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UNIT-2

Analytic Functions

Complex Variable:

If x and y are two Real Variables then $z = x+iy$ is called a Complex Variable. where x is called the Read Part of z and is denoted by $Re(z)$, y is called the imaginary Part of z and is denoted by $Im(z)$

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Function of a complex variable

If $z = x+iy$ and $w = u+iv$ are two complex numbers such that their exists one or more values of w corresponding to each value of z in a certain region of the z-plane then w is called the function of z. and is written as

$$w = u+iv = f(z) = f(x+iy)$$

$\rightarrow$ $w = (\bar{z})^2$

$$= (x-iy)^2$$

$$= x^2 - i2xy - y^2$$

$$w = (x^2-y^2) + i(-2xy)$$

$$u+iv = x^2-y^2 + i(-2xy)$$

Here $u(x,y) = x^2-y^2$ &

$v(x,y) = -2xy$

$\therefore w = f(z) = u(x,y) + iv(x,y)$

Single valued function:-

For every value of z their corresponds a unique value of w then w is called a single valued function of z.

Eg:- $w = (z)^2$, $w = 1/z$

Multiple valued function

If for every value of z their corresponds more than one value of w then w is called multiple valued function of z.

Eg:- $w = z^{1/4}$, $w = amp(z)$

Limit of a function of a complex variable

A single valued function $f(z)$ is said to have limit l for given $\epsilon > 0$ $\exists$ a $\delta > 0$ $\ni$ $|f(z) - l| < \epsilon$ whenever

$$0 < |z - z_0| < \delta$$

Symbollicaly it can be written as

$$\lim_{z \to z_0} f(z) = l$$

Continuity of $f(z)$ at a point $z = z_0$

for given $\epsilon > 0$ $\exists$ a $\delta < 0$ $\ni$ $|f(z) - f(z_0)| < \epsilon$.

A single valued function $f(z)$ is said to be continuous at a point z_0 if for any $\epsilon > 0$

whenever $0 < |z - z_0| < \delta$

Symbollically we can write as

$$\lim_{z \to z_0} f(z) = f(z_0)$$

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Derivability of $f(z)$ at a point $z = z_0$

The single valued function $f(z)$ is said to be derived at a point $z = z_0$ if for given $\epsilon > 0 \ \exists$ a $\delta > 0$

$$\rightarrow \left| \frac{f(z) - f(z_0)}{z - z_0} \right| < \epsilon \text{ whenever}$$

$$0 < |z - z_0| < \delta$$

symbolically it can be written as

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = f'(z_0)$$

Properties:-

1) If $f(z)$ and $g(z)$ are differentiable functions then their sum, Product, difference and quotient are also differentiable.

(1) $$\frac{d}{dz}[f(z) \pm g(z)] = \frac{d}{dz}[f(z)] \pm \frac{d}{dz}[g(z)]$$

(2) $$\frac{d}{dz}[c \cdot f(z)] = c \cdot \frac{d}{dz} f(z)$$

(3) $$\frac{d}{dz}[f(z) \cdot g(z)] = f(z)\frac{d}{dz} g(z) + g(z)\frac{d}{dz} f(z)$$

(4) $$\frac{d}{dz}\left[\frac{f(z)}{g(z)}\right] = \frac{g(z) f'(z) - f(z) g'(z)}{g(z)^2}$$

Note:-
Every derivable function is continuous but converse need not be true i.e every continuous function may or may not be derivable.

but every discontinuous systems is not derivable

Analytic function:- [Regular function or Holomorphic]

→ A function $f(z)$ is said to be Analytic at a point $z=z_0$ if there exists a neighbourhood of z_0 in that neighbourhood the given function $f(z)$ is derivable at everywhere.

i.e a single valued function $f(z)$ is said to be analytic at a point z_0 if it is differentiable at every point in some neighbourhood of z_0.

→ Analytic function is also called Regular function or Holomorphic function.

→ If a function $f(z)$ is differentiable for all values of z then it is called Entire function

Note:-

A function which is differentiable at a point is not always analytic function at that point.

→ A function fails to be analytic at a point known as singular point

Properties of Analytic functions:-

If $f(z)$ and $g(z)$ are two analytic functions on a domain 'D' then

① $f+g$ is analytic on 'D'

② $f-g$ is analytic on 'D'

③ fg is analytic on 'D'

④ f/g " " " " except at zero of g

⑤ cf " " " " where c is constant

⑥ cg " " " " where c is constant.

⑦ composition of f & g is analytic on 'D'

⑧ $f^2, f^3, f^4 \ldots f^n$ are also analytic on 'D'

Necessary and sufficient condition for $f(z)$ to be analytic

The necessary and sufficient condition for function $f(z) = w = u(x,y) + iv(x,y)$ to be analytic in a region R are as follows

1) $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are continuous functions of x & y in the region R

2) $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

i.e $u_x = v_y$ & $u_y = -v_x$

3) The equations in condition ② are known as Cauchy-Rieman equations (or) C-R equations.

2) Given function, $f(z) = z^3$.

$f(z) = (x+iy)^3$.

$= x^3 + (iy)^3 + 3(x^2)(iy) + 3(x)(iy)^2$.

$= x^3 - iy^3 + i3x^2y - 3xy^2$.

$\Rightarrow u+iv = (x^3 - 3x^2y) + i(3x^2y - y^3)$

$u = x^3 - 3x^2y$; $V = 3x^2y - y^3$.

Let $u_x = \frac{\partial u}{\partial x} = 3x^2 - 3y^2$.

$u_y = \frac{\partial u}{\partial y} = -6xy$.

Let $V_x = \frac{\partial v}{\partial x} = 6xy$

$V_y = \frac{\partial v}{\partial y} = 3x^2 - 3y^2$.

W.K.T C-R eq's.

$u_x = V_y$ & $u_y = -V_x$.

Consider $u_x = V_y$.

$3x^2 - 3y^2 = 3x^2 - 3y^2$.

Let $u_y = -V_x$.

$-6xy = -6xy$.

$\therefore$ C-R eq's are satisfied

$f(z)$ is analytic for all "z".

3) S.T z^2 is analytic for all "z".

Sol: Given, $f(z) = z^2$.

$f(z) = (x+iy)^2$.

$= x^2 - y^2 + 2xy$.

$u+iv = (x^2 - y^2) + i(2xy)$

$u = x^2 - y^2$

$u_x = 2x$.

$u_y = -2y$

$V = 2xy$.

$V_x = 2y$.

$V_y = 2x$.

$u_x = V_y$ & $u_y = -V_x$.

$\therefore$ C-R eq's are satisfied. $f(z)$ is analytic for all "z".

4) S.T $f(z) = z + 2\bar{z}$ is not analytic anywhere in the complex plane.

Sol: $f(z) = z + 2\bar{z}$.

$f(z) = (x+iy) + 2(x-iy)$

$= x + iy + 2x - 2iy$

$= 3x + i(-y)$

$u = 3x.$	$v = -y.$
$u_x = 3.$	$v_x = 0.$
$u_y = 0$	$v_y = -1.$

W.K.T C-R eq's are

$u_x = v_y$ & $u_y = -v_x.$

$3 \neq -1$ & $0 = 0$

$\therefore$ C-R eq's are not satisfied

$f(z) = z + 2\bar{z}$ is not analytic anywhere in the complex plane.

5) Find whether $f(z) = \sin x \cdot \sin y - i \cos x \cos y$ is analytic (or) not.

Sol:- $f(z) = \sin x \cdot \sin y - \cos x \cos y.$

$u = \sin x \cdot \sin y.$	$v = -\cos x \cos y.$
$u_x = \cos x \sin y.$	$v_x = \sin x \cos y$
$u_y = \sin x \cos y$	$v_y = \cos x \sin y$

W.K.T C-R eq's are

$u_x = v_y$ & $u_y \neq -v_x.$

$\therefore$ C-R eq's are not satisfied.

$f(z)$ is not analytic.

6) Find whether $f(z) = \frac{x - iy}{x^2+y^2}$ is analytic (or) not.

Sol: $f(z) = \frac{x}{x^2+y^2} - i \cdot \frac{y}{x^2+y^2}$

$u = \frac{x}{x^2+y^2}$

$u_x = \frac{x(2x) - (x^2+y^2)(1)}{(x^2+y^2)^2}$

$= \frac{2x^2 - x^2 - y^2}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$

$u_y = \frac{\cancel{} -4xy}{(x^2+y^2)^3}$

$v = \frac{-y}{x^2+y^2}$

$\cancel{v_x = \frac{-y(2x) - (x^2+y^2)}{(x^2+y^2)^2}}$

$= \frac{\cancel{-2xy - x^2} - y^2}{}$

$v_x = \frac{+4xy}{(x^2+y^2)^3}$

$v_y = \frac{x^2-y^2}{(x^2+y^2)^2}$

W.K.T C-R eq's are.

$u_x = v_y$ & $u_y = -v_x$.

$\therefore$ C-R eq's are not satisfied.

f(z) is analytic.

7) Find all values of "k", such that ~~f(z) = e^x~~

$f(z) = e^x(\cos ky + i \sin ky)$ is analytic.

Sol: $f(z) = e^x \cos ky + i e^x \sin ky$.

$u = e^x . \cos ky$

$u_x = e^x . \cos ky$.

$u_y = -e^x \sin ky \cdot k$

$v = e^x \sin ky$.

$v_x = \cancel{e^x \cos ky}\, e^x \sin ky$

$v_y = e^x \cos ky \cdot k$

$u_x = v_y$ & $u_y = -v_x$.

$\therefore$ C-R eq's are satisfied for $\boxed{k=1}$

~~k=~~

8) Determine "p" such that the fun. ~~f(z)=~~ $f(z) = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}\left(\frac{px}{y}\right)$ be an analytic function.

Sol:- $u = \frac{1}{2}\log(x^2+y^2)$

$$u_x = \frac{1}{\cancel{2}} \cdot \frac{\cancel{2}x}{x^2+y^2}$$

$$= \frac{x}{x^2+y^2}$$

$$u_y = \frac{1}{\cancel{2}} \cdot \frac{\cancel{2}y}{x^2+y^2}$$

$$= \frac{y}{x^2+y^2}$$

$v = \tan^{-1}\left(\frac{px}{y}\right)$

$$v_x = \frac{1}{1+\left(\frac{px}{y}\right)^2}\left(\frac{p}{y}\right) - \frac{py}{y^2+(px)^2}$$

$$v_y = \frac{1}{1+\left(\frac{px}{y}\right)^2}\left(\frac{-px}{y^2}\right) - \frac{px}{(px)^2+y^2}$$

W.K.T C-R eq's are.

$u_x = v_y$.

$$\frac{x}{x^2+y^2} = \frac{-px}{(px)^2+y^2}$$

$$\boxed{p=-1}$$

$u_y = -v_x$.

$$\frac{y}{x^2+y^2} = \frac{-py}{y^2+(px)^2}$$

$$\boxed{p=-1}$$

$\therefore$ $f(z)$ is analytic when $p=-1$.

9) Determine whether the function $2xy + i(x^2-y^2)$ is analytic.

Sol:- $u = 2xy$

$u_x = 2y$.

$u_y = 2x$.

$v = x^2-y^2$.

$v_x = 2x$.

$v_y = -2y$.

$u_x \neq v_y$ & $u_y \neq -v_x$.

$\therefore$ The function is not analytic.

10) P.T the function $f(z) = \bar{z}$. is not analytic at any point.

Sol; $f(z) = x - iy$.

$u = x$ $\quad$ $v = -y$.

$u_x = 1$ $\quad$ $v_x = 0$.

$u_y = 0$ $\quad$ $v_y = -1$.

$u_x \neq -v_y$ & $u_y \neq -v_x$.

$\therefore f(z)$ is not analytic at any point.

* ORTHOGONAL SYSTEMS:-

The two family of curves $u(x,y) = c_1$ & $v(x,y) = c_2$ are said to form an orthogonal system if they intersect at right angle at each point of their intersection.

Theorem:-

If $f(z) = u(x,y) + iv(x,y)$ is an analytic function then $u(x,y) = c_1$ and $v(x,y) = c_2$ are orthogonal.

Proof.

Given that $u(x,y) = c_1$ —— ①

$v(x,y) = c_2$ —— ②

Now diff eq ① w.r.to "x" partially.

$$\frac{\partial u}{\partial x} \cdot \cancel{\phi} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial x} = 0.$$

$$\frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial x} = -\frac{\partial u}{\partial x}$$

$$\frac{\partial y}{\partial x} = \frac{-\partial u/\partial x}{\partial u/\partial y}$$

$$m_1 = \frac{-u_x}{u_y} \quad \text{——} ③$$

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Diff. ② w.r. to 'x' Partially.

$$\frac{\partial v}{\partial x}+\frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial x}=0.$$

$$\frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial x}=-\frac{\partial v}{\partial x}$$

$$\frac{\partial y}{\partial x}=\frac{-\partial v/\partial x}{\partial v/\partial y}=-\frac{V_x}{V_y}$$

$$m_2=-\frac{V_x}{V_y} \longrightarrow ④$$

But $F(z)=u(x,y)+iv(x,y)$ is an analytic function.

It satisfies C.R eq's.

$u_x=V_y$ & $u_y=-V_x$.

From ③ & ④ $\Rightarrow$ $m_1m_2=-\frac{u_x}{u_y}\times\frac{-V_x}{V_y}$

$$=-\frac{u_x}{u_y}\times\frac{u_y}{u_x}$$

$$m_1m_2=-1.$$

$\therefore$ $u(x,y)=C_1$ & $v(x,y)=C_2$ are orthogonal.

1) Show that $x^2-y^2=C_1$ & $2xy=C_2$ are orthogonal systems.

SOL:- $u(x,y)=x^2-y^2=C_1 \longrightarrow ①$

$v(x,y)=2xy=C_2 \longrightarrow ②$

Diff ① w.r.to "x" partially.

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial x}=0.$$

$$2x+(-2y)\frac{\partial y}{\partial x}=0.$$

$$x-y\cdot\frac{\partial y}{\partial x}=0.$$

$x = y \cdot \frac{\partial y}{\partial x}$

$\frac{\partial y}{\partial x} = \frac{x}{y}$

$m_1 = \frac{x}{y}$ ——— (3)

Diff (2) w.r.to "x" partially

$\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y} \cdot \frac{\partial u}{\partial x} = 0$

$2x \cdot \frac{\partial x}{\partial x} + 2x \cdot \frac{\partial y}{\partial x} = 0$

$y + x \cdot \frac{\partial y}{\partial x} = 0$

$\frac{\partial y}{\partial x} = -\frac{y}{x}$

$m_2 = -\frac{y}{x}$ ——— (4)

From (3) & (4)

$m_1 m_2 = \frac{\cancel{x}}{\cancel{y}} \times \frac{-\cancel{y}}{\cancel{x}}$

$m_1 m_2 = -1$

$\therefore x^2 - y^2 = c_1$ & $2xy = c_2$ are orthogonal systems.

* LAPLACE EQUATION :-

Let $u(x,y)$ & $v(x,y)$ are said to be satisfied laplace eq. if $\nabla^2 u = 0$ & $\nabla^2 v = 0$ i.e.,

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ & $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$

Where "∇^2" → Laplasion operator.

$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$

*HARMONIC FUNCTION :-

A function $u(x,y)$ is said to be harmonic function if there exists continuous 2nd order partial derivatives and satisfies the laplace eq. i.e.,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

(or)

A fun. $u(x,y)$ is said to be H.F if the solutions of laplace eq's having continuous 2nd order partial derivatives.

1) S.T the fun. $u(x,y) = e^x . \cos y$ is harmonic.

Sol:- $u(x,y) = e^x \cos y$.

Diff w.r.to "x" & "y" partially.

then $\frac{\partial u}{\partial x} = e^x \cos y$ $\quad \frac{\partial u}{\partial y} = -e^x \sin y$

Now Diff $\frac{\partial u}{\partial x}$ w.r.to "x" partially.

$$\therefore \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{\partial u}{\partial x}\right] = \frac{\partial}{\partial x}(e^x \cos y) = e^x \cos y.$$

1) Now Diff $\frac{\partial u}{\partial y}$ w.r.to "y" partially.

$$\text{So } \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}(-e^x \sin y) = -e^x \cos y.$$

$\therefore$ 1st & 2nd order partial ~~deri~~ derivatives are continuous.

Consider $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = e^x \cos y - e^x \cos y$

$= 0.$

$\therefore u(x,y) = e^x \cos y$ is Harmonic function.

2) Find "K" if $F(x,y) = x^3+3kxy^2$ may harmonic.

Sol: $F(x,y) = x^3+3kxy^2$.

$$\frac{\partial F}{\partial x} = 3x^2+3ky^2.$$

$$\frac{\partial F}{\partial y} = 6kxy.$$

$$\frac{\partial^2 F}{\partial x^2} = 6x \quad ; \quad \frac{\partial^2 F}{\partial y^2} = 6kx.$$

$$\therefore \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} = 0.$$

$$6x+6kx=0.$$

$$\boxed{k=-1}$$

3) S.T fun. $U(x,y) = 2\log(x^2+y^2)$ is harmonic.

Sol: $U(x,y) = 2\log(x^2+y^2)$.

Let, $$\frac{\partial u}{\partial x} = 2\cdot\frac{2x}{x^2+y^2} = \frac{4x}{x^2+y^2}$$

$$\frac{\partial u}{\partial y} = 2\cdot\frac{1}{x^2+y^2}\cdot 2y = \frac{4y}{x^2+y^2}.$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{4x}{x^2+y^2}\right)$$

$$= \frac{(x^2+y^2)(4) - 4x(2x)}{(x^2+y^2)^2} = \frac{4y^2-4x^2}{(x^2+y^2)^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{4y}{x^2+y^2}\right)$$

$$= \frac{(x^2+y^2)(4) - 4y(2y)}{(x^2+y^2)^2} = \frac{4x^2-4y^2}{(x^2+y^2)^2}.$$

1st & 2nd Order Partial derivatives are continuous

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

$\therefore$ $U(x,y)$ is harmonic function.

4) P.T $U(x,y) = x^2 - y^2 - 2xy - 2x + 3y$ is harmonic.

Sol:- $U(x,y) = x^2 - y^2 - 2xy - 2x + 3y$.

$$\frac{\partial u}{\partial x} = 2x - 2y - 2$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}(2x - 2y - 2)$$

$$= 2$$

$$\frac{\partial u}{\partial y} = -2y - 2x + 3.$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}(-2y - 2x + 3)$$

$$= -2.$$

1st & 2nd order partial derivatives are continuous.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

$\therefore$ The given fun. is Harmonic.

* HARMONIC CONJUGATE:-

The function $V(x,y)$ is said to be harmonic conjugate of $U(x,y)$. If "U" & "V" are harmonic and the 1st order partial derivatives of "U" & "V" satisfies CR eq's.

(OR)

If two given functions "U" & "V" are harmonic in a domain (D) and their 1st order partial derivatives satisfies CR eq's through "D" then "V" is said to be harmonic conjugate of "U". ~~so t~~

1) If $V(x,y) = 3x^2y - y^3$ and $u(x,y) = x^3 - 3xy^2$ then S.T "V" is harmonic conjugate of "u".

Sol: Consider $u(x,y) = x^3 - 3xy^2$.

Now Diff "u" w.r.to x & y partially.

$u_x = 3x^2 - 3y^2$; $u_y = -6xy$.

Diff u_x & u_y w.r.to x & y partially.

$$u_{xx} = \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}(3x^2 - 3y^2) = 6x.$$

$$u_{yy} = \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}(-6xy) = -6x.$$

1st & 2nd order partial derivatives are continuous

then $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6x - 6x = 0$.

$\therefore$ $u(x,y)$ is Harmonic function.

Consider $V(x,y) = 3x^2y - y^3$.

Now Diff "V" w.r.to x & y partially.

$V_x = 6xy$; $V_y = 3x^2 - 3y^2$.

Diff V_x & V_y w.r.to x & y partially.

$$V_{xx} = \frac{\partial^2 V}{\partial x^2} = \frac{\partial}{\partial x}(6xy) = 6y.$$

$$V_{yy} = \frac{\partial^2 V}{\partial y^2} = \frac{\partial}{\partial y}(3x^2 - 3y^2) = -6y.$$

1st & 2nd Order partial derivatives are continuous.

then $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6y - 6y = 0$.

$\therefore$ $V(x,y)$ is Harmonic function.

But, we have C-R eq's,

$u_x = v_y$ & $u_y = -v_x$.

$3x^2 - 3y^2 = 3x^2 - 3y^2$ & $-6xy = -6xy$.

$\therefore$ $v(x,y)$ is harmonic conjugate of $u(x,y)$.

2) If "u" is a harmonic fun., S.T $w = u^2$ is not a harmonic fun. unless "u" is a constant.

<u>Sol</u>:- Given that "u" is a harmonic function

i.e., $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$. ——— ①

Consider, $w = u^2$.

$\dfrac{\partial w}{\partial x} = 2u \cdot \dfrac{\partial u}{\partial x}$; $\dfrac{\partial w}{\partial y} = 2u \cdot \dfrac{\partial u}{\partial y}$

$$\frac{\partial^2 w}{\partial x^2} = \frac{\partial}{\partial x}\left(2u \cdot \frac{\partial u}{\partial x}\right) = 2\left[u \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} \cdot \frac{\partial u}{\partial x}\right] = 2\left[u \cdot \frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial u}{\partial x}\right)^2\right]$$

$$\frac{\partial^2 w}{\partial y^2} = \frac{\partial}{\partial y}\left(2u \cdot \frac{\partial u}{\partial y}\right) = 2\left[u \cdot \frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y} \cdot \frac{\partial u}{\partial y}\right] = 2\left[u \cdot \frac{\partial^2 u}{\partial y^2} + \left(\frac{\partial u}{\partial y}\right)^2\right]$$

Consider,

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 2\left[u \cdot \frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial u}{\partial x}\right)^2\right] + 2\left[u \cdot \frac{\partial^2 u}{\partial y^2} + \left(\frac{\partial u}{\partial y}\right)^2\right]$$

$$= 2u\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right] + 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right]$$

$$= 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right] \quad (\because \text{From ①})$$

$\neq 0$.

$\therefore$ $w = u^2$ is not a Harmonic function.

If $u = k$ is a constant.

then $\frac{\partial u}{\partial x} = 0$ & $\frac{\partial u}{\partial y} = 0$.

$\therefore \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 0$.

$w = u^2$ is also a Harmonic function.

* If $f(z) = u(x,y) + iv(x,y)$ is an analytic fun. If and only if "v" is harmonic in a Domain(D) conjugate of "u".

~~Sol: Consider~~

Proof:-

consider $f(z)$ is an analytic fun.

i.e., $f(z) = u(x,y) + iv(x,y)$.

The C.R eq's are satisfied

i.e., $u_x = v_y$ —① & $u_y = -v_x$ —②

Now diff ① w.r.to "x".

$u_{xx} = v_{yx}$.

Now diff ② w.r.to "y".

$u_{yy} = -v_{xy}$.

then $u_{xx} + u_{yy} = 0$.

$\therefore u(x,y)$ is Harmonic function.

Illy $v(x,y)$ is also Harmonic function.

NOTE:-

→ If $f(z) = u(x,y) + iv(x,y)$ is an analytic function in a Domain(D) if and only if $v(x,y)$ is harmonic Conjugate of $u(x,y)$.

→ If $f(z) = u + iv$ is an analytic function then "u" need not be harmonic conjugate of "v".

→ Let $u(x,y)$ be harmonic in some neighbourhood of a point (x_0,y_0) then there exists a conjugate harmonic $V(x,y)$ defined in that neighbourhood such that $f(z) = u(x,y) + iv(x,y)$ is an analytic function. (i.e., $V(x,y)$ is harmonic conjugate of $u(x,y)$)

Construct an analytic function if real part is known

1) If $f(z) = u+iv$ is an analytic function and the real part $u(x,y) = x^3 - 3xy^2$. then find img. part of $f(z)$ and also find $f(z)$.

SOL: Consider, $u(x,y) = x^3 - 3xy^2$.

$$\frac{\partial u}{\partial x} = u_x = 3x^2 - 3y^2 \qquad \frac{\partial u}{\partial y} = u_y = -6xy$$

$$\frac{\partial^2 u}{\partial x^2} = 6x \qquad \frac{\partial^2 u}{\partial y^2} = -6x.$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6x - 6x = 0.$$

$u(x,y)$ is Harmonic function.

∴ Then ∃ $V(x,y)$ is harmonic conjugate of $f(z) = u+iv$ is an analytic function i.e.,

$$u_x = v_y \ \& \ u_y = -v_x.$$

$$v_y = 3x^2 - 3y^2 \text{ — (1)} \ \& \ v_x = 6xy \text{ — (2)}.$$

Now integrate (1) w.r.to "y".

$$V = 3x^2y - y^3 + \phi(x) \text{ — (3)}.$$

Now Diff. (3) w.r.to "x".

$$V_x = 6xy + \phi'(x) \text{ — (4)}.$$

From (2) & (4),

$$\phi'(x) = 0$$
$$\phi(x) = C.$$

eq ③ ⇒ $V(x,y) = 3x^2y - y^3 + C$.

$\therefore f(z) = U(x,y) + iV(x,y)$.

$f(z) = x^2 - 3xy^2 + i(3x^2y - y^3) + C$

2) Prove that $u(x,y) = e^{x^2-y^2}$ is a harmonic function and find it's harmonic conjugate.

SOL: $u(x,y) = e^{x^2-y^2}$.

$u_x = e^{x^2-y^2}\cdot(2x)$ $\qquad u_y = e^{x^2-y^2}(-2y)$

$u_{xx} = e^{x^2-y^2}(2) + (2x)(e^{x^2-y^2})(2x)$
$= 2e^{x^2-y^2} + 4x e^{x^2-y^2}$.

$u_{yy} = e^{x^2-y^2}(4y^2-2)$.

$\therefore u_{xx} + u_{yy} \neq 0$. $u(x,y) = e^{x^2-y^2}$ is not a harmonic fun.

3) P.T $u(x,y) = x^2 - y^2 - 2xy - 2x + 3y$ is harmonic function and Find $f(z) = u + iv$.

(I) If $f(z)$ is an analytic function then show that

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|Re\, f(z)|^2 = 2|f'(z)|^2$$

PROOF:-

Consider $f(z)=u+iv$ is an analytic function.

It satisfies C-R eq's

i.e., $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$

Let $f(z)=u+iv \Rightarrow Re f(z)=u$.

$|Re\, f(z)|^2=u^2$.

$f(z)=u+iv$.

$f'(z)=u_x+iv_x$.

$|f'(z)|=\sqrt{u_x^2+v_x^2}$

$|f'(z)|^2=u_x^2+v_x^2$

$=\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)$

Consider, $\frac{\partial^2}{\partial x^2}|Re\, f(z)|^2=\frac{\partial^2}{\partial x^2}(u^2)$.

$=\frac{\partial}{\partial x}\left[\frac{\partial}{\partial u}(u^2)\right]$

$=\frac{\partial}{\partial x}\left[2u\cdot\frac{\partial u}{\partial x}\right]$

$$\frac{\partial^2}{\partial x^2}|Re\, f(z)|^2 = 2\left[u\cdot\frac{\partial^2 u}{\partial x^2}+\left(\frac{\partial u}{\partial x}\right)^2\right]$$

$\frac{\partial^2}{\partial y^2}|Re\, f(z)|^2=\frac{\partial}{\partial y}\left[\frac{\partial}{\partial y}(u^2)\right]$

$=\frac{\partial}{\partial y}\left[2u\cdot\frac{\partial u}{\partial y}\right]$

$=2\left[u\cdot\frac{\partial^2 u}{\partial y^2}+\left(\frac{\partial y}{\partial x}\right)^2\right]$.

Then, $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|Re\, f(z)|^2=\frac{\partial^2}{\partial x^2}|Re\, f(z)|^2+\frac{\partial^2}{\partial y^2}|Re\, f(z)|^2$

$=2\left[u\frac{\partial^2 u}{\partial x^2}+\left(\frac{\partial u}{\partial x}\right)^2\right]+2\left[u\cdot\frac{\partial^2 u}{\partial y^2}+\left(\frac{\partial u}{\partial y}\right)^2\right]$

$$= 2\left[u\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)+\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2\right]$$

But, $f(z)$ is analytic $\Rightarrow u(x,y)$ is a Harmonic function i.e., 1st & 2nd order derivatives are continuous and it satisfies $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$.

$$\therefore \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|R_e f(z)|^2 = 2\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2\right]$$

$$= 2\,|f'(z)|^2.$$

(II)

If $f(z)$ is an analytic function then show that

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4\,|f'(z)|^2.$$

PROOF:-

Consider $f(z)=u+iv$ is an analytic function

It satisfies C-R eq's.

i.e., $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$

Let $f(z)=u+iv$

$$|f(z)|^2 = u^2+v^2.$$

consider, $\frac{\partial^2}{\partial x^2}|f(z)|^2 = \frac{\partial^2}{\partial x^2}(u^2+v^2).$

$$= \frac{\partial^2}{\partial x^2}(u^2)+\frac{\partial^2}{\partial x^2}(v^2).$$

$$= \frac{\partial}{\partial x}\left[2u\cdot\frac{\partial u}{\partial x}\right]+\frac{\partial}{\partial x}\left[2v\cdot\frac{\partial v}{\partial x}\right]$$

$$= 2\left[u\cdot\frac{\partial^2 u}{\partial x^2}+\left(\frac{\partial u}{\partial x}\right)^2\right]+2\left[v\cdot\frac{\partial^2 v}{\partial x^2}+\left(\frac{\partial v}{\partial x}\right)^2\right]$$

$$\cancel{= 2\left[u\frac{\partial^2 u}{\partial x^2}\right]}$$

Illy, $\frac{d^2}{dy^2}|f(z)|^2 = 2\left[u\cdot\frac{\partial^2 u}{\partial y^2}+\left(\frac{\partial u}{\partial y}\right)^2\right] + 2\left[v\cdot\frac{\partial^2 v}{\partial y^2}+\left(\frac{\partial v}{\partial y}\right)^2\right]$.

Now, $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$.

$$\therefore \left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right]|f(z)|^2 = 2\left[u\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)+v\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)\right.$$

$$\left.+\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2\right]$$

$$= 2\left[f'(z)+f'(z)\right].$$

$$= 4|f'(z)|^2$$

1) S.T $u(x,y) = x^2-y^2+xy$ is harmonic function and Find it's harmonic conjugate.

Sol: consider, $u(x,y) = x^2-y^2+xy$.

$\frac{\partial u}{\partial x} = 2x+y$ $\quad\quad$ $\frac{\partial u}{\partial y} = -2y+x$

$\frac{\partial^2 u}{\partial x^2} = 2$ $\quad\quad$ $\frac{\partial^2 u}{\partial y^2} = -2$.

1st & 2nd order partial derivatives are continuous.

$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} = 2-2=0$.

$u(x,y)$ is a Harmonic function.

Then $\exists$ a Harmonic Conjugate $v(x,y)$ of of $u(x,y)$

$f(z) = u+iv$

$f(z)$ is analytic function & satisfies C-R eq's.

i.e., $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

Let, $\frac{\partial v}{\partial y} = 2x + y$.

Integrate on b.s w.r.to 'y'

$V = 2xy + \frac{y^2}{2} + \phi(x)$

Diff "V" w.r.to "x"

$V_x = 2y + \phi'(x)$.

But $V_x = -U_y = -(-2y + x) = 2y - x$.

$2y - x = 2y + \phi'(x)$

$\phi'(x) = -x$.

Integrating we get

$\phi(x) = -\frac{x^2}{2} + ic$.

$\therefore V(x,y) = 2xy + \frac{y^2}{2} - \frac{x^2}{2} + ic$.

* MILNE - THOMSON METHOD

Method of constructing Analytic fun. $f(z)$ when $u(x,y)$ (or) $v(x,y)$ are given.

Let $f(z) = u(x,y) + iv(x,y)$.

$$f'(z) = \frac{\partial u}{\partial x}(x,y) + i \cdot \frac{\partial v}{\partial x}(x,y)$$

By Milne-Thomson method,

$$f'(z) = \frac{\partial u}{\partial x}(x,y) - i \cdot \frac{\partial u}{\partial y}(x,y).$$

Put $x = z$ & $y = 0$.

$$f'(z) = \frac{\partial u}{\partial x}(z,0) - i \cdot \frac{\partial u}{\partial y}(z,0)$$

I.O.B.s w.r.to "z".

$$\therefore f(z) = \int \left[\frac{\partial u}{\partial x}(z,0) - i \cdot \frac{\partial u}{\partial y}(z,0) \right] \cdot dz + ic.$$

1) Find the analytic function whose real part $u(x,y) = x^3 - 3xy^2$

SOL: Let $f(z) = u + iv$

$f'(z) = u_x + iv_x$

$f'(z) = u_x - iu_y$

Now Diff. $u(x,y)$ w.r.to x & y partially.

$\frac{\partial u}{\partial x} = u_x = 3x^2 - 3y^2$

$\frac{\partial u}{\partial y} = u_y = -6xy$

$f'(z) = 3x^2 - 3y^2 - i(-6xy)$

$= 3x^2 - 3y^2 + i6xy$

By Milne-Thomson method.

put $x = z$ & $y = 0$

$f'(z) = 3z^2$

I.O.B.S w.r.to "z"

$f(z) = 3\left(\frac{z^3}{3}\right) + C = z^3 + C$

2) Determine the analytic function whose real part $u = e^{x^2-y^2} \cos 2xy$.

3) Determine the analytic function whose real part

$$u = \frac{\sin 2x}{\cosh 2y - \cos 2x}$$

4) Find the analytic function $f(z)$ whose real part

$u = e^x[(x^2 - y^2)\cos y - 2xy \sin y]$

$2\left(\cancel{z}\frac{e^{z^2}}{2z} - \int 1 \frac{e^{z^2}}{2z}\right)$

$2\left(\frac{ez^2}{2} - \frac{1}{2}\int \frac{e^{z^2}}{z}\right)$

2) $f(z) = u + iv$. $u(x,y) = e^{x^2-y^2} \cos 2xy$.

$f'(z) = u_x - i u_y$

$\cancel{\frac{\partial u}{\partial x} = (e^{x^2-y^2})(-\sin 2xy) - (\cos 2xy)(e^{x^2-y^2})(2x)}$

$\cancel{\frac{\partial u}{\partial y}}$

$= e^{x^2-y^2} \cos 2xy$

$\frac{du}{dx} = e^{x^2-y^2}(-\sin 2xy)(2y) + (\cos 2xy) e^{x^2-y^2}(2x)$

$= 2x \cos 2xy (e^{x^2-y^2}) - 2y e^{x^2-y^2} \sin 2xy$

$= 2 e^{x^2-y^2}(x \cos 2xy - y \sin 2xy)$

$x = z, \; y = 0$

$= 2e^{z^2}(z \cos 0) = 2e^{z^2} \cdot z$

$\frac{du}{dy} = e^{x^2-y^2}(-\sin 2xy)(2x) + (\cos 2xy) e^{x^2-y^2}(-2y)$

$= -2e^{x^2-y^2}(x \sin 2xy + y \cos 2xy)$

$x = z, \; y = 0$

$= -2e^{z^2}(z(0) + 0)$

$= 0$

I.O.B.S

$f(z) = \int 2z e^{z^2}$

$= 2 \int e^{z^2} dz$

$\frac{2e^{z^2}}{2z} = \frac{e^{z^2}}{z} + C$

$\int e^{x}$

$= 2 \int z e^{z^2}$

$\frac{e^x}{2}$

$= 2\left(e^{z^2}\frac{(z^2)}{2} - \int \frac{1 e^{z^2}}{2z}\right)$

$2\left(\frac{e^{z^2} z^2 + c}{z^2}\right) = e^{z^2} + C$

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5) Find the analytic function $f(z)$ where $u = \sin x \cosh y$.

SOL: $f(z) = u + iv$.

$f'(z) = u_x - iu_y$.

$\frac{\partial u}{\partial x} = \cancel{\sin x (\sinh y) = \cosh y (\sin x)}$. $\cosh y (\cos x)$.

$\frac{\partial u}{\partial y} = \sin x \cdot \sinh y$.

$f'(z) = \cosh y \cdot \cos x - i \sin x \cdot \sinh y$.

By Milne-Thomson, put $x = z, y = 0$.

$f'(z) = \cosh(0) \cdot \cos z - i \cdot \sin z \cdot \sinh(0)$

$= \cos z$.

I.O.B.S.

$f(z) = \int \cos z = \sin z + c$.

To find the $v(x,y)$

Put $z = x + iy$.

$\cos iy = \cosh y$

$\sin iy = i \sinh y$.

$f(x+iy) = \sin(x+iy) + c$.

$= \sin x \cos iy + \cos x \sin iy + c$.

$= \sin x \cosh y + i \cos x \sinh y + c$.

$\therefore v(x,y) = \cos x \sinh y$

3) $u(x,y) = \frac{\sin 2x}{\cosh 2y - \cos 2x}$

$$u_x = \frac{(\cosh 2y - \cos 2x) 2\cos 2x - (\sin 2x)(-2\sin 2x)}{(\cosh 2y - \cos 2x)^2}.$$

1) Find the analytic fun. whose img. Part. $V(x,y) = x^2-y^2$.

Sol: $f(z) = u+iv$.

$f'(z) = V_y + iV_x$.

$\frac{\partial v}{\partial x} = 2x$. ; $\frac{\partial v}{\partial y} = -2y$.

$f'(z) = -2y + i2x$

Put $x=z$, $y=0$.

$f'(z) = i \cdot 2z$.

I.O.B.S.

$f(z) = i \cdot z^2 + c$.

$f(x+iy) = i(x+iy)^2 + c = i(x^2-y^2+2ixy) + c$.

$= ix^2 - iy^2 - 2xy + c$

$= -2xy + i(x^2-y^2) + c$.

$\therefore u(x,y) = -2xy$.

* METHOD TO FIND $f(z) = u + iv$ HAS FUNCTION OF "z" WHEN $u-v$ (or) $u+v$ is GIVEN: (V. Imp)

~~considery~~ Let, $f(z) = u + iv$; $u-v$ & $u+v$.

$if(z) = iu - v$.

Consider, $f(z) + if(z) = u + iv + iu - v$.

$f(z)[1+i] = (u-v) + i(u+v)$ —— ①.

Here, $(1+i)f(z) = F(z)$; $u-v = U$; $u+v = V$.

① => $F(z) = U + iV$

$F'(z) = U_x + iV_x = U_x - iU_y$

$F'(z) = U_x(x,y) = -iU_y(x,y)$.

By Milne-thomson method,

Put $x = z, y = 0$.

$F'(z) = U_x(z,0) - iU_y(z,0)$.

I.O.B.S w.r.to "z":

$$F(z) = \int [U_x(z,0) - iU_y(z,0)]\,dz + c.$$

$$(1+i)f(z) = \int [U_x(z,0) - iU_y(z,0)]\,dz + c.$$

$$\therefore f(z) = \frac{1}{1+i}\int [U_x(z,0) - iU_y(z,0)]\cdot dz + c.$$

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1) If $u-v=(x-y)(x^2+4xy+y^2)$, S.T $f(z)$ is an analytic function. (V-Imp)

SOL: Let, $f(z)=u+iv$

$if(z)=iu-v$.

$f(z)+if(z)=u+iv+iu-v$.

$(1+i)f(z)=(u-v)+i(u+v)$.

$(1+i)f(z)=F(z)$; $u-v=U$; $u+v=V$.

$F(z)=U+iV$.

$F'(z)=U_x+iV_x=U_x-iU_y$ —(1)

$U_x=(u-v)_x=(x-y)(2x+4y)+(1)(x^2+4xy+y^2)$.

$U_x=(x-y)(2x+4y)+x^2+4xy+y^2$.

$U_x=3x^2-3y^2+6xy$.

$U_y=(u-v)_y=(x-y)(4x+2y)+(-1)(x^2+4xy+y^2)$.

$U_y=3x^2-3y^2-6xy$.

(1) $\Rightarrow$ $F'(z)=(3x^2-3y^2+6xy)-i(3x^2-3y^2-6xy)$.

By Milne-thomson method,

Put $x=z, y=0$.

$F'(z)=3z^2-i3z^2$

$(1+i)f'(z)=3z^2(1-i)$

$f'(z)=3z^2\dfrac{(1-i)}{(1+i)}$

Now I.O.B.S w.r.to "z".

$f(z)=\dfrac{3(1-i)}{1+i}\int z^2\,dz+c$.

$\therefore f(z)=\left(\dfrac{1-i}{1+i}\right)z^3+c$.

2) Find the analytic fun. $f(z)$ if $u+v = e^x(\cos y + \sin y)$.

SOL: $f(z) = u + iv$.

$if(z) = iu - v$.

$f(z) + if(z) = (u-v) + i(u+v)$

$(1+i) f(z) = (u-v) + i(u+v)$.

$(1+i) f(z) = F(z)$; $u - v = U$; $u+v = V$.

$F(z) = U + iV$

$F'(z) = V_y + iV_x$ —— (1).

$V_x = (u+v)_x = e^x(0) + (\cos y + \sin y) e^x$

$V_x = e^x(\cos y + \sin y)$.

$V_y = (u+v)_y = e^x(-\sin y + \cos y) + (0)(\cos y + \sin y)$.

$V_y = e^x(\cos y - \sin y)$

(1) $\Rightarrow$ $F'(z) = e^x(\cos y - \sin y) + ie^x(\cos y + \sin y)$.

put $x = z$, $y = 0$.

$(1+i) f'(z) = e^z[\cos 0 - \sin 0] + ie^z[\cos 0 + \sin 0]$.

$= e^z + ie^z$.

$(1+i) f'(z) = (1+i) e^z$.

I.O.B.S w.r.to "z":

$f(z) = \int e^z \, dz + \frac{c}{1+i}$

$f(z) = e^z + \frac{c}{1+i}$

3) Determine the analytic fun. $f(z)$ if $u-v = \dfrac{\cos x + \sin x - e^{-y}}{2(\cos x - \cosh y)}$

and $f(\pi/2) = 0$.

Sol:- $(1+i)f(z) = (u-v) + i(u+v)$. $v_x = -u_y$

$f(z) = u + iv$

$f'(z) = u_x - iu_y$

$$u_x = \frac{1}{4(\cos x - \cosh y)^2}\left[2(\cos x - \cosh y)(-\sin x + \cos x) - 2(\cos x + \sin x - e^{-y})(-\sin x)\right].$$

$$u_y = \frac{1}{4(\cos x - \cosh y)^2}\left[2(\cos x - \cosh y)(e^{-y}) - 2(\cos x + \sin x - e^{-y})(-\sinh y)\right]$$

~~$f'(z)$~~ = Φ

put $x = z$, $y = 0$.

$$(1+i)f'(z) = \frac{1}{4(\cos z - \cosh 0)^2}\left[2(\cos z - \cos 0)(-\sin z + \cos 0)\right]$$

$$+ i \cdot \frac{1}{4(\cos z - \cos 0)^2}\left[2(\cos z - \cos 0)(e^{-0}) - 2(\cos z + \sin z - e^{-0})(-\sin 0)\right].$$

$$= \frac{1}{4(\cos z - 1)^2}\left[-2\cos z \sin z\right] + i\frac{1}{4(\cos z - 1)^2}\left[2\cos z\right]$$

$$= \frac{1}{4(\cos z - 1)^2}\left[-2\sin z \cos z + i \cdot 2\cos z\right].$$

* CAUCHY - REMAN EQUATION IN POLAR FORM :-

If $f(z) = u(r,\theta) + iv(r,\theta)$ is differentiable at $z = re^{i\theta} \neq 0$ then

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{\partial v}{\partial r} = \frac{-1}{r}\cdot\frac{\partial u}{\partial \theta}$$

PROOF :-

Let, $z = x+iy$

$f(z) = u(x,y) + iv(x,y)$.

Put $x = r\cos\theta$ & $y = r\sin\theta$.

$z = r\cos\theta + ir\sin\theta$.

$z = r[\cos\theta + i\sin\theta]$

$z = re^{i\theta}$

But, $f(re^{i\theta}) = u(r,\theta) + iv(r,\theta)$ —— ①.

Diff ① w.r.to "r".

$f'(r\cdot e^{i\theta})e^{i\theta} = u_r(r,\theta) + iv_r(r,\theta)$ —— ②.

Diff ① w.r.to "θ".

$f'(re^{i\theta})\cdot ri\cdot e^{i\theta} = u_\theta(r,\theta) + iv_\theta(r,\theta)$ —— ③.

From ② & ③ $\Rightarrow$ $\dfrac{1}{r\cdot i} = \dfrac{u_r(r,\theta) + iv_r(r,\theta)}{u_\theta(r,\theta) + iv_\theta(r,\theta)}$

* THEOREM

1) An analytic fun. with constant real part is constant

PROOF:-

Let $f(z) = u + iv$ is analytic.

It satisfies C-R eq's i.e., $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$.

Consider Real part ~~u(x,y)~~ is constant i.e.,

$u(x,y) = c_1$.

$\Rightarrow \frac{\partial u}{\partial x} = 0$ & $\frac{\partial u}{\partial y} = 0$.

$\frac{\partial v}{\partial x} = 0$ & $\frac{\partial v}{\partial y} = 0$.

$v(x,y) = c_2$.

$\therefore f(z) = u(x,y) + iv(x,y) = c_1 + ic_2$ is a constant function.

2) An analytic fun. with constant imaginary part is constant.

PROOF:

Let $f(z) = u + iv$ is analytic.

It satisfies C-R eq's i.e., $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$.

Consider Img. part is constant i.e., $v(x,y) = c_1$.

$\frac{\partial v}{\partial x} = 0$ & $\frac{\partial v}{\partial y} = 0$.

$-\frac{\partial u}{\partial y} = 0$ & $\frac{\partial u}{\partial x} = 0$.

$u(x,y) = c_2$.

$\therefore f(z) = u(x,y) + iv(x,y) = c_1 + c_2$ is a constant function.