UNIT- III

Complex Integration:-

Curve/path:-

A continuous function $\gamma: [a,b] \to \mathbb{C}$ is called a curve or a path

Closed curve:- A curve γ is said to be closed curve if $\gamma(a) = \gamma(b)$

Jordan curve:-

A curve $\gamma: [a,b] \to \mathbb{C}$ is said to be Jordan if it doesn't across itself.

Note:- $\gamma: [a,b] \to \mathbb{C}$ is an arc then $\gamma(a)$ is called initial point and $\gamma(b)$ is called terminal point.

→ If $z = x+iy$ be a complex number then $dz = dx + idy$ suppose $f(z) = u+iv$ be complex function then the complex line integral $\int_C f(z)dz$ can be expressed as

$$\int_C f(z)dz = \int_C f(z)(dx+idy)$$

$$= \int_C (u+iv)(dx+idy)$$

$$= \int_C udx + iudy + ivdx - vdy$$

$$= \int_C (udx - vdy) + i(udy + vdx)$$

Properties of complex line integral

1) $\int_\gamma [f(z) + g(z)]dz = \int_\gamma f(z)dz + \int_\gamma g(z)dz$

2) $\int_\gamma f(z)dz = -\int_{-\gamma} f(z)dz$

3) $\int_\gamma cf(z)dz = c\int_\gamma f(z)dz$

4) $\left|\int_\gamma f(z)dz\right| \le \int_\gamma |f(z)|dz$

Evaluate $\int_C f(z)dz$ where $f(z) = y - x - 3x^2 i$ and C is the straight line segment from $z=0$ to $z=1+i$

Then $\int_C f(z)dz = \int_{OA} (y-x-3x^2 i)(dx+idy)$

A (1,1)

O (0,0)

$$= \int_{x=0}^{1} (x-x-3x^2 i)(dx+idx)$$

$(y-y_1) = m(x-x_1)$

$$\int_C f(z)dz = \int_{x=0}^{1} (-3x^2 i)(1+i)dx$$

$(y-0) = m(x-0)$

$y = mx$

$y = \frac{+1}{1}(x)$

$$= -3i(1+i)\int_{x=0}^{1} x^2 dx$$

$y = x$

$$= -3(i-1)\left[\frac{x^3}{3}\right]_0^1$$

$dy = dx$

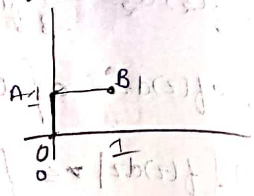
$$= -3(i-1)[1/3 - 0]$$

$$= 3(1-i)1/3$$

$$= 1-i$$

Evaluate $\int_C f(z)dz$ where $f(z) = y - x - 3x^2 i$ and C consists of two straight line segments one from $z=0$ to $z=i$ and other from $z=i$ to $z=1+i$

Given that $f(z) = y - x - 3x^2 i$ and the st line segment from $z=0$ to $z=i$ & other from $z=i$ to $z=1+i$



i.e; $z=0$ to $z=i$ and $z=i$ to $z=1+i$

$$\Rightarrow \int_C f(z)dz = \int_{OA} f(z)dz + \int_{AB} f(z)dz \rightarrow (1)$$

(i) Along the line segment $z=0$ to $z=i$ i.e OA:

Let eq^n of line segment OA is $x=0$

$\Rightarrow dx=0$

Then $\int_C f(z)dz = \int_{OA} f(z)dz$

$$= \int_{OA} (y - x - 3x^2 i)(dx + idy)$$

$$= \int_{y=0}^{1} (y)(idy)$$

$$= i\left[y^2/2\right]_{y=0}^{1}$$

$$= i[1/2 - 0] = i/2 \rightarrow (a)$$

(ii) The line segment $z=i$ to $z=1+i$

The eq^n of the line is $y=1$

$dy=0$

Then $\int_C f(z)dz = \int_{AB} (y - x - 3x^2 i)(dx + idy)$

$$= \int_{x=0}^{1} (1 - x - 3x^2 i)(dx)$$

$$= \left[\left(x - \frac{x^2}{2} - x^3 i\right)\right]_{x=0}^{1}$$

$$= (1 - 1/2 - i) - (0)$$

$$= \frac{1}{2} - i \text{ — (b)}$$

13/02/2020

Evaluate $\int_C f(z)dz$ where $f(z) = \frac{z+2}{z}$ and c is a semicircle $z = 2e^{i\theta}$ where $0 \le \theta \le \pi$

Let $z = 2e^{i\theta}$

$dz = 2ie^{i\theta}d\theta$

$|z| = 2|e^{i\theta}|$

$|z| = 2$

$|x+iy| = 2$

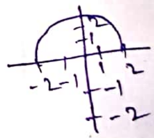
$\sqrt{x^2+y^2} = 2$

$x^2+y^2 = 2^2$

then $\int_C f(z)dz = \int_C \left(\frac{z+2}{z}\right) 2 \cdot ie^{i\theta} d\theta$

$$= \int_C \left(1 + \frac{2}{z}\right) 2ie^{i\theta} d\theta$$

$$= \int_{\theta=0}^{\pi} (1 + e^{-i\theta}) 2ie^{i\theta} d\theta$$



$$= 2i \int_{\theta=0}^{\pi} (e^{i\theta} + 1) d\theta$$

$$= 2i\left[\frac{e^{i\theta}}{i} + \theta\right]_{\theta=0}^{\pi}$$

$$= 2i\left[\frac{1}{i}(e^{i\pi}-1) + (\pi - 0)\right]$$

$$= 2i\left[\frac{1}{i}(-1-1) + \pi\right]$$

$$= 2i\left[-2/i + \pi\right]$$

$$= -4 + 2\pi i$$

Show that

1) $\int_C \frac{1}{z-a}\,dz = 2\pi i$

2) $\int_C (z-a)^n\,dz = 0$ where n is an integer, $n \neq -1$ and C is circle $|z-a| = r$

$|z-a| = r$

$|x+iy-a| = r$

$(x-a)^2 + y^2 = r^2$

$(x-a)^2 + y^2 = r^2$

$(a, 0)$

$|z-a| = r$

$z - a = re^{i\theta}$

$z = a + re^{i\theta}$

$dz = ri e^{i\theta} d\theta$

(i) consider $\int_C \frac{1}{z-a}\,dz = \int_C \frac{1}{re^{i\theta}}\, ire^{i\theta}\,d\theta$

$= i\int (1)\,d\theta$

$= i[\theta]_0^{2\pi}$

$= 2\pi i$

(ii) $\int_C (z-a)^n dz = 0$

$$= \int_0^{2\pi} (re^{i\theta})^n\, r i e^{i\theta}\, d\theta$$

$$= ir^{n+1}\int_{\theta=0}^{2\pi} e^{in\theta}\cdot e^{i\theta}\,d\theta$$

$$= ir^{n+1}\int_{\theta=0}^{2\pi} e^{i(n+1)\theta}\,d\theta$$

$$= ir^{n+1}\left[\frac{e^{i(n+1)\theta}}{i(n+1)}\right]_{\theta=0}^{2\pi}$$

$$= \frac{r^{n+1}}{n+1}[1-1]$$

$\therefore \int_C (z-a)^n dz = 0$ if $n \neq -1$

Note:- Let C be a circle $|z-a| = r$ and n is an integer then

$$\int_C \frac{1}{(z-a)^n}\,dz = \begin{cases} 2\pi i & \text{if } n = 1 \\ 0 & \text{if } n \neq 1 \end{cases}$$

Evaluate $\int_C \frac{z+2}{2}\,dz$ where C is the semicircle $z = 2e^{i\theta}$ and θ taken from 0 to π

$dz = 2ie^{i\theta}d\theta$

$$\int_0^{\pi} \frac{z+2}{2}\, 2ie^{i\theta}\,d\theta$$

$$\int_0^{\pi}\left(\frac{z}{2}+1\right) 2ie^{i\theta}\,d\theta = \int_0^{\pi} (e^{i\theta}+1)\, 2ie^{i\theta}\,d\theta$$

$$= \int_0^{\pi} 2ie^{2i\theta} + 2ie^{i\theta}\,d\theta$$

$$2i\int_0^{\pi} e^{2i\theta}+e^{i\theta}\,d\theta$$

$\cos 2\pi + i\sin 2\pi$

$-1+0$

$$2i\left[\frac{e^{2i\theta}}{2i}+\frac{e^{i\theta}}{i}\right]_0^{\pi}$$

$\cos 0 + i\sin 0$

$1+0$

$$2i\left[\frac{-1}{2i}-1+\frac{1}{i}(-1-1)\right]$$

$\cos\pi + i\sin\pi$

$\cos 0 + i\sin 0$

$$2i\left[\frac{-2}{2i}+\frac{1}{i}(-1-1)\right]$$

$$2i\left[\frac{-2}{\cancel{2i}}+\frac{-2}{i}\right] \quad -4$$

$\int_C \bar{z}\,dz$ $\quad z=2e^{i\theta}$ where c is right hand half where

$-\pi/2 \le \theta \le \pi/2$

$$=\int_{-\pi/2}^{\pi/2} \bar{z}\cdot 2ie^{i\theta}\,d\theta$$

$$=\int_{-\pi/2}^{\pi/2} e^{-i\theta}\cdot 2ie^{i\theta}\,d\theta$$

$$=\int_{-\pi/2}^{\pi/2} 2i\,d\theta$$

$$=2i\left[\theta\right]_{-\pi/2}^{\pi/2}$$

$=$

Evaluate $\int_C f(z)\,dz$ where $f(z)=z^{1/2}$ and c be the semicircle path $z=3e^{i\theta}$, $0\le\theta\le\pi$

$$\int_C z^{1/2}\, 3\cdot ie^{i\theta}\,d\theta$$

$$\int_0^{\pi}(3e^{i\theta})^{1/2}\cdot 3ie^{i\theta}\,d\theta$$

$$3^{3/2} i\int_0^{\pi}\left(e^{i\theta\cdot 3/2}\right)d\theta$$

$$=3^{3/2} i\left[\frac{e^{\frac{3i\theta}{2}}}{3i/2}\right]_0^{\pi}$$

$$=\frac{3^{3/2} i}{3i/2}\left[\cos 3\pi/2+i\sin 3\pi/2-\cos 0-\sin 0\right]$$

$+i(-1)$

$$=2\sqrt{3}\left[0-i-1+0\right]$$

$$=-2\sqrt{3}\left[+i+1\right]$$

$f(z)=z-1$ and the arc from $z=0$ to 2 consists of the semicircle $z=1+e^{i\theta}$

$1+e^{i\theta}=0$

$e^{i\theta}=-1$

$dz=ie^{i\theta}$

$x+iy=1+e^{i\theta}$

$$\int_C (z-1)\,ie^{i\theta}$$

$e^{i\theta}=1$

$\theta=$

$$\int_{\pi}^{2\pi} e^{i\theta}\,ie^{i\theta}\,d\theta$$

$e^{i\theta}=-1=\pi$ $\quad i\theta=e^{1}$

$e^{i\theta}=1\cdot=2\pi$

$$i\left[\frac{e^{2i\theta}}{2i}\right]_{\pi}^{2\pi} \quad i\left[\frac{1+0}{2i}-\left(\frac{-1+0}{2i}\right)\right]$$

$\frac{2i}{2i}=1$

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Find the value of $\int_C (x+y)dx + x^2 y\,dy$ along $y = x^2$ having $(0,0)$ & $(3,9)$ as end points

i) along $y = 3x$ b/w same points

$dy = 2x\,dx$

$$\int_0^3 (x+y)dx + x^2 y\, 2x\,dx$$

$$\int_0^3 x\,dx + y\,dx + x^2 y\, 2x\,dx$$

$$\int_0^3 (x + x^2 + 2x^5)\,dx$$

$$\left[\frac{x^2}{2} + \frac{x^3}{3} + \frac{2x^6}{6_3}\right]_0^3$$

$$\left[\frac{9}{2} + \frac{27}{3} + \frac{729}{3}\right].$$

$= 256.5$

$dy = 3dx$

$$\int_0^3 4x\,dx + 3x^2 \cdot 3dx$$

$$\left[\frac{4x^2}{2} + \frac{9x^4}{4}\right]_0^3$$

$= \frac{4\times 3^2}{2} + \frac{9\times 3^4}{4}$

$= 200.25$

Evaluate $\int_C f(z)\,dz$ where $f(z) = z-1$ and C is the line segment $0 \le x \le 2$ of the real axis

$x + iy - 1$

$x - 1 + iy$

$$\int_0^2 x - 1$$

$$\left[\frac{x^2}{2} - x\right]_0^2$$

$2 - 2 = 0$

Evaluate $\int_C f(z)dz$ where $f(z) = \frac{z+2}{z}$ and semicircle path $z = 5e^{i\theta}$, $\pi \le \theta \le 2\pi$

$$\int_\pi^{2\pi} \left(1 + \frac{2}{5e^{i\theta}}\right) \cdot 5ie^{i\theta}\,d\theta$$

$$5i\int_\pi^{2\pi} \left(e^{i\theta} + \frac{2e^{i\theta}}{5e^{i\theta}}\right) d\theta$$

$$5i\int_\pi^{2\pi} \left(e^{i\theta} + \frac{2}{5}\right) d\theta$$

$$5i\left[\frac{e^{i\theta}}{i} + \frac{2}{5}\theta\right]_\pi^{2\pi}$$

$$5i\left[\frac{1}{i}(-1) + \frac{2}{5}\times 2\pi -\right.$$

Find the value of $\int_C x^2 dx + y^2 dy$

i) Along the line segment from (0,0) to (0,2)

(ii) Along $y^2 = x^3$ (0,0) to (0,2)

$$\int_0^2 y^2 dy \qquad dy = 3x^2 dx$$

$$\left[\frac{y^3}{3}\right]_0^2 \qquad \int_0^2 3x^2$$

$8/3$

15/02/2020

Cauchy Integral theorem:-

Let $f(z) = u(x,y) + iv(x,y)$ be analytic on and within a simple closed contour C and let $f'(z)$ be continuous then, $\int_C f(z)dz = 0$. we have $f(z) =$

Proof:-

$$f(z) = u(x,y) + iv(x,y)$$

$$f = u + iv$$

$$z = x + iy$$

$$dz = dx + idy$$

$$f(z)dz = (u+iv)(dx+idy)$$

$$f(z)dz = udx - vdy + i(udy + vdx)$$

$$\int_C f(z)dz = \int udx - vdy + i(udy + vdx)$$

$$= \int_C udx - vdy + i\int_C udy + vdx$$

we know from Green's theorem in plane $\int Mdx + Ndy = \iint \frac{\partial N}{\partial x} - \left(\frac{\partial M}{\partial y}\right) dx\, dy$

since $u(x,y)$ & $v(x,y)$ ~~are~~ have continuous partial derivatives in D and $f(z)$ is analytic ~~and~~ $f'(z)$ is assumed to be continuous

Applying Green's theorem in plane to the above eqn we get.

$$= \iint \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy + i\iint \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy$$

$\therefore f(z)$ is analytic C-R eqn's are true.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\iint \left(\frac{\partial u}{\partial y} - \frac{\partial u}{\partial y}\right) dx\,dy + i\iint \left(\frac{\partial v}{\partial y} - \frac{\partial v}{\partial y}\right) dx\,dy = 0$$

$$\int_C f(z)dz = 0$$

→ Extension of Cauchy's theorem:-

If $f(z)$ is analytic is region D between two simple closed curve C and C_1 then $\int_C f(z)dz = \int_{C_1} f(z)dz$

To prove this we need to introduce the cross cut AB then $\int_C f(z)dz = 0$ where the path is as indicated by arrows.

Along AB and C_1 the direction is in clockwise sense and along BA and C the direction is anti clockwise sense

i.e $\int_C f(z)dz = \int_C f(z)dz + \int_{AB} f(z)dz + \int_{C_1} f(z)dz + \int_{BA} f(z)dz$

since $\int_{AB} f(z)dz$ and $\int_{BA} f(z)dz$ cancel

$$\int_C f(z)dz + \int_{C_1} f(z)dz = 0$$

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Reversing the direction of the integral around c_1 we get $\int_C f(z)dz - \int_{C_1} f(z)dz = 0$

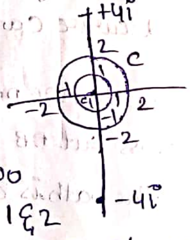
$\therefore \int_C f(z)dz = \int_{C_1} f(z)dz = 0$

Note:- If c is a simple closed contour and $c_1, c_2, \dots c_n$ are closed contours within c and if $f(z)$ is analytic ~~b/w~~ within c but on and outside c_j's then

$$\int_C f(z)dz = \int_{C_1} f(z)dz = \int_{C_2} f(z)dz \dots \int_{C_n} f(z)dz$$

Consider the region $1 \le |z| \le 2$ if B is positive oriented boundary of this region then show that

$$\int_B \frac{dz}{z^2(z^2+16)} = 0$$



$|z|=1$ and $|z|=2$ are two circles with centre at $(0,0)$ radius 1 & 2 respectively. Then the region $1 \le |z| \le 2$ is inside $|z|=2$ and outside $|z|=1$

The singular points of $f(z) = 1/z^2(z^2+16)$ are

$$z=0, \; z=\pm 4i$$

These three points are outside the region under consideration

Hence $f(z)$ is analytic on and within $|z| \le 2$ and on and outside $|z| \le 1$

Hence Extension to Cauchy theorem

$$\int_B \frac{dz}{z^2(z^2+16)} = 0$$

B is the +ve oriented boundary of region b/w the circle $|z|=4$ & the square with sides along the lines $x=\pm 1, y=\pm 1$. Evaluate $\int_B 1/3z^2+1 \, dz$

The singularities is $z = 1/\sqrt{3}\,i$ and the region under consideration outside the square and inside the circle. since the singularities are not in considered region

Using Cauchy integral theorem

$f(z)$ is analytic b/w the region $x=\pm1, y=\pm1$ and $|z|=4$

Hence $\int_B \frac{1}{3z^2+1} dz = 0$

17/02/2020

Cauchy's Integral ~~theorem~~ Formula:

If $f(z)$ be an analytic function everywhere on and within a closed contour C, let $z = z_0$ is any point within C then $\int_C f(z)/z-z_0\, dz = 2\pi i f(z_0)$

where the integral is taken in the +ve sense around c.

$$\rightarrow \int_C \frac{f(z)}{z-z_0} dz = 2\pi i f(z_0)$$

let $f(z)$ be an analytic function within a closed contour $'c'$

Here $c: |z-z_0| = r$

$$z - z_0 = re^{i\theta}$$

$$z = z_0 + re^{i\theta}$$

$$\text{and } dz = ire^{i\theta} d\theta$$

then $$\int_C \frac{f(z)dz}{z-z_0} = \int_{\theta=0}^{2\pi} \frac{f(z_0+re^{i\theta})}{re^{i\theta}} (ire^{i\theta}) d\theta$$

$$\int_C \frac{f(z)}{z-z_0} dz = i\int_{\theta=0}^{2\pi} f(z_0+re^{i\theta}) d\theta$$

let $r \to 0$ then the circle shrinks to $'z_0'$

$$\int_C \frac{f(z)}{z-z_0} dz = i\int_{\theta=0}^{2\pi} f(z_0) d\theta$$

$$= if(z_0)[\theta]_0^{2\pi}$$

$$\therefore \int_C \frac{f(z)}{z-z_0} dz = 2\pi i f(z_0)$$

Generalization of Cauchy's Integral formula

If $f(z)$ is analytic on and within a simple closed curve c and if z_0 is any point within c then

$$\int_C \frac{f(z)}{(z-z_0)^{n+1}} dz = \frac{2\pi i}{n!} f^n(z_0)$$

$$\int_C \frac{f(z)}{z-z_0} dz = 2\pi i f(z_0) \rightarrow (1)$$

differentiate w.r.t z_0

$$\int_C \frac{f(z)}{(z-z_0)^2} dz = 2\pi i f'(z_0) \rightarrow (2)$$

differentiate w.r.to z_0

$$\int_C 2\frac{f(z)}{(z-z_0)^3} dz = 2\pi i f''(z_0) \rightarrow (3)$$

$$\int_C \frac{f(z)}{(z-z_0)^3} dz = \frac{2\pi i}{2!} f''(z_0) \rightarrow (3)$$

Now diff eq^n-(3) w.r.to $'z_0'$

$$\int_C \frac{2.3 f(z)}{(z-z_0)^4} dz = 2\pi i f'''(z_0)$$

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2.3 $\int_C \frac{f(z)}{(z-z_0)^4}\, dz = 2\pi i f'''(z_0)$

$\int_C \frac{f(z)}{(z-z_0)^4}\, dz = \frac{2\pi i}{3!} f'''(z_0)$

$\int_C \frac{f(z)}{(z-z_0)^{n+1}}\, dz = \frac{2\pi i}{n!} f^{n}(z_0)$

Singular point:-
A point at which a function $f(z)$ is not analytic is called a singular point (or) singularity of $f(z)$.

Evaluate $\oint_C \frac{z^2+4}{(z-3)} dz$ where c is

i) a) $|z|=5$
b) $|z|=2$

$\int_C \frac{z^2+4}{z-3} dz$

$|z|=5$

$z = 5e^{i\theta}$ $dz = 5ie^{i\theta}$

$x^2+y^2 = 5$

$\int_C \frac{25e^{2i\theta}+4}{5e^{i\theta}-3} dz$

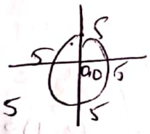
Given integral is $\int_C \frac{z^2+4}{z-3} dz$

let $f(z) = \frac{z^2+4}{z-3}$ is analytic except at $x=3$

$z=3$ is called singular point of $f(z) = z^2+4$

(a) $= |z|=5$

let $z=3$ is a singularity of $f(z)$ and is inside of $|z|=5$



Now, we can Apply the Cauchy's integral theorem

The $\int_C \frac{z^2+4}{z-3} dz = 2\pi i\,[z^2+4]$ at $z=3$

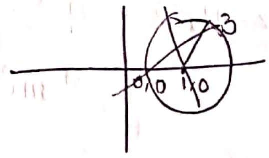
$= 2\pi i\,[3^2+4]$

$= 26\pi i$

(b) $\int_C \frac{z^2+4}{z-3} = 0$

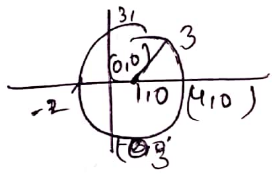
→ Evaluate $\int_C e^{2z}/(z+1)^4\, dz$ around $C: |z-1|=3$

$C=(1,0), \Rightarrow r=3$



$= \frac{2\pi i(8e^{2z})}{3!}$

$= \frac{\cancel{16}^{8}\pi i e^{2z}}{\cancel{6}_{3}}$

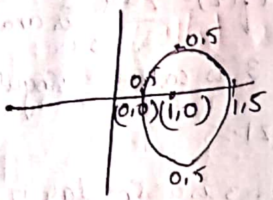


$= \left| \frac{8\pi i e^{2z}}{3} \right|_{z=-1}$

$= \frac{8\pi i e^{-2}}{3}$

→ Evaluate $\int_C \frac{z^3 \cdot e^{z}}{(z-1)^3} dz$ where c is $|z-1| = 1/2$

inside



$\int_C \frac{z^3 e^{-z}}{(z-1)^3} = \frac{2\pi i}{2!}\left(6z e^{-z} - e^{-z}3z^2 - 3z^2 e^{-z} + z^3 e^{-z}\right)_{z=1}$

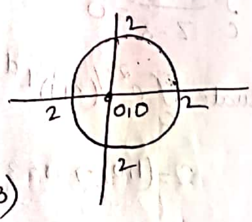
$= \frac{2\pi i (1) e^{-1}}{2}$

$= \frac{\pi i}{e}$

→ $\int_C \frac{z^3 - \sin 3z}{(z-\pi/2)^3} dz$ where c is $|z| = 2$

$\int_C \frac{z^3 - \sin 3z}{(z-\pi/2)^3} = \frac{2\pi i}{2!}(3\pi - 9)$

$= 3\pi i(\pi - 3)$

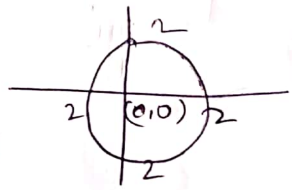


Evaluate $\int_C \left[\frac{\sin^2 z}{(z-\pi/6)^2}\right] dz$ $C = |z| = 1$

$\sin^2 z$ $2\sin z \cos z$

Evaluate $\int_C \frac{\cos z - \sin z}{(z+i)^3} dz$ $|z| = 2$

$z = -i$



$\frac{2\pi i}{2!}(\cos i - \sin i)$

Evaluate $\int_C \frac{1}{e^z (z-1)^3} = \int_C \frac{e^{-z}}{(z-1)^3}$ $|z| = 2$

$\frac{2\pi i}{2!}(e^{-1}) = \pi i (e^{-1})$ $\frac{\pi i}{e}$

20/02/2020

Evaluate $\int_C \frac{z}{z^2+1}\,dz$ where c is $\left|\frac{2z+1}{2}\right| = 2$

$|z+1/2| = 2$

$\cancel{\frac{2z+1}{2}} = 2e^{i\theta}$

$z^2+1=0$

$z = \pm i$

$\cancel{2z+1=4}$

$z + \frac{1}{2} = 2e^{i\theta}$

$(x+1/2) + iy = 2$

$z = 2e^{i\theta} - \frac{1}{2}$

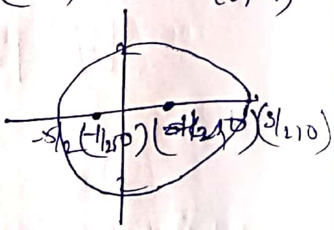
$(-1/2, 0)$

$(0,1)$

$(0,-1)$

$z = \frac{4e^{i\theta}-1}{2}$

$dz = \frac{4}{2} i e^{i\theta} d\theta$



$-\frac{1}{2}+2$

$-\frac{1}{2}-2$

$\frac{z}{z^2+1} =$

$\frac{z}{(z+i)(z-i)} = \frac{-i}{-2i}$

$= \frac{1}{i}$

$= -i = A$

$\frac{2\pi i}{1}$

$B = \frac{i}{2i}$

$= 1/i$

$= -i$

$x^2+1=0$

$x^2 = -1$

$\int_C \frac{i}{z^2+1} + \frac{i}{z^2+1}\,dz$

$\frac{2\pi i}{0!}\left(\frac{i}{z^2+1} + \frac{i}{z^2+1}\right)$

$\Downarrow -i \qquad \Downarrow i$

$2\pi i\left[\frac{i}{i^2+1} + \frac{i}{i^2+1}\right]$

$2\pi i\left[\frac{2i}{i^2+1}\right]$

$\int_C \frac{z}{z^2+1}\,dz = \int_C z\left[\frac{A}{z+i} + \frac{B}{z-i}\right]dz$

$\int_C z\left(\frac{A}{z+i} + \frac{B}{z-i}\right]dz$

$\frac{1}{-2i} = A \qquad B = \frac{1}{2i}$

$\int_C z\left(\frac{-1}{2i(z+i)} + \frac{1}{2i(z+i)}\right)dz$

$\int_C \frac{-z}{2i(z+i)} + \frac{z}{2i(z+i)}\,dz$

$$= \frac{-1}{2i}\int_C \frac{z\,dz}{z^2+i} + \frac{1}{2i}\int_C \frac{z}{z-i}\,dz$$

$$\int_C \frac{f(z)}{z-z_0}\,dz = 2\pi i f(z_0)$$

$$eq^n\text{-}(2)\ \int_C \frac{z}{z^2+1}\,dz = \frac{-1}{2i}\left[2\pi i(z) \text{ at } z=-i\right] + \frac{1}{2i}\left[2\pi i(z) \text{ at } z=i\right]$$

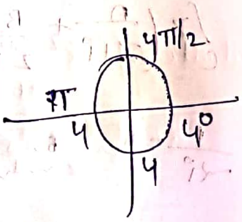
$$= \frac{-1}{2i}\left[2\pi\right] + \frac{1}{2i}\left[-2\pi\right]$$

$$= \pi i + \pi i$$

$$= 2\pi i$$

→ Evaluate $\int_C e^z/(z^2+\pi^2)^2\,dz$ where c is $|z|=4$

$(0,0), r=4$



$z^2+\pi^2=0$

$z^2=-\pi^2$ $\quad z=4e^{i\theta}$

$z=\pm\pi i \quad x+iy=4e^{i\theta}$

$(0,\pi)$

$(0,-\pi)$

$(0,3.14)$

$(0,-3.14)$

$$\int_C \frac{e^z}{(z^2+\pi^2)^2}\,dz = \int_C e^z\left(\frac{1}{(z^2+\pi^2)^2}\right)dz$$

$$= \int_C e^z$$

$$\frac{A}{z^2+\pi^2} + \frac{B}{(z^2+\pi^2)^2} \qquad \frac{A}{(z+\pi i)} + \frac{B}{(z-\pi i)}$$

$A(z^2+\pi^2)+B=1 \qquad A(z-\pi i)+B(z+\pi i)=1$

$A=0$

$$\frac{A}{z+\pi i} + \frac{B}{(z+\pi i)^2} + \frac{C}{z-\pi i} + \frac{D}{(z-\pi i)^2}$$

$$A(z+\pi i)(z-\pi i)^2 + B(z-\pi i)^2 + C(z+\pi i)^2(z-\pi i) + D(z+\pi i)^2$$

$$\frac{1}{(z+\pi i)^2(z-\pi i)^2} \qquad z=-\pi i$$

$$\frac{1}{(-\pi i-\pi i)^2} = \frac{1}{(-2\pi i)^2}$$

$$B = \frac{-1}{4\pi^2}$$

$$D = \frac{-1}{4\pi^2}$$

$A+C=0$

$-2\pi i A+\pi i A+B+2\pi i C-\pi i C+D$

$A(-\pi i)+C(\pi i)-\frac{2}{4\pi^2}$

$\pi i(C-A)-\frac{2}{4\pi^2}=0$

$C-A=\frac{2}{4\pi^2}\times\pi$

$=\frac{2}{4\pi^3 i}$

$A-C=\frac{-2}{4\pi^3}$

$=\frac{2}{4\pi^3}$

$=\frac{1}{2\pi^3}$

$A+C=0$

$A-C=\frac{1}{2\pi^3}$

$2A=\frac{1}{2\pi^3}$

$A=\frac{1}{4\pi^3}$

$C=\frac{-1}{4\pi^3}$

$\frac{1}{4\pi^3(z+\pi i)}+\frac{-1}{4\pi^2(z+\pi i)^2}+\frac{-1}{4\pi^3(z-\pi i)}-\frac{1}{4\pi^2(z-\pi i)^2}$

$\frac{e^z}{4\pi^3(z+\pi i)}-\frac{e^z}{4\pi^2(z+\pi i)^2}-\frac{e^z}{4\pi^3(z-\pi i)}-\frac{e^z}{4\pi^2(z-\pi i)^2}$

$\Downarrow \quad 0 \qquad \Downarrow \quad 0$

$\frac{2\pi i}{4\pi^3}(e^{-\pi i})-\frac{2\pi i}{4\pi^2}(e^{\pi i})-$

$\frac{2}{4\pi}\cdot\frac{1}{2\pi}$

$\frac{1}{2\pi}+\frac{1}{2\pi}=\frac{2}{2\pi}$

$=1/\pi$

$=$

Evaluate $\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz$ where c is the $|z| = 3$

$f(z) = \sin \pi z^2 + \cos \pi z^2$

singularities $= 1, 2$

$\frac{1}{(z-1)(z-2)} = \frac{-1}{(z-1)} + \frac{+1}{(z-2)}$

$= \frac{-1}{z-1} + \frac{1}{z-2}$

$\frac{-\sin \pi z^2 - \cos \pi z^2}{z-1} + \frac{\sin \pi z^2 + \cos \pi z^2}{z-2}$

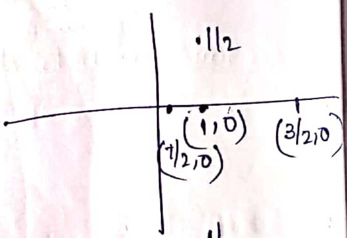
$2\pi i(-\sin \pi - \cos \pi) + 2\pi i(\sin 4\pi + \cos 4\pi)$

$2\pi i(0+1) + 2\pi i(0+1)$

$= 4\pi i$

→ Evaluate $\int_C \log z/(z-1)^3 \, dz$ c is $|z-1| = 1/2$

$z = 1$



$\frac{2\pi i}{2!}(-1)$

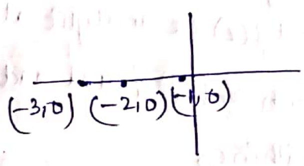
$= -\pi i$

$|x + iy - 1| = 1/2$

$(x-1) + iy()$ $x - 1 = 0$, $x = 1$

⇒ Evaluate $\int_C \frac{e^{-2z} z^2}{(z-1)^3 (z+2)} dz$ where c is $|z+2| = 1$

$z = 1, -2$



$\frac{A}{z-1} + \frac{B}{(z-1)^2} + \frac{C}{(z-1)^3} + \frac{D}{z+2} = 1$

$A(z-1)^2(z+2) + B(z-1)(z+2) + C(z+2) + D(z-1)^3$

$A(z^2+1-2z)(z+2) + B(z^2+z-2) + C(z+2) + D(z^3 - 1 - 3z^2 + 3z)$

$A + D = 0$

$-2A + 2A$

B(

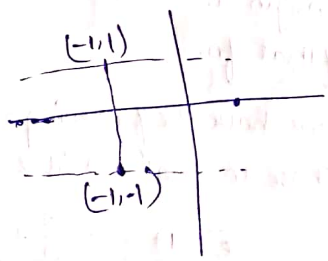
$D = (-8-1-12-6)$

$D = (-27)$

$\int_C z+1 / z^2+2z+4 \, dz$ c⇒ $|z+1+i| = 2$

$(z+2)^2$

$z = -2$



22/02/2020

Taylor's theorem:-

Let $f(z)$ be analytic at all points within a circle C_0 with center 'a' and radius 'r' then each point z within C_0

$$f(z) = f(a) + f'(a)(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \dots + \frac{f^n(a)}{n!}(z-a)^n \quad \text{eq}①$$

i.e. the series on the right hand side in eq① converges to $f(z)$ whenever $|z-a| < r_0$

C_0

Note:-

Let $f(z)$ be an entire function then we have

$$f(z) = \sum_{n=0}^{\infty} \frac{f^n(a)}{n!}(z-a)^n$$

where $|z-a| < \infty$

→ Taylor's series of an analytic function about the origin i.e. $z=0$ is classified as maclaurin's series of that fn.

→ we have the following results which are same as those in dual variable theory.

$$e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} \dots \frac{z^n}{n!} + \dots$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \dots$$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \dots$$

$$\frac{1}{1+z} = [1+z]^{-1} = 1 - z + z^2 - z^3 + z^4 - \dots$$

$$\frac{1}{1-z} = [1-z]^{-1} = 1 + z + z^2 + z^3 + z^4 \dots$$

→ Here e^z, $\sin z$, $\cos z$, $\sinh z$, $\cosh z$ are all entire functions,

→ $$\sinh z = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \frac{z^7}{7!} + \dots$$

$$\cosh z = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \frac{z^6}{6!} + \dots$$

→ To obtain the Taylor series expansion of $f(z)$ around/about $z=a$, put $z-a=\omega$. Then $f(z) = f(\omega+a) = \phi(\omega)$ (say) then substitute $\omega = z-a$ we get required Taylor series expansion.

Obtain the taylor series expansion of $f(z) = 1/z$ about the point $z=1$

Given function $f(z) = \frac{1}{z}$ about $z=1$

put $z-1=\omega$

$z = 1+\omega$

then $f(z) = \frac{1}{1+\omega}$

$= [1+\omega]^{-1}$

$= 1 - \omega + \omega^2 - \omega^3 + \omega^4 \dots$

$f(z) = 1 - (z-1) + (z-1)^2 + (z-1)^3 + (z-1)^4 + \dots$

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This is the required expansion

Expand e^z as taylor series about $z=1$

$z-1=\omega$

$z=1+\omega$

$f(z) = e^{1+\omega}\ e^z$

$= e^{1+\omega}$

$= e\, e^{\omega}$

$= e\left[1+\frac{\omega}{1!}+\frac{\omega^2}{2!}+\frac{\omega^3}{3!}+\cdots\right]$

$= e\left[1+\frac{z-1}{1!}+\frac{(z-1)^2}{2!}+\frac{(z-1)^3}{3!}+\cdots\right]$

It is the required expansion.

Find the taylor series expansion of e^z about $z=3$

$z-3=\omega$

$z=\omega+3$

$f(z) = e^{\omega+3}$

$= e^3 e^{\omega}$

$= e^3\left[1+\frac{\omega}{1!}+\frac{\omega^2}{2!}+\frac{\omega^3}{3!}+\cdots\right]$

$= e^3\left[1+\frac{z-3}{1!}+\frac{(z-3)^2}{2!}+\frac{(z-3)^3}{3!}\cdots\right]$

It is required expansion

$\rightarrow [1+z]^{-2} = 1-2z+3z^2-4z^3+5z^4+\cdots$

$\rightarrow [1-z]^{-2} = 1+2z+3z^2+4z^3+5z^4+\cdots$

Expand $f(z) = 1/z^2$ in powers of $z+1$

$z+1=\omega$

$z=\omega-1$

$f(z) = \frac{1}{z^2}$

$= \frac{1}{(\omega-1)^2}$

$= (\omega-1)^{-2}$

$= (-1)^{-2}(1-\omega)^{-2}$

$= 1+2\omega+3\omega^2+4\omega^3+5\omega^4+\cdots$

$= 1+2(z+1)+3(z+1)^2+4(z+1)^3+5(z+1)^4$

Expand $f(z) = 1/z^2$ in powers of $z-2$

$z-2=\omega$

$z=\omega+2$

$f(z) = \frac{1}{(\omega+2)^2}$

$= (\omega+2)^{-2}$

$= (2)^{-2}(\omega/2+1)^{-2}$

$= 1/4\left[1-2(\omega/2)+3(\omega/2)^2-4(\omega/2)^3+\cdots\right]$

$$\frac{1}{4}\left[1-2\left(\frac{z-2}{2}\right)+3\left(\frac{z-2}{2}\right)^2-4\left(\frac{z-2}{2}\right)^3-\cdots\right]$$

is the required expansion

(1) Expand $\sinh z$ by taylor series about $z=\pi i$

$z-\pi i=\omega$

$z=\omega+\pi i$

$f(z)=\sinh z$

$=\sinh(\omega+\pi i)$

$=\omega+\pi i+\frac{(\omega+\pi i)^3}{3!}+\frac{(\omega+\pi i)^5}{5!}+\frac{(\omega+\pi i)^7}{7!}+\cdots$

$\cosh\pi i=\cos\pi$

$\sinh\pi i=i\sin\pi$

$=\sinh\omega\cosh\pi i+\cosh\omega\sinh\pi i$

$=\sinh\omega(\cos\pi)+\cosh\omega(i\sin\pi)$

$=-\sinh\omega$

$=-\left[\omega+\frac{\omega^3}{3!}+\frac{\omega^5}{5!}+\cdots\right]$

$=-\left[(z-\pi i)+\frac{(z-\pi i)^3}{3!}+\frac{(z-\pi i)^5}{5!}+\cdots\right]$

24/02/2020

$f(z)=\frac{z-1}{z+1}$ about the point

(i) $z=0$

(ii) $z=1$

$f(z)=\frac{z-1}{z+1}$

(i) Taylor series of $f(z)$ about $z=0$

consider $f(z)=\frac{z-1}{z+1}$

$=\frac{z+1-2}{z+1}$

$=1-\frac{2}{z+1}$

$=1-2[1+z]^{-1}$

$f(z)=1-2[1-z+z^2-z^3+\cdots]$

(ii) $z-1=\omega$

$z=\omega+1$

$f(z)=\frac{\omega+1-1}{\omega+1+1}$

$=\frac{\omega}{\omega+2}$

$=\frac{\omega+2-2}{\omega+2}$

$=1-\frac{2}{\omega+2}$

$=1-2[\omega+2]^{-1}$

$=1-\frac{2}{2[1+\omega/2]}$

$$f(z) = 1 - \left[1 + \frac{w}{2}\right]^{-1}$$

$$= 1 - \left[1 - \left(\frac{w}{2}\right) + \left(\frac{w}{2}\right)^2 - \left(\frac{w}{2}\right)^3 + \cdots\right]$$

$$f(z) = \frac{z-1}{2} - \frac{(z-1)^2}{2^2} - \frac{(z-1)^3}{2^3} + \cdots$$

Expand $f(z) = \frac{z-1}{z^2}$ ~~about~~ in a Taylor series in powers of $(z-1)$.

$$z - 1 = w$$

$$z = 1 + w$$

$$f(z) = \frac{w}{(1+w)^2}$$

$$= w(1+w)^{-2}$$

$$= w\left(1 - 2w + 3w^2 - 4w^3 + 5w^4 - \cdots\right)$$

$$= (z-1)\left[1 - 2(z-1) + 3(z-1)^2 - 4(z-1)^3 + 5(z-1)^4 + \cdots\right)$$

Find the taylor's expansion for the fn $f(z) = \frac{1}{(1+z)^2}$ with center at $-i$

$$C = (0, -1)$$

Given function $f(z) = \frac{1}{(1+z)^2}$ about $z = -i$

we know, the Taylor's series expansion about $z = a$

$$f(z) = f(a) + \frac{f'(a)}{1!}(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \cdots$$

let $f(z) = \frac{1}{(1+z)^2} \Rightarrow f(a) = f(-i) = \frac{1}{(1-i)^2}$

$$= \frac{1}{1-1-2i}$$

$$= i/2$$

$$f'(z) = \frac{-2(1+z)}{(1+z)^4} = \frac{-2}{(1+z)^3} \Rightarrow f'(a) = f'(-i)$$

$$= \frac{-2}{(1-i)^3}$$

$$= \frac{-2}{1+i^2-3i+3}$$

$$= \frac{-2}{-2-2i}$$

$$= \frac{1}{1+i}$$

$$f''(z) = \frac{6(1+z)^2}{(1+z)^6} \Rightarrow f''(a) = f''(-i) = \frac{+6}{(1-i)^4}$$

$$= \frac{+6}{(-2i)(-2i)}$$

$$= \frac{-6}{4i^2} = \frac{+3}{2i^2}$$

$$= -3/2$$

$$f'''(z) = \frac{-4(1+z)^3}{(1+z)^8} \Rightarrow f'''(a) = f'''(-i) = \frac{-4}{(1-i)^5}$$

$$\frac{-4}{4i^2(1-i)} = \frac{-4}{4i^2-4i^3}$$

$$= \frac{-1}{-i}$$

$$= \frac{-4}{-4+4i}$$

$$= \frac{1}{1-i}$$

$$f(z) = \frac{i}{2} + \frac{1}{1+i}(z-i) + \frac{3}{2(2)}(z-i)^2 - \frac{1}{1-i(3!)}(z-i)^3 \ldots$$

(2) Expand $f(z) = \sin z$ in taylor series about

(i) $z = \pi/4$

(ii) $z = \pi/2$

$$z - \pi/4 = \omega$$

$$z = \omega + \pi/4$$

Given function $f(z) = \sin z$ about $z = \pi/4$

we know the Taylor's series expansion about $z = a$

$$f(z) = f(a) + \frac{f'(a)}{1!}(z-a) + \frac{f''(a)}{2!}(z-a)^2 - - - - -$$

(i) about $z = \pi/4$

$$f(a) = f(\pi/4) = \sin \pi/4 = 1/\sqrt{2}$$

$$f'(a) = f'(\pi/4) = \cos \pi/4 = 1/\sqrt{2}$$

$$f''(a) = f''(\pi/4) = -\sin \pi/4 = -1/\sqrt{2}$$

$$f(z) = 1/\sqrt{2} + \frac{1/\sqrt{2}}{1!}(z-\pi/4) - \frac{1/\sqrt{2}}{2!}(z-\pi/4)^3$$

Obtain the Taylor series expansion of $f(z) = \frac{e^z}{z(z+1)}$

(3) $z=2$

$z-2=\omega$

$z=2+\omega$

$$f(z) = \frac{e^z}{z(z+1)}$$

$$f(2+\omega) = \frac{e^{2+\omega}}{(2+\omega)(2+\omega+1)}$$

$$= \frac{e^{2+\omega}}{(2+\omega)(3+\omega)}$$

$$= \frac{e^2}{2+\omega} \cdot \frac{e^\omega}{3+\omega}$$

$$= \frac{1}{2\times 3} e^2 (1+\omega/2)^{-1} \cdot e^\omega (1+\omega/3)^{-1}$$

$$= \frac{e^2}{6} (1+\omega/2)^{-1} e^\omega (1+\omega/3)^{-1}$$

$$= \frac{e^2}{6}\left[1+\frac{\omega}{1!}+\frac{\omega^2}{2!}+\frac{\omega^3}{3!}\cdots\right]\left[1-\frac{\omega}{2}+\left(\frac{\omega}{2}\right)^2-\left(\frac{\omega}{2}\right)^3\cdots\right]$$

$$\left[1-\omega/3+(\omega/3)^2-(\omega/3)^3+\cdots\right]$$

$$= \frac{e^2}{6}\left[1+\frac{z-2}{1!}+\frac{(z-2)^2}{2!}+\frac{(z-3)^3}{3!}\cdots\right]$$

$$\left[1-\frac{z-2}{2}+\left(\frac{z-2}{2}\right)^2-\left(\frac{z-2}{2}\right)^3+\cdots\right]$$

$$\left[1-\frac{z-2}{3}+\left(\frac{z-2}{3}\right)^2-\left(\frac{z-2}{3}\right)^3+\cdots\right]$$

(4) $f(z) = \frac{1}{z^2-z-6}$ about (i) $z=-1$ (ii) $z=1$

$z+1=\omega$

$z=-1+\omega$

$$\frac{1}{(z-3)(z+2)} = \frac{A}{z-3}+\frac{B}{z+2}$$

$A = 1/5$

$B = -1/5$

$$= \frac{1}{5(z-3)} - \frac{1}{5(z+2)}$$

$$= \frac{1}{5}\left[\frac{1}{z-3} - \frac{1}{z+2}\right] \longrightarrow (1)$$

$z+1=\omega$

$z=\omega-1$

$$= \frac{1}{5}\left[\frac{1}{\omega-4} - \frac{1}{\omega+1}\right]$$

$$= \frac{1}{5}\left[(\omega-4)^{-1} - (\omega+1)^{-1}\right]$$

$$\frac{1}{5}\left[-1/4(1-\omega/4)^{-1}\right]-\left[(1+\omega)^{-1}\right]$$

$$= \frac{-1}{20}(1-\omega/4)^{-1} - \frac{1}{5}(1+\omega)^{-1}$$

$$-\frac{1}{20}\left[1+\omega/4+\left(\frac{\omega}{4}\right)^2+\left(\frac{\omega}{4}\right)^3-\cdots\right]\frac{1}{5}\left[1-\omega+\omega^2-\omega^3+\cdots\right]$$

$$f(z)=\frac{-1}{20}\left[1+\frac{z+1}{4}+\left(\frac{z+1}{4}\right)^2+\left(\frac{z+1}{4}\right)^3-\cdots\right]$$

$$-\frac{1}{5}\left[1-(z+1)+(z+1)^2-(z+1)^3+\cdots\right]$$

$z=1$

$z-1=\omega$

$z=\omega+1$

$$f(z)=\frac{1}{5(\omega+1-3)}-\frac{1}{5(\omega+1+2)}$$

$$=\frac{1}{5}(\omega-2)^{-1}-\frac{1}{5}(\omega+3)^{-1}$$

$$=\frac{1}{5}(-2)^{-1}(1-\omega/2)^{-1}-\frac{1}{5}(3)^{-1}(1+\omega/3)^{-1}$$

$$=-\frac{1}{10}(1-\omega/2)^{-1}-\frac{1}{15}(1+\omega/3)^{-1}$$

$$=\frac{-1}{10}\left(1+\omega/2+(\omega/2)^2+(\omega/2)^3+\cdots\right)$$

$$-1/15\left(1-\omega/3+(\omega/3)^2+(\omega/3)^3-\cdots\right]$$

$$=-\frac{1}{10}\left(1+\frac{z-1}{2}+\left(\frac{z-1}{2}\right)^2+\left(\frac{z-1}{2}\right)^3-\cdots\right]$$

$$-1/15\left(1-\frac{z-1}{3}+\left(\frac{z-1}{3}\right)^2-\left(\frac{z-1}{3}\right)^3\cdots\right]$$

→ (5) Expand $f(z)=\frac{z}{z^4+9}$ in Taylor series about $z=0$

$$\frac{z}{z^4+9}$$

$$z\left[z^4+9\right]^{-1}$$

$$\frac{z}{9}\left[1+\frac{z^4}{9}\right]^{-1}$$

$$\frac{z}{9}\left[1-\frac{z^4}{9}+\left(\frac{z^4}{9}\right)^2-\left(\frac{z^4}{9}\right)^3+\cdots\right]$$

(6) Expand $f(z)=\cancel{\frac{1}{(1+z)^2}}$ $\log z$ by taylor series about $z=1$

$f(z)=f(a)+\cdots$

$f(a)=\log 1$

$=0$

$f'(z)=\frac{1}{z}$ $\quad f'(a)=\frac{1}{1}=1$

$f''(z)=\frac{-1}{z^2}$ $\quad f''(a)=-1$

$f(z)=$ $\cancel{0+}$ $\frac{1}{1}(z-1)-\frac{1}{2}(z-1)^2+\frac{1}{}\cdots$

Expand $f(z) = \frac{z}{(z+1)(z+2)}$ about $z=1$

⑧

$$z\left[\frac{1}{(z+1)(z+2)}\right] = z\left[\frac{A}{z+1} + \frac{B}{z+2}\right]$$

$$= z\left[\frac{1}{z+1} + \frac{-1}{z+2}\right]$$

$$= z[(z+1)^{-1}] - z[z+2]^{-1}$$

$$= z(z+1)^{-1} - \frac{z}{2}(1+z/2)^{-1}$$

$$= z(1-z+z^2-z^3 \ldots) - \frac{z}{2}\left(1 - \frac{z}{2} + \left(\frac{z}{2}\right)^2 - \left(\frac{z}{2}\right)^3 \ldots\right]$$

⑨

$\rightarrow f(z) = \log(1+e^z)$ about $z=0$

$$f(z) = f(a) + \frac{f'(a)}{1!}(z-a) \ldots$$

$$f(a) = \log(1+e^0)$$

$$= \log 2$$

$$f'(z) = \frac{e^z}{1+e^z} \Rightarrow f'(a) = \frac{1}{2}$$

$$f''(z) = \frac{-1(e^z)}{(1+e^z)^2} \Rightarrow f''(a) = \frac{-1}{4}$$

$$f(z) = \log 2 + \frac{1}{2\times 1}(z) + \frac{-1}{4\times 2}(z^2) \ldots$$

$$= \log 2 + \frac{z}{2} - \frac{z^2}{8} \ldots$$

⑦

$f(z) = ze^z$ about $z=1$

$$z-1 = \omega$$

$$z = 1+\omega$$

$$f(z) = ze^z$$

$$= (1+\omega)e^{1+\omega}$$

$$= e^{1+\omega} + \omega e^{1+\omega}$$

$$f(z) = ze^z \qquad f'(z) = ze^z + e^z$$

$$f(a) = e^1 \qquad = e^1 + e^1$$

$$= 2e^1$$

$$f''(z) = ze^z + e^z + e^z$$

$$= e^1 + e^1 + e^1$$

$$= 3e.$$

$f(z) = e^{1+z}$ in powers of $z-1$.

$f(a) = e^2$

$f'(z) = e^{1+z}$

$f''(a) = e^2$

$f(z) = e^2 + \frac{e^2}{1!}(z-1) + \frac{e^2}{2!}(z-1)^2 + \cdots$

$f(z) = \frac{z^2-1}{(z+2)(z+3)}$ about $z=-2$

$f(z) = \frac{z^2-1}{(z+2)(z+3)}$

$\frac{z^2-1}{(z+2)(z+3)} = 1 + \frac{A}{z+2} + \frac{B}{z+3}$

$= (z+2)(z+3) + A(z+3) + B(z+2)$

$z^2-1 = z^2+5z+6+Az+3A+Bz+2B$

$5+A+B=0$

$6+3A+2B=-1$

$-6-3A-2B=1$

$15+3A+3B=0$

$-6-3A-2B=1$

$9-B=1$

$\boxed{B=9}$

$A=-14$

Laurent's series:-

If $f(z)$ is analytic inside and on the boundary of ringshaped region R bounded by two concentric circles C_1 & C_2 of radii r_1 & r_2 ($r_1>r_2$) respectively having center at 'a' then for all z in Region 'R'.

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-a)^n}$$

where $a_n = \frac{1}{2\pi i} \int_{C_1} \frac{f(z)}{(z-a)^{n+1}} dz$ $n=0,1,2,\ldots$

and $b_n = \frac{1}{2\pi i} \int_{C_2} \frac{f(z)}{(z-a)^{-n+1}} dz$ $n=1,2,3,\ldots$

→ Here $\sum_{n=0}^{\infty} a_n(z-a)^n$ is called the regular part and $\sum_{n=1}^{\infty} b_n/(z-a)^n$ is called the principle part of Laurent's series.

Find Laurent's series of $f(z) = \frac{e^{2z}}{(z-1)^2}$ about $z=1$

Put $z-1=\omega \Rightarrow z=\omega+1$

consider $f(z) = \frac{e^{2z}}{(z-1)^2}$

$f(\omega+1) = \frac{e^{2(\omega+1)}}{\omega^2}$

$= \frac{e^{2\omega} \cdot e^2}{\omega^2}$

$= e^2 \cdot 1/\omega^2 \cdot e^{2\omega}$

$= e^2 \frac{1}{\omega^2}\left[1 + \frac{2\omega}{1!} + \frac{(2\omega)^2}{2!} + \frac{(2\omega)^3}{3!} - - - \right]$

$f(\omega+1) = e^2\left[\frac{1}{\omega^2} + \frac{2}{\omega 1!} + \frac{2^2}{2!} + \frac{2^3}{3!}\omega + \frac{2^4}{4!}\omega^2 + \frac{2^5}{5!}\omega^3 + \frac{2^6}{6!}\omega^4 + - - - \right]$

Replace ω with $z-1$

$\therefore f(z) = e^2\left[\frac{1}{(z-1)^2} + \frac{2}{(z-1)} + \frac{2^2}{2!} + \frac{2^3}{3!}(z-1) + \frac{2^4}{4!}(z-1)^2 + \frac{2^5}{5!}(z-1)^3 + \frac{2^6}{6!}(z-1)^4 + - - - \right]$

Find the Laurent's series of $f(z) = \frac{1}{z^2-4z+3}$ for $1<|z|<3$

$z^2 - z - 3z + 3$

$z(z-1) - 3(z-1)$

$(z-1)(z-3)$

$\frac{1}{(z-1)(z-3)} = \frac{A}{z-1} + \frac{B}{z-3}$

$= \frac{1}{-2(z-1)} + \frac{1}{2(z-3)}$

$f = \frac{-1}{2}(z-1)^{-1} + \frac{1}{2}(z-3)^{-1}$

In the region of $1<|z|<3$

$1<|z|$ & $|z|<3$

$\frac{1}{|z|}<1$ & $\frac{|z|}{3}<1$

then $f(z) = \frac{-1}{2z[1-1/z]} - \frac{1}{2}\frac{1}{3[1-z/3]}$

$= \frac{-1}{2z}[1-1/z]^{-1} - \frac{1}{6}\left[1-\frac{z}{3}\right]^{-1}$

$= -\frac{1}{2z}\left[1 + (1/z) + (1/z)^2 + (1/z)^3 + (1/z)^4 + \cdots\right]$

$-1/6\left[1 + z/3 + \left(\frac{z}{3}\right)^2 + \left(\frac{z}{3}\right)^3 + \cdots\right]$

$f(z) = -\frac{1}{2}\left[\frac{1}{z} + \left(\frac{1}{z}\right)^2 + \left(\frac{1}{z}\right)^3 + (1/z)^4 \cdots\right]$

$\frac{1}{6}\left[1 + z/3 + \left(\frac{z}{3}\right)^2 + \left(\frac{z}{3}\right)^3 + \cdots\right]$

is required Laurent's series of $f(z)$.

→ obtain laurent's series for $f(z) = \frac{4z+3}{z(z-3)(z+2)}$ in the region of (i) $|z| = 1$ (ii) $2 < |z| < 3$ (iii) $|z| > 3$

$\frac{4z+3}{z(z-3)(z+2)} = \frac{3}{-6(z)} + \frac{15}{15(z-3)} - \frac{-5}{-10(z+2)}$

$= \frac{-1}{2z} + \frac{1}{z-3} - \frac{1}{2(z+2)}$

let $|z| = 1$; $\frac{|z|}{2} < 1$, $\frac{|z|}{3} < 1$

consider $f(z) = \frac{-1}{2} \cdot \frac{1}{z} + \frac{1}{(-3)\left[1 - \frac{z}{3}\right]} - \frac{1}{2} \cdot \frac{1}{2\left[\frac{z}{2} + 1\right]}$

$f(z) = \frac{-1}{2z} - \frac{1}{3}\left[1 - \frac{z}{3}\right]^{-1} - \frac{1}{4}\left[1 + \frac{z}{2}\right]^{-1}$

$\therefore f(z) = \frac{-1}{2z} - \frac{1}{3}\left[1 + \left(\frac{z}{3}\right) + \left(\frac{z}{3}\right)^2 + \left(\frac{z}{3}\right)^3 + \cdots\right]$

$-\frac{1}{4}\left[1 - \left(\frac{z}{2}\right) + \left(\frac{z}{2}\right)^2 - \left(\frac{z}{2}\right)^3 + \cdots\right]$

let $\cancel{\frac{|z|}{2}}$ & $\cancel{\frac{|z|}{3} < 1}$

$2 < |z|$ $|z| < 3$

$\frac{2}{|z|} < 1$ $\frac{|z|}{3} < 1$

$= \frac{-1}{2z} + \frac{1}{z-3} - \frac{1}{2(z+2)}$

$\frac{-1}{2z} + \frac{1}{(-3)\left(1 - \frac{z}{3}\right)} - \frac{1}{2z(1 + 2/z)}$

$\frac{-1}{2z} - \frac{1}{3}(1 - z/3)^{-1} - \frac{1}{2z}(1 + 2/z)^{-1}$

$\frac{-1}{2z} - \frac{1}{3}\left[1 + \left(\frac{z}{3}\right) + \left(\frac{z}{3}\right)^2 + \left(\frac{z}{3}\right)^3 + \cdots\right]$

$-\frac{1}{2z}\left[1 - \frac{2}{z} + \left(\frac{2}{z}\right)^2 - \left(\frac{2}{z}\right)^3 + \cdots\right]$

$-\frac{1}{2z} - \frac{1}{3}\left[1 + \left(\frac{z}{3}\right) + \left(\frac{z}{3}\right)^2 + \left(\frac{z}{3}\right)^3 + \cdots\right]$

$-\frac{1}{2}\left[\frac{1}{z} - \frac{2}{z^2} + \frac{2^2}{z^3} - \frac{2^3}{z^4} + \cdots\right]$

(iii) $\cancel{|z| < 3}$ $\cancel{\frac{|z|}{3} < 1}$ $|z| > 3$

$1 > 3/|z|$

$$= \frac{-1}{2z} - \frac{1}{2z(1+2/z)} + \frac{1}{z(1-3/|z|)}$$

$$= \frac{-1}{2z} - \frac{1}{2z}(1+2/z)^{-1} + \frac{1}{z}(1-3/|z|)^{-1}$$

$$= \frac{-1}{2z} - \frac{1}{2z}\left(1 - \frac{2}{z} + \left(\frac{2}{z}\right)^2 - \left(\frac{2}{z}\right)^3 - \cdots\right)$$

$$+ \frac{1}{z}\left(1 + 3/z + (3/z)^2 + (3/z)^3 - \cdots\right)$$

$$= \frac{-1}{2z} - \frac{1}{2}\left(1/z - \frac{2}{z^2} + \frac{2^2}{z^3} - \frac{2^3}{z^4} - \cdots\right)$$

$$+ \left(\frac{1}{z} + \frac{3}{z^2} + \frac{3^2}{z^3} + \frac{3^3}{z^4} - \cdots\right)$$

Expand $f(z) = \frac{1}{(z+1)(z+3)}$ valid for the region

i) $|z| < 1$

(ii) $1 < |z| < 3$

(iii) $|z| > 3$

(iv) $1 < |z+1| < 2$

$$\frac{1}{(z+1)(z+3)} = \frac{A}{z+1} + \frac{B}{z+3}$$

$$= \frac{1}{2(z+1)} + \frac{1}{-2(z+3)}$$

$|z| < 1$

$$\frac{1}{2}(z+1)^{-1} + \frac{-1}{2}(z+3)^{-1}$$

$$\frac{1}{2}(z+1)^{-1} - \frac{1}{6}(1+z/3)^{-1}$$

$$\frac{1}{2}(1 - z + z^2 - z^3 \ldots) - 1/6\left(1 - \frac{z}{3} + \left(\frac{z}{3}\right)^2 - \cdots\right)$$

(ii) $1 < |z|$ $\quad |z| < 3$

$|z| < 1$ $\quad \frac{|z|}{3} < 1$

$1 < 1/|z|$

$$= \frac{1}{2z(1+1/z)} - \frac{1}{2(3)(1+z/3)}$$

$$= \frac{1}{2z}(1+1/z)^{-1} - \frac{1}{6}(1+z/3)^{-1}$$

$$= \frac{1}{2z}\left(1 - 1/z + (1/z)^2 - (1/z)^3 - \cdots\right)$$

$$-1/6\left(1 - z/3 + (z/3)^2 - (z/3)^3 - \cdots\right)$$

$$= \frac{1}{2}\left(1/z - 1/z^2 + \frac{1}{z^3} - \frac{1}{z^4} \cdots\right)$$

$$-1/6\left(1 - z/3 + (z/3)^2 - (z/3)^3 - \cdots\right)$$

(iii) $|z| > 3$

$1 > 3/|z|$

$\frac{3}{|z|} < 1$

$$= \frac{1}{2z}(1+1/z)^{-1} - \frac{1}{2z(1+3/z)}$$

$$= \frac{1}{2z}(1+1/z)^{-1} - \frac{1}{2z}(1+3/z)^{-1}$$

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$= \frac{1}{2}\left(\frac{1}{z} - \frac{1}{z^2} - \frac{1}{z^3} - \cdots\right)$

$- \frac{1}{2z}\left(1 - 3/z + (3/z)^2 - (3/z)^3 - \cdots\right)$

$= \frac{1}{2}\left(1/z - 1/z^2 - 1/z^3 - \cdots\right)$

$- \frac{1}{2}\left(1/z - 3/z^2 + \frac{3^2}{z^3} - \frac{3^3}{z^4} \cdots\right)$

(iv) $1 < |z+1| < 2$

$1 < |z+1|$ $\quad |z+1| < 2$

$z+1 = \omega$

$z = \omega - 1$

$1 < \omega$ $\quad \omega < 2$

$\frac{1}{\omega} < 1$ $\quad 1 < 2/\omega$

$1 > \omega/2$

$f(z) = \frac{1}{2(z+1)} - \frac{1}{2(z+3)}$

$= \frac{1}{2(z+1)} - \frac{1}{2((z+1)+2)}$

$= \frac{1}{2\omega} - \frac{1}{2(\omega+2)}$

$= \frac{1}{2\omega} - \frac{1}{2\times 2(1+\omega/2)}$

$= \frac{1}{2}\omega^{-1} - \frac{1}{4}(1+\omega/2)^{-1}$

$= \frac{1}{2}\omega^{-1} - \frac{1}{4}\left(1 - \omega/2 + \left(\frac{\omega}{2}\right)^2 - \left(\frac{\omega}{2}\right)^3 - \cdots\right)$

$= \frac{1}{2(z+1)} - \frac{1}{4}\left[1 - \frac{z+1}{2} + \frac{(z+1)^2}{2^2} - \frac{(z+1)^3}{2^3} \cdots\right]$

Is the laurent's series of $f(z)$

→ $f(z) = \frac{2}{(z-1)(z-3)}$ $\quad 0 < |z-1| < 2$

$\frac{2}{(z-1)(z-3)} = \frac{A}{z-1} + \frac{B}{z-3}$

$= \frac{-1}{z-1} + \frac{1}{z-3}$

$0 < |z-1|$ $\quad |z-1| < 2$

$z-1 = \omega$ $\quad 0 < \omega$ $\quad \omega < 2$

$z = 1+\omega$ $\quad \frac{0}{\omega} < 1$ $\quad \frac{\omega}{2} < 1$

$0 < 1$

$= \frac{-1}{1+\omega-1} + \frac{1}{1+\omega-3}$

$= \frac{-1}{\omega} + \frac{1}{\omega-2}$

$$\frac{-1}{\omega} + \frac{1}{(-2)(1-\omega/2)}$$

$$\frac{-1}{\omega} - \frac{1}{2}(1-\omega/2)^{-1}$$

$$\frac{-1}{\omega} - \frac{1}{2}\left(1+\omega/2+\left(\frac{\omega}{2}\right)^2+\left(\frac{\omega}{2}\right)^3+\left(\frac{\omega}{2}\right)^4 - \cdots\right)$$

29/02/2020.

Find the laurent's series for $f(z) = \frac{1}{z(1+z^2)}$ in the region of $0<|z|<1$

$$\frac{1}{z(1+z^2)} = \frac{A}{z} + \frac{Bz+C}{1+z^2}$$

$0<z$

$A(1+z^2)+(Bz+C)z = 1$

$|z|<1$

$A+Az^2+Bz^2+Cz = 1$

$A+B=0$ $\boxed{B=-1}$

$\boxed{C=0}$

$\boxed{A=1}$

$$\frac{1}{z(1+z^2)} = \frac{1}{z} + \frac{-z}{1+z^2}$$

$$= \frac{1}{z} - \frac{z}{1+z^2}$$

$$= \frac{1}{z} - \frac{z}{z^2(1+1/z^2)}$$

$$= \frac{1}{z} - \frac{z}{z^2}(1+1/z^2)^{-1}$$

$$= \frac{1}{z} - \frac{1}{z}(1+1/z^2)^{-1}$$

$$= \frac{1}{z} - \frac{1}{z}\left[1-(1/z^2)+(1/z^2)^2+(1/z^2)^4 + \cdots\right]$$

$$= \frac{1}{z} - \left[1/z - \frac{1}{z^3} + \frac{1}{z^5} - \frac{1}{z^7} + \cdots\right]$$

$$f(z) = \frac{1}{z^3} - \frac{1}{z^5} + \frac{1}{z^7} + \cdots$$

(or)

$$= \frac{1}{z} + z(1+z^2)^{-1}$$

$$= \frac{1}{z} + z(1-z^2+(z^2)^2-(z^2)^3 \cdots]$$

$$= \frac{1}{z} + z(1-z^2+z^4-z^8 \cdots]$$

$$= \frac{1}{z} + (z-z^3+z^5-z^9 \cdots]$$

Evaluate the laurent's series expansion for $f(z) = \frac{7z-2}{z(z-2)(z+1)}$ in the region of $1<|z+1|<3$

$$\frac{7z-2}{z(z-2)(z+1)} = \frac{A}{z} + \frac{B}{z-2} + \frac{C}{z+1}$$

$$= \frac{1}{z} + \frac{2}{z-2} + \frac{-3}{z+1}$$

$$= 1/z + \frac{2}{z-2} - \frac{3}{z+1}$$

$\frac{12}{2(3)}$ $\frac{5}{}$ $\frac{-9}{(-1)(-3)}$

$z+1=\omega$

$\omega < 3$

$1<\omega$

$\frac{\omega}{3} < 3$

$\frac{1}{\omega} < 1$

$$= \frac{1}{\omega+1} + \frac{2}{\omega-1-2} - \frac{3}{\omega-1+1}$$

$$= \frac{1}{\omega-1} + \frac{2}{\omega-3} - \frac{3}{\omega}$$

$$= \frac{1}{\omega(1-1/\omega)}$$

$$= \frac{1}{\omega}(1-1/\omega)^{-1} + \frac{2}{3(\omega/3-1)} - \frac{3}{\omega}$$

$$= \frac{1}{\omega}(1-1/\omega)^{-1} - \frac{2}{3}(1-\omega/3)^{-1} - 3/\omega$$

$$= \frac{1}{\omega}(1+1/\omega + (1/\omega)^2 + (1/\omega)^3 - -)$$

$$- \frac{2}{3}(1+(\omega/3)+(\omega/3)^2+(\omega/3)^3 \cdots)$$

$$-3/\omega$$

$$= \left(1/\omega + \frac{1}{\omega^2} + \frac{1}{\omega^3} + \frac{1}{\omega^4} - - - \right)$$

$$- \frac{2}{3}\left(1+\frac{\omega}{3}+\left(\frac{\omega}{3}\right)^2+\left(\frac{\omega}{3}\right)^3 - - - \right) - 3/\omega$$

$$= \left(\frac{1}{z+1} + \frac{1}{(z+1)^2} + \frac{1}{(z+1)^3} + \frac{1}{(z+1)^4} - - - \right)$$

$$- \frac{2}{3}\left(1+\frac{z+1}{3}+\left(\frac{z+1}{3}\right)^2+\left(\frac{z+1}{3}\right)^3 - - - \right)$$

$$- 3/z+1$$

→ Expand $f(z) = e^{1+z}$ in Laurent series ~~for~~ powers of $z-1$.

$$f(z) = e^{1+z}$$

$$= e^{z-1} \cdot e^2$$

$$= e^2\left[1+\frac{z-1}{1!} + \frac{(z-1)^2}{2!} + \frac{(z-1)^3}{3!} - - \right]$$

Expand Laurent's series for $f(z) = e^z \sin z$ about $z=0$

$$f(z) = e^z \sin z$$

$$= \left(1+\frac{z}{1!}+\frac{z^2}{2!}+\frac{z^3}{3!} - - \right)\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} - - \right)$$

$$= \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - - - \right) + \left(z^2 - \frac{z^4}{3!} + \frac{z^6}{5!} - - - \right)$$

$$+ \left(\frac{z^3}{2!} - \frac{z^5}{3!2!} - - - \right)$$

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$$= \left(z + z^2 + \frac{z^3}{2!} - \frac{z^3}{3!} - \frac{z^4}{3!} + \frac{z^5}{5!} - \frac{z^5}{3!\,2!} \cdots \right)$$

Expand the laurent's series of $f(z) = \frac{z^2 - 1}{(z+2)(z+3)}$

in the region (i) $|z| < 2$ (ii) $2 < |z| < 3$

(iii) $|z| > 3$

$\frac{|z|}{2} < 1$ $\qquad \frac{2}{|z|} < 1$, $\frac{|z|}{3} < 1$

$1 > 3/|z|$

$\frac{3}{|z|} < 1$

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