ME6505 DYNAMICS OF MACHINES L T P C
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OBJECTIVES:

- To understand the force-motion relationship in components subjected to external forces and analysis of standard mechanisms.
- To understand the undesirable effects of unbalances resulting from prescribed motions in mechanism.
- To understand the effect of Dynamics of undesirable vibrations.
- To understand the principles in mechanisms used for speed control and stability control.

UNIT I FORCE ANALYSIS 9

Dynamic force analysis – Inertia force and Inertia torque– D Alembert's principle –Dynamic Analysis in reciprocating engines – Gas forces – Inertia effect of connecting rod– Bearing loads – Crank shaft torque – Turning moment diagrams –Fly Wheels – Flywheels of punching presses- Dynamics of Cam-follower mechanism.

UNIT II BALANCING 9

Static and dynamic balancing – Balancing of rotating masses – Balancing a single cylinder engine – Balancing of Multi-cylinder inline, V-engines – Partial balancing in engines – Balancing of linkages – Balancing machines-Field balancing of discs and rotors.

UNIT III SINGLE DEGREE FREE VIBRATION 9

Basic features of vibratory systems – Degrees of freedom – single degree of freedom – Free vibration – Equations of motion – Natural frequency – Types of Damping – Damped vibration– Torsional vibration of shaft – Critical speeds of shafts – Torsional vibration – Two and three rotor torsional systems.

UNIT IV FORCED VIBRATION 9

Response of one degree freedom systems to periodic forcing – Harmonic disturbances –Disturbance caused by unbalance – Support motion –transmissibility – Vibration isolation vibration measurement.

UNIT V MECHANISM FOR CONTROL 9

Governors – Types – Centrifugal governors – Gravity controlled and spring controlled centrifugal governors – Characteristics – Effect of friction – Controlling force curves. Gyroscopes –Gyroscopic forces and torques – Gyroscopic stabilization – Gyroscopic effects in Automobiles, ships and airplanes.

TOTAL : 45 PERIODS

OUTCOMES:

- Upon completion of this course, the Students can able to predict the force analysis in mechanical system and related vibration issues and can able to solve the problem.

TEXT BOOK:

1. Uicker, J.J., Pennock G.R and Shigley, J.E., "Theory of Machines and Mechanisms" ,3rd Edition, Oxford University Press, 2009.
2. Rattan, S.S, "Theory of Machines", 3rd Edition, Tata McGraw-Hill, 2009

REFERENCES:

1. Thomas Bevan, "Theory of Machines", 3rd Edition, CBS Publishers and Distributors, 2005.
2. Cleghorn. W. L, "Mechanisms of Machines", Oxford University Press, 2005
3. Benson H. Tongue, "Principles of Vibrations", Oxford University Press, 2nd Edition, 2007
4. Robert L. Norton, "Kinematics and Dynamics of Machinery", Tata McGraw-Hill, 2009.
5. Allen S. Hall Jr., "Kinematics and Linkage Design", Prentice Hall, 1961
6. Ghosh. A and Mallick, A.K., "Theory of Mechanisms and Machines", Affiliated East-West Pvt.Ltd., New Delhi, 1988.
7. Rao.J.S. and Dukkipati.R.V. "Mechanisms and Machine Theory", Wiley-Eastern Ltd., New Delhi, 1992.
8. John Hannah and Stephens R.C., "Mechanics of Machines", Viva Low-Prices Student Edition, 1999.
9. Grover. G.T., "Mechanical Vibrations", Nem Chand and Bros., 1996
10. William T. Thomson, Marie Dillon Dahleh, Chandramouli Padmanabhan, "Theory of Vibration with Application", 5th edition, Pearson Education, 2011
11. V.Ramamurthi, "Mechanics of Machines", Narosa Publishing House, 2002.
12. Khurmi, R.S.,"Theory of Machines", 14th Edition, S Chand Publications, 2005.

CONTENTS

UNIT – I - FORCE ANALYSIS

1.1 INTRODUCTION

The subject Dynamics of Machines may be defined as that branch of Engineering-science, which deals with the study of relative motion between the various parts of a machine, and forces which act on them. The knowledge of this subject is very essential for an engineer in designing the various parts of a machine.

A machine is a device which receives energy in some available form and utilises it to do some particular type of work.

If the acceleration of moving links in a mechanism is running with considerable amount of linear and/or angular accelerations, inertia forces are generated and these inertia forces also must be overcome by the driving motor as an addition to the forces exerted by the external load or work the mechanism does.

1.2 NEWT ON' S LAW :

First Law

Everybody will persist in its state of rest or of uniform motion (constant velocity) in a straight line unless it is compelled to change that state by forces impressed on it. This means that in the absence of a non-zero net force, the center of mass of a body either is at rest or moves at a constant velocity.

Second Law

A body of mass *m* subject to a force **F** undergoes an acceleration **a** that has the same direction as the force and a magnitude that is directly proportional to the force and inversely proportional to the mass, i.e., **F** = *m***a**. Alternatively, the total force applied on a body is equal to the time derivative of linear momentum of the body.

Third Law

The mutual forces of action and reaction between two bodies are equal, opposite and collinear. This means that whenever a first body exerts a force **F** on a second body, the second body exerts a force −**F** on the first body. **F** and −**F** are equal in magnitude and opposite in direction. This law is sometimes referred to as the *action-reaction law*, with **F** called the "action" and −**F** the "reaction"

1.3 TYPES OF FORCE ANALYSIS:

Equilibrium of members with two forces
Equilibrium of members with three forces
Equilibrium of members with two forces and torque
Equilibrium of members with two couples.
Equilibrium of members with four forces.

1.3.1 Principle of Super Position:

Sometimes the number of external forces and inertial forces acting on a mechanism are too much for graphical solution. In this case we apply the method of superposition. Using superposition the entire system is broken up into (n) problems, where n is the number of forces, by considering the external and inertial forces of each link individually. Response of a linear system to several forces acting simultaneously is equal to the sum of responses of the system to the forces individually. This approach is useful because it can be performed by graphically.

1.3.2 Free Body Diagram:

A free body diagram is a pictorial representation often used by physicists and engineers to analyze the forces acting on a body of interest. A free body diagram shows all forces of all types acting on this body. Drawing such a diagram can aid in solving for the unknown forces or the equations of motion of the body. Creating a free body diagram can make it easier to understand the forces, and torques or moments, in relation to one another and suggest the proper concepts to apply in order to find the solution to a problem. The diagrams are also used as a conceptual device to help identify the internal forces—for example, shear forces and bending moments in beams—which are developed within structures.

1.4 DYNAMIC ANALYSIS OF FOUR BAR MECHANISM:

A **four-bar linkage** or simply a **4-bar** or **four-bar** is the simplest movable linkage. It consists of four rigid bodies (called bars or links), each attached to two others by single joints or pivots to form closed loop. Four-bars are simple mechanisms common in mechanical engineering machine design and fall under the study of kinematics.

Dynamic Analysis of Reciprocating engines.
Inertia force and torque analysis by neglecting weight of connecting rod.
Velocity and acceleration of piston.
Angular velocity and Angular acceleration of connecting rod.
Force and Torque Analysis in reciprocating engine neglecting the weight of connecting rod.
Equivalent Dynamical System
Determination of two masses of equivalent dynamical system

The inertia force is an imaginary force, which when acts upon a rigid body, brings it in an equilibrium position. It is numerically equal to the accelerating force in magnitude, but ***opposite*** in direction. Mathematically,

$$\text{Inertia force} = -\text{Accelerating force} = -m.a$$

where m = Mass of the body, and

a = Linear acceleration of the centre of gravity of the body.

Similarly, the inertia torque is an imaginary torque, which when applied upon the rigid body, brings it in equilib-rium position. It is equal to the accelerating couple in magni-tude but ***opposite*** in direction.

1.4.1 D-Alembert's Principle

Consider a rigid body acted upon by a system of forces. The system may be reduced to a single resultant force acting on the body whose magnitude is given by the product of the mass of the body and the linear acceleration of the centre of mass of the body. According to Newton's second law of motion,

$F = m.a$

F = Resultant force acting on the body,

m = Mass of the body, and

a = Linear acceleration of the centre of mass of the body.

The equation **(*i*)** may also be written as:

$F – m.a = 0$

A little consideration will show, that if the quantity – $m.a$ be treated as a force, equal, opposite and with the same line of action as the resultant force F, and include this force with the system of forces of which F is the resultant, then the complete system of forces will be in equilibrium. This principle is known as ***D-Alembert's principle.*** The equal and opposite force – $m.a$ is known as ***reversed effective force*** or the ***inertia force*** (briefly written as F_I). The equation **(*ii*)** may be written as

$F + F_I = 0$...**(*iii*)**

Thus, D-Alembert's principle states that ***the resultant force acting on a body together with the reversed effective force (or inertia force), are in equilibrium.***

This principle is used to reduce a dynamic problem into an equivalent static problem.

1.4.2 Velocity and Acceleration of the Reciprocating Parts in Engines

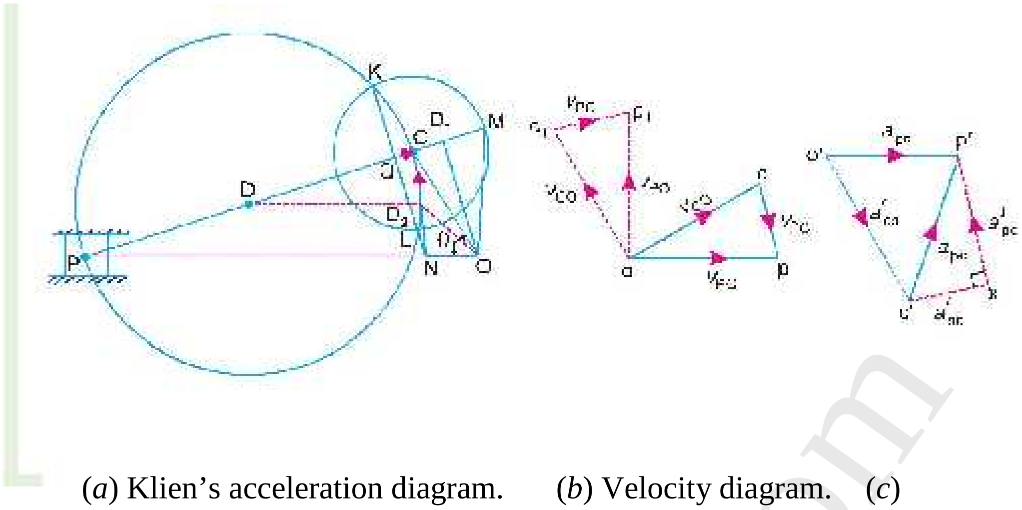
The velocity and acceleration of the reciprocating parts of the steam engine or internal combustion engine (briefly called as I.C. engine) may be determined by graphical method or analytical method. The velocity and acceleration, by graphical method, may be determined by one of the following constructions:

1. Klien's construction, **2.** Ritterhaus's construction, and **3.** Bennett's construction.

We shall now discuss these constructions, in detail, in the following pages.

1.5 KLIEN'S CONSTRUCTION

Let OC be the crank and PC the connecting rod of a reciprocating steam engine, as shown in Fig. 15.2 (*a*). Let the crank makes an angle θ with the line of stroke PO and rotates with uniform angular velocity ω rad/s in a clockwise direction. The Klien's velocity and acceleration diagrams are drawn as discussed below:



(*a*) Klien's acceleration diagram. (*b*) Velocity diagram. (*c*) Acceleration diagram.

Fig. 15.2. Klien's construction.

1.5.1 Klien's velocity diagram

First of all, draw *OM* perpendicular to *OP*; such that it intersects the line *PC* produced at *M*. The triangle *OCM* is known as ***Klien's velocity diagram***. In this triangle *OCM*,

OM may be regarded as a line perpendicular to *PO*,

CM may be regarded as a line parallel to *PC*, and ...(Q It is the same line.)

CO may be regarded as a line parallel to *CO*.

We have already discussed that the velocity diagram for given configuration is a triangle *ocp* as shown in Fig. 15.2 (*b*). If this triangle is revolved through 90°, it will be a triangle $oc_1\ p_1$, in which oc_1 represents v_{CO} (*i.e.* velocity of *C* with respect to *O* or velocity of crank pin *C*) and is paralel to *OC*,

op_1 represents v_{PO} (*i.e.* velocity of *P* with respect to *O* or velocity of cross-head or piston *P*) and is perpendicular to *OP*, and

c_1p_1 represents v_{PC} (*i.e.* velocity of *P* with respect to *C*) and is parallel to *CP*.

A little consideration will show, that the triangles oc_1p_1 and *OCM* are similar. Therefore,

$$\frac{oc_1}{OC} = \frac{op_1}{OM} = \frac{c_1p_1}{CM} = \omega \text{ (a constant)}$$

or
$$\frac{v_{CO}}{OC} = \frac{v_{PO}}{OM} = \frac{v_{PC}}{CM} = \omega$$

$$\therefore \quad v_{CO} = \omega \times OC\ ;\ v_{PO} = \omega \times OM, \text{ and } v_{PC} = \omega \times CM$$

Thus, we see that by drawing the Klien's velocity diagram, the velocities of various points may be obtained without drawing a separate velocity diagram.

1.5.2 Klien's acceleration diagram

The Klien's acceleration dia-gram is drawn as discussed below:

1. First of all, draw a circle with *C* as centre and *CM* as radius.

2. Draw another circle with *PC* as diameter. Let this circle inter-sect the previous circle at *K* and *L*.

3. Join *KL* and produce it to intersect *PO* at *N*. Let *KL* intersect *PC* at *Q*.

This forms the quadrilateral *CQNO*, which is known as ***Klien's acceleration diagram*.**

We have already discussed that the acceleration diagram for the given configuration is as shown in Fig. 15. 2 (*c*). We know that

(*i*) *o'c'* represents a^r_{CO} (*i.e.* radial component of the acceleration of crank pin *C* with respect to *O*) and is parallel to *CO*;

(*ii*) *c'x* represents a^r_{PC} (*i.e.* radial component of the acceleration of crosshead or piston *P* with respect to crank pin *C*) and is parallel to *CP* or *CQ*;

(*iii*) *xp'* represents a^t_{PC} (*i.e.* tangential component of the acceleration of *P* with respect to *C*) and is parallel to *QN* (because *QN* is perpendicular to *CQ*); and

(*iv*) *o'p'* represents a_{PO} (*i.e.* acceleration of *P* with respect to *O* or the acceleration of piston *P*) and is parallel to *PO* or *NO*.

A little consideration will show that the quadrilateral *o'c'x p'* [Fig. 15.2 (*c*)] is similar to quadrilateral *CQNO* [Fig. 15.2 (*a*)]. Therefore,

$$\frac{o'c'}{OC} = \frac{c'x}{CQ} = \frac{xp'}{QN} = \frac{o'p'}{NO} = \omega^2 \text{ (a constant)}$$

or

$$\frac{a^r_{CO}}{OC} = \frac{a^r_{PC}}{CQ} = \frac{a^t_{PC}}{QN} = \frac{a_{PO}}{NO} = \omega^2$$

$$\therefore \quad a^r_{CO} = \omega^2 \times OC; \; a^r_{PC} = \omega^2 \times CQ$$

$$a^t_{PC} = \omega^2 \times QN; \text{ and } a_{PO} = \omega^2 \times NO$$

Thus we see that by drawing the Klien's acceleration diagram, the acceleration of various points may be obtained without drawing the separate acceleration diagram.

1.6 SOLVED PROBLEMS

1. *The crank and connecting rod of a reciprocating engine are 200 mm and 700 mm respectively. The crank is rotating in clockwise direction at 120 rad/s. Find with the help of Klein's construction:* **1.** *Velocity and acceleration of the piston,* **2.** *Velocity and acceleration of the mid point of the connecting rod, and* **3.** *Angular velocity and angular acceleration of the connecting rod, at the instant when the crank is at 30° to I.D.C.* (*inner dead centre*).

Solution. Given: $OC = 200$ mm $= 0.2$ m ; $PC = 700$ mm $= 0.7$ m ; $\omega = 120$ rad/s

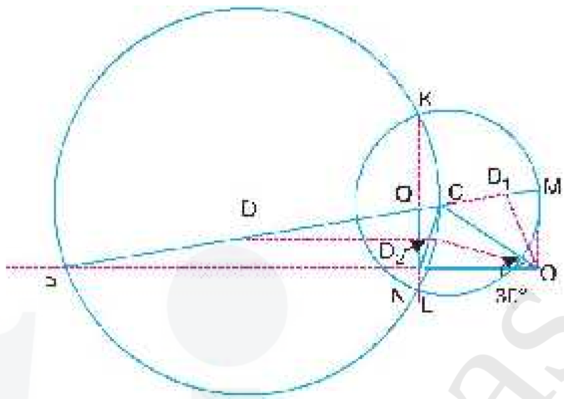


Fig. 15.5

The Klein's velocity diagram *OCM* and Klein's acceleration diagram *CQNO* as shown in Fig. 15.5 is drawn to some suitable scale, in the similar way as discussed in Art. 15.5. By measurement, we find that

$OM = 127$ mm $= 0.127$ m ; $CM = 173$ mm $= 0.173$ m ; $QN = 93$ mm $= 0.093$ m ; $NO = 200$ mm $= 0.2$ m

1. ***Velocity and acceleration of the piston***

We know that the velocity of the piston *P,*

$$v_P = \omega \times OM = 120 \times 0.127 = 15.24 \text{ m/s} \quad \textbf{Ans.}$$

and acceleration of the piston *P*,

$$a_P = \omega^2 \times NO = (120)^2 \times 0.2 = 2880 \text{ m/s}^2 \quad \textbf{Ans.}$$

2. ***Velocity and acceleration of the mid-point of the connecting rod***

In order to find the velocity of the mid-point *D* of the connecting rod, divide *CM* at D_1 in the same ratio as *D* divides *CP*. Since *D* is the mid-point of *CP*, therefore D_1 is the mid-point of *CM*, *i.e.* $CD_1 = D_1M$. Join OD_1. By measurement,

$$OD_1 = 140 \text{ mm} = 0.14 \text{ m}$$

$$\text{Velocity of } D,\ v_D = \omega \times OD_1 = 120 \times 0.14 = 16.8 \text{ m/s} \quad \textbf{Ans.}$$

In order to find the acceleration of the mid-point of the connecting rod, draw a line DD_2 parallel to the line of stroke *PO* which intersects *CN* at D_2. By measurement,

$$OD_2 = 193 \text{ mm} = 0.193 \text{ m}$$

$\therefore$ Acceleration of *D*,

$$a_D = \omega^2 \times OD_2 = (120)^2 \times 0.193 = 2779.2 \text{ m/s}^2 \quad \textbf{Ans.}$$

3. ***Angular velocity and angular acceleration of the connecting rod***

We know that the velocity of the connecting rod *PC* (*i.e.* velocity of *P* with respect to *C*), $v_{PC} = \omega \times CM = 120 \times 0.173 = 20.76$ m/s

∴ Angular acceleration of the connecting rod PC,

$$\omega_{PC} = \frac{v_{PC}}{PC} = \frac{20.76}{0.7} = 29.66 \text{ rad/s} \quad \textbf{Ans.}$$

We know that the tangential component of the acceleration of P with respect to C,

$$a^t_{PC} = \omega^2 \times QN = (120)^2 \times 0.093 = 1339.2 \text{ m/s}^2$$

∴ Angular acceleration of the connecting rod PC,

$$\alpha_{PC} = \frac{a^t_{PC}}{PC} = \frac{1339.2}{0.7} = 1913.14 \text{ rad/s}^2 \quad \textbf{Ans.}$$

1.7 APPROXIMATE ANALYTICAL METHOD FOR VELOCITY AND ACCELERATION OF THE PISTON

Consider the motion of a crank and connecting rod of a reciprocating steam engine as shown in Fig. 15.7. Let OC be the crank and PC the connecting rod. Let the crank rotates with angular velocity of ω rad/s and the crank turns through an angle θ from the inner dead centre (briefly written as I.D.C). Let x be the displacement of a reciprocating body P from I.D.C. after time t seconds, during which the crank has turned through an angle θ .

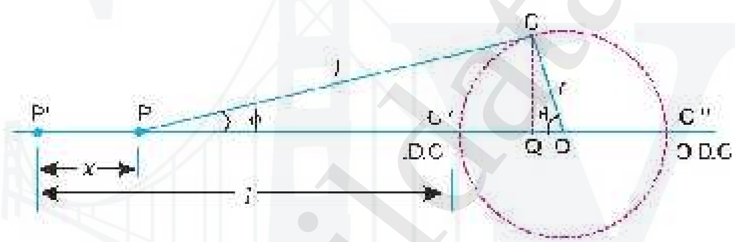


Fig. 15.7. Motion of a crank and connecting rod of a reciprocating steam engine.

Let l = Length of connecting rod between the centres,

r = Radius of crank or crank pin circle,

φ = Inclination of connecting rod to the line of stroke PO, and

n = Ratio of length of connecting rod to the radius of crank = l/r.

Velocity of the piston

From the geometry of Fig. 15.7,

$$x = P'P = OP' - OP = (P'C' + C'O) - (PQ + QO)$$

$$= (l + r) - (l \cos \phi + r \cos \theta) \qquad \begin{pmatrix} \because & PQ = l \cos \phi, \\ \text{and } QO = r \cos \theta \end{pmatrix}$$

$$= r(1 - \cos \theta) + l(1 - \cos \phi) = r \left[(1 - \cos \theta) + \frac{l}{r}(1 - \cos \phi) \right]$$

$$= r[(1 - \cos \theta) + n(1 - \cos \phi)] \qquad \ldots(i)$$

From triangles CPQ and CQO,

$$CQ = l \sin \phi = r \sin \theta \text{ or } l/r = \sin \theta / \sin \phi$$

$$\therefore \quad n = \sin \theta / \sin \phi \quad \text{or} \quad \sin \phi = \sin \theta / n \qquad \ldots(ii)$$

We know that, $\cos\phi = \left(1 - \sin^2\phi\right)^{\frac{1}{2}} = \left(1 - \frac{\sin^2\theta}{n^2}\right)^{\frac{1}{2}}$

Expanding the above expression by binomial theorem, we get

$$\cos\phi = 1 - \frac{1}{2} \times \frac{\sin^2\theta}{n^2} + \ldots\ldots \qquad \text{...(Neglecting higher terms)}$$

or

$$1 - \cos\phi = \frac{\sin^2\theta}{2n^2} \qquad \text{...}(iii)$$

Substituting the value of $(1 - \cos\phi)$ in equation (i), we have

$$x = r\left[(1 - \cos\theta) + n \times \frac{\sin^2\theta}{2n^2}\right] = r\left[(1 - \cos\theta) + \frac{\sin^2\theta}{2n}\right] \qquad \text{...}(iv)$$

Differentiating equation (iv) with respect to θ,

$$\frac{dx}{d\theta} = r\left[\sin\theta + \frac{1}{2n} \times 2\sin\theta.\cos\theta\right] = r\left(\sin\theta + \frac{\sin 2\theta}{2n}\right) \qquad \text{...}(v)$$

$$(\because 2\sin\theta.\cos\theta = \sin 2\theta)$$

$\therefore$ Velocity of P with respect to O or velocity of the piston P,

$$v_{PO} = v_P = \frac{dx}{dt} = \frac{dx}{d\theta} \times \frac{d\theta}{dt} = \frac{dx}{d\theta} \times \omega$$

$$\text{...}(\because \text{Ratio of change of angular velocity} = d\theta/dt = \omega)$$

Substituting the value of $dx/d\theta$ from equation (v), we have

$$v_{PO} = v_P = \omega r\left(\sin\theta + \frac{\sin 2\theta}{2n}\right) \qquad \text{...}(vi)$$

Acceleration of the piston

Since the acceleration is the rate of change of velocity, therefore acceleration of the piston P,

$$a_P = \frac{dv_P}{dt} = \frac{dv_P}{d\theta} \times \frac{d\theta}{dt} = \frac{dv_P}{d\theta} \times \omega$$

Differentiating equation (vi) with respect to θ,

$$\frac{dv_P}{d\theta} = \omega.r\left[\cos\theta + \frac{\cos 2\theta \times 2}{2n}\right] = \omega.r\left[\cos 0 + \frac{\cos 2\theta}{n}\right]$$

Substituting the value of $\frac{dv_P}{d\theta}$ in the above equation, we have

$$a_P = \omega r\left[\cos\theta + \frac{\cos 2\theta}{n}\right] \times \omega = \omega^2 . r\left[\cos\theta + \frac{\cos 2\theta}{n}\right] \qquad \text{...}(vii)$$

1.8 ANGULAR VELOCITY AND ACCELERATION OF THE CONNECTING ROD

Consider the motion of a connecting rod and a crank as shown in Fig. 15.7.From the geometry of the figure, we find that

$$CQ = l \sin \phi = r \sin \theta$$

$$\therefore \quad \sin \phi = \frac{r}{l} \times \sin \theta = \frac{\sin \theta}{n} \qquad \ldots\left(\because n = \frac{l}{r}\right)$$

Differentiating both sides with respect to time t,

$$\cos \phi \times \frac{d\phi}{dt} = \frac{\cos \theta}{n} \times \frac{d\theta}{dt} = \frac{\cos \theta}{n} \times \omega \qquad \left(\because \frac{d\theta}{dt} = \omega\right)$$

Since the angular velocity of the connecting rod PC is same as the angular velocity of point P with respect to C and is equal to $d\phi/dt$, therefore angular velocity of the connecting rod

$$\omega_{PC} = \frac{d\phi}{dt} = \frac{\cos \theta}{n} \times \frac{\omega}{\cos \phi} = \frac{\omega}{n} \times \frac{\cos \theta}{\cos \phi}$$

We know that, $\cos \phi = \left(1 - \sin^2 \phi\right)^{\frac{1}{2}} = \left(1 - \frac{\sin^2 \theta}{n^2}\right)^{\frac{1}{2}}$ $\qquad \ldots\left(\because \sin\phi = \frac{\sin \theta}{n}\right)$

$$\therefore \quad \omega_{PC} = \frac{\omega}{n} \times \frac{\cos \theta}{\left(1 - \frac{\sin^2 \theta}{n^2}\right)^{\frac{1}{2}}} = \frac{\omega}{n} \times \frac{\cos\theta}{\frac{1}{n}(n^2 - \sin^2 \theta)^{1/2}}$$

$$= \frac{\omega \cos \theta}{(n^2 - \sin^2 \theta)^{1/2}} \qquad \ldots(i)$$

Angular acceleration of the connecting rod PC,

$$\alpha_{PC} = \text{Angular acceleration of } P \text{ with respect to } C = \frac{d\,(\omega_{PC})}{dt}$$

We know that

$$\frac{d\,(\omega_{PC})}{dt} = \frac{d\,(\omega_{PC})}{d\theta} \times \frac{d\theta}{dt} = \frac{d\,(\omega_{PC})}{d\theta} \times \omega \qquad \ldots(ii)$$

$$\ldots(\because d\theta / dt = \omega)$$

Now differentiating equation (i), we get

$$\frac{d\,(\omega_{PC})}{d\theta} = \frac{d}{d\theta}\left[\frac{\omega \cos \theta}{(n^2 - \sin^2 \theta)^{1/2}}\right]$$

$$= \omega\left[\frac{(n^2 - \sin^2 \theta)^{1/2}(-\sin \theta)] - [(\cos\theta) \times \frac{1}{2}(n^2 - \sin^2 \theta)^{-1/2} \times -2 \sin \theta \cos \theta}{n^2 - \sin^2 \theta}\right]$$

$$= \omega\left[\frac{(n^2 - \sin^2 \theta)^{1/2}(-\sin \theta) + (n^2 - \sin^2 \theta)^{-1/2} \sin \theta \cos^2 \theta}{n^2 - \sin^2 \theta}\right]$$

$$= -\omega \sin\theta \left[\frac{(n^2 - \sin^2\theta)^{1/2} - (n^2 - \sin^2\theta)^{-1/2} \cos^2\theta}{n^2 - \sin^2\theta} \right]$$

$$= -\omega \sin\theta \left[\frac{(n^2 - \sin^2\theta) - \cos^2\theta}{(n^2 - \sin^2\theta)^{3/2}} \right] \quad \text{...[Dividing and multiplying by } (n^2 - \sin^2\theta)^{1/2}]$$

$$= \frac{-\omega\sin\theta}{(n^2 - \sin^2\theta)^{3/2}} \left[n^2 - (\sin^2\theta + \cos^2\theta) \right] = \frac{-\omega\sin\theta\,(n^2 - 1)}{(n^2 - \sin^2\theta)^{3/2}}$$

$$\text{...}(\because \sin^2\theta + \cos^2\theta = 1)$$

$$\therefore\ \alpha_{PC} = \frac{d(\omega_{PC})}{d\theta} \times \omega = \frac{-\omega^2 \sin\theta\,(n^2 - 1)}{(n^2 - \sin^2\theta)^{3/2}} \quad \text{...[From equation } (ii)] \quad \text{...}(iii)$$

The negative sign shows that the sense of the acceleration of the connecting rod is such that it tends to reduce the angle φ..

2.In a slider crank mechanism, the length of the crank and connecting rod are 150 mm and 600 mm respectively. The crank position is 60° from inner dead centre. The crank shaft speed is 450 r.p.m. (*clockwise*). *Using analytical method, determine: 1. Velocity and acceleration of the slider, and 2. Angular velocity and angular acceleration of the connecting rod.*

Solution. Given : r = 150 mm = 0.15 m ; l = 600 mm = 0.6 m ; θ = 60°; N = 400 r.p.m or ω = π × 450/60 = 47.13 rad/s

1. ***Velocity and acceleration of the slider***

We know that ratio of the length of connecting rod and crank, n = l / r = 0.6 / 0.15 = 4

∴ Velocity of the slider,

$$v_P = \omega r \left(\sin\theta + \frac{\sin 2\theta}{2n} \right) = 47.13 \times 0.15 \left(\sin 60^\circ + \frac{\sin 120^\circ}{2 \times 4} \right) \text{m/s}$$

$$= 6.9 \text{ m/s} \quad \textbf{Ans.}$$

and acceleration of the slider,

$$a_P = \omega^2 r \left(\cos\theta + \frac{\cos 2\theta}{n} \right) = (47.13)^2 \times 0.15 \left(\cos 60^\circ + \frac{\cos 120^\circ}{4} \right) \text{m/s}^2$$

$$= 124.94 \text{ m/s}^2 \quad \textbf{Ans.}$$

2. ***Angular velocity and angular acceleration of the connecting rod***

We know that angular velocity of the connecting rod,

$$\omega_{PC} = \frac{\omega\cos\theta}{n} = \frac{47.13 \times \cos 60^\circ}{4} = 5.9 \text{ rad/s} \quad \textbf{Ans.}$$

and angular acceleration of the connecting rod,

$$\alpha_{PC} = \frac{\omega^2 \sin\theta}{n} = \frac{(47.13)^2 \times \sin 60^\circ}{4} = 481 \text{ rad/s}^2 \quad \textbf{Ans.}$$

1.9 FORCES ON THE RECIPROCATING PARTS OF AN ENGINE, NEGLECTING THE WEIGHT OF THE CONNECTING ROD

The various forces acting on the reciprocating parts of a horizontal engine are shown in Fig. 15.8. The expressions for these forces, neglecting the weight of the connecting rod, may be derived as discussed below :

1. ***Piston effort***. It is the net force acting on the piston or crosshead pin, along the line of stroke. It is denoted by F_P in Fig. 15.8.

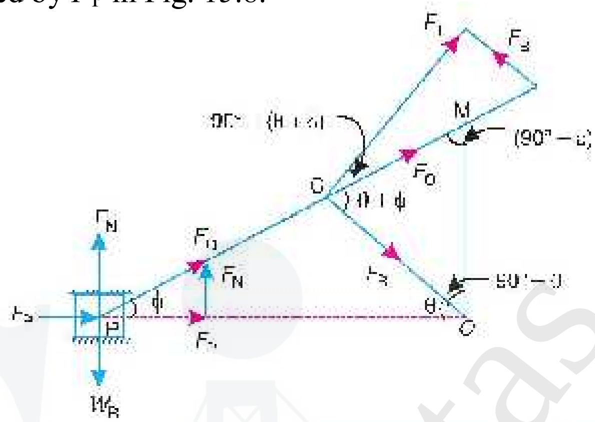


Fig. 15.8. Forces on the reciprocating parts of an engine.

Let m_R = Mass of the reciprocating parts, *e.g.* piston, crosshead pin or gudgeon pin etc., in kg, and

W_R = Weight of the reciprocating parts in newtons = $m_R.g$

We know that acceleration of the reciprocating parts,

$$a_R = a_P = \omega^2.r\left(\cos\theta + \frac{\cos 2\theta}{n}\right)$$

Accelerating force or inertia force of the reciprocating parts,

$$F_I = m_R.a_R = m_R.\omega^2.r\left(\cos\theta + \frac{\cos 2\theta}{n}\right)$$

It may be noted that in a horizontal engine, the reciprocating parts are accelerated from rest, during the latter half of the stroke (*i.e.* when the piston moves from inner dead centre to outer dead centre). It is, then, retarded during the latter half of the stroke (*i.e.* when the piston moves from outer dead centre to inner dead centre). The inertia force due to the acceleration of the reciprocating parts, opposes the force on the piston due to the difference of pressures in the cylinder on the two sides of the piston. On the other hand, the inertia force due to retardation of the reciprocating parts, helps the force on the piston.

Therefore,

Piston effort, F_P = Net load on the piston $\mp$ Inertia force

$= F_L \mp F_I$ ...(Neglecting frictional resistance)

$= F_L \mp F_I - R_F$ ...(Considering frictional resistance)

where R_F = Frictional resistance.

The −ve sign is used when the piston is accelerated, and +ve sign is used when the piston is retarded.

In a double acting reciprocating steam engine, net load on the piston,

$$F_L = p_1 A_1 - p_2 A_2 = p_1 A_1 - p_2 (A_1 - a)$$

where p_1, A_1 = Pressure and cross-sectional area on the back end side of the piston,

p_2, A_2 = Pressure and cross-sectional area on the crank end side of the piston,

a = Cross-sectional area of the piston rod.

2. ***Force acting along the connecting rod.*** It is denoted by F_Q in Fig. 15.8. From the geom-etry of the figure, we find that

$$F_Q = \frac{F_P}{\cos\varphi}$$

We know that $\cos\phi = \sqrt{1 - \frac{\sin^2 0}{n^2}}$

$$\therefore \quad F_Q = \frac{F_P}{\sqrt{1 - \frac{\sin^2\theta}{n^2}}}$$

3. ***Thrust on the sides of the cylinder walls or normal reaction on the guide bars.*** It is denoted by F_N in Fig. 15.8. From the figure, we find that

$$F_N = F_Q \sin\phi = \frac{F_P}{\cos\phi} \times \sin o = F_P \tan\phi \qquad \ldots\left[\because F_Q = \frac{F_P}{\cos\phi}\right]$$

4. ***Crank-pin effort and thrust on crank shaft bearings*.** The force acting on the connecting rod F_Q may be resolved into two components, one perpendicular to the crank and the other along the crank. The component of F_Q perpendicular to the crank is known as ***crank-pin effort*** and it is ***denoted by F_T*** in Fig. 15.8. The component of F_Q along the crank produces a thrust on the crank shaft bearings and it is denoted by F_B in Fig. 15.8.

Resolving F_Q perpendicular to the crank,

$$F_T = F_Q \sin(\theta + \phi) = \frac{F_P}{\cos\phi} \times \sin(\theta + o)$$

and resolving F_Q along the crank,

$$F_B = F_Q \cos(\theta + \phi) = \frac{F_P}{\cos\phi} \times \cos(\theta + \phi)$$

5. ***Crank effort*** or ***turning moment or torque on the crank shaft.*** The product of the crank-pin effort (F_T) and the crank pin radius (r) is known as ***crank effort*** or ***turning moment*** or ***torque on the crank shaft***. Mathematically,

Crank effort,
$$T = F_T \times r = \frac{F_P \sin(\theta + \phi)}{\cos\phi} \times r$$

$$= \frac{F_P (\sin\theta \cos\phi + \cos\theta \sin\phi)}{\cos\phi} \times r$$

$$= F_P \left(\sin\theta + \cos\theta \times \frac{\sin\phi}{\cos\phi} \right) \times r$$

$$= F_P (\sin\theta + \cos\theta \tan\phi) \times r \qquad ...(i)$$

We know that $l \sin\phi = r \sin\theta$

$$\sin\phi = \frac{r}{l} \sin\theta = \frac{\sin\theta}{n} \qquad ...\left(\because n = \frac{l}{r}\right)$$

and
$$\cos\phi = \sqrt{1 - \sin^2\phi} = \sqrt{1 - \frac{\sin^2\theta}{n^2}} = \frac{1}{n}\sqrt{n^2 - \sin^2\theta}$$

$\therefore$
$$\tan\phi = \frac{\sin\phi}{\cos\phi} = \frac{\sin\theta}{n} \times \frac{n}{\sqrt{n^2 - \sin^2\theta}} = \frac{\sin\theta}{\sqrt{n^2 - \sin^2\theta}}$$

Substituting the value of $\tan\phi$ in equation *(i)*, we have crank effort,

$$T = F_P \left(\sin\theta + \frac{\cos\theta \sin\theta}{\sqrt{n^2 - \sin^2\theta}} \right) \times r$$

$$= F_P \times r \left(\sin\theta + \frac{\sin 2\theta}{2\sqrt{n^2 - \sin^2\theta}} \right) \qquad ...(ii)$$

$$...(\because 2\cos\theta \sin\theta = \sin 2\theta)$$

3. *The crank-pin circle radius of a horizontal engine is 300 mm. The mass of the reciprocating parts is 250 kg. When the crank has travelled 60° from I.D.C., the difference between the driving and the back pressures is 0.35 N/mm^2. The connecting rod length between centres is 1.2 m and the cylinder bore is 0.5 m. If the engine runs at 250 r.p.m. and if the effect of piston rod diameter is neglected, calculate :* ***1.*** *pressure on slide bars,* ***2.*** *thrust in the connecting rod,* ***3.*** *tangential force on the crank-pin, and* ***4.*** *turning moment on the crank shaft.*

Solution. Given: r = 300 mm = 0.3 m ; m_R = 250 kg; θ = 60°; $p_1 - p_2$ = 0.35 N/mm^2; l =

1.2 m ; D = 0.5 m = 500 mm ; N = 250 r.p.m. or $\omega = 2\pi \times 250/60 = 26.2$ rad/s

First of all, let us find out the piston effort (F_P).

We know that net load on the piston,

$$F_L = (p_1 - p_2)\frac{\pi}{4} \times D^2 = 0.35 \times \frac{\pi}{4}(500)^2 = 68730 \text{ N}$$

...($\because$ Force = Pressure × Area)

Ratio of length of connecting rod and crank,

$$n = l / r = 1.2 / 0.3 = 4$$

and accelerating or inertia force on reciprocating parts,

$$F_I = m_R \cdot \omega^2 r \left(\cos\theta + \frac{\cos 2\theta}{n}\right)$$

$$= 250\,(26.2)^2\ 0.3\left(\cos 60° + \frac{\cos 120°}{4}\right) = 19306 \text{ N}$$

$\therefore$ Piston effort, $F_P = F_L - F_I = 68\,730 - 19\,306 = 49\,424 \text{ N} = 49.424 \text{ kN}$

1. *Pressure on slide bars*

Let φ = Angle of inclination of the connecting rod to the line of stroke.

We know that, $\sin\varphi = \frac{\sin\theta}{n} = \frac{\sin 60°}{4} = \frac{0.866}{4} = 0.2165$

$\therefore$ $\varphi = 12.5°$

We know that pressure on the slide bars,

$$F_N = F_P \tan\varphi = 49.424 \times \tan 12.5° = 10.96 \text{ kN} \quad \textbf{Ans.}$$

2. *Thrust in the connecting rod*

We know that thrust in the connecting rod,

$$F_Q = \frac{F_P}{\cos\varphi} = \frac{49.424}{\cos 12.5°} = 50.62 \text{ kN} \quad \textbf{Ans.}$$

3. *Tangential force on the crank-pin*

We know that tangential force on the crank pin,

$$F_T = F_Q \sin(\theta + \varphi) = 50.62 \sin(60° + 12.°5) = 48.28 \text{ kN} \quad \textbf{Ans.}$$

4. *Turning moment on the crank shaft*

We know that turning moment on the crank shaft,

$$T = F_T \cdot r = 48.28 \cdot 0.3 = 14.484 \text{ kN-m} \quad \textbf{Ans.}$$

4. *The crank and connecting rod of a petrol engine, running at 1800 r.p.m.are 50 mm and 200 mm respectively. The diameter of the piston is 80 mm and the mass of the reciprocat-ing parts is 1 kg. At a point during the power stroke, the pressure on the piston is 0.7 N/mm², when it has moved 10 mm from the inner dead centre. Determine :* **1.** *Net load on the gudgeon pin,* **2.** *Thrust in the connecting rod,* **3.** *Reaction between the piston and cylinder, and* **4.** *The engine speed at which the above values become zero.*

Solution. Given : $N = 1800$ r.p.m. or $\omega = 2\pi \times 1800/60 = 188.52$ rad/s ; $r = 50$ mm = 0.05 m; $l = 200$ mm ; $D = 80$ mm ; $m_R = 1$ kg ; $p = 0.7$ N/mm² ; $x = 10$ mm

1. ***Net load on the gudgeon pin***

We know that load on the piston,

$$F_L = \frac{\pi}{4} D^2 \times p = \frac{\pi}{4} \times (80)^2 \times 0.7 = 3520 \text{ N}$$

When the piston has moved 10 mm from the inner dead centre, *i.e.* when $P_1P = 10$ mm, the crank rotates from OC_1 to OC through an angle θ as shown in Fig. 15.10.

By measurement, we find that *θ = 33°.

We know that ratio of lengths of connecting rod and crank,

$$n = l/r = 200/50 = 4$$

and inertia force on the reciprocating parts,

$$F_I = m_R \cdot a_R = m_R \cdot \omega^2 \cdot r \left(\cos\theta + \frac{\cos 2\theta}{n} \right)$$

$$= 1 \times (188.52)^2 \times 0.05 \left(\cos 33° + \frac{\cos 66°}{4} \right) = 1671 \text{ N}$$

We know that net load on the gudgeon pin,

$$F_P = F_L - F_I = 3520 - 1671 = 1849 \text{ N Ans.}$$

2. ***Thrust in the connecting rod***

Let ϕ = Angle of inclination of the connecting rod to the line of stroke.

We know that, $\sin\phi = \frac{\sin\theta}{n} = \frac{\sin 33°}{4} = \frac{0.5446}{4} = 0.1361$

$\therefore$ $\phi = 7.82°$

We know that thrust in the connecting rod,

$$F_Q = \frac{F_P}{\cos \varphi} = \frac{1849}{\cos 7.82°} = 1866.3\text{N} \quad \textbf{Ans.}$$

3. ***Reaction between the piston and cylinder***

We know that reaction between the piston and cylinder,

$$F_N = F_P \tan \varphi = 1849 \tan 7.82° = 254 \text{ N} \quad \textbf{Ans.}$$

4. ***Engine speed at which the above values will become zero***

A little consideration will show that the above values will become zero, if the inertia force on the reciprocating parts (F_I) is equal to the load on the piston (F_L). Let ω_1 be the speed in rad/s, at which $F_I = F_L$.

$$\therefore \quad m_R (\omega_1)^2 r \left(\cos \theta + \frac{\cos 2\theta}{n} \right) = \frac{\pi}{4} D^2 \times p$$

$$1 (\omega_1)^2 \times 0.03 \left(\cos 33° + \frac{\cos 66°}{4} \right) = \frac{\pi}{4} \times (80)^2 \times 0.7 \quad \text{or} \quad 0.047 (\omega_1)^2 = 3520$$

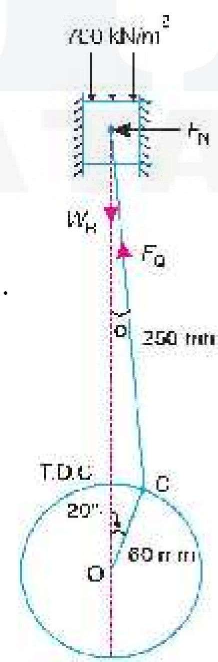
$$\therefore \quad (\omega_1)^2 = 3520 / 0.047 = 74\,894 \text{ or } \omega_1 = 273.6 \text{ rad/s}$$

$\therefore$ Corresponding speed in r.p.m.,

$$N_1 = 273.6 \times 60 / 2\pi = 2612 \text{ r.p.m.} \quad \textbf{Ans.}$$

5. *A vertical petrol engine 100 mm diameter and 120 mm stroke has a connecting rod 250 mm long. The mass of the piston is 1.1 kg. The speed is 2000 r.p.m. On the expansion stroke with a crank 20° from top dead centre, the gas pressure is 700 kN/m^2. Determine:*

1. *Net force on the piston,* ***2.*** *Resultant load on the gudgeon pin,* ***3.*** *Thrust on the cylinder walls, and* ***4.*** *Speed above which, other things remaining same, the gudgeon pin load would be reversed in direction.*



Solution. Given: $D = 100$ mm $= 0.1$ m ; $L = 120$ mm $= 0.12$ m or $r = L/2 = 0.06$ m ; $l = 250$ mm $= 0.25$ m ; $m_R = 1.1$ kg ; $N = 2000$ r.p.m. or $\omega = 2\pi \times 2000/60 = 209.5$ rad/s ; $\theta = 20°$; $p = 700$ kN/m^2

1. ***Net force on the piston***

The configuration diagram of a vertical engine is shown in Fig. 15.11.

We know that force due to gas pressure,

$$F_L = p \times \frac{\pi}{4} \times D^2 = 700 \times \frac{\pi}{4} \times (0.1)^2 = 5.5 \text{ kN} = 5500 \text{ N}$$

and ratio of lengths of the connecting rod and crank,

$$n = l/r = 0.25 / 0.06 = 4.17$$

$\therefore$ Inertia force on the piston, We know that for a vertical engine, net force on the piston,

$$F_P = F_L - F_I + W_R = F_L - F_I + m_R .g$$
$$= 5500 - 3254 + 1.1 \cdot 9.81 = 2256.8 \text{ N } \textbf{Ans.}$$

2. ***Resultant load on the gudgeon pin***

Let φ = Angle of inclination of the connecting rod to the line of stroke.

We know that,

$$\sin \varphi = \sin \theta / n = \sin 20°/4.17 = 0.082$$

$\therefore \varphi = 4.7°$

We know that resultant load on the gudgeon pin,

$$F_Q = \frac{F_P}{\cos \phi} = \frac{2256.8}{\cos 4.7°} = 2265 \text{ N}$$

3. ***Thrust on the cylinder walls***

We know that thrust on the cylinder walls,

$$F_N = F_P \tan \varphi = 2256.8 \cdot \tan 4.°7 = 185.5 \text{ N } \textbf{Ans.}$$

4. ***Speed, above which, the gudgeon pin load would be reversed in direction***

Let N_1 = Required speed, in r.p.m.

The gudgeon pin load *i.e.* F_Q will be reversed in direction, if F_Q becomes negative. This is only possible when F_P is negative. Therefore, for F_P to be negative, F_I must be greater than (F_L + W_R),

i.e. $$m_R (\omega_1)^2 r \left(\cos \theta + \frac{\cos 2\theta}{n}\right) > 5500 + 1.1 \times 9.81$$

$$1.1 \times (\omega_1)^2 \times 0.06 \left(\cos 20° + \frac{\cos 40°}{4.17}\right) > 5510.8$$

$$0.074 (\omega_1)^2 > 5510.8 \quad \text{or} \quad (\omega_1)^2 > 5510.8/0.074 \text{ or } 74\,470$$

or $$\omega_1 > 273 \text{ rad/s}$$

∴ Corresponding speed in r.p.m.,

$$N_1 > 273 \times 60/2\pi \quad \text{or} \quad 2606 \text{ r.p.m. } \textbf{Ans.}$$

6. *A horizontal steam engine running at 120 r.p.m. has a bore of 250 mm and a stroke of 400 mm. The connecting rod is 0.6 m and mass of the reciprocating parts is 60 kg. When the crank has turned through an angle of 45° from the inner dead centre, the steam pressure on the cover end side is 550 kN/m^2 and that on the crank end side is 70 kN/m^2. Considering the diameter of the piston rod equal to 50 mm, determine:*

1. *turning moment on the crank shaft,* ***2.*** *thrust on the bearings, and* ***3.*** *acceleration of the flywheel, if the power of the engine is 20 kW, mass of the flywheel 60 kg and radius of gyration 0.6 m.*

Solution. Given : N = 120 r.p.m. or $\omega = 2\pi \times 120/60 = 12.57$ rad/s ; D = 250 mm = 0.25 m ;
L = 400 mm = 0.4 m or $r = L/2 = 0.2$ m ; l = 0.6 m ; m_R = 60 kg ; $\theta = 45°$; d = 50 mm = 0.05 m ;
$p_1 = 550$ kN/m^2 = 550×10^3 N/m^2 ; $p_2 = 70$ kN/m^2 = 70×10^3 N/m^2

1. ***Turning moment on the crankshaft***

First of all, let us find the net load on the piston (F_P).

We know that area of the piston on the cover end side,

$$A_1 = \frac{\pi}{4} \times D^2 = \frac{\pi}{4} \times (0.25)^2 = 0.049 \text{ m}^2$$

and area of piston rod, $$a = \frac{\pi}{4} \times d^2 = \frac{\pi}{4} \times (0.05)^2 = 0.00196 \text{ m}^2$$

$\therefore$ Net load on the piston,

$$F_L = p_1 . A_1 - p_2 . A_2 = p_1 . A_1 - p_2 \, (A_1 - a)$$
$$= 550 \times 10^3 \times 0.049 - 70 \times 10^3 \, (0.049 - 0.001\,96) = 23\,657 \text{ N}$$

We know that ratio of lengths of the connecting rod and crank,

$$n = l/r = 0.6/0.2 = 3$$

and inertia force on the reciprocating parts,

$$F_I = m_R . \omega^2 . r \left(\cos\theta + \frac{\cos 2\theta}{n} \right)$$
$$= 60 \times (12.57)^2 \times 0.2 \left(\cos 45° + \frac{\cos 90°}{3} \right) = 1340 \text{ N}$$

$\therefore$ Net force on the piston or piston effort,

$$F_P = F_L - F_I = 23\,657 - 1340 = 22\,317 \text{ N} = 22.317 \text{ kN}$$

Let ϕ = Angle of inclination of the connecting rod to the line of stroke.

We know that, $\sin\phi = \sin\theta/n = \sin 45°/3 = 0.2357$

$\therefore$ $\phi = 13.6°$

We know that turning moment on the crankshaft,

$$T = \frac{F_P \sin(\theta + \phi)}{\cos\phi} \times r = \frac{22.317 \times \sin(45° + 13.6°)}{\cos 13.6°} \times 0.2 \text{ kN-m}$$
$$= 3.92 \text{ kN-m} = 3920 \text{ N-m} \quad \textbf{Ans.}$$

2. ***Thrust on the bearings***

We know that thrust on the bearings,

$$F_B = \frac{F_P \cos(\theta + \phi)}{\cos\phi} = \frac{22.317 \times \cos(45° + 13.6°)}{\cos 13.6°} = 11.96 \text{ kN} \quad \textbf{Ans.}$$

3. ***Acceleration of the flywheel***

Given: P = 20 kW = 20×10^3 W; m = 60 kg ; k = 0.6 m

Let α = Acceleration of the flywheel in rad/s^2.

We know that mass moment of inertia of the flywheel,

$$I = m.k^2 = 60 \times (0.6)^2 = 21.6 \text{ kg-m}^2$$

$\therefore$ Accelerating torque, $T_A = I.\alpha = 21.6\ \alpha$ N-m ...(*i*)

and resisting torque, $$T_R = \frac{P \times 60}{2\pi N} = \frac{20 \times 10^3 \times 60}{2\pi \times 120} = 1591 \text{ N-m} \qquad \left(\because P = \frac{2\pi NT}{60}\right)$$

Since the accelerating torque is equal to the difference of torques on the crankshaft or turning moment (T) and the resisting torque (T_R), therefore, accelerating torque,

$$T_A = T - T_R = 3920 - 1591 = 2329 \text{ N-m} \qquad ...(ii)$$

From equation (*i*) and (*ii*),

$$\alpha = 2329/21.6 = 107.8 \text{ rad/s}^2 \quad \textbf{Ans.}$$

1.10 EQUIVALENT DYNAMICAL SYSTEM

In order to determine the motion of a rigid body, under the action of external forces, it is usually convenient to replace the rigid body by two masses placed at a fixed distance apart, in such a way that,

1. the sum of their masses is equal to the total mass of the body ;
2. the centre of gravity of the two masses coincides with that of the body ; and
3. the sum of mass moment of inertia of the masses about their centre of gravity is equal to the mass moment of inertia of the body.

When these three conditions are satisfied, then it is said to be an ***equivalent dynamical system.*** Consider a rigid body, having its centre of gravity at G, as shown in Fig. 15.14.

Let m = Mass of the body,

k_G = Radius of gyration about its centre of gravity G,

m_1 and m_2 = Two masses which form a dynamical equivalent system,

l_1 = Distance of mass m_1 from G,

l_2 = Distance of mass m_2 from G,

$$m_1 + m_2 = m \qquad ...(i)$$

$$m_1.l_1 = m_2.l_2 \qquad ...(ii)$$

$$m_1 (l_1)^2 + m_2 (l_2)^2 = m (k_G)^2 \qquad ...(iii)$$

From equations (*i*) and (*ii*),

$$m_1 = \frac{l_2.m}{l_1 + l_2} \qquad ...(iv)$$

and

$$m_2 = \frac{l_1.m}{l_1 + l_2} \qquad ...(v)$$

Substituting the value of m_1 and m_2 in equation (*iii*), we have

$$\frac{l_2.m}{l_1 + l_2}(l_1)^2 + \frac{l_1.m}{l_1 + l_2}(l_2)^2 = m (k_G)^2 \quad \text{or} \quad \frac{l_1 l_2 (l_1 + l_2)}{l_1 + l_2} = (k_G)^2$$

$$\therefore \quad l_1 l_2 = (k_G)^2 \qquad ...(vi)$$

This equation gives the essential condition of placing the two masses, so that the system becomes dynamical equivalent. The distance of one of the masses (*i.e.* either l_1 or l_2) is arbitrary chosen and the other distance is obtained from equation **(*vi*).**

7 *A connecting rod is suspended from a point 25 mm above the centre of small end, and 650 mm above its centre of gravity, its mass being 37.5 kg. When permitted to oscil- late, the time period is found to be 1.87 seconds. Find the dynamical equivalent system constituted of two masses, one of which is located at the small end centre.*

Solution. Given : h = 650 mm = 0.65 m ; l_1 = 650 – 25 = 625 mm = 0.625 m ; m = 37.5 kg ; t_p = 1.87 s

First of all, let us find the radius of gyration (k_G) of the connect- ing rod (considering it is a compound pendulum), about an axis passing through its centre of gravity, G.

We know that for a compound pendulum, time period of oscillation (t_p),

$$1.87 = 2\pi\sqrt{\frac{(k_G)^2 + h^2}{g.h}} \quad \text{or} \quad \frac{1.87}{2\pi} = \sqrt{\frac{(k_G)^2 + (0.65)^2}{9.81 \times 0.65}}$$

Squaring both sides, we have

$$0.0885 = \frac{(k_G)^2 + 0.4225}{6.38}$$

$$(k_G)^2 = 0.0885 \times 6.38 - 0.4225 = 0.1425 \text{ m}^2$$

$$\therefore \quad k_G = 0.377 \text{ m}$$

It is given that one of the masses is located at the small end centre. Let the other mass is located at a distance l_2 from the centre of gravity G, as shown in Fig. 15.19. We know that, for a dynamically equivalent system,

$$l_1.l_2 = (k_G)^2$$

$$\therefore \quad l_2 = \frac{(k_G)^2}{l_1} = \frac{0.1425}{0.625} = 0.228 \text{ m}$$

Let m_1 = Mass placed at the small end centre A, and

m_2 = Mass placed at a distance l_2 from G, *i.e.* at B.

We know that, for a dynamically equivalent system,

$$m_1 = \frac{l_2.m}{l_1 + l_2} = \frac{0.228 \times 37.5}{0.625 + 0.228} = 10 \text{ kg} \quad \textbf{Ans.}$$

and

$$m_2 = \frac{l_1.m}{l_1 + l_2} = \frac{0.625 \times 37.5}{0.625 + 0.228} = 27.5 \text{ kg} \quad \textbf{Ans.}$$

1.11 CORRECTION COUPLE TO BE APPLIED TO MAKE TWO MASS SYSTEM DYNAMICALLY EQUIVALENT

In Art. 15.11, we have discussed the conditions for equivalent dynamical system of two bodies. A little consideration will show that when two masses are placed arbitrarily*, then the conditions (*i*) and (*ii*) as given in Art. 15.11 will only be satisfied. But the condition (*iii*) is not possible to satisfy. This means that the mass moment of inertia of these two masses placed arbitrarily, will differ than that of mass moment of inertia of the rigid body.

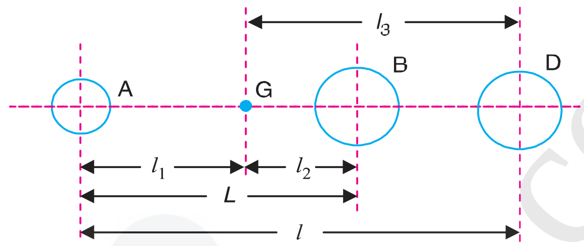


Fig. 15.21. Correction couple to be applied to make the two-mass system dynamically equivalent.

Consider two masses, one at A and the other at D be placed arbitrarily, as shown in Fig. 15.21. Let l_3 = Distance of mass placed at D from G,

I_1 = New mass moment of inertia of the two masses;

k_1 = New radius of gyration;

α = Angular acceleration of the body;

I = Mass moment of inertia of a dynamically equivalent system;

k_G = Radius of gyration of a dynamically equivalent system. We know that the torque required to accelerate the body,

$$T = I.\alpha = m\,(k_G)^2\,\alpha \quad \text{...}(i)$$

Similarly, the torque required to accelerate the two-mass system placed arbitrarily,

$$T_1 = I_1.\alpha = m\,(k_1)^2\,\alpha \quad \text{...}(ii)$$

∴ Difference between the torques required to accelerate the two-mass system and the torque required to accelerate the rigid body,

$$T' = T_1 - T = m\,(k_1)^2\,\alpha - m\,(k_G)^2\,\alpha = m\,[(k_1)^2 - (k_G)^2]\,\alpha \quad \text{...}(iii)$$

The difference of the torques T' is known as ***correction couple***. This couple must be applied, when the masses are placed arbitrarily to make the system dynamical equivalent. This, of course, will satisfy the condition (*iii*)

8.A connecting rod of an I.C. engine has a mass of 2 kg and the distance between the centre of gudgeon pin and centre of crank pin is 250 mm. The C.G. falls at a point 100 mm from the gudgeon pin along the line of centres. The radius of gyration about an axis through the C.G. perpendicular to the plane of rotation is 110 mm. Find the equivalent dynamical system if only one of the masses is located at gudgeon pin.

If the connecting rod is replaced by two masses, one at the gudgeon pin and the other at the crank pin and the angular acceleration of the rod is 23 000 rad/s^2 clockwise, determine the correc- tion couple applied to the system to reduce it to a dynamically equivalent system.

Solution. Given : $m = 2$ kg ; $l = 250$ mm $= 0.25$ m ; $l_1 = 100$ mm $= 0.1$m ; $k_G = 110$ mm $= 0.11$ m ; $\alpha = 23\ 000$ rad/s^2

Equivalent dynamical system

It is given that one of the masses is located at the gudgeon pin. Let the other mass be located at a distance l_2 from the centre of gravity. We know that for an equivalent dynamical system.

$$l_1 l_2 = (k_G)^2 \quad \text{or} \quad l_2 = \frac{(k_G)^2}{l_1} = \frac{(0.11)^2}{0.1} = 0.121 \text{ m}$$

Let m_1 = Mass placed at the gudgeon pin, and

m_2 = Mass placed at a distance l_2 from $C.G.$

We know that
$$m_1 = \frac{l_2 . m}{l_1 + l_2} = \frac{0.121 \times 2}{0.1 + 0.121} = 1.1 \text{ kg} \quad \textbf{Ans.}$$

and
$$m_2 = \frac{l_1 . m}{l_1 + l_2} = \frac{0.1 \times 2}{0.1 + 0.121} = 0.9 \text{ kg} \quad \textbf{Ans.}$$

Correction couple

Since the connecting rod is replaced by two masses located at the two centres (*i.e.* one at the gudgeon pin and the other at the crank pin), therefore,

$$l = 0.1 \text{ m}, \quad \text{and} \quad l_3 = l - l_1 = 0.25 - 0.1 = 0.15 \text{ m}$$

Let k_1 = New radius of gyration.

We know that $(k_1)^2 = l_1 . l_3 = 0.1 \times 0.15 = 0.015 \text{ m}^2$

$\therefore$ Correction couple,

$$T' = m\left(k_1^2 - k_G^2\right)\alpha = 2\left[0.015 - (0.11)^2\right] 23\ 000 = 133.4 \text{ N-m} \quad \textbf{Ans.}$$

1.12 INERTIA FORCES IN A RECIPROCATING ENGINE, CONSIDERING THE WEIGHT OFCONNECTING ROD

In a reciprocating engine, let *OC* be the crank and *PC*, the connecting rod whose centre of gravity lies at *G*. The inertia forces in a reciprocating engine may be obtained graphically as discussed below:

1. First of all, draw the acceleration diagram *OCQN* by Klien's construction. We know that the acceleration of the piston *P* with respect to *O*,

$$a_{PO} = a_P = \omega^2 \times NO,$$

acting in the direction from *N* to *O*. Therefore, the inertia force F_I of the reciprocating parts will act in the opposite direction as shown in Fig. 15.22.

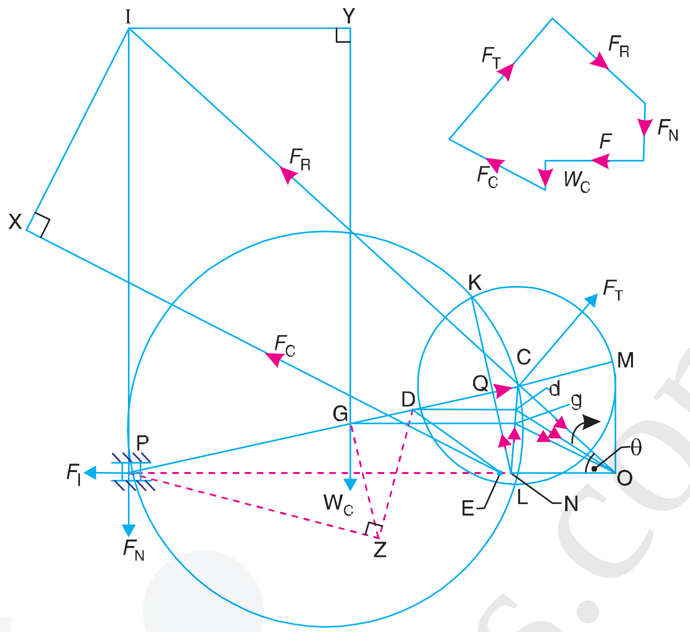


Fig. 15.22. Inertia forces is reciprocating engine, considering the weight of connecting rod.

2. Replace the connecting rod by dynamically equivalent system of two masses as discussed in Art. 15.12. Let one of the masses be arbitrarily placed at *P*. To obtain the position of the other mass, draw *GZ* perpendicular to *CP* such that *GZ* = *k*, the radius of gyration of the connecting rod. Join *PZ* and from *Z* draw perpendicular to *DZ* which intersects *CP* at *D*. Now, *D* is the position of the second mass.

Note: The position of the second mass may also be obtained from the equation,

$$GP \times GD = k^2$$

3. Locate the points *G* and *D* on *NC* which is the acceleration image of the connecting rod. This is done by drawing parallel lines from *G* and *D* to the line of stroke *PO*. Let these parallel lines intersect *NC* at *g* and *d* respectively. Join *gO* and *dO*. Therefore, acceleration of *G* with respect to *O*, in the direction from *g* to *O*,

$$a_{GO} = a_G = \omega^2 \times gO$$

and acceleration of *D* with respect to *O*, in the direction from *d* to *O*,

$$a_{DO} = a_D = \omega^2 \times dO$$

4. From *D*, draw *DE* parallel to *dO* which intersects the line of stroke *PO* at *E*. Since the accelerating forces on the masses at *P* and *D* intersect at *E*, therefore their resultant must also pass through *E*. But their resultant is equal to the accelerang force on the rod, so that the line of action of the accelerating force on the rod, is given by a line drawn through *E* and parallel to *gO*, in the direc- tion from *g* to *O*. The inertia force of the connecting rod F_C therefore acts through *E* and in the opposite direction as shown in Fig. 15.22. The inertia force of the connecting rod is given by

$$F_C = m_C \times \omega^2 \times gO \qquad \text{...}(i)$$

where m_C = Mass of the connecting rod.

A little consideration will show that the forces acting on the connecting rod are :

(*a*) Inertia force of the reciprocating parts (F_I) acting along the line of stroke *PO*,

(*b*) The side thrust between the crosshead and the guide bars (F_N) acting at *P* and right angles to line of stroke *PO*,

(c) The weight of the connecting rod

$(W_C = m_C.g)$,

(d) Inertia force of the connecting rod (F_C),

(e) The radial force (F_R) acting through *O* and parallel to the crank *OC*,

(f) The force (F_T) acting perpen- dicular to the crank *OC*.

Now, produce the lines of action of F_R and F_N to intersect at a point *I*, known as instantaneous centre. From *I* draw *I X* and *I Y* , perpendicular to the lines of action of F_C and W_C. Taking moments about *I*, we have

$$F_T \times IC = F_I \times IP + F_C \times I\,X + W_C \times I\,Y \qquad ...(ii)$$

The value of F_T may be obtained from this equation and from the force polygon as shown in Fig. 15.22, the forces F_N and F_R may be calculated. We know that, torque exerted on the crankshaft to overcome the inertia of the moving parts = $F_T \times OC$

1.12.1 Analytical Method for Inertia Torque

The effect of the inertia of the connecting rod on the crankshaft torque may be obtained as discussed in the following steps:

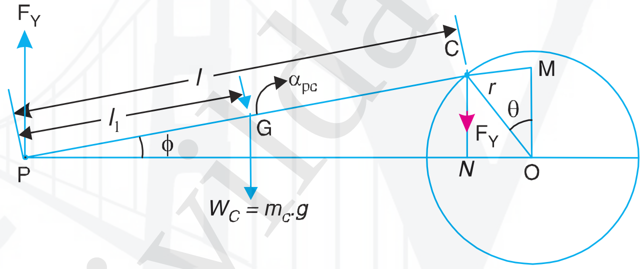


Fig. 15.23. Analytical method for inertia torque.

1. The mass of the connecting rod (m_C) is divided into two masses. One of the mass is placed at the crosshead pin *P* and the other at the crankpin *C* as shown in Fig. 15.23, so that the centre of gravity of these two masses coincides with the centre of gravity of the rod *G*.

2. Since the inertia force due to the mass at *C* acts radially outwards along the crank *OC*, therefore the mass at *C* has no effect on the crankshaft torque.

3. The inertia force of the mass at *P* may be obtained as follows: Let m_C = Mass of the connecting rod,

l = Length of the connecting rod,

l_1 = Length of the centre of gravity of the connecting rod from *P*.

∴ Mass of the connecting rod at P,

$$= \frac{l - l_1}{l} \times m_C$$

The mass of the reciprocating parts (m_R) is also acting at P. Therefore,

Total equivalent mass of the reciprocating parts acting at P

$$= m_R + \frac{l - l_1}{l} \times m_C$$

∴ Total inertia force of the equivalent mass acting at P,

$$F_I = \left(m_R + \frac{l - l_1}{l} \times m_C \right) a_R \qquad (i)$$

where a_R = Acceleration of the reciprocating parts

$$= \omega^2 r \left(\cos\theta + \frac{\cos 2\theta}{n} \right)$$

$$\therefore \quad F_I = \left[m_R + \frac{l - l_1}{l} \times m_C \right] \omega^2 r \left(\cos\theta + \frac{\cos 2\theta}{n} \right)$$

and corresponding torque exerted on the crank shaft,

$$T_I = F_I \times OM = F_I r \left(\sin\theta + \frac{\sin 2\theta}{2\sqrt{n^2 - \sin^2\theta}} \right) \qquad \ldots(ii)$$

4. In deriving the equation ***(ii)*** of the torque exerted on the crankshaft, it is assumed that one of the two masses is placed at C and the other at P. This assumption does not satisfy the condition for kinetically equivalent system of a rigid bar. Hence to compensate for it, a correcting torque is neces- sary whose value is given by

$$T' = m_C \left[(k_1)^2 - (k_G)^2 \right] \alpha_{PC} = m_C\, l_1\, (l - L)\, \alpha_{PC}$$

where L = Equivalent length of a simple pendulum when swung about an axis through P

$$= \frac{(k_G)^2 + (l_1)^2}{l_1}$$

α_{PC} = Angular acceleration of the connecting rod PC

$$= \frac{-\omega^2 \sin\theta}{n} \qquad \ldots\text{(From Art. 15.9)}$$

The correcting torque T' may be applied to the system by two equal and opposite forces F_Y acting through P and C. Therefore,

$$F_Y \times PN = T' \quad \text{or} \quad F_Y = T'/PN$$

and corresponding torque on the crankshaft,

$$T_C = F_Y \times NO = \frac{T'}{PN} \times NO \qquad \text{...}(iii)$$

We know that, $NO = OC \cos\theta = r\cos\theta$

and $PN = PC\cos\phi = l\cos\phi$

$$\therefore \quad \frac{NO}{PN} = \frac{r\cos\theta}{l\cos\phi} = \frac{\cos\theta}{n\cos\phi} \qquad \text{...}\left(\because n = \frac{l}{r}\right)$$

$$= \frac{\cos\theta}{n\sqrt{1-\frac{\sin^2\theta}{n^2}}} = \frac{\cos\theta}{\sqrt{n^2-\sin^2\theta}} \qquad \left(\because \cos\phi = \sqrt{1-\frac{\sin^2\theta}{n^2}}\right)$$

Since $\sin^2\theta$ is very small as compared to n^2, therefore neglecting $\sin^2\theta$, we have

$$\frac{NO}{PN} = \frac{\cos\theta}{n}$$

Substituting this value in equation *(iii)*, we have

$$T_C = T' \times \frac{\cos\theta}{n} = m_C \times l_1 (l-L)\,\alpha_{PC} \times \frac{\cos\theta}{n}$$

$$= -m_C \times l_1 (l-L)\frac{\omega^2\sin\theta}{n} \times \frac{\cos\theta}{n} \qquad \text{...}\left(\because \alpha_{PC} = \frac{-\omega^2\sin\theta}{n}\right)$$

$$= -m_C \times l_1 (l-L)\frac{\omega^2\sin 2\theta}{2n^2} \qquad \text{...}(\because 2\sin\theta\cos\theta = \sin 2\theta)$$

5. The equivalent mass of the rod acting at C,

$$m_2 = m_C \times \frac{l_1}{l}$$

$\therefore$ Torque exerted on the crank shaft due to mass m_2,

$$T_W = -m_2 \times g \times NO = -m_C \times g \times \frac{l_1}{l} \times NO = -m_C \times g \times \frac{l_1}{l} \times r\cos\theta$$

$$\text{...}(\because NO = r\cos\theta)$$

$$= -m_C \times g \times \frac{l_1}{n} \times \cos\theta \qquad \text{...}(\because l/r = n)$$

9. *The crank and connecting rod lengths of an engine are 125 mm and 500 mm respectively. The mass of the connecting rod is 60 kg and its centre of gravity is 275 mm from the crosshead pin centre, the radius of gyration about centre of gravity being 150 mm.*

If the engine speed is 600 r.p.m. for a crank position of 45° from the inner dead centre, determine, using Klien's or any other construction ***1.*** *the acceleration of the piston;* ***2.*** *the magni- tude, position and direction of inertia force due to the mass of the connecting rod.*

Solution. Given : $r = OC = 125$ mm ; $l = PC = 500$ mm; $m_C = 60$ kg ; $PG = 275$ mm ;
$m_C = 60$ kg ; $PG = 275$ mm ; $k_G = 150$ mm ; $N = 600$ r.p.m. or $\omega = 2\pi \times 600/60 = 62.84$ rad/s ; $\theta = 45°$

1. ***Acceleration of the piston***

Let a_P = Acceleration of the piston.

First of all, draw the configuration diagram *OCP*, as shown in Fig. 15.24, to some suitable scale, such that

$$OC = r = 125 \text{ mm} ; PC = l = 500 \text{ mm} ; \text{and } \theta = 45°.$$

Now, draw the Klien's acceleration diagram *OCQN*, as shown in Fig. 15.24, in the same manner as already discussed. By measurement,

$$NO = 90 \text{ mm} = 0.09 \text{ m}$$

∴ Acceleration of the piston,

$$a_P = \omega^2 \times NO = (62.84)^2 \times 0.09 = 355.4 \text{ m/s} \quad \textbf{Ans.}$$

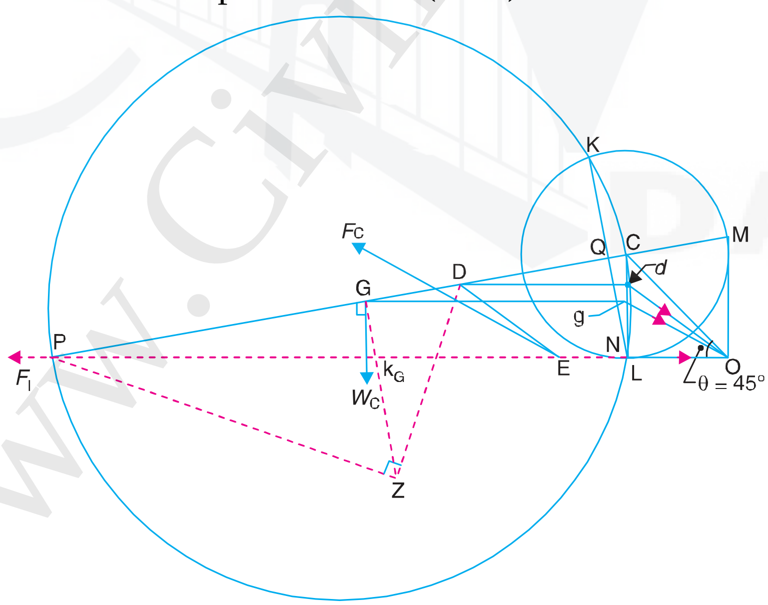


Fig. 15.24

2. ***The magnitude, position and direction of inertia force due to the mass of the connecting rod***

The magnitude, postition and direction of the inertia force may be obtained as follows:

(*i*) Replace the connecting rod by dynamical equivalent system of two masses, assuming that one of the masses is placed at *P* and the other mass at *D*. The position of the point *D* is obtained as discussed in Art. 15.12.

(*ii*) Locate the points *G* and *D* on *NC* which is the acceleration image of the connecting rod. Let these points are *g* and *d* on *NC*. Join *gO* and *dO*. By measurement,

$$gO = 103 \text{ mm} = 0.103 \text{ m}$$

∴ Acceleration of *G*, $a_G = \omega^2 \times gO$, acting in the direction from *g* to *O*.

(*iii*) From point *D*, draw *DE* parallel to *dO*. Now *E* is the point through which the inertia force of the connecting rod passes. The magnitude of the inertia force of the connecting rod is given by

$$F_C = m_C \times \omega^2 \times gO = 60 \times (62.84)^2 \times 0.103 = 24\ 400 \text{ N} = 24.4$$

kN **Ans.** **(*iv*)** From point *E*, draw a line parallel to *gO*, which shows the position of the inertia force of the connecting rod and acts in the opposite direction of *gO*.

10. *The following data refer to a steam engine:*

Diameter of piston = 240 mm; stroke = 600 mm ; length of connecting rod = 1.5 m ; mass of reciprocating parts = 300 kg; mass of connecting rod = 250 kg; speed = 125 r.p.m ; centre of gravity of connecting rod from crank pin = 500 mm ; radius of gyration of the connecting rod about an axis through the centre of gravity = 650 mm.

Determine the magnitude and direction of the torque exerted on the crankshaft when the crank has turned through 30° from inner dead centre.

Solution. Given : D = 240 mm = 0.24 m ; L = 600 mm or $r = L/2$ = 300 mm = 0.3 m ; l = 1.5 m ; m_R = 300 kg ; m_C = 250 kg ; N = 125 r.p.m. or $\omega = 2\pi \times 125/60$ = 13.1 rad/s ; GC = 500 mm = 0.5 m ; k_G = 650 mm = 0.65 m ; $\theta = 30°$

The inertia torque on the crankshaft may be determined by graphical method or analytical method as discussed below:

1. *Graphical method*

First of all, draw the configuration diagram *OCP*, as shown in Fig. 15.25, to some suitable scale, such that

$$OC = r = 300 \text{ mm} ; PC = l = 1.5 \text{ m} ; \text{and angle } POC = \theta = 30°.$$

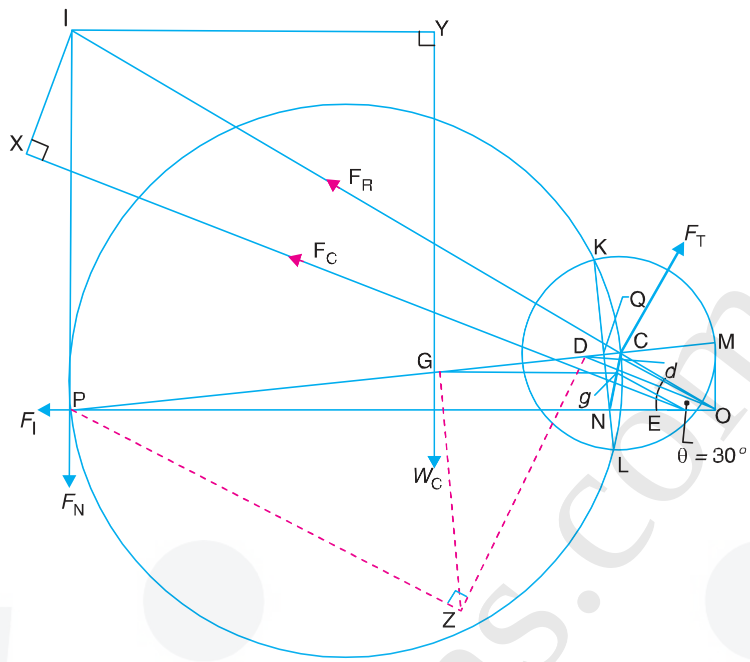


Fig. 15.25

Now draw the Klien's acceleration diagram *OCQN*, as shown in Fig. 15.25, and complete the figure in the similar manner as discussed in Art. 15.14.

By measurement; $NO = 0.28$ m ; $gO = 0.28$ m ; $IP = 1.03$ m ; $IX = 0.38$ m ; $IY = 0.98$ m, and $IC = 1.7$ m.

We know that inertia force of reciprocating parts,

$$F_I = m_R \times \omega^2 \times NO = 300 \times (13.1)^2 \times 0.28 = 14\ 415 \text{ N}$$

and inertia force of connecting rod,

$$F_C = m_C \times \omega^2 \times gO = 250 \times (13.1)^2 \times 0.28 = 12\ 013 \text{ N}$$

Let F_T = Force acting perpendicular to the crank *OC*.

Taking moments about point *I*.

$$F_T \times IC = F_I \times IP + W_C \times IY + F_C \times IX$$

$$F_T \times 1.7 - 14\ 415 \times 1.03 + 250 \times 9.81 \times 0.98 + 12013 \times 0.38 - 21816$$

$$\therefore \quad F_T = 2.816 / 1.7 = 12\ 833 \text{ N} \qquad \ldots(\because W_C = m_C.g)$$

We know that torque exerted on the crankshaft

$$= F_T \times r = 12\ 833 \times 0.3 = 3850 \text{ N-m } \textbf{Ans.}$$

11. *The connecting rod of an internal combustion engine is 225 mm long and has a mass 1.6 kg. The mass of the piston and gudgeon pin is 2.4 kg and the stroke is 150 mm. The cylinder bore is 112.5 mm. The centre of gravity of the connection rod is 150 mm from the small end. Its radius of gyration about the centre of gravity for oscillations in the plane of swing of the connect- ing rod is 87.5 mm. Determine the magnitude and direction of the resultant force on the crank pin when the crank is at 40° and the piston is moving away from inner dead centre under an effective gas presure of 1.8 MN/m^2. The engine speed is 1200 r.p.m.*

Solution. Given : $l = PC = 225$ mm $= 0.225$ m; $m_C = 1.6$ kg; $m_R = 2.4$ kg; $L = 150$ mm or $r = L/2 = 75$ mm $= 0.075$ m ; $D = 112.5$ mm $= 0.1125$ m ; $PG = 150$ mm ; $k_G = 87.5$ mm $= 0.0875$ m ; $\theta = 40°$; $p = 1.8$ MN/m^2 $= 1.8 \times 10^6$ N/m^2 ; $N = 1200$ r.p.m. or $\omega = 2\pi \times 1200/60 = 125.7$ rad/s

First of all, draw the configuration diagram *OCP*, as shown in Fig. 15.27 to some suitable scale, such that $OC = r = 75$ mm ; $PC = l = 225$ mm ; and $\theta = 40°$.

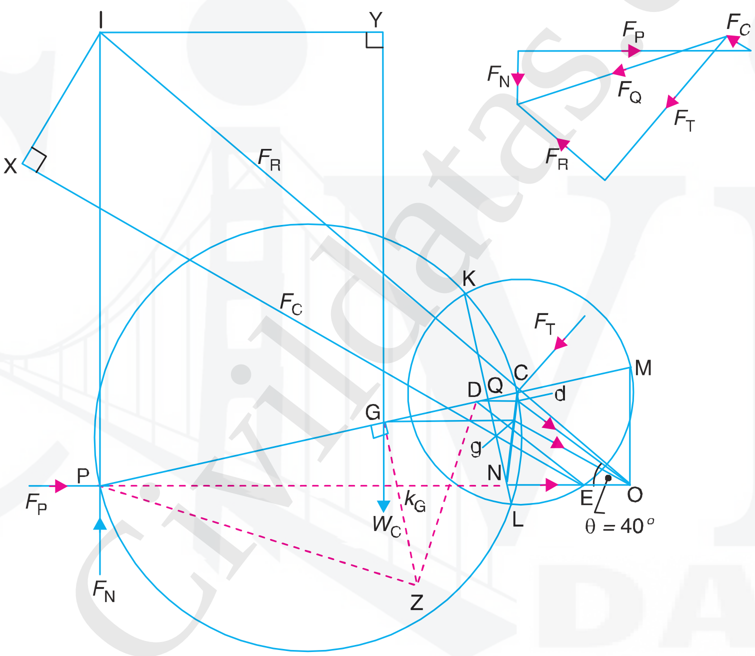


Fig. 15.27

Now, draw the Klien's acceleration diagram *OCQN*. Complete the diagram in the same manner as discussed earlier. By measurement,

$NO = 0.0625$ m ; $gO = 0.0685$ m ; $IC = 0.29$ m ; $IP = 0.24$ m ; $IY = 0.148$ m ; and $IX = 0.08$ m

We know that force due to gas pressure,

$$F_L = \frac{\pi}{4} \times D^2 \times p = \frac{\pi}{4} \times (0.1125)^2 \times 1.8 \times 10^6 = 17\ 895 \text{ N}$$

Inertia force due to mass of the reciprocating parts,

$$F_I = m_R \times \omega^2 \times NO = 2.4\ (125.7)^2 \times 0.0625 = 2370 \text{ N}$$

∴ Net force on the piston,

$$F_P = F_L - F_I = 17\ 895 - 2370 = 15\ 525 \text{ N}$$

Inertia force due to mass of the connecting rod,

$$F_C = m_C \times \omega^2 \times gO = 1.6 \times (125.7)^2 \times 0.0685 = 1732 \text{ N}$$

Let F_T = Force acting perpendicular to the crank OC.

Now, taking moments about point I,

$$F_P \times IP = W_C \times IY + F_C \times IX + F_T \times IC$$

$$15\,525 \times 0.24 = 1.6 \times 9.81 \times 0.148 + 1732 \times 0.08 + F_T \times 0.29$$

$$\therefore \quad F_T = 12\,362 \text{ N} \quad \ldots(\because W_C = m_C.g)$$

Let us now find the values of F_N and F_R in magnitude and direction. Draw the force polygon as shown in Fig. 15.25.

By measurement, F_N = 3550 N; and F_R = 7550 N

The magnitude and direction of the resultant force on the crank pin is given by F_Q , which is the resultant of F_R and F_T.

By measurement, F_Q = 13 750 N **Ans.**

1.13 TURNING MOMENT DIAGRAM

The turning moment diagram (also known as ***crank-effort diagram***) is the graphical representation of the turning moment or crank-effort for various positions of the crank. It is plotted on cartesian co-ordinates, in which the turning moment is taken as the ordinate and crank angle as abscissa

1.13.1 Turning Moment Diagram for a Single Cylinder Double Acting Steam Engine

A turning moment diagram for a single cylinder double acting steam engine is shown in Fig. 16.1. The vertical ordinate represents the turning moment and the horizontal ordinate represents the crank angle.

We have discussed in Chapter 15 (Art. 15.10.) that the turning moment on the crankshaft,

$$T = F_P \times r\left(\sin\theta + \frac{\sin 2\theta}{2\sqrt{n^2 - \sin^2\theta}}\right)$$

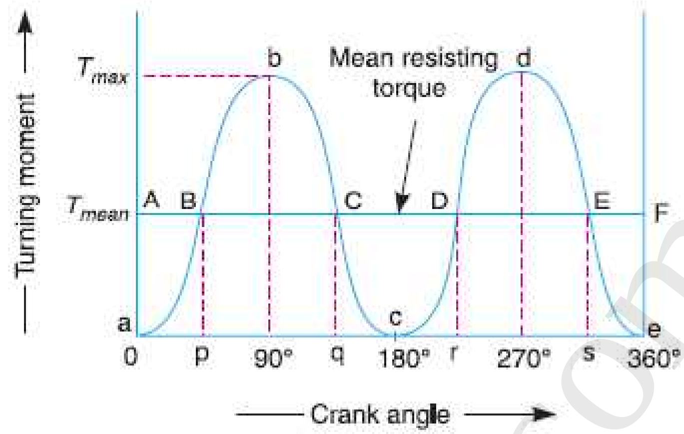


Fig. 16.1. Turning moment diagram for a single cylinder, double acting steam engine.

where F_p = Piston effort,

r = Radius of crank,

n = Ratio of the connecting rod length and radius of crank, and

θ = Angle turned by the crank from inner dead centre.

is maximum when the crank angle is 90° and it is again zero when crank angle is 180°.

This is shown by the curve *abc* in Fig. 16.1 and it represents the turning moment diagram for outstroke. The curve *cde* is the turning moment diagram for instroke and is somewhat similar to the curve *abc*.

Since the work done is the product of the turning moment and the angle turned, therefore the area of the turning moment diagram represents the work done per revolution. In actual practice, the engine is assumed to work against the mean resisting torque, as shown by a horizontal line *AF*. The height of the ordinate *a A* represents the mean height of the turning moment diagram. Since it is assumed that the work done by the turning moment per revolution is equal to the work done against the mean resisting torque, therefore the area of the rectangle *aAFe* is proportional to the work done against the mean resisting torque.

1.13.2 Turning Moment Diagram for a Four Stroke Cycle Internal Combustion Engine

A turning moment diagram for a four stroke cycle internal combustion engine is shown in Fig. 16.2. We know that in a four stroke cycle internal combustion engine, there is one working stroke after the crank has turned through two revolutions, *i.e.* 720° (or 4 π radians).

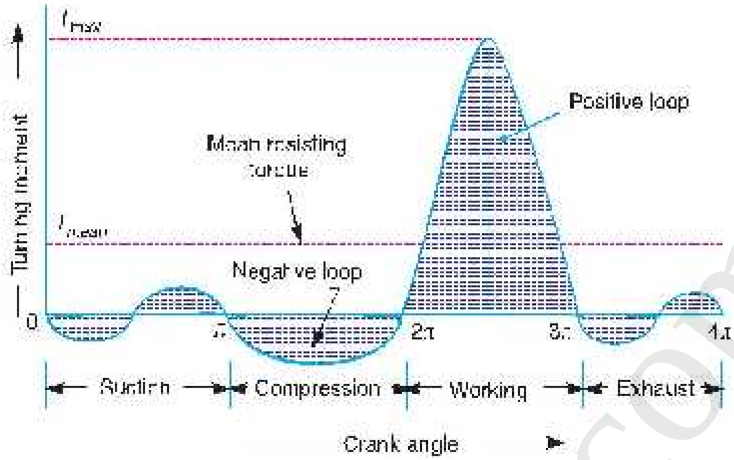
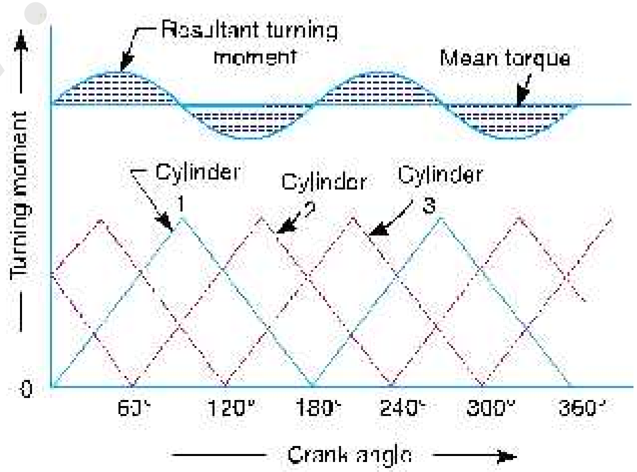


Fig. 16.2. Turning moment diagram for a four stroke cycle internal combustion engine.

Since the pressure inside the engine cylinder is less than the atmospheric pressure during the suction stroke, therefore a negative loop is formed as shown in Fig. 16.2. During the compression stroke, the work is done on the gases, therefore a higher negative loop is obtained. During the expansion or working stroke, the fuel burns and the gases expand, therefore a large positive loop is obtained. In this stroke, the work is done by the gases. During exhaust stroke, the work is done on the gases, therefore a negative loop is formed. It may be noted that the effect of the inertia forces on the piston is taken into account in Fig. 16.2.

1.13.3. Turning Moment Diagram for a Multi-cylinder Engine

A separate turning moment diagram for a compound steam engine having three cylinders and the resultant turning moment diagram is shown in Fig. 16.3. The resultant turning moment diagram is the sum of the turning moment diagrams for the three cylinders. It may be noted that the first cylinder is the high pressure cylinder, second cylinder is the intermediate cylinder and the third cylinder is the low pressure cylinder. The cranks, in case of three cylinders, are usually placed at 120° to each other.



1.14 FLUCTUATION OF ENERGY

The fluctuation of energy may be determined by the turning moment diagram for one complete cycle of operation. Consider the turning moment diagram for a single cylinder double acting steam engine as shown in Fig. 16.1. We see that the mean resisting torque line *AF* cuts the turning moment diagram at points *B*, *C*, *D* and *E*. When the crank moves from *a* to *p*, the work done by the engine is equal to the area *aBp*, whereas the energy required is represented by the area *aABp*. In other words, the engine has done less work (equal to the area *a AB*) than the requirement. This amount of energy is taken from the flywheel and hence the speed of the flywheel decreases. Now the crank moves from *p* to *q*, the work done by the engine is equal to the area *pBbCq*, whereas the requirement of energy is represented by the area *pBCq*. Therefore, the engine has done more work than the requirement. This excess work (equal to the area *BbC*) is stored in the flywheel and hence the speed of the flywheel increases while the crank moves from *p* to *q*.

Similarly, when the crank moves from *q* to *r*, more work is taken from the engine than is developed. This loss of work is represented by the area *C c D*. To supply this loss, the flywheel gives up some of its energy and thus the speed decreases while the crank moves from *q* to *r*. As the crank moves from *r* to *s*, excess energy is again developed given by the area *D d E* and the speed again increases. As the piston moves from *s* to *e*, again there is a loss of work and the speed decreases. The variations of energy above and below the mean resisting torque line are called ***fluctuations of energy*.** The areas *BbC*, *CcD*, *DdE*, etc. represent fluctuations of energy.

A little consideration will show that the engine has a maximum speed either at *q* or at *s*. This is due to the fact that the flywheel absorbs energy while the crank moves from *p* to *q* and from *r* to *s*. On the other hand, the engine has a minimum speed either at *p* or at *r*. The reason is that the flywheel gives out some of its energy when the crank moves from *a* to *p* and *q* to *r*. The difference between the maximum and the minimum energies is known as ***maximum fluctuation of energy*.**

1.14.1 Determination of Maximum Fluctuation of Energy

A turning moment diagram for a multi-cylinder engine is shown by a wavy curve in Fig. 16.4. The horizontal line *A G* represents the mean torque line. Let a_1, a_3, a_5 be the areas above the mean torque line and a_2, a_4 and a_6 be the areas below the mean torque line. These areas represent some quantity of energy which is either added or subtracted from the energy of the moving parts of the engine

Let the energy in the flywheel at $A = E$, then from Fig. 16.4, we have

Energy at $B = E + a_1$

Energy at $C = E + a_1 - a_2$

Energy at $D = E + a_1 - a_2 + a_3$ Energy at $E = E + a_1 - a_2 + a_3 - a_4$

Energy at $F = E + a_1 - a_2 + a_3 - a_4 + a_5$ Energy at $G = E + a_1 - a_2 + a_3 - a_4 + a_5 - a_6$

= Energy at *A* (*i.e.* cycle repeats after *G*)

Let us now suppose that the greatest of these energies is at *B* and least at *E*. Therefore,

Maximum energy in flywheel

$= E + a_1$ Minimum energy in the flywheel

$= E + a_1 - a_2 + a_3 - a_4$

$\therefore$ Maximum fluctuation of energy,

ΔE = Maximum energy – Minimum energy

$= (E + a_1) - (E + a_1 - a_2 + a_3 - a_4) = a_2 - a_3 + a_4$

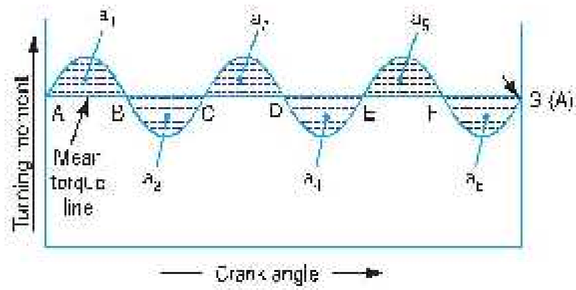


Fig. 16.4. Determination of maximum fluctuation of energy.

1.14.2 Coefficient of Fluctuation of Energy

It may be defined as the **ratio of the maximum fluctuation of energy to the work done per cycle.** Mathematically, coefficient of fluctuation of energy,

$$C_E = \frac{\text{Maximum fluctuation of energy}}{\text{Work done per cycle}}$$

The work done per cycle (in N-m or joules) may be obtained by using the following two relations :

1. Work done per cycle $= T_{mean} \times \theta$

where T_{mean} = Mean torque, and

θ = Angle turned (in radians), in one revolution.

$= 2\pi$, in case of steam engine and two stroke internal combustion engines

$= 4\pi$, in case of four stroke internal combustion engines.

The mean torque (T_{mean}) in N-m may be obtained by using the following relation :

$$T_{mean} = \frac{P \cdot 60}{2\pi N} = \frac{P}{\omega}$$

where P = Power transmitted in watts,

N = Speed in r.p.m., and

ω = Angular speed in rad/s $= 2\pi N/60$

2. The work done per cycle may also be obtained by using the following relation :

$$\text{Work done per cycle} = \frac{P \cdot 60}{n}$$

where n = Number of working strokes per minute,

$= N$, in case of steam engines and two stroke internal combustion engines,

= $N/2$, in case of four stroke internal combustion engines.

The following table shows the values of coefficient of fluctuation of energy for steam engines and internal combustion engines.

Coefficient of fluctuation of energy (C_E) for steam and internal combustion engines.

S.No.	*Type of engine*	*Coefficient of fluctuation of energy (C_E)*
1.	Single cylinder, double acting steam engine	0.21
2.	Cross-compound steam engine	0.096
3.	Single cylinder, single acting, four stroke gas engine	1.93
4.	Four cylinders, single acting, four stroke gas engine	0.066
5.	Six cylinders, single acting, four stroke gas engine	0.031

1.15 FLYWHEEL

A flywheel used in machines serves as a reservoir, which stores energy during the period when the supply of energy is more than the requirement, and releases it during the period when the requirement of energy is more than the supply.

In case of steam engines, internal combustion engines, reciprocating compressors and pumps, the energy is developed during one stroke and the engine is to run for the whole cycle on the energy produced during this one stroke. For example, in internal combustion engines, the energy is developed only during expansion or power stroke which is much more than the engine load and no energy is being developed during suction, compression and exhaust strokes in case of four stroke engines and during compression in case of two stroke engines. The excess energy developed during power stroke is absorbed by the flywheel and releases it to the crankshaft during other strokes in which no energy is developed, thus rotating the crankshaft at a uniform speed. A little consideration will show that when the flywheel absorbs energy, its speed increases and when it releases energy, the speed decreases. Hence a flywheel does not maintain a constant speed, it simply reduces the fluctuation of speed. In other words, a flywheel controls the speed variations caused by the fluctuation of the engine turning moment during each cycle of operation.

In machines where the operation is intermittent like *punching machines, shearing machines, rivetting machines, crushers, etc., the flywheel stores energy from the power source during the greater portion of the operating cycle and gives it up during a small period of the cycle. Thus, the energy from the power source to the machines is supplied practically at a constant rate throughout the operation.

1.16 COEFFICIENT OF FLUCTUATION OF SPEED

The difference between the maximum and minimum speeds during a cycle is called the ***maximum fluctuation of speed***. The ratio of the maximum fluctuation of speed to the mean speed is called the ***coefficient of fluctuation of speed***.

Let N_1 and N_2 = Maximum and minimum speeds in r.p.m. during the cycle, and

$$N = \text{Mean speed in r.p.m.} = \frac{N_1 + N_2}{2}$$

∴ Coefficient of fluctuation of speed,

$$C_S = \frac{N_1 - N_2}{N} = \frac{2(N_1 - N_2)}{N_1 + N_2}$$

$$= \frac{\omega_1 - \omega_2}{\omega} = \frac{2(\omega_1 - \omega_2)}{\omega_1 + \omega_2} \qquad \text{...(In terms of angular speeds)}$$

$$= \frac{v_1 - v_2}{v} = \frac{2(v_1 - v_2)}{v_1 + v_2} \qquad \text{...(In terms of linear speeds)}$$

The coefficient of fluctuation of speed is a limiting factor in the design of flywheel. It varies depending upon the nature of service to which the flywheel is employed.

1.17 ENERGY STORED IN A FLYWHEEL

A flywheel is shown in Fig. 16.5. We have discussed in Art. 16.5 that when a flywheel absorbs energy, its speed increases and when it gives up energy, its speed decreases.

Let m = Mass of the flywheel in kg,

k = Radius of gyration of the flywheel in metres,

I = Mass moment of inertia of the flywheel about its axis of rotation in kg-m^2 = $m.k^2$,

N_1 and N_2 = Maximum and minimum speeds during the cycle in r.p.m.,

ω_1 and ω_2 = Maximum and minimum angular speeds during the cycle in rad/s,

N = Mean speed during the cycle in r.p.m. $= \dfrac{N_1 + N_2}{2}$,

ω = Mean angular speed during the cycle in rad/s $= \dfrac{\omega_1 + \omega_2}{2}$,

C_S = Coefficient of fluctuation of speed, $= \dfrac{N_1 - N_2}{N}$ or $\dfrac{\omega_1 - \omega_2}{\omega}$

We know that the mean kinetic energy of the flywheel,

$$E = \frac{1}{2} \times I.\omega^2 = \frac{1}{2} \times m.k^2.\omega^2 \quad \text{(in N-m or joules)}$$

As the speed of the flywheel changes from ω_1 to ω_2, the maximum fluctuation of energy,

$$\Delta E = \text{Maximum K.E.} - \text{Minimum K.E.}$$

$$= \frac{1}{2} \times I (\omega_1)^2 - \frac{1}{2} \times I (\omega_2)^2 = \frac{1}{2} \times I \left[(\omega_1)^2 - (\omega_2)^2 \right]$$

$$= \frac{1}{2} \times I (\omega_1 + \omega_2)(\omega_1 - \omega_2) = I.\omega(\omega_1 - \omega_2) \quad \text{...(i)}$$

$$= I.\omega^2 \left(\frac{\omega_1 - \omega_2}{\omega} \right) \quad \text{... (Multiplying and dividing by } \omega)$$

$$= I.\omega^2.C_S = m.k^2.\omega^2.C_S \quad \text{... } (\because I = m.k^2) \quad \text{...(ii)}$$

$$= 2.E.C_S \text{ (in N–m or joules)} \quad \text{...} \left(\because E = \frac{1}{2} \times I.\omega^2 \right) \text{... (iii)}$$

The radius of gyration (k) may be taken equal to the mean radius of the rim (R), because the thickness of rim is very small as compared to the diameter of rim. Therefore, substituting $k = R$,

$$\Delta E = m.R^2.\omega^2.C_S = m.v^2.C_S$$

where v = Mean linear velocity (*i.e.* at the mean radius) in m/s = $\omega.R$

12. *The turning moment diagram for a multicylinder engine has been drawn to a scale 1 m m = 600 N-m vertically and 1 m m = 3° horizontally. The intercepted areas between the output torque curve and the mean resistance line, taken in order from one end, are as follows :*

+ 52, – 124, + 92, – 140, + 85, – 72 and + 107 m m^2*, when the engine is running at a speed of 600 r.p.m. If the total fluctuation of speed is not to exceed ± 1.5% of the mean, find the necessary mass of the flywheel of radius 0.5 m.*

Solution. Given : $N = 600$ r.p.m. or $\omega = 2\pi \times 600 / 60 = 62.84$ rad / s ; $R = 0.5$ m

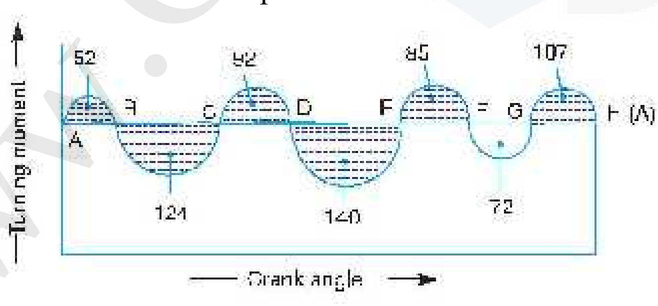


Fig. 16.7

Since the total fluctuation of speed is not to exceed ± 1.5% of the mean speed, therefore $\omega_1 - \omega_2 = 3\% \ \omega = 0.03 \ \omega$

and coefficient of fluctuation of speed,

$$C_S = \frac{\omega_1 - \omega_2}{\omega} = 0.03$$

The turning moment diagram is shown in Fig. 16.7.

Since the turning moment scale is 1 mm = 600 N-m and crank angle scale is 1 mm = 3° = 3° × π/180 = π / 60 rad, therefore

1 mm² on turning moment diagram

$$= 600 \times \pi/60 = 31.42 \text{ N-m}$$

Let the total energy at $A = E$, then referring to Fig. 16.7,

Energy at $B = E + 52$ ...**(Maximum energy)**

Energy at $C = E + 52 - 124 = E - 72$

Energy at $D = E - 72 + 92 = E + 20$

Energy at $E = E + 20 - 140 = E - 120$ ...**(Minimum energy)**

Energy at $F = E - 120 + 85 = E - 35$

Energy at $G = E - 35 - 72 = E - 107$

Energy at $H = E - 107 + 107 = E$ = Energy at A

We know that maximum fluctuation of energy,

ΔE = Maximum energy – Minimum energy

$= (E + 52) - (E - 120) = 172 = 172 \times 31.42 = 5404$ N-m

Let m = Mass of the flywheel in kg. We know that maximum fluctuation of energy (ΔE),

$$5404 = m.R^2.\omega^2.C_S = m \times (0.5)^2 \times (62.84)^2 \times 0.03 = 29.6\, m$$

$m = 5404 / 29.6 = 183$ kg **Ans.**

13. *A shaft fitted with a flywheel rotates at 250 r.p.m. and drives a machine. The torque of machine varies in a cyclic manner over a period of 3 revolutions. The torque rises from 750 N-m to 3000 N-m uniformly during 1/2 revolution and remains constant for the following revolution. It then falls uniformly to 750 N-m during the next 1/2 revolution and remains constant for one revolution, the cycle being repeated thereafter.*

Determine the power required to drive the machine and percentage fluctuation in speed, if the driving torque applied to the shaft is constant and the mass of the flywheel is 500 kg with radius of gyration of 600 m m.

Solution. Given : N = 250 r.p.m. or $\omega = 2\pi \times 250/60 = 26.2$ rad/s ; m = 500 kg ; k = 600 mm = 0.6 m

The turning moment diagram for the complete cycle is shown in Fig. 16.8.

We know that the torque required for one complete cycle

= Area of figure $OABCDEF$

=Area $OAEF$ + Area ABG + Area $BCHG$ + Area CDH

$$= OF \times OA + \frac{1}{2} \times AG \times BG + GH \times CH + \frac{1}{2} \times HD \times CH$$

$$= 6\pi \times 750 + \frac{1}{2} \times \pi(3000 - 750) + 2\pi(3000 - 750)$$
$$+ \frac{1}{2} \times \pi(3000 - 750)$$
$$= 11\,250\,\pi \text{ N-m} \qquad ...(i)$$

If T_{mean} is the mean torque in N-m, then torque required for one complete cycle

$$= T_{mean} \times 6\pi \text{ N-m} \qquad (ii)$$

From equations (*i*) and (*ii*),

$$T_{mean} = 11\,250\,\pi / 6\pi = 1875 \text{ N-m}$$

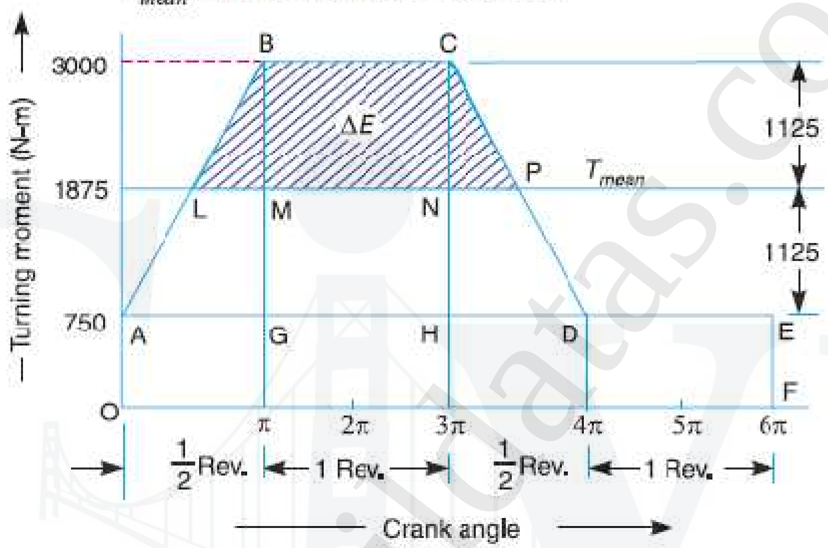


Fig. 16.8

We know that power required to drive the machine,

$$P = T_{mean} \times \omega = 1875 \times 26.2 = 49\,125 \text{ W} = 49.125 \text{ kW} \quad \textbf{Ans.}$$

Coefficient of fluctuation of speed

Let C_S = Coefficient of fluctuation of speed.

First of all, let us find the values of *L M* and *NP*. From similar triangles *ABG* and *BLM*,

$$\frac{LM}{AG} = \frac{BM}{BG} \quad \text{or} \quad \frac{LM}{\pi} = \frac{3000 - 1875}{3000 - 750} = 0.5 \quad \text{or} \quad LM = 0.5\,\pi$$

Now, from similar triangles *CHD* and *CNP*,

$$\frac{NP}{HD} = \frac{CN}{CH} \quad \text{or} \quad \frac{NP}{\pi} = \frac{3000 - 1875}{3000 - 750} = 0.5 \quad \text{or} \quad NP = 0.5\,\pi$$

From Fig. 16.8, we find that

$$BM = CN = 3000 - 1875 = 1125 \text{ N-m}$$

Since the area above the mean torque line represents the maximum fluctuation of energy, therefore, maximum fluctuation of energy,

$$\Delta E = \text{Area } LBCP = \text{Area } LBM + \text{Area } MBCN + \text{Area } PNC$$

$$= \frac{1}{2} \times LM \times BM + MN \times BM + \frac{1}{2} \times NP \times CN$$

$$= \frac{1}{2} \times 0.5\pi \times 1125 + 2\pi \times 1125 + \frac{1}{2} \times 0.5\pi \times 1125$$
$$= 8837 \text{ N-m}$$

We know that maximum fluctuation of energy (ΔE),

$$8837 = m.k^2.\omega^2.C_S = 500 \times (0.6)^2 \times (26.2)^2 \times C_S = 123\,559\, C_S$$

$$C_S = \frac{8837}{123\,559} = 0.071 \textbf{ Ans.}$$

14. *A single cylinder, single acting, four stroke gas engine develops 20 kW at 300 r.p.m. The work done by the gases during the expansion stroke is three times the work done on the gases during the compression stroke, the work done during the suction and exhaust strokes being negligible. If the total fluctuation of speed is not to exceed ± 2 per cent of the mean speed and the turning moment diagram during compression and expansion is assumed to be triangular in shape, find the moment of inertia of the flywheel.*

Solution. Given : $P = 20$ kW $= 20 \times 10^3$ W; $N = 300$ r.p.m. or $\omega = 2\pi \times 300/60 = 31.42$ rad/s

Since the total fluctuation of speed ($\omega_1 - \omega_2$) is not to exceed ± 2 per cent of the mean speed ( ), therefore

$$\omega_1 - \omega_2 = 4\%\ \omega$$

and coefficient of fluctuation of speed,

$$C_S = \frac{\omega_1 - \omega_2}{\omega} = 4\% = 0.04$$

The turning moment-crank angle diagram for a four stroke engine is shown in Fig. 16.11. It is assumed to be triangular during compression and expansion strokes, neglecting the suction and exhaust strokes.

We know that for a four stroke engine, number of working strokes per cycle,

$$n = N/2 = 300 / 2 = 150$$

$\therefore$ Work done/cycle $= P \times 60/\ n = 20 \times 10^3 \times 60/150 = 8000$ N-m ... (*i*)

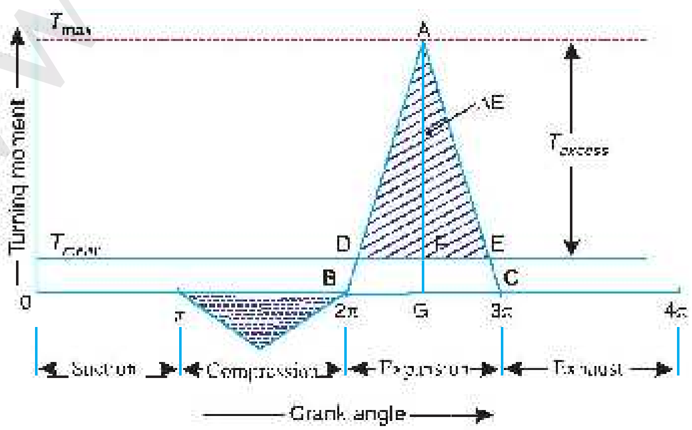


Fig. 16.11

Since the work done during suction and exhaust strokes is negligible, therefore net work done per cycle (during compression and expansion strokes)

$$= W_E - W_C = W_E - \frac{W_E}{3} = \frac{2}{3}W_E \qquad \text{... } (\because W_E = 3W_C) \text{ ...}(ii)$$

Equating equations (*i*) and (*ii*), work done during expansion stroke,

$$W_E = 8000 \times 3/2 = 12\,000 \text{ N-m}$$

We know that work done during expansion stroke (W_E),

$$12\,000 = \text{Area of triangle } ABC = \frac{1}{2} \times BC \times AG = \frac{1}{2} \times \pi \times AG$$

$$\therefore \qquad AG = T_{max} = 12\,000 \times 2/\pi = 7638 \text{ N-m}$$

and mean turning moment,

$$*T_{mean} = FG = \frac{\text{Work done/cycle}}{\text{Crank angle/cycle}} = \frac{8000}{4\pi} = 637 \text{ N-m}$$

$\therefore$ Excess turning moment,

$$T_{excess} = AF = AG - FG = 7638 - 637 = 7001 \text{ N-m}$$

Now, from similar triangles ADE and ABC,

$$\frac{DE}{BC} = \frac{AF}{AG} \quad \text{or} \quad DE = \frac{AF}{AG} \times BC = \frac{7001}{7638} \times \pi = 2.88 \text{ rad}$$

Since the area above the mean turning moment line represents the maximum fluctuation of energy, therefore maximum fluctuation of energy,

$$\Delta E = \text{Area of } \Delta ADE = \frac{1}{2} \times DE \times AF = \frac{1}{2} \times 2.88 \times 7001 = 10\,081 \text{ N-m}$$

Let I = Moment of inertia of the flywheel in kg-m^2 .

We know that maximum fluctuation of energy (ΔE),

$$10\,081 = I.\omega^2.C_S = I \times (31.42)^2 \times 0.04 = 39.5\, I$$

$$\therefore \qquad I = 10081/\,39.5 = 255.2 \text{ kg-m}^2 \quad \textbf{Ans.}$$

15. *The turning moment curve for an engine is represented by the equation, $T = (20\,000 + 9500 \sin 2\theta - 5700 \cos 2\theta)$ N-m, where θ is the angle moved by the crank from inner dead centre. If the resisting torque is constant, find:*

***1.** Power developed by the engine ; **2.** Moment of inertia of flywheel in kg-m^2, if the total fluctuation of speed is not exceed 1% of mean speed which is 180 r.p.m; and **3.** Angular acceleration of the flywheel when the crank has turned through 45° from inner dead centre.*

Solution. Given : $T = (20\,000 + 9500 \sin 2\theta - 5700 \cos 2\theta)$ N-m ; $N = 180$ r.p.m. or $\omega = 2\pi \times 180/60 = 18.85$ rad/s

Since the total fluctuation of speed $(\omega_1 - \omega_2)$ is 1% of mean speed (ω), therefore coefficient of fluctuation of speed,

$$C_S = \frac{\omega_1 - \omega_2}{\omega} = 1\% = 0.01$$

1. *Power developed by the engine*

We know that work done per revolution

$$= \int_0^{2\pi} T\,d\theta = \int_0^{2\pi} (20\,000 + 9500\sin 2\theta - 5700\cos 2\theta)\,d\theta$$

$$= \left[20\,000\,\theta - \frac{9500\cos 2\theta}{2} - \frac{5700\sin 2\theta}{2}\right]_0^{2\pi}$$

$$= 20\,000 \times 2\pi = 40\,000\,\pi \text{ N-m}$$

and mean resisting torque of the engine,

$$T_{mean} = \frac{\text{Work done per revolution}}{2\pi} = \frac{40\,000}{2\pi} = 20\,000 \text{ N-m}$$

We know that power developed by the engine

$$= T_{mean}\,.\,\omega = 20\,000 \times 18.85 = 377\,000 \text{ W} = 377 \text{ kW} \quad \textbf{Ans.}$$

2. *Moment of inertia of the flywheel*

Let I = Moment of inertia of the flywheel in kg-m^2.

The turning moment diagram for one stroke (*i.e.* half revolution of the crankshaft) is shown in Fig. 16.13. Since at points *B* and *D*, the torque exerted on the crankshaft is equal to the mean resisting torque on the flywheel, therefore,

$$T = T_{mean}$$

$$20\,000 + 9500\sin 2\theta - 5700\cos 2\theta - 20\,000$$

or $\quad 9500\sin 2\theta - 5700\cos 2\theta$

$$\tan 2\theta = \sin 2\theta/\cos 2\theta = 5700/9500 = 0.6$$

$\therefore \quad 2\theta = 31°$ or $\theta = 15.5°$

$\therefore \quad \theta_B = 15.5°$ and $\theta_D = 90° + 15.5° = 105.5°$

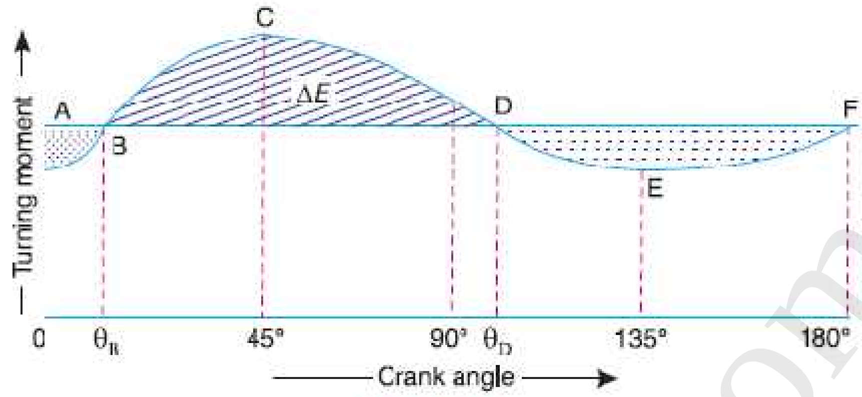


Fig. 16.13

Maximum fluctuation of energy,

$$\Delta E = \int_{\theta_B}^{\theta_D} \left(T - T_{mean}\right) d\theta$$

$$= \int_{15.5°}^{105.5°} (20\,000 + 9500 \sin 2\theta - 5700 \cos 2\theta - 20\,000)\, d\theta$$

$$= \left[-\frac{9500 \cos 2\theta}{2} - \frac{5700 \sin 2\theta}{2} \right]_{15.5°}^{105.5°} = 11\,078 \text{ N m}$$

3. ***Angular acceleration of the flywheel***

Let α = Angular acceleration of the flywheel, and

θ = Angle turned by the crank from inner dead centre = 45° . . . (Given)

The angular acceleration in the flywheel is produced by the excess torque over the mean torque. We know that excess torque at any instant,

$$T_{excess} = T - T_{mean}$$
$$= 20000 + 9500 \sin 2\theta - 5700 \cos 2\theta - 20000$$
$$= 9500 \sin 2\theta - 5700 \cos 2\theta$$

$\therefore$ Excess torque at 45°

$$= 9500 \sin 90° - 5700 \cos 90° = 9500 \text{ N-m} \quad \ldots (i)$$

We also know that excess torque

$$= I.\alpha = 3121 \times \alpha \quad \ldots (ii)$$

From equations (*i*) and (*ii*),

$$\alpha = 9500/3121 = 3.044 \text{ rad/s}^2 \textbf{ Ans.}$$

1.18 DIMENSIONS OF THE FLYWHEEL RIM

Consider a rim of the flywheel as shown in Fig. 16.17.

Let D = Mean diameter of rim in metres,

R = Mean radius of rim in metres,

A = Cross-sectional area of rim in m^2,

ρ = Density of rim material in kg/m^3,

N = Speed of the flywheel in r.p.m.,

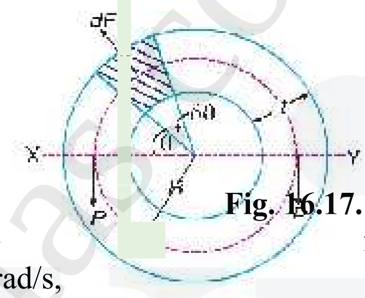


Fig. 16.17. Rim of a flywheel.

ω = Angular velocity of the flywheel in rad/s,

v = Linear velocity at the mean radius in m/s

$= \omega.R = \pi D.N/60$, and

σ = Tensile stress or hoop stress in N/m^2 due to the centrifugal force.

Consider a small element of the rim as shown shaded in Fig. 16.17. Let it subtends an angle δθ at the centre of the flywheel.

Volume of the small element

$= A \times R.\delta\theta$ ∴ Mass of the small element

dm = Density × volume = $\rho .A .R.\delta\theta$

and centrifugal force on the element, acting radially outwards,

$dF = dm.\omega^2.R = \rho .A .R^2.\omega^2.\delta\theta$

$$= 2\rho.A.R^2.\omega^2 \quad \ldots (i)$$

This vertical upward force will produce tensile stress or hoop stress (also called centrifugal stress or circumferential stress), and it is resisted by $2P$, such that

$$2P = 2\,\sigma.A \quad \ldots (ii)$$

Equating equations (*i*) and (*ii*),

$$2.\rho.A.R^2.\omega^2 = 2\sigma A$$

or

$$\sigma = \rho.R^2.\omega^2 = \rho.v^2 \quad \ldots (\because v = \omega.R)$$

$$\therefore \quad v = \sqrt{\frac{\sigma}{\rho}} \quad \ldots (iii)$$

We know that mass of the rim,

$$m = \text{Volume} \times \text{density} = \pi\, D.A.\rho$$

$$\therefore \qquad A = \frac{m}{\pi.D.\rho} \qquad ...(iv)$$

From equations *(iii)* and *(iv)*, we may find the value of the mean radius and cross-sectional area of the rim.

16 *The turning moment diagram for a multi-cylinder engine has been drawn to a scale of 1 mm to 500 N-m torque and 1 mm to 6° of crank displacement. The intercepted areas between output torque curve and mean resistance line taken in order from one end, in sq. mm are – 30, + 410, – 280, + 320, – 330, + 250, – 360, + 280, – 260 sq. mm, when the engine is running at 800 r.p.m.The engine has a stroke of 300 mm and the fluctuation of speed is not to exceed ± 2% of the mean speed. Determine a suitable diameter and cross-section of the flywheel rim for a limiting value of the safe centrifugal stress of 7 MPa. The material density may be assumed as 7200 kg/m^3. The width of the rim is to be 5 times the thickness.*

Solution. Given : N = 800 r.p.m. or $\omega = 2\pi \times 800 / 60 = 83.8$ rad/s; *Stroke = 300 mm ; $\sigma = 7$ MPa $= 7 \times 10^6$ N/m^2 ; $\rho = 7200$ kg/m^3

Since the fluctuation of speed is ± 2% of mean speed, therefore total fluctuation of speed,

$$\omega_1 - \omega_2 = 4\%\ \omega = 0.04\ \omega$$

and coefficient of fluctuation of speed,

$$C_S = \frac{\omega_1 - \omega_2}{\omega} = 0.04$$

Diameter of the flywheel rim

Let D = Diameter of the flywheel rim in metres, and

v = Peripheral velocity of the flywheel rim in m/s.

We know that centrifugal stress (σ),

$$7 \times 10^6 = \rho.v^2 = 7200\, v^2 \quad \text{or} \quad v^2 = 7 \times 10^6/7200 = 972.2$$

$$\therefore \qquad v = 31.2 \text{ m/s}$$

We know that $v = \pi\, D.N/60$

$$\therefore \qquad D = v \times 60 / \pi N = 31.2 \times 60/\pi \times 800 = 0.745 \text{ m} \quad \textbf{Ans.}$$

Cross-section of the flywheel rim

Let t = Thickness of the flywheel rim in metres, and

b = Width of the flywheel rim in metres = $5\,t$ ...(Given)

$\therefore$ Cross-sectional area of flywheel rim,

$$A = b.t = 5\,t \times t = 5\,t^2$$

First of all, let us find the mass (m) of the flywheel rim. The turning moment diagram is shown in Fig 16.18.

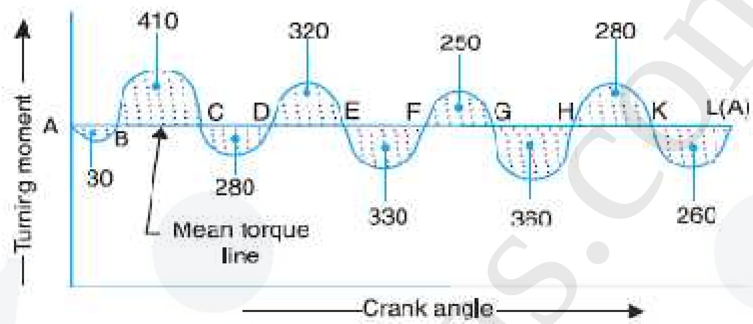


Fig. 16.18

Since the turning moment scale is 1 mm = 500 N-m and crank angle scale is 1 mm = 6° = π /30 rad, therefore

Let the energy at $A = E$, then referring to Fig. 16.18,

Energy at $B = E \quad 30$ . . . **(Minimum energy)**

Energy at $C = E - 30 + 410 = E + 380$

Energy at $D = E + 380 - 280 = E + 100$

Energy at $E = E + 100 + 320 = E + 420$ . . . **(Maximum energy)**

Energy at $F = E + 420 - 330 = E + 90$

Energy at $G = E + 90 + 250 = E + 340$

Energy at $H = E + 340 - 360 = E - 20$

$$\Delta E = \text{Maximum energy} - \text{Minimum energy}$$

$$= (E + 420) - (E - 30) = 450 \text{ mm}^2$$

$$= 450 \times 52.37 = 23\,566 \text{ N-m}$$

We also know that maximum fluctuation of energy (ΔE),

$$23\,566 = m.v^2.C_S = m \times (31.2)^2 \times 0.04 = 39\,m$$

$$\therefore \quad m = 23566 / 39 = 604 \text{ kg}$$

We know that mass of the flywheel rim (m),

$$604 = \text{Volume} \times \text{density} = \pi\,D.A.\rho$$

$$= \pi \times 0.745 \times 5t^2 \times 7200 = 84\,268\,t^2$$

$$\therefore \quad t^2 = 604 / 84\,268 = 0.007\,17 \text{ m}^2 \quad \text{or} \quad t = 0.085 \text{ m} = 85 \text{ mm}$$ **Ans.**

and $b = 5t = 5 \times 85 = 425$ mm **Ans.**

17. *The turning moment diagram of a four stroke engine may be assumed for the sake of simplicity to be represented by four triangles in each stroke. The areas of these triangles are as follows: Suction stroke = 5×10^{-5} m^2; Compression stroke = 21×10^{-5} m^2; Expansion stroke = 85×10^{-5} m^2; Exhaust stroke = 8×10^{-5} m^2.*

All the areas excepting expression stroke are negative. Each m^2 of area represents 14 MN-m of work.

Assuming the resisting torque to be constant, determine the moment of inertia of the flywheel to keep the speed between 98 r.p.m. and 102 r.p.m. Also find the size of a rim-type flywheel based on the minimum material criterion, given that density of flywheel material is 8150 kg/m^3 ; the allowable tensile stress of the flywheel material is 7.5 MPa. The rim cross-section is rectangular, one side being four times the length of the other.

Solution. Given: $a_1 = 5 \times 10^{-5}$ m^2; $a_2 = 21 \times 10^{-5}$ m^2; $a_3 = 85 \times 10^{-5}$ m^2; $a_4 = 8 \times 10^{-5}$ m^2; N_2 = 98 r.p.m.; N_1 = 102 r.p.m.; ρ = 8150 kg/m^3; σ = 7.5 MPa = 7.5×10^6 N/m^2

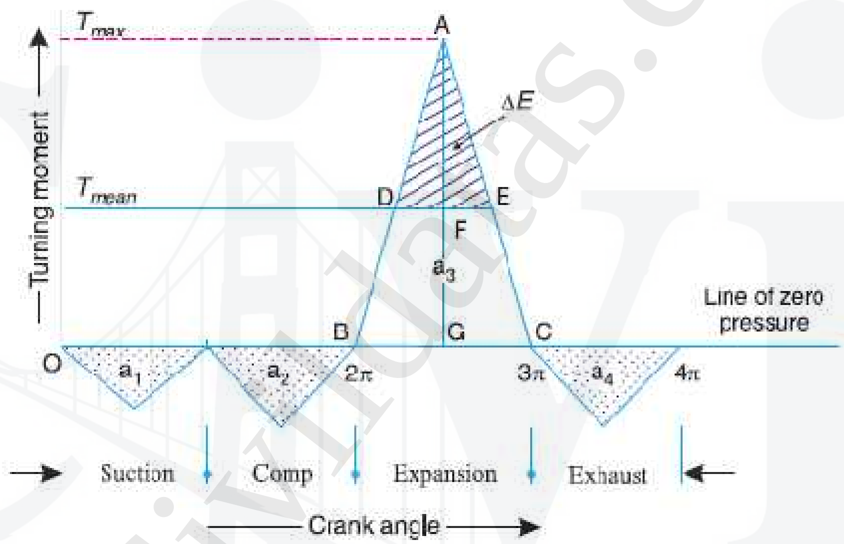


Fig. 16.20

$\therefore$ Net area $= a_3 - (a_1 + a_2 + a_4)$

$= 85 \times 10^5 - (5 \times 10^5 + 21 \times 10^{-5} + 8 \times 10^{-5}) = 51 \times 10^{-5}$ m^2

Since 1m^2 = 14 MN-m = 14×10^6 N-m of work, therefore

Net work done per cycle

$$= 51 \times 10^{-5} \times 14 \times 10^6 = 7140 \text{ N-m} \quad ...(i)$$

We also know that work done per cycle

$$= T_{mean} \times 4\pi \text{ N-m} \quad ...(ii)$$

From equation (*i*) and (*ii*),

$$T_{mean} = FG = 7140 / 4\pi = 568 \text{ N-m}$$

Work done during expansion stroke

$$= a_3 \times \text{Work scale} = 85 \times 10^{-5} \times 14 \times 10^6 = 11\,900 \text{ N-m} \quad ...(iii)$$

Also, work done during expansion stroke

$$= \frac{1}{2} \times BC \times AG = = \frac{1}{2} \times \pi \times AG = 1.571\ AG \quad ...(iv)$$

From equations *(iii)* and *(iv)*,

$$AG = 11\ 900/1.571 = 7575 \text{ N-m}$$

$\therefore$ Excess torque $= AF = AG - FG = 7575 - 568 = 7007$ N-m

Now from similar triangles *ADE* and *ABC*,

$$\frac{DE}{BC} = \frac{AF}{AG} \quad \text{or} \quad DE = \frac{AF}{AG} \times BC = \frac{7007}{7575} \times \pi = 2.9 \text{ rad}$$

We know that maximum fluctuation of energy,

$$\Delta E = \text{Area of } \Delta ADE = \frac{1}{2} \times DE \times AF$$

$$= \frac{1}{2} \times 2.9 \times 7007 = 10\ 160 \text{ N-m}$$

Moment of Inertia of the flywheel

Let I = Moment of inertia of the flywheel in kg-m^2.

We know that mean speed during the cycle

$$N = \frac{N_1 + N_2}{2} = \frac{102 + 98}{2} = 100 \text{ r.p.m.}$$

$\therefore$ Corresponding angular mean speed,

$$\omega = 2\pi N / 60 = 2\pi \times 100/60 = 10.47 \text{ rad/s}$$

and coefficient of fluctuation of speed,

$$C_S = \frac{N_1 - N_2}{N} = \frac{102 - 98}{100} = 0.04$$

We know that maximum fluctuation of energy (ΔE),

$$10\ 160 = I.\omega^2.C_S = I\,(10.47)^2 \times 0.04 = 4.385\ I$$

$\therefore$ $I = 10160 / 4.385 = 2317$ kg m^2 **Ans.**

Size of flywheel

Let t = Thickness of the flywheel rim in metres,

b = Width of the flywheel rim in metres = $4\,t$ ...(Given)

D = Mean diameter of the flywheel in metres, and

v = Peripheral velocity of the flywheel in m/s.

We know that hoop stress (σ),

$$7.5 \times 10^6 = \rho \,.\, v^2 = 8150\, v^2$$

$$\therefore \quad v^2 = \frac{7.5 \times 10^6}{8150} = 920 \quad \text{or} \quad v = 30.3 \text{ m/s}$$

and $$v = \pi DN/60 \quad \text{or} \quad D = v \times 60/\pi N = 30.3 \times 60/\pi \times 100 = 5.786 \text{ m}$$

Also $$m = \text{Volume} \times \text{density} = \pi D \times A \times \frac{\rho}{g} = \pi D \times b \times t \times \frac{\rho}{g}$$

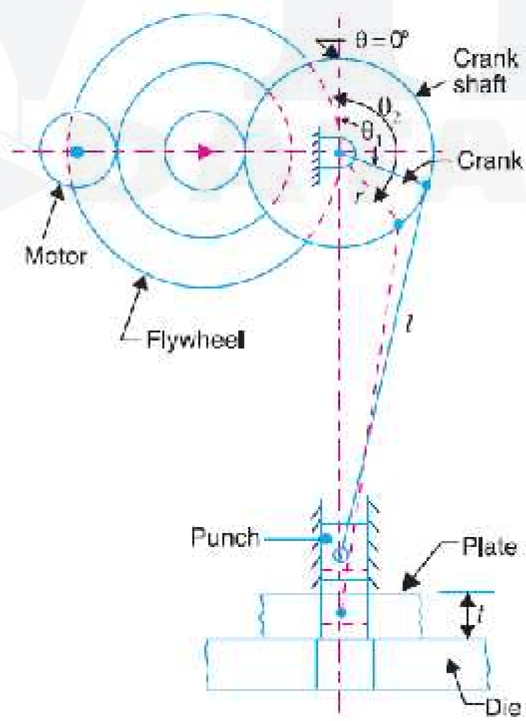
$$276.7 = \pi \times 5.786 \times 4t \times t \times \frac{8 \times 10^4}{9.81} = 59.3 \times 10^4\, t^2$$

$$\therefore \quad t^2 = 276.7/59.3 \times 10^4 = 0.0216 \text{ m or } 21.6 \text{ mm } \textbf{Ans.}$$

and $$b = 4t = 4 \times 21.6 = 86.4 \text{ mm } \textbf{Ans.}$$

1.19 FLYWHEEL IN PUNCHING PRESS

We have discussed in Art. 16.8 that the function of a flywheel in an engine is to reduce the fluctuations of speed, when the load on the crankshaft is constant and the input torque varies during the cycle. The flywheel can also be used to perform the same function when the torque is constant and the load varies during the cycle. Such an application is found in punching press or in a rivetting machine. A punching press is shown diagrammatically in Fig. 16.22. The crank is driven by a motor which supplies constant torque and the punch is at the position of the slider in a slider-crank mechanism. From Fig. 16.22, we see that the load acts only during the rotation of the crank from $\theta = \theta_1$ to $\theta = \theta_2$, when the actual punching takes place and the load is zero for the rest of the cycle. Unless a flywheel is used, the speed of the crankshaft will increase too much during the rotation of crankshaft will increase too much during the rotation of crank from $\theta = \theta_2$ to $\theta = 2\pi$ or $\theta = 0$ and again from $\theta = 0$ to $\theta = \theta_1$, because there is no load while



$\theta - \theta_1$ to $\theta - \theta_2$ due to much more load than the energy supplied. Thus the flywheel has to absorb excess energy available at one stage and has to make up the deficient energy at the other stage to keep to fluctuations of speed within permissible limits. This is done by choosing the suitable moment of inertia of the flywheel.

Punching press and flywheel.

Let E_1 be the energy required for punching a hole. This energy is determined by the size of the hole punched, the thickness of the material and the physical properties of the material.

Let d_1 = Diameter of the hole punched,

t_1 = Thickness of the plate, and

τ_u = Ultimate shear stress for the plate material.

$\therefore$ Maximum shear force required for punching,

$$F_S = \text{Area sheared} \times \text{Ultimate shear stress} = \pi d_1 t_1 \tau_u$$

It is assumed that as the hole is punched, the shear force decreases uniformly from maximum value to zero.

$\therefore$ Work done or energy required for punching a hole,

$$E_1 = \frac{1}{2} \times F_s \times t$$

Assuming one punching operation per revolution, the energy supplied to the shaft per revolution should also be equal to E_1. The energy supplied by the motor to the crankshaft during actual punching operation,

$$E_2 = E_1\left(\frac{\theta_2 - \theta_1}{2\pi}\right)$$

$\therefore$ Balance energy required for punching

$$= E_1 - E_2 = E_1 - E_1\left(\frac{\theta_2 - \theta_1}{2\pi}\right) = E_1\left(1 - \frac{\theta_2 - \theta_1}{2\pi}\right)$$

This energy is to be supplied by the flywheel by the decrease in its kinetic energy when its speed falls from maximum to minimum. Thus maximum fluctuation of energy,

$$\Delta E = E_1 - E_2 = E_1\left(1 - \frac{\theta_2 - \theta_1}{2\pi}\right)$$

The values of θ_1 and θ_2 may be determined only if the crank radius (r), length of connecting rod (l) and the relative position of the job with respect to the crankshaft axis are known. In the absence of relevant data, we assume that

$$\frac{\theta_2 - \theta_1}{2\pi} = \frac{t}{2s} = \frac{t}{4r}$$

where t = Thickness of the material to be punched,

s = Stroke of the punch = 2 × Crank radius = $2r$.

By using the suitable relation for the maximum fluctuation of energy (ΔE) as discussed in the previous articles, we can find the mass and size of the flywheel.

18 *A punching press is driven by a constant torque electric motor. The press is provided with a flywheel that rotates at maximum speed of 225 r.p.m. The radius of gyration of the flywheel is 0.5 m. The press punches 720 holes per hour; each punching operation takes 2 second and requires 15 kN-m of energy. Find the power of the motor and the minimum mass of the flywheel if speed of the same is not to fall below 200 r. p. m.*

Solution. Given N_1 = 225 r.p.m ; k = 0.5 m ; Hole punched = 720 per hr; E_1 = 15 kN-m = 15×10^3 N-m ; N_2 = 200 r.p.m.

Power of the motor

We know that the total energy required per second

= Energy required / hole × No. of holes / s

= $15 \times 10^3 \times 720/3600 = 3000$ N-m/s

∴ Power of the motor = 3000 W = 3 kW **Ans.** (∵ 1 N-m/s = 1 W)

Minimum mass of the flywheel

Let m = Minimum mass of the flywheel.

Since each punching operation takes 2 seconds, therefore energy supplied by the motor in 2 seconds,

$$E_2 = 3000 \times 2 = 6000 \text{ N-m}$$

∴ Energy to be supplied by the flywheel during punching or maximum fluctuation of energy,

$$\Delta E = E_1 - E_2 = 15 \times 10^3 - 6000 = 9000 \text{ N-m}$$

Mean speed of the flywheel,

$$N = \frac{N_1 + N_2}{2} = \frac{225 + 200}{2} = 212.5 \text{ r.p.m}$$

We know that maximum fluctuation of energy (ΔE),

$$9000 = \frac{\pi^2}{900} \times m.k^2.N(N_1 - N_2)$$

$$= \frac{\pi^2}{900} \times m \times (0.5)^2 \times 212.5 \times (225 - 200) = 14.565\, m$$

∴ m = 9000/14.565 = 618 kg **Ans.**

19. *A machine punching 38 mm holes in 32 mm thick plate requires 7 N-m of energy per sq. mm of sheared area, and punches one hole in every 10 seconds. Calculate the power of the motor required. The mean speed of the flywheel is 25 metres per second. The punch has a stroke of 100 mm.*

Find the mass of the flywheel required, if the total fluctuation of speed is not to exceed 3% of the mean speed. Assume that the motor supplies energy to the machine at uniform rate.

Solution. Given : d = 38 mm ; t = 32 mm ; E_1 = 7 N-m/mm^2 of sheared area ; v = 25 m/s ; s = 100 mm ; $v_1 - v_2$ = 3% v = 0.03 v

Power of the motor required

We know that sheared area,

$$A = \pi\, d.t = \pi \times 38 \times 32 = 3820 \text{ mm}^2$$

Since the energy required to punch a hole is 7 N m/mm^2 of sheared area, therefore total energy required per hole,

$$E_1 = 7 \times 3820 = 26\,740 \text{ N-m}$$

Also the time required to punch a hole is 10 second, therefore energy required for punching work per second

$$= 26\,740/10 = 2674 \text{ N m/s}$$

$\therefore$ Power of the motor required

$$= 2674 \text{ W} = 2.674 \text{ kW} \quad \textbf{Ans.}$$

Mass of the flywheel required

Let m = Mass of the flywheel in kg.

Since the stroke of the punch is 100 mm and it punches one hole in every 10 seconds, therefore the time required to punch a hole in a 32 mm thick plate

$$= \frac{10}{2 \times 100} \times 32 = 1.6 \text{ s}$$

$\therefore$ Energy supplied by the motor in 1.6 seconds,

$$E_2 = 2674 \times 1.6 = 4278 \text{ N-m}$$

Energy to be supplied by the flywheel during punching or the maximum fluctuation of energy,

$$\Delta E = E_1 - E_2 = 26\,740 - 4278 = 22\,462 \text{ N-m}$$

Coefficient of fluctuation of speed,

$$C_S = \frac{v_1 - v_2}{v} = 0.03$$

We know that maximum fluctuation of energy (ΔE),

$$22\,462 = m.v^2.C_S = m \times (25)^2 \times 0.03 = 18.75\, m$$

$$\therefore \quad m = 22\,462 / 18.75 = 1198 \text{ kg} \quad \textbf{Ans.}$$

20. *A riveting machine is driven by a constant torque 3 kW motor. The moving parts including the flywheel are equivalent to 150 kg at 0.6 m radius. One riveting operation takes 1 second and absorbs 10 000 N-m of energy. The speed of the flywheel is 300 r.p.m. before riveting. Find the speed immediately after riveting. How many rivets can be closed per minute?*

Solution. Given : P = 3 kW ; m = 150 kg ; k = 0.6 m ; N_1 = 300 r.p.m. or $\omega_1 = 2\pi \times 300/60 = 31.42$ rad/s

Speed of the flywheel immediately after riveting

Let ω_2 = Angular speed of the flywheel immediately after riveting.

We know that energy supplied by the motor,

$$E_2 = 3\text{kW} = 3000 \text{ W} = 3000 \text{ N-m/s} \qquad (\because 1 \text{ W} = 1 \text{ N-m/s})$$

But energy absorbed during one riveting operation which takes 1 second,

$$E_1 = 10\,000 \text{ N-m}$$

$\therefore$ Energy to be supplied by the flywheel for each riveting operation per second or the maximum fluctuation of energy,

$$\Delta E = E_1 - E_2 = 10\,000 - 3000 = 7000 \text{ N-m}$$

We know that maximum fluctuation of energy (ΔE),

$$7000 = \frac{1}{2} \times m.k^2 \left[(\omega_1)^2 - (\omega_2)^2\right] = \frac{1}{2} \times 150 \times (0.6)^2 \times \left[(31.42)^2 - (\omega_2)^2\right]$$

$$= 27\left[987.2 - (\omega_2)^2\right]$$

$\therefore$ $(\omega_2)^2 = 987.2 - 7000/27 = 728$ or $\omega_2 = 26.98$ rad/s

Corresponding speed in r.p.m.,

$$N_2 = 26.98 \times 60/2\pi = 257.6 \text{ r.p.m. } \textbf{Ans.}$$

Number of rivets that can be closed per minute

Since the energy absorbed by each riveting operation which takes 1 second is 10 000 N-m, therefore, number of rivets that can be closed per minute,

$$= \frac{E_2}{E_1} \times 60 = \frac{3000}{10\,000} \times 60 = 18 \text{ rivets } \textbf{Ans.}$$

21. *A punching press is required to punch 40 mm diameter holes in a plate of 15 mm thickness at the rate of 30 holes per minute. It requires 6 N-m of energy per mm² of sheared area. If the punching takes 1/10 of a second and the r.p.m. of the flywheel varies from 160 to 140, determine the mass of the flywheel having radius of gyration of 1 metre.*

Solution. Given: d = 40 mm; t = 15 mm; No. of holes = 30 per min.; Energy required = 6 N-m/mm^2; Time = 1/10 s = 0.1 s; N_1 = 160 r.p.m.; N_2 = 140 r.p.m.; k = 1m

We know that sheared area per hole

$$= \pi\, d.t = \pi \times 40 \times 15 = 1885 \text{ mm}^2$$

$\therefore$ Energy required to punch a hole,

$$E_1 = 6 \times 1885 = 11\,310 \text{ N-m}$$

and energy required for punching work per second

= Energy required per hole × No. of holes per second

= 11 310 × 30/60 = 5655 N-m/s

Since the punching takes 1/10 of a second, therefore, energy supplied by the motor in 1/10 second,

$$E_2 = 5655 \times 1/10 = 565.5 \text{ N-m}$$

$\therefore$ Energy to be supplied by the flywheel during punching a hole or maximum fluctuation of energy of the flywheel,

$$\Delta E = E_1 - E_2 = 11\,310 - 565.5 = 10\,744.5 \text{ N-m}$$

Mean speed of the flywheel,

$$N = \frac{N_1 + N_2}{2} = \frac{160 + 140}{2} = 150 \text{ r.p.m.}$$

We know that maximum fluctuation of energy (ΔE),

$$10\,744.5 = \frac{\pi^2}{900} \times m.k^2\, N(N_1 - N_2)$$

$$= 0.011 \times m \times 1^2 \times 150\,(160 - 140) = 33\,m$$

$$\therefore \quad m = 10744.5 / 33 = 327 \text{ kg} \quad \textbf{Ans.}$$

1.20 CAM DYNAMICS:

Mechanism provides a non-linear I/O relationship. Different mechanism like single or multi-degree of freedom, intermittent motion mechanisms and linkages etc. have different I/O Relationship. When we can not obtain a certain functions from the well known mechanisms, we use a cam mechanism. It is a one degree of freedom mechanism of two moving links. One is cam and the other is follower.

Rigid and elastic body cam system.
Analysis of eccentric cam
Problems on Cam –follower system.

1.21 REVIEW QUESTIONS:

1. When the crank is at the inner dead centre, in a horizontal reciprocating steam engine, then the velocity of the piston will be ?
2. A rigid body, under the action of external forces, can be replaced by two masses placed at a fixed distance apart. The two masses form an equivalent dynamical system, if?
3. The essential condition of placing the two masses, so that the system becomes dynamically equivalent is **?**
4. In an engine, the work done by inertia forces in a cycle is ?
5. In a turning moment diagram, the variations of energy above and below the mean resisting torque line is called?

1.22 TUTORIAL PROBLEMS

1. The stroke of a steam engine is 600 mm and the length of connecting rod is 1.5 m. The crank rotates
at 180 r.p.m. Determine: 1. velocity and acceleration of the piston when crank has travelled through an angle of 40° from inner dead centre, and 2. the position of the crank for zero acceleration of the piston. **[Ans. 4.2 m/s, 85.4 m/s^2; 79.3° from I.D.C]**

2. The following data refer to a steam engine :

Diameter of piston = 240 mm; stroke = 600 mm; length of connecting rod = 1.5 m; mass of reciprocat-ing parts = 300 kg; speed = 125 r.p.m.

Determine the magnitude and direction of the inertia force on the crankshaft when the crank has turned through 30° from inner dead centre. **[Ans. 14.92 kN]**

3. A vertical petrol engine 150 mm diameter and 200 mm stroke has a connecting rod 350 mm long. The
mass of the piston is 1.6 kg and the engine speed is 1800 r.p.m. On the expansion stroke with crank angle 30° from top dead centre, the gas pressure is 750 kN/m^2. Determine the net thrust on the piston. **[Ans. 7535 N]**

4. A certain machine tool does work intermittently. The machine is fitted with a flywheel of mass 200 kg and radius of gyration of 0.4 m. It runs at a speed of 400 r.p.m. between the operations. The machine is driven continuously by a motor and each operation takes 8 seconds. When the machine is doing its work, the speed drops from 400 to 250 r.p.m. Find 1. minimum power of the motor, when there are 5 operations performed per minute, and 2. energy expanded in performing each operation.

[Ans. 4.278 kW; 51.33 kN-m]

5. A constant torque 4 kW motor drives a riveting machine. A flywheel of mass 130 kg and radius of gyration 0.5 m is fitted to the riveting machine. Each riveting operation takes 1 second and requires 9000 N-m of energy. If the speed of the flywheel is 420 r.p.m. before riveting, find: 1. the fall in speed of the flywheel after riveting; and 2. the number of rivets fitted per hour.

[Ans. 385.15 r.p.m.; 1600]

UNIT – II BALANCING

2.1 INTRODUCTION:

Balancing is the process of eliminating or at least reducing the ground forces and/or moments. It is achieved by changing the location of the mass centres of links. Balancing of rotating parts is a well known problem. A rotating body with fixed rotation axis can be fully balanced i.e. all the inertia forces and moments. For mechanism containing links rotating about axis which are not fixed, force balancing is possible, moment balancing by itself may be possible, but both not possible. We generally try to do force balancing. A fully force balance is possible, but any action in force balancing severe the moment balancing.

2.2 BALANCING OF ROTATING MASSES:

The process of providing the second mass in order to counteract the effect of the centrifugal force of the first mass is called balancing of rotating masses.

2.2.1 Static balancing:

The net dynamic force acting on the shaft is equal to zero. This requires that the line of action of three centrifugal forces must be the same. In other words, the centre of the masses of the system must lie on the axis of the rotation. This is the condition for static balancing.

2.2.2 Dynamic balancing:

The net couple due to dynamic forces acting on the shaft is equal to zero. The algebraic sum of the moments about any point in the plane must be zero.

2.2.3 Various cases of balancing of rotating masses:

- Balancing of a single rotating mass by single mass rotating in the same plane.
- Balancing of a single rotating mass by two masses rotating in the different plane.
- Balancing of a several masses rotating in single plane.
- Balancing of a several masses rotating in different planes.

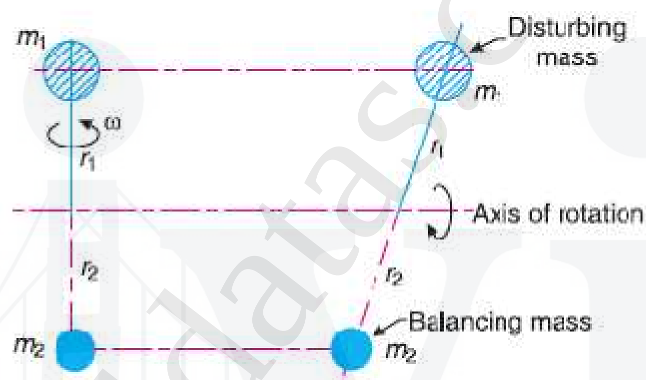
2.3 BALANCING OF A SINGLE ROTATING MASS BY SINGLE MASS ROTATING IN THE SAME PLANE:

Consider a disturbing mass m_1 attached to a shaft rotating at ω rad/s as shown in Fig. Let r_1 be the radius of rotation of the mass m_1 (*i.e.* distance between the axis of rotation of the shaft and the centre of gravity of the mass m_1).

We know that the centrifugal force exerted by the mass m_1 on the shaft,

$$F_{C1} = m_1 \cdot \omega^2 \cdot r_1 \quad \ldots (i)$$

This centrifugal force acts radially outwards and thus produces bending moment on the shaft. In order to counteract the effect of this force, a balancing mass (m_2) may be attached in the same plane of rotation as that of disturbing mass (m_1) such that the centrifugal forces due to the two masses are equal and opposite.



Balancing of a single rotating mass by a single mass rotating in the same plane.

Let r_2 – Radius of rotation of the balancing mass m_2 (*i.e.* distance between the axis of rotation of the shaft and the centre of gravity of mass m_2).

$\therefore$ Centrifugal force due to mass m_2,

$$F_{C2} = m_2 \cdot \omega^2 \cdot r_2 \quad \ldots (ii)$$

Equating equations (*i*) and (*ii*),

$$m_1 \cdot \omega^2 \cdot r_1 = m_2 \cdot \omega^2 \cdot r_2 \quad \text{or} \quad m_1 \cdot r_1 = m_2 \cdot r_2$$

Notes : 1. The product $m_2 . r_2$ may be split up in any convenient way. But the radius of rotation of the balancing mass (m_2) is generally made large in order to reduce the balancing mass m_2.

2. The centrifugal forces are proportional to the product of the mass and radius of rotation of respective masses, because ω^2 is same for each mass.

2.4 BALANCING OF A SINGLE ROTATING MASS BY TWO MASSES ROTATING IN THE DIFFERENT PLANE:

We have discussed in the previous article that by introducing a single balancing mass in the same plane of rotation as that of disturbing mass, the centrifugal forces are balanced. In other words, the two forces are equal in magnitude and opposite in direction. But this type of arrangement for balancing gives rise to a couple which tends to rock the shaft in its bearings. Therefore in order to put the system in complete balance, two balancing masses are placed in two different planes, parallel to the plane of rotation of the disturbing mass, in such a way that they satisfy the following two conditions of equilibrium.

1. The net dynamic force acting on the shaft is equal to zero. This requires that the line of action of three centrifugal forces must be the same. In other words, the centre of the masses of the system must lie on the axis of rotation. This is the condition for ***static balancing.***
2. The net couple due to the dynamic forces acting on the shaft is equal to zero. In other words, the algebraic sum of the moments about any point in the plane must be zero.

The conditions **(1)** and **(2)** together give ***dynamic balancing.*** The following two possibilities may arise while attaching the two balancing masses :

1. The plane of the disturbing mass may be in between the planes of the two balancing masses, and
2. The plane of the disturbing mass may lie on the left or right of the two planes containing the balancing masses.

We shall now discuss both the above cases one by one.

Let l_1 = Distance between the planes A and L,

l_2 = Distance between the planes A and M, and

l = Distance between the planes L and M.

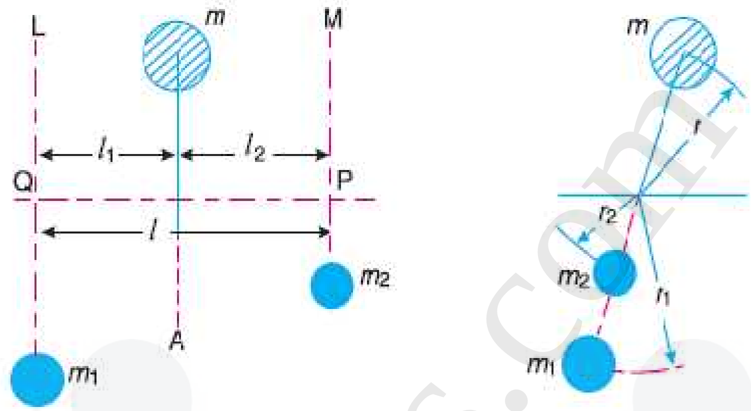


Fig. 21.2. Balancing of a single rotating mass by two rotating masses in different planes when the plane of single rotating mass lies in between the planes of two balancing masses.

We know that the centrifugal force exerted by the mass m in the plane A,

$$F_C = m \cdot \omega^2 \cdot r$$

Similarly, the centrifugal force exerted by the mass m_1 in the plane L,

$$F_{C1} = m_1 \cdot \omega^2 \cdot r_1$$

and, the centrifugal force exerted by the mass m_2 in the plane M,

$$F_{C2} = m_2 \cdot \omega^2 \cdot r_2$$

Since the net force acting on the shaft must be equal to zero, therefore the centrifugal force on the disturbing mass must be equal to the sum of the centrifugal forces on the balancing masses, therefore

$$F_C = F_{C1} + F_{C2} \quad \text{or} \quad m \cdot \omega^2 \cdot r = m_1 \cdot \omega^2 \cdot r_1 + m_2 \cdot \omega^2 \cdot r_2$$

$$\therefore \quad m \cdot r = m_1 \cdot r_1 + m_2 \cdot r_2 \quad \ldots (i)$$

Now in order to find the magnitude of balancing force in the plane *L* (or the dynamic force at the bearing *Q* of a shaft), take moments about *P* which is the point of intersection of the plane *M* and the axis of rotation. Therefore

$$F_{C1} \times l = F_C \times l_2 \quad \text{or} \quad m_1 \cdot \omega^2 \cdot r_1 \times l = m \cdot \omega^2 \cdot r \times l_2$$

$$\therefore \quad m_1 \cdot r_1 \cdot l = m \cdot r \cdot l_2 \quad \text{or} \quad m_1 \cdot r_1 = m \cdot r \times \frac{l_2}{l} \quad \ldots (ii)$$

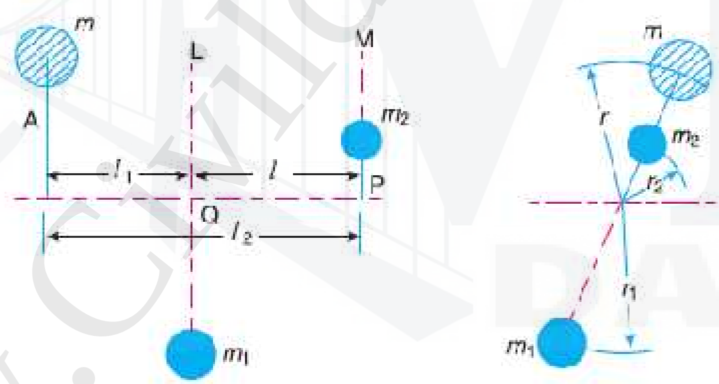
Similarly, in order to find the balancing force in plane *M* (or the dynamic force at the bearing *P* of a shaft), take moments about *Q* which is the point of intersection of the plane *L* and the axis of rotation. Therefore

$$F_{C2} \times l = F_C \times l_1 \quad \text{or} \quad m_2 \cdot \omega^2 \cdot r_2 \times l = m \cdot \omega^2 \cdot r \times l_1$$

$$\therefore \quad m_2 \cdot r_2 \cdot l = m \cdot r \cdot l_1 \quad \text{or} \quad m_2 \cdot r_2 = m \cdot r \times \frac{l_1}{l} \quad (iii)$$

It may be noted that equation (*i*) represents the condition for static balance, but in order to achieve dynamic balance, equations (*ii*) or (*iii*) must also be satisfied.

2. *When the plane of the disturbing mass lies on one end of the planes of the balancing masses*



Balancing of a single rotating mass by two rotating masses in different planes, when the plane of single rotating mass lies at one end of the planes of balancing masses

In this case, the mass *m* lies in the plane *A* and the balancing masses lie in the planes *L* and *M*, as shown in Fig. 21.3. As discussed above, the following conditions must be satisfied in order to balance the system, *i.e.*

$$F_C + F_{C2} = F_{C1} \quad \text{or} \quad m \cdot \omega^2 \cdot r + m_2 \cdot \omega^2 \cdot r_2 = m_1 \cdot \omega^2 \cdot r_1$$

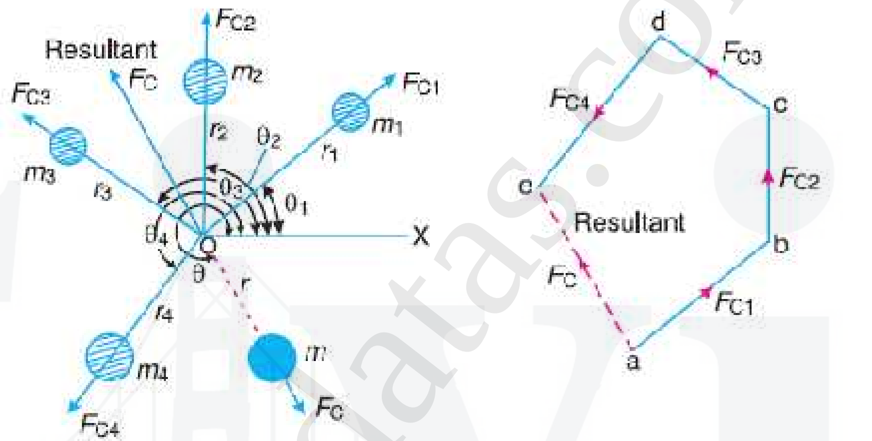
$$\therefore \quad m \cdot r + m_2 \cdot r_2 = m_1 \cdot r_1 \quad \ldots (iv)$$

Now, to find the balancing force in the plane *L* (or the dynamic force at the bearing *Q* of a shaft), take moments about *P* which is the point of intersection of the plane *M* and the axis of rotation. Therefore

2.5 BALANCING OF A SEVERAL MASSES ROTATING IN SAME PLANE:

Consider any number of masses (say four) of magnitude m_1, m_2, m_3 and m_4 at distances of r_1, r_2, r_3 and r_4 from the axis of the rotating shaft. Let $\theta_1, \theta_2, \theta_3$ and θ_4 be the angles of these masses with the horizontal line OX, as shown in Fig. . Let these masses rotate about an axis through O and perpendicular to the plane of paper, with a constant angular velocity of ω rad/s.

The magnitude and position of the balancing mass may be found out analytically or graphically as discussed below :



(a) Space diagram. (b) Vector diagram.

Fig. 21.4. Balancing of several masses rotating in the same plane.

1. *Analytical method*

The magnitude and direction of the balancing mass may be obtained, analytically, as discussed below :

1. First of all, find out the centrifugal force* (or the product of the mass and its radius of rotation) exerted by each mass on the rotating shaft.

2. Resolve the centrifugal forces horizontally and vertically and find their sums, *i.e.* ΣH and ΣV. We know that

Sum of horizontal components of the centrifugal forces,

$$\Sigma H = m_1 \cdot r_1 \cos\theta_1 + m_2 \cdot r_2 \cos\theta_2 + \ldots\ldots$$

and sum of vertical components of the centrifugal forces,

$$\Sigma V = m_1 \cdot r_1 \sin\theta_1 + m_2 \cdot r_2 \sin\theta_2 + \ldots\ldots$$

3. Magnitude of the resultant centrifugal force,

$$F_C = \sqrt{(\Sigma H)^2 + (\Sigma V)^2}$$

4. If θ is the angle, which the resultant force makes with the horizontal, then

$$\tan\theta = \Sigma V / \Sigma H$$

5. The balancing force is then equal to the resultant force, but in *opposite direction*.
6. Now find out the magnitude of the balancing mass, such that

$$F_C = m \cdot r$$

where m = Balancing mass, and

r = Its radius of rotation.

2. *Graphical method*

The magnitude and position of the balancing mass may also be obtained graphically as discussed below :

1. First of all, draw the space diagram with the positions of the several masses, as shown in Fig.
2. Find out the centrifugal force (or product of the mass and radius of rotation) exerted by each mass on the rotating shaft.
3. Now draw the vector diagram with the obtained centrifugal forces (or the product of the masses and their radii of rotation), such that *ab* represents the centrifugal force exerted by the mass m_1 (or $m_1.r_1$) in magnitude and direction to some suitable scale. Similarly, draw *bc*, *cd* and *de* to represent centrifugal forces of other masses m_2, m_3 and m_4 (or $m_2.r_2$, $m_3.r_3$ and $m_4.r_4$).
4. Now, as per polygon law of forces, the closing side *ae* represents the resultant force in magnitude and direction, as shown in Fig.
5. The balancing force is, then, equal to the resultant force, but in *opposite direction.*
6. Now find out the magnitude of the balancing mass (m) at a given radius of rotation (r), such that

$$m \cdot \omega^2 \cdot r = \text{Resultant centrifugal force}$$

or

$$m.r = \text{Resultant of } m_1.r_1,\ m_2.r_2,\ m_3.r_3 \text{ and } m_4.r_4$$

2.6 BALANCING OF SEVERAL MASSES ROTATING DIFFERENT PLANE:

When several masses revolve in different planes, they may be transferred to a ***reference plane*** (briefly written as ***R.P.***), which may be defined as the plane passing through a point on the axis of rotation and perpendicular to it. The effect of transferring a revolving mass (in one plane) to a reference plane is to cause a force of magnitude equal to the centrifugal force of the revolving mass to act in the reference plane, together with a couple of magnitude equal to the product of the force and the distance between the plane of rotation and the reference plane. In order to have a complete balance of the several revolving masses in different planes, the following two conditions must be satisfied :

1. The forces in the reference plane must balance, *i.e.* the resultant force must be zero.
2. The couples about the reference plane must balance, *i.e.* the resultant couple must be zero.

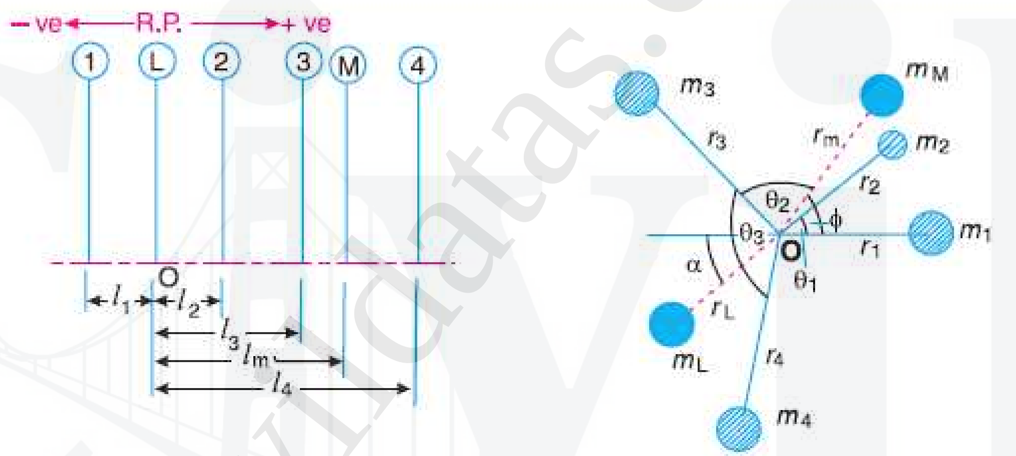
Diesel engine.

Let us now consider four masses m_1, m_2, m_3 and m_4 revolving in planes 1, 2, 3 and 4 respectively as shown in Fig. (*a*). The relative angular positions of these masses are shown in the end view [Fig. 21.7 (*b*)]. The magnitude of the balancing masses m_L and m_M in planes *L* and *M* may be obtained as discussed below :

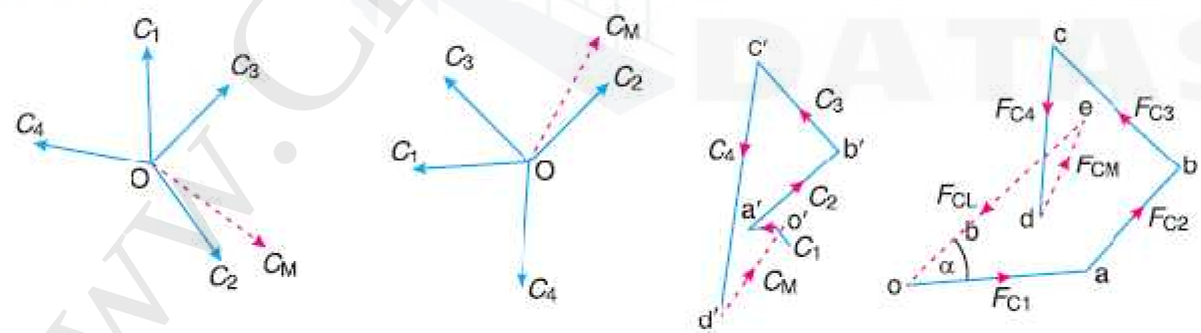
1. Take one of the planes, say *L* as the reference plane (*R.P.*). The distances of all the other planes to the left of the reference plane may be regarded as ***negative***, and those to the right as ***positive***.
2. Tabulate the data as shown in Table 21.1. The planes are tabulated in the same order in which they occur, reading from left to right.

Table 21.1

Plane (1)	Mass (m) (2)	Radius(r) (3)	Cent.force $\div \omega^2$ (m.r) (4)	Distance from Plane L (l) (5)	Couple $\div \omega^2$ (m.r.l) (6)
1	m_1	r_1	$m_1.r_1$	$-l_1$	$-m_1.r_1.l_1$
L(R.P.)	m_L	r_L	$m_L.r_L$	0	0
2	m_2	r_2	$m_2.r_2$	l_2	$m_2.r_2.l_2$
3	m_3	r_3	$m_3.r_3$	l_3	$m_3.r_3.l_3$
M	m_M	r_M	$m_M.r_M$	l_M	$m_M.r_M.l_M$
4	m_4	r_4	$m_4.r_4$	l_4	$m_4.r_4.l_4$



(a) Position of planes of the masses. (b) Angular position of the masses.



(c) Couple vector. (d) Couple vectors turned counter clockwise through a right angle. (e) Couple polygon. (f) Force polygon.

Fig. Balancing of several masses rotating in different planes.

3. A couple may be represented by a vector drawn perpendicular to the plane of the couple. The couple C_1 introduced by transferring m_1 to the reference plane through O is propor-

tional to $m_1.r_1.l_1$ and acts in a plane through Om_1 and perpendicular to the paper. The vector representing this couple is drawn in the plane of the paper and perpendicular to Om_1 as shown by OC_1 in Fig. (*c*). Similarly, the vectors OC_2, OC_3 and OC_4 are drawn perpendicular to Om_2, Om_3 and Om_4 respectively and in the plane of the paper.

4. The couple vectors as discussed above, are turned counter clockwise through a right angle for convenience of drawing as shown in Fig. 21.7 (*d*). We see that their relative positions remains unaffected. Now the vectors OC_2, OC_3 and OC_4 are parallel and in the same direction as Om_2, Om_3 and Om_4, while the vector OC_1 is parallel to Om_1 but in *opposite direction. Hence the ***couple vectors are drawn radially outwards for the masses on one side of the reference plane and radially inward for the masses on the other side of the reference plane.***

5. Now draw the couple polygon as shown in Fig. (*e*). The vector $d'\,o'$ represents the balanced couple. Since the balanced couple C_M is proportional to $m_M.r_M.l_M$, therefore

$$C_M = m_M \cdot r_M \cdot l_M = \text{vector } d'o' \quad \text{or} \quad m_M = \frac{\text{vector } d'\,o'}{r_M \cdot l_M}$$

From this expression, the value of the balancing mass m_M in the plane *M* may be obtained, and the angle of inclination ϕ of this mass may be measured from Fig. 21.7 (*b*).

6. Now draw the force polygon as shown in Fig. (*f*). The vector *eo* (in the direction from *e* to *o*) represents the balanced force. Since the balanced force is proportional to $m_L.r_L$, therefore,

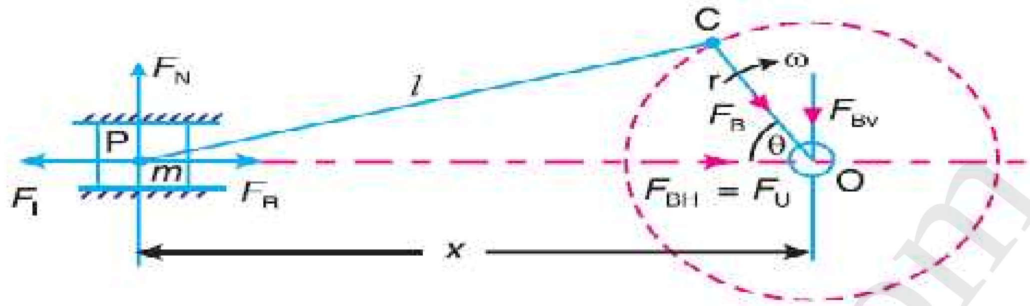
$$m_L \cdot r_L = \text{vector } eo \quad \text{or} \quad m_L = \frac{\text{vector } eo}{r_L}$$

From this expression, the value of the balancing mass m_L in the plane *L* may be obtained the angle of inclination α of this mass with the horizontal may be measured from Fig. (*b*).

2.7 BALANCING OF RECIPROCATING MASSES:

Mass balancing encompasses a wide array of measures employed to obtain partial or complete compensation for the inertial forces and moments of inertia emanating from the crankshaft assembly. All masses are externally balanced when no free inertial forces or moments of inertia are transmitted through the block to the outside. However, the remaining internal forces and moments subject the engine mounts and block to various loads as well as deformities and vibratory stresses. The basic loads imposed by gas-based and inertial forces

2.7.1 Primary and secondary unbalanced forces of reciprocating parts:



Reciprocating engine mechanism.

Let F_R = Force required to accelerate the reciprocating parts,

Let m = Mass of the reciprocating parts,

l = Length of the connecting rod PC,

r = Radius of the crank OC,

θ = Angle of inclination of the crank with the line of stroke PO,

ω = Angular speed of the crank,

n = Ratio of length of the connecting rod to the crank radius = l / r.

We have already discussed in Art. 15.8 that the acceleration of the reciprocating parts is approximately given by the expression,

$$a_R = \omega^2 \cdot r\left(\cos\theta + \frac{\cos 2\theta}{n}\right)$$

∴ Inertia force due to reciprocating parts or force required to accelerate the reciprocating parts,

$$F_I = F_R = \text{Mass} \times \text{acceleration} = m \cdot \omega^2 \cdot r\left(\cos\theta + \frac{\cos 2\theta}{n}\right)$$

We have discussed in the previous article that the horizontal component of the force exerted on the crank shaft bearing (*i.e.* F_{BH}) is equal and opposite to inertia force (F_I). This force is an unbalanced one and is denoted by F_U.

∴ Unbalanced force,

$$F_U = m \cdot \omega^2 \cdot r\left(\cos\theta + \frac{\cos 2\theta}{n}\right) = m.\omega^2 \cdot r\cos\theta + m \cdot \omega^2 \cdot r \times \frac{\cos 2\theta}{n} = F_P + F_S$$

The expression $(m \cdot \omega^2 \cdot r\cos\theta)$ is known as ***primary unbalanced force*** and $\left(m \cdot \omega^2 \cdot r \times \frac{\cos 2\theta}{n}\right)$ is called ***secondary unbalanced force***.

2.8 BALANCING OF SINGLE CYLINDER ENGINE:

A single cylinder engine produces three main vibrations. In describing them we will assume that the cylinder is vertical. Firstly, in an engine with no balancing counterweights, there would be an enormous vibration produced by the change in momentum of the piston, gudgeon pin, connecting rod and crankshaft once every revolution. Nearly all single-cylinder crankshafts incorporate balancing weights to reduce this. While these weights can balance the crankshaft completely, they cannot completely balance the motion of the piston, for two reasons. The first reason is that the balancing weights have horizontal motion as well as vertical motion, so balancing the purely vertical motion of the piston by a crankshaft weight adds a horizontal vibration. The second reason is that, considering now the vertical motion only, the smaller piston end of the connecting rod (little end) is closer to the larger crankshaft end (big end) of the connecting rod in mid-stroke than it is at the top or bottom of the stroke, because of the connecting rod's angle. So during the 180° rotation from mid-stroke through top-dead-center and back to mid-stroke the minor contribution to the piston's up/down movement from the connecting rod's change of angle has the same direction as the major contribution to the piston's up/down movement from the up/down movement of the crank pin. By contrast, during the 180° rotation from mid-stroke through bottom-dead-center and back to mid-stroke the minor contribution to the piston's up/down movement from the connecting rod's change of angle has the opposite direction of the major contribution to the piston's up/down movement from the up/down movement of the crank pin. The piston therefore travels faster in the top half of the cylinder than it does in the bottom half, while the motion of the crankshaft weights is sinusoidal. The vertical motion of the piston is therefore not quite the same as that of the balancing weight, so they can't be made to cancel out completely.

Secondly, there is a vibration produced by the change in speed and therefore kinetic energy of the piston. The crankshaft will tend to slow down as the piston speeds up and absorbs energy, and to speed up again as the piston gives up energy in slowing down at the top and bottom of the stroke. This vibration has twice the frequency of the first vibration, and absorbing it is one function of the flywheel.

Thirdly, there is a vibration produced by the fact that the engine is only producing power during the power stroke. In a four-stroke engine this vibration will have half the frequency of the first vibration, as the cylinder fires once every two revolutions. In a two-stroke engine, it will have the same frequency as the first vibration. This vibration is also absorbed by the flywheel.

2.9 BALANCING OF INERTIAL FORCES IN THE MULTI-CYLINDER ENGINE:

In multi-cylinder engines the mutual counteractions of the various components in the Crank shaft assembly are one of the essential factors determining the selection of the Crank shafts configuration and with it the design of the engine itself. The inertial forces are Balanced if the common centre of gravity for all moving crankshaft-assembly components lies at the crankshaft's midpoint, i.e. if the crankshaft is symmetrical (as viewed from the front). The crankshaft's symmetry level can be defined using geometrical representations of 1st- and 2nd-order forces (star diagrams). The 2nd order star diagram for the four-cylinder in-line engine is asymmetrical, meaning that this order is characterized by substantial free inertial Forces. These forces can be balanced using two countershafts rotating in opposite directions at double the rate of the crankshaft (Lanchester system).

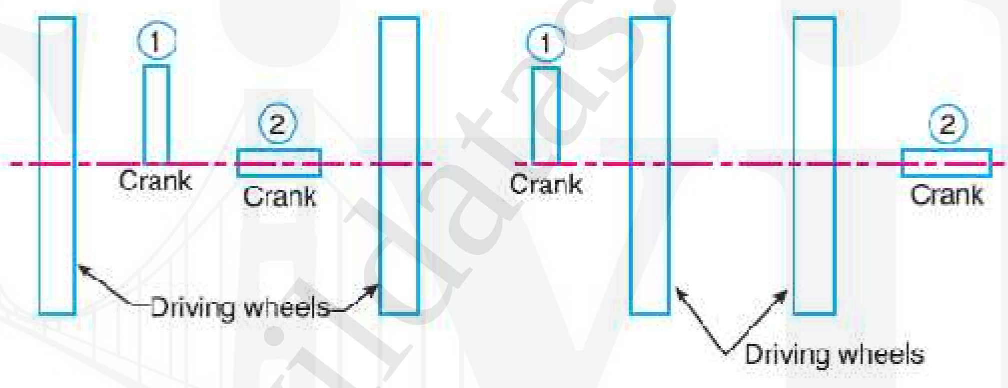
2.10 PARTIAL BALANCING OF LOCOMOTIVES:

The locomotives, usually, have two cylinders with cranks placed at right angles to each other in order to have uniformity in turning moment diagram. The two cylinder locomotives may be classified as :

1. Inside cylinder locomotives ; and 2. Outside cylinder locomotives.

In the ***inside cylinder locomotives***, the two cylinders are placed in between the planes of two driving wheels as shown in Fig. (*a*) ; whereas in the ***outside cylinder locomotives***, the two cylinders are placed outside the driving wheels, one on each side of the driving wheel, as shown in Fig. (*b*). The locomotives may be

(*a*) Single or uncoupled locomotives ; and (*b*) Coupled locomotives.



(*a*) Inside cylinder locomotives. (*b*) Outside cylinder locomotives.

2.10.1 Variation of Tractive force:

The resultant unbalanced force due to the cylinders, along the line of stroke, is known as tractive force.

2.10.2 Swaying Couple:

The couple has swaying effect about a vertical axis, and tends to sway the engine alternately in clock wise and anticlockwise directions. Hence the couple is known as swaying couple.

The unbalanced forces along the line of stroke for the two cylinders constitute a couple about the centre line *YY* between the cylinders as shown in Fig. 22.5.

This couple has swaying effect about a vertical axis, and tends to sway the engine alternately in clockwise and anticlockwise directions. Hence the couple is known as ***swaying couple.***

Let a = Distance between the centre lines of the two cylinders.

$\therefore$ Swaying couple

$$= (1-c)m.\omega^2.r\cos\theta\times\frac{a}{2} - (1-c)m.\omega^2.r\cos(90^\circ+\theta)\frac{a}{2}$$

$$= (1-c)m\omega^2.r\times\frac{a}{2}(\cos\theta+\sin\theta)$$

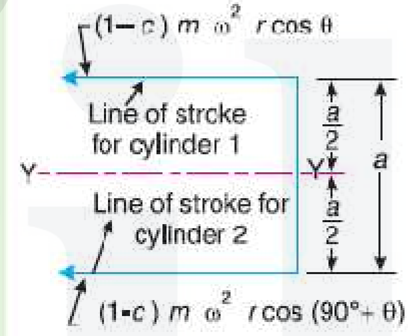


Fig. Swaying couple.

The swaying couple is maximum or minimum when $(\cos\theta+\sin\theta)$ is maximum or minimum. For $(\cos\theta+\sin\theta)$ to be maximum or minimum,

$$\frac{d}{d\theta}(\cos\theta+\sin\theta)=0 \quad \text{or} \quad -\sin\theta+\cos\theta=0 \quad \text{or} \quad -\sin\theta=-\cos\theta$$

$$\therefore \quad \tan\theta=1 \quad \text{or} \quad \theta=45^\circ \quad \text{or} \quad 225^\circ$$

Thus, the swaying couple is maximum or minimum when $\theta = 45^\circ$ or 225°.

$\therefore$ Maximum and minimum value of the swaying couple

$$= \pm(1-c)m.\omega^2.r\times\frac{a}{2}(\cos 45^\circ+\sin 45^\circ)=\pm\frac{a}{\sqrt{2}}(1-c)m.\omega^2.r$$

2.10.3 Hammer blow:

The maximum magnitude of the unbalanced force along the perpendicular to the line of stroke is known as Hammer blow.

We have already discussed that the maximum magnitude of the unbalanced force along the perpendicular to the line of stroke is known as ***hammer blow***.

We know that the unbalanced force along the perpendicular to the line of stroke due to the balancing mass B, at a radius b, in order to balance reciprocating parts only is $B.\omega^2.b \sin\theta$. This force will be maximum when $\sin\theta$ is unity, *i.e.* when $\theta = 90°$ or $270°$.

$\therefore$ Hammer blow $= B.\omega^2.b$ (Substituting $\sin\theta = 1$)

The effect of hammer blow is to cause the variation in pressure between the wheel and the rail. This variation is shown in Fig. 22.6, for one revolution of the wheel.

Let P be the downward pressure on the rails (or static wheel load).

$\therefore$ Net pressure between the wheel and the rail

$$= P \pm B.\omega^2.b$$

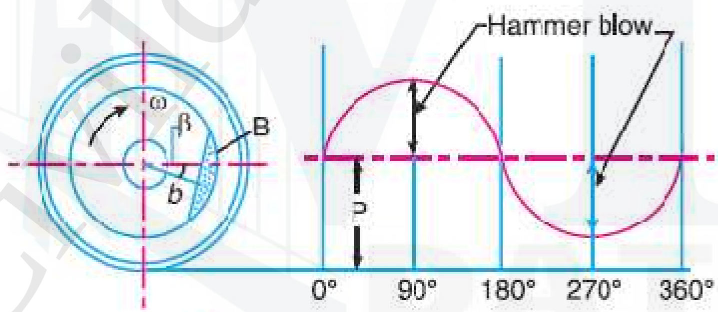


Fig. Hammer blow.

If $(P - B.\omega^2.b)$ is ***negative***, then the wheel will be lifted from the rails. Therefore the limiting condition in order that the wheel does not lift from the rails is given by

$$P = B.\omega^2.b$$

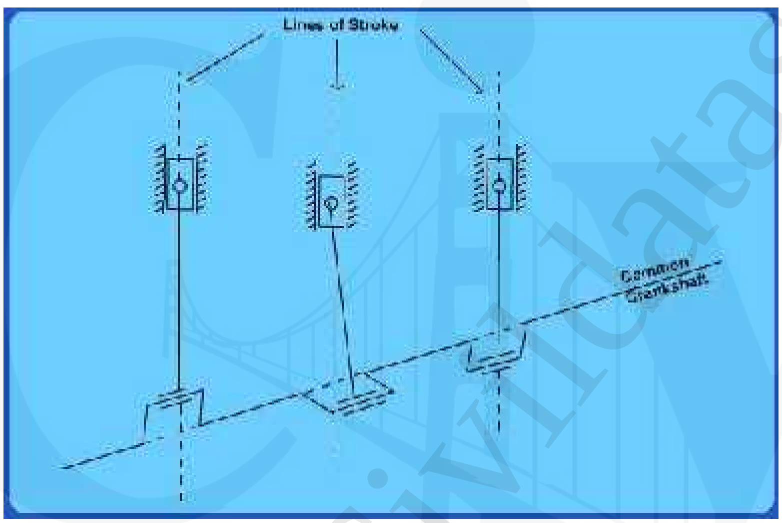
and the permissible value of the angular speed,

$$\omega = \sqrt{\frac{P}{B.b}}$$

2.11 BALANCING OF INLINE ENGINES:

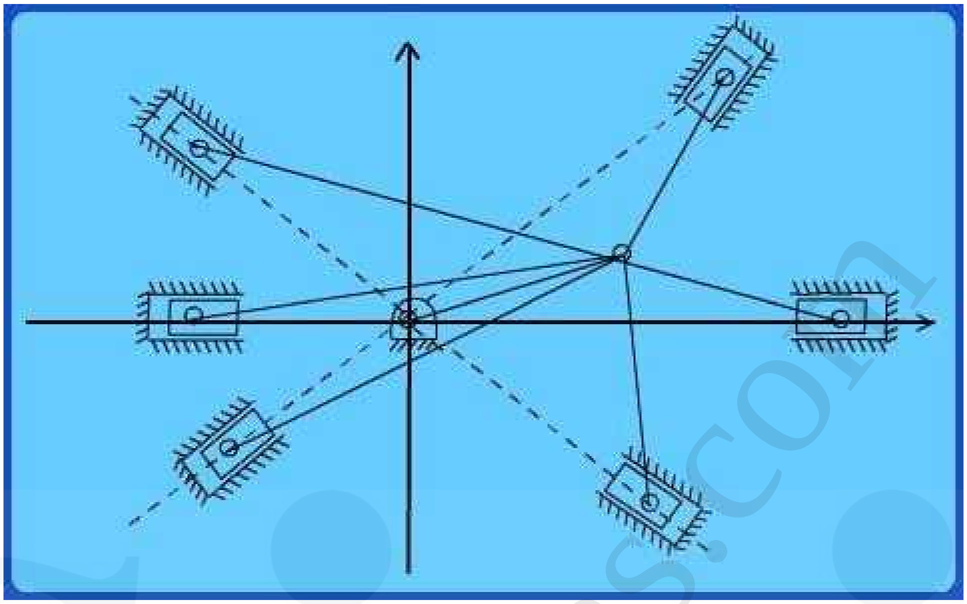
An in-line engine is one wherein all the cylinders are arranged in a single line, one behind the other as schematically indicated in Fig. Many of the passenger cars found on Indian roads such as Maruti 800, Zen, Santro, Honda City, Honda CR-V, and Toyota Corolla all have four cylinder in-line engines. Thus this is a commonly employed engine and it is of interest to us to understand the analysis of its state of balance.

For the sake of simplicity of analysis, we assume that all the cylinders are identical viz., r, ℓ, m_{rec} and m_{rot} are same. Further we assume that the rotating masses have been balanced out for all cylinders and we are left with only the forces due to the reciprocating masses.



2.12 BALANCING OF RADIAL ENGINES:

A radial engine is one in which all the cylinders are arranged circumferentially as shown in Fig.These engines were quite popularly used in aircrafts during World War II. Subsequent developments in steam/gas turbines led to the near extinction of these engines. However it is still interesting to study their state of balance in view of some elegant results we shall discuss shortly. Our method of analysis remains identical to the previous case i.e., we proceed with the assumption that all cylinders are identical and the cylinders are spaced at uniform interval $\left(\frac{2\pi}{n}\right)$ around the circumference.



2.13 SOLVED PROBLEMS

1. A shaft has three eccentrics, each 75 mm diameter and 25 mm thick, machined in one piece with the shaft. The central planes of the eccentric are 60 mm apart. The distance of the centres from the axis of rotation are 12 mm, 18 mm and 12 mm and their angular positions are 120° apart. The density of metal is 7000 kg/m³. Find the amount of out-of-balance force and couple at 600 r.p.m. If the shaft is balanced by adding two masses at a radius 75 mm and at distances of 100 mm from the central plane of the middle eccentric, find the amount of the masses and their angular positions. (AU-MAY/JUNE-2013)

Solution. Given : $D = 75$ mm $= 0.075$ m ; $t = 25$ mm $= 0.025$ m ; $r_A = 12$ mm $= 0.012$ m ; $r_B = 18$ mm $= 0.018$ m ; $r_C = 12$ mm $= 0.012$ mm ; $\rho = 7000$ kg/m³ ; $N = 600$ r.p.m. or $\omega = 2\pi \times 600/60 = 62.84$ rad/s ; $r_L = r_M = 75$ mm $= 0.075$ m

We know that mass of each eccentric,

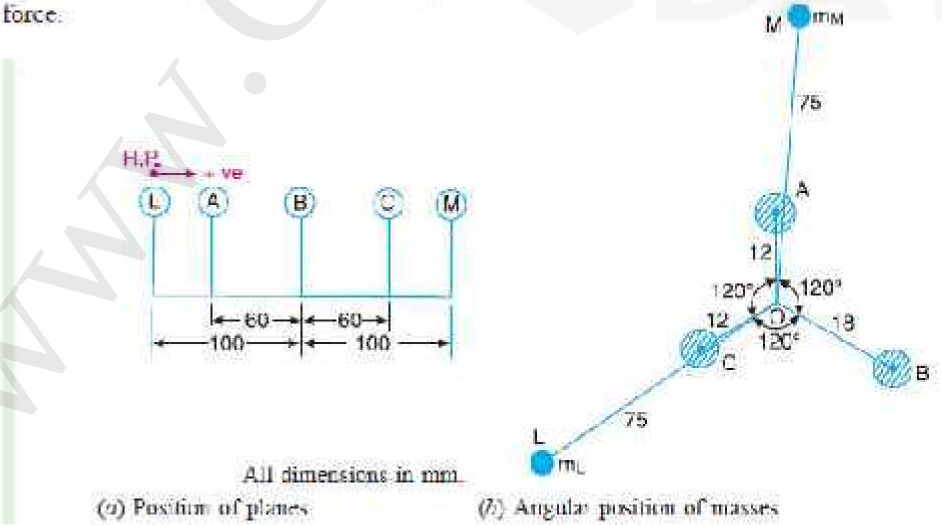
$$m_A = m_B = m_C = \text{Volume} \times \text{Density} = \frac{\pi}{4} \times D^2 \times t \times \rho$$
$$= \frac{\pi}{4}(0.075)^2(0.025)7000 = 0.77 \text{ kg}$$

Let *L* and *M* be the planes at distances of 100 mm from the central plane of middle eccentric. The position of the planes and the angular position of the three eccentrics is shown in Fig. 21.12 (*a*) and (*b*) respectively. Assuming *L* as the reference plane and mass of the eccentric *A* in the vertical direction, the data may be tabulated as below :

Plane (1)	*Mass* (*m*) *kg* (2)	*Radius* (*r*) *m* (3)	*Cent. force* ÷ ω^2 (*m.r*) *kg-m* (4)	*Distance from plane L* (*l*) *m* (5)	*Couple* ÷ ω^2 (*m.r.l*) *kg-m²* (6)
L (*R.P.*)	m_L	0.075	$75 \times 10^{-3}\ m_L$	0	0
A	0.77	0.012	9.24×10^{-3}	0.04	0.3696×10^{-3}
B	0.77	0.018	13.86×10^{-3}	0.1	1.386×10^{-3}
C	0.77	0.012	9.24×10^{-3}	0.16	1.4784×10^{-3}
M	m_M	0.075	$75 \times 10^{-3}\ m_M$	0.20	$15 \times 10^{-3}\ m_M$

Out-of-balance force

The out-of-balance force is obtained by drawing the force polygon, as shown in Fig. 21.12 (*c*), from the data given in Table 21.6 (column 4). The resultant *oc* represents the out-of-balance force.



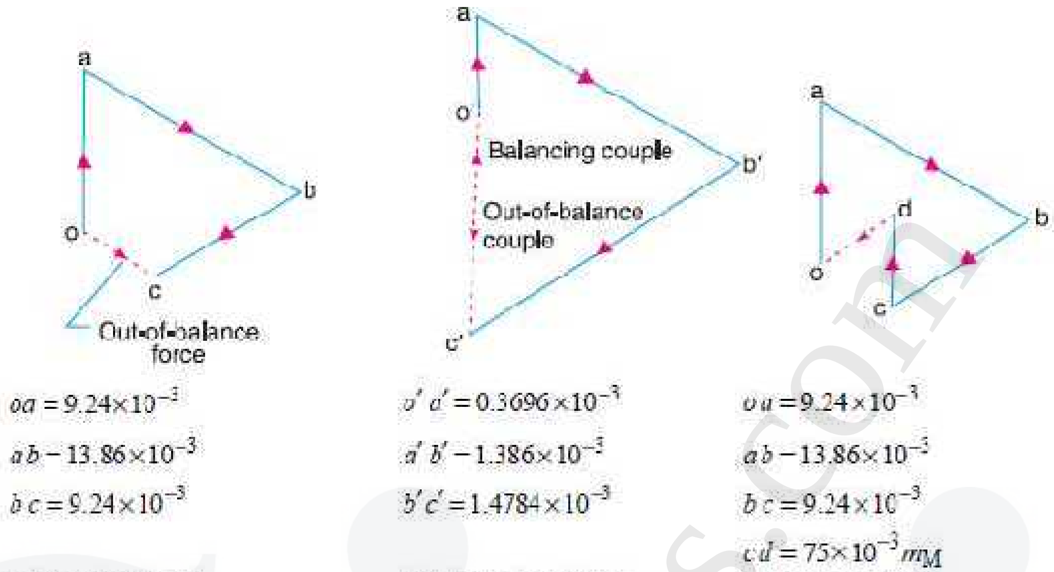
All dimensions in mm.

(*a*) Position of planes (*b*) Angular position of masses

Since the centrifugal force is proportional to the product of mass and radius (*i.e.* *m.r*), therefore by measurement.

Out-of-balance force = vector oc = 4.75×10^{-3} kg-m

$$= 4.75 \times 10^{-3} \times \omega^2 = 4.75 \times 10^{-3} (62.84)^2 = 18.76 \text{ N}$$ **Ans.**



(*c*) Force polygon. (*d*) Couple polygon. (*e*) Force polygon.

Fig. 21.12

Out-of-balance couple

The out-of-balance couple is obtained by drawing the couple polygon from the data given in Table 21.6 (column 6), as shown in Fig. 21.12 (d). The resultant $o'c'$ represents the out-of-balance couple. Since the couple is proportional to the product of force and distance (m.r.l), there-fore by measurement,

Out-of-balance couple = vector $o'c'$ = 1.1×10^{-3} kg-m^2

$$= 1.1\times10^{-3}\times\omega^2 = 1.1\times10^{-3}(62.84)^2 = 4.34 \text{ N-m}$$ **Ans.**

Amount of balancing masses and their angular positions

The vector $c'o'$ (in the direction from c' to o'), as shown in Fig. 21.12 (*d*) represents the balancing couple and is proportional to $15 \times 10^{-3}\, m_M$, *i.e.*

$$15 \times 10^{-3}\, m_M = \text{vector } c'o' = 1.1 \times 10^{-3} \text{ kg-m}^2$$

or $m_M = 0.073$ kg **Ans.**

Draw *OM* in Fig. 21.12 (*b*) parallel to vector $c'o'$. By measurement, we find that the angular position of balancing mass (m_M) is 5° from mass *A* in the clockwise direction. **Ans.**

In order to find the balancing mass (m_L), a force polygon as shown in Fig. 21.12 (*e*) is drawn. The closing side of the polygon *i.e.* vector *do* (in the direction from *d* to *o*) represents the balancing force and is proportional to $75 \times 10^{-3}\, m_L$. By measurement, we find that,

$$75 \times 10^{-3}\, m_L = \text{vector } do = 5.2 \times 10^{-3} \text{ kg-m or } m_L = 0.0693 \text{ kg}$$ **Ans.**

Draw *OL* in Fig. 21.12 (*b*), parallel to vector *do*.

By measurement, we find that the angular position of mass (m_L) is 124° from mass *A* in the clockwise direction. **Ans.**

2.(i) A, B, C and D are four masses carried by a rotating shaft at radii 100, 125, 200 and 150 mm respectively. The planes in which the masses revolve are spaced 600 mm apart and the mass of B, C and D are 10 kg, 5 kg, and 4 kg respectively.Find the required mass A and the relative angular settings of the four masses so that the shaft shall be in complete balance.

(AU-NOV/DEC-2012) *(8)*

Solution. Given : r_A = 100 mm = 0.1 m ; r_B = 125 mm = 0.125 m ; r_C = 200 mm = 0.2 m ; r_D = 150 mm = 0.15 m ; m_B = 10 kg ; m_C = 5 kg ; m_D = 4 kg

1. The position of planes is shown in Fig. 21.10 (*a*). Assuming the plane of mass *A* as the reference plane (*R.P.*), the data may be tabulated as below :

Plane (1)	Mass (m) kg (2)	Radius (r) m (3)	Cent. Force ÷ ω^2 (m.r) kg-m (4)	Distance from plane A (l) m (5)	Couple ÷ ω^2 (m.r.l) kg-m^2 (6)
A(R.P.)	m_A	0.1	0.1 m_A	0	0
B	10	0.125	1.25	0.5	0.75
C	5	0.2	1	1.2	1.2
D	4	0.15	0.6	1.8	1.08

First of all, the angular setting of masses *C* and *D* is obtained by drawing the couple polygon from the data given in Table 21.4 (column 6). Assume the position of mass *B* in the horizontal direction *OB* as shown in Fig. 21.10 (*b*). Now the couple polygon as shown in Fig.(*c*) is drawn as discussed below :

1. Draw vector $o'b'$ in the horizontal direction (*i.e.* parallel to *OB*) and equal to 0.75 kg-m^2, to some suitable scale.

2. From points o' and b', draw vectors $o'c'$ and $b'c'$ equal to 1.2 kg-m^2 and 1.08 kg-m^2 respectively. These vectors intersect at c'.

3. Now in Fig. 21.10 (*b*), draw *OC* parallel to vector $o'c'$ and *OD* parallel to vector $b'c'$. By measurement, we find that the angular setting of mass *C* from mass *B* in the anticlockwise direction, *i.e.* $\angle BOC = 240°$ **Ans.**

and angular setting of mass *D* from mass *B* in the anticlockwise direction, *i.e.* $\angle BOD = 100°$ **Ans.**

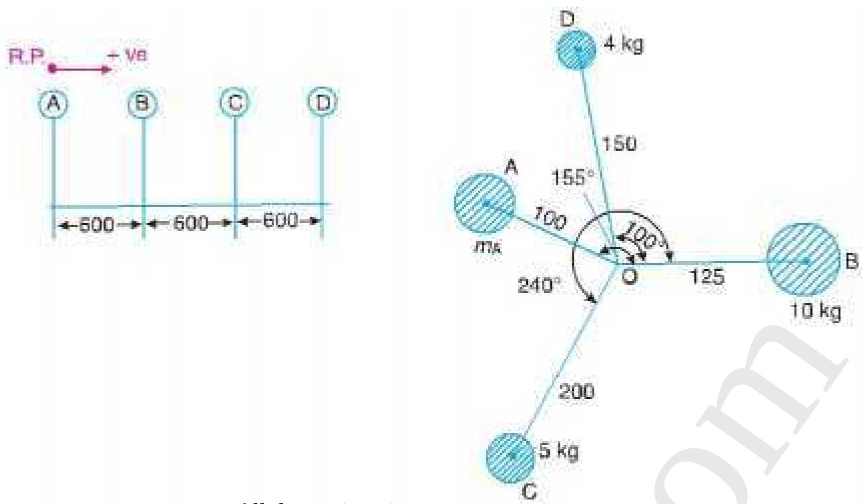
In order to find the required mass *A* (m_A) and its angular setting, draw the force polygon to some suitable scale, as shown in Fig. 21.10 (*d*), from the data given in Table 21.4 (column 4).

Since the closing side of the force polygon (vector *do*) is proportional to 0.1 m_A, therefore by measurement,

$0.1\, m = 0.7$ kg-m^2 or $m_A = 7$ kg **Ans.**

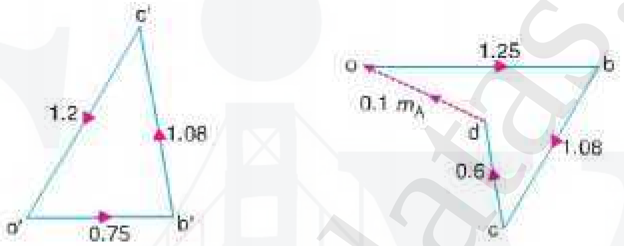
Now draw *OA* in Fig. 21.10 (*b*), parallel to vector *do*. By measurement, we find that the angular setting of mass *A* from mass *B* in the anticlockwise direction, *i.e.*

$\angle BOA = 155°$ **Ans.**



All dimensions in mm

(*a*) Position of planes. (*b*) Angular position of masses.



(*c*) Couple polygon. (*d*) Force polygon.

Fig. 21.10

2(ii) Derives the expressions for the following: (i) Variation in tractive force and (ii) Swaying couple. (8) (AU-NOV/DEC-2009)

Variation in tractive force

The resultant unbalanced force due to the two cylinders, along the line of stroke, is known as ***tractive force***. Let the crank for the first cylinder be inclined at an angle θ with the line of stroke, as shown in Fig. 22.4.

Since the crank for the second cylinder is at right angle to the first crank, therefore the angle of inclination for the second crank will be (90° + θ).

Let m = Mass of the reciprocating parts per cylinder, and

c = Fraction of the reciprocating parts to be balanced.

$$= (1-c)\,m\omega^2.r\cos\theta$$

Similarly, unbalanced force along the line of stroke for cylinder 2,

$$= (1-c)\,m.\omega^2.r\cos(90^\circ+\theta)$$

∴ As per definition, the tractive force,

F_T = Resultant unbalanced force along the line of stroke

$$= (1-c)\,m\omega^2.r\cos\theta + (1-c)\,m\omega^2.r\cos(90^\circ+\theta)$$

$$= (1-c)\,m\omega^2.r(\cos\theta - \sin\theta)$$

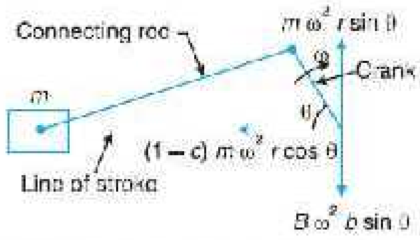


Fig. 22.4 Variation of tractive force.

We know that unbalanced force along the line of stroke for cylinder 1

The tractive force is maximum or minimum when (cos θ − sin θ) is maximum or minimum. For (cos θ − sin θ) to be maximum or minimum,

$$\frac{d}{d\theta}(\cos\theta - \sin\theta) = 0 \quad \text{or} \quad -\sin\theta - \cos\theta = 0 \quad \text{or} \quad -\sin\theta = \cos\theta$$

$$\therefore \quad \tan\theta = -1 \quad \text{or} \quad \theta = 135^\circ \quad \text{or} \quad 315^\circ$$

Thus, the tractive force is maximum or minimum when θ = 135° or 315°.

∴ Maximum and minimum value of the tractive force or the variation in tractive force

$$= \pm(1-c)\, m.\omega^2.r(\cos 135^\circ - \sin 135^\circ) = \pm\sqrt{2}\,(1-c)\, m.\omega^2.r$$

The unbalanced forces along the line of stroke for the two cylinders constitute a couple about the centre line *YY* between the cylinders as shown in Fig. 22.5.

This couple has swaying effect about a vertical axis, and tends to sway the engine alternately in clockwise and anticlockwise directions. Hence the couple is known as ***swaying couple.***

a = Distance between the centre lines of the two cylinders.

Let *a* = Distance between the centre lines of the two cylinders.

∴ Swaying couple

$$= (1-c)m\omega^2.r\cos\theta \times \frac{a}{2}$$

$$-(1-c)m\omega^2.r\cos(90^\circ+\theta)\frac{a}{2}$$

$$= (1-c)m\omega^2.r \times \frac{a}{2}(\cos\theta + \sin\theta)$$

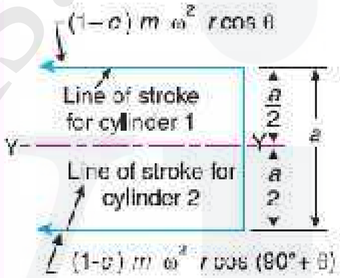


Fig. 22.5. Swaying couple.

The swaying couple is maximum or minimum when (cosθ + sinθ) is maximum or minimum. For (cosθ + sinθ) to be maximum or minimum,

$$\frac{d}{d\theta}(\cos\theta + \sin\theta) = 0 \quad \text{or} \quad -\sin\theta + \cos\theta = 0 \quad \text{or} \quad -\sin\theta = -\cos\theta$$

$$\therefore \quad \tan\theta = 1 \quad \text{or} \quad \theta = 45^\circ \quad \text{or} \quad 225^\circ$$

Thus, the swaying couple is maximum or minimum when θ = 45° or 225°.

∴ Maximum and minimum value of the swaying couple

$$= \pm(1-c)m\omega^2.r \times \frac{a}{2}(\cos 45^\circ + \sin 45^\circ) = \pm\frac{a}{\sqrt{2}}(1-c)m\omega^2.r$$

3. A shaft carries four masses A, B, C and D of magnitude 200 kg, 300 kg, 400 kg and 200 kg respectively and revolving at radii 80 mm, 70 mm, 60 mm and 80 mm in planes measured from A at 300 mm, 400 mm and 700 mm. The angles between the cranks measured anticlockwise are A to B 45°, B to C 70° and C to D 120°. The balancing masses are to be placed in planes X and Y. The distance between the planes A and X is 100 mm, between X and Y is 400 mm and between Y and D is 200 mm. If the balancing masses revolve at a radius of 100 mm, find their magnitudes and angular positions. (AU-MAY/JUNE-2012)

Solution. Given : m_A = 200 kg ; m_B = 300 kg ; m_C = 400 kg ; m_D = 200 kg ; r_A = 80 mm =0.08m ;

r_B = 70 mm = 0.07 m ; r_C = 60 mm = 0.06 m ; r_D = 80 mm = 0.08 m ; r_X = r_Y = 100 mm = 0.1 m

Let m_X = Balancing mass placed in plane *X*, and m_Y = Balancing mass placed in plane *Y*.

The position of planes and angular position of the masses (assuming the mass A as horizontal) are shown in Fig. 21.8 (*a*) and (*b*) respectively.

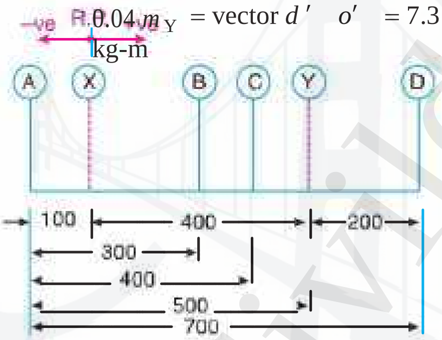
Assume the plane *X* as the reference plane (*R.P.*). The distances of the planes to the right of plane *X* are taken as + ve while the distances of the planes to the left of plane *X* are taken as – ve. The data may be tabulated as shown in Table 21.2.

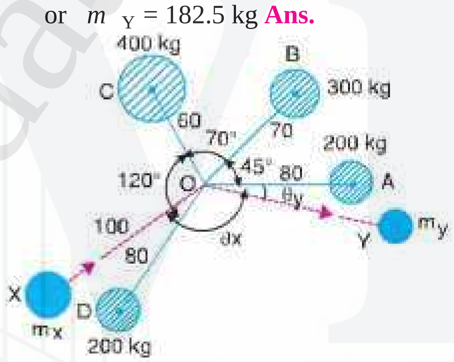
Plane (1)	Mass (m) kg (2)	Radius (r) m (3)	Cent.force ÷ ω^2 (m.r) kg-m (4)	Distance from Plane x(l) m (5)	Couple ÷ ω^2 (m.r.l) kg-m² (6)
A	200	0.08	16	– 0.1	– 1.6
X(R.P.)	m_X	0.1	0.1 m_X	0	0
B	300	0.07	21	0.2	4.2
C	400	0.06	24	0.3	7.2
Y	m_Y	0.1	0.1 m_Y	0.4	0.04 m_Y
D	200	0.08	16	0.6	9.6

The balancing masses m_X and m_Y and their angular positions may be determined graphi-cally as discussed below :

1. First of all, draw the couple polygon from the data given in Table 21.2 (column 6) as shown in Fig. 21.8 (*c*) to some suitable scale. The vector *d′ o′* represents the balanced couple. Since the balanced couple is proportional to 0.04 m_Y, therefore by measurement,

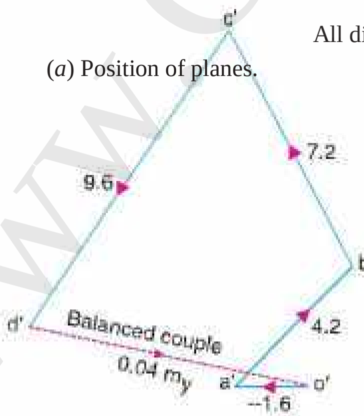
$0.04\, m_Y$ = vector $d'o'$ = 7.3 kg-m^2 or m_Y = 182.5 kg **Ans.**



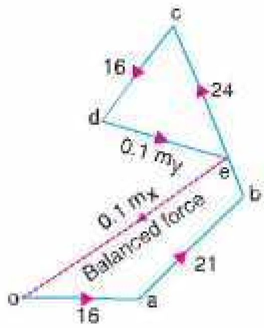


All dimensions in mm.

(*a*) Position of planes. (*b*) Angular position of masses.



(*c*) Couple polygon.



(*d*) Force polygon

The angular position of the mass m_Y is obtained by drawing Om_Y in Fig. 21.8 (*b*), parallel to vector *d′ o′*. By measurement, the angular position of m_Y is θ_Y = 12° in the clockwise direction from mass m_A

(*i.e.* 200 kg). **Ans**.

2. Now draw the force polygon from the data given in Table 21.2 (column 4) as shown in Fig. 21.8 (*d*). The vector *eo* represents the balanced force. Since the balanced force is proportional to 0.1 m_X, therefore by measurement,

0.1 m_X = vector *eo* = 35.5 kg-m or m_X = **355 kg Ans.**

The angular position of the mass m_X is obtained by drawing Om_X in Fig. 21.8 (*b*), parallel to vector *eo*. By measurement, the angular position of m_X is θ_X = 145° in the clockwise direction from mass m_A (*i.e.* 200 kg). **Ans**.

4. Four masses A, B, C and D as shown below are to be completely balanced.

	A	B	C	D
Mass (kg)	**—**	**30**	**50**	**40**
Radius (mm)	**180**	**240**	**120**	**150**

The planes containing masses B and C are 300 mm apart. The angle between planes containing B and C is 90°. B and C make angles of 210° and 120° respectively with D in the same sense. Find :

1. The magnitude and the angular position of mass A ; and

2. The position of planes A and D. (AU-NOV/DEC-2011)

Solution. Given : r_A = 180 mm = 0.18 m ; m_B = 30 kg ; r_B = 240 mm = 0.24 m ; m_C = 50 kg ; r_C = 120 mm = 0.12 m ; m_D = 40 kg ; r_D = 150 mm = 0.15 m ; ∠ *BOC* = 90° ; ∠ *BOD* = 210° ; ∠ *COD* = 120°

1. *The magnitude and the angular position of mass A*

Let m_A = Magnitude of Mass *A*,

x = Distance between the planes *B* and *D*, and *y* = Distance between the planes *A* and *B*.

The position of the planes and the angular position of the masses is shown in Fig. 21.9 (*a*) and (*b*) respectively.

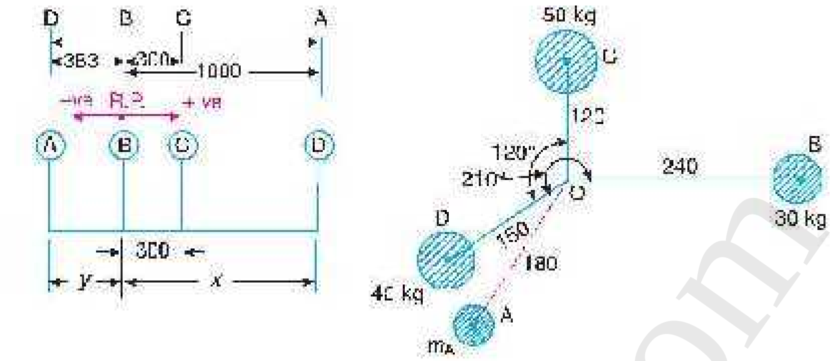
Assuming the plane *B* as the reference plane (*R.P.*) and the mass *B* (m_B) along the horizon-tal line as shown in Fig. 21.9 (*b*), the data may be tabulated as below :

Plane (1)	Mass (m) kg (2)	Radius (r) m (3)	Cent.force ÷ ω^2 (m.r) kg-m (4)	Distance from plane B (l) m (5)	Couple ÷ ω^2 (m.r.l) kg-m^2 (6)
A	m_A	0.18	0.08 m_A	− y	− 0.18 m_A y
B (R.P.)	30	0.24	7.2	0	0
C	50	0.12	6	0.3	1.8
D	40	0.15	6	x	6x

The magnitude and angular position of mass *A* may be determined by drawing the force polygon from the data given in Table 21.3 (Column 4), as shown in Fig. 21.9 (*c*), to some suitable scale. Since the masses are to be completely balanced, therefore the force polygon must be a closed figure. The closing side (*i.e.* vector *do*) is proportional to 0.18 m_A. By measurement,

0.18 m_A = Vector *do* = 3.6 kg-m or m_A = 20 kg **Ans.**

In order to find the angular position of mass *A*, draw *OA* in Fig. 21.9 (*b*) parallel to vector *do*. By measurement, we find that the angular position of mass *A* from mass *B* in the anticlockwise direction is $\angle$ *AOB* = **236° Ans**.



All dimensions in mm.

(*c*) Force polygon. (*d*) Couple polygon.

Fig. 21.9.

2. *Position of planes A and D*

The position of planes *A* and *D* may be obtained by drawing the couple polygon, as shown in Fig. 21.9 (*d*), from the data given in Table 21.3 (column 6). The couple polygon is drawn as discussed below :

1. Draw vector $o'c'$ parallel to *OC* and equal to 1.8 kg-m^2, to some suitable scale.

2. From points c' and o', draw lines parallel to *OD* and *OA* respectively, such that they intersect at point d'. By measurement, we find that

$$6\,x = \text{vector } c'd' = 2.3 \text{ kg-m}^2 \text{ or } x = 0.383 \text{ m}$$

We see from the couple polygon that the direction of vector $c'd'$ is opposite to the direction of mass *D*. Therefore the plane of mass *D* is 0.383 m or 383 mm towards left of plane *B* and not towards right of plane *B* as already assumed. **Ans**.

Again by measurement from couple polygon,

$$-0.18\, m_A . y = \text{vector } o'd' = 3.6 \text{ kg-m}^2$$

$$-0.18 \times 20\, y = 3.6 \quad \text{or} \quad y = -1 \text{ m}$$

The negative sign indicates that the plane *A* is not towards left of *B* as assumed but it is 1 m or 1000 mm towards right of plane *B*. **Ans.**

5. An inside cylinder locomotive has its cylinder centre lines 0.7 m apart and has a stroke of 0.6 m. The rotating masses per cylinder are equivalent to 150 kg at the crank pin, and the reciprocating masses per cylinder to 180 kg. The wheel centre lines are 1.5 m apart. The cranks are at right angles.

The whole of the rotating and 2/3 of the recipro-cating masses are to be balanced by masses placed at a radius of 0.6 m. Find the magnitude and direction of the balancing masses.

Find the fluctuation in rail pressure under one wheel, variation of tractive effort and the magnitude of swaying couple at a crank speed of 300 r.p.m. (AU-NOV/DEC-2008)

Solution. Given : $a = 0.7$ m; $l_B = l_C = 0.6$ m or
$r_B = r_C = 0.3$ m; $m_1 = 150$ kg; $m_2 = 180$ kg; $c = 2/3$; $r_A = r_D = 0.6$ m; $N = 300$ r.p.m. or
$\omega = 2 \cdot 300 / 60 = 31.42$ rad/s

We know that the equivalent mass of the rotating parts to be balanced per cylinder at the crank pin,

$m = m_B = m_C = m_1 + c.m_2 = 150 + \frac{2}{3} \times 180 = 270$ kg

Magnitude and direction of the balancing masses

Let m_A and m_D = Magnitude of the balancing masses

θ_A and θ_D = Angular position of the balancing masses m_A and m_D from the first crank B.

The magnitude and direction of the balancing masses may be determined graphically as discussed below :

1. First of all, draw the space diagram to show the positions of the planes of the wheels and the cylinders, as shown in Fig. 22.7 (*a*). Since the cranks of the cylinders are at right angles, therefore assuming the position of crank of the cylinder *B* in the horizontal direc-tion, draw *OC* and *OB* at right angles to each other as shown in Fig. 22.7 (*b*).

Tabulate the data as given in the following table. Assume the plane of wheel *A* as the reference plane.

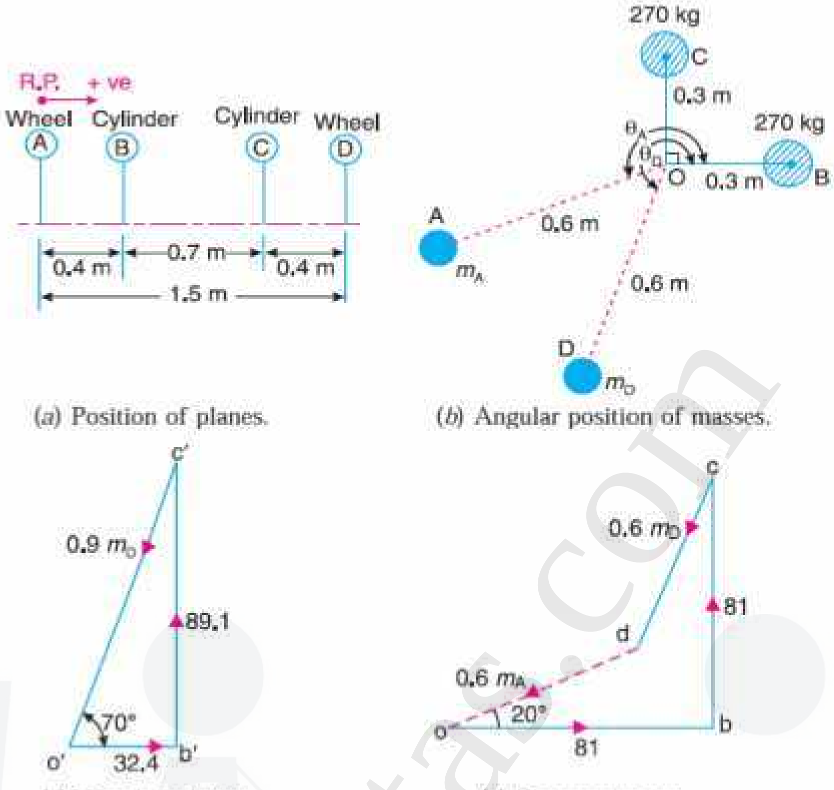
Plane (1)	mass. (m) kg (2)	Radius (r)m (3)	Cent. force ÷ ω^2 (m.r) kg-m (4)	Distance from plane A (l)m (5)	Couple ÷ ω^2 (m.r.l) kg-m² (6)
A (R.P.)	m_A	0.6	0.6 m_A	0	0
B	270	0.3	81	0.4	32.4
C	270	0.3	81	1.1	89.1
D	m_D	0.6	0.6m_D	1.5	0.9 m_D

3. Now, draw the couple polygon from the data given in Table 22.1 (column 6), to some suitable scale, as shown in Fig 22.7 (*c*). The closing side *c 'o'* represents the balancing couple and it is proportional to 0.9 m_D. Therefore, by measurement,

$$0.9\ m_D = \text{vector } c'o' = 94.5 \text{ kg-m}^2 \quad \text{or} \quad m_D = 105 \text{ kg} \quad \textbf{Ans.}$$

$$D = \frac{m_1}{m} \times 105 = \frac{150}{270} \times 105 = 58.3 \text{ kg}$$

and balancing mass for reciprocating masses.



(*a*) Position of planes. (*b*) Angular position of masses.

(*c*) Couple polygon. (*d*) Force polygon.

Fig. 22.7

4. To determine the angular position of the balancing mass *D*, draw *OD* in Fig. 22.7 (*b*) parallel to vector $c'o'$. By measurement,

$$\theta_D = 250°$$

Ans.

5. In order to find the balancing mass *A*, draw the force polygon from the data given in Table 22.1 (column 4), to some suitable scale, as shown in Fig. 22.7 (*d*), The vector *do* represents the balancing force and it is proportional to 0.6 m_A. Therefore by measurement,

$0.6m_A$ = vector *do* = 63 kg-m or m_A = 105 kg **Ans.**

6. To determine the angular position of the balancing mass *A*, draw *OA* in Fig. 22.7 (*b*) parallel to vector *do*. By measurement,

$$\theta_A = 200°$$

Ans.

Fluctuation in rail pressure

We know that each balancing mass105 kg

∴ Balancing mass for rotating masses,

$$B = \frac{c\,m_2}{m} \times 105 = \frac{2}{3} \times \frac{180}{270} \times 105 = 46.6 \text{ kg}$$

This balancing mass of 46.6 kg for reciprocating masses gives rise to the centrifugal force.

∴ Fluctuation in rail pressure or hammer blow

$$= B\omega^2.b = 46.6\,(31.42)^2\,0.6 = 27\,602 \text{ N. Ans.} \qquad \ldots (\because b = r_A = r_D)$$

Variation of tractive effort

We know that maximum variation of tractive effort

,

$$= \pm\sqrt{2}(1-c)\,m_2.\omega^2.r = \pm\sqrt{2}\left(1-\frac{2}{3}\right)180(31.42)^2\,0.3\,\text{N}$$

$$= \pm 25\ 127\ \text{N Ans.} \quad \ldots (\because\ r = r_B = r_C)$$

Swaying couple

We know that maximum swaying couple

$$= \frac{a(1-c)}{\sqrt{2}} \times m_2.\omega^2.r = \frac{0.7\left(1-\frac{2}{3}\right)}{\sqrt{2}} \times 180(31.42)^2\,0.3\ \text{N-m}$$

$$= 8797\ \text{N-m Ans.}$$

6. The following data apply to an outside cylinder uncoupled locomotive :
Mass of rotating parts per cylinder = 360 kg ; Mass of reciprocating parts per cylinder = 300 kg ; Angle between cranks = 90° ; Crank radius = 0.3 m ; Cylinder centres = 1.75 m ; Radius of balance masses = 0.75 m ; Wheel centres = 1.45 m. If whole of the rotating and two-thirds of reciprocating parts are to be balanced in planes of the driving wheels, find :

1. Magnitude and angular positions of balance masses,

2. Speed in kilometres per hour at which the wheel will lift off the rails when the load on each driving wheel is 30 kN and the diameter of tread of driving wheels is 1.8 m, and

3. Swaying couple at speed arrived at in (2) above. (AU-NOV/DEC-2013)

Solution : Given : m_1 = 360 kg ; m_2 = 300 kg ; ∠ AOD = 90° ; $r_A = r_D$ = 0.3 m ; a = 1.75 m ; $r_B = r_C$ = 0.75 m ; c = 2 / 3.

We know that the equivalent mass of the rotating parts to be balanced per cylinder,

$$m = m_A = m_D = m_1 + c.m_2 = 360 + \frac{2}{3} \times 300 = 560\ \text{kg}$$

1. ***Magnitude and angular position of balance masses***

Let m_B and m_C = Magnitude of the balance masses, and

θ_B and θ_C = angular position of the balance masses m_B and m_C from the crank A.

The magnitude and direction of the balance masses may be determined, graphically, as discussed below :

1. First of all, draw the positions of the planes of the wheels and the cylinders as shown in Fig. 22.11 (a). Since the cranks of the two cylinders are at right angles, therefore assum-ing the position of the cylinder A in the horizontal direction, draw OA and OD at right angles to each other as shown in Fig. 22.11 (b).

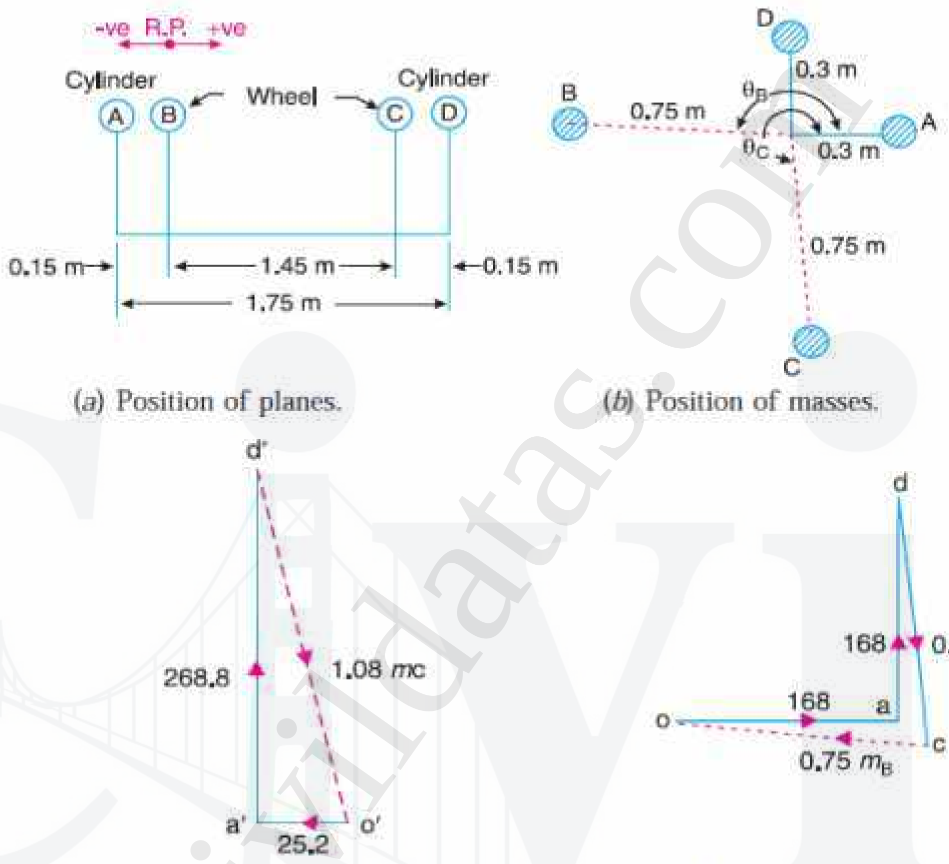
Plane (1)	*Mass (m) kg* (2)	*Radius (r) m* (3)	*Cent. force ÷ ω^2 (m.r) kg-m* (4)	*Distance from plane B(l) m* (5)	*Couple ÷ ω^2 (m.r.l) kg-m²* (6)
A	560	0.3	168	– 0.15	– 25.2
B (R.P.)	m_B	0.75	0.75 m_B	0	0
C	m_C	0.75	0.75 m_C	1.45	1.08 m_C
D	560	0.3	168	1.6	268.8

2. Assuming the plane of wheel B as the reference plane, the data may be tabulated as be-low:

3. Now draw the couple polygon with the data given in Table 22.4 column (6), to some suitable scale as shown in Fig. 22.11(*c*). The closing side $d'\ o'$ represents thebalancing couple

and it is proportional to 1.08 m_C. Therefore, by measurement,

$$1.08\ m_C = 269.6 \text{ kg-m}^2 \quad \text{or} \quad m_C = 249 \text{ kg} \quad \textbf{Ans.}$$



(*a*) Position of planes. (*b*) Position of masses.

(*c*) Couple polygon. (*d*) Force polygon.

Fig. 22.11

4. To determine the angular position of the balancing mass *C*, draw *OC* parallel to vector $d'\ o'$ as shown in Fig. 22.11 (*b*). By measurement, $\theta_C = 275°$ **Ans.**

5. In order to find the balancing mass *B*, draw the force polygon with the data given in Tablecolumn (4), to some suitable scale, as shown in Fig. 22.11 (*d*). The vector *co* represents the balancing force and it is proportional to 0.75 m_B. Therefore, by measurement, 0.75 m_B = 186.75 kg-m or m_B = 249 kg **Ans.**

4. To determine the angular position of the balancing mass *B*, draw *OB* parallel to vector *oc* as shown Fig. 22.11 (*b*). By measurement, $\theta_B = 174.5°$ **Ans.**

2. ***Speed at which the wheel will lift off the rails***

Given : $P = 30 \text{ kN} = 30 \times 10^3 \text{ N}$; $D = 1.8$ m

Let ω = Angular speed at which the wheels will lift off the rails in rad/s,

and v = Corresponding linear speed in km/h.

We know that each balancing mass, $m_B = m_C = 249$ kg

∴ Balancing mass for reciprocating parts,

$$B = \frac{c.m_l}{m} \times 219 = \frac{2}{3} \times \frac{300}{560} \times 249 = 89 \text{ kg}$$

We know that $\omega = \sqrt{\frac{P}{Rb}} = \sqrt{\frac{30 \times 10^3}{89 \times 0.75}} = 21.2 \text{ rad/s}$ ...(∵ $b = r_B = r_C$)

and $v = \omega \times D/2 = 21.2 \times 1.8/2 = 19.08 \text{ m/s}$

$= 19.08 \times 3600/1000 = 68.7$ km/h **Ans.**

3. Swaying couple at speed ω = 21.1 rad/s

We know that the swaying couple

$$= \frac{a(1-c)}{\sqrt{2}} \times m_l.\omega^2.r = \frac{1.75\left[1-\frac{2}{3}\right]}{\sqrt{2}} \times 300(21.2)^2 0.3 \text{ N-m}$$

$= 16\,687$ N m $= 16.687$ kN m **Ans.**

7. The three cranks of a three cylinder locomotive are all on the same axle and are set at 120°. The pitch of the cylinders is 1 meter and the stroke of each piston is 0.6 m. The reciprocating masses are 300 kg for inside cylinder and 260 kg for each outside cylinder and the planes of rotation of the balance masses are 0.8 m from the inside crank. If 40% of the reciprocating parts are to be balanced, find :

1. the magnitude and the position of the balancing masses required at a radius of 0.6 m ; And

2. the hammer blow per wheel when the axle makes 6 r.p.s. (AU-MAY/JUNE-2013)

Solution. Given : $\angle AOB = \angle BOC = \angle COA = 120°$; $l_A = l_B = l_C = 0.6$ m or $r_A = r_B = r_C = 0.3$ m ; $m_I = 300$ kg ; $m_O = 260$ kg ; $c = 40\% = 0.4$; $b_1 = b_2 = 0.6$ m ; $N = 6$ r.p.s.

$= 6 \times 2\pi = 37.7$ rad/s

Since 40% of the reciprocating masses are to be balanced, therefore mass of the reciprocat-ing parts to be balanced for each outside cylinder,

$m_A = m_C = c \times m_O = 0.4 \times 260 = 104$ kg and mass of the reciprocating parts to be balanced for inside cylinder,

$m_B = c \times m_1 = 0.4 \times 300 = 120$ kg

1. ***Magnitude and position of the balancing masses***

Let B_1 and B_2 = Magnitude of the balancing masses in kg,

θ_1 and θ_2 = Angular position of the balancing masses B_1 and B_2 from crank *A*.

The magnitude and position of the balancing masses may be determined graphically as discussed below :

1. First of all, draw the position of planes and cranks as shown in Fig. 22.8 (*a*) and (*b*) respectively. The position of crank *A* is assumed in the horizontal direction.
2. Tabulate the data as given in the following table. Assume the plane of balancing mass B_1 (*i.e.* plane 1) as the reference plane.

Table 22.2

Plane (1)	Mass (m)kg (2)	Radius (r) m (3)	Cent. force ÷ ω^2 (m.r) kg-m (4)	Distance from plane1 (l)m (5)	Couple ÷ ω^2 (m.r.l) kg-m^2 (6)
A	104	0.3	31.2	0.2	6.24
1 (R.P.)	B_1	0.6	0.6 B_1	0	0
B	120	0.3	36	0.8	28.8
2	B_2	0.6	0.6 B_2	1.6	0.96 B_2
C	104	0.3	31.2	1.8	56.16

3. Now draw the couple polygon with the data given in Table 22.2 (column 6), to some suitable scale, as shown in Fig. 22.8 (*c*). The closing side $c'\ o'$ represents the balancing couple
and it is proportional to 0.96 B_2. Therefore, by measurement,

$$0.96\ B_2 = \text{vector } c'\ o' = 55.2 \text{ kg-m}^2 \text{ or } B_2 = 57.5 \text{ kg}$$ **Ans.**

4. To determine the angular position of the balancing mass B_2, draw OB_2 parallel to vector $c'\ o'$ as shown in Fig. 22.8 (*b*). By measurement,

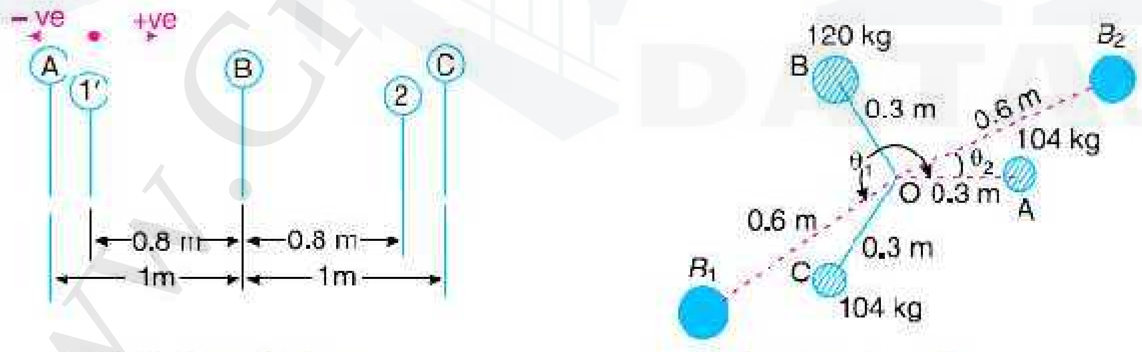
$\theta_2 = 24°$ **Ans.**

5. In order to find the balance mass B_1, draw the force polygon with the data given in Table 22.2(column 4), to some suitable scale, as shown in Fig. 22.8 (*d*). The closing side *co* represents the balancing force and it is proportional to 0.6 B_1. Therefore, by measurement,

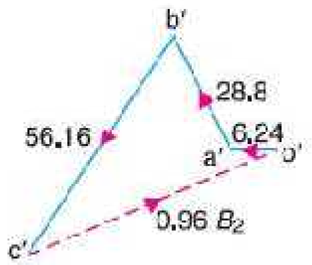
$0.6\ B_1 = \text{vector } co = 34.5$ kg-m or $B_1 = 57.5$ kg **Ans.**

6. To determine the angular position of the balancing mass B_1, draw OB_1 parallel to vector *co*, as shown in Fig. 22.8 (*b*). By measurement,

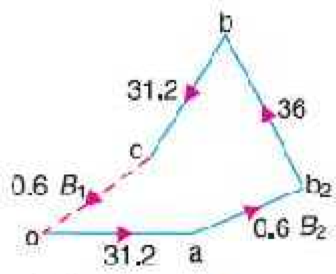
$\theta_1 = 215°$ **Ans.**



(*a*) Position of planes. (*b*) Position of cranks.



(c) Couple polygon.



(d) Force polygon.

Fig. 22.8

2. *Hammer blow per wheel*

We know that hammer blow per wheel

$$= B_1.\omega^2.b_1 = 57.5\ (37.7)^2\ 20.6 = 49\ 035\ \text{N}\ \textbf{Ans.}$$

8. The following data refer to two cylinder locomotive with cranks at 90° :

Reciprocating mass per cylinder = 300 kg ; Crank radius = 0.3 m ; Driving wheel diameter = 1.8 m ; Distance between cylinder centre lines = 0.65 m ; Distance between the driving wheel central planes = 1.55 m.

Determine : 1. the fraction of the reciprocating masses to be balanced, if the hammer blow is not to exceed 46 kN at 96.5 km. p.h. ; 2. the variation in tractive effort ; and 3. the maximum swaying couple. (AU-MAY/JUNE-2009)

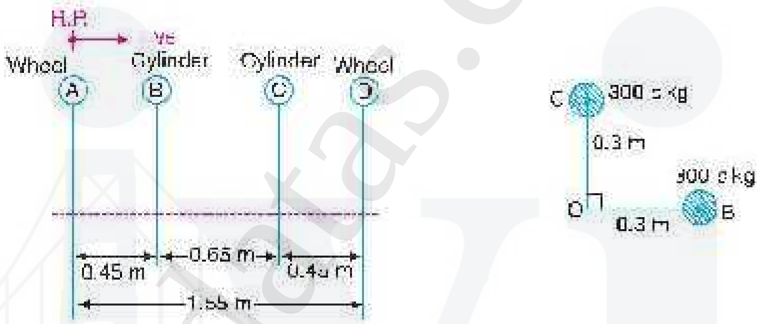
Solution. Given : m = 300 kg ; r = 0.3 m ; D = 1.8 m or R = 0.9 m ; a = 0.65 m ; Hammer blow = 46 kN = 46 × 10^3 N ; v = 96.5 km/h = 26.8 m/s

1. *Fraction of the reciprocating masses to be balanced*

Let c = Fraction of the reciprocating masses to be balanced, and

B = Magnitude of balancing mass placed at each of the driving wheels at radius b.

We know that the mass of the reciprocating parts to be balanced = $c.m$ = 300c kg



(*a*) Position of planes. (*b*) Position of cranks.

Fig. 22.9

The position of planes of the wheels and cylinders is shown in Fig. 22.9 (*a*), and the position of

Plane (1)	Mass (m) kg (2)	Radius (r) m (3)	Cent. force ÷ ω^2 ($m.r$) kg-m (4)	Distance from plane A (l) m (5)	Couple ÷ ω^2 ($m.r.l$) kg-m^2 (6)
A (R.P.)	B	b	B.b	0	0
B	300 c	0.3	90 c	0.45	40.5 c
C	300 c	0.3	90 c	1.1	99 c
D	B	b	B.b	1.55	1.55 B.b

cranks is shown in Fig 22.9 (*b*). Assuming the plane of wheel A as the reference plane, the data may be tabulated as below :

Now the couple polygon, to some suitable scale, may be drawn with the data given in Table 22.3 (column 6), as shown in Fig. 22.10. The closing side of the polygon (vector $c'\ o'$) represents the
balancing couple and is proportional to 1.55 $B.b$.

From the couple polygon,

$$1.55\ B.b = \sqrt{(40.5c)^2 + (99\ c)^2} = 107c$$

$\therefore\ B.b = 107\ c / 1.55 = 69\ c$

We know that angular speed,

$\omega = v/R = 26.8/0.9 = 29.8$ rad/s $\therefore$ Hammer blow,

$$46 \times 10^3 = B.\,\omega^2.b$$
$$= 69\,c\,(29.8)^2 = 61\,275\,c$$
$$\therefore \quad c = 46 \times 10^3/61\,275 = 0.751$$

Ans.

2. Variation in tractive effort

We know that variation in tractive effort,

$$= \pm\sqrt{2}(1-c)\,m\omega^2.r = \pm\sqrt{2}(1-0.751)\,300\,(29.8)^2\,0.3$$
$$= 28\,140 \text{ N} = 28.14 \text{ kN Ans.}$$

3.Maximum swaying couple

We know the maximum swaying couple

$$= \frac{a(1-c)}{\sqrt{2}} \times m\omega^2.r = \frac{0.65(1-0.751)}{\sqrt{2}} \times 300\,(29.8)^2\,0.3 = 9148 \text{ N-m}$$
$$= 9.148 \text{ kN-m Ans.}$$

2.14 REVIEW QUESTIONS

1. Discuss how a single revolving mass is balanced by two masses revolving in different planes.

2.How the different masses rotating in different planes are balanced ?
3. In order to have a complete balance of the several revolving masses in different planes?
4. What are in-line engines ? How are they balanced ? It is possible to balance them completely ?
5. The primary unbalanced force is maximum when the angle of inclination of the crank with the line of stroke is ____________
6. In order to facilitate the starting of locomotive in any position, the cranks of a locomotive, with two cylinders, are placed at . . . . . . to each other
7. When the primary direct crank of a reciprocating engine makes an angle θ with the line of stroke, then the secondary direct crank will make an angle of . . . . . with the line of stroke.

2.15 TUTORIAL PROBLEMS:

1. Four masses *A*, *B*, *C* and *D* revolve at equal radii and are equally spaced along a shaft. The mass *B* is 7 kg and the radii of *C* and *D* make angles of 90° and 240° respectively with the radius of *B*. Find the magnitude of the masses *A*, *C* and *D* and the angular position of *A* so that the system may be completely balanced.

[Ans. 5 kg ; 6 kg ; 4.67 kg ; 205° from mass *B* in anticlockwise direction]

2. A rotating shaft carries four masses *A*, *B*, *C* and *D* which are radially attached to it. The mass centres are 30 mm, 38 mm, 40 mm and 35 mm respectively from the axis of rotation. The masses *A*, *C* and *D* are 7.5 kg, 5 kg and 4 kg respectively. The axial distances between the planes of rotation of *A* and *B* is 400 mm and between *B* and *C* is 500 mm. The masses *A* and *C* are at right angles to each other. Find for a complete balance,

1. the angles between the masses *B* and *D* from mass *A*,
2. the axial distance between the planes of rotation of *C* and *D*,
3. the magnitude of mass *B*. **[Ans. 162.5°, 47.5° ; 511 mm : 9.24 kg]**

3. A three cylinder radial engine driven by a common crank has the cylinders spaced at 120°. The stroke is 125 mm, length of the connecting rod 225 mm and the mass of the reciprocating parts per cylinder 2 kg. Calculate the primary and secondary forces at crank shaft speed of 1200 r.p.m.

[Ans. 3000 N ; 830 N]

4. The pistons of a 60° twin *V*-engine has strokes of 120 mm. The connecting rods driving a common crank has a length of 200 mm. The mass of the reciprocating parts per cylinder is 1 kg and the

speed of the crank shaft is 2500 r.p.m. Determine the magnitude of the primary and secondary forces. **[Ans. 6.3 kN ; 1.1 kN]**

5. A twin cylinder *V*-engine has the cylinders set at an angle of 45°, with both pistons connected to the single crank. The crank radius is 62.5 mm and the connecting rods are 275 mm long. The reciprocating mass per line is 1.5 kg and the total rotating mass is equivalent to 2 kg at the crank radius. A balance mass fitted opposite to the crank, is equivalent to 2.25 kg at a radius of 87.5 mm. Determine for an engine speed of 1800 r.p.m. ; the maximum and minimum values of the primary and secondary forces due to the inertia of reciprocating and rotating masses.

[Ans. Primary forces : 3240 N (max.) and 1830 N (min.) Secondary forces : 1020 N (max.) and 470 N (min.)]

UNIT-III SINGLE DEGREE FREE VIBRATION

3.1 INTRODUCTION:

When a system is subjected to an initial disturbance and then left free to vibrate on its own, the resulting vibrations are referred to as free vibrations .**Free vibration** occurs when a mechanical system is set off with an initial input and then allowed to vibrate freely. Examples of this type of vibration are pulling a child back on a swing and then letting go or hitting a tuning fork and letting it ring. The mechanical system will then vibrate at one or more of its "natural frequencies" and damp down to zero.

3.2 BASIC ELEMENTS OF VIBRATION SYSTEM:

- Mass or Inertia
- Springiness or Restoring element
- Dissipative element (often called damper)
- External excitation

3.3 CAUSES OF VIBRATION:

Unbalance: This is basically in reference to the rotating bodies. The uneven distribution of mass in a rotating body contributes to the unbalance. A good example of unbalance related vibration would be the ―vibrating alert‖ in our mobile phones. Here a small amount of unbalanced weight is rotated by a motor causing the vibration which makes the mobile phone to vibrate. You would have experienced the same sort of vibration occurring in your front loaded washing machines that tend to vibrate during the ―spinning‖ mode.

Misalignment: This is an other major cause of vibration particularly in machines that are driven by motors or any other prime movers.

Bent Shaft: A rotating shaft that is bent also produces the the vibrating effect since it losses it rotation capability about its center.

Gears in the machine: The gears in the machine always tend to produce vibration, mainly due to their meshing. Though this may be controlled to some extent, any problem in the gearbox tends to get enhanced with ease.

Bearings: Last but not the least, here is a major contributor for vibration. In majority of the cases every initial problem starts in the bearings and propagates to the rest of the members of the machine. A bearing devoid of lubrication tends to wear out fast and fails quickly, but before this is noticed it damages the remaining components in the machine and an initial look would seem as if something had gone wrong with the other components leading to the bearing failure.

3.3.1 Effects of vibration:

(a)Bad Effects:

The presence of vibration in any mechanical system produces unwanted noise, high stresses, poor reliability, wear and premature failure of parts. Vibrations are a great source of human discomfort in the form of physical and mental strains.

(b)Good Effects:

A vibration does useful work in musical instruments, vibrating screens, shakers, relive pain in physiotherapy.

3.4 METHODS OF REDUCTION OF VIBRATION:

- -unbalance is its main cause, so balancing of parts is necessary.
- -using shock absorbers.
- -using dynamic vibration absorbers.
- -providing the screens (if noise is to be reduced)

3.5 TYPES OF VIBRATORY MOTION:

- Free Vibration
- Forced Vibration

3.6 TERMS USED VIBRATORY MOTION:

(a)Time period (or)period of vibration:

It is the time taken by a vibrating body to repeat the motion itself.time period is usually expressed in seconds.

(b) Cycle:

It is the motion completed in one time period.

(c) Periodic motion:

A motion which repeats itself after equal interval of time.

(d)Amplitude (X)

The maximum displacement of a vibrating body from the mean position.it is usually expressed in millimeter.

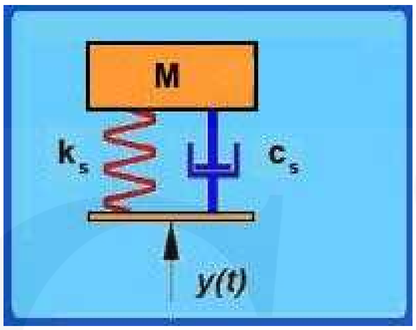
(e) Frequency (f)

The number of cycles completed in one second is called frequency

3.7 DEGREES OF FREEDOM:

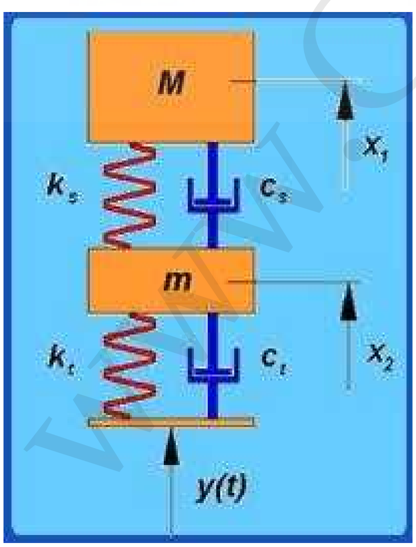
The minimum number of independent coordinates required to specify the motion of a system at any instant is known as D.O.F of the system.

3.7.1 Single degree of freedom system:



The system shown in this figure is what is known as a Single Degree of Freedom system. We use the term degree of freedom to refer to the number of coordinates that are required to specify completely the configuration of the system. Here, if the position of the mass of the system is specified then accordingly the position of the spring and damper are also identified. Thus we need just one coordinate (that of the mass) to specify the system completely and hence it is known as a single degree of freedom system.

3.7.2 Two degree of freedom system:



A two degree of freedom system With reference to automobile applications, this is referred as –quarter car‖ model. The bottom mass refers to mass of axle, wheel etc components which are below the suspension spring and the top mass refers to the mass of the portion of the car and passenger. Since we need to specify both the top and bottom mass positions to completely specify the system, this becomes a two degree of freedom system.

3.8 TYPES OF VIBRATORY MOTION:

The following types of vibratory motion are important from the subject point of view :

1. ***Free or natural vibrations.*** When no external force acts on the body, after giving it an initial displacement, then the body is said to be under *free or natural vibrations*. The frequency of the free vibrations is called *free or natural frequency*.

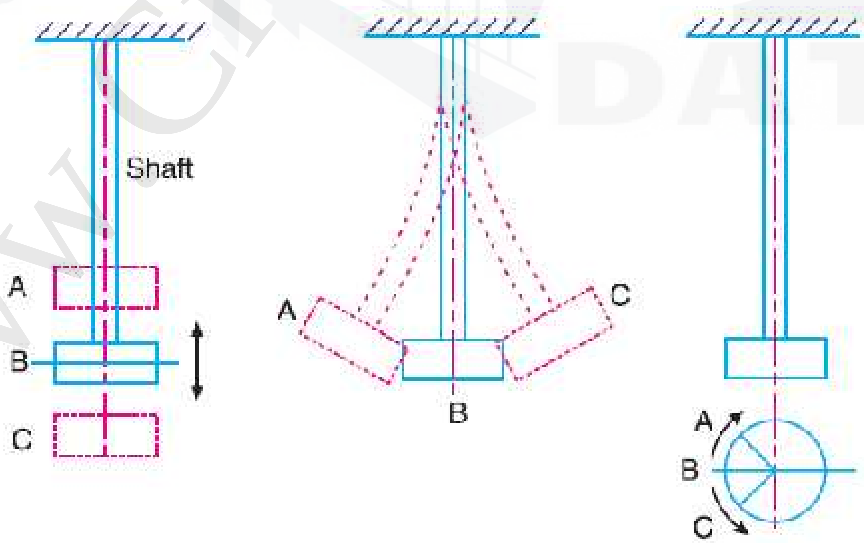
2. ***Forced vibrations.*** When the body vibrates under the influence of external force, then the body is said to be under *forced vibrations*. The external force applied to the body is a periodic disturbing force created by unbalance. The vibrations have the same frequency as the applied force.

Note : When the frequency of the external force is same as that of the natural vibrations, resonance takes place.

3. ***Damped vibrations.*** When there is a reduction in amplitude over every cycle of vibration, the motion is said to be *damped vibration*. This is due to the fact that a certain amount of energy possessed by the vibrating system is always dissipated in overcoming frictional resistances to the motion.

Types of Vibration:

(a)Longitudinal vibration (b)Transverse Vibration (c)Torsional Vibration.



B = Mean position ; A and C = Extreme positions.

(*a*) Longitudinal vibrations. (*b*) Transverse vibrations. (*c*) Torsional vibrations.

Longitudinal Vibration:

When the particles of the shaft or disc moves parallel to the axis of the shaft, then the vibrations known as longitudinal vibrations.

Free undamped longitudinal vibrations;

When a body is allowed to vibrate on its own, after giving it an initial displacement, then the ensuring vibrations are known as free or natural vibrations. When the vibrations take place parallel to the axis of constraint and no damping is provided, then it is called free undamped longitudinal vibrations.

3.9 NATURAL FREQUENCY OF FREE UNDAMPED LONGITUDINAL VIBRATION:

3.9.1 Eq u ilib riu m m et h od or Newt on ' s met h od :

Consider a constraint (*i.e.* spring) of negligible mass in an unstrained position.

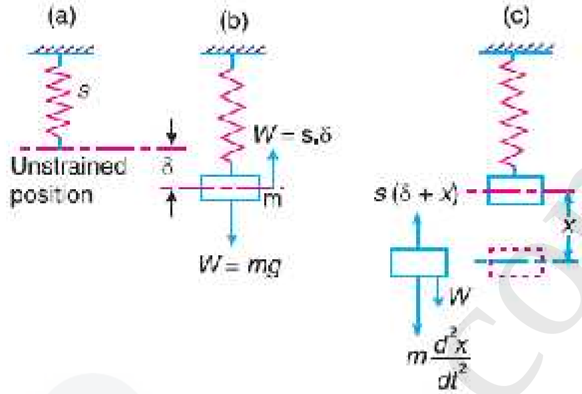
Let s = Stiffness of the constraint. It is the force required to produce unit displacement in the direction of vibration. It is usually expressed in N/m.

m = Mass of the body suspended from the constraint in kg,

W = Weight of the body in newtons = $m.g$.

δ = Static deflection of the spring in metres due to weight W newtons, and

x = Displacement given to the body by the external force, in metres.



Natural frequency of free longitudinal vibrations.

In the equilibrium position, as shown in Fig. 23.2 (*b*), the gravitational pull $W = m.g$, is balanced by a force of spring, such that $W = s.\delta$.

Since the mass is now displaced from its equilibrium position by a distance x, as shown in Fig. (*c*), and is then released, therefore after time t,

Restoring force $= W - s(\delta + x) = W - s.\delta - s.x$

$= s.\delta - s.\delta - s.x = -s.x$ . . . ($\because W = s.\delta$) . . . (*i*)

and Accelerating force = Mass × Acceleration

$$= m \times \frac{d^2x}{dt^2}$$ . . . (Taking downward force as positive) . . . (*ii*)

Equating equations (*i*) and (*ii*), the equation of motion of the body of mass m after time t is

$$m \times \frac{d^2x}{dt^2} = -s.x \quad \text{or} \quad m \times \frac{d^2x}{dt^2} + s.x = 0$$

$$\therefore \quad \frac{d^2x}{dt^2} + \frac{s}{m} \times x = 0 \qquad \text{. . . (iii)}$$

We know that the fundamental equation of simple harmonic motion is

$$\frac{d^2x}{dt^2} + \omega^2.x = 0 \qquad \text{. . . (iv)}$$

Comparing equations (*iii*) and (*iv*), we have

$$\omega = \sqrt{\frac{s}{m}}$$

$\therefore$ Time period, $t_p = \dfrac{2\pi}{\omega} = 2\pi\sqrt{\dfrac{m}{s}}$

and natural frequency, $f_n = \frac{1}{t_p} = \frac{1}{2\pi}\sqrt{\frac{s}{m}} = \frac{1}{2\pi}\sqrt{\frac{g}{\delta}}$ . . . ($\because$ $m.g = s.\delta$)

Taking the value of g as 9.81 m/s^2 and δ in metres,

$$f_n = \frac{1}{2\pi}\sqrt{\frac{9.81}{\delta}} = \frac{0.4985}{\sqrt{\delta}}\ \text{Hz}$$

Note : The value of static deflection δ may be found out from the given conditions of the problem. For longitudinal vibrations, it may be obtained by the relation,

$$\frac{\text{Stress}}{\text{Strain}} = E \quad \text{or} \quad \frac{W}{A} \times \frac{l}{\delta} = E \quad \text{or} \quad \delta = \frac{W.l}{E.A}$$

where δ = Static deflection *i.e.* extension or compression of the constraint,

W = Load attached to the free end of constraint,

l = Length of the constraint,

E = Young's modulus for the constraint, and

A = Cross-sectional area of the constraint.

3.9.2 Energy Method

In free vibrations, no energy is transferred into the system or from the system. Therefore, the total energy (sum of KE and PE)is constant and is same all the times.

We know that the kinetic energy is due to the motion of the body and the potential energy is with respect to a certain datum position which is equal to the amount of work required to move the body from the datum position. In the case of vibrations, the datum position is the mean or equilibrium position at which the potential energy of the body or the system is zero.

In the free vibrations, no energy is transferred to the system or from the system. Therefore the summation of kinetic energy and potential energy must be a constant quantity which is same at all the times. In other words,

$$\therefore \quad \frac{d}{dt}(K.E.+P.E.)=0$$

This industrial compressor uses compressed air to power heavy-duty construction tools. Compressors are used for jobs, such as breaking up concrete or paving, drilling, pile driving, sand-blasting and tunnelling. A compressor works on the same principle as a pump. A piston moves backwards and forwards inside a hollow cylinder, which compresses the air and forces it into a hollow chamber. A pipe or hose connected to the chamber channels the compressed air to the tools.

Note : This picture is given as additional information and is not a direct example of the current chapter.

We know that kinetic energy,

$$K.E.=\frac{1}{2}\times m\left(\frac{dx}{dt}\right)^2$$

and potential energy,

$$P.E.=\left(\frac{0+s.x}{2}\right)x=\frac{1}{2}\times s.x^2$$

... ($\because$ *P.E.* = Mean force × Displacement)

$$\therefore \quad \frac{d}{dt}\left[\frac{1}{2}\times m\left(\frac{dx}{dt}\right)^2+\frac{1}{2}\times s.x^2\right]=0$$

$$\frac{1}{2}\times m\times 2\times\frac{dx}{dt}\times\frac{d^2x}{dt^2}+\frac{1}{2}\times s\times 2x\times\frac{dx}{dt}=0$$

or $\quad m\times\frac{d^2x}{dt^2}+s.x=0 \quad$ or $\quad \frac{d^2x}{dt^2}+\frac{s}{m}\times x=0$ ... (Same as before)

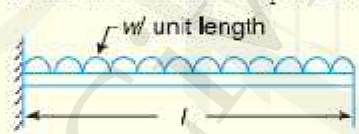
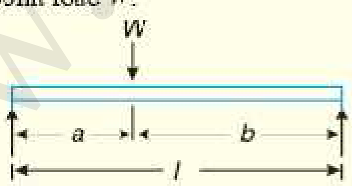
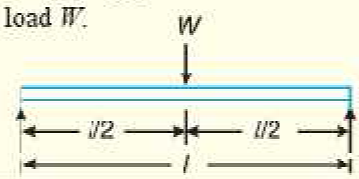
The time period and the natural frequency may be obtained as discussed in the previous method.

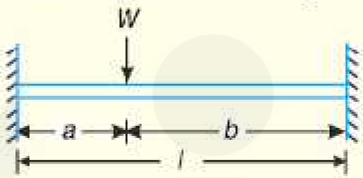
3.9.3 Rayleigh's method

In this method, the maximum kinetic energy at mean position is made equal to the maximum potential energy at the extreme position.

3.10 EQUIVALENT STIFFNESS OF SPRING.

(1) Springs in series
(2) Springs in parallel
(3) Combined springs
(4) Inclined springs

S.No.	Type of beam	Deflection (δ)
1.	Cantilever beam with a point load W at the free end. 	$\delta = \frac{Wl^3}{3EI}$ (at the free end)
2.	Cantilever beam with a uniformly distributed load of w per unit length. 	$\delta = \frac{wl^4}{8EI}$ (at the free end)
3.	Simply supported beam with an eccentric point load W. 	$\delta = \frac{Wa^2b^2}{3EIl}$ (at the point load)
4.	Simply supported beam with a central point load W. 	$\delta = \frac{Wl^3}{48EI}$ (at the centre)

S.No.	Type of beam	Deflection (δ)
5.	Simply supported beam with a uniformly distributed load of w per unit length. 	$\delta = \frac{5}{384} \times \frac{wl^4}{EI}$ (at the centre)
6.	Fixed beam with an eccentric point load W. 	$\delta = \frac{Wa^3b^3}{3EIl}$ (at the point load)
7.	Fixed beam with a central point load W. 	$\delta = \frac{Wl^3}{192EI}$ (at the centre)
8.	Fixed beam with a uniformly distributed load of w per unit length. 	$\delta = \frac{wl^4}{384EI}$ (at the centre)

3.11 DAMPING:

It is the resistance to the motion of a vibrating body. The vibrations associated with this resistance are known as damped vibrations.

3.11.1 Types of damping:

(1) Viscous damping
(2) Dry friction or coulomb damping
(3) Solid damping or structural damping
(4) Slip or interfacial damping.

3.11.2 Damping Coefficient:

The damping force per unit velocity is known as damping coefficient.

3.11.3 Equivalent damping coefficient:

Dampers may be connected either in series or in parallel to provide required damping.

3.12 DAMPED VIBRATION:

The vibrations associated with this resistance are known as damped vibrations.

3.12.1 Damping factor:

Damping factor can be defined as the ratio of actual damping coefficient to critical damping coefficient.

The ratio of the actual damping coefficient (c) to the critical damping coefficient (c_c) is known as *damping factor* or *damping ratio*. Mathematically,

$$\text{Damping factor} = \frac{c}{c_c} = \frac{c}{2m.\omega_n} \quad \ldots (\because\ c_c = 2\pi.\omega_n)$$

The damping factor is the measure of the relative amount of damping in the existing system with that necessary for the critical damped system.

Thus mainly three cases arise depending on the value of ξ

$\xi > 1 \Leftrightarrow$ *Overdamped System*

$\xi = 1 \Leftrightarrow$ *Critically damped System*

$\xi < 1 \Leftrightarrow$ *Underdamped System*

When $\xi \geq 1$ the system undergoes aperiodically decaying motion and hence such systems are said to be **Overdamped Systems**.

An example of such a system is a door damper – when we open a door and enter a room, we want the door to gradually close rather than exhibit oscillatory motion and bang into the person entering the room behind us! So the damper is designed such that $\xi \geq 1$

Critically damped motion ($\xi = 1$ a hypothetical borderline case separating oscillatory decay from a periodic decay) I the fastest decaying aperiodic motion.

When –$\xi < 1$‖, x(t) is a damped sinusoid and the system exhibits a vibratory motion whose amplitude keeps diminishing. This is the most common vibration case and we will spend most of our time studying such systems. These are referred to as **Underdamped systems**.

3.12.2 Logarithmic decrement:

It is defined as the natural logarithm of ratio of any two successive amplitudes of an under damped system. It is a dimensionless quantity.

We define Damping factor ξ as

$$\xi = \frac{c}{2\sqrt{km}}$$

$$\text{such that } \frac{4km}{c^2} = \frac{1}{\xi^2}$$

$$\text{And} \qquad \xi = \frac{c}{2m\omega_n}$$

$$\therefore \frac{c}{2m} = \xi\omega_n$$

3.13 TRANSVERSE VIBRATION:

When the particles of the shaft or disc moves approximately perpendicular to the axis of the shaft, then the vibrations known as transverse vibrations.

Consider a shaft of negligible mass, whose one end is fixed and the other end carries a body of weight *W*, as shown in Fig. 23.3.

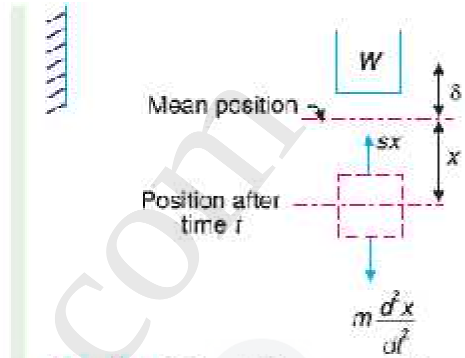


Fig. 23.3. Natural frequency of free transverse vibrations.

Let s = Stiffness of shaft,

δ = Static deflection due to weight of the body,

x = Displacement of body from mean position after time t.

m = Mass of body = W/g

As discussed in the previous article,

Restoring force $= s.x$ . . . (*i*)

and accelerating force $= m \times \frac{d^2x}{dt^2}$ . . . (*ii*)

Equating equations (*i*) and (*ii*), the equation of motion becomes

$$m \times \frac{d^2x}{dt^2} = -s.x \quad \text{or} \quad m \times \frac{d^2x}{dt^2} + s.x = 0$$

$$\therefore \quad \frac{d^2x}{dt^2} + \frac{s}{m} \times x = 0 \qquad \text{. . . (Same as before)}$$

Hence, the time period and the natural frequency of the transverse vibrations are same as that of longitudinal vibrations. Therefore

Time period, $t_p = 2\pi\sqrt{\frac{m}{s}}$

and natural frequency, $f_n = \frac{1}{t_p} = \frac{1}{2\pi}\sqrt{\frac{s}{m}} = \frac{1}{2\pi}\sqrt{\frac{g}{\delta}}$

Note : The shape of the curve, into which the vibrating shaft deflects, is identical with the static deflection curve of a cantilever beam loaded at the end. It has been proved in the text book on Strength of Materials, that the static deflection of a cantilever beam loaded at the free end is

$$\delta = \frac{Wl^3}{3EI} \text{ (in metres)}$$

where W = Load at the free end, in newtons,

l = Length of the shaft or beam in metres,

E = Young's modulus for the material of the shaft or beam in N/m^2, and

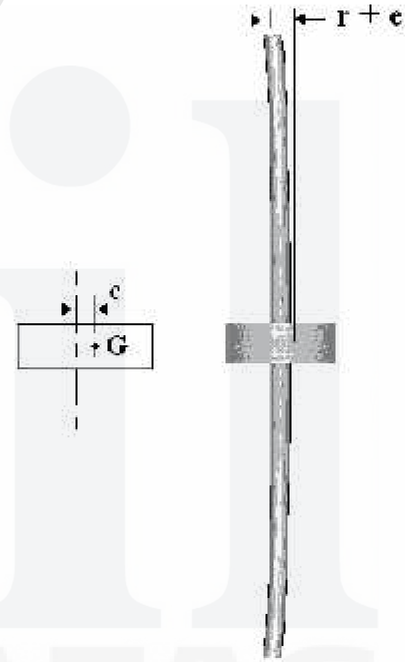
I = Moment of inertia of the shaft or beam in m^4.

3.13.1 Whirling speed of shaft:

The speed, at which the shaft runs so that the additional deflection of the shaft from the axis of rotation becomes infinite, is known as critical or whirling speed.

No shaft can ever be perfectly straight or perfectly balanced. When an element of mass is a distance from the axis of rotation, centrifugal force, will tend to pull the mass outward. The elastic properties of the shaft will act to restore the –straightness‖. If the frequency of rotation is equal to one of the resonant frequencies of the shaft, whirling will occur. In order to save the machine from failure, operation at such whirling speeds must be avoided.

When a shaft rotates, it may well go into transverse oscillations. If the shaft is out of balance, the resulting centrifugal force will induce the shaft to vibrate. When the shaft rotates at a speed equal to the natural frequency of transverse oscillations, this vibration becomes large and shows up as a whirling of the shaft. It also occurs at multiples of the resonant speed. This can be very damaging to heavy rotary machines such as turbine generator sets and the system must be carefully balanced to reduce this effect and designed to have a natural frequency different to the speed of rotation. When starting or stopping such machinery, the critical speeds must be avoided to prevent damage to the bearings and turbine blades. Consider a weightless shaft as shown with a mass M at the middle. Suppose the centre of the mass is not on the centre line.



The whirling frequency of a symmetric cross section of a given length between two points is given by:

$$N = 94.25\sqrt{\frac{E\ I}{m\ L^3}}\ \text{RPM}$$

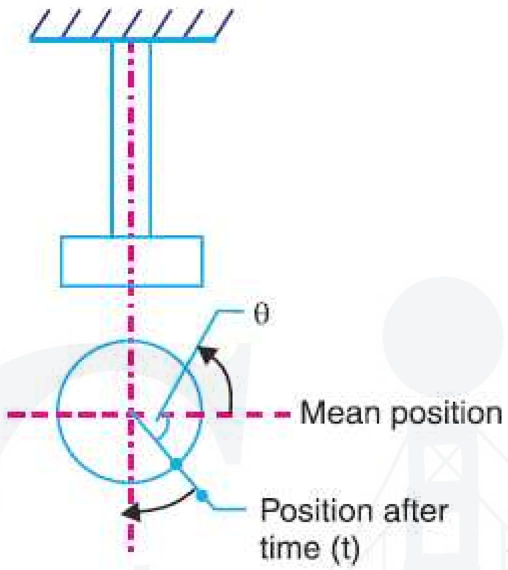
Where E = young's modulus, I = Second moment of area, m = mass of the shaft, L= length of the shaft between points

A shaft with weights added will have an angular velocity of N (rpm) equivalent as follows:

$$\frac{1}{N_N^2} = \frac{1}{N_A^2} + \frac{1}{N_B^2} + \cdots + \frac{1}{N_n^2}$$

3.14 TORSIONAL VIBRATION:

When the particles of the shaft or disc move in a circle about the axis of the shaft, then the vibrations known as tensional vibration



Natural frequency of free torsional vibrations.

Let θ = Angular displacement of the shaft from mean position after time t in radians,

m = Mass of disc in kg,

I = Mass moment of inertia of disc in kg-m^2 $= m.k^2$,

k = Radius of gyration in metres,

q = Torsional stiffness of the shaft in N-m.

$\therefore$ Restoring force $= q.\theta$ ... (*i*)

and accelerating force $= I \times \dfrac{d^2\theta}{dt^2}$ ... (*ii*)

Equating equations (*i*) and (*ii*), the equation of motion is

$$I \times \frac{d^2\theta}{dt^2} = -q.\theta$$

or

$$I \times \frac{d^2\theta}{dt^2} + q.\theta = 0$$

$$\therefore \quad \frac{d^2\theta}{dt^2} + \frac{q}{I} \times \theta = 0 \qquad \text{. . . (iii)}$$

The fundamental equation of the simple harmonic motion is

$$\frac{d^2\theta}{dt^2} + \omega^2 .x = 0 \qquad \text{. . . (iv)}$$

Comparing equations (*iii*) and (*iv*),

$$\omega = \sqrt{\frac{q}{I}}$$

$\therefore$ Time period, $t_p = \dfrac{2\pi}{\omega} = 2\pi\sqrt{\dfrac{I}{q}}$

and natural frequency , $f_n = \dfrac{1}{t_p} = \dfrac{1}{2\pi}\sqrt{\dfrac{q}{I}}$

The value of the torsional stiffness q may be obtained from the torsion equation,

$$\frac{T}{J} = \frac{C\theta}{l} \quad \text{or} \quad \frac{T}{\theta} = \frac{C.J}{l}$$

$$q = \frac{C.J}{l} \qquad \ldots \left(\because \frac{T}{\theta} = q \right)$$

where C = Modulus of rigidity for the shaft material,

J = Polar moment of inertia of the shaft cross-section,

$= \frac{\pi}{32} d^4$; d is the diameter of the shaft, and

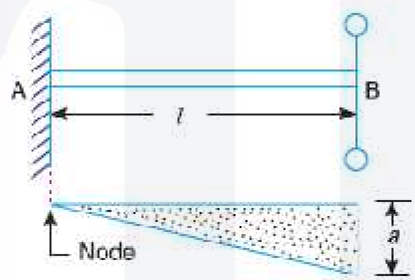
l = Length of the shaft.

3.14.1 Torsional vibration of a single rotor system:

We have already discussed that for a shaft fixed at one end and carrying a rotor at the free end as shown in Fig. the natural frequency of torsional vibration.

$$f_n \quad \frac{1}{2}\sqrt{\frac{q}{I}} \quad \frac{1}{2}\sqrt{\frac{C.J}{l.I}}$$

$$\ldots \quad q \quad \frac{C.J}{l}$$



Free torsional vibrations of a single rotor system.

where C = Modulus of rigidity for shaft material,

J = Polar moment of inertia of shaft

$= \frac{}{32} d^4$

d = Diameter of shaft,

l = Length of shaft,

m = Mass of rotor,

---

k = Radius of gyration of rotor, and

I = Mass moment of inertia of rotor = $m.k^2$

A little consideration will show that the amplitude of vibration is zero at A and maximum at B, as shown in Fig. It may be noted that the point or the section of the shaft whose amplitude of torsional vibration is zero, is known as ***node.*** In other words, at the node, the shaft remains unaffected by the vibration.

3.14.2 Torsional vibration of a two rotor system:

Consider a two rotor system as shown in Fig. It consists of a shaft with two rotors at its ends. In this system, the torsional vibrations occur only when the two rotors *A* and *B* move in opposite directions *i.e.* if *A* moves in anticlockwise direction then *B* moves in clockwise direction at the same instant and *vice versa*. It may be noted that the two rotors must have the same frequency.

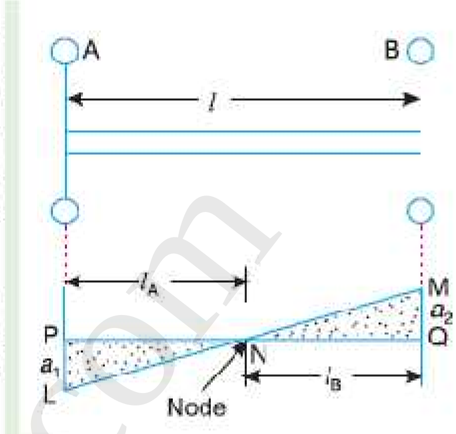


Fig Free torsional vibrations of a two rotor system.

We see from Fig. that the node lies at point *N*. This point can be safely assumed as a fixed end and the shaft may be considered as two separate shafts *N P* and *N Q* each fixed to one of its ends and carrying rotors at the free ends.

Let l = Length of shaft,

l_A = Length of part *NP* *i.e.* distance of node from rotor *A*,

l_B = Length of part *NQ*, *i.e.* distance of node from rotor *B*,

I_A = Mass moment of inertia of rotor *A*,

I_B = Mass moment of inertia of rotor *B*,

d = Diameter of shaft,

J = Polar moment of inertia of shaft, and

C = Modulus of rigidity for shaft material.

Natural frequency of torsional vibration for rotor *A*,

$$f_{nA} \quad \frac{1}{2}\sqrt{\frac{CJ}{l_A \cdot I_A}} \qquad \ldots (i)$$

and natural frequency of torsional vibration for rotor *B*,

$$f_{nB} \quad \frac{1}{2}\sqrt{\frac{C.J}{l_B \cdot I_B}} \qquad \ldots (ii)$$

Since $f_{nA} = f_{nB}$, therefore

$$\frac{1}{2}\sqrt{\frac{C.J}{l_A \cdot I_A}} \quad \frac{1}{2}\sqrt{\frac{C.J}{l_B \cdot I_B}} \quad \text{or} \quad l_A \cdot I_A = l_B \cdot I_B \qquad \ldots (iii)$$

$$l_A \quad \frac{l_B \cdot I_B}{I_A}$$

We also know that

$$l \quad l_A \quad l_B \qquad \ldots (iv)$$

3.14.3 Torsionally equivalent shaft:

we have assumed that the shaft is of uniform diameter. But in actual practice, the shaft may have variable diameter for different lengths. Such a shaft may, theoretically, be replaced by an equivalent shaft of uniform diameter.

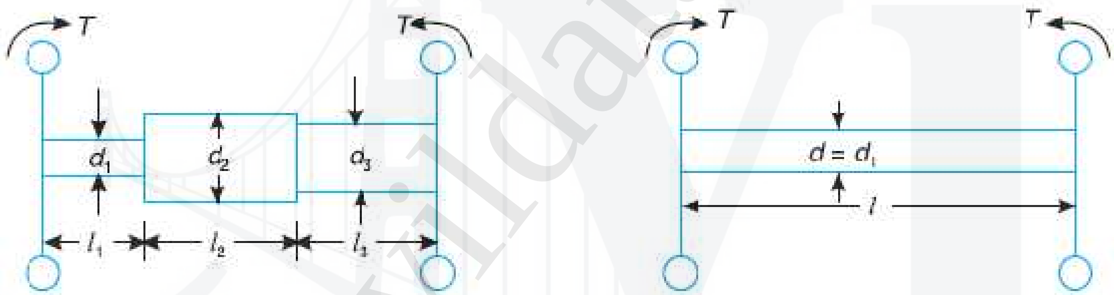
Consider a shaft of varying diameters as shown in Fig. (*a*). Let this shaft is replaced by an equivalent shaft of uniform diameter *d* and length *l* as shown in Fig. (*b*).These two shafts must have the same total angle of twist when equal opposing torques *T* are applied at their opposite ends.

Let d_1, d_2 and d_3 = Diameters for the lengths l_1, l_2 and l_3 respectively,

$_1$, $_2$ and $_3$ = Angle of twist for the lengths l_1, l_2 and l_3 respectively,

= Total angle of twist, and

J_1, J_2 and J_3 = Polar moment of inertia for the shafts of diameters d_1, d_2 and d_3 respectively.



(*a*) Shaft of varying diameters. (*b*) Torsionally equivalent shaft.

Fig 24.8

Since the total angle of twist of the shaft is equal to the sum of the angle of twists of different lengths, therefore

$$\quad _1 \quad _2 \quad _3$$

or $$\frac{T.l}{C.J} \quad \frac{T.l_1}{C.J_1} \quad \frac{T.l_2}{C.J_2} \quad \frac{T.l_3}{C.J_3}$$

$$\frac{l}{J} \quad \frac{l_1}{J_1} \quad \frac{l_2}{J_2} \quad \frac{l_3}{J_3}$$

$$\frac{l}{\frac{}{32} d^4} \quad \frac{l_1}{\frac{}{32}(d_1)^4} \quad \frac{l_2}{\frac{}{32}(d_2)^4} \quad \frac{l_3}{\frac{}{32}(d_3)^4}$$

$$\frac{l}{d^4} \quad \frac{l_1}{(d_1)^4} \quad \frac{l_2}{(d_2)^4} \quad \frac{l_3}{(d_3)^4}$$

In actual calculations, it is assumed that the diameter d of the equivalent shaft is equal to one of the diameter of the actual shaft. Let us assume that $d = d_1$.

$$\frac{l}{(d_1)^4} \quad \frac{l_1}{(d_1)^4} \quad \frac{l_2}{(d_2)^4} \quad \frac{l_3}{(d_3)^4}$$

or $$l \quad l_1 \quad l_2 \left(\frac{d_1}{d_2}\right)^4 \quad l_3 \left(\frac{d_1}{d_3}\right)^4$$

This expression gives the length l of an equivalent shaft.

3.15 SOLVED ROBLEMS

1. A machine of mass 75 kg is mounted on springs and is fitted with a dashpot to damp out vibrations. There are three springs each of stiffness 10 N/mm and it is found that the amplitude of vibration diminishes from 38.4 mm to 6.4 mm in two complete oscillations. Assuming that the damping force varies as the velocity, determine : 1. the resistance of the dash-pot at unit velocity ; 2. the ratio of the frequency of the damped vibration to the frequency of the undamped vibration ; and 3. the periodic time of the damped vibration.

Solution. Given : m = 75 kg ; s = 10 N/mm = 10 ×10^3 N/m ; x_1 = 38.4 mm = 0.0384 m ; x_3 = 6.4 mm = 0.0064 m

Since the stiffness of each spring is 10 × 10^3 N/m and there are 3 springs, therefore total stiffness,

$$s = 3\times10\times10^3 = 30\times10^3 \text{ N/m}$$

We know that natural circular frequency of motion,

$$\omega_n = \sqrt{\frac{s}{m}} = \sqrt{\frac{30\times10^3}{75}} = 20 \text{ rad/s}$$

1. ***Resistance of the dashpot at unit velocity***

Let c = Resistance of the dashpot in newtons at unit velocity *i.e.* in N/m/s,

x_2 = Amplitude after one complete oscillation in metres, and

x_3 = Amplitude after two complete oscillations in metres.

We know that $$\frac{x_1}{x_2} = \frac{x_2}{x_3}$$

$$\therefore \quad \left(\frac{x_1}{x_2}\right)^2 = \frac{x_1}{x_3} \qquad \ldots \left[\because \frac{x_1}{x_3} = \frac{x_1}{x_2} \times \frac{x_2}{x_3} = \frac{x_1}{x_2} \times \frac{x_1}{x_2} = \left(\frac{x_1}{x_2}\right)^2\right]$$

or
$$\frac{x_1}{x_2} = \left(\frac{x_1}{x_3}\right)^{1/2} = \left(\frac{0.0384}{0.0064}\right)^{1/2} = 2.45$$

We also know that

$$\log_e\left(\frac{x_1}{x_2}\right) = a \times \frac{2\pi}{\sqrt{(\omega_n)^2 - a^2}}$$

2. 2. Ratio of the frequency of the damped vibration to the frequency of undamped vibration

Let
$$f_{n1} = \text{Frequency of damped vibration} = \frac{\omega_d}{2\pi}$$

$$f_{n2} = \text{Frequency of undamped vibration} = \frac{\omega_n}{2\pi}$$

$$\therefore \quad \frac{f_{n1}}{f_{n2}} = \frac{\omega_d}{2\pi} \times \frac{2\pi}{\omega_n} = \frac{\omega_d}{\omega_n} = \frac{\sqrt{(\omega_n)^2 - a^2}}{\omega_n} = \frac{\sqrt{(20)^2 - (2.8)^2}}{20} = 0.99 \text{ Ans.}$$

3. Periodic time of damped vibration

We know that periodic time of damped vibration

$$= \frac{2\pi}{\omega_d} = \frac{2\pi}{\sqrt{(\omega_n)^2 - a^2}} = \frac{2\pi}{\sqrt{(20)^2 - (2.8)^2}} = 0.32 \text{ s Ans.}$$

2. The mass of a single degree damped vibrating system is 7.5 kg and makes 24 free oscillations in 14 seconds when disturbed from its equilibrium position. The amplitude of vibration reduces to 0.25 of its initial value after five oscillations. Determine : 1. stiffness of the spring, 2. logarithmic decrement, and 3. damping factor, i.e. the ratio of the system damping to critical damping.

Solution. Given : $m = 7.5$ kg

Since 24 oscillations are made in 14 seconds, therefore frequency of free vibrations,

$$f_n = 24/14 = 1.7$$

and
$$\omega_n = 2\pi \times f_n = 2\pi \times 1.7 = 10.7 \text{ rad/s}$$

1. Stiffness of the spring

Let s = Stiffness of the spring in N/m.

We know that $(\omega_n)^2 = s/m$ or $s = (\omega_n)^2 m = (10.7)^2 7.5 = 860$ N/m Ans.

$$\log_e 2.45 = a \times \frac{2\pi}{\sqrt{(20)^2 - a^2}}$$

$$0.8951 = \frac{a \times 2\pi}{\sqrt{400 - a^2}} \quad \text{or} \quad 0.8 = \frac{a^2 \times 39.5}{400 - a^2} \quad \ldots \text{(Squaring both sides)}$$

$\therefore$ $a^2 = 7.94$ or $a = 2.8$

We know that $a = c / 2m$

$\therefore$ $c = a \times 2m = 2.8 \times 2 \times 75 = 420$ N/m/s **Ans.**

3. Damping factor

Let c = Damping coefficient for the actual system, and

c_c = Damping coefficient for the critical damped system. ... (Given)

We know that logarithmic decrement (δ),

$$0.28 = \frac{a \times 2\pi}{\sqrt{(\omega_n)^2 - a^2}} = \frac{a \times 2\pi}{\sqrt{(10.7)^2 - a^2}}$$

$$\frac{x_4}{x_5} = \frac{x_5}{x_6}$$

or
$$\frac{x_1}{x_2} = \left(\frac{x_1}{x_5}\right)^{1/5} = \left(\frac{x_1}{0.25\, x_1}\right)^{1/5} = (4)^{1/5} = 1.32$$

We know that logarithmic decrement,

$$\delta = \log_e \left(\frac{x_1}{x_2}\right) = \log_e 1.32 = 0.28 \text{ Ans.}$$

$$0.0784 = \frac{a^2 \times 39.5}{114.5 - a^2} \quad \ldots \text{(Squaring both sides)}$$

$8.977 - 0.0784\, a^2 = 39.5\, a^2$ or $a^2 = 0.227$ or $a = 0.476$

We know that $a = c / 2m$ or $c = a \times 2m = 0.476 \times 2 \times 7.5 = 7.2$ N/m/s **Ans.**

and $c_c = 2m\omega_n = 2 \times 7.5 \times 10.7 = 160.5$ N/m/s **Ans.**

$\therefore$ Damping factor $= c/c_c = 7.2 / 160.5 = 0.045$ **Ans.**

3(i) The measurements on a mechanical vibrating system show that it has a mass of 8 kg and that the springs can be combined to give an equivalent spring of stiffness 5.4 N/mm. If the vibrating system have a dashpot attached which exerts a force of 40 N when the mass has a velocity of 1 m/s, find : 1. critical damping coefficient, 2. damping factor, 3. logarithmic decrement, and 4. ratio of two consecutive amplitudes.

Solution. Given : m = 8 kg ; s = 5.4 N/mm = 5400 N/m

Since the force exerted by dashpot is 40 N, and the mass has a velocity of 1 m/s , therefore Damping coefficient (actual),

1. Critical damping coefficient

We know that critical damping coefficient,

$$c_c = 2m\omega_n = 2m \times \sqrt{\frac{s}{m}} = 2 \times 8\sqrt{\frac{5400}{8}} = 416 \text{ N/m/s Ans.}$$

2. *Damping factor*

We know that damping factor

$$= \frac{c}{c_c} = \frac{40}{416} = 0.096 \text{ Ans.}$$

c = 40 N/m/s

3. *Logarithmic decrement*

We know that logarithmic decrement,

$$\delta = \frac{2\pi c}{\sqrt{(c_c)^2 - c^2}} = \frac{2\pi \times 40}{\sqrt{(416)^2 - (40)^2}} = 0.6 \text{ Ans.}$$

4. *Ratio of two consecutive amplitudes*

Let x_n and x_{n-1} = Magnitude of two consecutive amplitudes,

We know that logarithmic decrement,

$$\delta = \log_e\left[\frac{x_n}{x_{n+1}}\right] \text{ or } \frac{x_n}{x_{n+1}} = e^{\delta} = (2.7)^{0.6} = 1.82 \text{ Ans.}$$

3 (ii) An instrument vibrates with a frequency of 1 Hz when there is no damping. When the damping is provided, the frequency of damped vibrations was observed to be 0.9 Hz. Find 1. the damping factor, and 2. logarithmic decrement.

Solution. Given : f_n = 1 Hz ; f_d = 0.9 Hz

1. *Damping factor*

Let m = Mass of the instrument in kg,

c = Damping coefficient or damping force per unit velocity in N/m/s, and

c_c = Critical damping coefficient in N/m/s.

We know that natural circular frequency of undamped vibrations,

$$\omega_n = 2\pi \times f_n = 2\pi \times 1 = 6.284 \text{ rad/s}$$

and circular frequency of damped vibrations,

$$\omega_d = 2\pi \times f_d = 2\pi \times 0.9 = 5.66 \text{ rad/s}$$

We also know that circular frequency of damped vibrations (ω_d),

$$5.66 = \sqrt{(\omega_n)^2 - a^2} = \sqrt{(6.284)^2 - a^2}$$

Squaring both sides,

$$(5.66)^2 = (6.284)^2 - a^2 \text{ or } 32 = 39.5 - a^2$$

$\therefore$ $a^2 = 7.5$ or $a = 2.74$

We know that, $a = c/2m$ or $c = a \times 2m = 2.74 \times 2m = 5.48\, m$ N/m/s

and $c_c = 2m\omega_n = 2m \times 6.284 = 12.568\, m$ N/m/s

$\therefore$ Damping factor,

$$c/c_c = 5.48m/12.568m = 0.436 \text{ Ans.}$$

4(i) A coil of spring stiffness 4 N/mm supports vertically a mass of 20 kg at the free end. The motion is resisted by the oil dashpot. It is found that the amplitude at the beginning of the fourth cycle is 0.8 times the amplitude of the previous vibration. Determine the damping force per unit velocity. Also find the ratio of the frequency of damped and undamped vibrations.

Solution. Given : s = 4 N/mm = 4000 N/m ; m = 20 kg

Damping force per unit velocity

Let c = Damping force in newtons per unit velocity *i.e.* in N/m/s

x_n = Amplitude at the beginning of the third cycle,

x_{n+1} = Amplitude at the beginning of the fourth cycle = 0.8 x_n

We know that natural circular frequency of motion,

$$\omega_n = \sqrt{\frac{s}{m}} = \sqrt{\frac{4000}{20}} = 14.14 \text{ rad/s}$$

and
$$\log_e\left(\frac{x_n}{x_{n+1}}\right) = a \times \frac{2\pi}{\sqrt{(\omega_n)^2 - a^2}}$$

or
$$\log_e\left(\frac{x_n}{0.8\,x_n}\right) = a \times \frac{2\pi}{\sqrt{(14.14)^2 - a^2}}$$

$$\log_e 1.25 = a \times \frac{2\pi}{\sqrt{200 - a^2}} \quad \text{or} \quad 0.223 = a \times \frac{2\pi}{\sqrt{200 - a^2}}$$

Squaring both sides

$$0.05 = \frac{a^2 \times 4\pi^2}{200 - a^2} = \frac{39.5\,a^2}{200 - a^2}$$

$$0.05 \times 200 - 0.05\,a^2 = 39.5 a^2 \quad \text{or} \quad 39.55\,a^2 = 10$$

$\therefore$
$$a^2 = 10 / 39.55 = 0.25 \quad \text{or} \quad a = 0.5$$

We know that
$$a = c / 2m$$

$\therefore$
$$c = a \times 2m = 0.5 \times 2 \times 20 = 20 \text{ N/m/s} \quad \textbf{Ans.}$$

Ratio of the frequencies

Let
$$f_{n1} = \text{Frequency of damped vibrations} = \frac{\omega_d}{2\pi}$$

$$f_{n2} = \text{Frequency of undamped vibrations} = \frac{\omega_n}{2\pi}$$

$\therefore$

$$\frac{f_{n1}}{f_{n2}} = \frac{\omega_d}{2\pi} \times \frac{2\pi}{\omega_n} = \frac{\omega_d}{\omega_n} = \sqrt{\frac{(\omega_n)^2 - a^2}{\omega_n}} = \sqrt{\frac{(14.14)^2 - (0.5)^2}{14.14}}$$

$$\ldots \left(\because\ \omega_d = \sqrt{(\omega_n)^2 - a^2}\right)$$

$$= 0.999 \quad \textbf{Ans.}$$

4(ii) Derive an expression for the natural frequency of single degrees of freedom system.

We know that the kinetic energy is due to the motion of the body and the potential energy is with respect to a certain datum position which is equal to the amount of work required to move the body from the datum position. In the case of vibrations, the datum position is the mean or equilibrium position at which the potential energy of the body or the system is zero.

In the free vibrations, no energy is transferred to the system or from the system. Therefore the summation of kinetic energy and potential energy must be a constant quantity which is same at all the times. In other words,

$$\therefore \quad \frac{d}{dt}(K.E. + P.E.) = 0$$

We know that kinetic energy,

$$K.E. = \frac{1}{2} \times m\left(\frac{dx}{dt}\right)^2$$

and potential energy,

$$P.E. = \left(\frac{0 + s.x}{2}\right)x = \frac{1}{2} \times s.x^2$$

$$\ldots (\because P.E. = \text{Mean force} \times \text{Displacement})$$

$$\therefore \quad \frac{d}{dt}\left[\frac{1}{2} \times m\left(\frac{dx}{dt}\right)^2 + \frac{1}{2} \times s.x^2\right] = 0$$

$$\frac{1}{2} \times m \times 2 \times \frac{dx}{dt} \times \frac{d^2x}{dt^2} + \frac{1}{2} \times s \times 2x \times \frac{dx}{dt} = 0$$

or

$$m \times \frac{d^2x}{dt^2} + s.x = 0 \quad \text{or} \quad \frac{d^2x}{dt^2} + \frac{s}{m} \times x = 0$$

We know that the fundamental equation of simple harmonic motion is

$$\frac{d^2x}{dt^2} + \omega^2 . x = 0$$

Comparing equations,

$$\omega = \sqrt{\frac{s}{m}}$$

$$\therefore \quad \text{Time period,} \quad t_p = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{s}}$$

and natural frequency,

$$f_n = \frac{1}{t_p} = \frac{1}{2\pi}\sqrt{\frac{s}{m}} = \frac{1}{2\pi}\sqrt{\frac{g}{\delta}} \qquad \ldots (\because m.g = s.\delta)$$

Taking the value of g as 9.81 m/s^2 and δ in metres,

$$f_n = \frac{1}{2\pi}\sqrt{\frac{9.81}{\delta}} = \frac{0.4985}{\sqrt{\delta}} \text{ Hz}$$

5. A vertical shaft of 5 mm diameter is 200 mm long and is supported in long bearings at its ends. A disc of mass 50 kg is attached to the centre of the shaft. Neglecting any increase in stiffness due to the attachment of the disc to the shaft, find the critical speed of rotation and the maximum bending stress when the shaft is rotating at 75% of the critical speed. The centre of the disc is 0.25 mm from the geometric axis of the shaft. E = 200 GN/m^2.

Solution. Given : d = 5 mm = 0.005 m ; l = 200 mm = 0.2 m ; m = 50 kg ; e = 0.25 mm = 0.25×10^{-3} m ; E = 200 GN/m^2 = 200×10^9 N/m^2

Critical speed of rotation

We know that moment of inertia of the shaft,

$$I = \frac{\pi}{64} \times d^4 = \frac{\pi}{64}(0.005)^4 = 30.7 \times 10^{-12}\,\text{m}^4$$

Since the shaft is supported in long bearings, it is assumed to be fixed at both ends. We know that the static deflection at the centre of the shaft due to a mass of 50 kg,

$$\delta = \frac{Wl^3}{192\,EI} = \frac{50 \times 9.81 (0.2)^3}{192 \times 200 \times 10^9 \times 30.7 \times 10^{-12}} = 3.33 \times 10^{-3}\,\text{m}$$

$$\ldots (\because W = m.g)$$

We know that critical speed of rotation (or natural frequency of transverse vibrations),

$$N_c = \frac{0.4985}{\sqrt{3.33 \times 10^{-3}}} = 8.64 \text{ r.p.s. } \textbf{Ans.}$$

Maximum bending stress

Let σ = Maximum bending stress in N/m^2, and

N = Speed of the shaft = 75% of critical speed = 0.75 N_c . . . (Given)

When the shaft starts rotating, the additional dynamic load (W_1) to which the shaft is subjected, may be obtained by using the bending equation,

$$\frac{M}{I} = \frac{\sigma}{y_1} \quad \text{or} \quad M = \frac{\sigma.I}{y_1}$$

We know that for a shaft fixed at both ends and carrying a point load (W_1) at the centre, the maximum bending moment

$$M = \frac{W_1.l}{8}$$

$$\therefore \quad \frac{W_1.l}{8} = \frac{\sigma \cdot I}{d/2} \qquad \ldots (\because y_1 = d/2)$$

and

$$W_1 = \frac{\sigma.I}{d/2} \times \frac{8}{l} = \frac{\sigma \times 30.7 \times 10^{-12}}{0.005/2} \times \frac{8}{0.2} = 0.49 \times 10^{-6}\,\sigma\,\text{N}$$

$\therefore$ Additional deflection due to load W_1,

$$y = \frac{W_1}{W} \times \delta = \frac{0.49 \times 10^{-6}\sigma}{50 \times 9.81} \times 3.33 \times 10^{-3} = 3.327 \times 10^{-12}\sigma$$

We know that

$$y = \frac{\pm e}{\left(\frac{\omega_c}{\omega}\right)^2 - 1} = \frac{\pm e}{\left(\frac{N_c}{N}\right) - 1} \qquad \ldots (\text{Substituting } \omega_c = N_c \text{ and } \omega = N)$$

$$3.327\times10^{-12}\,\sigma = \frac{\pm0.25\times10^{-3}}{\left(\dfrac{N_c}{0.75\,N_c}\right)^2 - 1} = \pm0.32\times10^{-3}$$

$$\sigma = 0.32\times10^{-3}/3.327\times10^{-12} = 0.0962\times10^{9}\ \text{N/m}^2 \quad \text{...(Taking + ve sign)}$$

$$= 96.2\times10^{6}\ \text{N/m}^2 = 96.2\ \text{MN/m}^2 \ \ \textbf{Ans.}$$

6.(i) A shaft 50 mm diameter and 3 metres long is simply supported at the ends and carries three loads of 1000 N, 1500 N and 750 N at 1 m, 2 m and 2.5 m from the left support. The Young's modulus for shaft material is 200 GN/m^2. Find the frequency of transverse vibration.

Solution. Given : d = 50 mm = 0.05 m ; l = 3 m, W_1 = 1000 N ; W_2 = 1500 N ; W_3 = 750 N; E = 200 GN/m^2 = 200 × 10^9 N/m^2

The shaft carrying the loads is shown in Fig. 23.13

We know that moment of inertia of the shaft,

$$I = \frac{\pi}{64}\times d^4 = \frac{\pi}{64}(0.05)^4 = 0.307\times10^{-6}\ \text{m}^4$$

and the static deflection due to a point load W,

$$\delta = \frac{Wa^2b^2}{3\,EIl}$$

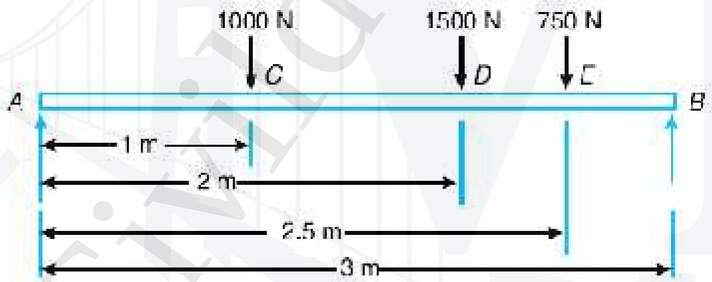


Fig. 23.13

∴ Static deflection due to a load of 1000 N,

$$\delta_1 = \frac{1000\times1^2\times2^2}{3\times200\times10^9\times0.307\times10^{-6}\times3} = 7.24\times10^{-3}\ \text{m}$$

. . . (Here a = 1 m, and b = 2 m)

Similarly, static deflection due to a load of 1500 N,

$$\delta_2 = \frac{1500\times2^2\times1^2}{3\times200\times10^9\times0.307\times10^{-6}\times3} = 10.86\times10^{-3}\ \text{m}$$

. . . (Here a = 2 m, and b = 1 m)

and static deflection due to a load of 750 N,

$$\delta_3 = \frac{750\,(2.5)^2\,(0.5)^2}{3\times200\times10^9\times0.307\times10^{-6}\times3} = 2.12\times10^{-3}\ \text{m}$$

. . . (Here a = 2.5 m, and b = 0.5 m)

We know that frequency of transverse vibration,

$$f_n = \frac{0.4985}{\sqrt{\delta_1 + \delta_2 + \delta_3}} = \frac{0.4985}{\sqrt{7.24\times10^{-3} + 10.86\times10^{-3} + 2.12\times10^{-3}}}$$

$$= \frac{0.4985}{0.1422} = 3.5 \text{ Hz} \quad \textbf{Ans.}$$

6.(ii) Calculate the whirling speed of a shaft 20 mm diameter and 0.6 m long carrying a mass of 1 kg at its mid-point. The density of the shaft material is 40 Mg/m³, and Young's modulus is 200 GN/m². Assume the shaft to be freely supported.

Solution. Given : d = 20 mm = 0.02 m ; l = 0.6 m ; m_1 = 1 kg ; ρ = 40 Mg/m³ = 40 × 10⁶ g/m³ = 40 × 10³ kg/m³ ; E = 200 GN/m² = 200 × 10⁹ N/m²

The shaft is shown in Fig. 23.15.

We know that moment of inertia of the shaft,

$$I = \frac{\pi}{64}\times d^4 = \frac{\pi}{64}(0.02)^4 \text{ m}^4$$
$$= 7.855 \times 10^{-9} \text{ m}^4$$

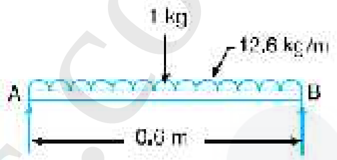


Fig. 23.15

Since the density of shaft material is 40 × 10³ kg/m³, therefore mass of the shaft per metre length,

$$m_S = \text{Area} \times \text{length} \times \text{density} = \frac{\pi}{4}(0.02)^2 \times 1 \times 40 \times 10^3 = 12.6 \text{ kg/m}$$

We know that static deflection due to 1 kg of mass at the centre,

$$\delta = \frac{Wl^3}{48EI} = \frac{1\times 9.81(0.6)^3}{48\times200\times10^9\times7.855\times10^{-9}} = 28\times10^{-6} \text{ m}$$

and static deflection due to mass of the shaft,

$$\delta_S = \frac{5wl^4}{384EI} = \frac{5\times12.6\times9.81(0.6)^4}{384\times200\times10^9\times7.855\times10^{-9}} = 0.133\times10^{-3} \text{ m}$$

∴ Frequency of transverse vibration,

$$f_n = \frac{0.4985}{\sqrt{\delta + \frac{\delta_S}{1.27}}} = \frac{0.4985}{\sqrt{28\times10^{-6} + \frac{0.133\times10^{-3}}{1.27}}}$$

$$= \frac{0.4985}{11.52\times10^{-3}} = 43.3 \text{ Hz}$$

Let N_c = Whirling speed of a shaft.

We know that whirling speed of a shaft in r.p.s. is equal to the frequency of transverse vibration in Hz , therefore

$$N_c = 43.3 \text{ r.p.s.} = 43.3 \times 60 = 2598 \text{ r.p.m.} \quad \textbf{Ans.}$$

3.16 REVIEW QUESTIONS

1. When a body is subjected to transverse vibrations, the stress induced in a body will be ?

2. In under damped vibrating system, if x_1 and x_2 are the successive values of the amplitude on the same side of the mean position, then the logarithmic decrement is equal to?

3. Discuss the effect of inertia of the shaft in longitudinal and transverse vibrations?

4. How the natural frequency of torsional vibrations for a two rotor system is obtained ?

5. . At a nodal point in a shaft, the amplitude of torsional vibration is?

3.17 TUTORIAL PROBLEMS

1. A beam of length 10 m carries two loads of mass 200 kg at distances of 3 m from each end together with a central load of mass 1000 kg. Calculate the frequency of transverse vibrations. Neglect the mass of the beam and take $I = 10^9$ mm^4 and $E = 205\times10^3$ N/mm^2. **[Ans. 13.8 Hz]**

2. A vertical shaft 25 mm diameter and 0.75 m long is mounted in long bearings and carries a pulley of mass 10 kg midway between the bearings. The centre of pulley is 0.5 mm from the axis of the shaft. Find (*a*) the whirling speed, and (*b*) the bending stress in the shaft, when it is rotating at 1700 r.p.m. Neglect the mass of the shaft and E = 200 GN/m^2. **[Ans. 3996 r.p.m ; 12.1 MN/m^2]**

3. A shaft of 100 mm diameter and 1 metre long is fixed at one end and the other end carries a flywheel of mass 1 tonne. The radius of gyration of the flywheel is 0.5 m. Find the frequency of torsional vibrations, if the modulus of rigidity for the shaft material is 80 GN/m^2. [Ans. 8.9 Hz]

4. The flywheel of an engine driving a dynamo has a mass of 180 kg and a radius of gyration of 30 mm. The shaft at the flywheel end has an effective length of 250 mm and is 50 mm diameter. The armature mass is 120 kg and its radius of gyration is 22.5 mm. The dynamo shaft is 43 mm diameter and 200 mm effective length. Calculate the position of node and frequency of torsional oscillation. C = 83 kN/mm^2. [Ans. 205 mm from flywheel, 218 Hz]

5. The two rotors A and B are attached to the end of a shaft 500 mm long. The mass of the rotor A is 300 kg and its radius of gyration is 300 mm. The corresponding values of the rotor B are 500 kg and 450 mm respectively. The shaft is 70 mm in diameter for the first 250 mm ; 120 mm for the next 70 mm and 100 mm diameter for the remaining length. The modulus of rigidity for the shaft material is 80 GN/m^2. Find : 1. The position of the node, and 2. The frequency of torsional vibration. [Ans. 225 mm from A ; 27.3 Hz]

UNIT-IV FORCED VIBRATION

4.1 INTRODUCTION:

When a system is subjected continuously to time varying disturbances, the vibrations resulting under the presence of the external disturbance are referred to as forced vibrations.

Forced vibration is when an alternating force or motion is applied to a mechanical system. Examples of this type of vibration include a shaking washing machine due to an imbalance, transportation vibration (caused by truck engine, springs, road, etc), or the vibration of a building during an earthquake. In forced vibration the frequency of the vibration is the frequency of the force or motion applied, with order of magnitude being dependent on the actual mechanical system.

When a vehicle moves on a rough road, it is continuously subjected to road undulations causing the system to vibrate (pitch, bounce, roll etc). Thus the automobile is said to undergo forced vibrations. Similarly whenever the engine is turned on, there is a resultant residual unbalance force that is transmitted to the chassis of the vehicle through the engine mounts, causing again forced vibrations of the vehicle on its chassis. A building when subjected to time varying ground motion (earthquake) or wind loads, undergoes forced vibrations. Thus most of the practical examples of vibrations are indeed forced vibrations.

4.2 CAUSES RESONANCE:

Resonance is simple to understand if you view the spring and mass as energy storage elements – with the mass storing kinetic energy and the spring storing potential energy. As discussed earlier, when the mass and spring have no force acting on them they transfer energy back and forth at a rate equal to the natural frequency. In other words, if energy is to be efficiently pumped into both the mass and spring the energy source needs to feed the energy in at a rate equal to the natural frequency. Applying a force to the mass and spring is similar to pushing a child on swing, you need to push at the correct moment if you want the swing to get higher and higher. As in the case of the swing, the force applied does not necessarily have to be high to get large motions; the pushes just need to keep adding energy into the system.

The damper, instead of storing energy, dissipates energy. Since the damping force is proportional to the velocity, the more the motion, the more the damper dissipates the energy. Therefore a point will come when the energy dissipated by the damper will equal the energy being fed in by the force. At this point, the system has reached its maximum amplitude and will continue to vibrate at this level as long as the force applied stays the same. If no damping exists, there is nothing to dissipate the energy and therefore theoretically the motion will continue to grow on into infinity.

4.3 FORCED VIBRATION OF A SINGLE DEGREE-OF-FREEDOM SYSTEM:

We saw that when a system is given an initial input of energy, either in the form of an initial displacement or an initial velocity, and then released it will, under the right conditions, vibrate freely. If there is damping in the system, then the oscillations die away. If a system is given a continuous input of energy in the form of a continuously applied force or a continuously applied displacement, then the consequent vibration is called forced vibration. The energy input can overcome that dissipated by damping mechanisms and the oscillations are sustained.

We will consider two types of forced vibration. The first is where the ground to which the system is attached is itself undergoing a periodic displacement, such as the vibration of a building in an earthquake. The second is where a periodic force is applied to the mass, or object performing the motion; an example might be the forces exerted on the body of a car by the forces produced in the engine. The simplest form of periodic force or displacement is sinusoidal, so we will begin by considering forced vibration due to sinusoidal motion of the ground. In all real systems, energy will be dissipated, i.e. the system will be damped, but often the damping is very small. So let us first analyze systems in which there is no damping.

4.4 STEADY STATE RESPONSE DUE TO HARMONIC OSCILLATION:

Consider a spring-mass-damper system as shown in figure 4.1. The equation of motion of this system subjected to a harmonic force $F \sin \omega t$ can be given by

$$m\ddot{x} + kx + c\dot{x} = F \sin \omega t \tag{4.1}$$

where, *m* , *k* and *c* are the mass, spring stiffness and damping coefficient of the system, *F* is the amplitude of the force, *w* is the excitation frequency or driving frequency.

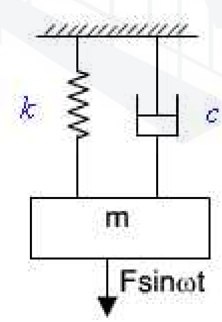


Figure 4.1 Harmonically excited system

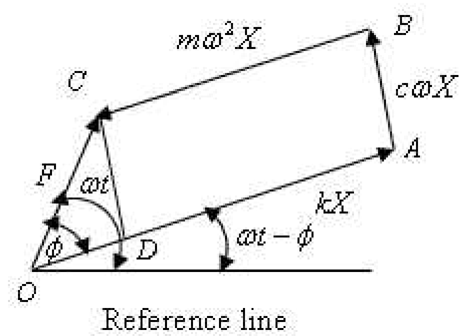


Figure 4.2: Force polygon

The steady state response of the system can be determined by solving equation(4.1) in many different ways. Here a simpler graphical method is used which will give physical understanding to this dynamic problem. From solution of differential equations it is known that the steady state solution (particular integral) will be of the form

$$x = X\sin(\omega t - \phi) \qquad (4.2)$$

As each term of equation (4.1) represents a forcing term viz., first, second and third terms, represent the inertia force, spring force, and the damping forces. The term in the right hand side of equation (4.1) is the applied force. One may draw a close polygon as shown in figure 4.2 considering the equilibrium of the system under the action of these forces. Considering a reference line these forces can be presented as follows.

- Spring force = $kx = kX\sin(\omega t - \phi)$ (This force will make an angle $\omega t - \phi$ with the reference line, represented by line OA).
- Damping force = $c\dot{x} = c\omega X\cos(\omega t - \phi)$ (This force will be perpendicular to the spring force, represented by line AB).
- Inertia force = $m\ddot{x} = -m\omega^2 X\sin(\omega t - \phi)$ (this force is perpendicular to the damping force and is in opposite direction with the spring force and is represented by line BC) .
- Applied force = $F\sin\omega t$ which can be drawn at an angle ωt with respect to the reference line and is represented by line OC.

From equation (1), the resultant of the spring force, damping force and the inertia force will be the applied force, which is clearly shown in figure 4.2.

It may be noted that till now, we don't know about the magnitude of *X* and ϕ which can be easily computed from Figure 2. Drawing a line CD parallel to AB, from the triangle OCD of Figure 2,

$$F^2 = (c\omega X)^2 + (kX - m\omega^2 X)^2$$

$$X = \frac{F}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}}$$

$$\phi = \tan^{-1}\frac{c\omega}{k - m\omega^2}$$

From the previous module of free-vibration it may be recalled that
• Natural frequency $\omega_n = \sqrt{\frac{k}{m}}$ • Critical damping $c_c = 2m\omega_n$ • Damping factor or damping ratio $\zeta = \frac{c}{c_c}$ • Hence, $\frac{c\omega}{k} = \frac{c}{c_c}\frac{c_c\omega}{k} = \zeta\frac{2m\omega_n\omega}{k} = 2\zeta\frac{\omega}{\omega_n}$

$$X = \frac{F/k}{\sqrt{\left(1 - \frac{m}{k}\omega^2\right)^2 + \left(\frac{c\omega}{k}\right)^2}}$$

$$\Rightarrow \quad \frac{Xk}{F} = \frac{1}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right] + \left[2\zeta\left(\frac{\omega}{\omega_n}\right)\right]^2}}$$

$$\tan\phi = \frac{2\zeta(\omega/\omega_n)}{1 - (\omega/\omega_n)^2}$$

or

$$\phi = \tan^{-1}\left(\frac{2\zeta(\omega/\omega_n)}{1 - (\omega/\omega_n)^2}\right)$$

As the ratio $\frac{F}{k}$ is the static deflection (X_0) of the spring, $\frac{Xk}{F} = \frac{X}{X_0}$ is known as the magnification factor or amplitude ratio of the system

4.5 FORCED VIBRATION WITH DAMPING:

In this section we will see the behaviour of the spring mass damper model when we add a harmonic force in the form below. A force of this type could, for example, be generated by a rotating imbalance.

$$F = F_0 \cos(2\pi f t).$$

If we again sum the forces on the mass we get the following ordinary differential equation:

$$m\ddot{x} + c\dot{x} + kx = F_0 \cos(2\pi f t).$$

The steady state solution of this problem can be written as:

$$x(t) = X \cos(2\pi f t - \phi).$$

The result states that the mass will oscillate at the same frequency, f, of the applied force, but with a phase shift ϕ.

The amplitude of the vibration –X‖ is defined by the following formula.

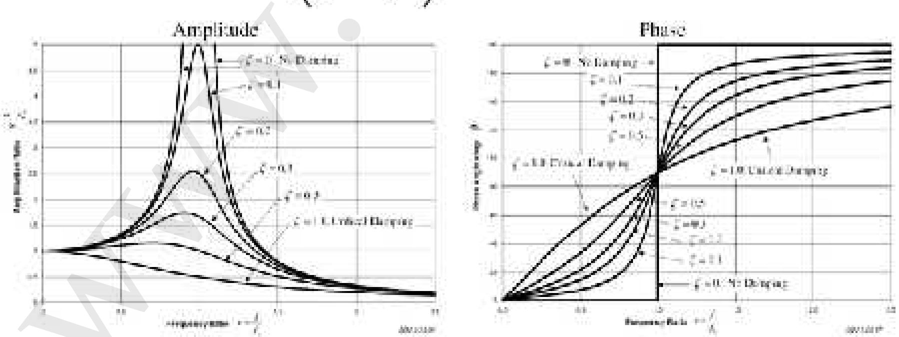
$$X = \frac{F_0}{k} \frac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}.$$

Where –r‖ is defined as the ratio of the harmonic force frequency over the undamped natural frequency of the mass−spring−damper model.

$$r = \frac{f}{f_n}.$$

The phase shift , ϕ, is defined by the following formula.

$$\phi = \arctan\left(\frac{2\zeta r}{1-r^2}\right).$$



The plot of these functions, called "the frequency response of the system", presents one of the most important features in forced vibration. In a lightly damped system when the forcing frequency nears the natural frequency ($r \approx 1$) the amplitude of the vibration can get extremely high. This phenomenon is called **resonance** (subsequently the natural frequency of a system is often referred to as the resonant frequency). In rotor bearing systems any rotational speed that excites a resonant frequency is referred to as a critical speed.

If resonance occurs in a mechanical system it can be very harmful – leading to eventual failure of the system. Consequently, one of the major reasons for vibration analysis is to predict when this type of resonance may occur and then to determine what steps to take to prevent it from occurring. As the amplitude plot shows, adding damping can significantly reduce the magnitude of the vibration. Also, the magnitude can be reduced if the natural frequency can be shifted away from the forcing frequency by changing the stiffness or mass of the system. If the system cannot be changed, perhaps the forcing frequency can be shifted (for example, changing the speed of the machine generating the force).

The following are some other points in regards to the forced vibration shown in the frequency response plots.

At a given frequency ratio, the amplitude of the vibration, X, is directly proportional to the amplitude of the force F_0 (e.g. if you double the force, the vibration doubles)

With little or no damping, the vibration is in phase with the forcing frequency when the frequency ratio $r < 1$ and 180 degrees out of phase when the frequency ratio $r > 1$

When $r \ll 1$ the amplitude is just the deflection of the spring under the static force F_0. This deflection is called the static deflection δ_{st}. Hence, when $r \ll 1$ the effects of the damper and the mass are minimal.

When $r \gg 1$ the amplitude of the vibration is actually less than the static deflection δ_{st}. In this region the force generated by the mass ($F = ma$) is dominating because the acceleration seen by the mass increases with the frequency. Since the deflection seen in the spring, X, is reduced in this region, the force transmitted by the spring ($F = kx$) to the base is reduced. Therefore the mass–spring–damper system is isolating the harmonic force from the mounting base – referred to as vibration isolation. Interestingly, more damping actually reduces the effects of vibration isolation when $r \gg 1$ because the damping force ($F = cv$) is also transmitted to the base.

4.6 ROTATING UNBALANCE FORCED VIBRATION:

One may find many rotating systems in industrial applications. The unbalanced force in such a system can be represented by a mass *m* with eccentricity *e* , which is rotating with angular velocity as shown in Figure 4.1.

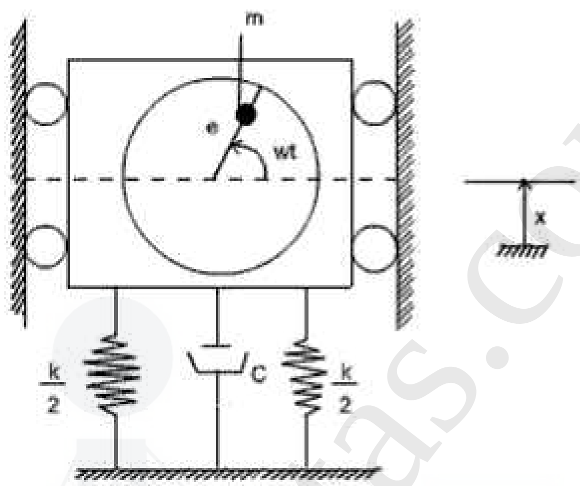


Figure 4.1 : Vibrating system with rotating unbalance

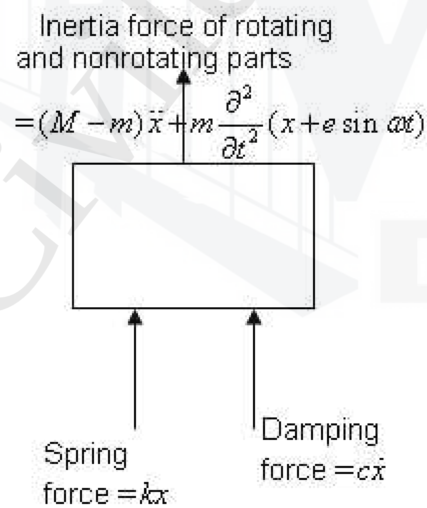


Figure 4.2. Freebody diagram of the system

Let *x* be the displacement of the nonrotating mass (*M*-*m*) from the static equilibrium position, then the displacement of the rotating mass *m* is $x + e\sin\omega t$

From the freebody diagram of the system shown in figure 4.2, the equation of motion is

$$(M - m)\ddot{x} + m\frac{\partial^2}{\partial t^2}(x + e\sin\omega t) + kx + c\dot{x} = 0 \quad (4.1)$$

or $$M\ddot{x} + k\dot{x} + cx = me\omega^2 \sin\omega t \quad (4.2)$$

This equation is same as equation (1) where *F* is replaced by $me\omega^2$. So from the force polygon as shown in figure 4.3

$$me\omega^2 = \sqrt{\left\{(-M\omega^2 + k)^2 + c\omega^2\right\}}\, X^2 \quad (4.3)$$

or $$X = \frac{me\omega^2}{\sqrt{(k - M\omega^2)^2 + (c\omega)^2}} \quad (4.4)$$

or $$\frac{X}{e} = \frac{\frac{m\omega}{M}}{\sqrt{\left(\frac{k}{M} - \omega^2\right)^2 + \left(\frac{c}{M}\omega\right)^2}} \quad (4.5)$$

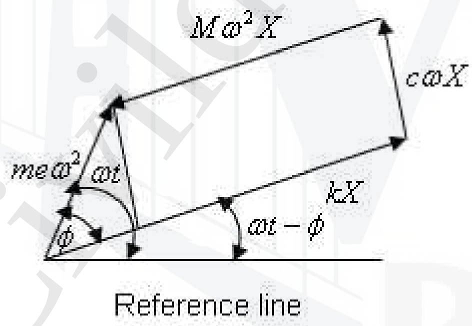


Figure 4.3: Force polygon

or $$\frac{X}{e} = \frac{\frac{m\omega}{M}}{\sqrt{\left(\frac{k}{M} - \omega^2\right)^2 + \left(\frac{c}{M}\omega\right)^2}} \quad (4.6)$$

and $$\tan\phi = \frac{2\zeta\frac{\omega}{\omega_n}}{1-\left(\frac{\omega}{\omega_n}\right)^2} \tag{4.7}$$

So the complete solution becomes

$$x(t) = x_1 e^{-\zeta\omega_n t}\sin\left(\sqrt{1-\zeta^2}\,\omega_n t + \phi_1\right) + \frac{me\omega^2}{\sqrt{(k-M\omega^2)^2+(c\omega)^2}}\sin\left(\omega t - \phi\right) \tag{4.8}$$

4.7 VIBRATION ISOLATION AND TRANSMISSIBILITY:

When a machine is operating, it is subjected to several time varying forces because of which it tends to exhibit vibrations. In the process, some of these forces are transmitted to the foundation – which could undermine the life of the foundation and also affect the operation of any other machine on the same foundation. Hence it is of interest to minimize this force transmission. Similarly when a system is subjected to ground motion, part of the ground motion is transmitted to the system as we just discussed e.g., an automobile going on an uneven road; an instrument mounted on the vibrating surface of an aircraft etc. In these cases, we wish to minimize the motion transmitted from the ground to the system. Such considerations are used in the design of machine foundations and in order to understand some of the basic issues involved, we will study this problem based on the single d.o.f model discussed so far.

we get the expression for force transmitted to the base as follows:

$$F_T = \sqrt{(kX_0)^2 + (c\Omega X_0)^2}$$

$$X_0 = X_e\sqrt{\frac{k^2+(c\Omega)^2}{(k-m\Omega^2)^2+(c\Omega)^2}}$$

4.7.1 Vibration Isolators:

Consider a vibrating machine; bolted to a rigid floor (Figure 2a).The force transmitted to the floor is equal to the force generated in the machine. The transmitted force can be decreased by adding a suspension and damping elements (often called vibration isolators) Figure 2b , or by adding what is called an inertia block, a large mass (usually a block of cast concrete), directly attached to the machine (Figure 2c).Another option is to add an additional level of mass (sometimes called a seismic mass, again a block of cast concrete) and suspension (Figure 2d).

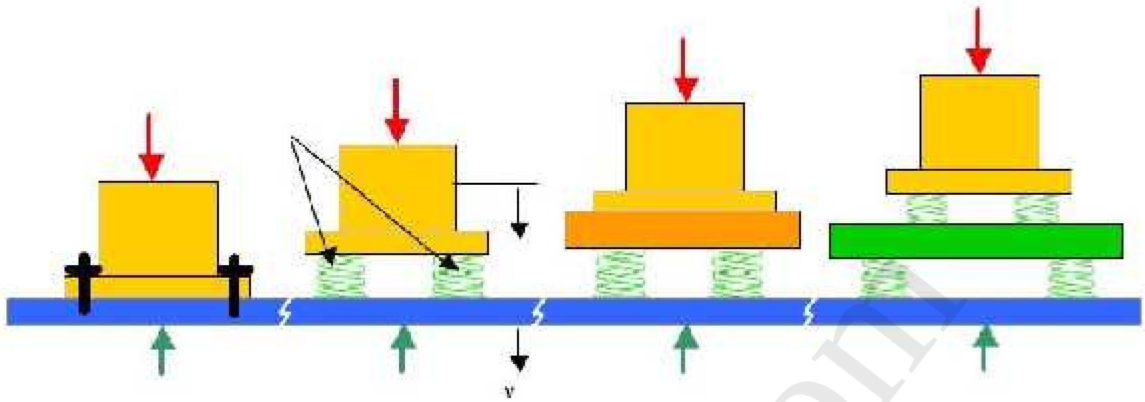


Figure 2.Vibration isolation systems: a) Machine bolted to a rigid foundation b) Supported on isolation springs, rigid foundation c) machine attached to an inertial block. d) Supported on isolation springs, non-rigid foundation (such as a floor); or machine on isolation springs, seismic mass and second level of isolator springs

When oscillatory forces arise unavoidably in machines it is usually desired to prevent these forces from being transmitted to the surroundings. For example, some unbalanced forces are inevitable in a car engine, and it is uncomfortable if these are wholly transmitted to the car body. The usual solution is to mount the source of vibration on sprung supports. Vibration isolation is measured in terms of the motion or force transmitted to the foundation. The lesser the force or motion transmitted the greater the vibration isolation

Suppose that the foundation is effectively rigid and that only one direction of movement is effectively excited so that the system can be treated as having only one degree of freedom.

4.8 RESPONSE WITHOUT DAMPING:

The amplitude of the force transmitted to the foundations is Where ***k*** is the Stiffness of the support and ***x(t)*** is the displacement of the mass ***m***.
The governing equation can be determined by considering that the total forcing on the machine is equal to its mass multiplied by its acceleration (Newton's second law)

The ratio (transmitted force amplitude) / (applied force amplitude) is called the **transmissibility**.

$$\text{Transmissibility} = \left|\frac{F_T}{F}\right| = \frac{1}{\left|1-\frac{\omega^2}{{\omega_n}^2}\right|} = \frac{1}{\left|1-\frac{f^2}{{f_n}^2}\right|}$$

The transmissibility can never be zero but will be less than 1 providing $\frac{\omega}{\omega_n} > \sqrt{2}$ or $\frac{f}{f_n} > \sqrt{2}$ otherwise it will be greater than 1.

4.9 SOLVED PROBLEMS

1. Derive the relation for the displacement of mass from the equilibrium position of the damped vibration system with harmonic forcing.

Consider a system consisting of spring, mass and damper as shown in Fig. 23.19. Let the system is acted upon by an external periodic (*i.e.* simple harmonic) disturbing force,

$$F_x = F \cos \omega .t$$

where F = Static force, and

ω = Angular velocity of the periodic disturbing force.

When the system is constrained to move in vertical guides, it has only one degree of freedom. Let at sometime t, the mass is displaced downwards through a distance x *from its mean position.*

The equation of motion may be written as,

$$m \times \frac{d^2x}{dt^2} = -c \times \frac{dx}{dt} - s.x + F \cos \omega t$$

or

$$m \times \frac{d^2x}{dt^2} + c \times \frac{dx}{dt} + s.x = F \cos \omega t$$

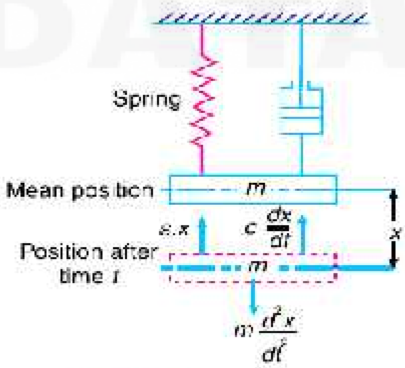


Fig. 23.19. Frequency of under damped forced vibrations.

This equation of motion may be solved either by differential equation method or by graphi-cal method as discussed below :

1. ***Differential equation method***

The equation **(*i*)** is a differential equation of the second degree whose right hand side is some function in t. The solution of such type of differential equation consists of two parts ;

one part is the complementary function and the second is particular integral. Therefore the solution may be written as

$$x = x_1 + x_2$$

where x_1 = Complementary function, and x_2 = Particular integral.

The complementary function is same as discussed in the previous article, *i.e.*

$x_1 = Ce^{-at} \cos(\omega_d t - \theta)$ . . . ***(ii)*** where C and θ are constants. Let us now find the value of particular integral as discussed below :

Let the particular integral of equation ***(i)*** is given by

$$x_2 = B_1 \sin \omega.t + B_2 \cos \omega.t \qquad \text{. . (where } B_1 \text{ and } B_2 \text{ are constants)}$$

$$\therefore \qquad \frac{dx}{dt} = B_1.\omega \cos \omega.t - B_2.\omega \sin \omega.t$$

and $$\frac{d^2x}{dt^2} = -B_1.\omega^2 \sin \omega.t - B_2.\omega^2 \cos \omega.t$$

Substituting these values in the given differential equation *(i)*, we get

$$m(-B_1.\omega^2 \sin \omega.t - B_2.\omega^2 \cos \omega.t) + c(B_1.\omega \cos \omega.t - B_2.\omega \sin \omega.t) + s(B_1 \sin \omega.t + B_2 \cos \omega.t) = F \cos \omega.t$$

or $$(-m.B_1.\omega^2 - c.\omega.B_2 + s.B_1)\sin \omega.t + (-m.\omega^2.B_2 + c.\omega.B_1 + s.B_2)\cos \omega.t = F \cos \omega.t$$

or $$\left[(s - m.\omega^2)B_1 - c.\omega.B_2\right]\sin \omega.t + \left[c.\omega.B_1 + (s - m.\omega^2)B_2\right]\cos \omega.t = F \cos \omega.t + 0 \sin \omega.t$$

Comparing the coefficients of sin *ωt* and cos *ωt* on the left hand side and right hand side separately, we get

$$(s - m.\omega^2)B_1 - c.\omega.B_2 = 0 \qquad \text{. . . } (iii)$$

and $$c.\omega.B_1 + (s - m.\omega^2)B_2 = F \qquad \text{. . . } (iv)$$

Now from equation *(iii)*

$$(s - m.\omega^2)B_1 = c.\omega.B_2$$

$$\therefore \qquad B_2 = \frac{s - m.\omega^2}{c.\omega} \times B_1 \qquad \text{. . . } (v)$$

Substituting the value of B_2 in equation *(iv)*

$$c.\omega.B_1 + \frac{(s - m.\omega^2)(s - m.\omega^2)}{c.\omega} \times B_1 = F$$

$$c^2.\omega^2.B_1 + (s - m.\omega^2)^2 B_1 = c.\omega.F$$

$$B_1\left[c^2.\omega^2 + (s - m.\omega^2)^2\right] = c.\omega.F$$

$$\therefore \qquad B_1 = \frac{c.\omega.F}{c^2.\omega^2 + (s - m.\omega^2)^2}$$

and $$B_2 = \frac{s - m\omega^2}{c.\omega} \times \frac{c.\omega.F}{c^2.\omega^2 + (s - m\omega^2)^2} \quad \text{. . . [From equation } (v)]$$

$$= \frac{F(s - m\omega^2)}{c^2.\omega^2 + (s - m\omega^2)^2}$$

∴ The particular integral of the differential equation (i) is

$$x_2 = B_1 \sin \omega.t + B_2 \cos \omega.t$$

$$= \frac{c.\omega.F}{c^2.\omega^2 + (s - m\omega^2)^2} \times \sin \omega t + \frac{F(s - m\omega^2)}{c^2.\omega^2 + (s - m\omega^2)^2} \times \cos \omega t$$

$$= \frac{F}{c^2.\omega^2 + (s - m\omega^2)^2}\left[c.\omega \sin \omega.t + (s - m\omega^2) \cos \omega.t\right] \quad \text{. . . } (vi)$$

Let $c.\omega = X \sin \phi$; and $s - m.\omega^2 = X \cos \phi$

∴ $$X = \sqrt{c^2.\omega^2 + (s - m\omega^2)^2} \quad \text{. . . (By squaring and adding)}$$

and $$\tan \phi = \frac{c.\omega}{s - m\omega^2} \quad \text{or} \quad \phi = \tan^{-1}\left(\frac{c.\omega}{s - m.\omega^2}\right)$$

Now the equation (vi) may be written as

$$x_2 = \frac{F}{c^2.\omega^2 + (s - m\omega^2)^2}\left[X \sin \phi. \sin \omega t + X \cos \phi \cos \omega t\right]$$

$$= \frac{F.X}{c^2.\omega^2 + (s - m.\omega^2)^2} \times \cos(\omega.t - \phi)$$

$$= \frac{F\sqrt{c^2.\omega^2 + (s - m\omega^2)^2}}{c^2.\omega^2 + (s - m\omega^2)^2} \times \cos(\omega.t - \phi)$$

$$= \frac{F}{\sqrt{c^2.\omega^2 + (s - m.\omega^2)^2}} \times \cos(\omega.t - \phi)$$

∴ The complete solution of the differential equation (i) becomes

$$x = x_1 + x_2$$

$$= C.e^{-at} \cos(\omega_d.t - \theta) + \frac{F}{\sqrt{c^2.\omega^2 + (s - m.\omega^2)^2}} \times \cos(\omega t - \phi)$$

In actual practice, the value of the complementary function x_1 at any time t is much smaller as compared to particular integral x_2. Therefore, the displacement x, at any time t, is given by the particular integral x_2 only.

∴ $$x = \frac{F}{\sqrt{c^2.\omega^2 + (s - m\omega^2)^2}} \times \cos(\omega.t - \phi) \quad \text{... } (vii)$$

A little consideration will show that the frequency of forced vibration is equal to the angular velocity of the periodic force and the amplitude of the forced vibration is equal to the maximum displacement of vibration.

∴ Maximum displacement or the amplitude of forced vibration,

$$x_{max} = \frac{F}{\sqrt{c^2.\omega^2 + (s - m.\omega^2)^2}} \quad \text{. . . } (viii)$$

This equation shows that motion is simple harmonic whose circular frequency is ω and the amplitude is $\dfrac{F}{\sqrt{c^2.\omega^2+(s-m.\omega^2)^2}}$.

2. A mass of 10 kg is suspended from one end of a helical spring, the other end being fixed. The stiffness of the spring is 10 N/mm. The viscous damping causes the amplitude to decrease to one-tenth of the initial value in four complete oscillations. If a periodic force of 150 cos 50 t N is applied at the mass in the vertical direction, find the amplitude of the forced vibrations. What is its value of resonance ?

Solution. Given : $m = 10$ kg ; $s = 10$ N/mm $= 10 \times 10^3$ N/m ; $x_5 = \dfrac{x_1}{10}$

Since the periodic force, $F_x = F\cos\omega t = 150\cos 50t$, therefore

Static force, $F = 150$ N

and angular velocity of the periodic disturbing force,

$$\omega = 50 \text{ rad/s}$$

We know that angular speed or natural circular frequency of free vibrations,

$$\omega_n = \sqrt{\frac{s}{m}} = \sqrt{\frac{10\times10^3}{10}} = 31.6 \text{ rad/s}$$

Amplitude of the forced vibrations

Since the amplitude decreases to 1/10th of the initial value in four complete oscillations, therefore, the ratio of initial amplitude (x_1) to the final amplitude after four complete oscillations (x_5) is given by

$$\frac{x_1}{x_5} = \frac{x_1}{x_2}\times\frac{x_2}{x_3}\times\frac{x_3}{x_4}\times\frac{x_4}{x_5} = \left(\frac{x_1}{x_2}\right)^4 \quad \ldots \left(\because \frac{x_1}{x_2} = \frac{x_2}{x_3} = \frac{x_3}{x_4} = \frac{x_4}{x_5}\right)$$

$$\therefore \quad \frac{x_1}{x_2} = \left(\frac{x_1}{x_5}\right)^{1/4} = \left(\frac{x_1}{x_1/10}\right)^{1/4} = (10)^{1/4} = 1.78 \quad \ldots \left(x_5 = \frac{x_1}{10}\right)$$

We know that

$$\log_e\left(\frac{x_1}{x_2}\right) = a\times\frac{2\pi}{\sqrt{(\omega_n)^2-a^2}}$$

$$\log_e 1.78 = a\times\frac{2\pi}{\sqrt{(31.6)^2-a^2}} \quad \text{or} \quad 0.576 = \frac{a\times2\pi}{\sqrt{1000-a^2}}$$

We know that amplitude of the forced vibrations,

$$x_{max} = \frac{x_o}{\sqrt{\frac{c^2.\omega^2}{s^2}+\left[1-\frac{\omega^2}{(\omega_n)^2}\right]^2}}$$

$$= \frac{0.015}{\sqrt{\frac{(57.74)^2(50)^2}{(10\times10^3)^2}+\left[1-\left(\frac{50}{31.6}\right)^2\right]^2}} = \frac{0.015}{\sqrt{0.083+2.25}}$$

Squaring both sides and rearranging,

$$39.832\ a^2 = 332 \quad \text{or} \quad a^2 = 8.335 \quad \text{or} \quad a = 2.887$$

We know that $a = c/2m$ or $c = a \times 2m = 2.887 \times 2 \times 10 = 57.74$ N/m/s

and deflection of the system produced by the static force F,

$$x_0 = F/s = 150/10 \times 10^3 = 0.015 \text{ m}$$

3. The mass of an electric motor is 120 kg and it runs at 1500 r.p.m. The armature mass is 35 kg and its C.G. lies 0.5 mm from the axis of rotation. The motor is mounted on five springs of negligible damping so that the force transmitted is one-eleventh of the impressed force. Assume that the mass of the motor is equally distributed among the five springs.

Determine : 1. stiffness of each spring; 2. dynamic force transmitted to the base at the operating speed; and 3. natural frequency of the system.

Solution. Given $m_1 = 120$ kg ; $m_2 = 35$ kg; $r = 0.5$ mm $= 5 \times 10^{-4}$ m; $\varepsilon = 1/11$;
$N = 1500$ r.p.m. or $\omega = 2\pi \times 1500 / 60 = 157.1$ rad/s ;

1. Stiffness of each spring

Let s = Combined stiffness of the spring in N-m, and

ω_n = Natural circular frequency of vibration of the machine in rad/s.

$$= \frac{0.015}{1.53} = 9.8\times10^{-3} \text{ m} = 9.8 \text{ mm } \textbf{Ans.}$$

Amplitude of forced vibrations at resonance

We know that amplitude of forced vibrations at resonance,

$$x_{max} = x_0 \times \frac{s}{c.\omega_n} = 0.015\times\frac{10\times10^3}{57.54\times31.6} = 0.0822 \text{ m} = 82.2 \text{ mm } \textbf{Ans.}$$

We know that transmissibility ratio (ε),

$$\frac{1}{11} = \frac{1}{\left(\frac{\omega}{\omega_n}\right)^2-1} = \frac{(\omega_n)^2}{\omega^2-(\omega_n)^2} = \frac{(\omega_n)^2}{(157.1)^2-(\omega_n)^2}$$

or $$(157.1)^2-(\omega_n)^2 = 11(\omega_n)^2 \quad \text{or} \quad (\omega_n)^2 = 2057 \quad \text{or} \quad \omega_n = 45.35 \text{ rad/s}$$

2. Dynamic force transmitted to the base at the operating speed (i.e. 1500 r.p.m. or 157.1 rad/s)

We We know that maximum unbalanced force on the motor due to armature mass,

$$F = m_1\,\omega^2 . r = 35\,(157.1)^2\, 5\times10^{-4} = 432\text{ N}$$

Sin $\therefore$ Dynamic force transmitted to the base,

$$F_T = \varepsilon.F = \frac{1}{11}\times 432 = 39.27\text{ N} \quad \textbf{Ans}$$

3. Natural frequency of the system

We have calculated above that the natural frequency of the system,

$$\omega_n = 45.35 \text{ rad/s} \quad \textbf{Ans.}$$

4. What do you understand by transmissibility? Describe the method of finding the transmissibility ratio from unbalanced machine supported with foundation.

A little consideration will show that when an unbalanced machine is installed on the foundation, it produces vibration in the foundation. In order to prevent these vibrations or to minimize the transmission of forces to the foundation, the machines are mounted on springs and dampers or on some vibration isolating material, as shown in Fig. 23.22. The arrangement is assumed to have one degree of freedom, *i.e.* it can move up and down only.

It may be noted that when a periodic (*i.e.* simple harmonic) disturbing force $F \cos \omega t$ is applied to a machine of mass m supported by a spring of stiffness s, then the force is transmitted by means of the spring and the damper or dashpot to the fixed support or foundation.

The ratio of the force transmitted (F_T) to the force applied (F) is known as the ***isolation factor*** or ***transmissibility ratio*** of the spring support.

We have discussed above that the force transmitted to the foundation consists of the fol-lowing two forces :

1. Spring force or elastic force which is equal to $s.\,x_{max}$, and

2. Damping force which is equal to $c.\,\omega\,.x_{max}$.

Since these two forces are perpendicular to one another, as shown in Fig.23.23, therefore the force transmitted,

$$F_T = \sqrt{(s.x_{max})^2 + (c.\omega.x_{max})^2}$$

$$= x_{max}\sqrt{s^2 + c^2.\omega^2}$$

$\therefore$ Transmissibility ratio,

$$\varepsilon = \frac{F_T}{F} = \frac{x_{max}\sqrt{s^2 + c^2.\omega^2}}{F}$$

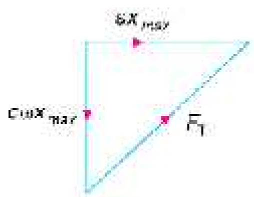


Fig. 23.23

We know that

$$x_{max} = x_o \times D = \frac{F}{s} \times D \qquad \ldots \left(\because \; x_o = \frac{F}{s} \right)$$

$$\therefore \qquad \varepsilon = \frac{D}{s}\sqrt{s^2 + c^2.\omega^2} = D\sqrt{1 + \frac{c^2.\omega^2}{s^2}}$$

$$= D\sqrt{1 + \left(\frac{2c}{c_c} \times \frac{\omega}{\omega_n} \right)^2} \qquad \ldots \left(\because \; \frac{c.\omega}{s} = \frac{2c}{c_c} \times \frac{\omega}{\omega_n} \right)$$

magnification factor,

$$D = \frac{1}{\sqrt{\left(\frac{2c.\omega}{c_c.\omega_n} \right)^2 + \left(1 - \frac{\omega^2}{(\omega_n)^2} \right)^2}}$$

$$\therefore \qquad \varepsilon = \frac{\sqrt{1 + \left(\frac{2c.\omega}{c_c.\omega_n} \right)^2}}{\sqrt{\left(\frac{2c.\omega}{c_c.\omega_n} \right)^2 + \left(1 - \frac{\omega^2}{(\omega_n)^2} \right)^2}}$$

When the damper is not provided, then $c = 0$, and

$$\varepsilon = \frac{1}{1 - (\omega/\omega_n)^2}$$

From above, we see that when $\omega/\omega_n > 1$, ε is negative. This means that there is a phase difference of 180° between the transmitted force and the disturbing force $(F \cos \omega t)$. The value of ω/ω_n must be greater than $\sqrt{2}$ if ε is to be less than 1 and it is the numerical value of ε, independent of any phase difference between the forces that may exist which is important. It is therefore more convenient to use equation *(ii)* in the following form, *i.e.*

$$\varepsilon = \frac{1}{(\omega/\omega_n)^2 - 1} \quad \ldots (iii)$$

Fig. 23.24 is the graph for different values of damping factor c/c_c to show the variation of transmissibility ratio (ε) against the ratio ω/ω_n

1. When $\omega/\omega_n = \sqrt{2}$, then all the curves pass through the point $\varepsilon = 1$ for all values of damping factor c/c_c.

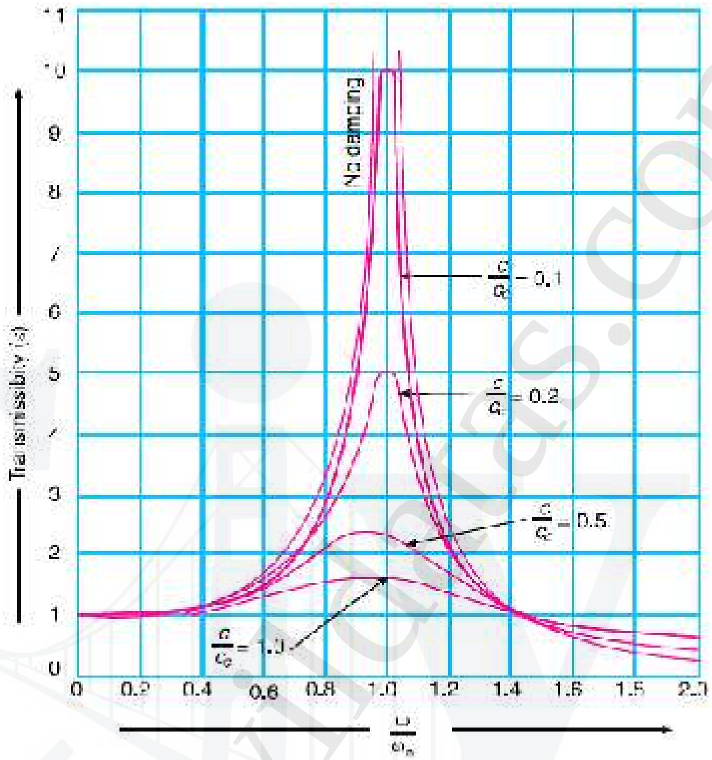


Fig. 23.24. Graph showing the variation of transmissibility ratio.

2. When $\omega/\omega_n < \sqrt{2}$, then $\varepsilon > 1$ for all values of damping factor c/c_c. This means that the force transmitted to the foundation through elastic support is greater than the force applied.

3. When $\omega/\omega_n > \sqrt{2}$, then $\varepsilon < 1$ for all values of damping factor c/c_c. This shows that the force transmitted through elastic support is less than the applied force. Thus vibration isolation is possible only in the range of $\omega/\omega_n > \sqrt{2}$.

5. A machine has a mass of 100 kg and unbalanced reciprocating parts of mass 2 kg which move through a vertical stroke of 80 mm with simple harmonic motion. The machine is mounted on four springs, symmetrically arranged with respect to centre of mass, in such a way that the machine has one degree of freedom and can undergo vertical displacements only.

Neglecting damping, calculate the combined stiffness of the spring in order that the force transmitted to the foundation is 1 / 25 th of the applied force, when the speed of rotation of ma-chine crank shaft is 1000 r.p.m.

When the machine is actually supported on the springs, it is found that the damping reduces the amplitude of successive free vibrations by 25%. Find : 1. the force transmitted to foundation at 1000 r.p.m., 2. the force transmitted to the foundation at resonance, and 3. the amplitude of the forced vibration of the machine

at resonance.

Solution. Given : $m_1 = 100$ kg ; $m_2 = 2$ kg ; $l = 80$ mm $= 0.08$ m ; $\varepsilon = 1/25$; $N = 1000$ r.p.m. or $\omega = 2\pi \times 1000/60 = 104.7$ rad/s

Combined stiffness of springs

Let s = Combined stiffness of springs in N/m, and

ω_n = Natural circular frequency of vibration of the machine in rad/s

We know that transmissibility ratio (ε),

$$\frac{1}{25} = \frac{1}{\left(\frac{\omega}{\omega_n}\right)^2 - 1} = \frac{(\omega_n)^2}{\omega^2 - (\omega_n)^2} = \frac{(\omega_n)^2}{(104.7)^2 - (\omega_n)^2}$$

or $(104.7)^2 - (\omega_n)^2 = 25(\omega_n)^2$ or $(\omega_n)^2 = 421.6$ or $\omega_n = 20.5$ rad/s

We know that $\omega_n = \sqrt{s/m}$

$\therefore$ $s = m_1 (\omega_n)^2 = 100 \times 421.6 = 42\ 160$ N/m **Ans.**

1. Force transmitted to the foundation at 1000 r.p.m.

Let F_T = Force transmitted, and

x_1 = Initial amplitude of vibration.

Since the damping reduces the amplitude of successive free vibrations by 25%, therefore final amplitude of vibration,

$$x_2 = 0.75\, x_1$$

We know that

$$\log_e\left(\frac{x_1}{x_2}\right) = \frac{a \times 2\pi}{\sqrt{(\omega_n)^2 - a^2}} \quad \text{or} \quad \log_e\left(\frac{x_1}{0.75 x_1}\right) = \frac{a \times 2\pi}{\sqrt{421.6 - a^2}}$$

Squaring both sides,

$$(0.2877)^2 = \frac{a^2 \times 4\pi^2}{421.6 - a^2} \quad \text{or} \quad 0.083 = \frac{39.5\,a^2}{421.6 - a^2}$$

$$\ldots\left[\because \log_e\left(\frac{1}{0.75}\right) = \log_e 1.333 = 0.2877\right]$$

$$35 - 0.083\,a^2 = 39.5\,a^2 \quad \text{or} \quad a^2 = 0.884 \quad \text{or} \quad a = 0.94$$

We know that damping coefficient or damping force per unit velocity,

$$c = a \times 2m_1 = 0.94 \times 2 \times 100 = 188 \text{ N/m/s}$$

and critical damping coefficient,

$$c_c = 2m.\omega_n = 2 \times 100 \times 20.5 = 4100 \text{ N/m/s}$$

$\therefore$ Actual value of transmissibility ratio,

$$\varepsilon = \frac{\sqrt{1+\left(\dfrac{2c.\omega}{c_c.\omega_n}\right)^2}}{\sqrt{\left(\dfrac{2c.\omega}{c_c.\omega_n}\right)^2+\left(1-\dfrac{\omega^2}{(\omega_n)^2}\right)^2}}$$

$$= \frac{\sqrt{1+\left(\dfrac{2\times188\times104.7}{4100\times20.5}\right)^2}}{\sqrt{\left(\dfrac{2\times188\times104.7}{4100\times20.5}\right)^2+\left[1-\left(\dfrac{104.7}{20.5}\right)^2\right]^2}} = \frac{\sqrt{1+0.22}}{\sqrt{0.22+629}}$$

$$= \frac{1.104}{25.08} = 0.044$$

We know that the maximum unbalanced force on the machine due to reciprocating parts,

$$F = m_2.\omega^2.r = 2(104.7)^2\,(0.08/2) = 877\text{ N} \qquad \ldots (\because\ r = l/2)$$

$\therefore$ Force transmitted to the foundation,

$$F_T = \varepsilon F = 0.044\times877 = 38.6\text{ N } \textbf{Ans.} \qquad \ldots (\because\ \varepsilon = F_T/F)$$

2. Force transmitted to the foundation at resonance

Since at resonance, $\omega = \omega_n$, therefore transmissibility ratio,

$$\varepsilon = \frac{\sqrt{1+\left(\dfrac{2c}{c_c}\right)^2}}{\sqrt{\left(\dfrac{2c}{c_c}\right)^2}} = \frac{\sqrt{1+\left(\dfrac{2\times188}{4100}\right)^2}}{\sqrt{\left(\dfrac{2\times188}{4100}\right)^2}} = \frac{\sqrt{1+0.0084}}{0.092} = 10.92$$

3. Amplitude of the forced vibration of the machine at resonance

We know that amplitude of the forced vibration at resonance

$$= \frac{\text{Force transmitted at resonance}}{\text{Combined stiffness}} = \frac{367}{42\,160} = 8.7\times10^{-3}\text{ m}$$

$= 8.7$ mm **Ans.**

6.(i) Derive the relation for magnification factor in case of forced vibration.

It is the ratio of *maximum displacement of the forced vibration (x_{max}) to the deflection due to the static force $F(x_o)$*. We have proved in the previous article that the maximum displacement or the amplitude of forced vibration.

$$x_{max} = \frac{x_o}{\sqrt{\dfrac{c^2.\omega^2}{s^2}+\left(1-\dfrac{\omega^2}{(\omega_n)^2}\right)^2}}$$

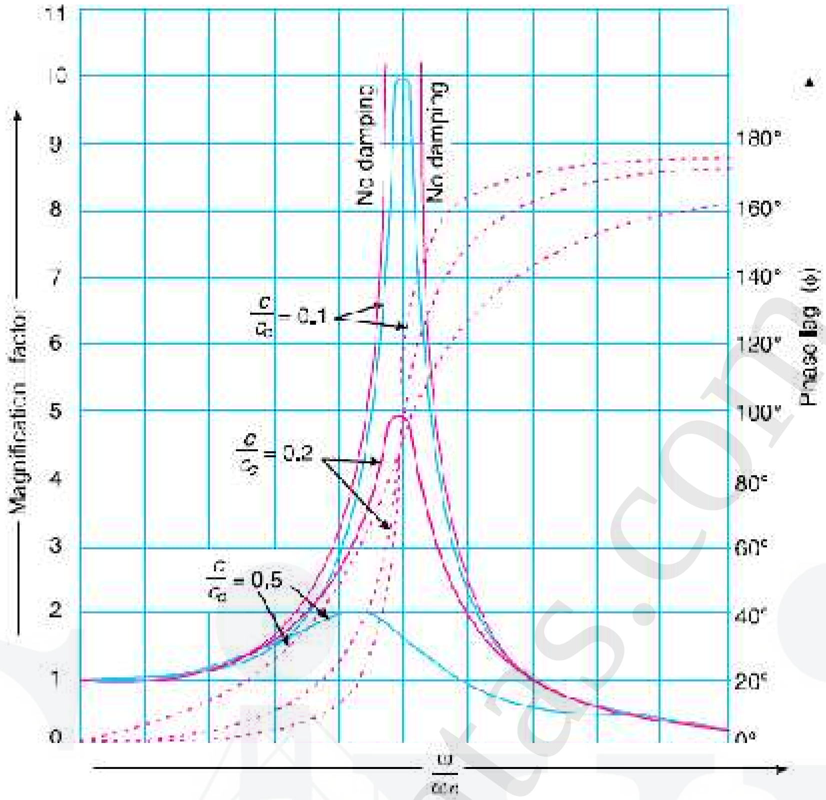


Fig. 23.21. Relationship between magnification factor and phase angle for different values of ω/ω_n.

$\therefore$ Magnification factor or dynamic magnifier,

$$D = \frac{x_{max}}{x_o} = \frac{1}{\sqrt{\frac{c^2.\omega^2}{s^2} + \left(1 - \frac{\omega^2}{(\omega_n)^2}\right)^2}} \quad \text{... } (i)$$

$$= \frac{1}{\sqrt{\left(\frac{2c.\omega}{c_c.\omega_n}\right)^2 + \left(1 - \frac{\omega^2}{(\omega_n)^2}\right)^2}}$$

$$\text{...} \left[\because \frac{c.\omega}{s} = \frac{2c.\omega}{2m \times \frac{s}{m}} = \frac{2c.\omega}{2m(\omega_n)^2} = \frac{2c.\omega}{c_c.\omega_n}\right]$$

The magnification factor or dynamic magnifier gives the factor by which the static deflection produced by a force F (*i.e.* x_o) must be multiplied in order to obtain the maximum amplitude of the forced vibration (*i.e.* x_{max}) by the harmonic force $F \cos \omega t$

$$\therefore \qquad x_{max} = x_o \times D$$

Fig. 23.21 shows the relationship between the magnification factor (D) and phase angle ϕ for different value of ω/ω_n and for values of damping factor c/c_c = 0.1, 0.2 and 0.5.

6.(ii) A single cylinder vertical petrol engine of total mass 300 kg is mounted upon a steel chassis frame and causes a vertical static deflection of 2 mm. The reciprocating parts of the engine has a mass of 20 kg and move through a vertical stroke of 150 mm with simple harmonic motion. A dashpot is provided whose damping resistance is directly proportional to the velocity and amounts to 1.5 kN per metre per second.

Considering that the steady state of vibration is reached ; determine : 1. the amplitude of forced vibrations, when the driving shaft of the engine rotates at 480 r.p.m., and 2. the speed of the driving shaft at which resonance will occur.

Solution : Given. m = 300 kg; δ = 2 mm = 2×10^{-3} m ; m_1 = 20 kg ; l = 150 mm = 0.15 m ; c = 1.5 kN/m/s = 1500 N/m/s ; N = 480 r.p.m. or ω = 2 · 480 / 60 = 50.3 rad/s

1. Amplitude of the forced vibrations

We know that stiffness of the frame,

$$s = m.g / \delta = 300 \times 9.81/2 \times 10^{-3} = 1.47 \times 10^{6} \text{ N/m}$$

Since the length of stroke (l) = 150 mm = 0.15 m, therefore radius of crank,

$$r = l / 2 = 0.15 / 2 = 0.075 \text{ m}$$

We know that the centrifugal force due to the reciprocating parts or the static force,

$$F = m_1 \omega^2 . r = 20\,(50.3)^2\, 0.075 = 3795 \text{ N}$$

∴ Amplitude of the forced vibration (maximum),

$$x_{max} = \frac{F}{\sqrt{c^2.\omega^2 + (s - m.\omega^2)^2}}$$

$$= \frac{3795}{\sqrt{(1500)^2 (50.3)^2 + [1.47 \times 10^6 - 300\,(50.3)^2]^2}}$$

$$= \frac{3795}{\sqrt{5.7 \times 10^9 + 500 \times 10^9}} = \frac{3795}{710 \times 10^3} = 5.3 \times 10^{-3} \text{ m}$$

= 5.3 mm **Ans.**

2. Speed of the driving shaft at which the resonance occurs

Let N = Speed of the driving shaft at which the resonance occurs in r.p.m.

We know that the angular speed at which the resonance occurs,

$$\omega = \omega_n = \sqrt{\frac{s}{m}} = \sqrt{\frac{1.47 \times 10^6}{300}} = 70 \text{ rad/s}$$

∴ $N = \omega \times 60 / 2\pi = 70 \times 60 / 2\pi$ = 668.4 r.p.m. **Ans.**

4.10 REVIEW QUESTIONS

1. Explain the term 'dynamic magnifier'
2. What are the materials used for vibration isolation?
3. In vibration isolation system, if $\omega / \omega_n > 1$, then the phase difference between the transmitted force and the disturbing force is ?
4. In under damped vibrating system, if x_1 and x_2 are the successive values of the amplitude on the ________________
5. same side of the mean position, then the logarithmic decrement is equal to

4.11 TUTORIAL PROBLEMS:

1. A machine of mass 100 kg is supported on openings of total stiffness 800 kN/m and has a rotating unbalanced element which results in a disturbing force of 400 N at a speed of 3000 r.p.m. Assum-ing the damping ratio as 0.25, determine : 1. the amplitude of vibrations due to unbalance ; and 2. the transmitted force. **[Ans. 0.04 mm ; 35.2 N]**

2. A mass of 500 kg is mounted on supports having a total stiffness of 100 kN/m and which provides viscous damping, the damping ratio being 0.4. The mass is constrained to move vertically and is subjected to a vertical disturbing force of the type $F \cos \omega t$. Determine the frequency at which resonance will occur and the maximum allowable value of F if the amplitude at resonance is to be restricted to 5 mm. **[Ans. 2.25 Hz ; 400 N]**

3. A machine of mass 75 kg is mounted on springs of stiffness 1200 kN/m and with an assumed damping factor of 0.2. A piston within the machine of mass 2 kg has a reciprocating motion with a stroke of 80 mm and a speed of 3000 cycles/min. Assuming the motion to be simple harmonic, find : 1. the amplitude of motion of the machine, 2. its phase angle with respect to the exciting force, 3. the force transmitted to the foundation, and 4. the phase angle of transmitted force with respect to the exciting force. **[Ans. 1.254 mm ; 169.05° ; 2132 N ; 44.8°]**

UNIT-V MECHANISMS FOR CONTROL

5.1 INTRODUCTION TO GOVERNOR:

A **centrifugal governor** is a specific type of governor that controls the speed of an engine by regulating the amount of fuel (or working_fluid) admitted, so as to maintain a near constant speed whatever the load or fuel supply conditions. It uses the principle of proportional control.

It is most obviously seen on steam engines where it regulates the admission of steam into the cylinder(s). It is also found on internal combustion engines and variously fuelled turbines, and in some modern striking_clocks.

5.2 PRINCIPLE OF WORKING:

Power is supplied to the governor from the engine's output shaft by (in this instance) a belt or chain (not shown) connected to the lower belt wheel. The governor is connected to a throttle valve that regulates the flow of working_fluid (steam) supplying the prime mover (prime mover not shown). As the speed of the prime mover increases, the central spindle of the governor rotates at a faster rate and the kinetic energy of the balls increases. This allows the two masses on lever arms to move outwards and upwards against gravity. If the motion goes far enough, this motion causes the lever arms to pull down on a thrust_bearing, which moves a beam linkage, which reduces the aperture of a throttle valve. The rate of working-fluid entering the cylinder is thus reduced and the speed of the prime mover is controlled, preventing over speeding.

Mechanical stops may be used to limit the range of throttle motion, as seen near the masses in the image at right.

The direction of the lever arm holding the mass will be along the vector_ sum of the reactive centrifugal force vector and the gravitational force.

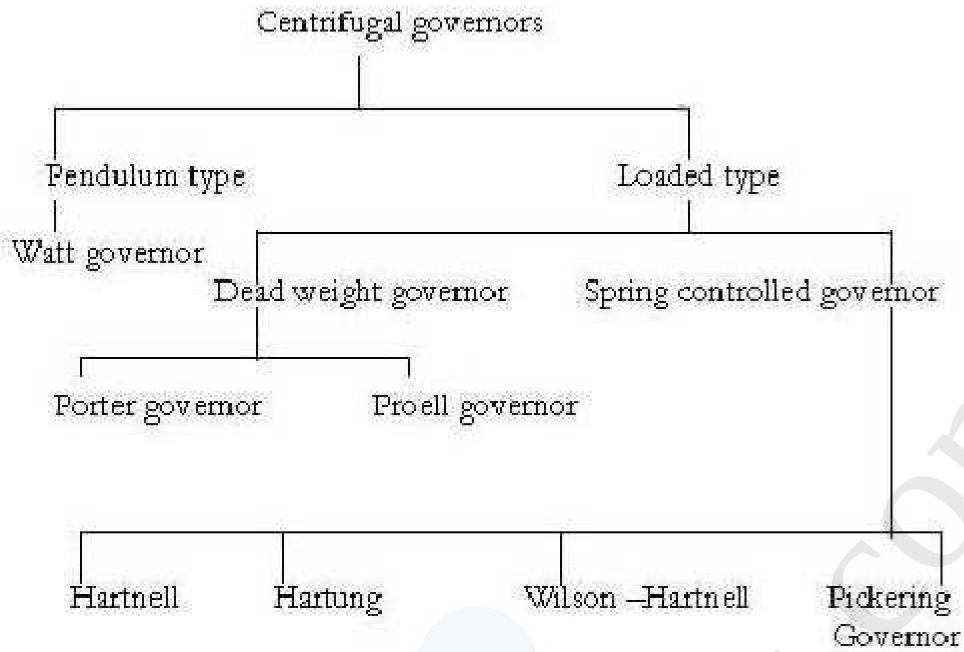
5.3 CLASSIFICATION OF GOVERNORS:

Governors are classified based upon two different principles. These are:

1. Centrifugal governors

2. Inertia governors

Centrifugal governors are further classified as –



(4) Height of governor

It is the vertical distance between the centre of the governor halls and the point of intersection between the upper arms on the axis of spindle is known as governor height. It is generally denoted by h.

(5) Sleeve lift

The vertical distance the sleeve travels due to change in the equilibrium Speed is called the sleeve lift. The vertical downward travel may be termed as Negative lift

(6) Isochronism

This is an extreme case of sensitiveness. When the equilibrium speed is constant for all radii of rotation of the balls within the working range, the governor is said to be in isochronism. This means that the difference between the maximum and minimum equilibrium speeds is zero and the sensitiveness shall be ininite.

(7) Stability

Stability is the ability to maintain a desired engine speed without Fluctuating. Instability results in hunting or oscillating due to over correction. Excessive stability results in a dead-beat governor or one that does not correct sufficiently for load changes

(8) Hunting

The phenomenon of continuous fluctuation of the engine speed above and below the mean speed is termed as hunting. This occurs in over- sensitive or isochronous governors. Suppose an isochronous governor is fitted to an engine running at a steady load. With a slight increase of load, the speed will fall and the sleeve will immediately fall to its lowest position. This shall open the control valve wide and excess supply of energy will be given, with the result that the speed will rapidly increase and the sleeve will rise to its higher position. As a result of this movement of the sleeve, the control valve will be cut off; the supply to the engine and the speed will again fall, the cycle being repeated indefinitely. Such a governor would admit either more or less amount of fuel and so effect would be that the engine would hunt.

5.4 SENSITIVENESS

A governor is said to be sensitive, if its change of speed s from no Load to full load may be as small a fraction of the mean equilibrium speed as possible and the corresponding sleeve lift may be as large as possible.

Suppose

$\omega 1$ = max. Equilibrium speed

$\omega 2$ = min. equilibrium speed

ω = mean equilibrium speed = $(\omega 1+ \omega 2)/2$

Therefore sensitiveness = $(\omega 1- \omega 2)/2$

5.5 CHARACTERISTICS AND QUALITIES OF CENTRIFUGAL GOVERNOR:

For satisfactory performance and working a centrifugal governor should possess The following qualities.

a. On the sudden removal of load its sleeve should reach at the top most position at Once.
b. Its response to the change of speed should be fast.
c. Its sleeve should float at some intermediate position under normal operating Conditions.
d. At the lowest position of sleeve the engine should develop maximum power.
e. It should have sufficient power, so that it may be able to exert the required force At the sleeve to operate the control & mechanism

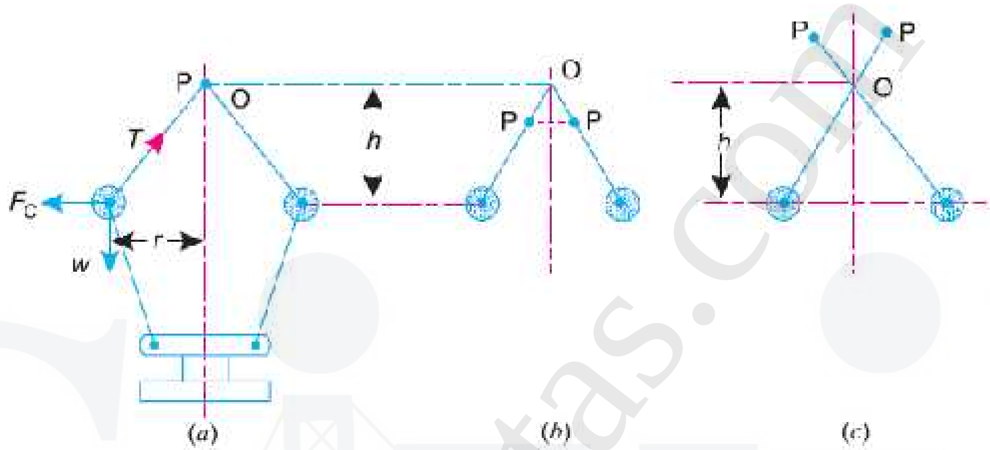
5.6 WATT GOVERNOR:

The simplest form of a centrifugal governor is a Watt governor, as shown in Fig. It is basically a conical pendulum with links attached to a sleeve of negligible mass. The arms of the governor may be connected to the spindle in the following three ways :

1. The pivot *P*, may be on the spindle axis as shown in Fig. (*a*).

2. The pivot *P*, may be offset from the spindle axis and the arms when produced intersect at *O*, as shown in Fig (*b*).

3. The pivot *P*, may be offset, but the arms cross the axis at *O*, as shown in Fig (*c*).



Watt governor.

Let m = Mass of the ball in kg,

w = Weight of the ball in newtons = $m.g$,

T = Tension in the arm in newtons,

ω = Angular velocity of the arm and ball about the spindle axis in rad/s,

r = Radius of the path of rotation of the ball *i.e.* horizontal distance from the centre of the ball to the spindle axis in metres,

F_C = Centrifugal force acting on the ball in newtons = $m.\omega^2.r$, and

h = Height of the governor in metres.

It is assumed that the weight of the arms, links and the sleeve are negligible as compared to the weight of the balls. Now, the ball is in equilibrium under the action of

1. the centrifugal force (F_C) acting on the ball, **2.** the tension (T) in the arm, and **3.** the weight (w) of the ball.

Taking moments about point O, we have

$$F_C \times h = w \times r = m.g.r$$

or

$$m.\omega^2.r.h = m.g.r \quad \text{or} \quad h = g/\omega^2 \qquad \ldots (i)$$

When g is expressed in m/s^2 and ω in rad/s, then h is in metres. If N is the speed in r.p.m., then

$$\omega = 2\pi N/60$$

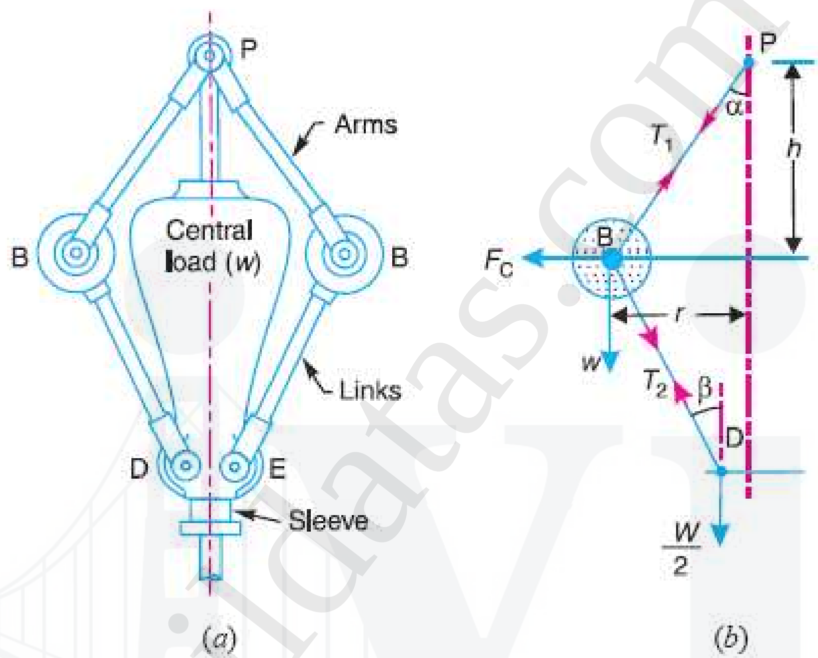
$$\therefore \quad h = \frac{9.81}{(2\pi N/60)^2} = \frac{895}{N^2} \text{ metres} \qquad \ldots (\because g = 9.81 \text{ m/s}^2) \ldots (ii)$$

Note : We see from the above expression that the height of a governor h, is inversely proportional to N^2. Therefore at high speeds, the value of h is small. At such speeds, the change in the value of h corresponding to a small change in speed is insufficient to enable a governor of this type to operate the mechanism to give the necessary change in the fuel supply. This governor may only work satisfactorily at relatively low speeds *i.e.* from 60 to 80 r.p.m.

5.7 PORTER GOVERNOR:

The Porter governor is a modification of a Watt's governor, with central load attached to the sleeve as shown in Fig. (*a*). The load moves up and down the central spindle. This additional downward force increases the speed of revolution required to enable the balls to rise to any pre-determined level.

Consider the forces acting on one-half of the governor as shown in Fig. (*b*).



Porter governor.

Let
m = Mass of each ball in kg,
w = Weight of each ball in newtons = $m.g$,
M = Mass of the central load in kg,
W = Weight of the central load in newtons = $M.g$,
r = Radius of rotation in metres,

---

h = Height of governor in metres ,
N = Speed of the balls in r.p.m .,
ω = Angular speed of the balls in rad/s = $2\pi N/60$ rad/s,
F_C = Centrifugal force acting on the ball in newtons = $m.\omega^2.r$,
T_1 = Force in the arm in newtons,
T_2 = Force in the link in newtons,
α = Angle of inclination of the arm (or upper link) to the vertical, and
β = Angle of inclination of the link (or lower link) to the vertical.

5.8 PROELL GOVERNOR:

The Proell governor has the balls fixed at B and C to the extension of the links DF and EG, as shown in Fig. (a). The arms FP and GQ are pivoted at P and Q respectively.

Consider the equilibrium of the forces on one-half of the governor as shown in Fig (b). The instantaneous centre (I) lies on the intersection of the line PF produced and the line from D drawn perpendicualr to the spindle axis. The prependicular BM is drawn on ID.

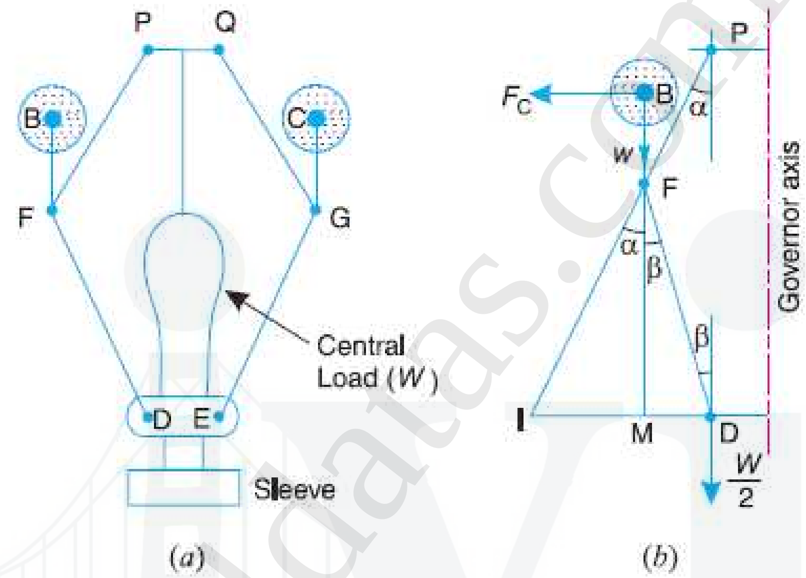
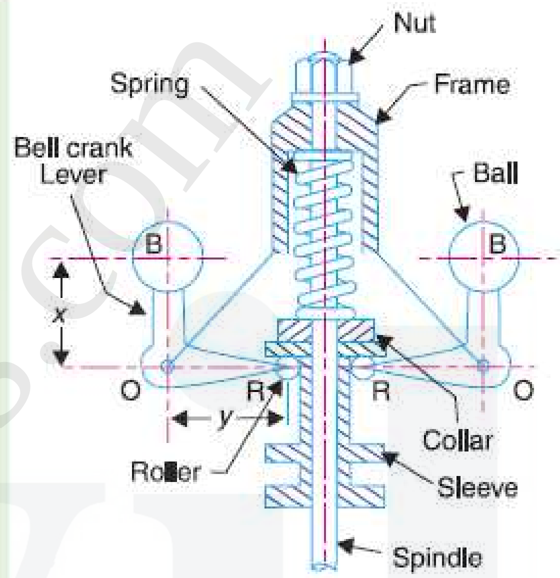


Fig. Proell governor.

5.9 HARTNELL GOVERNOR:

A Hartnell governor is a spring loaded governor as shown in Fig. It consists of two bell crank levers pivoted at the points O,O to the frame. The frame is attached to the governor spindle and therefore rotates with it. Each lever carries a ball at the end of the vertical arm OB and a roller at the end of the horizontal arm OR. A helical spring in compression provides equal downward forces on the two rollers through a collar on the sleeve. The spring force may be adjusted by screwing a nut up or down on the sleeve.



Hartnell governor.

Let m = Mass of each ball in kg,

M = Mass of sleeve in kg,

r_1 = Minimum radius of rotation in metres,

r_2 = Maximum radius of rotation in metres,

ω_1 = Angular speed of the governor at minimum radius in rad/s,

ω_2 = Angular speed of the governor at maximum radius in rad/s,

S_1 = Spring force exerted on the sleeve at ω_1 in newtons,

S_2 = Spring force exerted on the sleeve at ω_2 in newtons,

F_{C1} = Centrifugal force at ω_1 in newtons = $m\ (\omega_1)^2\ r_1$,

F_{C2} = Centrifugal force at ω_2 in newtons = $m\ (\omega_2)^2\ r_2$,

s = Stiffness of the spring or the force required to compress the spring by one mm,

x = Length of the vertical or ball arm of the lever in metres,

y = Length of the horizontal or sleeve arm of the lever in metres, and

r = Distance of fulcrum O from the governor axis or the radius of rotation when the governor is in mid-position, in metres.

5.10 HARTUNG GOVERNOR:

A spring controlled governor of the Hartung type is shown in Fig. (*a*). In this type of governor, the vertical arms of the bell crank levers are fitted with spring balls which compress against the frame of the governor when the rollers at the horizontal arm press against the sleeve.

Let S = Spring force,

F_C = Centrifugal force,

M = Mass on the sleeve, and

x and y = Lengths of the vertical and horizontal arm of the bell crank lever respectively.

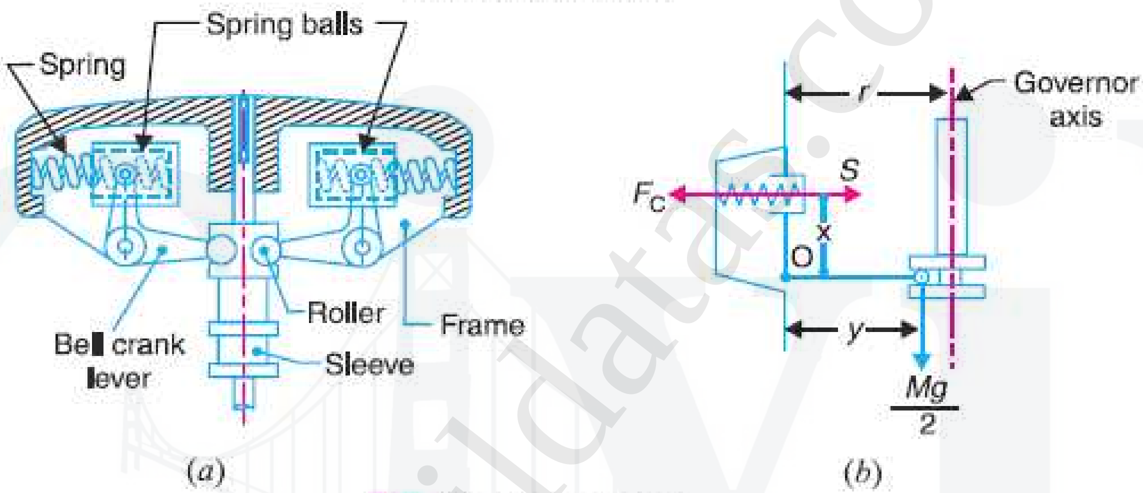


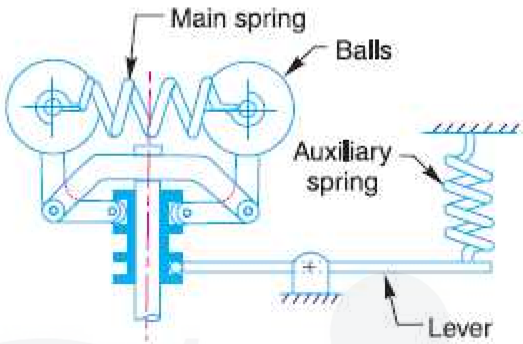
Fig. Hartung governor.

Fig. (*a*) and (*b*) show the governor in mid-position. Neglecting the effect of obliquity of the arms, taking moments about the fulcrum O,

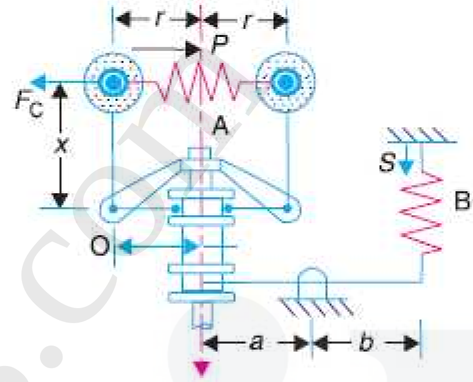
$$F_C \times x = S \times x + \frac{M.g}{2} \times y$$

5.11 WILSON HARTNELL GOVERNOR:

A Wilson-Hartnell governor is a governor in which the balls are connected by a spring in tension as shown in Fig. An auxiliary spring is attached to the sleeve mechanism through a lever by means of which the equilibrium speed for a given radius may be adjusted. The main spring may be considered of two equal parts each belonging to both the balls. The line diagram of a Wilson-Hartnell governor is shown in Fig.



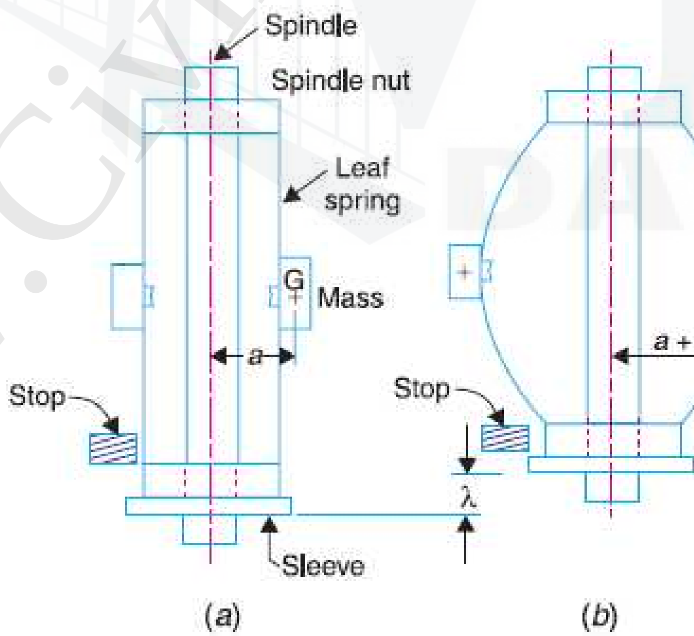
Wilson-Hartnell governor.



Line diagram of Wilson-Hartnell governor.

5.12 PICKERING GOVERNOR:

A Pickering governor is mostly used for driving gramophone. It consists of *three straight leaf springs arranged at equal angular intervals round the spindle. Each spring carries a weight at the centre. The weights move outwards and the springs bend as they rotate about the spindle axis with increasing speed.



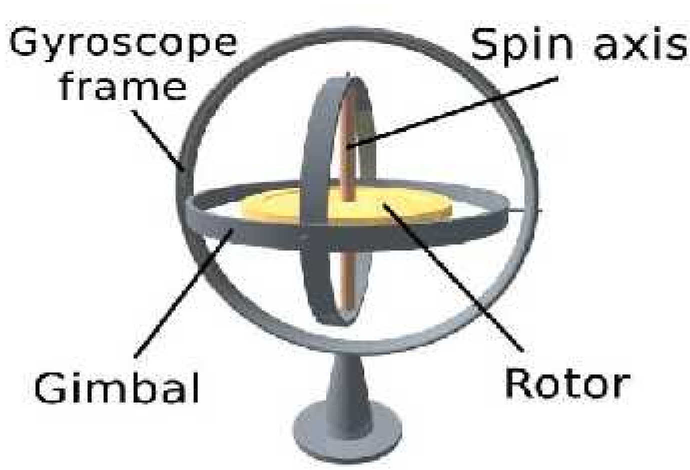
5.13 DIFFERENCE BETWEEN A FLYWHEEL AND A GOVERNOR:

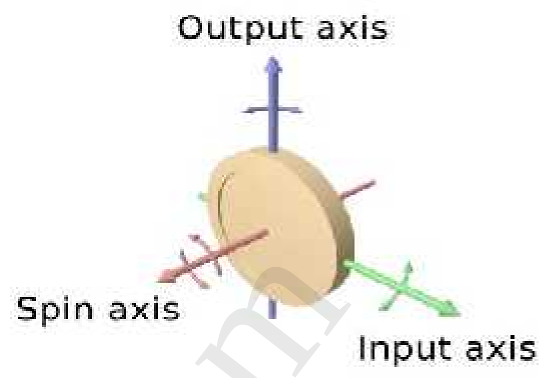
S.no	Flywheel	Governor
1.	It is provided on the engine and fabricating machines viz., rolling mills; punching machines; shear machines, presses etc.	It is provided on prime movers such as engines and turbines
2.	Its function is to store available mechanical energy when it is in excess of the load requirements and to part with the same when the available energy is less than that required by the load.	Its function is to regulate the supply of driving fluid producing energy, according to the load requirements so that at different loads almost a constant speed is maintained.
3.	In engines it takes care of fluctuations of speed during thermodynamic cycle.	It take care of fluctuation of speed due to variation of load over range of working of engines and turbines.
4.	It works continuously from cycle to cycle.	It works intermittently, i.e. only when there is change in the load.
5.	In fabrication machines it is very economical to use it as its use reduces capital investment on prime movers and their running expenses.	But for governor, there would have been unnecessarily more consumption of driving fluid thus it economizes its consumption

5.14 GYROSCOPE AND ITS APPLICATIONS

(19) Gyroscope

A gyroscope is a device for measuring or maintaining orientation, based on the principles of conservation of angular momentum. A mechanical gyroscope is essentially a spinning wheel or disk whose axle is free to take any orientation. This orientation changes much less in response to a given external torque than it would without the large angular momentum associated with the gyroscope's high rate of spin. Since external torque is minimized by mounting the device in gimbals, its orientation remains nearly fixed, regardless of any motion of the platform on which it is mounted. Gyroscopes based on other operating principles also Exit, such as the electronic, microchip-packaged MEMS gyroscope devices found in consumer electronic devices, solid state ring lasers, fiber optic gyroscopes and the extremely sensitive quantum gyroscope. Applications of gyroscopes include navigation (INS) when magnetic compasses do not work (as in the Hubble telescope) or are not precise enough (as in ICBMs) or for the stabilization of flying vehicles like radio-controlled helicopters or UAVs. Due to higher precision, gyroscopes are also used to maintain direction in tunnel mining.



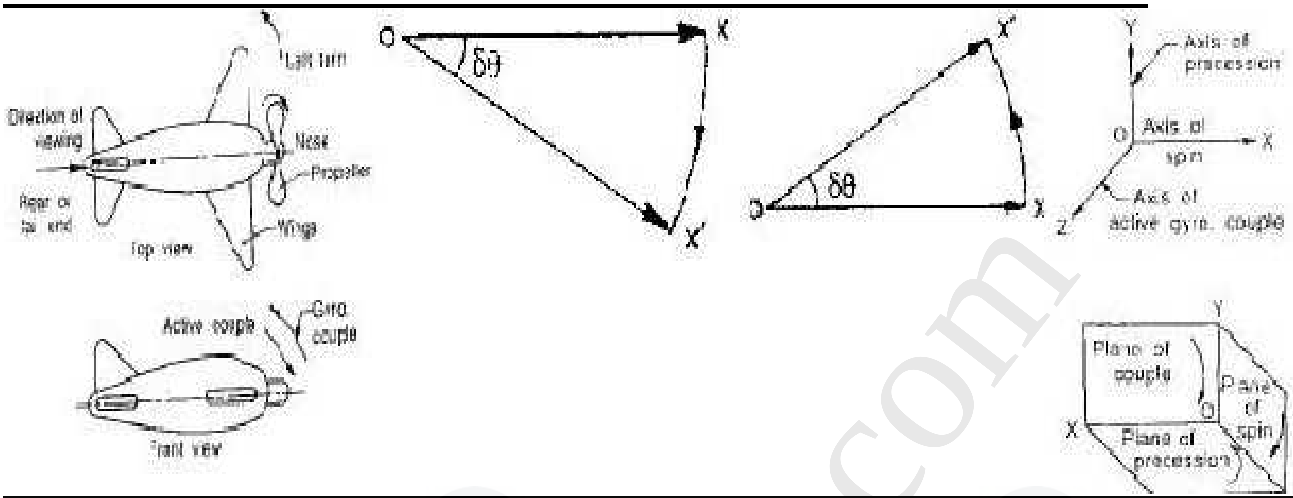


(20) Description and diagram:

Diagram of a gyro wheel. Reaction arrows about the output axis (blue) correspond to forces applied about the input axis (green), and vice versa. Within mechanical systems or devices, a conventional gyroscope is a mechanism comprising a rotor journal led to spin about one axis, the journals of the rotor being mounted in an inner gimbal or ring, the inner gimbal is journal led for oscillation in an outer gimbal which is journal led in another gimbal. So basically there are three gimbals. The **outer gimbal** or ring which is the gyroscope frame is mounted so as to pivot about an axis in its own plane determined by the support. This outer gimbal possesses one degree of rotational freedom and its axis possesses none. The next **inner gimbal** is mounted in the gyroscope frame (outer gimbal) so as to pivot about an axis in its own plane that is always perpendicular to the pivotal axis of the gyroscope frame (outer gimbal). This inner gimbal has two degrees of rotational freedom. Similarly, next **innermost gimbal** is attached to the inner gimbal which has three degree of rotational freedom and its axis posses two. The axle of the spinning wheel defines the spin axis. The rotor is journaled to spin about an axis which is always perpendicular to the axis of the innermost gimbal. So, the rotor possesses four degrees of rotational freedom and its axis possesses three. The wheel responds to a force applied about the input axis by a reaction force about the output axis.

The behavior of a gyroscope can be most easily appreciated by consideration of the front wheel of a bicycle. If the wheel is leaned away from the vertical so that the top of the wheel moves to the left, the forward rim of the wheel also turns to the left. In other words, rotation on one axis of the turning wheel produces rotation of the third axis.

5.15 EFFECT OF THE GYROSCOPIC COUPLE ON AN AERO PLANE



5.16 EFFECT OF GYROSCOPIC COUPLE

This couple is, therefore, to raise the nose and dip the tail of the aero plane.

Notes

1. When the aero plane takes a right turn under similar Conditions as discussed above, the effect of the reactive Couple will be to dip the nose and raise the tail of the aero plane.
2. When the engine or propeller rotates in anticlockwise direction when viewed from the rear or tail end and the aero plane takes a left turn, then the effect of reactive gyroscopic couple will be to dip the nose and raise the tail of the aero plane.

3. When the aero plane takes a right turn under similar Conditions as mentioned in note 2 above, the effect of Reactive gyroscopic couple will be to raise the nose and dip the of the aero plane.

4. When the engine or propeller rotates in clockwise direction when viewed from the front and the aero plane takes a left turn, then the effect of reactive gyroscopic couple will be to raise the tail and dip the nose of the aero plane.

5. When the aero plane takes a right turn under similar conditions as mentioned in note4 above, the effect of reactive gyroscopic couple will be to raise the nose and dip the tail of the aero plane.

5.17 EFFECT OF GYROSCOPIC COUPLE ON SHIP

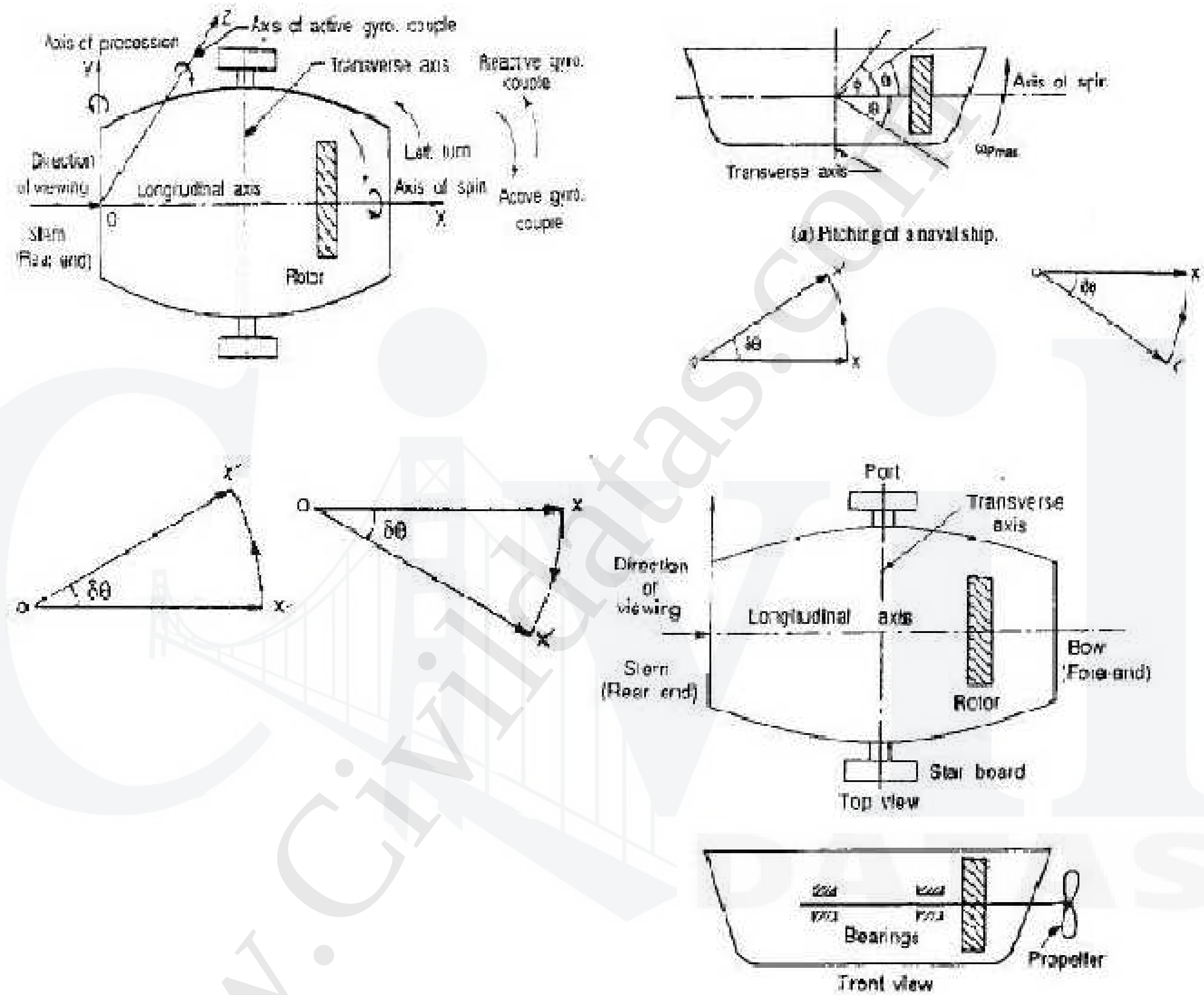
The top and front views of a naval ship are shown in fig. The fore end of the ship is called bow and the rear end is known as stern or aft. The left hand and the right hand sides of the ship, when viewed from the stern are called port and star board respectively. We shall now discuss the effect of gyroscopic couple in the naval ship in the following three cases:

1. Steering

2. Pitching, and

3. Rolling

5.17.1 EFFECT OF GYROSCOPIC COUPLE ON A NAVAL SHIP DURING PITCHING& STEERING



Steering is the turning of a complete ship in a curve towards left or right, while it moves forward, considers the ship taking a left turn, and rotor rotates in the clockwise direction when viewed from the stern, as shown in Fig. below. The effect of gyroscopic couple on a naval ship during steering taking left or right turn may be obtained in the similar way as for an aero plane as discussed in Art.

When the rotor of the ship rotates in the clockwise direction when viewed from the stern, it will have its angular momentum vector in the direction ox as shown in Fig. A1. As the ship steers to the left, the active gyroscopic couple will change the angular momentum vector from ox to ox'. The vector xx' now represents the active gyroscopic couple and is perpendicular to ox. Thus the plane of active gyroscopic couple is perpendicular to xx' and its direction in the axis OZ for left hand turn is clockwise as shown in Fig below. The reactive gyroscopic couple of the same magnitude will act in the opposite direction (i.e in anticlockwise direction). The effect of this reactive gyroscopic couple is to raise the bow and lower the stern.

Notes

1. When the ship steers to the right under similar condition as discussed above, the effect of the reactive gyroscopic couple, as shown in Fig. B1, will be to raise the stern and lower the bow.
2. When the rotor rotates in the anticlockwise direction, when viewed from the stern and the ship is steering to the left, then the effect of reactive gyroscopic couple will be to lower the bow and raise the stern.
3. When the ship is steering to the right under similar conditions as discussed in note 2 above, then the effect of reactive gyroscopic couple will be to raise the bow and lower the stern.
4. When the rotor rotates in the clockwise direction when viewed from the bow or fore end and the ship is steering to the left, then the effect of reactive gyroscopic couple will be to raise the stern and lower the bow.
5. When the ship is steering to the right under similar conditions as discussed in note 4 above, then the effect of reactive gyroscopic couple will be to raise the bow and lower the stern.
6. The effect of the reactive gyroscopic couple on a boat propelled by a turbine taking left or right turn.

5.17.2 Effect of Gyroscopic couple on a Naval Ship during Rolling:

We know that, for the effect of gyroscopic couple to occur, the axis of precession should always be perpendicular to the axis of spin. If, however, the axis of precession becomes parallel to the axis of spin, there will be no effect of the gyroscopic couple acting on the body of the ship. In case of rolling of a ship, the axis of precession (i.e. longitudinal axis) is always parallel to the axis of spin for all positions. Hence, there is no effect of the gyroscopic couple acting on the body of a ship.

5.18 EFFECT OF GYROSCOPIC COUPLE ON A 4-WHEEL DRIVE:

Consider the four wheels A, B, C and D of an automobile locomotive taking a turn towards left as shown in Fig. 14.11. The wheels A and C are inner wheels, whereas B and D are outer wheels. The centre of gravity ($C.G.$) of the vehicle lies vertically above the road surface.

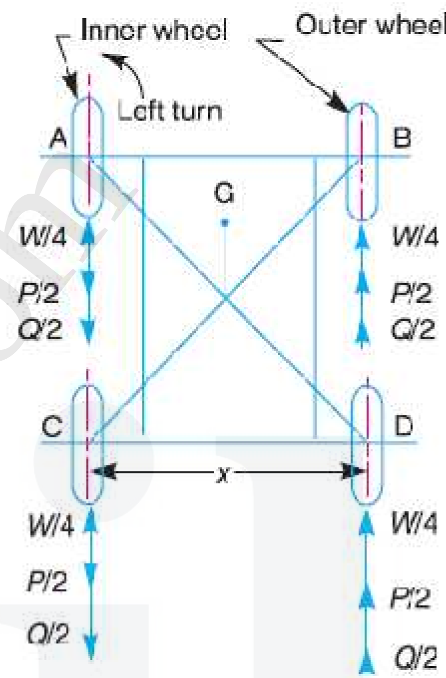


Fig. 14.11. Four wheel drive moving in a curved path.

Let m = Mass of the vehicle in kg,

W = Weight of the vehicle in newtons = $m.g$,

r_W = Radius of the wheels in metres,

R = Radius of curvature in metres ($R > r_W$),

h = Distance of centre of gravity, vertically above the road surface in metres,

x = Width of track in metres,

I_W = Mass moment of inertia of one of the wheels in kg-m^2,

ω_W = Angular velocity of the wheels or velocity of spin in rad/s,

I_E = Mass moment of inertia of the rotating parts of the engine in kg-m^2,

ω_E = Angular velocity of the rotating parts of the engine in rad/s,

G = Gear ratio = ω_E / ω_W,

v = Linear velocity of the vehicle in m/s = $\omega_W . r_W$

A little considereation will show, that the weight of the vehicle (W) will be equally distributed over the four wheels which will act downwards. The reaction between each wheel and the road surface of the same magnitude will act upwards. Therefore

Road reaction over each wheel

$$= W/4 = m.g/4 \text{ newtons}$$

Let us now consider the effect of the gyroscopic couple and centrifugal couple on the vehicle.

1. ***Effect of the gyroscopic couple***

Since the vehicle takes a turn towards left due to the precession and other rotating parts, therefore a gyroscopic couple will act.

We know that velocity of precession,

$$\omega_P = v/R$$

$\therefore$ Gyroscopic couple due to 4 wheels,

$$C_W = 4\,I_W.\omega_W.\omega_P$$

and gyroscopic couple due to the rotating parts of the engine,

$$C_E = I_E.\omega_E.\omega_P = I_E.G.\omega_W.\omega_P \qquad \ldots (\because G = \omega_E/\omega_W)$$

$\therefore$ Net gyroscopic couple,

$$C = C_W \pm C_E = 4\,I_W.\omega_W.\omega_P \pm I_E.G.\omega_W.\omega_P$$
$$= \omega_W.\omega_P\,(4\,I_W \pm G.I_E)$$

The ***positive*** sign is used when the wheels and rotating parts of the engine rotate in the same direction. If the rotating parts of the engine revolves in opposite direction, then ***negative*** sign is used.

Due to the gyroscopic couple, vertical reaction on the road surface will be produced. The reaction will be vertically upwards on the outer wheels and vertically downwards on the inner wheels. Let the magnitude of this reaction at the two outer or inner wheels be P newtons. Then

$$P \times x = C \quad \text{or} \quad P = C/x$$

∴ Vertical reaction at each of the outer or inner wheels,

$$P/2 = C/2x$$

Note: We have discussed above that when rotating parts of the engine rotate in opposite directions, then –ve sign is used, *i.e.* net gyroscopic couple,

$$C = C_W - C_E$$

When $C_E > C_W$, then C will be –ve. Thus the reaction will be vertically downwards on the outer wheels and vertically upwards on the inner wheels.

2. *Effect of the centrifugal couple*

Since the vehicle moves along a curved path, therefore centrifugal force will act outwardly at the centre of gravity of the vehicle. The effect of this centrifugal force is also to overturn the vehicle. We know that centrifugal force,

$$F_C = \frac{m \times v^2}{R}$$

∴ The couple tending to overturn the vehicle or overturning couple,

$$C_O = F_C \times h = \frac{m.v^2}{R} \times h$$

This overturning couple is balanced by vertical reactions, which are vertically upwards on the outer wheels and vertically downwards on the inner wheels. Let the magnitude of this reaction at the two outer or inner wheels be Q. Then

$$Q \times x = C_O \quad \text{or} \quad Q = \frac{C_O}{x} = \frac{m.v^2.h}{R.x}$$

∴ Vertical reaction at each of the outer or inner wheels,

$$\frac{Q}{2} = \frac{m.v^2.h}{2R.x}$$

∴ Total vertical reaction at each of the outer wheel,

$$P_O = \frac{W}{4} + \frac{P}{2} - \frac{Q}{2}$$

and total vertical reaction at each of the inner wheel,

$$P_I = \frac{W}{4} - \frac{P}{2} - \frac{Q}{2}$$

A little consideration will show that when the vehicle is running at high speeds, P_I may be zero or even negative. This will cause the inner wheels to leave the ground thus tending to overturn the automobile. In order to have the contact between the inner wheels and the ground, the sum of $P/2$ and $Q/2$ must be less than $W/4$.

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5.19. SOLVED PROBLEMS

1.(i) Explain the function of the proell governor with the help of a neat sketch. Derive that relationship among the various forces acting on the link. (12)

The Proell governor has the balls fixed at *B* and *C* to the extension of the links *DF* and *EG*, as shown in Fig. 18.12 (*a*). The arms *FP* and *GQ* are pivoted at *P* and *Q* respectively.

Consider the equilibrium of the forces on one-half of the governor as shown in Fig. 18.12 (*b*). The instantaneous centre (*I*) lies on the intersection of the line *PF* produced and the line from *D* drawn perpendicular to the spindle axis. The perpendicular *B M* is drawn on *ID*.

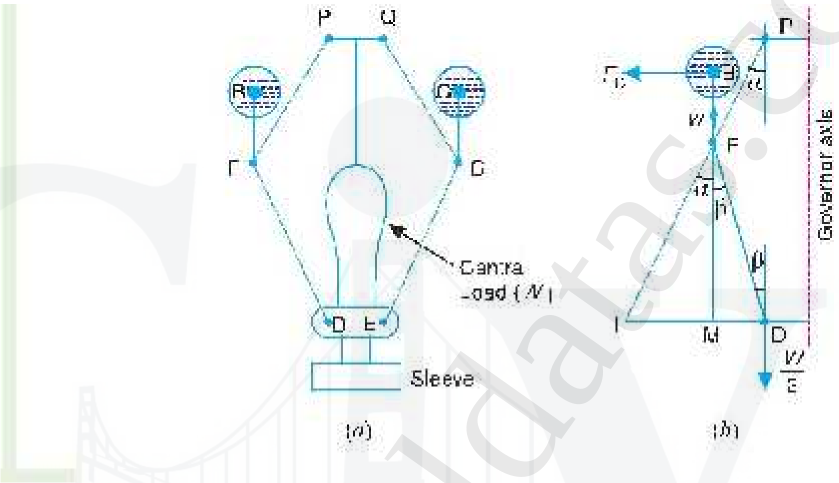


Fig. 18.12. Proell governor.

Taking moments about *I*,

$$F_C \times BM = w \times IM + \frac{W}{2} \times ID = m.g \times IM + \frac{M.g}{2} \times ID \quad \ldots (i)$$

$$\therefore \quad F_C = m.g \times \frac{IM}{BM} + \frac{M.g}{2}\left(\frac{IM + MD}{BM}\right) \quad \ldots (\because \; ID = IM + MD)$$

Multiplying and dividing by *FM*, we have

$$F_C = \frac{FM}{BM}\left[m.g \times \frac{IM}{FM} + \frac{M.g}{2}\left(\frac{IM}{FM} + \frac{MD}{FM}\right)\right]$$

$$= \frac{FM}{BM}\left[m.g \times \tan\alpha + \frac{M.g}{2}(\tan\alpha + \tan\beta)\right]$$

$$= \frac{FM}{BM} \times \tan\alpha\left[m.g + \frac{M.g}{2}\left(1 + \frac{\tan\beta}{\tan\alpha}\right)\right]$$

We know that $F_C = m.\omega^2 r$; $\tan\alpha = \frac{r}{h}$ and $q = \frac{\tan\beta}{\tan\alpha}$

$$\therefore \quad m.\omega^2.r = \frac{FM}{BM} \times \frac{r}{h}\left[m.g + \frac{M.g}{2}(1+q)\right]$$

and
$$\omega^2 = \frac{FM}{BM}\left[\frac{m + \frac{M}{2}(1+q)}{m}\right]\frac{g}{h} \quad \ldots (ii)$$

Substituting $\omega = 2\pi N/60$, and $g = 9.81$ m/s^2, we get

$$N^2 = \frac{FM}{BM}\left[\frac{m + \frac{M}{2}(1+q)}{m}\right]\frac{895}{h} \quad \ldots (iii)$$

1 (ii). What are centrifugal governors? how do they differ from inertia governor? (4)

The centrifugal governors are based on the balancing of centrifugal force on the rotating balls by an equal and opposite radial force, known as the ***controlling force***

In inertia governors the positions of the balls are affected by the rate of change of speed. i.e., angular acceleration or retardation of the governor shaft. The amount of displacement of governor balls is controlled by suitable springs and the fuel supply to the engine is controlled by governor mechanism.

Though the sensitiveness of the inertia governors is more, there is a practical difficulty of balancing the inertia forces caused by the revolving parts of the governor to the controlling force. Hence these governors are not preferred when compared with the centifugal governors.

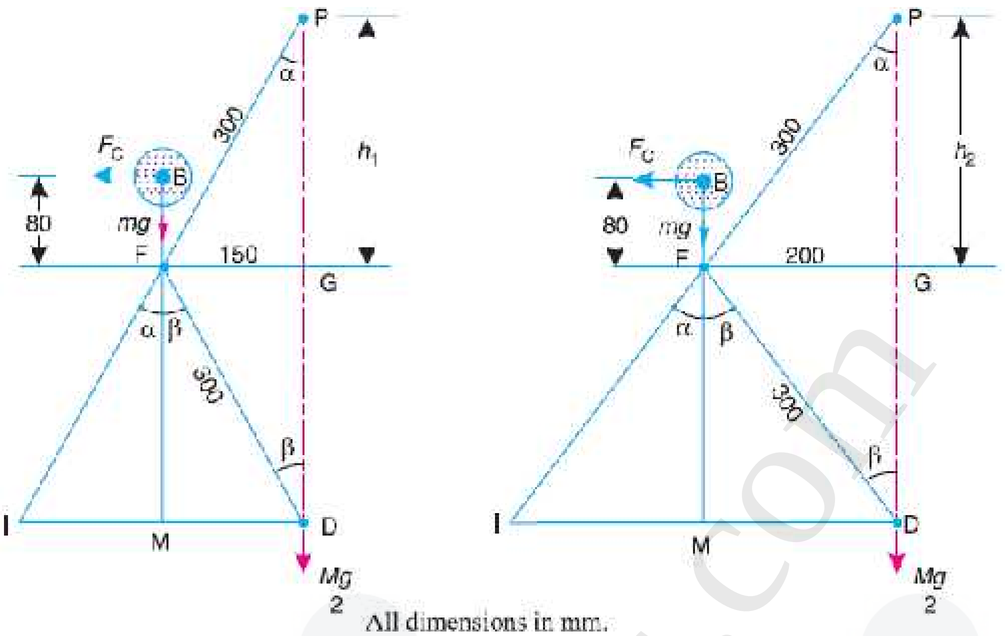
2. . A Proell governor has equal arms of length 300 m m. The upper and lower ends of the arms are pivoted on the axis of the governor. The extension arms of the lower links are each 80 mm long and parallel to the axis when the radii of rotation of the balls are 150 mm and 200 m m. The mass of each ball is 10 kg and the mass of the central load is 100 kg. Determine the range of speed of the governor.

Solution. Given : $PF = DF = 300$ mm ; $BF = 80$ mm ; $m = 10$ kg ; $M = 100$ kg ; $r_1 = 150$ mm; $r_2 = 200$ mm

First of all, let us find the minimum and maximum speed of the governor. The minimum and maximum position of the governor is shown in Fig. 18.13.

Let N_1 = Minimum speed when radius of rotation, $r_1 = FG = 150$ mm ; and

N_2 = Maximum speed when radius of rotation , $r_2 = FG =$ 200 mm. From Fig. (a), we find that height of the governor,



All dimensions in mm.

(a) Minimum position. (a) Maximum position.

From Fig. (a), we find that

$\sin\alpha = \sin\beta = 150 / 300 = 0.5$ or $\alpha = \beta = 30°$

and $MD = FG = 150 \text{ mm} = 0.15 \text{ m}$

$FM = FD\cos\beta = 300\cos 30° = 260 \text{ mm} = 0.26 \text{ m}$

$IM = FM\tan\alpha = 0.26\tan 30° = 0.15 \text{ m}$

$BM = BF + FM = 80 + 260 = 340 \text{ mm} = 0.34 \text{ m}$

$ID = IM + MD = 0.15 + 0.15 = 0.3 \text{ m}$

We know that centrifugal force,

$$F_C = m(\omega_1)^2 r_1 = 10\left(\frac{2\pi N_1}{60}\right)^2 0.15 = 0.0165\,(N_1)^2$$

Now taking moments about point I,

$$F_C \times BM = m.g \times IM + \frac{M.g}{2} \times ID$$

or $$0.0165\,(N_1)^2\,0.34 = 10 \times 9.81 \times 0.15 + \frac{100 \times 9.81}{2} \times 0.3$$

$$0.0056\,(N_1)^2 = 14.715 + 147.15 = 161.865$$

$$\therefore \quad (N_1)^2 = \frac{161.865}{0.0056} = 28\,904 \quad \text{or} \quad N_1 = 170 \text{ r.p.m.}$$

$$h = PG = \sqrt{(PF)^2 - (FG)^2} = \sqrt{(300)^2 - (200)^2} = 224 \text{ mm} = 0.224 \text{ m}$$

$FM = GD = PG = 224 \text{ mm} = 0.224 \text{ m}$

$BM = BF + FM = 80 + 224 = 304 \text{ mm} = 0.304 \text{ m}$

We know that $(N_2)^2 = \frac{\quad}{BM}\left(\frac{\quad}{m}\right)\frac{\quad}{h_2}$ $\quad \ldots (\because \alpha = \beta \text{ or } q = 1)$

$$= \frac{0.224}{0.304}\left(\frac{10+100}{10}\right)\frac{895}{0.224} = 32\,385 \quad \text{or} \quad N_2 = 180 \text{ r.p.m.}$$

We know that range of speed

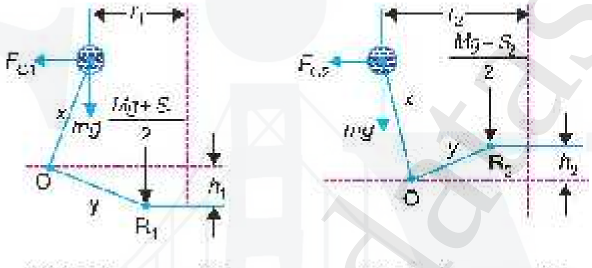
$$= N_2 - N_1 = 180 - 170 = 10 \text{ r.p.m. Ans.}$$

3. The radius of rotation of the balls of a Hartnell governor is 80 mm at the minimum speed of 300 r.p.m. Neglecting gravity effect, determine the speed after the sleeve has lifted by 60 m m. Also determine the initial compression of the spring, the governor effort and the power.

The particulars of the governor are given below:

Length of ball arm = 150 mm ; length of sleeve arm = 100 mm ; mass of each ball = 4 kg ; and stiffness of the spring = 25 N/m m.

Solution. Given : r_1 = 80 mm = 0.08 m ; N_1 = 300 r.p.m. or $\omega_1 = 2\pi \times 300/60 = 31.42$ rad/s ; h = 60 mm = 0.06 m ; x = 150 mm = 0.15 m ; y = 100 mm = 0.1 m ; m = 4 kg ; s = 25 N/mm



(*a*) Minimum position. (*b*) Maximum position.

The minimum and maximum position of the governor is shown in Fig. 18.36 (*a*) and (*b*) respectively. First of all, let us find the maximum radius of rotation (r_2). We know that lift of the sleeve,

$$h = (r_2 - r_1)\frac{y}{x}$$

or

$$r_2 = r_1 + h \times \frac{x}{y} = 0.08 + 0.06 \times \frac{0.15}{0.1} = 0.17 \text{ m} \quad \ldots (\because h = h_1 + h_2)$$

S_1 and S_2= Spring force at the minimum and maximum speed respectively, in newtons

.

We know centrifugal force at the minimum speed,

$$F_{C1} = m(\omega_1)^2 r_1 = 4\,(31.42)^2\,0.08 = 316 \text{ N}$$

Now taking moments about the fulcrum O of the bell crank lever when in minimum position as shown in Fig 18.36 (*a*). The gravity effect is neglected, *i.e.* the moment due to the weight of balls, sleeve and the bell crank lever arms is neglected.

$$\therefore \quad F_{C1} \times x = \frac{M.g + S_1}{2} \times y \quad \text{or} \quad S_1 = 2F_{C1} \times \frac{x}{y} = 2 \times 316 \times \frac{0.15}{0.1} = 948 \text{ N}$$

$$\ldots (\because M = 0)$$

We know that $S_2 - S_1 = h.s$ or $S_2 = S_1 + h.s = 948 + 60 \times 25 = 2448$ N

We know that centrifugal force at the maximum speed,

$$F_{C1} = m(\omega_2)^2 r_2 = \left(\frac{2\pi N_2}{60}\right)^2 r_2 = m\left(\frac{2\pi N_2}{60}\right)^2 0.17 = 0.007\,46\,(N_2)^2$$

Initial compression of the spring

We know that initial compression of the spring $= \dfrac{S_1}{s} = \dfrac{918}{25} = 37.92$ mm **Ans.**

Governor effort

We know that the governor effort,

$$P = \frac{S_2 - S_1}{2} = \frac{2418 - 918}{2} = 750 \text{ N} \quad \textbf{Ans.}$$

Now taking moments about the fulcrum *O* when in maximum position, as shown in Fig. 18.36 (*b*),

$$F_{C2} \times x = \frac{M \cdot g + S_2}{2} \times y$$

$$0.007\,46\,(N_2)^2\; 0.15 = \frac{2418}{2} \times 0.1 \quad \text{or} \quad 0.001\,12\,(N_2)^2 = 122.4 \qquad \ldots (\because M = 0)$$

$$(N_2)^2 = \frac{122.4}{0.001\,12} = 109\,286 \quad \text{or} \quad N_2 = 331 \text{ r.p.m.} \quad \textbf{Ans.}$$

Governor power

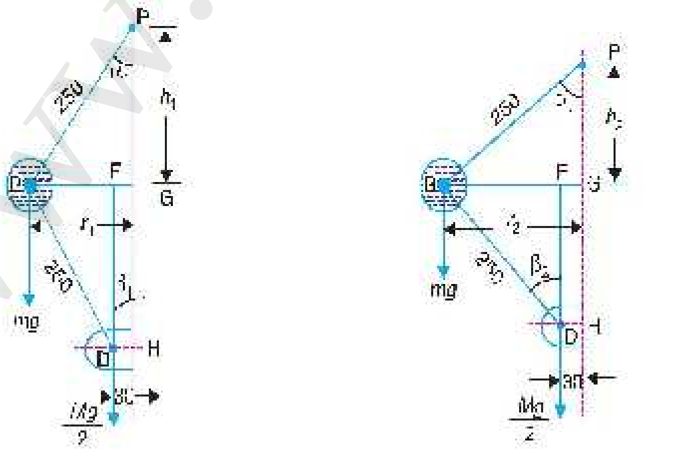
We know that the governor power

$= P \times h = 750 \times 0.06 = 45$ N-m **Ans.**

4. A Porter governor has all four arms 250 mm long. The upper arms are attached on the axis of rotation and the lower arms are attached to the sleeve at a distance of 30 m m from the axis. The mass of each ball is 5 kg and the sleeve has a mass of 50 kg. The extreme radii of rotation are 150 mm and 200 m m. Determine the range of speed of the governor.

Solution. Given : $BP = BD = 250$ mm ; $DH = 30$ mm ; $m = 5$ kg ; $M = 50$ kg ; $r_1 = 150$ mm ; $r_2 = 200$ mm

First of all, let us find the minimum and maximum speed of the governor. The minimum and maximum position of the governor is shown in Fig. 18.8 (*a*) and (*b*) respectively.



(*a*) Minimum position. (*b*) Maximum position.

Fig. 18.8

Let N_1 = Minimum speed when $r_1 = BG = 150$ mm ; and
N_2 = Maximum speed when $r_2 = BG = 200$ mm.

From Fig. 18.8 (*a*), we find that height of the governor,

$$h_1 = PG = \sqrt{(BP)^2 - (BG)^2} = \sqrt{(250)^2 - (150)^2} = 200 \text{ mm} = 0.2 \text{ m}$$

$$BF = BG - FG = 150 - 30 = 120 \text{ mm} \qquad \ldots (\because FG = DH)$$

and

$$DF = \sqrt{(DB)^2 - (BF)^2} = \sqrt{(250)^2 - (120)^2} = 219 \text{ mm}$$

$\therefore$ $\tan \alpha_1 = BG/PG = 150 / 200 = 0.75$

and $\tan \beta_1 = BF/DF = 120/219 = 0.548$

$$\therefore \quad q_1 = \frac{\tan \beta_1}{\tan \alpha_1} = \frac{0.548}{0.75} = 0.731$$

We know that

$$(N_1)^2 = \frac{m + \frac{M}{2}(1 + q_1)}{m} \times \frac{895}{h_1} = \frac{5 + \frac{50}{2}(1 + 0.731)}{5} \times \frac{895}{0.2} = 43206$$

$\therefore$ $N_1 = 208$ r.p.m.

From Fig. 18.8(*b*), we find that height of the governor,

$$h_2 = PG = \sqrt{(BP)^2 - (BG)^2} = \sqrt{(250)^2 - (200)^2} = 150 \text{ mm} = 0.15 \text{ m}$$

$$BF = BG - FG = 200 - 30 = 170 \text{ mm}$$

and

$$DF = \sqrt{(DB)^2 - (BF)^2} = \sqrt{(250)^2 - (170)^2} = 183 \text{ mm}$$

$\therefore$ $\tan \alpha_2 = BG/PG = 200/150 = 1.333$

and $\tan \beta_2 = BF/DF = 170/183 = 0.93$

$$\therefore \quad q_2 = \frac{\tan \beta_2}{\tan \alpha_2} = \frac{0.93}{1.333} = 0.7$$

We know that

$$(N_2)^2 = \frac{m + \frac{M}{2}(1 + q_2)}{m} \times \frac{895}{h_2} = \frac{5 + \frac{50}{2}(1 + 0.7)}{5} \times \frac{895}{0.15} = 56\,683$$

$\therefore$ $N_2 = 238$ r.p.m.

We know that range of speed

$$= N_2 - N_1 = 238 - 208 = 30 \text{ r.p.m. Ans.}$$

5**. A spring loaded governor of the Hartnell type has arms of equal length. The masses rotate in a circle of 130 mm diameter when the sleeve is in the mid position and the ball arms are vertical. The equilibrium speed for this position is 450 r.p.m., neglecting friction. The maximum sleeve movement is to be 25 mm and the maximum variation of speed taking in account the friction to be 5**

per cent of the mid position speed. The mass of the sleeve is 4 kg and the friction may be considered equivalent to 30 N at the sleeve. The power of the governor must be sufficient to over-come the friction by one per cent change of speed either way at mid-position. Determine, neglecting obliquity effect of arms ; 1. The value of each rotating mass : 2. The spring stiffness in N/mm ; and 3. The initial compression of spring.

Solution. Given : $x = y$; $d = 130$ mm or $r = 65$ mm $= 0.065$ m ; $N = 450$ r.p.m. or $\omega = 2\pi \times 450/60 = 47.23$ rad/s ; $h = 25$ mm $= 0.025$ m ; $M = 4$ kg ; $F = 30$ N

1. ***Value of each rotating mass***

Let m = Value of each rotating mass in kg, and

S = Spring force on the sleeve at mid position in newtons.

Since the change of speed at mid position to overcome friction is 1 per cent either way (*i.e.* ± 1%), therefore

Minimum speed at mid position,

$$\omega = \omega - 0.01\omega = 0.99\omega = 0.99 \times 47.13 = 46.66 \text{ rad/s and}$$

maximum speed at mid-position,

$$\omega_2 = \omega + 0.01\omega = 1.01\omega = 1.01 \times 47.13 = 47.6 \text{ rad/s}$$

∴ Centrifugal force at the minimum speed,

$$F_{C1} = m(\omega_1)^2 r = m(46.66)^2\ 0.065 = 141.5\ m \text{ N}$$

and centrifugal force at the maximum speed,

$$F_{C2} = m(\omega_2)^2 r = m(47.6)^2\ 0.0065 = 147.3$$

m N We know that for minimum speed at mid-position,

$$S + (M.g + F) - 2F_{C1} \times \frac{x}{y}$$

or $$S + (4 \times 9.81 - 30) = 2 \times 141.5\, m \times 1 \quad \ldots (\because x = y)$$

∴ $$S + 9.24 = 283\, m \quad \ldots (i)$$

and for maximum speed at mid-position,

$$S + (M.g + F) = 2F_{C2} \times \frac{x}{y}$$

$$S + (4 \times 9.81 + 30) = 2 \times 147.3\, m \times 1 \quad \ldots (\because x = y)$$

∴ $$S + 69.24 = 294.6\, m \quad \ldots (ii)$$

From equations (*i*) and (*ii*),

$$m = 5.2 \text{ kg } \textbf{Ans.}$$

2. ***Spring stiffness in N/mm***

Let s = Spring stiffness in N/mm.

Since the maximum variation of speed, considering friction is ± 5% of the mid-position speed, therefore,

Minimum speed considering

friction,

$$\omega_1' = \omega - 0.05\omega = 0.95\omega = 0.95 \times 47.13 = 44.8 \text{ rad/s}$$

and maximum speed considering friction,

$$\omega_2' = \omega + 0.05\omega = 1.05\omega = 1.05 \times 47.13 = 49.5 \text{ rad/s}$$

We know that minimum radius of rotation considering friction,

$$r_1 = r - h_1 \times \frac{x}{y} = 0.065 - \frac{0.025}{2} = 0.0525 \text{ m}$$

$$\ldots \left(\because x = y \text{ and } h_1 = \frac{h}{2}\right)$$

And maximum radius of rotation considering friction,

We know that for minimum speed considering friction,

$$S_1 + (M.g - F) = 2F_{C1}' \times \frac{x}{y}$$

$$S_1 + (4 \times 9.81 - 30) = 2 \times 548 \times 1 \qquad \ldots (\because x = y)$$

$$\therefore \quad S_1 + 9.24 = 1096 \quad \text{or} \quad S_1 = 1096 - 9.24 = 1086.76 \text{ N}$$

and for maximum speed considering friction,

$$S_2 + (M.g + F) = 2F_{C2}' \times \frac{x}{y}$$

3. Initial compression of the spring $\ldots (\because x = y)$

We know that initial compression of the spring

$$= \frac{S_1}{s} = \frac{1086.76}{32.72} = 33.2 \text{ mm} \quad \textbf{Ans.}$$

6. In an engine governor of the Porter type, the upper and lower arms are 200 m m and 250 mm respectively and pivoted on the axis of rotation. The mass of the central load is 15 kg, the mass of each ball is 2 kg and friction of the sleeve together with the resistance of the operating gear is equal to a load of 25 N at the sleeve. If the limiting inclinations of the upper arms to the vertical are 30° and 40°, find, taking friction into account, range of speed of the governor.

Solution . Given : BP = 200 mm = 0.2 m ; BD = 250 mm = 0.25 m ; M = 15 kg ; m = 2 kg ; F = 25 N ; $\alpha_1 = 30°$; $\alpha_2 = 40°$

First of all, let us find the minimum and maximum speed of the governor.

The minimum and maximum position of the governor is shown Fig. 18.7 (*a*) and (*b*) respectively.

Let N_1 = Minimum speed, and N_2 = Maximum speed.

From Fig. 18.7 (*a*), we find that minimum radius of rotation,

$r_1 = BG = BP \sin 30° = 0.2 \times 0.5 = 0.1$ m Height of the governor,

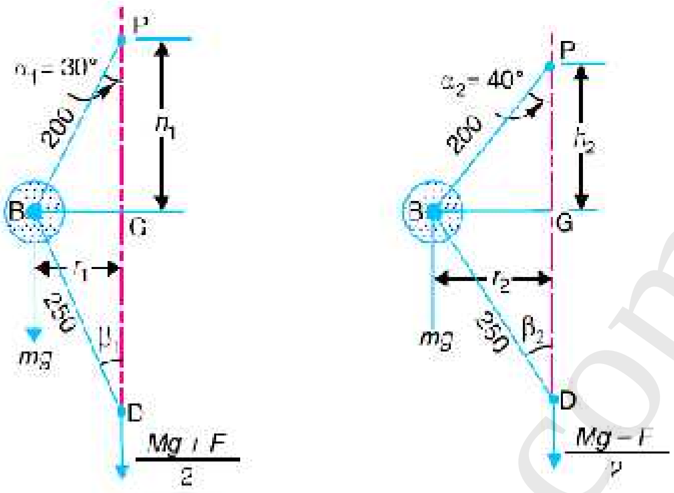
$h_1 = PG = BP \cos 30° = 0.2 \times 0.866 = 0.1732$ m

$$\text{and} \quad DG = \sqrt{(BD)^2 - (BG)^2} = \sqrt{(0.25)^2 - (0.1)^2} = 0.23 \text{ m}$$

$$\therefore \quad \tan\beta_1 = BG/DG = 0.1/0.23 = 0.4348$$

$$\text{and} \quad \tan\alpha_1 = \tan 30° = 0.5774$$

$$\therefore \quad q_1 = \frac{\tan\beta_1}{\tan\alpha_1} = \frac{0.4348}{0.5774} = 0.753$$



All dimensions in mm.

(*a*) Minimum position. (*b*) Maximum position.

Fig. 18.7

We know that when the sleeve moves downwards, the frictional force (F) acts upwards and the minimum speed is given by,

$$(N_1)^2 = \frac{m.g + \left(\frac{M.g - F}{2}\right)(1 + q_1)}{m.g} \times \frac{895}{h_1}$$

$$= \frac{2 \times 9.81 + \left(\frac{15 \times 9.81 - 24}{2}\right)(1 + 0.753)}{2 \times 9.81} \times \frac{895}{0.1732} = 33596$$

$\therefore \quad N_1 = 183.3$ r.p.m.

Now from Fig. 18.7 (*b*), we find that maximum radius of rotation,

$$r_2 = BG = BP \sin 40° = 0.2 \times 0.643 = 0.1268 \text{ m}$$

Height of the governor,

$$h_2 = PG = BP \cos 40° = 0.2 \times 0.766 = 0.1532 \text{ m}$$

and $\quad DG = \sqrt{(BD)^2 - (BG)^2} = \sqrt{(0.25)^2 - (0.1268)^2} = 0.2154$ m

$\therefore \quad \tan \beta_2 = BG/DG = 0.1268 / 0.2154 = 0.59$

and $\quad \tan \alpha_2 = \tan 40° = 0.839$

$$\therefore \quad q_2 = \frac{\tan \beta_2}{\tan \alpha_2} = \frac{0.59}{0.839} = 0.703$$

We know that when the sleeve moves upwards, the frictional force (F) acts downwards and the maximum speed is given by,

$$(N_2)^2 = \frac{m.g + \left(\frac{m.g + F}{2}\right)(1 + q_2)}{m.g} \times \frac{895}{h_2}$$

$$= \frac{2 \times 9.81 + \left(\frac{15 \times 9.81 + 24}{2}\right)(1 + 0.703)}{2 \times 9.81} \times \frac{895}{0.1532} = 49\ 236$$

$\therefore$ $N_2 = 222$ r.p.m.

We know that range of speed

$= N_2 - N_1 = 222 - 183.3 = 38.7$ r.p.m. **Ans.**

7.In a spring loaded governor of the Hartnell type, the mass of each ball is 1kg, length of vertical arm of the bell crank lever is 100 mm and that of the horizontal arm is 50 m m. The distance of fulcrum of each bell crank lever is 80 mm from the axis of rotation of the governor. The extreme radii of rotation of the balls are 75 mm and 112.5 m m. The maximum equilibrium speed is 5 per cent greater than the minimum equilibrium speed which is 360 r.p.m. Find, neglecting obliq-uity of arms, initial compression of the spring and equilibrium speed corresponding to the radius of rotation of 100 m m.

Solution. Given : $m = 1$ kg ; $x = 100$ mm $= 0.1$ m ; $y = 50$ mm $= 0.05$ m ; $r = 80$ mm $= 0.08$ m ; $r_1 = 75$ mm $= 0.075$ m ; $r_2 = 112.5$ mm $= 0.1125$ m ; $N_1 = 360$ r.p.m. or $\omega_1 = 2\pi \times 360/60 = 37.7$ rad/s

Since the maximum equilibrium speed is 5% greater than the minimum equilibrium speed (ω_1), therefore maximum equilibrium speed,

$$\omega_2 = 1.05 \times 37.7 = 39.6 \text{ rad/s}$$

We know that centrifugal force at the minimum equilibrium speed,

$$F_{C1} = m(\omega_1)^2 r_1 = 1 (37.7)^2 \, 0.075 = 106.6 \text{ N}$$

and centrifugal force at the maximum equilibrium speed,

$$F_{C2} = m(\omega_2)^2 r_2 = 1 (39.6)^2 \, 0.1125 = 176.4 \text{ N}$$

Initial compression of the spring

Let S_1 = Spring force corresponding to ω_1, and

S_2 = Spring force corresponding to ω_2.

Since the obliquity of arms is neglected, therefore for minimum equilibrium position

$$M.g + S_1 = 2F_{C1} \times \frac{x}{y} = 2 \times 106.6 \times \frac{0.1}{0.05} = 426.4 \text{ N}$$

$\therefore$ $S_1 = 426.4$ N ...($\because M = 0$)

and for maximum equilibrium position,

$$M.g + S_2 = 2F_{C2} \times \frac{x}{y} = 2 \times 176.4 \times \frac{0.1}{0.05} = 705.6 \text{ N}$$

$\therefore$ $S_2 = 705.6$ N ...($\because M = 0$)

We know that lift of the sleeve,

$$h = (r_2 - r_1)\frac{y}{x} = (0.1125 - 0.075)\frac{0.05}{0.1} = 0.018\,75 \text{ m}$$

and stiffness of the spring $s = \dfrac{S_2 - S_1}{h} = \dfrac{705.6 - 426.4}{0.018\,75} = 14\,890 \text{ N/m} = 14.89 \text{ N/mm}$

$\therefore$ Initial compression of the spring

$$= \frac{S_1}{s} = \frac{426.4}{14.89} = 28.6 \text{ mm} \quad \textbf{Ans.}$$

Equilibrium speed corresponding to radius of rotation r = 100 mm = 0.1 m

Let N = Equilibrium speed in r.p.m.

Since the obliquity of the arms is neglected, therefore the centrifugal force at any instant,

$$F_C = F_{C1} + (F_{C2} - F_{C1})\left(\frac{r - r_1}{r_2 - r_1}\right)$$

$$= 106.6 + (176.4 - 106.6)\left(\frac{0.1 - 0.075}{0.1125 - 0.075}\right) = 153 \text{ N}$$

We know that centrifugal force (F_C),

$$153 = m.\omega^2.r = 1\left(\frac{2\pi N}{60}\right)^2 0.1 = 0.0011\,N^2$$

$\therefore$ $N^2 = 153 / 0.0011 = 139\,090$ or $N = 373$ r.p.m. **Ans.**

5.20 REVIEW QUESTIONS

(1) What are the effects of friction and of adding a central weight to the sleeve of a Watt governor ?

(2) What is stability of a governor ? Sketch the controlling force *versus* radius diagrams for a stable, unstable and isochronous governor. Derive the conditions for stability.

(3) Prove that the sensitiveness of a Proell governor is greater than that of a Porter governor

(4) When the sleeve of a Porter governor moves upwards, the governor speed

(5) When the relation between the controlling force (F_C) and radius of rotation (r) for a spring controlled governor is $F_C = a.r + b$, then the governor will be

5.21 TUTORIAL PROBLEMS

1. In a governor of the Hartnell type, the mass of each ball is 1.5 kg and the lengths of the vertical and horizontal arms of the bell crank lever are 100 mm and 50 mm respectively. The fulcrum of the bell crank lever is at a distance of 90 mm from the axis of rotation. The maximum and minimum radii of rotation of balls are 120 mm and 80 mm and the corresponding equilibrium speeds are 325 and 300 r.p.m. Find the stiffness of the spring and the equilibrium speed when the radius of rotation is 100 mm.

[Ans. 18 kN/m, 315 r.p.m.]

2. A governor of the Hartnell type has equal balls of mass 3 kg, set initially at a radius of 200 mm. The arms of the bell crank lever are 110 mm vertically and 150 mm horizontally. Find : 1. the initial compressive force on the spring, if the speed for an initial ball radius of 200 mm is 240 r.p.m. ; and 2. the stiffness of the spring required to permit a sleeve movement of 4 mm on a fluctuation of 7.5 per cent in the engine speed. **[Ans. 556 N ; 23.75 N/mm]**

3. A four wheel trolley car of total mass 2000 kg running on rails of 1 m gauge, rounds a curve of 25 m

radius at 40 km / h. The track is banked at 10°. The wheels have an external diameter of 0.6 m and each pair of an axle has a mass of 200 kg. The radius of gyration for each pair is 250 mm. The height of C.G. of the car above the wheel base is 0.95 m. Allowing for centrifugal force and gyroscopic couple action, determine the pressure on each rail. **[Ans. 4328 N ; 16 704 N]**

4. Each paddle wheel of a steamer have a mass of 1600 kg and a radius of gyration of 1.2 m. The steamer turns to port in a circle of 160 m radius at 24 km / h, the speed of the paddles being 90 r.p.m. Find the magnitude and effect of the gyroscopic couple acting on the steamer. **[Ans. 905.6 N-m]**

5. The rotor of the turbine of a yacht makes 1200 r.p.m. clockwise when viewed from stern. The rotor has

a mass of 750 kg and its radius of gyration is 250 mm. Find the maximum gyroscopic couple transmit-ted to the hull (body of the yacht) when yacht pitches with maximum angular velocity of 1 rad /s. What is the effect of this couple ? **[Ans. 5892 N-m]**