Unit - 2

Probability Distribution

Consider a random experiment has only two possible outcomes.

for example, in the experiment of rolling a dice success corresponds to getting a number 1 on the face of a die and failure corresponds to getting a number 2, 3, 4, 5, 6 on the face of die.

Assume that this outcomes have probabilities p, q such that $p+q=1$ ie, $p = 1/6$ and $q = 5/6$

sometimes, we need to find probability of 'X' success among 'n' times under the following conditions.

1) these are only two possible outcomes for each trial ie success and failure
2) The probability of getting success is same for each trial.
3) There are 'n' no. of trials which are independent.

Bernoulli's distribution:

A R.V 'X' which takes two values 0 and 1 with probability q and p respectively ie,

$$P(X=0) = q$$

$$P(X=1) = p$$

where $q = 1-p$ is called a Bernoulli's discrete R.V and is said to have a Bernoulli's distribution.

→ The probability function of Bernoulli's distribution can be written as

$p^x q^{1-x}$

$P(x) = p^x q^{1-x}$

$P(x) = p^x (1-p)^{1-x}$ where $x = 0, 1$

Note: i) Mean of Bernoullis discrete RV 'X' is

i) $\mu = E(x) = \sum_{x=0}^{1} P_i x_i = P_0 x_0 + P_1 x_1$

$= q \times 0 + p \times 1$

$\boxed{\mu = p}$

ii) Variance $= \sigma^2 = V(x) = \sum_{x=0}^{\infty} P_i x_i^2 - \mu^2$

$= P_0 x_0^2 + P_1 x_1^2 - p^2$

$= q(0)^2 + P(1)^2 - p^2$

$= p - p^2$

$\sigma^2 = p(1-p)$

$\sigma^2 = pq.$

Standard deviation, $\boxed{\sigma = \sqrt{pq}}$

⇒ Generally probability distributions are classified into 2 types

1) Discrete 2) Continuous

Discrete P.D's: They are of 5 types

1) Binomial distribution ✓
2) Poisson distribution ✓
3) Geometrical distribution
4) Negative Binomial distribution
5) Rectangular distribution

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Continous P.D's : They are of 4 types

1) Normal distribution

2) Student's t-test distribution

3) Chi-square distribution

4) F-test distribution

→ In these we will discuss only 3 distributions

They are

1) Binomial distribution

2) Poisson distribution

3) Normal distribution

Binomial Distribution : This was discovered by "James Bernoulli" in the year 1700 and it is a discrete probability distribution

Consider a series of 'n' independent bernoulli's trials in which the probability of success 'p' and the probability of failure "q"

The probability of 'x' success and consequently "n-x" failures in 'n' independent trials in a specific order.

i) sssffssfsfs .....sfs

ppp....p(x times) qqq ....q(n-x) times

$= p^x q^{n-x}$

ii) fsssfffss .....fssfff

ppp....p(x times) qqq....q(n-x) times

$= p^x q^{n-x}$

⋮

so on

But 'x' success in 'n' trials can occur in x ways and probability of each ways is $p^x q^{n-x}$

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⇒ Therefore the probability of 'x' success in 'n' trials in any order is given by

$$P(X=x) = {}^{n}C_{x}\, p^{x} q^{n-x}$$

<u>Defination</u>: A R.V X' having a Binomial distribution if it assumes only non-negative values and its probability distribution is given by

$$P(X=x) = {}^{n}C_{x}\, p^{x} q^{n-x}$$

where $x = 0, 1, 2, \dots n$

or

$$P(X=x) = {}^{n}C_{x}\, p^{x} (1-p)^{n-x}$$

and it is ~~de~~ denoted by $X \sim B(x, p, n]$

* we read it as the R.V 'x' follows Binomial distribution with parameters x, n, p.

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→ A number of defective bolts in a box containing n bolts. A no. of machines lying

→ The number of post graduates in a group of 'n' men

Constants of Binomial Distribution

1) **Mean of the Binomial distribution:**

Proof: We know that, probability of a Binomial distribution is,

$$P(X=x) = {}^nC_x\, p^x q^{n-x}$$

Consider $\mu = E(X) = \sum_{x=0}^{n} P(x)\, x(x)$

$$= \sum_{x=0}^{n} P(x) \cdot x$$

$$= \sum_{x=0}^{n} {}^nC_x\, p^x q^{n-x} \cdot x$$

$$= \sum_{x=0}^{n} x\, {}^nC_x\, p^x q^{n-x}$$

$$= 0 + 1 \cdot {}^nC_1\, p\, q^{n-1} + 2 \cdot {}^nC_2\, p^2 q^{n-2} + \cdots + n\, {}^nC_n\, p^n q^{n-n}$$

$$= npq^{n-1} + 2\frac{n!}{(n-2)!\,2!} p^2 q^{n-2} + \cdots + np^n.$$

$$= npq^{n-1} + n(n-1)p^2q^{n-2} + \frac{n(n-1)(n-2)}{2}p^3q^{n-3} + \cdots + np^n$$

$$= np\left[q^{n-1} + (n-1)pq^{n-2} + \frac{(n-1)(n-2)}{2}p^2q^{n-3} + \cdots + p^{n-1}\right]$$

$$= np[q+p]^{n-1}$$ ($\because$ By Binomial expression)

$$= np(1)^{n-1}$$

$$E(X) = np$$

Variance of a Binomial distribution

Proof: We know that, the probability of a Binomial distribution is

$$P(X=x) = n_{C_x} p^x q^{n-x}$$

Consider variance $= V(X) = \sigma^2 = E(X^2) - [E(X)]^2$

$$= \sum_{x=0}^{n} x^2 p(x) - \mu^2$$

$$= \sum_{x=0}^{n} (x^2 + x - x)\, p(x) - \mu^2$$

$$= \sum_{x=0}^{n} [(x^2 - x) + x]\, p(x) - \mu^2$$

$$= \sum_{x=0}^{n} [x(x-1) + x]\, p(x) - \mu^2$$

$$= \sum_{x=0}^{n} x(x-1)p(x) + \sum_{x=0}^{n} x\, p(x) - \mu^2$$

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$$= 0+0+2(1)P(2)+3(2)P(3)+\cdots+n(n-1)p^n+\mu-\mu^2$$

$$= 2(1)\,{}^nC_2\,p^2q^{n-2}+3(2)\,{}^nC_3\,p^3q^{n-3}+\cdots+\mu-\mu^2$$

$$= \cancel{2}.\frac{n!}{(n-2)!\cancel{2!}}p^2q^{n-2}+\cancel{6}\frac{n!}{(n-3)!\cancel{3!}}p^3q^{n-3}+\cdots+\mu-\mu^2$$

$$= n(n-1)p^2q^{n-2}+n(n-1)(n-2)p^3q^{n-3}+\cdots+n(n-1)p^n+\mu-\mu^2$$

$$= n(n-1)p^2\left[q^{n-2}+(n-2)p\,q^{n-3}+\cdots+p^{n-2}\right]+\mu-\mu^2$$

$$= n(n-1)p^2\left[q+p\right]^{n-2}+\mu-\mu^2$$

$$= n(n-1)p^2(1)^{n-2}+(np)-n^2p^2$$

$$= n(n-1)p^2+np-n^2p^2$$

$$= np\left[(n-1)p+1-np\right]$$

$$= np\left[1-p\right]$$

$$V(X)=\sigma^2=npq$$

$\Rightarrow$ Hence the standard deviation of the Binomial distribution $=\sigma=\sqrt{npq}$

1/8/19 | Recurrence Relation of Binomial Distribution :

Proof : We know that

$$P(x) = {}^nC_x p^x q^{n-x} \quad \text{---} \quad (1)$$

$$P(x+1) = {}^nC_{x+1} p^{x+1} q^{n-x-1} \quad \text{---} \quad (2)$$

Now Divide eq^n (2) with eq^n (1)

$$\frac{P(x+1)}{P(x)} = \frac{{}^nC_{x+1} p^{x+1} q^{n-x-1}}{{}^nC_x p^x q^{n-x}}$$

$$\frac{P(x+1)}{P(x)} = \frac{\frac{\cancel{n!}}{(n-x-1)!\,(x+1)!}\cancel{p^x}\cdot p\,\cancel{q^{n-x}}\cdot q^{-1}}{\frac{\cancel{n!}}{(n-x)!\,x!}\cancel{p^x}\,\cancel{q^{n-x}}}$$

$$= \frac{p}{q}\,\frac{(n-x)\cancel{(n-x-1)!}\,\cancel{x!}}{\cancel{(n-x-1)!}\,(x+1)\cancel{x!}}$$

$$\frac{P(x+1)}{P(x)} = \frac{p}{q}\,\frac{n-x}{x+1}$$

$$P(x+1) = \frac{p}{q}\cdot\frac{n-x}{x+1}\cdot P(x).$$

Q: A fair coin is tossed 6 times. Find the probability of getting 4 heads

Sol Let p = probability of getting head $= \frac{1}{2}$

q = not getting head $= 1-p = \frac{1}{2}$

No. of trails $n = 6$ and $x = 4$

W

2Q:

Sol

WKT $P(x) = {}^{n}C_{x}\, p^{x}\, q^{n-x}$

$P(4) = {}^{6}C_{4}\, p^{4}\, q^{6-4}$

$= {}^{6}C_{4} \left(\frac{1}{2}\right)^{4} \left(\frac{1}{2}\right)^{2}$

$= \frac{6!}{2!\,4!} \left(\frac{1}{16}\right)\left(\frac{1}{4}\right)$

64) 150 (0.2
128
220

$= \frac{15}{64}$

$P(4) = 0.23$

2Q: Determine the probability of getting sum '6' exactly 3 times in 7 throws with a pair of fair dice.

Sol: Here, In a single throw of a pair of fair dice, a sum of '6' can occur in 5 ways

Those are (1,5) (2,4) (3,3) (4,2) (5,1) out of 36 outcomes.

Let p = getting sum is 6 in 1 throw $= 5/36$

q = probability of getting sum $\neq 6$ $= 1 - p = \frac{31}{36}$

Here no. of trials $= 7 = n$ and

$x = 3$

Therefore probability of getting a '6' exactly 3 times in 7 throws is

$\therefore P(x) = {}^{n}C_{x}\, p^{x}\, q^{n-x}$

$P(3) = {}^{7}C_{3} \left(\frac{5}{36}\right)^{3} \left(\frac{31}{36}\right)^{4}$

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$$= \frac{7\times 6\times 5}{3\times 2\times 1} \times \frac{(5)^3 (31)^4}{(36)^7}$$

$$= \frac{7\times (5)^4 (31)^4}{(36)^7} = \frac{4.040}{7.836}$$

$$= 0.516$$

3/8/19

Q: A die is thrown 6 times, if getting an even number is a success, find the probabilities of

i) atleast one success

ii) $\leq$ three successes

iii) 4 successes

Sol: In a single throw of a die, getting an even number is 3 ways.

i.e 2, 4, 6.

Let p = probability of getting an even number $= \frac{3}{6} = 1/2$

$q = 1-p = 1/2$

Here, no. of trails $= n = 6$

probability of a Binomial distribution is $P(X=x) = P(x) = n_{c_x} p^x q^{n-x}$

i) P(atleast one success) $= P(X \geq 1)$

$= 1 - P(X=0)$

$= 1 - n_{c_0} p^0 q^{n-0}$

WKT $P(x) = {}^{n}C_{x}\, p^{x}\, q^{n-x}$

$$P(4) = {}^{6}C_{4}\, p^{4}\, q^{6-4}$$

$$= {}^{6}C_{4} \left(\tfrac{1}{2}\right)^{4} \left(\tfrac{1}{2}\right)^{2}$$

$$= \frac{6!}{2!\,4!} \left(\frac{1}{16}\right)\left(\frac{1}{4}\right)$$

64) 150 (0.2
128
220

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Here no. of trials $= 7 = n$ and

$x = 3$

Therefore probability of getting '6' exactly 3 times in 7 throws is

$\therefore P(x) = {}^{n}C_{x}\, p^{n}\, q^{n-x}$

$$P(3) = {}^{7}C_{3} \left(\frac{5}{36}\right)^{3} \left(\frac{31}{36}\right)^{4}$$

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$$= \frac{7\times\cancel{6}\times5}{\cancel{3}\times\cancel{2}\times1} \times \frac{(5)^3\,(31)^4}{(36)^7}$$

$$= \frac{7\times(5)^4\,(31)^4}{(36)^7} = \frac{4.040}{7.836}$$

$$= 0.516$$

3/8/19

Q: A die is thrown 6 times, if getting an even number is a success., find the probabilities of

i) atleast one success

ii) $\leq$ three successes

iii) 4 successes

Sol: In a single throw of a die, getting an even number is 3 ways.

i.e. 2, 4, 6.

Let p = probability of getting an even number $= \frac{3}{6} = \frac{1}{2}$

$q = 1-p = \frac{1}{2}$

Here, no. of trails $= n = 6$

probability of a Binomial distribution is $P(X=x) = P(x) = {}^nC_x\, p^x\, q^{n-x}$

i) P(atleast one success) $= P(X \geq 1)$

$= 1 - P(X=0)$

$= 1 - {}^nC_0\, p^0\, q^{n-0}$

$$= 1 - {}^6C_0 (1/2)^0 (1/2)^6$$

$$= 1 - \frac{1}{64}$$

$$= \frac{63}{64} = 0.9844$$

ii) $P(\leq 3 \text{ success}) = P(X=0) + P(X=1) + P(X=2) + P(X=3)$

$$= {}^6C_0 (1/2)^0 (1/2)^6 + {}^6C_1 (1/2)^1 (1/2)^5 + {}^6C_2 (1/2)^2 (1/2)^4 + {}^6C_3 (1/2)^3 (1/2)^3$$

$$= \frac{1}{64} + \frac{6}{64} + \frac{15}{64} + \frac{20}{64}$$

$$= \frac{42}{64} = \frac{21}{32} = 0.6562$$

$\frac{6!}{3!}$

$\frac{6\times5\times4}{3\times2\times1}$

iii) $P(4 \text{ success}) = P(X=4)$

$$= {}^6C_4 (1/2)^4 (1/2)^2$$

$$= \frac{15}{64} = 0.2344$$

$\frac{6\times5\times4\times3}{4\times3\times2\times1}$

Q: 10 coins are thrown simultaneously, find the probability of getting atleast

i) 7 heads

ii) 6 heads.

Sol: let p = probability of getting a head $= 1/2$

q = probability of not getting a head $= 1 - p = 1/2$

Here no. of trials $= n = 10$

The probability of Binomial distribution is $p(X=x) = p(x) = {}^nC_x p^x q^{n-x}$

$\Rightarrow \quad P(X=x) = {}^{10}C_x \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{10-x}$

i) Consider $p(\text{atleast } 7 \text{ heads}) = p(X \geq 7)$

$= p(X=7) + p(X=8) + p(X=9) + p(X=10)$

$= {}^{10}C_7 \left(\frac{1}{2}\right)^7 \left(\frac{1}{2}\right)^3 + {}^{10}C_8 \left(\frac{1}{2}\right)^8 \left(\frac{1}{2}\right)^2 + {}^{10}C_9 \left(\frac{1}{2}\right)^9 \left(\frac{1}{2}\right)^1$

$+ {}^{10}C_{10} \left(\frac{1}{2}\right)^{10} \left(\frac{1}{2}\right)^0$

$= \frac{24}{7 \times 1024} + \frac{45}{1024} + \frac{10}{1024} + \frac{1}{1024}$

$= \frac{59.4}{1024}$

$= 0.1719$

$\frac{10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4}{7 \times 6 \times 5 \times 4}$

$\frac{10 \times 9 \times 8 \times 7}{7 \times 6 \times 5}$ 24

$\frac{10 \times 9}{2 \times 1}$

3.4 30 56

$\frac{10 \times 9 \times 8 \times 7}{4 \times 3 \times 2 \times 1}$

ii) $p(\text{atleast } 6 \text{ heads}) = p(X \geq 6)$

$= p(X=6) + p(X=7) + p(X=8) + p(X=9) + p(X=10)$

$= {}^{10}C_6 \left(\frac{1}{2}\right)^6 \left(\frac{1}{2}\right)^4 + \frac{24}{7 \times 1024} + \frac{45}{1024} + \frac{10}{1024} + \frac{1}{1024}$

$= \frac{210}{1024} + \frac{3.4}{1024} + \frac{45}{1024} + \frac{10}{1024} + \frac{1}{1024}$

$= 0.376$

Q_2: If the probability of a defective bolt is $\frac{1}{8}$. find i) the mean

ii) Variance

for the distribution of defective bolts of 640

Sol: Let p = probability of getting defective bolt = $1/8$

$$q = 1-p = 1-\frac{1}{8} = 7/8$$

Here $n = 640$

i) Mean $= \mu = np = E(X)$

$$= (640)\left(\frac{1}{8}\right)$$

$$\mu = E(X) = 80$$

ii) Variance $= \sigma^2 = npq$

$$= (640)\left(\frac{1}{8}\right)\left(7/8\right)$$

$$V(X) = \sigma^2 = 70$$

Q_2: Two dice are thrown 120 times, find the average number of times in which the number on the first die exceeds the number on the second die.

when two dice thrown, possible outcomes = $\{(1,1)(1,2) \cdots\cdots (6,6)\}$

Sol: Let p = probability of getting a number on first die exceeds the number on second die = 15

Total possible outcomes = 36

$\Rightarrow P = 15/36 = 5/12$

Those are ~~(1,2)~~ (2,1) (3,1) (3,2) (4,1) (4,2)
(4,3) (5,1) (5,2) (5,3) (5,4)
(6,1) (6,2) (6,3) (6,4) (6,5) ~~(6,6)~~

$\therefore q = 1 - p = 1 - \frac{15}{36}$

$= \frac{21}{36}$

$q = 7/12$

Here, $n = 120$

~~Probability of a~~

i) Average number of times = mean = μ

$\mu = np$

$= (120)\left(\frac{5}{12}\right)$

$= 50$

Q: 6 dice are thrown 729 times, how many times do you ~~expect~~ except atleast 3 dice to show a ~~face~~ 5 or 6?

Sol Here $n = 6$ = no. of trials

Let p = probability of getting '5' or '6' on a die $= \frac{2}{6} = 1/3$

Then, $q = 1 - p = 2/3$

$\therefore$ probability of getting atleast 3 dice to show 5 or '6' is $P(X \geq 3) = P(X=3) + P(X=4) + P(X=5) + P(X=6)$

$= {}^6C_3\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^3 + {}^6C_4\left(\frac{1}{3}\right)^4\left(\frac{2}{3}\right)^2 + {}^6C_5\left(\frac{1}{3}\right)^5\left(\frac{2}{3}\right)^1 + {}^6C_6\left(\frac{1}{3}\right)^6\left(\frac{1}{3}\right)^0$

$= \frac{1}{3^6}\left[160+60+12+1\right]$

$P(X \geq 3) = \frac{233}{729}$

$\therefore$ The number of such cases in 729 times is $729 \times \frac{233}{729}$

Q: If 10% of the rivets produced by a machine are defective, find the probability that out of 5 rivets chosen at random

i) none will be defective

ii) one will be defective

iii) atmost two rivets will be defective

Sol: Let p = probability of defective rives = 10%.

$p = \frac{10}{100} = 0.1$

$q = 1 - p = 0.9$

and $n = 5$

But $P(X = x) = P(x) = {}^5C_x (p)^x (q)^{n-x}$ (from B.D:)

i) $P(\text{none will be defective}) = P(X=0)$

$= {}^5C_0 (0.1)^0 (0.9)^5$

$= (0.9)^5$

$= 0.59$

ii) $p(X = \text{one will be defective}) = P(X=1)$

$$= 5_{C_1} (0.1)^1 (0.9)^4$$

$$= 0.5 (0.9)^4$$

$$P(X=1) = 0.32$$

iii) $p(\text{atmost 2 defective}) = p(X \leq 2)$

$$= p(X=0) + p(X=1) + p(X=2)$$

$$= 5_{C_0} (0.1)^0 (0.9)^5 + 5_{C_1} (0.1)^1 (0.9)^4 + 5_{C_2} (0.1)^2 (0.9)^3$$

$$P(X \leq 2) = 0.98$$

Q2: Out of 800 families with 5 children each, how many would you expect to have i) 3 boys

ii) 5 girls

iii) either 2 or 3 boys

iv) atleast one boy.

Assume equal probabilities for boys and girls.

Sol: Let the number of boys in each family $= X$.

Consider, p = probability of each boy $= 1/2$

$$q = 1 - p = 1/2$$

No. of children $= n = 5$

But the probability of a Binomial distribution $= n_{C_x} p^x q^{n-x}$

$= {}^5C_x (1/2)^x (1/2)^{5-x}$

i) Consider $P(3 \text{ boys}) = P(X = 3)$

$= {}^5C_3 (1/2)^3 (1/2)^2$

$\frac{5\times4\times3}{3\times2}$ $= \frac{10}{32} = 0.3125$

$\therefore$ Expected number of families having 3 boys $= 800 \times 0.3125 = 250$

ii) Consider $P(5 \text{ girls}) = P(X = 0)$

$= {}^5C_0 (1/2)^5 (1/2)^0$

$= \frac{5}{32} =$

$\therefore$ Expected number of families having 5 girls $= 800 \times \frac{5}{32} = 25$

iii) Consider $P(X = \text{either } 2 \text{ or } 3)$

$= P(X = 2) + P(X = 3)$

$= {}^5C_2 (1/2)^2 (1/2)^3 + {}^5C_3 (1/2)^3 (1/2)^2$

$\frac{5\times4\times3}{3\times2}$ $= \frac{2}{2^5} + \frac{10}{2^5}$

$= \frac{12}{32}$

Expected number of families having 2 or 3 boys $= \cancel{800} \times \frac{12}{32}$

$= 500$

iv) Consider $P(\text{atleast one boys}) = P(X \geq 1)$

$$1 - P(X=0)$$

$$= 1 - 5C_0 (1/2)^0 (1/2)^5$$

$$= 1 - \frac{5}{32}$$

$$= \frac{27}{32}$$

∴ Expected number of families having atleast one boy $= 800 \times \frac{27}{32}$

$$= 775$$

Q: The mean and variance of a B.D are 4 and 4/3 respectively. Find $P(X \geq 1)$

Sol Given, Mean = 4 variance = npq

$\Rightarrow np = 4$ —(1) $\Rightarrow npq = \frac{4}{3}$ —(2)

from (1) & (2)

$$q = 1/3$$

$$\therefore p = 1 - 1/3 = 2/3$$

W.K.T $np = 4$

$$n\left(\frac{2}{3}\right) = 4$$

$$\Rightarrow n = 6$$

$$\therefore P(X \geq 1) = 1 - P(X=0)$$

$$= 1 - 6C_0 (2/3)^0 (1/3)^6$$

$$= 1 - \frac{6}{729}$$

$$= \frac{723}{729}$$

$$= 0.998$$

Q: A discrete random variable 'x' has the mean '6' and variance '2'. If it is assumed that the distribution is binomial find the probability that $5 \le X \le 7$

Sol: Given, Mean $= 6$ variance $= 2$

$np = 6$ — (1) $\qquad npq = 2$ — (2)

$\Rightarrow$ from (1) & (2).

$$q = 1/3$$

$$p = 2/3$$

$$\therefore np = 6$$

$$n(2/3) = 6$$

$$n = 9$$

$$\therefore P(5 \le X \le 7) = P(X=5) + P(X=6) + P(X=7)$$

$$= {}^9C_5 \left(\tfrac{2}{3}\right)^5 \left(\tfrac{1}{3}\right)^4 + {}^9C_6 \left(\tfrac{2}{3}\right)^6 \left(\tfrac{1}{3}\right)^3 + {}^9C_7 \left(\tfrac{2}{3}\right)^7 \left(\tfrac{1}{3}\right)^2$$

$$= \frac{1}{3^9}\left[4032 + \right.$$

$$=$$

9×8×7×6
×8×3×2×1

26
32
252
78
4032

4
8×7
2×1
84

Q: In a Binomial distribution consisting of 5 independent trials, probabilities of 1 & 2 success are 0.4096 and 0.2048 respectively find the parameter 'p' of the distribution

Sol Given $n = 5$

$$P(X=1) = 0.4096$$

$$5_{C_4} (P)^1 (q)^4 = 0.4096 \quad \text{--- (1)}$$

$$P(X=2) = 0.2048$$

$$5_{C_2} (P)^2 (q)^3 = 0.2048 \quad \text{--- (2)}$$

Divide (2) with (1)

$$\frac{5_{C_2}\, p^2\, q^3}{5_{C_4}\, P\, q^4} = \frac{0.2048}{0.4096}$$

104
25
174
128
80
107
535
20048
40

Binomial frequency distribution.

If 'n' independent trials constitute 1 experiment and this experiment is repeated N times then the frequency of 'x' successes ~~is~~ is $N \cdot nc_x p^x q^{n-x}$. Since the probabilities of 0, 1, 3 .... x ... n successes

in 'n' trials are given by the terms of binomial expansion of $(q+p)^n$. Therefore, in N sets of 'n' trials the theoritical frequencies of 0, 1, 2, 3 .... x ..... n successes will be given by the terms of expansion of $N \cdot (q+p)^n$.

The possible number of successes and their frequencies is called a binomial frequency distribution.

Q: Fit a binomial distribution to the following frequency distribution

x	0	1	2	3	4	5	6
f	13	25	52	58	32	16	4

Sol Here, the no. of trials $= n = 6$

and $N = \Sigma f_i = 13+25+52+58+32+16+4$

$N = 200$

We know, Mean of a B.D $= \dfrac{\Sigma f_i x_i}{\Sigma f_i}$

$$np = \frac{0(13)+1(25)+2(52)+3(58)+4(32)+5(16)+6(4)}{200}$$

$$6p = \frac{25+104+174+128+80+24}{200}$$

$$6p = \frac{535}{200}$$

$$p = \frac{535}{200 \times 6}$$

$$p = 0.44$$

and $q = 1-p = 1-0.44$

$$= 0.56$$

$\therefore$ The Binomial frequency Distribution is $N(q+p)^n$ then,

$$N(q+p)^n = 200(\underset{q}{0.56}+\underset{p}{0.44})^6$$

$$(q+p)^n = {}^nC_0\, p^0 q^n + {}^nC_1\, p^1 q^{n-1} + {}^nC_2\, p^2 q^{n-2} + \dots$$

$$= 200\left[{}^6C_0 (0.44)^0 (0.56)^6 + {}^6C_1 (0.44)^1 (0.56)^5 + {}^6C_2 (0.44)^2 (0.56)^4 + {}^6C_3 (0.44)^3 (0.56)^3 + {}^6C_4 (0.44)^4 (0.56)^2 + {}^6C_5 (0.44)^5 (0.56)^1 + {}^6C_6 \dots\right]$$

$$= 200\left[(0.030) + (2.64)(0.055) + (2.904)(\quad) + 0.3016 + 0.1821 + 0.05864 + 0.007866\right]$$

$\frac{6 \times 5}{2}$ $= 5.782 + 27.92 + 56.18 + 60.32 + 36.42 + 11.728 + 1.5732$

Therefore the successive terms in the expansion gives the expected or theoritical frequencies which are

(integers always)

x	0	1	2	3	4	5	6
Expected freq	6	28	56	60	36	12	2

Q: Fit a binomial distribution to the following data

x	0	1	2	3	4	5	$\cancel{x}$
f	2	14	20	34	22	8	

Sol

Here no. of trials $= n = 5$

$$N = \Sigma f_i = 2 + 14 + 20 + 34 + 22 + 8$$

$$N = 100$$

$$\text{Mean of a B.D} = \frac{\Sigma f_i x_i}{\Sigma f_i}$$

$$np = \frac{0 + 14 + 40 + 102 + 88 + 40}{100}$$

196
88
284

$$5p = \frac{284}{100}$$

$$p = \frac{284}{500}$$

$$p = 0.568$$

5) 284 (568

1.000
0.568
0.432

$$q = 1 - p = 1 - 0.568$$

$$q = 0.432$$

∴ The Binomial frequency distribution is $N(q+p)^n$ then

$$N(q+p)^n = 100\left[\underset{q}{0.432} + \underset{p}{0.568}\right]^5$$

$$= 100\Big[{}^5C_0 (0.568)^0 (0.432)^5 + {}^5C_1 + (0.568)^1 (0.432)^4 + {}^5C_2 (0.568)^2 (0.432)^3$$
$$+ {}^5C_3 (0.568)^3 (0.432)^2 + {}^5C_4 (0.568)^4 (0.432)^1 + {}^5C_5 (0.568)^5 (0.432)^0 \Big]$$

$$= 100\left[0.015 + 0.0989 + 0.26 + 0.0341 + 0.224 + 0.059\right]$$

$$= 1.5 + 9.89 + 26 + 34.1 + 22.4 + 5.9$$

$\therefore$ The successive terms in the expansion gives the theoritical frequencies which are

x	0	1	2	3	4	5
Expected freq (f)	1	10		34	22	6

Q: four coins are tossed 160 times the no. of times 'x' heads occur is given below

x	0	1	2	3	4
no. of times	8	34	69	43	6

fit a Binomial distribution to this data on the hypothesis that coins are unbiased.

Sol: Coins are unbiased

$$p = 1/2 \quad q = 1/2 \quad \text{and } n = 4$$

and $N = \Sigma f_i = 8 + 34 + 69 + 43 + 6 = 160$

By B.D we have $P(x) = n_{c_x} p^x q^{n-x}$ and recurrence relation of B.D is

$$P(x+1) = \frac{n-x}{x+1} \cdot \frac{p}{q} P(x)$$

Consider $P(X=0) = 4c_0 (1/2)^0 (1/2)^{4-0} = (1/2)^4 = 1/16$

ii) $P(0+1) = \frac{4-0}{0+1} \frac{(1/2)}{(1/2)} \cdot P(0)$

$P(1) = 4 \cdot (1/16) = 1/4$

and $P(1+1) = \frac{4-1}{1+1} \frac{(1/2)}{(1/2)} P(1) \Rightarrow P(2) = (3/2)(1/4)$

$= 3/8$

$P(2+1) = \frac{4-2}{2+1} \frac{(1/2)}{(1/2)} P(2) \Rightarrow P(3) = 1/4$

$P(3+1) = \frac{4-3}{3+1} \frac{(1/2)}{(1/2)} P(3) = 1/4 \cdot 1/4 = 1/16$

x	observed freq (f_i)	probability $P(x)$	Expected freq $f(x) = N \cdot P(x)$
0	8	1/16	$160(1/16) = 10$
1	34	1/4	$160(1/4) = 40$
2	69	3/8	$160(3/8) = 60$
3	43	1/4	$160(1/4) = 40$
4	6	1/16	$160(1/16) = 10$

Q: 7 coins are tossed and the no. of heads are noted. The experiment is repeated 128 times and the following distribution is obtained.

no. of heads	0	1	2	3	4	5	6	7
frequency	7	6	19	35	30	23	7	1

fit a Binomial distribution assuming

a) the coin is unbiased

b) the nature of the coin is not known

Sol a) $N = \sum f_i = 7+6+19+35+30+23+7+1$

$= 128$

$p = 1/2 \quad q = 1/2 \quad n = 7$

$$P(x+1) = \frac{n-x}{x+1} \frac{p}{q} P(x)$$

$$P(0) = {}^7C_0 (1/2)^0 (1/2)^{7-0} = (1/2)^7 = \frac{1}{128}$$

$$P(0+1) = \frac{7-0}{0+1} \frac{(1/2)}{(1/2)} P(0) = 7(1/128) = 7/128$$

$$P(1+1) = \frac{7-1}{1+1} \frac{(1/2)}{(1/2)} P(1) = 21/128$$

$$P(2+1) = \frac{7-2}{2+1} \frac{(1/2)}{(1/2)} P(2) = 35/128$$

$$P(3+1) = P(4) = 35/128$$

$$P(4+1) = P(5) = 21/128$$

$$P(5+1) = P(6) = 7/128 \qquad P(6+1) = P(7) = 1/128$$

x	observed freq (f_i)	probability	Expected freq $= N \cdot P(x)$
0	7	1/128	1
1	6	7/128	7
2	19	21/128	21
3	35	35/128	35
4	30	35/128	35
5	23	21/128	21
6	7	7/128	7
7	1	1/128	1

b)

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Variance of the poison distribution.

Proof: We know that probability of P.D is $p(x) = \frac{e^{-\lambda} \lambda^x}{x!}$

Consider variance, $\sigma^2 = E(x^2) - [E(x)]^2$

$$= \sum_{x=0}^{\infty} x^2 p(x) - \lambda^2$$

$$\cancel{= \sum_{x=0}^{\infty} [(x^2-1)+1] p(x) - \lambda^2}$$

$$= \sum_{x=0}^{\infty} (x^2-1) p(x) + \cancel{\sum_{x=0}^{\infty} p(x) - \lambda^2}$$

$$= \sum_{x=0}^{\infty} \cancel{x} \cdot x \cdot \frac{e^{-\lambda} \lambda^x}{\cancel{x}(x-1)!} - \lambda^2$$

$$= \sum_{x=0}^{\infty} x \, \frac{e^{-\lambda} \lambda^x}{(x-1)!} - \lambda^2$$

$$= \sum_{x=0}^{\infty} [x-1+1] \frac{e^{-\lambda} \lambda^x}{(x-1)!} - \lambda^2$$

$$= \sum_{x=0}^{\infty} (x-1) \frac{e^{-\lambda} \lambda^x}{(x-1)!} + \sum_{x=0}^{\infty} \frac{e^{-\lambda} \lambda^x}{(x-1)!} - \lambda^2$$

$$= \sum_{x=0}^{\infty} \frac{e^{-\lambda} \lambda^x}{(x-2)!} + \sum_{x=0}^{\infty} \frac{e^{-\lambda} \lambda^x}{(x-1)!} - \lambda^2$$

put $x-2=y$ in the first term

$x-1=z$ in the second term

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$$= \sum_{y+2=2}^{\infty} \frac{e^{-\lambda} \lambda^{y+2}}{y!} + \sum_{z+1=1}^{\infty} \frac{e^{-\lambda} \lambda^{z+1}}{z!} - \lambda^2$$

$$= \sum_{y=0}^{\infty} e^{-\lambda} \frac{\lambda^y \lambda^2}{y!} + \sum_{z=0}^{\infty} e^{-\lambda} \frac{\lambda^z \lambda}{z!} - \lambda^2$$

$$= e^{-\lambda} \lambda^2 \sum_{y=0}^{\infty} \frac{\lambda^y}{y!} + e^{-\lambda} \lambda \sum_{z=0}^{\infty} \frac{\lambda^z}{z!} - \lambda^2$$

$$= e^{-\lambda} \lambda^2 e^{\lambda} + e^{-\lambda} \lambda e^{\lambda} - \lambda^2$$

$$= \lambda^2 + \lambda(1-\lambda)$$

$$V(X) = \lambda$$

Mode of the poisson distribution.

Proof: Mode is the value of x for which probability $p(x)$ is maximum.

i.e. $p(x) \geq p(x+1)$ and $p(x) \geq p(x-1)$

Consider $p(x) \geq p(x+1)$

$$\frac{e^{-\lambda} \lambda^x}{x!} \geq \frac{e^{-\lambda} \lambda^{x+1}}{(x+1)!}$$

$$\Rightarrow \cancel{e^{-\lambda}} \frac{\cancel{\lambda^x}}{\cancel{x!}} \geq \frac{\cancel{e^{-\lambda}} \cancel{\lambda^x} \cdot \lambda}{x+1 \cancel{(x)!}}$$

$$\Rightarrow 1 \geq \frac{\lambda}{x+1}$$

$$\Rightarrow x+1 \geq \lambda$$

$$\Rightarrow x \geq \lambda - 1 \quad \text{--- } (1)$$

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Similarly, consider $P(x) \geq P(x-1)$

$$\frac{e^{-\lambda}\lambda^x}{x!} \geq \frac{e^{-\lambda}\lambda^{x-1}}{(x-1)!}$$

$$\frac{\cancel{e^{-\lambda}}\,\cancel{\lambda^x}}{x\cancel{(x-1)!}} \geq \frac{\cancel{e^{-\lambda}}\,\cancel{\lambda^x}\cdot\lambda^{-1}}{\cancel{(x-1)!}}$$

$$\frac{1}{x} \geq \frac{1}{\lambda}$$

$$\Rightarrow x \leq \lambda \quad \text{--- (2)}$$

from (1) & (2)

$$\therefore \lambda - 1 \leq x \leq \lambda$$

$\therefore$ Mode of the poisson distribution lies b/w $\lambda-1$ and λ.

Case 1) If λ is an integer then $\lambda-1$ is also an integer, so we have two max values and the distribution is bimodal and two modes are $\lambda-1, \lambda$

Case 2) If λ is not an integer the mode of the poisson distribution is integral part of λ.

Recurrence Relation of the P.D :-

proof Consider : $P(x) = e^{-\lambda}\frac{\lambda^x}{x!}$ --- (1)

$$P(x+1) = e^{-\lambda}\frac{\lambda^{x+1}}{(x+1)!} \quad \text{--- (2)}$$

Divide eq^n ① by eq^n ②

$$\frac{P(x)}{P(x+1)} = \cancel{e^{-\lambda}} \frac{\lambda^x}{x!} \cdot \frac{(x+1)!}{\cancel{e^{-\lambda}} \lambda^{x+1}}$$

$$\frac{P(x)}{P(x+1)} = \frac{(x+1)\cancel{x!}}{\cancel{x!} \cdot \lambda}$$

$$\boxed{P(x+1) = \left(\frac{\lambda}{x+1}\right) P(x)}$$

put $x = x-1$

(or)

$$\boxed{P(x) = \frac{\lambda}{x} P(x-1)}$$

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Poisson Distribution :

Poisson distribution due to french mathematician Simen Denis . on 1837 is a Discrete probability distribution

Derivation of the P.D :

A p.d can be derived as a limiting case of binomial distribution under the conditions that

i) 'p' is very small ($p \to 0$)

ii) n is very large ($n \to \infty$)

iii) $np = \lambda$ (λ is finite)

proof : We know the probability of B-D is

$$P(x) = {}^nc_x\, p^x q^{n-x}$$

$$= {}^nc_x\, p^x (1-p)^{n-x}$$

$$= \frac{n!}{(n-x)!\, x!} p^x \frac{(1-p)^n}{(1-p)^x}$$

$$= \frac{n(n-1)(n-2)\cdots(n-x+2)(n-x+1)(n-x)!}{(n-x)!\,x!} p^x \frac{(1-p)^n}{(1-p)^x}$$

$$P(x) = \frac{n(n-1)(n-2)\cdots(n-x+1)}{x!} p^x \frac{(1-p)^n}{(1-p)^x}$$

Put $np = \lambda \Rightarrow n = \lambda/p$

$$P(x) = \frac{(\lambda/p)(\lambda/p - 1)(\lambda/p - 2)\cdots(\lambda/p - x+1)}{x!} p^x \frac{(1-p)^n}{(1-p)^x}$$

$$= \frac{\lambda(\lambda-p)(\lambda-2p)\cdots(\lambda-(x-1)p)}{\cancel{p^x}\cdot x!} \cancel{p^x} \frac{(1-\lambda/n)^n}{(1-p)^x}$$

We have i) $n \to \infty$

ii) $p \to 0$ iii) $np = \lambda$ (finite)

$$\Rightarrow p(x) = \lim_{n\to\infty} \lim_{p\to 0} \frac{\lambda(\lambda-p)(\lambda-2p)\cdots[\lambda-(x-1)p]}{x!} \frac{(1-\lambda/n)^n}{(1-p)^x}$$

$$= \frac{\lambda^x}{x!} \frac{\lim_{n\to\infty}(1-\lambda/n)^n}{\lim_{p\to 0}(1-p)^x}$$

$$= \frac{\lambda^x}{x!} e^{-\lambda}$$

$$\therefore p(x) = \frac{e^{-\lambda}\lambda^x}{x!}$$

$\therefore$ $p(x)$ = probability of "x" successes $= \frac{e^{-\lambda}\lambda^x}{x!}$

This is known as poisson distribution.

putting $x = 0, 1, 2, 3 \ldots$ the probabilities of $0, 1, 2, 3 \ldots r$ successes are given by $e^{-\lambda}$, $e^{-\lambda}.\lambda$, $e^{-\lambda}\frac{\lambda^2}{2!}$ --- respectively

where $\lambda > 0$ is a parameter

⇒ The sum of these probabilities is unity as it should be

$$P(x,\lambda) = P(X=\lambda) = \begin{cases} e^{-\lambda}\frac{\lambda^x}{x!} & ; x = 0,1,2 \ldots \\ 0 & ; \text{otherwise} \end{cases}$$

⇒ A Random variable 'X' is said to follow a poisson distribution if it assumes only non-negative values and its probability density funct' is given by

Here $\lambda > 0$ is called the parameter of the distribution

Eg: 1) the no. of printing mistakes per page in a large text.

2) The no. of telephone calls per minute at a switch board.

3) The no. of cars passing a certain point in 1 min.

4) the no. of persons born blind per year in a large city.

Conditions of poisson distribution:

1) The no. of trials (n) is large

2) The probability of successes is very small [very close to zero]

3) $np = \lambda$ is finite

Constants of the poisson distribution

1) Mean:

We know that $p(x) = e^{-\lambda} \frac{\lambda^x}{x!}$

$$\text{Mean} = f(x) = \sum_{x=0}^{\infty} x\, p(x)$$

$$= \sum_{x=0}^{\infty} x\, e^{-\lambda} \frac{\lambda^x}{x!}$$

$$= \sum_{x=0}^{\infty} x\, e^{-\lambda} \frac{\lambda^x}{x(x-1)!}$$

$$= \sum_{x=1}^{\infty} e^{-\lambda} \frac{\lambda^x}{(x-1)!}$$

put $x-1 = y$

$x = y+1$

$$\text{Mean} = f(x) = \sum_{y+1=1}^{\infty} e^{-\lambda} \frac{\lambda^{y+1}}{y!}$$

$$= \sum_{y=0}^{\infty} e^{-\lambda} \frac{\lambda^{y} \cdot \lambda}{y!}$$

$$= \lambda e^{-\lambda} e^{\lambda}$$

$$\text{Mean} = E(X) = \lambda$$

Q:

⟹ If the probability that an individual suffers a bad reaction from a certain injection is 0.001, Determine the probability that out of 2000 individuals

i) Exactly 3

ii) More than 2 individuals

iii) None

iv) More than 1 individual suffer a bad reaction.

Sol: Given that $n = 2000$ (very large)

$P = 0.0001$ (very small that is tends to '0')

and $\lambda = np = 2$ is finite

All the conditions of P.D holds.

$\therefore$ we can apply P.D.

The probability of P.D is

$$P(x) = \frac{e^{-\lambda} \lambda^x}{x!} = \frac{e^{-2}\, 2^x}{x!}$$

i) Consider, $P(x = \text{exactly } 3) = P(X = 3)$

$$= e^{-2} \frac{2^3}{3!}$$

$$= e^{-2}\left(\frac{4}{3}\right) = \frac{4}{3}(0.1353)$$

$$= (1.33)(0.1353)$$

$$P(X = 3) = 0.1804.$$

ii) Consider $P(X > 2) = 1 - P(X \le 2)$

$$= 1 - [P(X=0) + P(X=1) + P(X=2)]$$

$$= 1 - \left[e^{-2}\frac{2^0}{0!} + e^{-2}\frac{2^1}{1!} + e^{-2}\frac{2^2}{2!}\right]$$

$2e^{-2}$ $\quad = 1 - [0.1353 + 0.278 + 0.278]$

$\frac{2}{e^2}$ $\quad = 1 - 0.69$

$$= 0.31$$

iii) Consider $P(X = \text{none}) = P(X = 0)$

$$= e^{-2}\frac{2^0}{0!}$$

$$= \frac{1}{e^2} = 0.1353$$

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iv) Consider $P(X > 1) = 1 - P(X \leq 1)$

$$= 1 - [P(X=0) + P(X=1)]$$

$$= 1 - [0.1353 + e^{-2}\, 2]$$

$$= 1 - [0.1353 + 0.278]$$

$$= 1 - 0.413$$

$$= 0.587$$

Q: A Hospital switch board receives an average of 4 emergence calls in a 10 min interval. What is the probability that i) there are atmost 2 emergency calls in a 10 min interval

ii) There are exactly 3 calls in 10 min interval.

Sol Given that $\lambda = 4$ (per 10 min)

We have the probability of P.D is

$$P(x) = \frac{e^{-\lambda} \lambda^x}{x!} = \frac{e^{-4}\, 4^x}{x!}$$

i) Consider $P(x = \text{atmost 2 calls})$

$$= P(X \leq 2)$$

$$= P(X=0) + P(X=1) + P(X=2)$$

$$= e^{-4} + e^{-4}(4) + e^{-4}(8)$$

$$= 0.0183 + 0.073 + 0.1465$$

$$P(X \leq 2) = 0.238$$

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ii) $P(X = \text{exactly } 3) = P(X=3)$

$$= e^{-4}\left(\frac{64}{6}\right)$$

$$= \cancel{(0.0183)(7.33)}$$

$$= \cancel{0.0243}(0.0183)(10.66)$$

$$= 0.195.$$

Q: A manufacturer knows that the condensers he makes contain an average 1% defectives. He packs them in boxes of 100. What is the probability that a box picked at random will contain 3 or ~~more~~ more faulty condensers

Sol: Given that $n = 100$ $\qquad \lambda = np$

$p = 1\% = 0.01$ (very small)

We know, the probability of PD is

$$P(x) = \frac{e^{-\lambda}\lambda^x}{x!} = \frac{e^{-1}(1)^x}{x!}$$

i) $P(X \geq 3) = 1 - P(X<3)$

$$= 1 - [P(X=0) + P(X=1) + P(X=2)]$$

$$= 1 - [e^{-1} + e^{-1} + e^{-1}(0.5)]$$

$$= 1 - [0.3678 + 0.3678 + 0.1839]$$

$$= 1 - 0.9195$$

$$P(X \geq 3) = 0.0805$$

Q: If a Bank received on the average 6 bad checks per day, find the probability that it will receive 4 bad checks on any given day

Sol Given $\lambda = 6$, $x = 4$

P.D is $p(x) = e^{-\lambda} \frac{\lambda^x}{x!}$

$$= e^{-6} \frac{6^4}{4!}$$

$$= e^{-6} \left(\frac{1296}{24}\right)$$

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$$= e^{-6}(54) = 0.1338$$

Q: Average number of accidents on any day on a national Highway is 1.8. Determine the probability that the number of accidents are

i) Atleast 1 ii) Atmost 1

Sol Given $\lambda = 1.8$

We have the probability of poisson distribution $P.D = p(x) = e^{-\lambda} \cdot \frac{\lambda^x}{x!}$

$$p(x) = e^{-1.8} \frac{(1.8)^x}{x!}$$

i) $P(X \geq 1) = 1 - P(X = 0)$

$$= 1 - e^{-1.8} \cdot \frac{(1.8)^0}{0!}$$

$$= 1 - 0.1652$$

$P(X \geq 1) = 0.8348$

iii) $P(X \leq 1) = P(X=0) + P(X=1)$

$$= e^{-1.8} \frac{(1.8)^0}{0!} + e^{-1.8} \frac{(1.8)^1}{1!}$$

$$= 0.1652 + 0.2975$$

$$P(X \leq 1) = 0.4627$$

Q2: If a RV has a poisson distribution such that $P(1) = P(2)$: find i) Mean of the distribution ii) $P(4)$ iii) $P(X \geq 1)$ iv) $P(1 < X < 4)$

let $P(x) = \frac{e^{-\lambda} \lambda^x}{x!}$

Sol Given $P(1) = P(2)$

$$\cancel{e^{-\lambda}} \frac{\lambda^1}{1!} = \cancel{e^{-\lambda}} \frac{\lambda^2}{2!}$$

$$2\lambda = \lambda^2$$

$$\cancel{\lambda = 0} \quad \lambda^2 - 2\lambda = 0$$

$$\lambda(\lambda - 2) = 0$$

$$\lambda = 0, 2$$

$\because \lambda$ is $>$ zero.

$$\boxed{\lambda = 2}$$

i) Mean of the distribution $= \lambda = 2$

ii) $P(4) = e^{-2} \frac{2^4}{4!} = e^{-2}\left(\frac{16}{24}\right)$

$$= e^{-2}(0.6666)$$

$$P(4) = 0.0902$$

iii) $P(X \geq 1) = 1 - P(X=0)$

$$= 1 - e^{-2} \frac{2^0}{0!}$$

$$= 1 - e^{-2}$$

$$= 1 - 0.1353$$

$$P(X \geq 1) = 0.8647$$

iv) $P(1 < X < 4) = P(X=2) + P(X=3)$

$$= e^{-\lambda} \frac{\lambda^2}{2!} + e^{-\lambda} \frac{\lambda^3}{3!}$$

$$= e^{-2} \frac{2^2}{2!} + e^{-2} \frac{2^3}{6}$$

$$= e^{-2} \left[2 + \frac{4}{3} \right]$$

$$= e^{-2} (3.3)$$

$$P(1 < X < 4) = 0.4511$$

Q2: Using Recurrence formula. find the probabilities when $X = 0, 1, 2, 3, 4, 5,$ if the mean of the distribution is 3.

Given, $\lambda = 3$

Sol: We know that, Recurrence formula

$$P(x+1) = \frac{\lambda}{x+1} P(x) \quad \text{--- (1)}$$

Let probability of P.D is

$$P(x) = e^{-\lambda} \frac{\lambda^x}{x!} = e^{-3} \frac{3^x}{x!} \quad \text{--- (2)}$$

Put $x=0$ in eq^n (2)

$$P(0) = e^{-3} \frac{3^0}{0!}$$

$$P(0) = 0.0497$$

put $x=0$ in eq^n (1)

$$P(0+1) = \frac{3}{0+1} P(0)$$

$$\Rightarrow P(1) = 3(0.0497)$$

$$\Rightarrow P(1) = 0.1491$$

put $x=1$ in eq^n (1)

$$P(1+1) = \frac{3}{2} P(1)$$

$$\Rightarrow P(2) = \frac{3}{2}(0.1491)$$

$$\Rightarrow P(2) = 0.2236$$

put $x=2$ in eq^n (1)

$$\Rightarrow P(2+1) = \frac{3}{3} P(2)$$

$$\Rightarrow P(3) = 0.2236.$$

put $x=3$ in eq^n (1)

$$P(3+1) = \frac{3}{4} P(3)$$

$$\Rightarrow P(4) = 0.1677$$

put $x=4$ in eq^n (1)

$$P(4+1) = \frac{3}{5} P(4)$$

$$\Rightarrow P(5) = 0.10062$$

Q: If a poisson distribution is such that $P(X=1)\cdot\frac{3}{2} = P(X=3)$. find

i) $P(X \geq 1)$ ii) $P(X \leq 3)$ iii) $P(2 \leq X \leq 5)$

Sol: Given $P(X=1)\cdot\frac{3}{2} = P(X=3)$

We know that $P\cdot D = p(x) = e^{-\lambda}\frac{\lambda^x}{x!}$

$$\cancel{e^{-\lambda}}\frac{\lambda^1}{1!}\left(\frac{3}{2}\right) = \cancel{e^{-\lambda}}\frac{\lambda^3}{3!}$$

$$\frac{3}{\cancel{2}}\lambda = \frac{\lambda^3}{\cancel{6}\,3}$$

$$\lambda^3 - 9\lambda = 0$$

$$\lambda(\lambda^2 - 9) = 0$$

$$\lambda = 0 \quad \lambda = \pm 3$$

$\because$ since $\lambda > 0$

$$\boxed{\lambda = 3}$$

i) $P(X \geq 1) = 1 - P(X=0)$

$$= 1 - e^{-\lambda}\frac{\lambda^x}{x!}$$

$$= 1 - e^{-3}\frac{3^0}{0!}$$

$$= 1 - 0.0497$$

$$P(X \geq 1) = 0.9503$$

$\frac{\cancel{13}\times 12^3}{\cancel{12\times 11}}$

ii) $P(X \leq 3) = P(X=0) + P(X=1) + P(X=2) + P(X=3)$

$$= e^{-3} \frac{3^0}{0!} + e^{-3} \frac{3^1}{1!} + e^{-3} \frac{3^2}{2!} + e^{-3} \frac{3^3}{3!}$$

$$= e^{-3} \left[1 + 3 + 4.5 + 4.5\right]$$

$$= 13 (e^{-3})$$

$P(X \leq 3) = 0.6472$

iii) $P(2 \leq X \leq 5) = P(X=2) + P(X=3) + P(X=4) + P(X=5)$

$$= e^{-3} \frac{3^2}{2!} + e^{-3} \frac{3^3}{3!} + e^{-3} \frac{3^4}{4!} + e^{-3} \frac{3^5}{5!}$$

$$= e^{-3} \left[4.5 + 4.5 + 3.37 + 2.025\right]$$

$$= e^{-3} (14.395)$$

$$= 0.7166$$

Q_2: If two cards are drawn from a pack of 52 cards which are diamonds, using Poisson distribution, ~~of~~ find the probability of getting 2 diamonds atleast 3 times in 51 consecutive trials of 2 cards drawing each time.

Sol: Given $n = 51$ = no. of trials.

The probability of getting 2 diamonds from a pack of 52 cards $= p = \frac{^{13}C_2}{^{52}C_2}$

$$p = \frac{13 \times 12}{52 \times 51} = 0.058$$

$\lambda = np = \text{mean}$

$= 51(0.058)$

$\lambda = 3$

We know that probability of poisson distribution is $P(x) = \frac{e^{-\lambda} \lambda^{x}}{x!}$

i) $P(\text{atleast } 3) = P(X \geq 3)$

$= 1 - [P(X=0) + P(X=1) + P(X=2)]$

$= 1 - \left[\frac{e^{-3} 3^{0}}{0!} + \frac{e^{-3} 3^{1}}{1!} + \frac{e^{-3} 3^{2}}{2!}\right]$

$= 1 - e^{-3}[1 + 3 + 4.5]$

$= 1 - e^{-3}[8.5]$

$= 1 - 0.4231$

$P(X \geq 3) = 0.5769$

Q: If 'X' is a poisson variate such that $3P(x=4) = \frac{1}{2}P(x=2) + P(x=0)$, find i) Mean of 'X'.

ii) $P(X \leq 2)$

Sol: P.D $P(x) = \frac{e^{-\lambda} \lambda^{x}}{x!}$

Given $3P(x=4) = \frac{1}{2}P(x=2) + P(x=0)$

$6P(x=4) = P(x=2) + 2P(x=0)$

$6 \frac{e^{-\lambda} \lambda^{4}}{24} = \frac{e^{-\lambda} \lambda^{2}}{2} + 2e^{-\lambda}$

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$$\frac{\lambda^4}{4} = \frac{\lambda^2}{2} + 2$$

$$\frac{\lambda^4}{2} = \lambda^2 + 4$$

$$\lambda^4 = 2\lambda^2 + 8$$

$$\lambda^4 - 2\lambda^2 = 8$$

$$\cancel{\lambda^2(\lambda^2 - 2) = 8}$$

$$\cancel{\lambda^2 = 4} \quad \cancel{\lambda^2 - 2 = 2}$$

$$\boxed{\lambda = 2}$$

i) Mean $= \lambda = 2$

ii) $P(X \leq 2) = P(X=0) + P(X=1) + P(X=2)$

$$= e^{-2}\frac{2^0}{0!} + e^{-2}\frac{2^1}{1!} + e^{-2}\frac{2^2}{2!}$$

$$= e^{-2}[1 + 2 + 2]$$

$$= 5e^{-2}$$

$P(X \leq 2) = 0.6766$

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Q: Fit a poisson distribution for the following data and calculate the expected frequencies

x	0	1	2	3	4
$f(x)$	109	65	22	3	1

Sol let N = total frequency $= \Sigma f_i$

$$N = 109 + 65 + 22 + 3 + 1$$

$$N = 200$$

and Mean of the P.D is λ

$$\text{Mean} = \frac{\Sigma f_i x_i}{\Sigma f_i}$$

$$\lambda = \frac{0(109) + 65 + 2(22) + 3(3) + 4}{200}$$

$$\lambda = \frac{122}{200}$$

$$\lambda = 0.61$$

$\therefore$ the Expected frequency of P.D is

$$N\,P(x) = N \frac{e^{-\lambda} \lambda^x}{x!}$$

$$N \frac{e^{-\lambda} \lambda^x}{x!} = 200\, e^{-0.61} \frac{(0.61)^x}{x!}$$

put $x=0$; $200\, e^{-0.61} \frac{(0.61)^0}{0!} = 108.67$

put $x=1$; $200 e^{-0.61} \frac{(0.61)^1}{1!} = 66.28$

Put $x=2$; $200 e^{-0.61} \frac{(0.61)^2}{2!} = 20.21$

Put $x = 3$; $200\, e^{-0.61} \frac{(0.61)^3}{3!} = 4.11$

Put $x = 4$; $200\, e^{-0.61} \frac{(0.6)^4}{4!} = 0.62$

The expected frequencies are, 108.67, 66.28, 20.21, 4.11 & 0.62

X	0	1	2	3	4
f(x)	109	67	20	4	1

Here the frequencies are always integers, by converting them to integers we get

Q: fit a P.D to the following data.

X	0	1	2	3	4	5
f(x)	142	156	69	27	5	1

Sol: Let N = total frequency = Σf_i

$N = 142 + 156 + 69 + 27 + 5 + 1$

$N = 400$

Mean of P.D is

$$\text{Mean} = \frac{\Sigma f_i x_i}{\Sigma f_i}$$

$$\text{Mean} = \frac{0 + 156 + 138 + 81 + 20 + 5}{400}$$

$$\lambda = \frac{400}{400}$$

$$\lambda = 1$$

Expected frequency of P.D is

$$N\,P(x) = \frac{N e^{-\lambda} \lambda^x}{x!}$$

$$= 400\, e^{-1} \cdot \frac{(1)^x}{x!}$$

put $x=0$; $400\, e^{-1} \frac{(1)^0}{0!} = 147.15$

put $x=1$; $400\, e^{-1} \frac{(1)^1}{1!} = 147.15$

put $x=2$; $400\, e^{-1} \frac{(1)^2}{2!} = 73.57$

put $x=3$; $400\, e^{-1} \frac{(1)^3}{3!} = 24.52$

put $x=4$; $400\, e^{-1} \frac{(1)^4}{4!} = 6.12$

put $x=5$, $400\, e^{-1} \frac{(1)^5}{5!} = 1.2$

The expected frequencies are 147.15, 147.15, 73.57, 24.52, 6.12, 1.2

X	0	1	2	3	4	5
f(x)	148	148	74	25	6	1

Q: The distribution of typing mistakes committed by a typist is given below ~~Assuming~~ the distribution to be poisson find the expected frequencies.

X	0	1	2	3	4	5
f(x)	42	33	14	6	4	1

Sol: Let N = total frequency = 100

$$\text{Mean} = \frac{\Sigma f_i x_i}{\Sigma f_i} = \frac{33+28+18+16+5}{100}$$

$\lambda = 1$

The Expected frequency of P.D is

$$= N \cdot e^{-\lambda} \frac{\lambda^x}{x!}$$

$$= 100 \cdot e^{-1} \cdot \frac{(1)^x}{x!}$$

put $x = 0$; $100 e^{-1} \frac{(1)^0}{0!} = 36.78$

put $x = 1$; $100 e^{-1} \frac{(1)^1}{1!} = 36.78$

put $x = 2$; $100 e^{-1} \frac{(1)^2}{2!} = 18.39$

put $x = 3$; $100 e^{-1} \frac{(1)^3}{3!} = 6.1$

put $x = 4$; $100 e^{-1} \frac{(1)^4}{4!} = 1057$

put $x = 5$; $100 e^{-1} \frac{(1)^5}{5!} = 0.305$

The expected frequencies are 36.78, 36.78, 18.39, 6.1, 1.5, 0.3

x	0	1	2	3	4	5
f(x)	37	37	18	6	2	1

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Normal Distribution:

The Normal distribution was first discovered by English mathematician De-moivere in 1733. further defined by french mathematician Laplace in 1774 and independently by Karl- friedrich Gauss.

→ "Normal Distribution" is also known as "Gaussian Distribution".

→ It is another limiting form of the binomial distribution for large values of n when neither 'p' nor 'q' is very small.

Defination: A RV 'X' is said to have a normal distribution, if its density function or probability distribution is given by

$$f(x, \mu, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \quad \text{where} \quad (1)$$

$$-\infty < x < \infty, \quad -\infty < \mu < \infty, \quad \sigma > 0$$

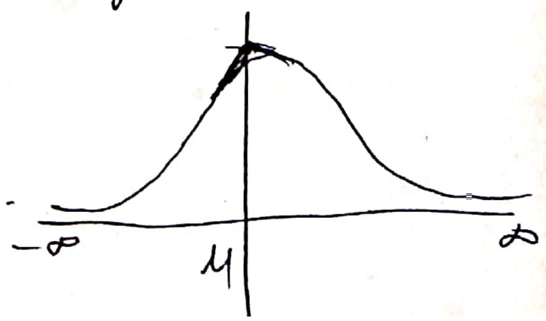
Where μ = Mean

σ = Standard deviation are two parameters of N.D

→ The RV 'X' is then said to be a Normal RV (or) Normal variate

→ The curve representing the N.D eqn (1) is called Normal curve and that total area bounded by the curve and x-axis is '1'. i.e,

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$



Thus the probability of $a < x < b$ is area under the vertical lines $x = a$ and $x = b$ i.e,

$$P(a < x < b) = \int_a^b f(x)\,dx$$

Note: A RV 'X' with mean 'μ' and variance 'σ^2' and following the probability law eqⁿ ① is expressed by

$$X \sim N(\mu, \sigma^2)$$

follows the   with parameters.

Normal distribution as a limiting form of Binomial distribution:

N.D is a limiting case of B.D under the following conditions.

i) The number of trials 'n' is independently large i.e, $n \to \infty$

ii) Neither 'p' nor 'q' is very small

Constants of N.D:-

① Mean of N.D:

proof: The probability function of N.D is

$$f(x,\mu,\sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{-1/2\left(\frac{x-\mu}{\sigma}\right)^2} \quad \text{— ①}$$

$$\text{Mean} = \mu = \int_{-\infty}^{\infty} x\, f(x)\, dx$$

$$= \int_{-\infty}^{\infty} x \frac{1}{\sigma\sqrt{2\pi}} e^{-1/2\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$\text{Mean} = \mu = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} x \, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx \quad \text{--- (2)}$$

let $\frac{x-\mu}{\sigma} = z$

diff w.r.to z, $x = \sigma z + \mu$

$$\frac{1}{\sigma}\frac{dx}{dz} - 0 = 1$$

$$dx = \sigma \, dz$$

eqⁿ (2) $\Rightarrow$ $\text{Mean} = \mu = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (\mu + \sigma z) e^{-\frac{z^2}{2}} \sigma \, dz$

$$= \frac{\sigma}{\sigma\sqrt{2\pi}} \left[\underbrace{\int_{-\infty}^{\infty} \mu e^{-z^2/2} dz}_{\text{even}} + \underbrace{\int_{-\infty}^{\infty} \sigma z e^{-z^2/2} dz}_{\text{odd}} \right]$$

$$= \frac{\mu}{\sqrt{2\pi}} \left[\int_{-\infty}^{\infty} e^{-z^2/2} dz \right]$$

$$= \frac{2\mu}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-z^2/2} dz$$

$\Rightarrow$ $\cancel{\text{put } \frac{z^2}{2} = t}$

$\cancel{z = \sqrt{2t}}$

$\cancel{\frac{dz}{dt}}$

By using γ function, $\int_{0}^{\infty} e^{-z^2/2} dz = \sqrt{\frac{\pi}{2}}$

$$\Rightarrow \frac{2\mu}{\sqrt{2\pi}} \left(\sqrt{\frac{\pi}{2}} \right)$$

$\boxed{\text{Mean} = \mu}$

$\boxed{\Gamma(n) = \int_{0}^{\infty} e^{-x} x^{n-1} dx}$

Therefore Mean $(\mu) = \mu$

Therefore, Now change the mean of the N.D with b

$\therefore$ Mean $= \mu = b$.

(2) Variance of a normal distribution :-

proof: The p.d.f of N.D is

$$f(x,\mu,\sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \quad \text{—— (1)}$$

Consider $V(x) = \sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx$

$$= \int_{-\infty}^{\infty} (x-b)^2 \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-b}{\sigma}\right)^2} dx$$

put $\frac{x-b}{\sigma} = z$

$x-b = \sigma z$

$\frac{1}{\sigma} \left| \frac{dx}{dz} \right. = 1 \Rightarrow dx = \sigma dz$

$$V(x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (\sigma z)^2 e^{-\frac{1}{2}(z^2)} \sigma dz$$

$$= \frac{\sigma^3}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} z^2 e^{-z^2/2} dz$$

even

$$V(x) = \frac{2\sigma^2}{\sqrt{2\pi}} \int_{0}^{\infty} z^2 e^{-z^2/2} dz$$

$\frac{z^2}{2} = t$

$z = \sqrt{2t}$

$\frac{2z}{2} \frac{dz}{dt} = 1 \Rightarrow dz = \frac{1}{z} dt$

$dz = \frac{1}{\sqrt{2t}} dt$

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$$\Rightarrow V(X) = \frac{2\sigma^2}{\sqrt{2\pi}} \int_0^\infty 2t\, e^{-t} \frac{1}{\sqrt{2t}}\, dt$$

$$= \frac{4\sigma^2}{\sqrt{2\pi}} \frac{1}{\sqrt{2}} \int_0^\infty e^{-t}\, t^{1-1/2}\, dt$$

$$= \frac{4\sigma^2}{\sqrt{2\pi}} \frac{1}{\sqrt{2}} \int_0^\infty e^{-t}\, t^{1/2}\, dt.$$

$$= \frac{4\sigma^2}{\sqrt{2\pi}} \frac{1}{\sqrt{2}} \int_0^\infty e^{-t}\, t^{3/2-1}\, dt$$

$$= \frac{4\sigma^2}{\sqrt{2\pi}} \frac{1}{\sqrt{2}} \Gamma\left(\tfrac{3}{2}\right)$$

$$= \frac{4\sigma^2}{\sqrt{2\pi}} \frac{1}{\sqrt{2}} \left(\frac{3}{2}-1\right) \Gamma\left(\frac{3}{2}-1\right)$$

$$= \frac{4\sigma^2}{\sqrt{2\pi}} \frac{1}{2\sqrt{2}} \Gamma\left(\tfrac{1}{2}\right)$$

$$= \frac{4\sigma^2}{4\sqrt{\pi}} \sqrt{\pi}$$

$$V(X) = \sigma^2$$

Mode of the N.D :

Proof : Let mode is the value of 'x' for which $f(x)$ is maximum.

i.e, for it sol^n, $f'(x)=0$ & $f''(x)<0$

Consider $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-1/2\left(\frac{x-\mu}{\sigma}\right)^2}$ ——— (1)

Diff w.r to x.

$$f'(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-1/2\left(\frac{x-\mu}{\sigma}\right)^2} \frac{-1}{2}\left(2\left(\frac{x-\mu}{\sigma}\right)\right)$$

$$f'(x) = \frac{-1}{\sigma\sqrt{2\pi}} e^{-1/2\left(\frac{x-\mu}{\sigma}\right)^2} \left(\frac{x-\mu}{\sigma}\right)$$

$$f'(x) = -f(x)\left(\frac{x-\mu}{\sigma}\right)$$

Consider $f'(x) = 0$

$$-f(x)\left(\frac{x-\mu}{\sigma}\right) = 0$$

$$\boxed{x = \mu}$$

and $f''(x) = -\left[f(x)\frac{1}{\sigma} + f'(x)\left(\frac{x-\mu}{\sigma}\right)\right]$

$$= -\left[\frac{1}{\sigma^2\sqrt{2\pi}} e^{-1/2\left(\frac{x-\mu}{\sigma}\right)^2} - \frac{1}{\sigma\sqrt{2\pi}} e^{-1/2\left(\frac{x-\mu}{\sigma}\right)^2} \left(\frac{x-\mu}{\sigma}\right)\right]$$

put $x = \mu$

$$f''(x) = -\frac{1}{\sigma^2\sqrt{2\pi}}$$ for any value of 'σ'

$$f''(x) < 0.$$

Hence $x = \mu$ is the mode of the N.D

Median of the N.D :

proof: Let M be the median of the N.D.

then Median $= \int_{-\infty}^{M} f(x)\,dx$

$$\Rightarrow \int_{-\infty}^{M} f(x)\,dx = 1/2$$

$$= \int_{-\infty}^{M} \frac{1}{\sigma\sqrt{2\pi}} e^{-1/2\left(\frac{x-\mu}{\sigma}\right)^2} dx = \frac{1}{2}$$

$$= \int_{-\infty}^{\mu} \frac{1}{\sigma\sqrt{2\pi}} e^{-1/2\left(\frac{x-\mu}{\sigma}\right)^2} dx + \int_{\mu}^{M} \frac{1}{\sigma\sqrt{2\pi}} e^{-1/2\left(\frac{x-\mu}{\sigma}\right)^2} dx = \frac{1}{2} \quad \text{— (1)}$$

Consider $\int_{-\infty}^{\mu} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$

let $\frac{x-\mu}{\sigma} = z$

$\frac{1}{\sigma}\frac{dx}{dz} = 1$

$dx = \sigma dz$

when $x = -\infty$ $z = -\infty$

$x = \mu$ $z = 0$

then $\int_{-\infty}^{\mu} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = \int_{-\infty}^{0} \frac{\cancel{\sigma}}{\cancel{\sigma}\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz$

$= \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-z^2/2} dz$

$\frac{z^2}{2} = t$

$z^2 = 2t$

$\cancel{2}z \frac{dz}{dt} = \cancel{2}$

$dz = \frac{dt}{z} = \frac{1}{\sqrt{2t}} dt$

$= \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-t} \frac{1}{\sqrt{2t}} dt$

$= \frac{1}{2\sqrt{\pi}} \int_{0}^{\infty} e^{-t} t^{-1/2} dt$

$= \frac{1}{2\sqrt{\pi}} \int_{0}^{\infty} e^{-t} t^{1/2-1} dt$

$= \frac{1}{2\sqrt{\pi}} \Gamma\frac{1}{2}$

$= \frac{1}{2\sqrt{\pi}} \sqrt{\pi} = \frac{1}{2}$

Eqn ① becomes

$$\frac{1}{2} + \int_{\mu}^{M} \frac{1}{\sigma\sqrt{2\pi}} e^{-1/2\left(\frac{x-\mu}{\sigma}\right)^2} dx = 1/2$$

$$\frac{1}{\sigma\sqrt{2\pi}} \int_{\mu}^{M} e^{-1/2\left(\frac{x-\mu}{\sigma}\right)^2} dx = 0$$

$$\int_{\mu}^{M} e^{-1/2\left(\frac{x-\mu}{\sigma}\right)^2} dx = 0$$

(here both limits should be same

$$\therefore \boxed{M = \mu}$$

Median of the N.D $= \mu = M$

⇒ Hence for the N.D, Mean = Median = Mode.

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Mean deviation from the Mean for N.D :-

Proof : By defination, Mean deviation (about Mean)

$$= \int_{-\infty}^{\infty} |x-\mu| f(x)\, dx$$

$$= \int_{-\infty}^{\infty} |x-\mu| \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} |x-\mu| \cdot e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx \quad \text{--- (1)}$$

$$\frac{x-\mu}{\sigma} = z \Rightarrow x-\mu = \sigma z$$

$$\frac{1}{\sigma} \frac{dx}{dz} = 1$$

$$dx = \sigma dz$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} |\sigma z| \, e^{-\frac{1}{2}\left(\frac{\sigma z}{\sigma}\right)^2} \sigma dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |\sigma| |z| \, e^{-\frac{z^2}{2}} dz$$

$$= \frac{2\sigma}{\sqrt{2\pi}} \int_{0}^{\infty} z \, e^{-z^2/2} dz$$

$$\frac{z^2}{2} = t \Rightarrow \frac{2z}{2} \frac{dz}{dt} = 1$$

$$dz = \frac{1}{z} dt$$

$$dz = \frac{1}{\sqrt{2t}} dt$$

$$= \sigma \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-t} dt$$

$$= \sigma\sqrt{\frac{2}{\pi}} \left[\frac{e^{-t}}{-1}\right]_0^{\infty}$$

$$= -\sigma\sqrt{\frac{2}{\pi}}\left[e^{-\infty} - e^{-0}\right]$$

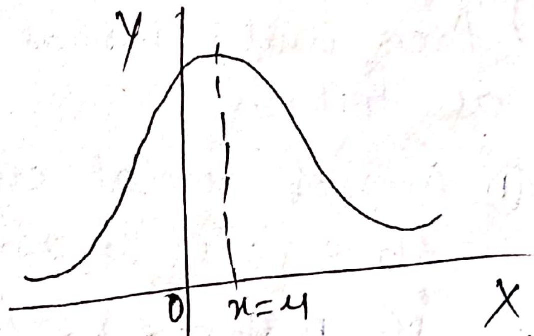
$$= \sigma\sqrt{\frac{2}{\pi}} = \sigma(0.797)$$

Mean deviation $= \sigma\left(\frac{4}{5}\right)$

Hence the mean deviation from the mean for N.D is equal to $\frac{4}{5}$ times of Standard deviation approx.

Chief characteristics of N.D:

1) The graph of the N.D $y=f(x)$ in the $x-y$ plane is known as the Normal curve



2) The curve is bell-shaped and symmetrical about the line $x=\mu$ and the two tails on the right and left sides of the mean extends to infinity

iii) Area under the normal curve represents the total population.

iv) Mean, Median, & Mode of this distribution coincide, so normal curve is unimodel. (has only one max. point)

v) x - axis is an asymptote to the curve
↓ tgt at infinity position

vi) Linear combination of independent normal variates is also a normal variate.

vii) The points of inflection of the curve are at $x = \mu \pm \sigma$

viii) The probability that the normal variate 'X' with mean μ and standard deviation σ lies b/w x_1 and x_2 is given by

$$P(x_1 \le X \le x_2) = \frac{1}{\sigma\sqrt{2\pi}} \int_{x_1}^{x_2} e^{-1/2\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

ix) Area under normal curve is distributed as follows

① Area of normal curve b/w $\mu - \sigma$ and $\mu + \sigma$ is 68.27%

i.e, $P(\mu - \sigma < X < \mu + \sigma) = 0.6827$

② Area of normal curve b/w $\mu - 2\sigma$ and $\mu + 2\sigma$ is 95.43%

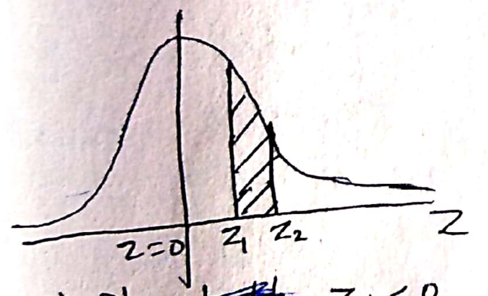
③ Area of normal curve b/w $\mu - 3\sigma$ and $\mu + 3\sigma$ is 99.73%

⇒ The N.D for $\mu = 0$ & $\sigma = 1$ is known as standard normal distribution

How to find probability density of normal curve :

1) The probability that the normal variate 'X' with mean μ and standard deviation 'σ' lies b/w two specific values x_1 and x_2 with $x_1 \leq x_2$ can be obtained using Area under the standard normal curve as follows.

Then $P(x_1 \leq X \leq x_2) = |A(z_2) - A(z_1)|$



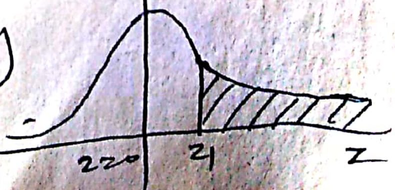
Case 2) If ~~both~~ $z_1 < 0$ and $z_2 > 0$ then

$P(x_1 \leq X \leq x_2) = A(z_2) + A(z_1)$

Step 2) b) To find probability of $z > z_1$

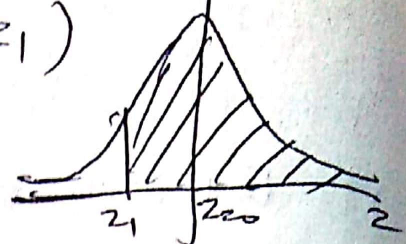
Case 1) If $z_1 > 0$ then

$P(z > z_1) = 0.5 - A(z_1)$



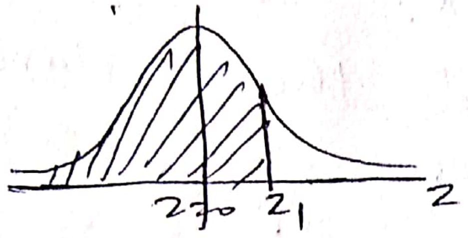
Case 2) If $z_1 < 0$ then

$$P(z > z_1) = 0.5 + A(z_1)$$

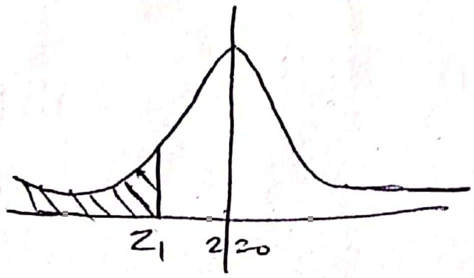


Step 2) c) To find $z < z_1$

Case 1) If $z_1 > 0$. then $P(z < z_1) = 0.5 + A(z_1)$



Case 2) If $z_1 < 0$ then $P(z < z_1) = 0.5 - A(z_1)$



Applications:

Normal distribution plays a very important role in statistical theory because of the following reasons.

1) Data obtained from Psychological, physical & Biological measurements approx. follow Normal distribution. I.Q. scores, heights and weights of individuals etc. are examples of measurements which are normally distributed or nearly so.

2) Most of the distributions that are encountered in practice, for example, Binomial, poisson, Hypergeometric etc can be approximated to Normal distribution. If Binomial distribution tends to Normal distribution. If the parameter $\lambda \to \infty$, then P.D tends to N.D.

3) Since the Normal distribn is a limiting case of the Binomial distribution for exceptionally large numbers, it is applicable to many applied problems in kinetic theory of gases and fluctations in the magnitude of an electric current

4) Even if a variable is normally distributed it can sometimes be brough to normal form by simple transformation of the variable

5) For large samples, any statistic (i.e sample mean, sample S.D etc) approx. follows N.D. and as such it can be studied with the help of normal curve.

6) Normal curve is used to find confidence limits of the populations parameters.

7) The proof of all tests of significance in sampling are based upon the fundamental assumption that the population from which the samples have been drawn is normal

8) Normal distribution finds large applications in statistical quality control in industry for finding control limits.

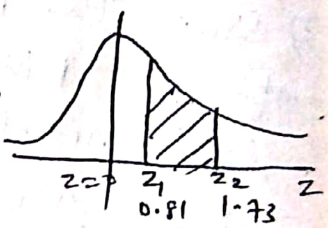
Problems:

1) For a, Normally distributed variate with mean '1' and standard deviation '3'. Find the probabilities that

i) $3.43 \le x \le 6.19$

ii) $-1.43 \le x \le 6.19$

Sol Given that $\mu = 1$ & $\sigma = 3$



i) when $x_1 = 3.43$:-

Then $z_1 = \dfrac{x_1 - \mu}{\sigma} = \dfrac{3.43 - 1}{3} = \dfrac{2.43}{3}$

$= 0.81$

when $x_2 = 6.19$:-

$z_2 = \dfrac{x_2 - \mu}{\sigma} = \dfrac{6.19 - 1}{3} = 1.73$

Consider $P(x_1 \le X \le x_2) = P(z_1 \le z \le z_2)$

$P(3.43 \le x \le 6.19) = P(0.81 \le z \le 1.73)$

$= |A(z_2) - A(z_1)|$ ($\because$ from tables)

$= |0.4582 - 0.2910|$

$= 0.1672$

ii) When $x_1 = -1.43$:-

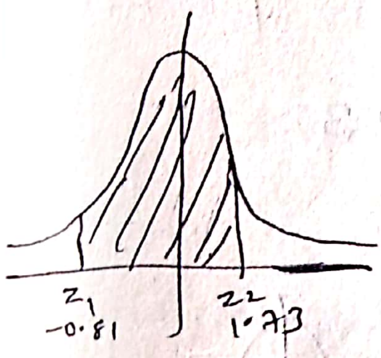
$$z_1 = \frac{x_1 - \mu}{\sigma} = \frac{-1.43 - 1}{3} = -0.81$$

When $x_2 = 6.19$

$$z_2 = \frac{x_2 - \mu}{\sigma} = 1.73$$

Consider $P(x_1 \le x \le x_2) = P(z_1 \le z \le z_2)$

$$P(-1.43 \le x \le 6.19) = P(-0.81 \le z \le 1.73)$$



$$= A(z_2) + A(z_1)$$

$$= A(1.73) + A(-0.81)$$

$$= 0.4582 + A(0.81)$$

$$= 0.4582 + 0.2910$$

$$= 0.9492$$

$\because (A(z_1) = A(-z_1))$

2) If 'x' is a normal variate with mean '30' and S.D '5'. find the probabilities that i) $26 \le x \le 40$
ii) $x \ge 45$.

Sol: Given $\mu = 30$, $\sigma = 5$

When $x_1 = 26$;

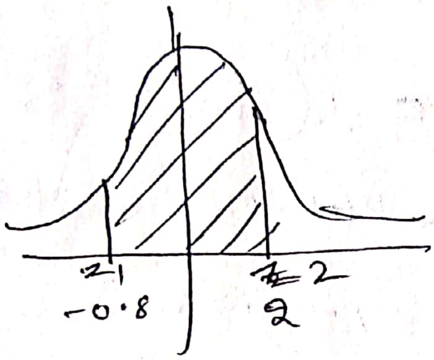
$$z_1 = \frac{x_1 - \mu}{\sigma} = \frac{26 - 30}{5} = \frac{-4}{5} = -0.8$$

When $x_2 = 40$

$$Z_2 = \frac{x_2 - \mu}{\sigma} = \frac{40-30}{5} = \frac{10}{5} = 2.$$

$$P(x_1 \leq x \leq x_2) = P(z_1 \leq z \leq z_2)$$

$$P(26 \leq x \leq 40) = P(-0.8 \leq z \leq 2)$$



$$= A(z_2) + A(z_1)$$

$$= A(2) + A(-0.8)$$

$$= A(2) + A(0.8)$$

$$= 0.4772 + 0.2881$$

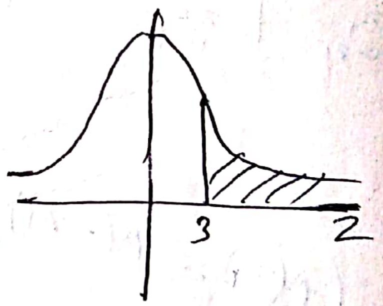
$$= 0.7653$$

$(\because A(-z) = A(z))$

ii) $P(X \geq 45)$

When $x = 45$.

$$z_0 = \frac{45-30}{5}$$



$$z_0 = \frac{15}{3}$$

$$z_0 = 3$$

$$P(X \geq 45) = P(z \geq 3)$$

$$= 0.5 - A(z)$$

$$= 0.5 - 0.4987$$

$$= 0.0013.$$

$$\begin{array}{r} 5000 \\ 4987 \\ \hline 0013 \end{array}$$

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3) In a sample of 1000 cases, the mean of a certain test is 14 and S.D = 2.5. Assuming distribution is normal. find

i) How many students score b/w. 12 and 15 ?

ii) How many score above 18 ?

iii) How many score below 18 ?

Sol Given $\mu = 14$, $\sigma = 2.5$

i) When $x_1 = 12$:

$$z_1 = \frac{12-14}{2.5} = \frac{-2}{2.5} = -0.8$$

When $x_2 = 15$

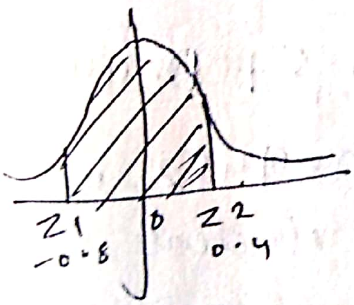
$$z_2 = \frac{15-14}{2.5} = \frac{+1}{2.5} = 0.4$$

Consider $P(x_1 \le x \le x_2) = P(z_1 \le z \le z_2)$

$$P(12 \le x \le 15) = P(-0.8 \le z \le 0.4)$$

$$= A(0.4) + A(-0.8)$$

$$= A(0.4) + A(0.8)$$



$$= 0.1554 + 0.2881$$

$$= 0.4435$$

$\therefore$ Number of students score between 12 and 15 $= 1000 \times 0.4435$

$$= 443.5$$

$$\simeq 443 \text{ approx.}$$

ii) $x > 18$

When $x = 18$

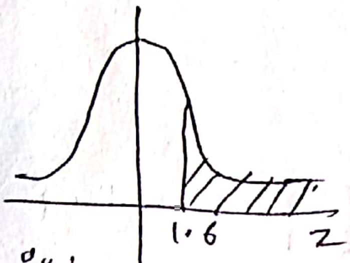
$$z = \frac{x-\mu}{\sigma} = \frac{18-14}{2.5} = \frac{40}{25} = \frac{8}{5}$$

$$z = 1.6$$

$$P(x>18) = P(z>1.6)$$

$$= 0.5 - A(1.6)$$

$$\begin{array}{r} 5000 \\ 4452 \\ \hline 0548 \end{array}$$



$$= 0.5 - 0.4452$$

$$= 0.0548$$

No. of students $= 54.8$

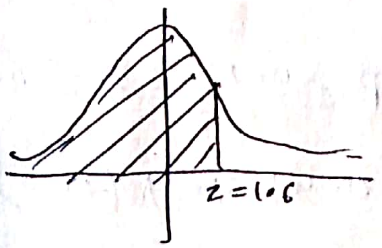
$\simeq 54$

iii) $x < 18$

When $x = 18$

$$z = \frac{x-\mu}{\sigma} = \frac{18-14}{2.5} = 1.6$$

$$P(x \leq 18) = P(z < 1.6)$$



$$= 0.5 + A(1.6)$$

$$= 0.5 + 0.4452$$

$$= 0.9452 \simeq 945$$

4) If the masses of 300 students are normally distributed with mean 68 pages and S.D 3 kg's, How many students have masses

i) > 72 kg

ii) $\leq$ ~~65 kg~~ 64 kg

iii) 65 and 71 kg inclusive

Sol Given $\mu = 68$ $\sigma = 3$

i) $x > 72$

When $x = 72$

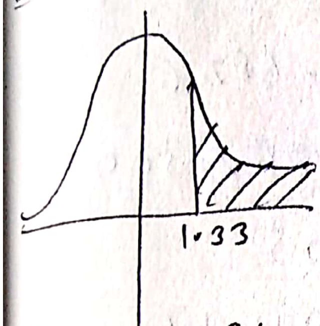
$$z = \frac{x-\mu}{\sigma} = \frac{72-68}{3} = \frac{4}{3} = 1.33$$

$$P(x > 72) = P(z > 1.33)$$

$$= 0.5 - A(1.33)$$

$$= 0.5 - 0.4082$$

$$= 0.0918$$



No. of student $= 0.0918 \times 300$

$\simeq 28$

ii) $x \le 64$

When $x = 64$

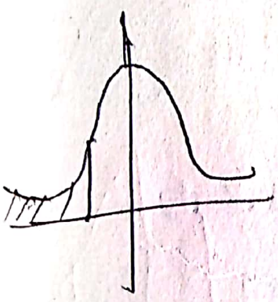
$$z = \frac{64-68}{3} = \frac{-4}{3} = -1.33$$

$$P(x \le 64) = P(z \le -1.33)$$

$$= 0.5 - A(-1.33)$$

$$= 0.5 - A(1.33)$$

$$= 0.0918$$



$$= 0.0918 \times 300$$

No. of student $\simeq 28$

iii) $65 \le x \le 71$

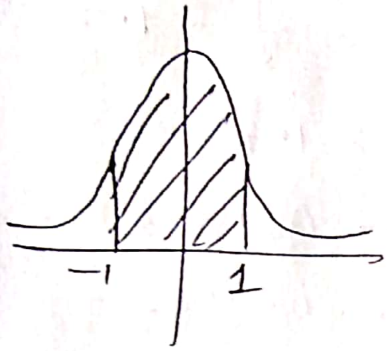
When $x_1 = 65$

$$z_1 = \frac{65-68}{3} = \frac{-3}{3} = -1$$

When $x_2 = 71$

$$z_2 = \frac{71-68}{3} = \frac{3}{3} = 1$$

$$P(65 \le x \le 71) = P(-1 \le z \le z_2)$$
$$= A(z_2) + A(1)$$
$$= A(1) + A(-1)$$
$$= A(1) + A(1)$$
$$= 0.3413 + 0.3413$$
$$= 0.6826$$



No. of students $= 0.6826 \times 300$

$\simeq 205$

**

5) In a N.D, 7% of items are under 35 and 89% are under 63. Determine the mean and variance of the distribution.

Sol: Let Mean $= \mu$, S.D $= \sigma$ are parameters of N.D

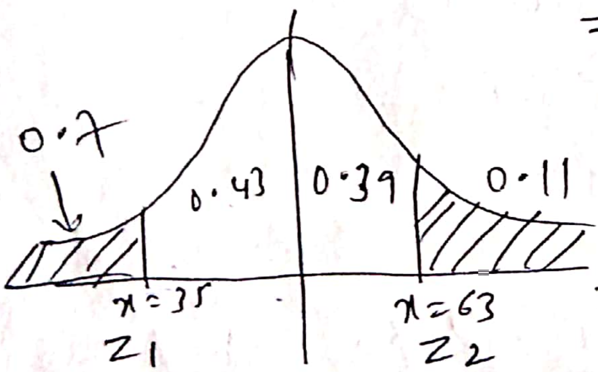
Given $P(X < 35) = 0.07$

$P(X < 63) = 0.89$

But $P(X > 63) = 1 - P(X < 63)$

$= 1 - 0.89$

$= 0.11$



z-axis.

When $x_1 = 35$

$$z_1 = \frac{35 - \mu}{\sigma} \quad \text{—— (1)}$$

When $x_2 = 63$, $z_2 = \frac{x_2 - \mu}{\sigma}$

$$= \frac{63-\mu}{\sigma} \quad \text{--- (2)}$$

from the fig,

Consider $P(0<z<z_1) = 0.43 = A(-z_1)$
$\Rightarrow z_1 = 1.48$

$P(0<z<z_2) = 0.39 = A(z_2)$
$\Rightarrow z_2 = 1.23$

then eqn (1) $\Rightarrow$ $1.48 = \frac{35-\mu}{\sigma}$ --- (3)

eqn (2) $\Rightarrow$ $1.23 = \frac{63-\mu}{\sigma}$ --- (4)

eqn (3) $-$ eqn (4)

$(1.48)\sigma - (1.23)\sigma = 35 - \mu - 63 + \mu$

$0.25\sigma = -28$