UNIT – IV

VECTOR DIFFERENTIATION

*** (1) Find the Directional Derivative of $x^2yz + 4xz^2$ at point $(1,-2,-1)$ in the direction of Normal to surface $x\log z - y^2$ at $(-1,2,1)$

Sol:-

Directional Derivative $= \nabla\phi \cdot \hat{n} = \nabla\phi \cdot \frac{\nabla f}{|\nabla f|}$

let $\phi = x^2yz + 4xz^2$

$f = x\log z - y^2$

$$\nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$$

$$= i\frac{\partial}{\partial x}(x^2yz + 4xz^2) + j\frac{\partial}{\partial y}(x^2yz + 4xz^2) + k\frac{\partial}{\partial z}(x^2yz + 4xz^2)$$

$$= i(2xyz + 4z^2) + j(x^2z + 0) + k(x^2y + 8xz)$$

$$\nabla\phi = i(2xyz + 4z^2) + j(x^2z) + k(x^2y + 8xz)$$

$$\nabla\phi_{at(1,-2,-1)} = i[2(1)(-2)(-1) + 4(-1)^2] + j[(1)^2(-1)] + k[(1)^2(-2) + 8(1)(-1)]$$

$$\nabla\phi_{at(1,-2,-1)} = 8i - j - 10k.$$

$$f = x\log z - y^2$$

$$\nabla f = i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z}$$

$$\nabla f = i\frac{\partial}{\partial x}(x\log z - y^2) + j\frac{\partial}{\partial y}(x\log z - y^2)$$

$$+ k\frac{\partial}{\partial z}(x\log z - y^2)$$

$$\nabla f = i(\log z) + j(-2y) + k\left(\frac{x}{z}\right)$$

$$\nabla f \text{ at } (-1, 2, 1) = i\log(1) + j(-2(2)) + k\left(\frac{-1}{1}\right)$$

$$\rightarrow \nabla f \text{ at } (-1, 2, 1) = -4j - k$$

$$|\nabla f| = \sqrt{(-4)^2 + (-1)^2} = \sqrt{16+1} = \sqrt{17}$$

$\therefore$ Directional Derivative

$$= \nabla\phi \cdot \frac{\nabla f}{|\nabla f|} = (8i - j - 10k)\cdot\frac{(-4j - k)}{\sqrt{17}}$$

$$= \frac{0 + 4 + 10}{\sqrt{17}} = \frac{14}{\sqrt{17}}$$

**

(2) Find the Directional Derivative of $f = xy + yz + zx$ in the Direction of vector $i + 2j + 2k$ at $(1, 2, 0)$

sol:- Directional Derivative $= \nabla f \cdot \hat{n}$

$= \nabla f \cdot \dfrac{\nabla \phi}{|\nabla \phi|}$.

let $f = xy + yz + zx$

$\nabla \phi = i + 2j + 2k$.

$\nabla f = i \dfrac{\partial f}{\partial x} + j \dfrac{\partial f}{\partial y} + k \dfrac{\partial f}{\partial z}$

$= i \dfrac{\partial}{\partial x}(xy + yz + zx) + j \dfrac{\partial}{\partial y}(xy + yz + zx)$

$+ k \dfrac{\partial}{\partial z}(xy + yz + zx)$

$\nabla f = i(y+z) + j(x+z) + k(x+y)$

$\rightarrow \nabla f$ at $(1, 2, 0) = 2\hat{i} + \hat{j} + 3K$

Given that $\nabla \phi = \hat{i} + 2\hat{j} + 2K$

$|\nabla \phi| = \sqrt{(1)^2 + (2)^2 + (2)^2} = \sqrt{9} = 3$

Directional Derivative $= \nabla f \cdot \hat{n} = \nabla f \cdot \dfrac{\nabla \phi}{|\nabla \phi|}$

$= (2\hat{i} + j + 3K) \cdot \dfrac{(\hat{i} + 2j + 2k)}{\sqrt{9}}$

$= \dfrac{2 + 2 + 6}{3} = 10/3$

(3) Find the Directional Derivative of $\phi = x^2yz + 4xz^2$ at $(1, -2, -1)$ in the Direction of vector $2\hat{i} - \hat{j} - 2k$

(4) Find the Directional Derivative of function $f = x^2 - y^2 + 2z^2$ at the point $(1, 2, 3)$ in the Direction of line PQ. where $Q = (5, 0, 4)$

sol: Directional Derivative $= \nabla f \cdot \hat{n}$

$$= \nabla f \cdot \frac{\nabla \phi}{|\nabla \phi|}$$

let $f = x^2 - y^2 + 2z^2$

$$\nabla f = i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z}$$

$$\nabla f = i\frac{\partial}{\partial x}(x^2 - y^2 + 2z^2) + j\frac{\partial}{\partial y}(x^2 - y^2 + 2z^2)$$

$$+ k\frac{\partial}{\partial z}(x^2 - y^2 + 2z^2)$$

$$\nabla f = i(2x) + j(-2y) + k(4z)$$

$$\rightarrow \nabla f_{\text{at } P(1,2,3)} = 2i - 4j + 12k.$$

Given that P, Q are position vectors w.r.t to origin.

$\overline{OQ} = 5i + 4k$, $\overline{OP} = i + 2j + 3k$

$$\overline{PQ} = \overline{OQ} - \overline{OP} = (5i + 4k) - (i + 2j + 3k)$$

$$\overline{PQ} = 4i - 2j + k$$

let $\overline{PQ} = 4i - 2j + k = \nabla \phi$

(1) If $\bar{f} = xy^2\bar{i} + 2x^2yz\bar{j} - 3yz^2\bar{k}$ Find $\operatorname{div}\bar{f}$ at $(1,-1,1)$

Sol:- Given that $\bar{f} = xy^2\bar{i} + 2x^2yz\bar{j} - 3yz^2\bar{k}$

To Find $\operatorname{div}\bar{f}$ at $(1,-1,1)$

$$\nabla\cdot f = \operatorname{div}\bar{f} = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\cdot\left(xy^2\bar{i} + 2x^2yz\bar{j} - 3yz^2\bar{k}\right)$$

$$= \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\cdot\left(xy^2\bar{i} + 2x^2yz\bar{j} - 3yz^2\bar{k}\right)$$

$$= \frac{\partial}{\partial x}(xy^2) + \frac{\partial}{\partial y}(2x^2yz) + \frac{\partial}{\partial z}(-3yz^2)$$

$$\nabla\cdot\bar{f} = y^2 + 2x^2z + (-6yz)$$

$$\nabla\cdot\bar{f} = \operatorname{div}\bar{f}_{\text{at}(1,-1,1)} = (-1)^2 + 2(1)^2 1 + (-6(-1)\cdot 1)$$

$$\Rightarrow \operatorname{div}\bar{f} = 1+2+6 = 9$$

** (2) Find $\operatorname{div}\bar{f}$ when $\bar{f} = \operatorname{grad}(x^3+y^3+z^3-3xyz)$

Sol. $\bar{f} = \operatorname{grad}(x^3+y^3+z^3-3xyz)$

$$\bar{f} = \nabla(x^3+y^3+z^3-3xyz)$$

$$\bar{f} = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)(x^3+y^3+z^3-3xyz)$$

$$\bar{f} = i\frac{\partial}{\partial x}(x^3+y^3+z^3-3xyz) + j\frac{\partial}{\partial y}(x^3+y^3+z^3-3xyz) + k\frac{\partial}{\partial z}(x^3+y^3+z^3-3xyz)$$

(4) Find div $\bar{r}$ where $\bar{r} = xi + yj + zk$

sol: Given that $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$$\text{div}\,\bar{r} = \nabla \cdot \bar{r} = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\cdot\left(x\bar{i} + y\bar{j} + z\bar{k}\right)$$

$$= \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 1+1+1 = 3$$

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(5) prove that $r^n \bar{r}$ is solenoidal if $n = -3$

(or)

Find div $\bar{f}$ where $\bar{f} = r^n \bar{r}$. Find n if it is solenoidal?

(or)

prove that $\text{div}(r^n \bar{r}) = (n+3)r^n$

sol:- let $\bar{f} = r^n \bar{r}$

To PT $r^n \bar{r}$ is solenoidal if $n = -3$

ie $\text{div}\,\bar{f} = 0$ if $n = -3$, $(\because \bar{f} = r^n \bar{r})$

(or)

$\text{div}(r^n \bar{r}) = 0$ if $n = -3$

let $\bar{r} = xi + yj + zk$

$$|\bar{r}| = r = \sqrt{x^2 + y^2 + z^2}$$

$$r^2 = x^2 + y^2 + z^2$$

(22)

$$r^2 = x^2 + y^2 + z^2$$

P.D. Diff w.r.t. 'x' on b.s

$$\frac{\partial}{\partial x}(r^2) = \frac{\partial}{\partial x}(x^2 + y^2 + z^2)$$

$$2r\frac{\partial r}{\partial x} = 2x + 0 + 0 \Rightarrow \cancel{2}r\frac{\partial r}{\partial x} = \cancel{2}x$$

$$\boxed{\frac{\partial r}{\partial x} = \frac{x}{r}} \quad \text{lly} \quad \boxed{\frac{\partial r}{\partial y} = \frac{y}{r}} \quad \text{lly} \quad \boxed{\frac{\partial r}{\partial z} = \frac{z}{r}}$$

For $r^n\bar{r} = r^n(xi + yj + zk)$

$$r^n\bar{r} = r^n x\bar{i} + r^n y\bar{j} + r^n z\bar{k}$$

To PT $\operatorname{div}(r^n\bar{r}) = 0$ if $n = -3$

$$\operatorname{div}(r^n\bar{r}) = \nabla\cdot(r^n\bar{r})$$

$$= \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\cdot\left(r^n x\bar{i} + r^n y\bar{j} + r^n z\bar{k}\right)$$

$$= \frac{\partial}{\partial x}r^n x + \frac{\partial}{\partial y}r^n y + \frac{\partial}{\partial z}r^n z$$

$$= \left[r^n\frac{\partial}{\partial x}\cdot x + x\cdot\frac{\partial}{\partial x}r^n\right] + \left[r^n\cdot\frac{\partial}{\partial y}y + y\frac{\partial}{\partial y}r^n\right]$$

$$+ \left[r^n\cdot\frac{\partial}{\partial z}\cdot z + z\cdot\frac{\partial}{\partial z}r^n\right]$$

$$= \left[r^n\cdot 1 + x\cdot n\cdot r^{n-1}\frac{\partial r}{\partial x}\right] + \left[r^n\cdot 1 + y\cdot n\, r^{n-1}\frac{\partial r}{\partial y}\right]$$

$$+ \left[r^n\cdot 1 + z\cdot n\cdot r^{n-1}\frac{\partial r}{\partial z}\right]$$

$$= \left[r^n + x\cdot n\, r^{n-1}\cdot\frac{x}{r}\right] + \left[r^n + y\cdot n\, r^{n-1}\cdot\frac{y}{r}\right]$$

$$+ \left[r^n + z\cdot n\cdot r^{n-1}\frac{z}{r}\right]$$

$$= r^n + x \cdot n r^{n-1} \cdot x \cdot r^{-1} + r^n + y \cdot n r^{n-1} \cdot y \cdot r^{-1}$$
$$+ r^n + z \cdot n \cdot r^{n-1} \cdot z \, r^{-1}$$

$$= 3r^n + x^2 n \cdot r^{n-2} + y^2 n \, r^{n-2} + z^2 n \cdot r^{n-2}$$

$$= 3r^n + n r^{n-2} (x^2 + y^2 + z^2)$$

$$= 3r^n + n \cdot r^{n-2} (r^2) \qquad (\because r^2 = x^2 + y^2 + z^2)$$

$$= 3r^n + n \cdot r^{n-2+2}$$

$$= 3r^n + n \cdot r^n = r^n (n+3)$$

$$\therefore \operatorname{div}(r^n \bar{r}) = (n+3) r^n$$

$r^n \bar{r}$ is solenoidal if $n = -3$ //

if $n = -3$
$$\operatorname{div}(r^n \bar{r}) = \nabla \cdot (r^n \bar{r}) = \nabla \cdot f = 0$$

***
(6) Show that $\dfrac{\bar{r}}{r^3}$ is solenoidal

(or)

Evaluate $\nabla \cdot \left(\dfrac{\bar{r}}{r^3}\right)$ where $\bar{r} = xi + yj + zk$
$r = |\bar{r}|$

Sol: To show that $\dfrac{\bar{r}}{r^3}$ is solenoidal

ie $\nabla \cdot \dfrac{\bar{r}}{r^3} = 0$

$$\bar{r} = xi + yj + zk$$
$$|\bar{r}| = r = \sqrt{x^2 + y^2 + z^2}$$

$$r^2 = x^2 + y^2 + z^2$$

$r^2 = x^2 + y^2 + z^2$

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\frac{\bar{r}}{r^3} = \bar{r}\, r^{-3} = (xi + yj + zk)\,(r^{-3})$$

$$\frac{\bar{r}}{r^3} = r^{-3} x\bar{i} + r^{-3} y\bar{j} + r^{-3} z\bar{k}$$

To show that $\frac{\bar{r}}{r^3}$ is solenoidal

ie $\nabla\cdot\left(\frac{\bar{r}}{r^3}\right) = 0$

$$\text{LHS} = \nabla\cdot\left(\frac{\bar{r}}{r^3}\right) = \nabla\cdot\left(r^{-3} x\bar{i} + r^{-3} y\bar{j} + r^{-3} z\bar{k}\right)$$

$$= \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\cdot\left(r^{-3} x\bar{i} + r^{-3} y\bar{j} + r^{-3} z\bar{k}\right)$$

$$= \frac{\partial}{\partial x}(x r^{-3}) + \frac{\partial}{\partial y}(r^{-3} y) + \frac{\partial}{\partial z}(r^{-3}\cdot z)$$

$$= \left[r^{-3}\cdot\frac{\partial}{\partial x}x + x\frac{\partial}{\partial x}r^{-3}\right] + \left[r^{-3}\cdot\frac{\partial}{\partial y}y + y\frac{\partial}{\partial y}r^{-3}\right]$$

$$+ \left[r^{-3}\frac{\partial}{\partial z}z + z\frac{\partial}{\partial z}r^{-3}\right]$$

$$= \left[r^{-3}\cdot 1 + x\cdot(-3)\, r^{-3-1}\frac{\partial r}{\partial x}\right] + \left[r^{-3}\cdot 1 + y\cdot(-3)\, r^{-3-1}\frac{\partial r}{\partial y}\right]$$

$$+ \left[r^{-3}\cdot 1 + z\cdot(-3)\, r^{-3-1}\frac{\partial r}{\partial z}\right]$$

$$= \left[r^{-3} - 3x\cdot r^{-4}\cdot\frac{x}{r}\right] + \left[r^{-3} - 3y\, r^{-4}\frac{y}{r}\right]$$

$$+ \left[r^{-3} - 3z\, r^{-4}\frac{z}{r}\right]$$

$$= r^{-3} - 3x \cdot r^{-4} \cdot x \cdot r^{-1} + r^{-3} - 3y\, r^{-4} \cdot y \cdot r^{-1}$$
$$+ r^{-3} - 3z\, r^{-4} \cdot z\, r^{-1}$$

$$= 3r^{-3} - 3x^2 r^{-5} - 3y^2 r^{-5} - 3z^2 r^{-5}$$

$$= 3r^{-3} - 3r^{-5}(x^2+y^2+z^2)$$

$$= 3r^{-3} - 3r^{-5}(r^2)$$

$$= 3r^{-3} - 3r^{-5+2} = 3r^{-3} - 3r^{-3} = 0$$

$$= 0$$

$$\therefore \operatorname{div}\left(\frac{\bar{r}}{r^3}\right) = 0$$

$$\nabla \cdot \left(\frac{\bar{r}}{r^3}\right) = \operatorname{div}\left(\frac{\bar{r}}{r^3}\right) = 0$$

$\Rightarrow \frac{\bar{r}}{r^3}$ is solenoidal vector.

(7) *** prove that if $\bar{r}$ is the position (26) vector any point in space, then $r^n \bar{r}$ is irrotational vector.

(or)

show that $curl(r^n \bar{r}) = 0$

(or)

show that $\nabla \times (r^n \bar{r}) = 0$

sol:- let $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$|\bar{r}| = r = \sqrt{x^2+y^2+z^2}$

$r^2 = x^2+y^2+z^2$

'p' Diff w.r.t 'x' on b.s

$\frac{\partial}{\partial x}(r^2) = \frac{\partial}{\partial x}(x^2+y^2+z^2)$

$2r\frac{\partial r}{\partial x} = 2x \Rightarrow \boxed{\frac{\partial r}{\partial x} = \frac{x}{r}}$

lly $\boxed{\frac{\partial r}{\partial y} = \frac{y}{r}}$ lly $\boxed{\frac{\partial r}{\partial z} = \frac{z}{r}}$

$\rightarrow r^n \bar{r} = r^n(x\hat{i} + y\hat{j} + z\hat{k})$

$r^n \bar{r} = x r^n \bar{i} + y r^n \bar{j} + z r^n \bar{k}$

To show that $r^n \bar{r}$ is irrotational

ie $\nabla \times (r^n \bar{r}) = 0$

(or)

$curl(r^n \bar{r}) = 0$

$LHS = \quad curl(r^n \bar{r}) = \nabla \times (r^n \bar{r})$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x r^n & y r^n & z r^n \end{vmatrix}$$

$$= i\left[\frac{\partial}{\partial y}(z r^n) - \frac{\partial}{\partial z}(y r^n)\right] - j\left[\frac{\partial}{\partial x}(z r^n) - \frac{\partial}{\partial z}(x r^n)\right] + k\left[\frac{\partial}{\partial x}(y r^n) - \frac{\partial}{\partial y}(x r^n)\right]$$

$$= i\left[z \cdot n r^{n-1}\frac{\partial r}{\partial y} - y \cdot n r^{n-1}\frac{\partial r}{\partial z}\right] - j\left[z \cdot n r^{n-1}\frac{\partial r}{\partial x} - x \cdot n r^{n-1}\frac{\partial r}{\partial z}\right] + k\left[y \cdot n r^{n-1}\frac{\partial r}{\partial x} - x n r^{n-1}\frac{\partial r}{\partial y}\right]$$

$$= i\left[z \cdot n r^{n-1} \cdot \frac{y}{r} - y \cdot n r^{n-1} \cdot \frac{z}{r}\right] - j\left[z n r^{n-1} \frac{x}{r} - x \cdot n r^{n-1} \frac{z}{r}\right] + k\left[y n r^{n-1} \frac{x}{r} - x n r^{n-1} \frac{y}{r}\right]$$

$$= i(0-0) - j(0-0) + k(0-0)$$

$$= \bar{0}$$

$\therefore \quad curl(r^n \bar{r}) = 0$

$\Rightarrow r^n \bar{r}$ is irrotational vector.