pmatrix} 1 \\ i \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -i \end{pmatrix}$ (C) $\begin{pmatrix} -1 \\ i \end{pmatrix} \begin{pmatrix} i \\ -i \end{pmatrix}$ (D) $\begin{pmatrix} i \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ i \end{pmatrix}$

Sol. $\lambda_1 = i,\ \lambda_2 = -i$

$$(P - \lambda_1 I)\underset{\sim}{X} = 0$$

$$\Rightarrow \begin{bmatrix} 0-i & 1 \\ -1 & -i \end{bmatrix} \underset{\sim}{X} = \underset{\sim}{0}$$

$$\Rightarrow \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{cases} -x_1 i + x_2 = 0 \\ -x_1 - ix_2 = 0 \end{cases}$$

$x_1 = 1,\ x_2 = i$

$\therefore \begin{bmatrix} 1 \\ i \end{bmatrix}$ is the eigen vector corresponding to the eigen value i.

$$(P - \lambda_2 I)\underset{\sim}{X}' = \underset{\sim}{0}$$

$$\Rightarrow \begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{cases} ix_1 + x_2 = 0 \\ -x_1 + ix_2 = 0 \end{cases}$$

$\therefore x_1 = -1,\ x_2 = i$

$\therefore \begin{bmatrix} -1 \\ i \end{bmatrix}$ is the eigenvector corresponding to the eigen value $-i$.

13) Let P, M, B be $n \times m$ matrices $\ni$ M and N are non-singular. If $\underset{\sim}{X}$ is an eigen vector of P corresponding to the eigen value λ, then an eigen vector of $N^{-1}MPM^{-1}N$ corresponding to the eigenvalue λ is

(A) $MN^{-1}\underset{\sim}{X}$ (B) $M^{-1}N\underset{\sim}{X}$ (C) $NM^{-1}\underset{\sim}{X}$ (D) $N^{-1}M\underset{\sim}{X}$

Sol. (C) Since λ is eigenvalue of P and $\underset{\sim}{X}$ be the eigen vector corresponding to it.

$$\Rightarrow P\underset{\sim}{x} = \lambda \underset{\sim}{x}$$

$$\begin{aligned} N^{-1}MPM^{-1}N(N^{-1}M\underset{\sim}{X}) &= N^{-1}MPM^{-1}(NN^{-1})M\underset{\sim}{X} \\ &= N^{-1}MPM^{-1}M\underset{\sim}{X} \\ &= N^{-1}MP(M^{-1}M)\underset{\sim}{X} \\ &= N^{-1}MP\underset{\sim}{X} \\ &= N^{-1}M\lambda\underset{\sim}{x} \\ &= \lambda N^{-1}M\underset{\sim}{x} \end{aligned}$$

$\therefore N^{-1}M\underset{\sim}{x}$ is eigenvector corresponding to λ.

~~14) Let $P=\begin{bmatrix} 1 & 1 \\ i & -1 \end{bmatrix}$. Then~~

~~(A) P has two linearly independent eigenvectors (B) P has an eigen vectors~~
~~(C) P is non-singular (D) $\exists$ a non-singular S $\ni$ $S^{-1}PS$ is a diagonal mtx~~

~~sol.~~

~~(A)~~

---

14) Let P be an $n\times n$ idempotent mtx, that is $P^2=P$, which of the following is FALSE?

(A) P^T is idempotent
(B) The possible eigenvalues of P are 0 and 1.
(C) The non-diagonal entries of P can be zero.
(D) There are infinite no. of $n\times n$ non-singular matrices that are idempotent

Solution:—

(D) Since P is idempotent matrix, i.e. $P^2=P$, then P^T is also idempotent, as $(P^2)^T=P^T \Rightarrow (P^T)^2=P^T$.

$P^2=P \Rightarrow P(P-I)=0 \Rightarrow$ eigen values of P are 0 and 1.

The non-diagonal entries of an idempotent mtx can be zero,

e.g. $P=\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, $P^2=\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}=P$.

So, (D) is FALSE.

15) Let A and B are any arbitrary square matrices of order 2. Then show that AB and BA have same eigen values but may have different eigen vectors.

Solution:— Let us ~~take~~ an example.

$$A=\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix},\ B=\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$AB=\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$$

$$|AB-\lambda I|=0 \Rightarrow (1-\lambda)(2-\lambda)=2$$

$$\Rightarrow \lambda^2-3\lambda=0$$

$$\Rightarrow \lambda(\lambda-3)=0$$

$$\Rightarrow \lambda=0, \lambda=3.$$

For $\lambda=0$, $[AB-0.I]\underset{\sim}{X}=\underset{\sim}{0}$

$$\Rightarrow \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\therefore x_1+x_2=0$

$\therefore x_1=1, x_2=-1$

$\therefore \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ be an eigen vector corresponding to $\lambda=0$.

For $\lambda=3$,

$[AB-3.I]\underset{\sim}{X}=\underset{\sim}{0}$

$$\Rightarrow \begin{bmatrix} 1-3 & 1 \\ 2 & 2-3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow -2x_1+x_2=0$, $2x_1-x_2=0$

$\therefore x_2=2, x_1=1$

$\therefore \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is the eigen vector.

Here $BA = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$

$\therefore |BA - \lambda I| = 0$

$\Rightarrow \begin{vmatrix} 1-\lambda & 2 \\ 1 & 2-\lambda \end{vmatrix} = 0$

$\Rightarrow \lambda = 0, 3.$

for $\lambda = 0$, $[BA - 0.I]\underset{\sim}{X} = \underset{\sim}{0}$.

$\Rightarrow \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow x_1 + 2x_2 = 0$

$\Rightarrow x_1 = -2x_2$

$\therefore x_2 = -1/2$, $x_1 = 1$

$\therefore \begin{bmatrix} 1 \\ -1/2 \end{bmatrix}$ be the eigenvector corresponding to $\lambda = 0$.

for $\lambda = 3$, $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ be the eigenvector corresponding to $\lambda = 3$.

$\therefore$ AB & BA have same eigen values but not same eigen vectors.

16) A real 3×3 mtx M has eigen values ±1 and 2. S.T. (i) M is invertible (ii) $M^3 - 2M^2$ is singular (iii) M is diagonalisable.

Sol. We have a mtx M whose 3 eigenvalues are $+1, -1$ and 2. it's ch. equation can be given as

$(M+I)(M-I)(M-2I) = 0$

$\Rightarrow (M^2 - I)(M-2I) = 0$

$\Rightarrow M^3 - 2M^2 = M - 2I$

(i) By the properties of eigenvalues: determinant of mtx = multiplication of eigenvalues

$|M| = (-1)\times(1)\times(2)$

$|M| = -2$

here $|M| \neq 0 \Rightarrow$ M is invertible.

(ii) $|M^3 - 2M^2| = |M - 2I|$

since $|M| = -2$

$|M - 2I| = 0$

$\Rightarrow |M^3 - 2M^2| = 0$

$\Rightarrow M^3 - 2M^2$ is singular.

(iii) Since M having 3 distinct eigenvalues, then the ch. vectors corresponding to distinct characteristics roots of a mtx are LIN.

And an $n\times n$ mtx is diagonalisable if and only if it possesses n linearly independent eigenvectors.

$\Rightarrow$ M is diagonalisable.