UNIT-IV
PRINCIPAL STRESSES & STRAINS

A structural member may be subjected to different types of stresses (normal and shearing stresses) simultaneously. It is therefore necessary to find the region where the effect of these stresses will be critical from the design point of view. When such stresses act at a point in a stressed material, there always exist three orthogonal planes carrying entirely normal stresses.

Principal Planes: At any point in a strained material, there are three planes mutually perpendicular to each other which carry direct stresses only. And no shear stress. Out of these three direct stresses, one will be maximum, the other is minimum and the third an intermediate between the two. These particle planes, which have no shear stress, are known as principal planes.

Principal stress: The magnitude of direct stress across a principal plane is known as principal stress. The determination of principal planes and the principal stress is an important factor in the design of various structures and machine components.

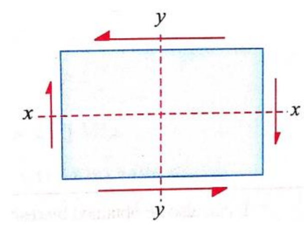
Methods for the stresses on an oblique section of a body: The following two methods for the determination of stresses on an oblique section of a strained body.

1. Analytical Method and 2. Graphical method.

Analytical method for the stresses on an oblique section of a body:
The analytical method for the determination of stresses on an oblique section is as follows.

1. A body subjected to a direct stress in one plane
2. A body subjected to a direct stress in two mutually perpendicular directions.

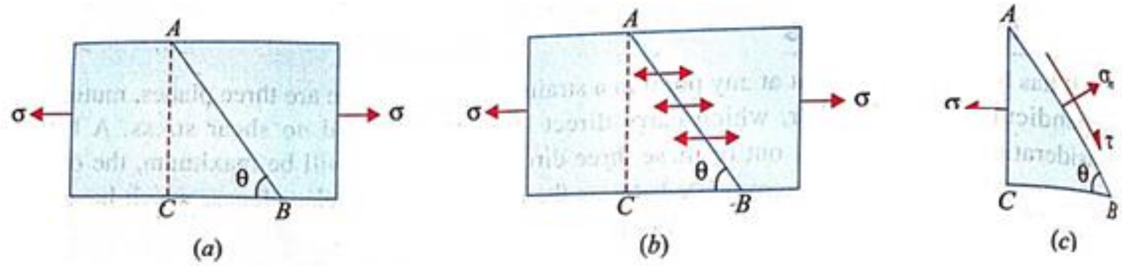
Sign Conventions for Analytical Method:



1. All the tensile stresses and strains are taken as positive, where as all the compressive stresses and strains are taken as negative.
2. The well-established principles of mechanics are used for the shear stress. The shear stress which tends to rotate the element in the clockwise direction is taken as positive, whereas that which tends to rotate in an anticlockwise direction as negative.

   In the element shown in fig., the shear stress on the vertical faces (or x-x axis) is taken as positive, whereas the shear stress on the horizontal faces (or y-y axis) is taken as negative.

Stresses on an oblique section of a body subjected to a direct stress in one plane:
Consider a rectangular body of uniform cross-sectional area and unit thickness subjected to a direct tensile stress along x-x axis as shown in fig. Now let us consider an oblique section AB inclined with the x-x axis (i.e., with the line of action of the tensile stress on which we required to find out the stresses as shown in fig.



Let σ = Tensile stress across the face AC and

θ = Angle, which the oblique section AB makes with BC *i.e.* with the x-x axis in the clockwise direction.

First of all, consider the equilibrium of an element or wedge ABC whose free body diagram is shown in fig 7.2 (*b*) and (*c*). We know that the horizontal force acting on the face AC,

$$P = \sigma . AC\ (\leftarrow)$$

Resolving the force perpendicular or normal to the section AB

$$P_n = P\sin\theta = \sigma . AC\sin\theta \quad \text{....}(i)$$

and now resolving the force tangential to the section AB,

$$P_t = P\cos\theta = \sigma . AC\cos\theta \quad \text{....}(ii)$$

We know that normal stress across the section AB*,

$$\sigma_n = \frac{P_n}{AB} = \frac{\sigma AC\sin\theta}{AB} = \frac{\sigma . AC\sin\theta}{\frac{AC}{\sin\theta}} = \sigma\sin^2\theta$$

$$= \frac{\sigma}{2}(1-\cos 2\theta) = \frac{\sigma}{2} - \frac{\sigma}{2}\cos 2\theta \quad \text{...}(iii)$$

and shear stress (*i.e.*, tangential stress) across the section AB,

$$\tau = \frac{P_t}{AB} = \frac{\sigma . AC\cos\theta}{AB} = \frac{\sigma . AC\cos\theta}{\frac{AC}{\sin\theta}} = \sigma\sin\theta\cos\theta$$

$$= \frac{\sigma}{2}\sin 2\theta \quad \text{...}(iv)$$

It will be interesting to know from equation (*iii*) above that the normal stress across the section AB will be maximum, when $\sin^2\theta = 1$ or $\sin\theta = 1$ or $\theta = 90°$. Or in other words, the face AC will carry the maximum direct stress. Similarly, the shear stress across the section AB will be maximum when $\sin 2\theta = 1$ or $2\theta = 90°$ or $270°$. Or in other words, the shear stress will be maximum on the planes inclined at 45° and 135° with the line of action of the tensile stress. Therefore maximum shear stress when θ is equal to 45°,

$$\tau_{max} = \frac{\sigma}{2}\sin 90° = \frac{\sigma}{2}\times 1 = \frac{\sigma}{2}$$

and maximum shear stress, when θ is equal to 135°,

$$\tau_{max} = -\frac{\sigma}{2}\sin 270° = -\frac{\sigma}{2}(-1) = \frac{\sigma}{2}$$

It is thus obvious that the magnitudes of maximum shear stress is half of the tensile stress. Now the resultant stress may be found out from the relation :

$$\sigma_R = \sqrt{\sigma_n^2 + \tau^2}$$

Note : The planes of maximum and minimum normal stresses (*i.e.* principal planes) may also be found out by equating the shear stress to zero. This happens as the normal stress is either maximum or minimum on a plane having zero shear stress. Now equating the shear stress to zero,

$$\sigma \sin\theta \cos\theta = 0$$

It will be interesting to know that in the above equation either $\sin\theta$ is equal to zero or $\cos\theta$ is equal to zero. We know that if sin is zero, then θ is equal to 0°. Or in other words, the plane coincides with the line of action of the tensile stress. Similarly, if $\cos\theta$ is zero, then θ is equal to 90°. Or in other words, the plane is at right angles to the line of action of the tensile stress. Thus we see that there are two principal planes, at right angles to each other, one of them coincides with the line of action of the stress and the other at right angles to it.

Problem-1: *A wooden bar is subjected to a tensile stress of 5 MPa. What will be the values of normal and shear stresses across a section, which makes an angle of 25° with the direction of the tensile stress.*

Solution. Given : Tensile stress (σ) = 5 MPa and angle made by section with the direction of the tensile stress (θ) = 25°.

Normal stress across the section

We know that normal stress across the section

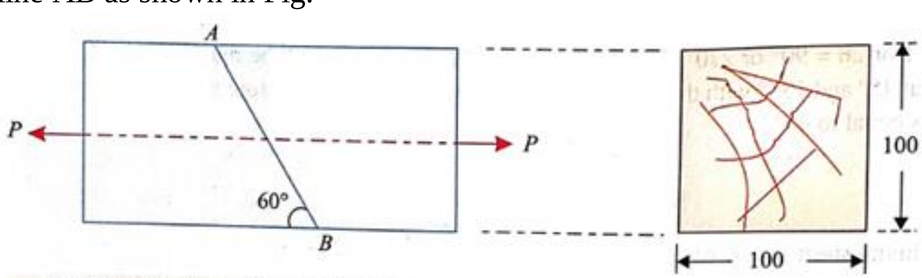
$$\sigma_n = \frac{\sigma}{2} - \frac{\sigma}{2}\cos 2\theta = \frac{5}{2} - \frac{5}{2}\cos(2\times 25°) \text{ MPa}$$
$$= 2.5 - 2.5\cos 50° = 2.5 - (2.5\times 0.6428) \text{ MPa}$$
$$= 2.5 - 1.607 = 0.89 \text{ MPa} \quad \textbf{Ans.}$$

Shear stress across the section

We also know that shear stress across the section,

$$\tau = \frac{\sigma}{2}\sin 2\theta = \frac{\sigma}{2}\sin(2\times 25°) = 2.5\sin 50° \text{ MPa}$$
$$= 2.5\times 0.766 = 1.915 \text{ MPa} \quad \textbf{Ans.}$$

Problem-2: Two wooden pieces 100 mm x 100 mm in cross section are joined together along a line AB as shown in Fig.



Find the maximum force (P), which can be applied if the shear stress along the joint AB is 1.3 MPa.

SOLUTION. Given : Section = 100 mm × 100 mm ; Angle made by section with the direction of tensile stress (θ) = 60° and permissible shear stress (τ) = 1.3 MPa = 1.3 N/mm^2.

Let σ = Safe tensile stress in the member

We know that cross- sectional area of the wooden member,

$$A = 100 \times 100 = 10\,000 \text{ mm}^2$$

and shear stress (τ),

$$1.3 = \frac{\sigma}{2}\sin 2\theta = \frac{\sigma}{2}\sin(2\times 60°) = \frac{\sigma}{2}\sin 120° = \frac{\sigma}{2}\times 0.866$$
$$= 0.433\,\sigma$$

or

$$\sigma = \frac{1.3}{0.433} = 3.0 \text{ N/mm}^2$$

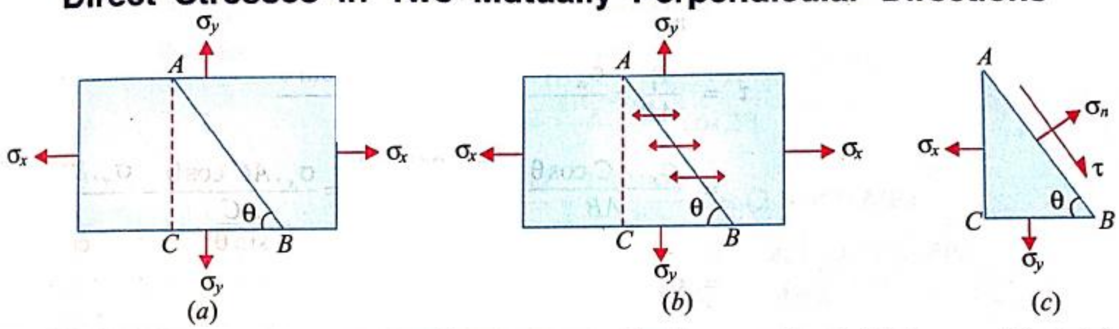
∴ Maximum axial force, which can be applied,

$$P = \sigma.A = 3.0 \times 10\,000 = 30\,000 \text{ N} = 30 \text{ kN} \quad \textbf{Ans.}$$

Problem-4: A bar is subjected to a tensile stress of 100 MPa,. Determine the normal and tangential stresses on a plane making an angle of 30^0 with the directions of tensile stress. [**Ans:** 75 MPa; 43.3 MPa]

Problem-5: A point in a strained material is subjected to a tensile stress of 50 MPa. Find the normal and shear stress at an angle of 50^0 with the direction of the stress. [**Ans:** 29.34 MPa; 24.62 MPa]

Stresses on an Oblique Section of a Body Subjected to Direct Stresses in Two Mutually Perpendicular Directions



Consider a rectangular body of uniform cross-sectional area and unit thickness subjected to direct tensile stresses in two mutually perpendicular directions along *x-x* and *y-y* axes as shown in Fig. 7.5. Now let us consider an oblique section *AB* inclined with *x-x* axis (*i.e.* with the line of action of the stress along *x-x* axis, termed as a major tensile stress on which we are required to find out the stresses as shown in the figure).

Let σ_x = Tensile stress along *x-x* axis (also termed as major tensile stress),

σ_y = Tensile stress along *y-y* axis (also termed as a minor tensile stress), and

θ = Angle which the oblique section *AB* makes with *x-x* axis in the clockwise direction.

First of all, consider the equilibrium of the wedge *ABC*. We know that horizontal force acting on the face *AC* (or *x-x* axis).

$$P_x = \sigma_x . AC\,(\leftarrow)$$

and vertical force acting on the face *BC* (or *y-y* axis),

$$P_y = \sigma_y . BC\,(\downarrow)$$

Resolving the forces perpendicular or normal to the section AB,

$$P_n = P_x \sin\theta + P_y \cos\theta = \sigma_x \,.\, AC \sin\theta + \sigma_y \,.\, BC \cos\theta \qquad ...(i)$$

and now resolving the forces tangential to the section AB,

$$P_t = P_x \cos\theta - P_y \sin\theta = \sigma_x \,.\, AC \cos\theta - \sigma_y \,.\, BC \sin\theta \qquad ....(ii)$$

We know that normal stress across the section AB,

$$\sigma_n = \frac{P_n}{AB} = \frac{\sigma_x . AC \sin\theta + \sigma_y \, BC \cos\theta}{AB}$$

$$= \frac{\sigma_x . AC \sin\theta}{AB} + \frac{\sigma_y . BC \cos\theta}{AB} = \frac{\sigma_x . AC \sin\theta}{\frac{AC}{\sin\theta}} + \frac{\sigma_y . BC \cos\theta}{\frac{BC}{\cos\theta}}$$

$$= \sigma_x \sin^2\theta + \sigma_y \,.\, \cos^2\theta = \frac{\sigma_x}{2}(1 - \cos 2\theta) + \frac{\sigma_y}{2}(1 + \cos 2\theta)$$

$$= \frac{\sigma_x}{2} - \frac{\sigma_x}{2}\cos 2\theta + \frac{\sigma_y}{2} + \frac{\sigma_y}{2}\cos 2\theta$$

$$= \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos 2\theta \qquad ...(iii)$$

and shear stress (*i.e.*, tangential stress) across the section AB,

$$\tau = \frac{P_t}{AB} = \frac{\sigma_x . AC \cos\theta - \sigma_y . BC \sin\theta}{AB}$$

$$= \frac{\sigma_x . AC \cos\theta}{AB} - \frac{\sigma_y . BC \sin\theta}{AB} = \frac{\sigma_x . AC \cos\theta}{\frac{AC}{\sin\theta}} - \frac{\sigma_y . BC \sin\theta}{\frac{BC}{\cos\theta}}$$

$$= \sigma_x \,.\, \sin\theta \cos\theta - \sigma_y \sin\theta \cos\theta$$

$$= (\sigma_x - \sigma_y) \sin\theta \cos\theta = \frac{\sigma_x - \sigma_y}{2} \sin 2\theta \qquad ...(iv)$$

It will be interesting to know from equation (*iii*) the shear stress across the section AB will be maximum when $\sin 2\theta = 1$ or $2\theta = 90°$ or $\theta = 45°$. Therefore maximum shear stress,

$$\tau_{max} = \frac{\sigma_x - \sigma_y}{2}$$

Now the resultant stress may be found out from the relation :

$$\sigma_R = \sqrt{\sigma_n^2 + \tau^2}$$

Problem-6: A point in a strained material is subjected to two mutually perpendicular tensile stresses of 200 MPa and 100 MPa. Determine the intensities of normal, shear and resultant stresses on a plane inclined at 300 with the axis of minor tensile stress.

Solution: Given. Tensile stress along x-x axis (σ_x) = 150 MPa. Tensile stress along y-y axis (σ_x) = 100 MPa and angle made by a plane with the axis of tensile stress $\theta = 30^0$.

Normal stress on the inclined plane

We know that normal stress on the inclined plane,

$$\sigma_n = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2} \cos 2\theta$$

$$= \frac{200 + 100}{2} - \frac{20 - 100}{2} \cos (2 \times 30°) \text{ MPa}$$

$$= 150 - (50 \times 0.5) = 125 \text{ MPa} \qquad \textbf{Ans.}$$

Shear stress on the inclined plane

We know that shear stress on the inclined plane,

$$\tau = \frac{\sigma_x - \sigma_y}{2} \sin 2\theta = \frac{200 - 100}{2} \times \sin (2 \times 30°) \text{ MPa}$$

$$= 50 \sin 60° = 50 \times 0.866 = 43.3 \text{ MPa} \qquad \textbf{Ans.}$$

Resultant stress on the inclined plane

We also know that resultant stress on the inclined plane,

$$\sigma_R = \sqrt{\sigma_n^2 + \tau^2} = \sqrt{(125)^2 + (43.3)^2} = 132.3 \text{ MPa} \qquad \textbf{Ans.}$$

Problem-7: *The stresses at point of a machine component are 150 MPa and 50 MPa both tensile. Find the intensities of normal, shear and resultant stresses on a plane inclined at an angle of 55° with the axis of major tensile stress. Also find the magnitude of the maximum shear stress in the component.*

SOLUTION. Given : Tensile stress along *x-x* axis (σ_x) = 150 MPa ; Tensile stress along *y-y* axis (σ_y) = 50 MPa and angle made by the plane with the major tensile stress (θ) = 55°.

Normal stress on the inclined plane

We know that the normal stress on the inclined plane,

$$\sigma_n = \frac{\sigma_x + \sigma_y}{2} \; \frac{\sigma_x - \sigma_y}{2} \cos 2\theta$$

$$= \frac{150 + 50}{2} - \frac{150 - 50}{2} \cos (2 \times 55°) \text{ MPa}$$

$$= 100 - 50 \cos 110° = 100 - 50 (-0.342) \text{ MPa}$$

$$= 10 + 17.1 = 117.1 \text{ MPa} \qquad \textbf{Ans.}$$

Shear stress on the inclined plane

We know that the shear stress on the inclined plane,

$$\tau = \frac{\sigma_x - \sigma_y}{2} \sin 2\theta = \frac{150 - 50}{2} \times \sin (2 \times 55°) \text{ MPa}$$

$$= 50 \sin 110° = 50 \times 0.9397 = 47 \text{ MPa} \qquad \textbf{Ans.}$$

Resultant stress on the inclined plane

We know that resultant stress on the inclined plane,

$$\sigma_R = \sqrt{\sigma_n^2 + \tau^2} = \sqrt{(117.1)^2 + (47.0)^2} = 126.2 \text{ MPa} \qquad \textbf{Ans.}$$

Magnitude of the maximum shear stress in the component

We also know that the magnitude of the maximum shear stress in the component,

$$\tau_{max} = \pm \frac{\sigma_x - \sigma_y}{2} = \pm \frac{150 - 50}{2} = \pm 50 \text{ MPa} \qquad \textbf{Ans.}$$

Problem-8: The stresses at a point in a component are 100 MPa (tensile) and 50 MPa (compressive). Determine the magnitude of the normal and shear stresses on a plane inclined at an angle of 25^0 with tensile stress. Also determine the direction of the resultant stress and the magnitude of the maximum intensity of shear stress.

Solution: Given: Tensile stress along x-x axis (σx) 100 MPa; Compressive stress along y-y axis (σy) = -50 MPa and angle made by the plane with tensile stress (θ) = 25^0.

Normal stress on the inclined plane

We know that the normal stress on the inclined plane,

$$\sigma_n = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos 2\theta$$

$$= \frac{100 + (-50)}{2} - \frac{100 - (-50)}{2}\cos(2 \times 25°) \text{ MPa}$$

$$= 25 - 75\cos 50° = 25 - (75 \times 0.6428) = -23.21 \text{ MPa} \qquad \textbf{Ans.}$$

Shear stress on the inclined plane

We know that the shear stress on the inclined plane,

$$\tau = \frac{\sigma_x - \sigma_y}{2}\sin 2\theta = \frac{100 - (-50)}{2}\sin(2 \times 25°) \text{ MPa}$$

$$= 75\sin 50° = 75 \times 0.766 = 57.45 \text{ MPa} \qquad \textbf{Ans.}$$

Direction of the resultant stress

Let θ = Angle, which the resultant stress makes with x-x axis.

We know that $\tan\theta = \dfrac{\tau}{\sigma_n} = \dfrac{57.45}{-23.21} = -2.4752$ or $\theta = -68°$ **Ans.**

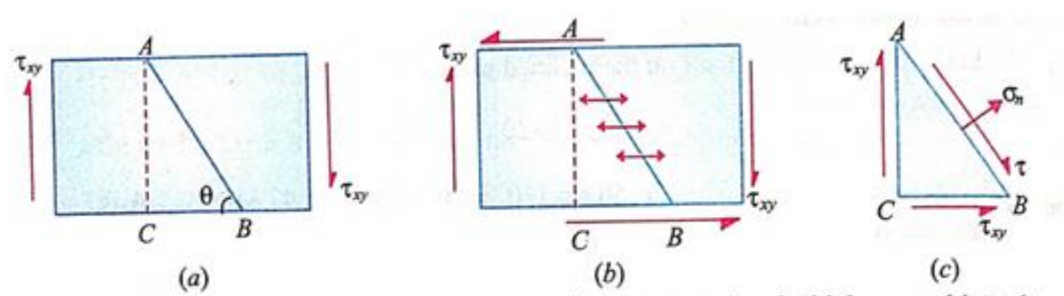
Magnitude of the maximum shear stress

We also know that magnitude of the maximum shear stress,

$$\tau_{max} = \pm\frac{\sigma_x - \sigma_y}{2} = \pm\frac{100 - (-50)}{2} = \pm 75 \text{ MPa} \qquad \textbf{Ans.}$$

Problem-9: At a point in a strained material, the principal stresses are 100 Mpa and 50 Mpa both tensile. Find the normal and shear stresses at a section inclined at 300 with the axis of the major principal stress. [**Ans:** 87.5 MPa, 21.65 MPa.]

Stresses on an oblique section of a body subjected to a simple shear stress:



Consider a rectangular body of uniform cross-sectional area and unit thickness subjected to a positive (*i.e.*, clockwise) shear stress along x-x axis as shown in Fig. (a). Now let us consider an oblique section AB inclined with x-x axis on which we are required to find out the stresses as shown in the figure (b).

Let τ_{xy} = Positive (*i.e.*, clockwise) shear stress along *x-x* axis, and

θ = Angle , which the oblique section AB makes with *x-x* axis in the anticlockwise direction.

First of all, consider the equilibrium of the wedge ABC. We know that as per the principle of simple shear, the face BC, of the wedge will be subjected to an anticlockwise shear stress equal to τ_{xy} as shown in the Fig. (*b*). We know that vertical force acting on the face AC,

$$P_1 = \tau_{xy} \,.\, AC\,(\uparrow)$$

and horizontal force acting on the face BC,

$$P_2 = \tau_{xy} \,.\, BC\,(\rightarrow)$$

Resolving the forces perpendicular or normal to the AB,

$$P_n = P_1 \cos\theta + P_2 \sin\theta = \tau_{xy} \,.\, AC\cos\theta + \tau_{xy} \,.\, BC\sin\theta$$

and now resolving the forces tangential to the section AB,

$$P_t = P_2 \sin\theta - P_1 \cos\theta = \tau_{xy} \,.\, BC\sin\theta - \tau_{xy} \,.\, AC\cos\theta$$

We know that normal stress across the section AB,

$$\sigma_n = \frac{P_n}{AB} = \frac{\tau_{xy}.AC\cos\theta + \tau_{xy}.BC\sin\theta}{AB}$$

$$= \frac{\tau_{xy}.AC\cos\theta}{AB} + \frac{\tau_{xy}.BC\sin\theta}{AB}$$

$$= \frac{\tau_{xy}.AC\cos\theta}{\dfrac{AC}{\sin\theta}} + \frac{\tau_{xy}.BC\sin\theta}{\dfrac{BC}{\cos\theta}}$$

$$= \tau_{xy} \,.\, \sin\theta\cos\theta + \tau_{xy} \,.\, \sin\theta\cos\theta$$

$$= 2\,\tau_{xy} \,.\, \sin\theta\cos\theta = \tau_{xy} \,.\, \sin 2\theta$$

and shear stress (*i.e.* tangential stress) across the section AB

$$\tau = \frac{P_t}{AB} = \frac{\tau_{xy} \,.\, BC\sin\theta - \tau_{xy} \,.\, AC\cos\theta}{AB}$$

$$= \frac{\tau_{xy}.BC\sin\theta}{AB} - \frac{\tau_{xy}.AC\cos\theta}{AB} = \frac{\tau_{xy}.BC\sin\theta}{\dfrac{BC}{\sin\theta}} - \frac{\tau_{xy}.AC\cos\theta}{\dfrac{AC}{\cos\theta}}$$

$$= \tau_{xy}\sin^2\theta - \tau_{xy}\cos^2\theta$$

$$= \frac{\tau_{xy}}{2}(1-\cos 2\theta) - \frac{\tau_{xy}}{2}(1+\cos 2\theta)$$

$$= \frac{\tau_{xy}}{2} - \frac{\tau_{xy}}{2}\cos 2\theta - \frac{\tau_{xy}}{2} - \frac{\tau_{xy}}{2}\cos 2\theta$$

$$= -\tau_{xy}\cos 2\theta$$

...(Minus sign means that normal stress is opposite to that across AC)

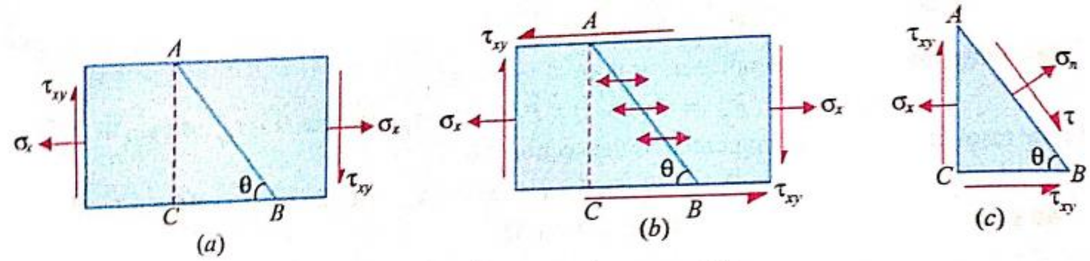
Now the planes of maximum and minimum normal stresses (*i.e.*, principal planes) may be found out by equating the shear stress to zero *i.e.*

$$-\tau_{xy}\cos 2\theta = 0$$

The above equation is possible only if $2\theta = 90°$ or $270°$ (because cos 90° or cos 270° = 0) or in other words, $\theta = 45°$ or $135°$.

Stresses on an oblique section of a body subjected to a direct stress in one plane and accompanied by a simple shear stress:

Consider a rectangular body of uniform cross-sectional area and unit thickness subjected to a tensile stress along x-x axis accompanied by a positive (i.e., clockwise) shear stress along x-x axis as shown in fig. Now let us consider an oblique section AB inclined with x-x axis on which we are required to find out the stress as shown in the figure.



Let σ_x = Tensile stress along *x-x* axis,

τ_{xy} = Positive (*i.e.* clockwise) shear stress along *x-x* axis, and

θ = Angle which the oblique section *AB* makes with *x-x* axis in clockwise direction.

First of all, consider the equilibrium of the wedge *ABC*. We know that as per the principle of simple shear, the face *BC* of the wedge will be subjected to an anticlockwise shear stress equal to τ_{xy} as shown in Fig. 7.7 (*b*). We know that horizontal force acting on the face *AC*,

$$P_x = \sigma_x \,.\, AC\ (\leftarrow) \qquad ...(i)$$

Similarly, vertical force acting on the face *AC*,

$$P_y = \tau_{xy} \,.\, AC\ (\uparrow) \qquad ...(ii)$$

and horizontal force acting on the face *BC*,

$$P = \tau_{xy} \,.\, BC\ (\rightarrow) \qquad ...(iii)$$

Resolving the forces perpendicular to the section *AB*,

$$\begin{aligned} P_n &= P_x \sin\theta - P_y \cos\theta - P \sin\theta \\ &= \sigma_x \,.\, AC \sin\theta - \tau_{xy} \,.\, AC \cos\theta - \tau_{xy} \,.\, BC \sin\theta \end{aligned}$$

and now resolving the forces tangential to the section *AB*,

$$\begin{aligned} P_t &= P_x \cos\theta + P_y \sin\theta - P \cos\theta \\ &= \sigma_x \,.\, AC \cos\theta + \tau_{xy} \,.\, AC \sin\theta - \tau_{xy} \,.\, BC \cos\theta \end{aligned}$$

We know that normal stress across the section *AB*,

$$\begin{aligned} \sigma_n &= \frac{P_n}{AB} = \frac{\sigma_x . AC \sin\theta - \tau_{xy} . AC \cos\theta - \tau_{xy} . BC \sin\theta}{AB} \\ &= \frac{\sigma_x . AC \sin\theta}{AB} - \frac{\tau_{xy} . AC \cos\theta}{AB} - \frac{\tau_{xy} . BC \sin\theta}{AB} \\ &= \frac{\sigma_x . AC \sin\theta}{\frac{AC}{\sin\theta}} - \frac{\tau_{xy} . AC \cos\theta}{\frac{AC}{\sin\theta}} - \frac{\tau_{xy} . BC \sin\theta}{\frac{BC}{\cos\theta}} \end{aligned}$$

$$= \frac{\sigma_x . AC \cos\theta}{\frac{AC}{\sin\theta}} + \frac{\tau_{xy}\, AC \sin\theta}{\frac{AC}{\sin\theta}} - \frac{\tau_{xy} . BC \cos\theta}{\frac{BC}{\cos\theta}}$$

$$= \sigma_x \sin\theta \cos\theta + \tau_{xy} \sin^2\theta - \tau_{xy} \cos^2\theta$$

$$= \frac{\sigma_x}{2} \sin 2\theta + \frac{\tau_{xy}}{2}(1 - \cos 2\theta) - \frac{\tau_{xy}}{2}(1 + \cos 2\theta)$$

$$= \frac{\sigma_x}{2} \sin 2\theta + \frac{\tau_{xy}}{2} - \frac{\tau_{xy}}{2} \cos 2\theta - \frac{\tau_{xy}}{2} - \frac{\tau_{xy}}{2} \cos 2\theta$$

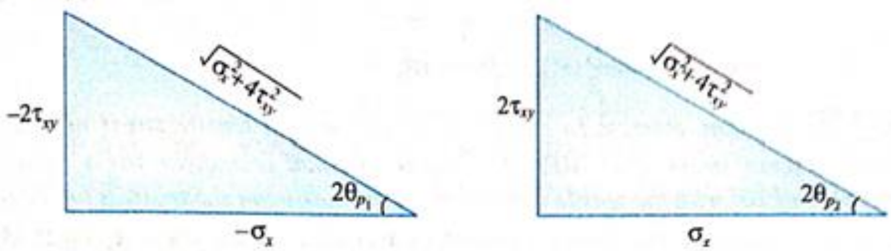
$$= \frac{\sigma_x}{2} \sin 2\theta - \tau_{xy} \cos 2\theta \qquad ...(v)$$

Now the planes of maximum and minimum normal stresses (*i.e.*, principal planes) may be found out by equating the shear stress to zero *i.e.*, from the above equation, we find that the shear stress on any plane is a function of σ_x, τ_{xy} and θ. A little consideration will show that the values of σ_x and τ_{xy} are constant and thus the shear stress varies with the angle θ. Now let θ_p be the value of the angle for which the shear stress is zero.

$$\therefore \quad \frac{\sigma_x}{2} \sin 2\theta_p - \tau_{xy} \cos 2\theta_p = 0 \quad \text{or} \quad \frac{\sigma_x}{2} \sin 2\theta_p = \tau_{xy} \cos 2\theta_p$$

$$\therefore \quad \tan 2\theta_p = \frac{2\tau_{xy}}{\sigma}$$

From the above equation we find that the following two cases satisfy this condition as shown in Fig.(a) and (b)



(a) case 1 (b) case 2

Thus we find that these are two principal planes at right angles to each other, their inclination with *x-x* axis being θ_{p_1} and θ_{p_2}.

Now for case 1,

$$\sin 2\theta_{p_1} = \frac{-2\tau_{xy}}{\sqrt{\sigma_x^2 + 4\tau_{xy}^2}} \quad \text{and} \quad \cos 2\theta_{p_1} = \frac{-\sigma_x}{\sqrt{\sigma_x^2 + 4\tau_{xy}^2}}$$

Similarly for case 2,

$$\sin 2\theta_{p_2} = \frac{2\tau_{xy}}{\sqrt{\sigma_x^2 + 4\tau_{xy}^2}} \quad \text{and} \quad \cos 2\theta_{p_2} = \frac{\sigma_x}{\sqrt{\sigma_x^2 + 4\tau_{xy}^2}}$$

Now the values of principal stresses may be found out by substituting the above values of $2\theta_{p_1}$ and $2\theta_{p_2}$ in equation (*iv*).

Maximum principal stress, $\sigma_{p_1} = \frac{\sigma_x}{2} - \frac{\sigma_x}{2}\cos 2\theta - \tau_{xy}\sin 2\theta$

$$= \frac{\sigma_x}{2} - \frac{\sigma_x}{2} \times \frac{-\sigma_x}{\sqrt{\sigma_x^2 + 4\tau_{xy}^2}} - \tau_{xy} \times \frac{-2\tau_{xy}}{\sqrt{\sigma_x^2 + 4\tau_{xy}^2}}$$

$$= \frac{\sigma_x}{2} + \frac{\sigma_x^2}{2\sqrt{\sigma_x^2 + 4\tau_{xy}^2}} + \frac{2\tau_{xy}^2}{\sqrt{\sigma_x^2 + 4\tau_{xy}^2}}$$

$$= \frac{\sigma_x}{2} + \frac{\sigma_x^2 + 4\tau_{xy}^2}{2\sqrt{\sigma_x^2 + 4\tau_{xy}^2}} = \frac{\sigma_x}{2} + \frac{\sqrt{\sigma_x^2 + 4\tau_{xy}^2}}{2}$$

$$= \frac{\sigma_x}{2} + \sqrt{\left(\frac{\sigma_x^2}{2}\right) + \tau_{xy}^2}$$

Minimum principal stress, $\sigma_{p_2} = \frac{\sigma_x}{2} - \frac{\sigma_x}{2}\cos 2\theta - \tau_{xy}\sin 2\theta$

$$= \frac{\sigma_x}{2} - \frac{\sigma_x}{2} \times \frac{\sigma_x}{\sqrt{\sigma_x^2 + 4\tau_{xy}^2}} - \tau_{xy} \times \frac{2\tau_{xy}}{\sqrt{\sigma_x^2 + 4\tau_{xy}^2}}$$

$$= \frac{\sigma_x}{2} - \frac{\sigma_x^2}{2\sqrt{\sigma_x^2 + 4\tau_{xy}^2}} - \frac{2\tau_{xy}^2}{\sqrt{\sigma_x^2 + 4\tau_{xy}^2}}$$

$$= \frac{\sigma_x}{2} - \frac{\sqrt{\sigma_x^2 + 4\tau_{xy}^2}}{2} = \frac{\sigma_x}{2} - \frac{\sigma_x^2 + 4\tau_{xy}^2}{2\sqrt{\sigma_x^2 + 4\tau_{xy}^2}}$$

$$= \frac{\sigma_x}{2} - \sqrt{\left(\frac{\sigma_x}{2}\right)^2 + \tau_{xy}^2}$$

Problem-10: A plane element in a body is subjected to a tensile stress of 100 MPa accompanied by a shear stress of 25 MPa. Find (i) the normal and shear stress on a plane inclined at an angle of 20^0 with the tensile stress and (ii) the maximum shear stress on the plane.

Solution: Given- Tensile stress along x-x axis (σ_x) 100 MPa. Shear stress (τxy) = 25 MPa and angle made by plane with tensile stress (θ) = 20^0.

Normal and shear stresses on inclined section

We know that the normal stress on the plane,

$$\sigma_n = \frac{\sigma_x}{2} - \frac{\sigma_x}{2}\cos 2\theta - \tau_{xy}\sin 2\theta$$

$$= \frac{100}{2} - \frac{100}{2}\cos(2 \times 20°) - 25\sin(2 \times 20°) \text{ MPa}$$

$$= 50 - 50\cos 40° - 25\sin 40° \text{ MPa}$$

$$= 50 - (50 \times 0.766) - (25 \times 0.6428) \text{ MPa}$$

$$= 50 - 38.3 - 16.07 = -4.37 \text{ MPa} \qquad \textbf{Ans.}$$

and shear stress on the plane,

$$\tau = \frac{\sigma_x}{2}\sin 2\theta - \tau_{xy}\cos 2\theta$$

$$= \frac{100}{2}\sin(2 \times 20°) - 25\cos(2 \times 20°) \text{ MPa}$$

$$= 50\sin 40° - 25\cos 40° \text{ MPa}$$

$$= (50 \times 0.6428) - (25 \times 0.766) \text{ MPa}$$

$$= 32.14 - 19.15 = 12.99 \text{ MPa} \qquad \textbf{Ans.}$$

Maximum shear stress on the plane

We also know that maximum shear stress on the plane,

$$\tau_{max} = \sqrt{\left(\frac{\sigma_x}{2}\right)^2 + \tau_{xy}^2} = \sqrt{\left(\frac{100}{2}\right)^2 + (25)^2} = 55.9 \text{ MPa} \qquad \textbf{Ans.}$$

Problem-11: *An element in a strained body is subjected to a tensile stress of 150 MPa and a shear stress of 50 MPa tending to rotate the element in an anticlockwise direction. Find (i) the magnitude of the normal and shear stresses on a section inclined at 40° with the tensile stress; and (ii) the magnitude and direction of maximum shear stress that can exist on the element.*

SOLUTION. Given : Tensile stress along horizontal x-x axis (σ_x) = 150 MPa ; Shear stress (τ_{xy}) – 50 MPa (Minus sign due to anticlockwise) and angle made by section with the tensile stress (θ) = 40°.

Normal and Shear stress on the inclined section

We know that magnitude of the normal stress on the section,

$$\sigma_n = \frac{\sigma_x}{2} - \frac{\sigma_x}{2}\cos 2\theta - \tau_{xy}\sin 2\theta$$

$$= \frac{150}{2} - \frac{150}{2}\cos(2 \times 40°) - (-50)\sin(2 \times 40°) \text{ MPa}$$

$$= 75 - (75 \times 0.1736) + (50 \times 0.9848) \text{ MPa}$$

$$= 75 - 13.02 + 49.24 = 111.22 \text{ MPa} \qquad \textbf{Ans.}$$

and shear stress on the section

$$\tau = \frac{\sigma_x}{2}\sin 2\theta - \tau_{xy}\cos 2\theta$$

$$= \frac{150}{2}\sin(2 \times 40°) - (-50)\cos(2 \times 40°) \text{ MPa}$$

$$= (75 \times 0.9848) + (50 \times 0.1736) \text{ MPa}$$

$$= 73.86 + 8.68 = 82.54 \text{ MPa} \qquad \textbf{Ans.}$$

(ii) *Maximum shear stress and its direction that can exist on the element*

We know that magnitude of the maximum shear stress.

$$\tau_{max} = \pm\sqrt{\left(\frac{\sigma_x}{2}\right)^2 + \tau_{xy}^2} = \pm\sqrt{\left(\frac{150}{2}\right)^2 + (-50)^2} = \pm 90.14 \text{ MPa} \quad \textbf{Ans.}$$

Let θ_s = Angle which plane of maximum shear stress makes with x-x axis.

We know that, $\tan 2\theta_s = \frac{\sigma_x}{2\tau_{xy}} = \frac{150}{2\times 50} = 1.5$ or $2\theta_s = 56.3°$

$\therefore$ $\theta_s = 28.15°$ or $118.15°$ **Ans.**

Problem-12: An element in a strained body is subjected to a compressive stress of 200 MPa and a clockwise shear stress of 50 MPa on the same plane. Calculate the values of normal and shear stresses on a plane inclined at 35^0 with the compressive stress. Also calculate the value of maximum shear stress in the element.

Solution: Given- Compressive stress along horizontal x-x axis (σ_x) = -200 MPa, shear stress (τ_{xy}) = 50 MPa and angle made by the plane with the compressive stress (θ) = 35^0.

Normal and shear stresses across inclined section

We know that normal stress on the plane,

$$\sigma_n = \frac{\sigma_x}{2} - \frac{\sigma_x}{2}\cos 2\theta - \tau_{xy}\sin 2\theta$$

$$= \frac{-200}{2} - \frac{-200}{2}\cos(2\times 35°) - 50\sin(2\times 35°) \text{ MPa}$$

$$= -100 + (10 \times 0.342) - (50 \times 0.94) \text{ MPa}$$

$$= -100 + 34.2 - 46.9 = -112.9 \text{ MPa} \quad \textbf{Ans.}$$

and shear stress on the plane,

$$\tau = \frac{\sigma_x}{2}\sin 2\theta - \tau_{xy}\cos 2\theta$$

$$= \frac{-200}{2}\sin(2\times 35°) - 50\cos(2\times 35°) \text{ MPa}$$

$$= (-100 \times 0.9397) - (50 \times 0.342) \text{ MPa}$$

$$= -93.97 - 17.1 = -111.07 \text{ MPa} \quad \textbf{Ans.}$$

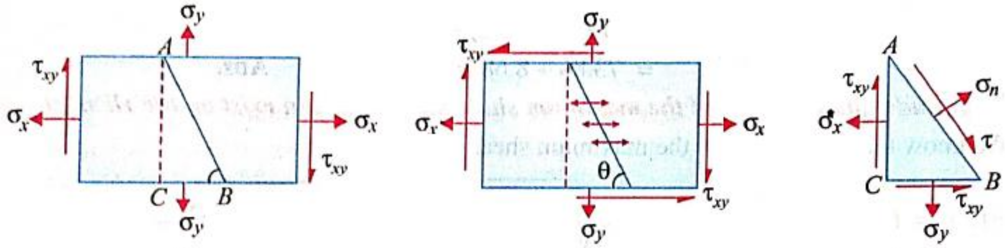
Maximum shear stress in the element

We also know that value of maximum shear stress in the element,

$$\tau_{max} = \sqrt{\left(\frac{\sigma_x}{2}\right)^2 + \tau_{xy}^2} = \sqrt{\left(\frac{-200}{2}\right)^2 + (50)^2} = 111.8 \text{ MPa} \quad \textbf{Ans.}$$

Problem-13: A point in a strained material is subjected to a tensile stress of 120 MPa and a clockwise shear stress of 40 MPa. What are the values of normal and shear stresses on a plane inclined at 450 with the normal to the tensile stress. [**Ans:** 20 MPa, 60 MPa]

Stresses on an oblique section of a body subjected to direct stresses in two mutually perpendicular directions accompanied by a simple shear stress:



(a) (b) (c)

Consider a rectangular body of uniform cross-sectional area and unit thickness subjected to tensile stresses along x-x and y-y axes and accompanied by a positive (*i.e.*, clockwise) shear stress along x-x axis as shown in Fig.7.9 (*b*). Now let us consider an oblique section AB inclined with x-x axis on which we are required to find out the stresses as shown in the figure.

Let
σ_x = Tensile stress along x-x axis,
σ_y = Tensile stress along y-y axis,
τ_{xy} = Positive (*i.e.* clockwise) shear stress along x-x axis. and
θ = Angle, which the oblique section AB makes with x-x axis in an anticlockwise direction.

First of all, consider the equilibrium of the wedge ABC. We know that as per the principle of simple shear, the face BC of the wedge will be subjected to an anticlockwise shear stress equal to τ_{xy} as shown in Fig. 7.9 (*b*). We know that horizontal force acting on the face AC,

$$P_1 = \sigma_x \,.\, AC\ (\leftarrow) \qquad \ldots (i)$$

and vertical force acting on the face AC,

$$P_2 = \tau_{xy} \,.\, AC\ (\uparrow) \qquad \ldots (ii)$$

Similarly, vertical force acting on the face BC,

$$P_3 = \sigma_y \,.\, BC\ (\downarrow) \qquad ..(iii)$$

and horizontal force on the face BC,

$$P_4 = \tau_{xy} \,.\, BC\ (\rightarrow) \qquad \ldots (iv)$$

Now resolving the forces perpendicular to the section AB,

$$\begin{aligned} P_n &= P_1 \sin\theta - P_2 \cos\theta + P_3 \cos\theta - P_4 \sin\theta \\ &= \sigma_x \,.\, AC \sin\theta - \tau_{xy} \,.\, AC \cos\theta + \sigma_y \,.\, BC \cos\theta - \tau_{xy} \,.\, BC \sin\theta \end{aligned}$$

and now resolving the forces tangential to AB,

$$\begin{aligned} P_t &= P_1 \cos\theta + P_2 \sin\theta - P_3 \sin\theta - P_4 \cos\theta \\ &= \sigma_x \,.\, AC \cos\theta + \tau_{xy} \,.\, AC \sin\theta - \sigma_y \,.\, BC \sin\theta - \tau_{xy} \,.\, BC \cos\theta \end{aligned}$$

Normal Stress (across the inclined section AB)

$$\begin{aligned} \sigma_n &= \frac{P_n}{AB} = \frac{\sigma_x . AC \sin\theta - \tau_{xy} . AC \cos\theta + \sigma_y . BC \cos\theta - \tau_{xy} . BC \sin\theta}{AB} \\ &= \frac{\sigma_x . AC \sin\theta}{AB} - \frac{\tau_{xy} . AC \cos\theta}{AB} + \frac{\sigma_y . BC \cos\theta}{AB} - \frac{\tau_{xy} . BC \sin\theta}{AB} \\ &= \frac{\sigma_x . AC \sin\theta}{\frac{AC}{\sin\theta}} - \frac{\tau_{xy} . AC \cos\theta}{\frac{AC}{\sin\theta}} + \frac{\sigma_y . BC \cos\theta}{\frac{BC}{\cos\theta}} \frac{\tau_{xy} . BC \sin\theta}{\frac{BC}{\cos\theta}} \end{aligned}$$

$$= \sigma_x \cdot \sin^2\theta - \tau_{xy}\sin\theta\cos\theta + \sigma_y \cdot \cos^2\theta - \tau_{xy} \cdot \sin\theta\cos\theta$$

$$= \frac{\sigma_x}{2}(1-\cos 2\theta) + \frac{\sigma_y}{2}(1+\cos 2\theta) - 2\tau_{xy} \cdot \sin\theta\cos\theta$$

$$= \frac{\sigma_x}{2} - \frac{\sigma_x}{2}\cos 2\theta + \frac{\sigma_y}{2} + \frac{\sigma_y}{2}\cos 2\theta - \tau_{xy}\sin 2\theta$$

or

$$\sigma_n = \frac{\sigma_x+\sigma_y}{2} - \frac{\sigma_x-\sigma_y}{2}\cos 2\theta - \tau_{xy}\sin 2\theta \qquad \ldots(v)$$

Shear Stress or Tangential Stress (across inclined the section *AB*)

$$\tau = \frac{P_t}{AB} = \frac{\sigma_x . AC\cos\theta + \tau_{xy} . AC\sin\theta - \sigma_y . BC\sin\theta - \tau_{xy}\, BC\cos\theta}{AB}$$

$$= \frac{\sigma_x . AC\cos\theta}{AB} + \frac{\tau_{xy} . AC\sin\theta}{AB} - \frac{\sigma_y . BC\sin\theta}{AB} - \frac{\tau_{xy}\, BC\cos\theta}{AB}$$

$$= \frac{\sigma_x . AC\cos\theta}{\frac{AC}{\sin\theta}} + \frac{\tau_{xy} . AC\sin\theta}{\frac{AC}{\sin\theta}} - \frac{\sigma_y . BC\sin\theta}{\frac{BC}{\cos\theta}} - \frac{\tau_{xy} . BC\cos\theta}{\frac{BC}{\cos\theta}}$$

$$= \sigma_x\sin\theta\cos\theta + \tau_{xy}\sin^2\theta - \sigma_y\sin\theta\cos\theta - \tau_{xy}\cos^2\theta$$

$$= (\sigma_x - \sigma_y)\sin\theta\cos\theta + \frac{\tau_{xy}}{2}(1-\cos 2\theta) - \frac{\tau_{xy}}{2}(1+\cos 2\theta)$$

or

$$\tau = \frac{\sigma_x-\sigma_y}{2}\sin 2\theta - \tau_{xy}\cos 2\theta \qquad \ldots(vi)$$

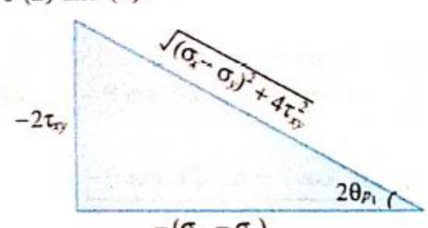
Now the planes of maximum and minimum normal stresses (*i.e.* principal planes) may be found out by equating the shear stress to zero. From the above equations, we find that the shear stress to any plane is a function of σ_y, σ_x, τ_{xy} and θ. A little consideration will show that the values of σ_y, σ_x and τ_{xy} are constant and thus the shear stress varies in the angle θ. Now let θ_p be the value of the angle for which the shear stress is zero.

$$\therefore \quad \frac{\sigma_x-\sigma_y}{2}\sin 2\theta_p - \tau_{xy}\cos 2\theta_p = 0$$

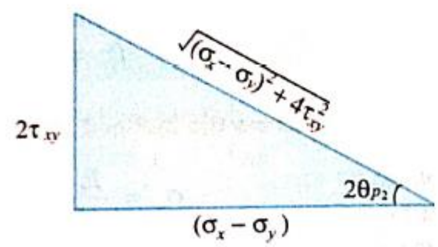
or

$$\frac{\sigma_x-\sigma_y}{2}\sin 2\theta_p = \tau_{xy}\cos 2\theta_p \qquad \text{or} \qquad \tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

From the above equation, we find that the following two cases satisfy this condition as shown in Fig 7.10 (*a*) and (*b*).



(*a*) Case 1



(*b*) Case 2

Thus we find that there are two principal planes, at right angles to each other, their inclinations with *x-x* axis being θ_{p_1} and θ_{p_2}.

Now for case 1,

$$\sin 2\theta_{p_1} = \frac{-2\tau_{xy}}{\sqrt{(\sigma_x-\sigma_y)^2+4\tau_{xy}^2}} \quad \text{and} \quad \cos 2\theta_{p_1} = \frac{-(\sigma_x-\sigma_y)}{\sqrt{(\sigma_x-\sigma_y)^2+4\tau_{xy}^2}}$$

Similarly for case 2,

$$\sin 2\theta_{p_2} = \frac{2\tau_{xy}}{\sqrt{(\sigma_x-\sigma_y)^2+4\tau_{xy}^2}} \quad \text{and} \quad \cos 2\theta_{p_2} = \frac{(\sigma_x-\sigma_y)}{\sqrt{(\sigma_x-\sigma_y)^2+4\tau_{xy}^2}}$$

Now the values of principal stresses may be found out by substituting the above values of $2\theta_{p_1}$ and $2\theta_{p_2}$ in equation (v).

Maximum Principal Stress,

$$\sigma_{p_1} = \frac{\sigma_x+\sigma_y}{2} - \frac{\sigma_x-\sigma_y}{2}\cos 2\theta - \tau_{xy}\sin 2\theta$$

$$= \frac{\sigma_x+\sigma_y}{2} - \left(\frac{\sigma_x-\sigma_y}{2} \times \frac{-(\sigma_x-\sigma_y)}{\sqrt{(\sigma_x-\sigma_y^2)+4\tau_{xy}^2}}\right) - \left(\tau_{xy} \times \frac{-2\tau_{xy}}{\sqrt{(\sigma_x-\sigma_y)^2+4\tau_{xy}^2}}\right)$$

$$= \frac{\sigma_x+\sigma_y}{2} + \frac{(\sigma_x-\sigma_y)^2+4\tau_{xy}^2}{2\sqrt{\sigma_x-\sigma_y)^2+4\tau_{xy}^2}} = \frac{\sigma_x+\sigma_y}{2} + \frac{\sqrt{(\sigma_x-\sigma_y)^2+4\tau_{xy}^2}}{2}$$

or

$$\boxed{\sigma_{p_1} = \frac{\sigma_x+\sigma_y}{2} + \sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2+\tau_{xy}^2}}$$

Minimum Principal Stress

$$\sigma_{p2} = \frac{\sigma_x+\sigma_y}{2} - \frac{(\sigma_x-\sigma_y)}{2}\cos 2\theta - \tau_{xy}\sin 2\theta$$

$$= \frac{\sigma_x+\sigma_y}{2} - \left(\frac{\sigma_x-\sigma_y}{2} \times \frac{(\sigma_x-\sigma_y)}{\sqrt{(\sigma_x-\sigma_y)^2+4\tau_{xy}^2}}\right) - \left(\tau_{xy} \times \frac{2\tau_{xy}}{\sqrt{(\sigma_x-\sigma_y)^2+4\tau_{xy}^2}}\right)$$

$$= \frac{\sigma_x+\sigma_y}{2} - \frac{(\sigma_x-\sigma_y)^2+4\tau_{xy}^2}{2\sqrt{(\sigma_x-\sigma_y)^2+4\tau_{xy}^2}} = \frac{\sigma_x-\sigma_y}{2} - \frac{\sqrt{(\sigma_x-\sigma_y)^2+4\tau_{xy}^2}}{2}$$

or

$$\boxed{\sigma_{p_2} = \frac{\sigma_x+\sigma_y}{2} - \sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2+\tau_{xy}^2}}$$

Problem-14: *A point is subjected to a tensile stress of 250 MPa in the horizontal direction and another tensile stress of 100 MPa in the vertical direction. The point is also subjected to a simple shear stress of 25 MPa, such that when it is associated with the major tensile stress, it tends to rotate the element in the clockwise direction. What is the magnitude of the normal and shear stresses on a section inclined at an angle of 20° with the major tensile stress?*

SOLUTION. Given : Tensile stress in horizontal x-x direction $(\sigma_x) = 250$ MPa ; Tensile stress in vertical y-y direction $(\sigma_y) = 100$ MPa ; Shear stress $(\tau_{xy}) = 25$ MPa and angle made by section with the major tensile stress $(\theta) = 20°$.

Magnitude of normal stress

We know that magnitude of normal stress,

$$\begin{aligned}\sigma_n &= \frac{\sigma_x+\sigma_y}{2}-\frac{\sigma_x-\sigma_y}{2}\cos 2\theta-\tau_{xy}\sin 2\theta\\ &= \frac{250+100}{2}-\frac{250-100}{2}\cos(2\times 20°)-25\sin(2\times 20°)\\ &= 175-75\cos 40°-25\sin 40° \text{ MPa}\\ &= 175-(75\times 0.766)-(25\times 0.6428)\text{ MPa}\\ &= 175-57.45-16.07=101.48\text{ MPa} \quad \textbf{Ans.}\end{aligned}$$

Magnitude of shear stress

We also know that magnitude of shear stress,

$$\begin{aligned}\tau &= \frac{\sigma_x-\sigma_y}{2}\sin 2\theta-\tau_{xy}\cos 2\theta\\ &= \frac{250-100}{2}\sin(2\times 20°)-25\cos(2\times 20°)\\ &= 75\sin 40°-25\cos 40° \text{ MPa}\\ &= (75\times 0.6428)-(25\times 0.766)\text{ MPa}\\ &= 48.21-19.15=29.06\text{ MPa} \quad \textbf{Ans.}\end{aligned}$$

Problem-15: *A plane element in a boiler is subjected to tensile stresses of 400 MPa on one plane and 150 MPa on the other at right angles to the former. Each of the above stresses is accompanied by a shear stress of 100 MPa such that when associated with the minor tensile stress tends to rotate the element in anticlockwise direction. Find*

(*a*) *Principal stresses and their directions.*

(*b*) *Maximum shearing stresses and the directions of the plane on which they act.*

SOLUTION. Given : Tensile stress along *x-x* axis (σ_x) = 400 MPa ; Tensile stress along y-y axis (σ_y) = 150 MPa and shear stress (τ_{xy}) = – 100 MPa (Minus sign due to anticlockwise on *x-x* direction)

(*a*) *Principal stresses and their directions*

We know that maximum principal stress.

$$\begin{aligned}\sigma_{max} &= \frac{\sigma_x+\sigma_y}{2}+\sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2+\tau_{xy}^2}\\ &= \frac{400+150}{2}+\sqrt{\left(\frac{400-150}{2}\right)^2+(-100)^2}\text{ MPa}\\ &= 275+160.1=435.1\text{ MPa} \quad \textbf{Ans.}\end{aligned}$$

and minimum principal stress,

$$\begin{aligned}\sigma_{min} &= \frac{\sigma_x+\sigma_y}{2}-\sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2+\tau_{xy}^2}\\ &= \frac{400+150}{2}-\sqrt{\left(\frac{400-150}{2}\right)^2+(-100)^2}\text{ MPa}\\ &= 275-160.1=114.9\text{ MPa} \quad \textbf{Ans.}\end{aligned}$$

Let θ_p = Angle which plane of principal stress makes with *x-x* axis.

We know that, $\tan 2\theta_p = \dfrac{2\tau_{xy}}{\sigma_x-\sigma_y}=\dfrac{2\times 100}{400-150}=0.8 \quad \text{or} \quad 2\theta_p = 38.66°$

$\therefore \quad \theta_p = 19.33° \quad \text{or} \quad 109.33°$ **Ans.**

(b) ***Maximum shearing stresses and their directions***

We also know that maximum shearing stress

$$\tau_{max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \sqrt{\left(\frac{400-150}{2}\right)^2 + (-100)^2}$$

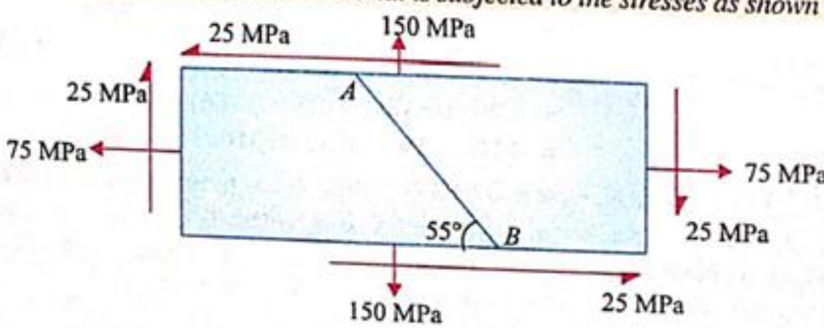
$$= 160.1 \text{ MPa} \quad \textbf{Ans.}$$

Let θ_s = Angle which plane of maximum shearing stress makes with x-x axis.

We know that, $$\tan 2\theta_s = \frac{\sigma_x - \sigma_y}{2\tau_{xy}} = \frac{400-150}{2 \times 100} = 1.25 \quad \text{or} \quad 2\theta_s = 51.34°$$

$$\theta_s = 25.67° \quad \text{or} \quad 115.67° \quad \textbf{Ans.}$$

Problem-16: *A point in a strained material is subjected to the stresses as shown in Fig.*



Find graphically, or otherwise, the normal and shear stresses on the section AB.

Solution. Given : Tensile stress along horizontal x-x axis (σ_x) = 75 MPa ; Tensile stress along vertical y-y axis (σ_y) = 150 MPa ; Shear stress (τ_{xy}) = 25 MPa and angle made by section with the horizontal direction (θ) = 55°.

Normal stress on the section AB

We know that normal stress on the section AB,

$$\sigma_n = \frac{\sigma_x - \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2} \cos 2\theta - \tau_{xy} \sin 2\theta$$

$$= \frac{75+150}{2} - \frac{75-150}{2} \cos (2 \times 55°) - 25 \sin (2 \times 55°) \text{ MPa}$$

$$= 112.5 + 37.5 \cos 110° - 25 \sin 110° \text{ MPa}$$

$$= 11.25 + 37.5 \times (-0.342) - (25 \times 0.9397) \text{ MPa}$$

$$= 112.5 - 12.83 - 23.49 = 76.18 \text{ MPa} \quad \textbf{Ans.}$$

Shear stress on the section AB

We also know that shear stress on the section AB.

$$\tau = \frac{\sigma_x - \sigma_y}{2} \sin 2\theta - \tau_{xy} \cos 2\theta$$

$$= \frac{75-150}{2} \sin (2 \times 55°) - 25 \cos (2 \times 55°) \text{ MPa}$$

$$= -37.5 \sin 110° - 25 \cos 110° \text{ MPa}$$

$$= -37.5 \times 0.9397 - 25 \times (-0.342) \text{ MPa}$$

$$= -35.24 + 8.55 = -26.69 \text{ MPa} \quad \textbf{Ans.}$$

Problem-17: *A plane element of a body is subjected to a compressive stress of 300 MPa in x-x direction and a tensile stress of 200 MPa in the y-y direction. Each of the above stresses is subjected to a shear stress of 100 MPa such that when it is associated with the compressive stress, it tends to rotate the element in an anticlockwise direction. Find graphically, or otherwise, the normal and shear stresses on a plane inclined at an angle of 30° with the x-x axis.*

SOLUTION. Given : Compressive stress in *x-x* direction $(\sigma_x) = -300$ MPa (Minus sign due to compressive stress) ; Tensile stress in *y-y* direction $(\sigma_y) = 200$ MPa ; Shear stress $(\tau_{xy}) = -100$ MPa (Minus sign due to anticlockwise direction along the compressive stress *i.e.*, σ_x) and angle made by section with the *x-x* axis $(\theta) = 30°$.

Normal stress on the plane

We know that normal stress on the plane,

$$\sigma_n = \frac{\sigma_x+\sigma_y}{2} - \frac{\sigma_x-\sigma_y}{2}\cos 2\theta - \tau_{xy}\sin 2\theta$$

$$= \frac{-300+200}{2} - \frac{-300-200}{2}\cos(2\times 30°) - [-100\sin(2\times 30°]$$

$$= -50 - (-250\cos 60°) + 100\sin 60° \text{ MPa}$$

$$= -50 + (250 \times 0.5) + (10 \times 0.866) \text{ MPa}$$

$$= -50 + 125 + 86.6 = 161.6 \text{ MPa} \quad \textbf{Ans.}$$

Shear stress on the plane

We also know that shear stress on the plane.

$$\tau = \frac{\sigma_x-\sigma_y}{2}\sin 2\theta - \tau_{xy}\cos 2\theta$$

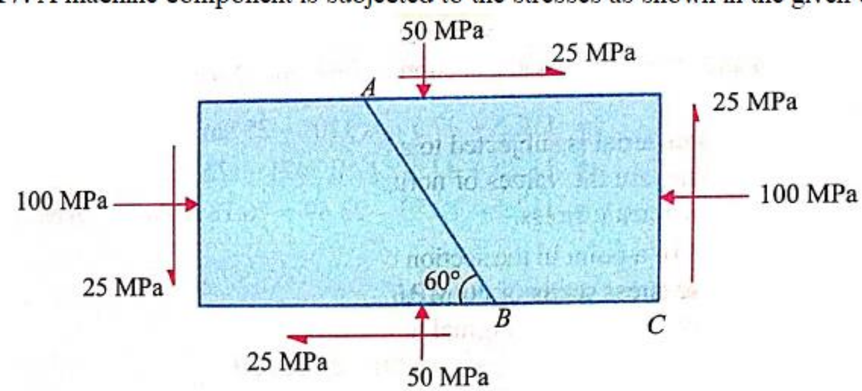
$$= \frac{-300-200}{2}\sin(2\times 30°) - [-100\cos(2\times 30°)] \text{ MPa}$$

$$= -250\sin 60° + 100\cos 60° \text{ MPa}$$

$$= -250 \times 0.866 + 100 \times 0.5 \text{ MPa}$$

$$= -216.5 + 50 = -166.5 \text{ MPa} \quad \textbf{Ans.}$$

Problem-17: A machine component is subjected to the stresses as shown in the given below figure.



Find the normal and shearing stresses on the section AB inclined at an angle of 60° with x-x axis. Also find the resultant stress on the section.

SOLUTION. Given : Compressive stress along horizontal *x-x* axis $(\sigma_x) = -100$ MPa (Minus sign due to compressive stress) ; Compressive stress along vertical *y-y* axis $(\sigma_y) = -50$ MPa (Minus sign due to compressive stress) ; Shear stress $(\tau_{xy}) = -25$ MPa (Minus sign due to anticlockwise on *x-x* axis) and angle made by section *AB* with *x-x* axis $(\theta) = 60°$.

Normal stress on the section AB

We know that normal stress on the section AB,

$$\sigma_n = \frac{\sigma_x+\sigma_y}{2} - \frac{\sigma_x-\sigma_y}{2}\cos 2\theta - \tau_{xy}\sin 2\theta$$

$$= \frac{-100+(-50)}{2} - \frac{-100-(-50)}{2}\cos(2\times 60^\circ) - [-25\sin(2\times 60^\circ)]$$

$$= -75 + 25\cos 120^\circ + 25\sin 120^\circ \text{ MPa}$$

$$= -75 + [25\times(-0.5)] + (25\times 0.866) \text{ MPa}$$

$$= -75 - 12.5 + 21.65 = -65.85 \text{ MPa} \qquad \textbf{Ans.}$$

Shearing stress on the section AB

We know that shearing stress on the section AB,

$$\tau = \frac{\sigma_x-\sigma_y}{2}\sin 2\theta - \tau_{xy}\cos 2\theta$$

$$= \frac{-100-(-50)}{2}\sin(2\times 60^\circ) - [-25\cos(2\times 60^\circ)]$$

$$= -25\sin 120^\circ + 25\cos 120^\circ = -25\times 0.866 + [25\times(-0.5)] \text{ MPa}$$

$$= -21.65 - 12.5 = -34.15 \text{ MPa} \qquad \textbf{Ans.}$$

Resultant stress on the section AB

We also know that resultant stress on the section AB,

$$\sigma_R = \sqrt{\sigma_n^2 + \tau^2} = \sqrt{(-65.85)^2 + (-34.15)^2} = 74.2 \text{ MPa} \qquad \textbf{Ans.}$$

Problem-18: The principal stresses or a point in the section of a member are 50 MPa or 20 MPa both tensile. If there is a clockwise shear stress of 30 MPa, find the normal and shear stresses on a section inclined at an angle of 150 with the normal to the major tensile stress.

Graphical method for the stresses on an Oblique section of a body: The determination of normal, shear and resultant stresses across a section by drawing a Mohr's circle of stresses. The construction of Mohr's circle of stresses as well as determination of normal, shear and resultant stresses is very easier than the analytical method. Moreover, there is a little chance of committing any error in this method. We shall draw the Mohr's circle of stresses for the following cases.

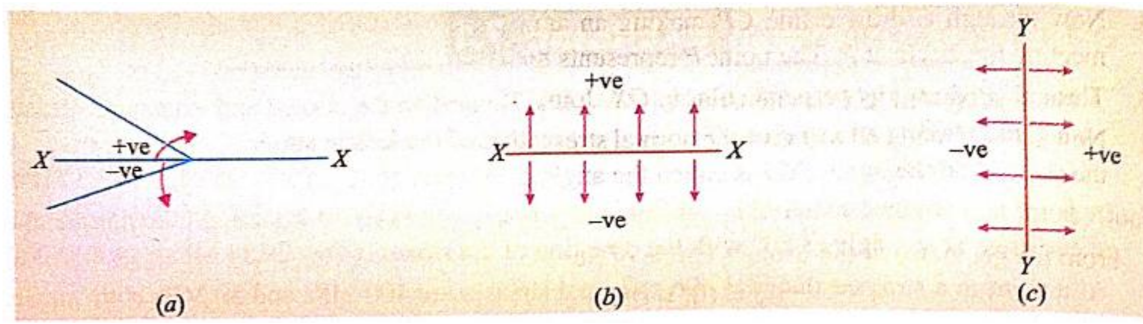
1. A body subjected to a direct stress in one plane
2. A body subjected to a direct stress in two mutually perpendicular directions.
3. A body subjected to a simple shear stress
4. A body subjected to a direct stress in one plane accompanied bya simple shear stress
5. A body subjected to a direct stress in two mutually perpendicular directions accompanied by a simple shear stress

Sign Conventions for Graphical Method: The following sign conventions which are widely used and internationally recognized.

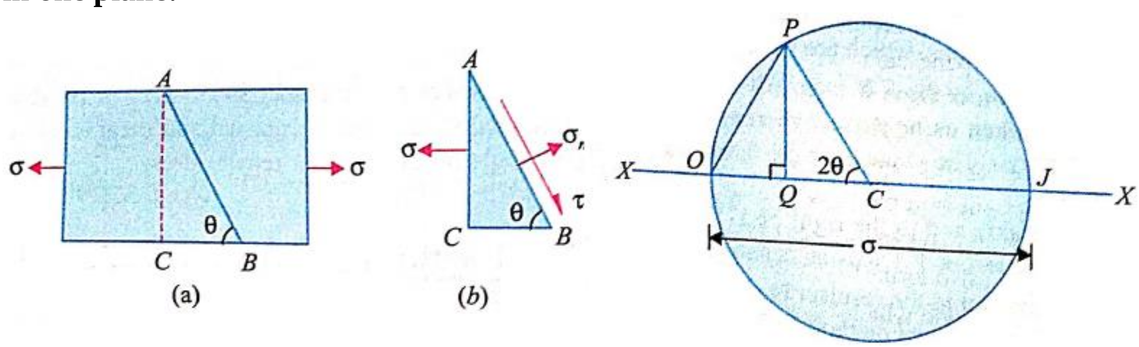
1. The angle is taken with reference to the X-X axis. All the angles traced in the anticlockwise direction to the X-X axis are taken as negative, whereas those in the

clockwise direction as positive as shown in fig. The value of angle θ, until and unless mentioned is taken as positive and draw clockwise.

2. The measurements above X-X axis and to the right of Y-Y axis are taken as positive, whereas those below X-X axis and to the left of T-Y axis as negative as shown in fig.
3. Sometimes there is a slight variation in the results obtained by analytical method and graphical method. The values obtained by graphical method are taken to be correct if they agree upto the first decimal point with values obtained by analytical method , e.g., 5.66 (Analytical) = 5.7 (Graphical), similarly 3.24 (Analytical) = 3.2 (Graphical)



Mohr's Circle for stresses on an oblique section of a body subjected to a direct stress in one plane:



Consider a rectangular body of uniform cross-sectional area and unit thickness subjected to a direct tensile stress along x-x axis as shown in fig (a) and (b). Now let us consider an oblique section AB inclined with x-x axis, on which we are required to find out the stresses as shown in fig.

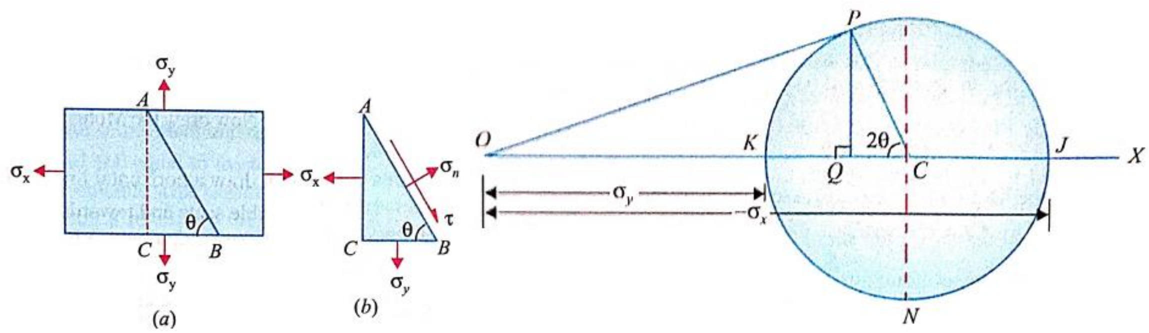
Let σ = Tensile stress, in x-x direction and

θ = Angle which the oblique section AB makes with the x-x axis in clock wise direction

First of all consider the equilibrium of the wedge ABC. Now draw the Mohr's circle of stresses as shown in fig. and as discussed below.

2. Cut off OJ equal to the tensile stress (σ) to some suitable scale and towards right (because σ is tensile). Bisect OJ at C. Now the point O represents the stress system on plane BC and the point J represents the stress system on plane AC.
3. Now with C as centre and radius equal to CO and or CJ draw a circle. It is known as Mohr's Circle for stress.
4. Now through C draw a line CP making an angle of 2θ with CO in the clockwise direction meeting the circle at P. The point P represents the section AB.
5. Through P, draw PQ perpendicular to OX. Join OP.
6. Now OQ, QP and OP will give the normal stress, shear stress and resultant stress respectively to the scale. And the angle POJ is called the angle of obliquity (θ).

Mohr's Circle for stresses on an oblique section of a body subjected to a direct stress in Two mutually perpendicular directions:



Consider a rectangular body of uniform cross-sectional area and unit thickness subjected to direct tensile stresses in two mutually perpendicular directions along x-x and y-y axis as shown in fig (a) and (b). Now let us consider an oblique section AB inclined with x-x axis, on which we are required to find out the stresses as shown in fig.

Let σ_x = Tensile stress, in x-x direction (also termed as major tensile stress)

σ_y = Tensile stress, in y-y direction (also termed as minor tensile stress), and

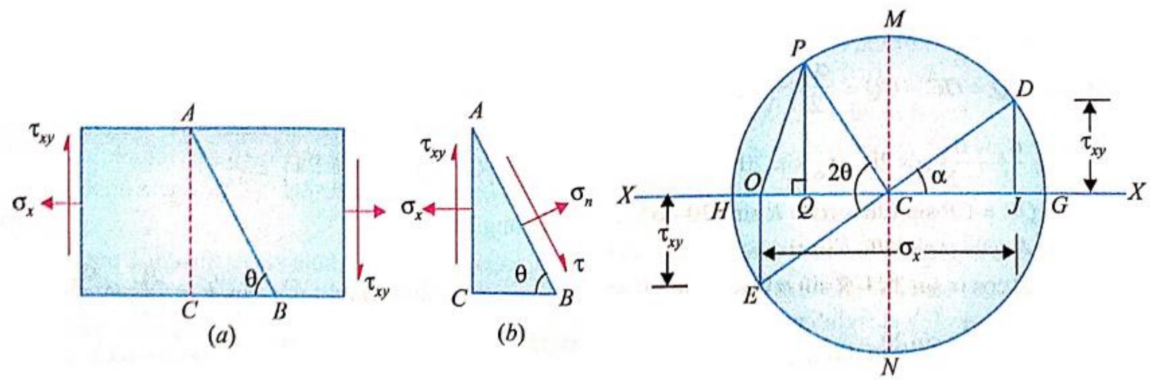
θ = Angle which the oblique section AB makes with the x-x axis in clock wise direction

First of all consider the equilibrium of the wedge ABC. Now draw the Mohr's circle of stresses as shown in fig. and as discussed below.

1. First of all, take some suitable point O and through it draw a horizontal line OX.
2. Cut off OJ and OK equal to the tensile stress σ_x and σ_y to some suitable scale and towards right (because both the stresses are tensile). The point J represents the stress system on plane AC and the point K represents the stress system on plane BC. Bisect JK at C.
3. Now with C as centre and radius equal to CK or CJ draw a circle. It is known as Mohr's Circle for stress.

4. Now through C, draw a line CP making an angle of 2θ with CK in the clockwise direction meeting the circle at P. The point P represents the stress system on the section AB.
5. Through P, draw PQ perpendicular to the line OX. Join OP.
6. Now OQ, QP and OP will give the normal stress, shear stress and resultant stress respectively to the scale. Similarly CM or CN will give the maximum shear stress to the scale. The angle POC is called the angle of obliquity (θ).

Mohr's Circle for stresses on an oblique section of a body subjected to a direct stress in one plane accompanied by a simple shear stress.



Consider a rectangular body of uniform cross-sectional area and unit thickness subjected to direct tensile stress along x-x axis accompanied by a positive (i.e., clockwise) shear stress along x-x axis as shown in fig (a) and (b). Now let us consider an oblique section AB inclined with x-x axis, on which we are required to find out the stresses as shown in fig.

Let σ_x = Tensile stress, in x-x direction

σ_{xy} = Positive (i.e., clockwise) shear stress along x-x axis and

θ = Angle which the oblique section AB makes with the x-x axis in clock wise direction.

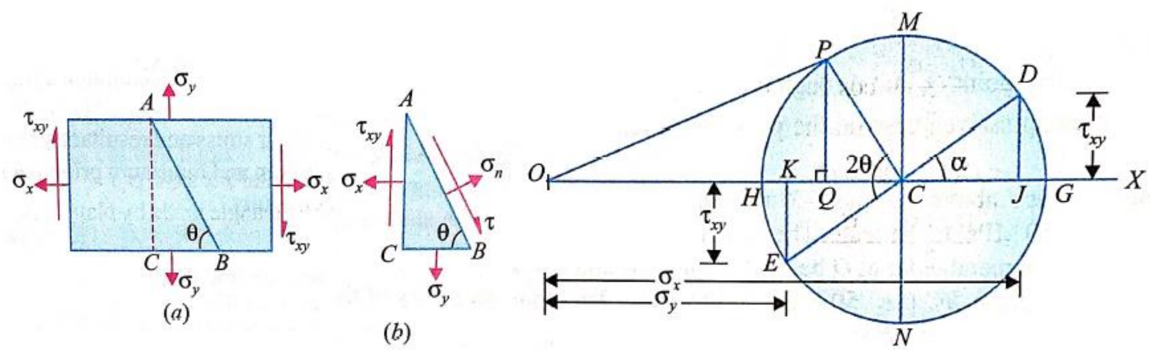
First of all consider the equilibrium of the wedge ABC. **We know that as per the principle of simple shear the face BC of the wedge will also be subjected to an anticlockwise shear stress.** Now draw the Mohr's circle of stresses as shown in fig. and as discussed below.

1. First of all, take some suitable point O and through it draw a horizontal line XOX.
2. Cut off OJ equal to the tensile stress σ_x to some suitable scale and towards right (because σ_x is tensile).
3. Now erect a perpendicular at J above the line X-X (because τ_{xy} is positive along x-x axis) and cut off JD equal to the shear stress τ_{xy} to the scale. The point D represents the stress system on plane AC. Similarly, erect a perpendicular below the line x-x (because τ_{xy} is negative along y-y axis) and cut off OE equal to the

shear stress τ_{xy} to the scale. The point E represents the stress system on plane BC. Join DE and bisect it at C.)

4. Now with C as centre and radius equal to CD or CE draw a circle. It is known as Mohr's Circle for stress.
5. Now through C, draw a line CP making an angle of 2θ with CE in the clockwise direction meeting the circle at P. The point P represents the stress system on the section AB.
6. Through P, draw PQ perpendicular to the line OX. Join OP.
7. Now OQ, QP and OP will give the normal stress, shear stress and resultant stress respectively to the scale. And the angle POC is called the angle of obliquity (θ).

Mohr's Circle for stresses on an oblique section of a body subjected to a direct stress in Two Mutually perpendicular directions accompanied by a simple shear stress.



Consider a rectangular body of uniform cross-sectional area and unit thickness subjected to direct tensile stresses in two mutually perpendicular directions along x-x and y-y axes accompanied by a positive (i.e., clockwise) shear stress along x-x axis as shown in fig (a) and (b). Now let us consider an oblique section AB inclined with x-x axis, on which we are required to find out the stresses as shown in fig.

Let σ_x = Tensile stress, in x-x direction

σ_y = Tensile stress, in y-y direction

σ_{xy} = Positive (i.e., clockwise) shear stress along x-x axis and

θ = Angle which the oblique section AB makes with the x-x axis in clock wise direction.

First of all consider the equilibrium of the wedge ABC. **We know that as per the principle of simple shear the face BC of the wedge will be subjected to an anticlockwise shear stress equal to σ_{xy} as shown in fig.** Now draw the Mohr's circle of stresses as shown in fig. and as discussed below.

1. First of all, take some suitable point O and through it draw a horizontal line OX.
2. Cut off OJ and OK equal to the tensile stresses σ_x and σ_y respectively to some suitable scale and towards right (because both the stresses are tensile).
3. Now erect a perpendicular at J above the line X-X (because τ_{xy} is positive along x-x axis) and cut off JD equal to the shear stress τ_{xy} to the scale. The point D represents the stress system on plane AC. Similarly, erect a perpendicular below the line X-X (because τ_{xy} is negative along y-y axis) and cut off KE equal to the shear stress τ_{xy} to the scale. The point E represents the stress system on plane BC. Join DE and bisect it at C.)
4. Now with C as centre and radius equal to CD or CE draw a circle. It is known as Mohr's Circle for stress.
5. Now through C, draw a line CP making an angle of 2θ with CE in the clockwise direction meeting the circle at P. The point P represents the stress system on the section AB.
6. Through P, draw PQ perpendicular to the line OX. Join OP.
7. Now OQ, QP and OP will give the normal stress, shear stress and resultant stress respectively to the scale. Similarly OG and OH will give the maximum and minimum principle shear stresses to the scale. The angle POC is called the angle of obliquity (θ).

Note: Students are advised to solve the all analytical problems by using Mohr's Circle method and compare the results.