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# Random Variables

## Introduction :

So far we discussed Sample space 's' is a set of all possible outcomes of an experiment.

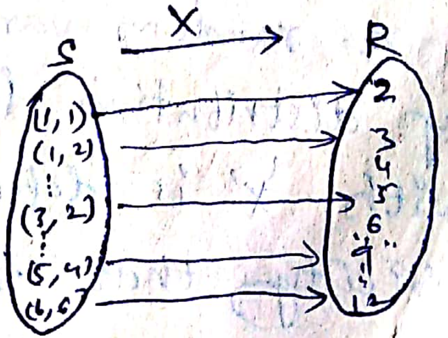
WKT, outcomes of the experiment are the elements of a sample space 's' and they need not be numbers. Sometimes we wish to assign a specific number to each and every outcome. Such assignment is called random variable.

**Defi :** A Random variable 'X' on a sample space 's' is a function $X : S \to R$, that assigns a real number $X(s)$ to each sample point $s \in$ 's'. Thus $X(s)$ represents the real number which the random variable 'X' associates with outcome 's'.

Eg: If two dice are thrown, then

$$S = \{(1,1)\ (1,2) \ldots \ldots (6,6)\}$$

Now the RV 'X' defined as sum of the faces of two dice.



⇒ Random variables are usually classify according to the number of values, which they can take.

RV are classified as 1) discrete RV
2) continuous RV

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**Discrete RV**: The RV ~~is said to be discret~~ which can take only a finite number of discrete values, in ~~an~~ interval domain

In other words, if the random variable takes the values only on the set $\{0, 1, 2, 3 \ldots n\}$ is called a discrete RV.

Eg: A number of defectives in a sample of electric bulbs, number of printing mistakes in each page of a book. The number of telephone calls received by the telephone operator.

→ **Continuous RV**: A RV 'X' which can take values continuously ie, which takes all possible values in a given interval is called a continuous RV.

Eg: Height, age, weight of individuals and also temp, time.

**Probability distribution function**:

Let 'X' be a random variable, then the probability distribution function associated with 'X' is defined as the probability that the outcome of an experiment will be one of the outcomes for which $X(S) \le x$ where $x \in R$ is the function, $F(x)$ or $F_X(x)$ defined by $F_X(x) = P(X \le x)$ where $x$ lies b/w $-\infty$ to $\infty$

**Properties of distribution function**

i) If 'F' is the distribution function of R.V 'X' and if $a < b$ then

i) $P(a < X \le b) = F(b) - F(a)$

ii) $P(a \le X \le b) = P(X = a) + [F(b) - F(a)]$

iii) $P(a < X < b) = [F(b) - F(a)] - P(X = b)$

iv) $P(a \le X < b) = [F(b) - F(a)] - P(X = b) + P(X = a)$

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**Discrete probability distribution:** Probability distribution of a random variable is a set of its possible values together with their respective probabilities. Suppose 'X' is a discrete RV with possible outcomes $x_1, x_2, x_3 \ldots x_n$, the probability of each possible outcome "$x_i$" is $P_i = P(X = x_i) = P(x_i)$ for $i = 1, 2, 3 \ldots$

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If the numbers $p(x_i)$, $i = 1, 2, \cdots$ satisfy the two conditions

i) $p(x_i) \geq 0$

ii) $\Sigma p(x_i) = 1$ where $i = 1, 2 \cdots\cdots$

Here, the function 'p' is called the probability mass function of RV 'X' and the set $\{p(x_i)\}$ $i = 1, 2, 3 \cdots\cdots$ is called the discrete probability distribution of the discrete RV 'X'.

The probability distribution of RV 'X' is given by means of the following table

| | | | | | |
|---|---|---|---|---|---|
| $x_i$ | $x_1$ | $x_2$ | $x_3$ | $x_4 \cdots\cdots$ | $x_n$ |
| $p(x_i)$ | $p(x_1)$ | $p(x_2)$ | $p(x_3)$ | | $p(x_n)$ |

$\Rightarrow$ further

$$P(X < x_i) = p(x_1) + p(x_2) + \cdots\cdots p(x_{i-1})$$

$$P(X \leq x_i) = p(x_1) + p(x_2) + \cdots + p(x_{i-1}) + p(x_i)$$

$$P(X > x_i) = 1 - P(X \leq x_i)$$

Q: Let 'X' denote the number of heads in a single toss of 4 fair coins. Determine
i) $P(X<2)$ ii) $P(1<X\leq 3)$

Sol i) The required probability distribution table is

| | | | | | |
|---|---|---|---|---|---|
| $x_i$ | 0 | 1 | 2 | 3 | 4 |
| $P(x_i)$ | 1/16 | 1/16 | 1/16 | 1/16 | 1/16 |

$P(X<2) = P(X=0) + P(X=1)$
$= \frac{1}{16} + \frac{1}{16}$
$= 1/8$

$P(1<X\leq 3) = P(X=2) + P(X=3)$
$= \frac{1}{16} + \frac{1}{16}$
$= 1/8$

Q: A RV 'X' has the following probability distribution function

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $x$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| $P(x)$ | 0 | $K$ | $2K$ | $2K$ | $3K$ | $K^2$ | $2K^2$ | $7K^2+K$ |

i) Determine i) K, ii) Evaluate probability of $P(X<6)$, $P(X\geq 6)$ $P(0<X<5)$ and $P(0\leq X\leq 4)$
iii) If $P(X\leq K) > 1/2$. find the minimum value of 'K'.

Sol Sum of probabilities = 1 ie $\sum_{K=0}^{7} P(x_K) = 1$

$8K + 10K^2 + K = 1$

$10K^2 + 9K = 1$

$K(10K+9)=1$

$K=1$ or, $10K=-8$

$K=-4/5$

ii) $P(X<6)=P(X=0)+P(X=1)+P(X=2)+P(X=3)+P(X=4)+P(X=5)$

$=0+K+2K+2K+3K+K^2$

$=8K+K^2$

$=9$ (take positive 'K')

$P(X\geq 6)=1-P(X<6)$

$=1-9$

$=-8$

$P(0<X<5)=P(X=1)+P(X=2)+P(X=3)+P(X=4)$

$=K+2K+2K+3K$

$=8K$

$=8$

iii) $P(0\leq X\leq 4)=P(X=0)+P(X=1)+P(X=2)+P(X=3)+P(X=4)$

$=0+K+2K+2K+3K$

$=8$

$\Rightarrow P(X\leq K)>1/2$

$P(X=0)+P(X=1)>1/2$

$K>1/2$

The min. value of 'K' is $>\frac{1}{2}$

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Q: A RV 'X' has the following probability distribution,

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $x_i$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| $P(x_i)$ | K | 2K | 3K | 4K | 5K | 6K | 7K | 8K |

find the value of i) K ii) $P(X \geq 2)$ $P(2 \leq X \leq 5)$

Sol i) sum of the probabilities = 1

$$36K = 1$$

$$K = 1/36$$

ii) $P(X \geq 2) = 1 - (P(x=1))$

$$= 1 - K \Rightarrow 1 - \frac{1}{36} = \frac{35}{36}$$

iii) $P(2 \leq X \leq 5) = P(X=2) + P(X=3) + P(X=4) + P(X=5)$

$$= 2K + 3K + 4K + 5K$$

$$= 14\left(\frac{1}{36}\right) = 7/12$$

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Q: The probability density function of variable 'X' is

| $X$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| $P(X)$ | $K$ | $3K$ | $5K$ | $7K$ | $9K$ | $11K$ | $13K$ |

i) Find $(X<4)$, $P(X \geq 5)$, $P(3<X\leq 6)$

ii) What will be the min. value of 'K' so that $P(X\leq 2) > 0.3$ ?

Sol

$$\Sigma P(X_i) = 1$$

$$49K = 1$$

$$K = 1/49$$

i) let $P(X<4) = P(X=0)+P(X=1)+P(X=2)+P(X=3)$

$$= K+3K+5K+7K$$

$$= 16K$$

$$= \frac{16}{49}$$

$$P(X\geq 5) = P(X=5)+P(X=6)$$

$$= 11K+13K$$

$$= 24\left(\frac{1}{49}\right) = \frac{24}{49}$$

$$P(3<X\leq 6) = P(X=4)+P(X=5)+P(X=6)$$

$$= 9K+11K+13K$$

$$= \frac{33}{49}$$

ii) $P(X\leq 2) = P(X=0)+P(X=1)+P(X=2)$

$$= 9K \cancel{= \frac{49}{49}}$$

$$\Rightarrow 9K > 0.3$$

$$= 0.18 > 0.3 \text{ false}$$

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$$K > \frac{0.3}{9}$$

$$K > \frac{1}{30}$$

The min. value of 'K' is $\frac{1}{30}$

Q: Two dice are thrown let 'X' assign to each point $(a, b)$ in 'S'. The max. of its numbers i.e. $X(a,b) = \max(a, b)$ Find the probability distribution 'X' is a RV with $X(S) = \{1, 2, 3, 4, 5, 6\}$

Sol: Two dice are thrown, then

sample space $S = \{(1,1)(1,2)(1,3)(1,4)(1,5)(1,6)$
$(2,1)(2,2)(2,3)(2,4)(2,5)(2,6)$
$(3,1)(3,2)(3,3)(3,4)(3,5)(3,6)$
$(4,1)(4,2)(4,3)(4,4)(4,5)(4,6)$
$(5,1)(5,2)(5,3)(5,4)(5,5)(5,6)$
$(6,1)(6,2)(6,3)(6,4)(6,5)(6,6)\}$

$\Rightarrow n(S) = 36$

The max. numbers could be 1, 2, 3, 4, 5, 6

Here $X(S) = X(a,b) = \max(a, b)$

i) The max. number '1' will appear in only one case i.e. $(1,1)$

$$\max(a,b) = 1$$

$$P(X=1) = 1/36$$

ii) The max. number '2' will appear in 3 cases $(1,2)(2,1)(2,2)$

$\max(a,b) = 2$

$$P(X=2) = 3/36$$

iii) The max. number '3' will appear in 5 cases $(1,3)\ (2,3)\ (3,1)\ (3,2)\ (3,3)$

$\max(a,b) = 3$

$$P(X=3) = 5/36$$

iv) The max. num '4' will appear in 7 cases $(1,4)\ (2,4)\ (3,4)\ (4,1)\ (4,2)\ (4,3)\ (4,4)$

$\max(a,b) = 4$

$$P(X=4) = 7/36$$

v) The max. num '5' will appear in 9 cases $(1,5)\ (2,5)\ (3,5)\ (4,5)\ (5,1)\ (5,2)\ (5,3)\ (5,4)\ (5,5)$ $\max(a,b) = 5$

$$P(X=5) = 9/36$$

vi) The max. num '6' will appear in 11 cases $(1,6)\ (2,6)\ (3,6)\ (4,6)\ (5,6)\ (6,1)\ (6,2)\ (6,3)\ (6,4)\ (6,5)\ (6,6)$

$\max(a,b) = 6$

$$P(X=6) = \frac{11}{36}$$

The req. probability Distribution table is

| $X$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $P(X)$ | $1/36$ | $3/36$ | $5/36$ | $7/36$ | $9/36$ | $11/36$ |

**Expectation :**

Suppose a random variable 'X' assumes the values $x_1, x_2, x_3 \dots x_n$ with respective probabilities $P_1, P_2 \dots P_n$ then the expectatio (or) Mean (or) Expected value of 'X' denoted by $E(X)$ and is defined as the sum of products of different values of 'x' and the corresponding probabilities i.e.,

$$E(X) = \sum_{i=1}^{n} P_i x_i$$

⇒ Result 1 ; $E(X+K) = E(X) + K$

**Proof :** By defination, we have

$$E(X) = \sum_{i=1}^{n} P_i x_i$$

Consider $E(X+K) = \sum_{i=1}^{n} P_i (x_i + K)$

$$= \sum_{i=1}^{n} P_i x_i + P_i K$$

$$= \sum_{i=1}^{n} P_i x_i + \sum_{i=1}^{n} P_i K$$

$$E(X+K) = E(X) + K \sum_{i=1}^{n} P_i \qquad \boxed{\sum_{i=1}^{n} P_i = 1}$$

$$E(X+K) = E(X) + K$$

2) $E(ax+b) = a E(x) + b$

**proof:** By the defi, $E(X) = \sum_{i=1}^{n} P_i x_i$

Consider $E(ax+b) = \sum_{i=1}^{n} P_i (ax_i + b)$

$$= \sum_{i=1}^{n} (P_i a x_i + P_i b)$$

$$= \sum_{i=1}^{n} P_i a x_i + \sum_{i=1}^{n} P_i b$$

$$= a \sum_{i=1}^{n} P_i x_i + b \sum_{i=1}^{n} P_i$$

$$E(ax+b) = a E(X) + b$$

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3) $E(K) = K$

4) $E(XY) = E(X) \cdot E(Y)$

**Note:** Expectation or Mean of a RV 'X' is denoted by $E(X)$ or $\mu$ or $\bar{X}$.

$\Rightarrow$ ~~Variance~~ : The RV 'X' and its variance is denoted by $V(X)$ or $\sigma^2$ and if defined as

$$V(X) = \sigma^2 = \sum_{i=1}^{n} (x_i - \bar{X})^2 p_i$$

~~Proof~~ : (WKT)

~~E(X) ⇒~~

** Imp

$\Rightarrow$ Now, we will write variance of 'X' in terms of expectation as

Proof: WKT, $E(X) = \bar{X} = \mu = \sum_{i=1}^{n} p_i x_i$

Consider $V(X) = \sigma^2 = \sum_{i=1}^{n} (x_i - \bar{X})^2 p_i$

$$= \sum_{i=1}^{n} \left[x_i^2 - 2x_i\bar{X} + \bar{X}^2\right] p_i$$

$$\sigma^2 = \sum_{i=1}^{n} \left[x_i^2 p_i - 2x_i\bar{X} p_i + \bar{X}^2 p_i\right]$$

$$= \sum_{i=1}^{n} \left[x_i^2 p_i\right] - \sum_{i=1}^{n} \left[2x_i \bar{X} p_i\right] + \sum_{i=1}^{n} \left[\bar{X}^2 p_i\right]$$

$$= E(X^2) - 2\bar{X} \sum_{i=1}^{n} x_i p_i + \bar{X}^2 \sum_{i=1}^{n} p_i$$

$$= E(X^2) - 2\bar{X} \cdot \bar{X} + \bar{X}^2$$

$$= E(X^2) - 2\bar{X}^2 + \bar{X}^2$$

$$= E(X^2) - \bar{X}^2$$

$$V(X) = \sigma^2 = E(X^2) - (E(X))^2$$

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**Note:** Variance can be expressed in the form of expectation is given by:

$$V(X) = \sigma^2 = E\left[X - E(X)\right]^2$$

***Imp.

**Result:** If 'X' is a discrete RV then

$$V(aX+b) = a^2 V(X), \quad a, b \text{ are const}$$

**Proof** Let $Y = aX+b$ — ①

Now apply expectation on both sides

$$E(Y) = E(aX+b)$$

$$E(Y) = aE(X) + b \quad — ②$$

Now subtract ② from ①

$$Y - E(Y) = aX + b - aE(X) - b$$

$$Y - E(Y) = a\left[X - E(X)\right]$$

Now squaring on both sides

$$\left[Y - E(Y)\right]^2 = a^2\left[X - E(X)\right]^2$$

Apply Expectation on both sides

$$E\left(Y - E(Y)\right)^2 = a^2 E\left[X - E(X)\right]^2$$

$$V(Y) = a^2 V(X)$$

$$V(aX+b) = a^2 V(X)$$

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→ If $a=0$ then, $V(b)=0$

If $b=0$ then, $V(aX) = a^2 V(X)$

If $a=1$ then, $V(X+b) = V(X)$

→ Standard deviation : It is the positive square root of the variance

$$S.D = \sigma = \sqrt{\sum_{i=1}^{n} P_i x_i^2 - \mu^2}$$

Problems

① A RV 'X' has the following probability function

| X | -3 | -2 | -1 | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|
| P(X) | K | 0.1 | K | 0.2 | 2K | 0.4 | 2K |

find i) K ii) Mean iii) variance

Sol i) Sum of probabilities = 1

$$6K + 0.7 = 1$$

$$6K = 0.3$$

$$K = \frac{\cancel{3}\,1}{\cancel{60}\,20}$$

$$K = 0.05$$

ii) Let Mean $= \sum_{i=1}^{n} P_i x_i$

$$= P_1 X_1 + P_2 X_2 + \cdots + P_6 X_6 + P_7 X_7$$

$$= -3K - 0.2 - K + 2K + 0.8 + 6K$$

$$= 4K + 0.6$$

Mean $= 0.2 + 0.6$

Mean $= 0.8$

$0.05 \times 4 = 0.20$

iii) Variance $= V(X) = \sigma^2 = \sum_{i=1}^{n} P_i x_i^2 - \mu^2$

$= 9K + 0.4 + K + 2K + 1.6 + 18K - (0.8)^2$

$= 30K + 2 - 0.64$

$=$

Q: For the discrete probability distribution

| X | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| P(X) | 0 | 2K | 2K | 3K | $K^2$ | $2K^2$ | $7K^2+K$ |

i) Find K ii) Mean iii) variance iv) standard deviation.

Sol i) Sum of probabilities $= 1$

$10K^2 + 8K - 1 = 0 \Rightarrow 10K^2 + 8K = 1$

$K(10K+8) = 1$

$K = \frac{-4+\sqrt{26}}{10}, \frac{-4-\sqrt{26}}{10}$

$K = 0.109, \frac{-9.09}{10}$

ii) Mean $= E(X) = \bar{X} = \mu = \sum_{i=0}^{6} P_i x_i$

$\mu = P_0x_0 + P_1x_1 + P_2x_2 + P_3x_3 + P_4x_4 + P_5x_5 + P_6x_6$

$= 0 + 2K + 4K + 9K + 4K^2 + 10K^2 + (7K^2+K)6$

$\mu = 56K^2 + 21K$

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$\mu = 56(0.109)^2 + 21(0.109)$
$\mu = 2.94$

iii) Variance $= V(X) = \sigma^2$

$= \sum_{i=0}^{6} P_i x_i^2 - \mu^2$

$= 0 + 2k + 8k + 27k + 16k^2 + 50k^2 + 252k^2 + 36k - (2.94)^2$

$= 318k^2 + 73k - (2.94)^2$

$= 3.778 + 7.957 - 8.643$

$\sigma^2 = 3.092$

iv) ~~Let x mean~~ Standard deviation

$\sigma = \sqrt{V(X)}$

$= \sqrt{3.092}$

$\sigma = 1.75$

Q: Let 'X' denote the minimum of the two numbers that appear when a pair of fair dice is thrown once. Determine the i) discrete probability distribution ii) Expectation iii) Variance

Sol When two dice are thrown simultaneously possible outcomes are $6 \times 6 = 36$

S = {(1,1) (1,2) (1,3) (1,4) (1,5) (1,6)
(2,1) (2,2) (2,3) (2,4) (2,5) (2,6)
(3,1)
(4,1)
(5,1)
(6,1) (6,6) }

i)
ii)
iii)
iv

The minimum numbers could be 1, 2, 3, 4, 5, 6

i) For min. 1, favourable cases are (1, 1) (1, 2) (1, 3) (1, 4) (1, 5) (1, 6) (2, 1) (3, 1) (4, 1) (5, 1) (6, 1).

$$P(x = 1) = \frac{11}{36}$$

ii) for min 2, favourable cases are (2, 2) (2, 3) (2, 4) (2, 5) (2, 6) (3, 2) (4, 2) (5, 2) (6, 2)

$$P(X = 2) = \frac{9}{36}$$

iii) for min 3, favourable cases are (3, 3) (3, 4) (3, 5) (3, 6) (4, 3) (5, 3) (6, 3)

$$P(X = 3) = 7/36$$

iv) for min 4, favourable cases are (4, 4) (4, 5) (4, 6) (5, 4) (6, 4)

$$P(x = 4) = 5/36$$

v) for min 5, favourable cases are (5, 5) (5, 6) (6, 5)

$$P(X = 5) = 3/36$$

vi) for min 6, favourable cases are (6, 6).

$$P(x = 6) = 1/36$$

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The required probability distribution table is

| X | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| P(x) | 11/36 | 9/36 | 7/36 | 5/36 | 3/36 | 1/36 |

ii) Expectation = Mean $= \sum_{i=1}^{6} P_i x_i$

$$= \frac{11}{36} + \frac{18}{36} + \frac{21}{36} + \frac{20}{36} + \frac{15}{36} + \frac{6}{36}$$

$$= \frac{91}{36}$$

Mean $= 2.52 = \mu$

iii) Variance $= \sum_{i=1}^{6} P_i x_i^2 - \mu^2$

$$= \frac{11}{36} + \frac{36}{36} + \frac{63}{36} + \frac{80}{36} + \frac{75}{36} + \frac{36}{36} - (2.52)^2$$

$$= 8.36 - 6.35$$

Variance $= 2.01$

⇒ for the following probability distribution

| x | -3 | 6 | 9 |
|---|---|---|---|
| P(x) | 1/6 | 1/2 | 1/3 |

Find i) $E(x)$ ii) $E(x^2)$ iii) $E[(2x+1)^2]$

Sol i) $E(x) = \sum_{i=1}^{3} P_i x_i$

$$= P_1 x_1 + P_2 x_2 + P_3 x_3$$

$$= \frac{-3}{6} + \frac{6}{2} + \frac{9}{3}$$

$$= \frac{-3+18+18}{6} = \frac{33}{6} = \frac{11}{2}$$

ii) $E(x^2) = \sum_{i=1}^{3} p_i x_i^2$

$= p_1x_1^2 + p_2x_2^2 + p_3x_3^2$

$= 9\left(\frac{1}{6}\right) + 36\left(\frac{1}{2}\right) + 81\left(\frac{1}{3}\right)$

$= \frac{3}{2} + \frac{18}{1} + 27$

$E(x^2) = \frac{93}{2}$

iii) Let $E\left[(2x+1)^2\right] = E\left[4x^2 + 1 + 4x\right]$

$= 4E(x^2) + E(1) + 4E(x)$

$= 4\left(\frac{93}{2}\right) + 1 + 4\left(\frac{11}{2}\right)$

$= \frac{372+44}{2} + 1$

$= 209$

Q: calculate Expectation and variance of 'X' if the probability distribution of the RV 'X' is given by.

| X | -1 | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|
| f | 0.3 | 0.1 | 0.1 | 0.3 | 0.2 |

Sol: $E(X) = \sum_{i=1}^{5} f_i x_i$

$= -0.3 + 0 + 0.1 + 0.6 + 0.6$

$\mu = E(X) = 1.00$

$V(X) = \sum_{i=1}^{5} f_i x_i^2 - \mu^2$

$= 0.3 + 0.1 + 1.2 + 1.8 - 1$

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$= 3 \cdot 4 - 1$

$= 2 \cdot 4$

Q: Given that $f(x) = \frac{k}{2x}$ is a probability distribution for a RV 'X' that can take on the values $x = 0, 1, 2, 3, 4$. find i) k

ii) Mean iii) variance

Sol: Given that $f(x) = \frac{k}{2x}$

Here when $x = 0$ then $f(x)$ is not defined

$\therefore$ Delete $x = 0$ from RV 'X' and workout the problem

The probability distribution is

| X | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| f(x) | $\frac{k}{2}$ | $\frac{k}{4}$ | $\frac{k}{6}$ | $\frac{k}{8}$ |

i) $\sum_{i=1}^{4} p_i = 1$

$$\frac{k}{2} + \frac{k}{4} + \frac{k}{6} + \frac{k}{8} = 1$$

$$k\left[\frac{12+6+4+3}{24}\right] = 1$$

$$k = \frac{24}{25}$$

ii) Mean $= E(X) = \sum_{i=1}^{4} p_i x_i$

$$= \frac{k}{2} + \frac{k}{2} + \frac{k}{2} + \frac{k}{2}$$

$$= 2k = \frac{48}{25}$$

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iii) Variance $= \sum_{i=1}^{4} p_i x_i^2 - \mu^2$

$$= (1)\left(\frac{k}{2}\right) + 4\left(\frac{k}{4}\right) + 9\left(\frac{k}{6}\right) + 16\left(\frac{k}{8}\right) - \left(\frac{48}{25}\right)^2$$

$$= k\left(\frac{1}{2} + 1 + \frac{3}{2} + 2\right) - \left(\frac{48}{25}\right)^2$$

$$= 5k - \left(\frac{48}{25}\right)^2$$

$$= 5\left(\frac{24}{25}\right) - \left(\frac{48}{25}\right)^2$$

$$= \frac{120}{25} - \left(\frac{48}{25}\right)^2$$

$$= 4.8 - 3.6$$

$$V(x) = 1.2$$

Q: Find the mean, variance of the uniform probability distribution is given by $f(x) = \frac{1}{n}$ for $x = 1, 2, 3, \ldots n$

Sol Given that $f(x) = \frac{1}{n}$ ; $x = 1, 2, 3 \ldots n$

Probability distribution is.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $X$ | 1 | 2 | 3 | 4 | 5 | $\ldots$ | $n$ |
| $f(x) = \frac{1}{n}$ | $\frac{1}{n}$ | $\frac{1}{n}$ | $\frac{1}{n}$ | $\frac{1}{n}$ | $\frac{1}{n}$ | | $\frac{1}{n}$ |

i) Expectation $= \mu = E(x) = \sum_{i=1}^{n} f_i x_i$

$$= f_1 x_1 + f_2 x_2 + f_3 x_3 \ldots + f_n x_n$$

$$= \left(\frac{1}{n}\right)(1) + \frac{1}{n}(2) + \frac{1}{n}(3) + \ldots + \left(\frac{1}{n}\right) n$$

$$\mu = \frac{1}{n}\left[\frac{n(n+1)}{2}\right] = \frac{n+1}{2}$$

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iii) Variance $= \sigma^2 = \sum_{i=1}^{n} f_i x_i^2 - \mu^2$

$= 1\left(\frac{1}{n}\right) + \frac{1}{n}(4) + \frac{1}{n}(9) + \cdots \frac{1}{n}(n^2) - \left(\frac{n+1}{2}\right)^2$

$= \frac{1}{n}\left[1 + 2^2 + 3^2 + \cdots + n^2\right] - \left(\frac{n+1}{2}\right)^2$

$= \frac{1}{\cancel{n}}\left(\frac{\cancel{n}(n+1)(2n+1)}{6}\right) - \left(\frac{n+1}{2}\right)^2$

$= \left[\frac{2n^2+3n+1}{6} - \frac{n^2+1+2n}{4}\right]$

$= \frac{8n^2 + \cancel{12n} + 4 - 6n^2 - 6 - \cancel{12n}}{24}$

$V(X) = \frac{2n^2 - 2}{24} = \frac{n^2-1}{12}$

(P) A sample of 4 items is selected at Random from a box containg 12 items of which 5 are defective. Find the expected number 'E' of defective items.

Soln:- let 'x' is the no. of defective items among a box containing 12 items.

let 'x' values are $x = 0, 1, 2, 3, 4$

5 → are defective

7 → are Good

consider

i) $p(x=0) = p(\text{no defective item}) = \frac{7c_4}{12c_4} = \frac{7}{99}$

ii) $p(x=1) = p(\text{1 is defective and 3 are good})$

$= \frac{5c_1 \times 7c_3}{12c_4} = 35/99$

iii) $p(x=2) = p(\text{2 are defective and 2 are good})$

$= \frac{5c_2 \times 7c_2}{12c_4} = \frac{42}{99}$

iv) $P(x=3) = p(\text{3 are defective and 1 is good})$

$= \frac{5c_3 \times 7c_1}{12c_4} = \frac{14}{99}$

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v) $p(X=4) = p$ ( All are defective) $= \frac{5C_4}{12C_4} = \frac{1}{99}$

| $x_i$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $p(X=x_i)$ | $7/99$ | $35/99$ | $42/99$ | $14/99$ | $1/99$ |

$$E(X) = \sum_{i=0}^{4} p_i x_i = 0 + \frac{35}{99} + \frac{84}{99} + \frac{42}{99} + \frac{4}{99}$$

$$E(X) = \frac{165}{99} \Rightarrow 1.66$$

(P) From a box of 10 items containing 3 defectives; a sample 4 items is drawn at random let the random variable 'X' denotes the no. of defective items in the sample. Find the probability distribution of 'X' when the sample is drawn without replacement.

Soln:-

let 'X' be the no. of defective items (i.e., 3) among a box containing 10 items

let 'X' values are $X = 0, 1, 2, 3$

Where

7 → are good

3 → are defective

i) $p(X=0) = p$ (no defective) $= \frac{7C_4}{10C_4} = \frac{1}{6}$

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ii> $P(X=1) = P($ 1 is defective & 3 are good $)$

$$\Rightarrow \frac{{}^7C_3 \times {}^3C_1}{{}^{10}C_4} \Rightarrow 1/2$$

iii> $P(X=2) = P($ 2 are defective & 2 are good

$$= \frac{{}^7C_2 \times {}^3C_2}{{}^{10}C_4} \Rightarrow 3/10$$

iv) $P(X=3) = P($3 are defective and 1 good$)$

$$\Rightarrow \frac{{}^3C_3 \times {}^7C_1}{{}^{10}C_4} \Rightarrow 1/30$$

| $x_i$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $P(X=x_i)$ | 1/6 | 1/2 | 3/10 | 1/30 |

(P) For the following probability distribution

| X | -3 | -2 | -1 | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|
| P(X) | 0.001 | 0.01 | 0.1 | ? | 0.1 | 0.01 | 0.001 |

Find i> The Missing probability

ii> Mean

iii> Variance

iv) $E[X^2+2X+3]$

Sol$^n$: i> let the missing probability be $x$

~~the~~ $\sum_{i=-3}^{3} p_i x_i = 1$ then sum of the probabilities is 1.

$\Rightarrow 2(0.001) + 2(0.01) + 2(0.1) + x = 1$

$x = 1 - 0.002 - 0.02 - 0.2$

$x = 0.778$.

ii> Mean $= \sum_{i=-3}^{3} p_i x_i$

$\bar{x} = -0.003 - 0.02 - 0.1 + 0.1 + 0.02 + 0.003$

$\bar{x} = 0$ [zero]

iii> Variance $= \sum_{i=1}^{7} p_i x_i^2 - \mu^2$

$\Rightarrow 0.009 + 0.04 + 0.1 + 0.1 + 0.04 + 0.009$

Variance $(V(X)) = 0.298$.

iv) $E(X^2 + 2X + 3) = E(X^2) + 2E(X) + 3$

$\rightarrow$ We already calculated $E(x^2)$ and $E(x)$

Then $E(X^2 + 2X + 3) = 0.298 + 2(0) + 3$

$\Rightarrow 3.298$.

# Continuous Probability Distribution

23/7/19

When a RV 'X' takes every value in an interval, it gives rise to continuous distribution of 'X'. The distributions defined by the variates like temp, height, weight, are continuous distributions.

Probability density function: Consider the small interval $x-\frac{dx}{2}, x+\frac{dx}{2}$ of length $dx$ round the point 'x'. Let $f(x)$ be any continuous function of 'x' so that $f(x)\,dx$ represents the probability that the variable 'X' falls in the infinite interval $\left[x-\frac{dx}{2}, x+\frac{dx}{2}\right]$

→ Symbolically it can be expressed as

$$P\left(x-\frac{dx}{2} \leq X \leq x+\frac{dx}{2}\right) = f(x)\,dx$$

then $f(x)$ is called probability density function (or) simply density function of the variable 'X'.

Properties of probability density function (pdf) of $f(x)$:

A function $f(x)$ is said to be p.d.f if

1) $f(x) \geq 0$

2) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

3) $P(a \leq X \leq b) = \int_a^b f(x)\, dx.$

⇒ ~~Mean~~ On replacing $P_i$ by $f(x)\,dx$, $x_i$ by $x$ and $\sum_i$ by $\int$ over specified range of the variable 'x' in the formulae of discrete probability distribution.

Now we obtain, the corresponding formula for the continuous p. d.

⇒ Let $f(x)$ be the p.d.f of a RV 'X' then,

① Mean :

Mean of a distribution is given by

$$\mu = E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx.$$

If 'x' is defined from 'a' to 'b' then

$$\mu = E(X) = \int_a^b x\, f(x)\, dx$$

② Median :

Median is the point which divides the entire distribution into 2 equal parts.

In case of continuous distribution, median is the point which divides the total area into two equal parts.

Thus if 'x' is defined from a to b and 'M' is the median then,

$$\int_a^M f(x)\, dx = \int_M^b f(x)\, dx = \frac{1}{2}$$

→ Solving for M, to get the Median

③ Mode : Mode is the value of 'x' for which f(x) is maximum

Mode is thus given by $f'(x) = 0$ & $f''(x) < 0$ for $a < X < b$.

④ Variance : Variance of a distribution is given by $\sigma^2 = v(x) = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\, dx$

(or)

$$\sigma^2 = v(x) = \int_{-\infty}^{\infty} x^2 f(x) dx - \mu^2$$

---

Q: If a RV has the probability density f(x) as

$$f(x) = \begin{cases} 2e^{-2x} & \text{for } x > 0 \\ 0 & \text{for } x \le 0 \end{cases}$$

find the probability that it will take on a value i) between 1 & 3

ii) greater than 0.5

Sol Given

$$f(x) = \begin{cases} 2e^{-2x} & \text{for } x > 0 \\ 0 & x \le 0 \end{cases}$$

i) The probability that the variate 'x' can take the values b/n 1 and 3

i.e. $P(1 \le X \le 3) = \int_1^3 f(x)\, dx$

$$= \int_1^3 2e^{-2x}\, dx$$

$$= 2\left(\frac{e^{-2x}}{-2}\right)_1^3$$

$$= -\left[e^{-6} - e^{-2}\right]$$

$$= e^{-2} - e^{-6}$$

ii) The probability that the variate 'x' can take, the values greather than 0.5.

$$P(x > 0.5) = \int_{0.5}^{\infty} f(x)\,dx$$

$$= \int_{0.5}^{\infty} 2e^{-2x}\,dx$$

$$2\left(\frac{e^{-2x}}{-2}\right)_{0.5}^{\infty}$$

$$= -\left[e^{-2(\infty)} - e^{-2(0.5)}\right]$$

$$= -(0 - e^{-1})$$

$$= e^{-1}$$

Q: If the p.d. of a RV is given by

$$f(x) = \begin{cases} k(1-x^2), & \text{for } 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

find the value of 'k' and probabilities that a RV having this p.d. will take on a value i) b/w 0.1 and 0.2

ii) greater than 0.5.

Sol: WKT $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\int_{-\infty}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx + \int_{1}^{\infty} f(x)\,dx = 1$$

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$$0 + \int_0^1 f(x)\,dx + 0 = 1$$

$$\int_0^1 k(1-x^2)\,dx = 1$$

$$k\left[x - \frac{x^3}{3}\right]_0^1 = 1$$

$$k\left[\frac{2}{3}\right] = 1$$

$$k = 3/2$$

ii) The probability that the variate 'x' can take values b/w 0.1 and 0.2 is

$$P(0.1 \le x \le 0.2) = \int_{0.1}^{0.2} k(1-x^2)\,dx$$

$$= \frac{3}{2}\int_{0.1}^{0.2} (1-x^2)\,dx$$

$$= \frac{3}{2}\left[x - \frac{x^3}{3}\right]_{0.1}^{0.2}$$

$$= \frac{3}{2}\left[0.2 - \frac{0.008}{3} - 0.1 + \frac{0.001}{3}\right]$$

$$= \frac{3}{2}\left[0.1 - \frac{0.007}{3}\right]$$

$$\begin{array}{r} 0.300 \\ 0.007 \\ \hline 0.293 \end{array}$$

$$= \frac{0.293}{2}$$

$$= 0.1465$$

iii) The probability that the variate 'x' takes values greater than 0.5

0.3125

$$P(x > 0.5) = \int_{0.5}^{1} \frac{3}{2}(1-x^2)\,dx$$

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$$= \frac{3}{2}\left[x - \frac{x^3}{3}\right]_{0.5}^{\infty}$$

$$= \frac{3}{2}\left[\infty - 0.5 + \frac{0.125}{3}\right] =$$

Q: Probability density function of RV 'X' is

$$f(x) = \begin{cases} \frac{1}{2}\sin x & \text{for } 0 \le x \le \pi \\ 0 & \text{elsewhere} \end{cases}$$

Find the mean, mode and median of the distribution and also find the probability b/w 0 and $\pi/2$.

Sol Given $f(x) = \begin{cases} \frac{1}{2}\sin x & \text{for } 0 \le x \le \pi \\ 0 & \text{else where.} \end{cases}$

i) Mean $= \mu = E(x) = \int_{-\infty}^{\infty} x f(x)\, dx$

$$= \int_{-\infty}^{0} x f(x)\, dx + \int_{0}^{\pi} x f(x)\, dx + \int_{\pi}^{\infty} x f(x)\, dx$$

$$= 0 + \int_{0}^{\pi} x.\left(\frac{1}{2}\sin x\right) dx + 0$$

$$= \frac{1}{2}\int_{0}^{\pi} x \sin x\, dx$$

$$= \frac{1}{2}\left[x(-\cos x)\right]_0^{\pi} + \left(\sin x\right)_0^{\pi}\Big)$$

$$= \frac{1}{2}\left(-\pi(-1-1) + 0\right)$$

$$= \pi$$

ii) Let M be the median of the distribution.

$$\int_{a}^{M} f(x)\, dx = \int_{M}^{b} f(x)\, dx = 1/2$$

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i.e $\int_0^M f(x)\,dx = \int_M^{\pi} f(x)\,dx = 1/2$

Consider $\int_0^M f(x)\,dx = 1/2$

$$\int_0^M \frac{1}{2}\sin x = 1/2$$

$$(-\cos x)_0^M = 1$$

$$-\cos M + 1 = 1$$

$$\cos M = 0$$

$$\cos M = \cos \pi/2$$

Median $= M = \pi/2$

iii) let $f(x) = \frac{1}{2}\sin x$

$f'(x) = \frac{1}{2}\cos x$

consider $f'(x) = 0 \Rightarrow \frac{1}{2}\cos x = 0$

$$\cos x = 0$$

$$\boxed{x = \pi/2}$$

and $f''(x) = -\frac{1}{2}\sin x$

$f''(\pi/2) = -\frac{1}{2}\sin\frac{\pi}{2}$

$f''(\pi/2) = -1/2 < 0$

$\therefore$ at $x = \pi/2$ the function $f(x)$ has maximum value

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Q: A continuous RV has the probability density function $f(x) = \begin{cases} kxe^{-\lambda x}, & x \geq 0, \lambda > 0 \\ 0 & \text{otherwise} \end{cases}$

Determine i) K ii) Mean; iii) Variance

Sol i)
WKT, $\int_{-\infty}^{\infty} f(x)\, dx = 1$

$$= \int_{-\infty}^{0} f(x)\, dx + \int_{0}^{\infty} f(x)\, dx = 1$$

$$= \int_{0}^{\infty} k \underset{u}{x} \underset{v}{e^{-\lambda x}}\, dx = 1$$

$$= k\left[x\left(\frac{e^{-\lambda x}}{-\lambda}\right)_0^{\infty} - \int_0^{\infty} 1 \cdot \frac{e^{-\lambda x}}{-\lambda}\, dx\right] = 1$$

$$= k\left[\frac{-1}{\lambda}\left[\overset{\text{L-H rule.}}{0}\right] + \frac{1}{\lambda}\left(\frac{e^{-\lambda x}}{-\lambda}\right)_0^{\infty}\right] = 1$$

$$= k\left[\frac{-1}{\lambda^2}\left[0-1\right]\right] = 1$$

$$\Rightarrow k\left(\frac{1}{\lambda^2}\right) = 1$$

$$\boxed{k = \lambda^2}$$

∴ Required functn is $f(x) = \begin{cases} \lambda^2 x e^{-\lambda x} & x \geq 0, \lambda > 0 \\ 0, & \text{otherwise} \end{cases}$

(ii) Mean $= \int_{-\infty}^{\infty} x f(x)\, dx$.

$$= \int_{-\infty}^{0} x f(x)\, dx + \int_{0}^{\infty} x f(x)\, dx$$

$$\mu = \int_0^\infty x \cdot \lambda^2 x e^{-\lambda x} dx$$

$$\mu = \lambda^2 \int_0^\infty x^2 e^{-\lambda x} dx$$

$$= \lambda^2 \left[ x^2 \left( \frac{e^{-\lambda x}}{-\lambda} \right)_0^\infty - \int_0^\infty 2x \cdot \frac{e^{-\lambda x}}{-\lambda} dx \right]$$

$$= \lambda^2 \left[ 0 + \frac{2}{\lambda} \int_0^\infty x e^{-\lambda x} dx \right]$$

$$= \frac{2\lambda^2}{\lambda} \left[ x \left( \frac{e^{-\lambda x}}{-\lambda} \right)_0^\infty - \int_0^\infty 1 \cdot \frac{e^{-\lambda x}}{-\lambda} dx \right]$$

$$= 2\lambda \left[ 0 + \frac{1}{\lambda} \left( \frac{e^{-\lambda x}}{-\lambda} \right)_0^\infty \right]$$

$$= \frac{-2}{\lambda} \left[ e^{-\infty} - e^{-0} \right]$$

Mean $= 2/\lambda$

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Leibnitz rule:

$$u v^{I} - \frac{du}{dx} v^{II} + \frac{d^2u}{dx^2} v^{III}$$

diff integration

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ii) Variance : $\int_{-\infty}^{\infty} x^2 f(x) dx - \mu^2$

$$= \int_{-\infty}^{0} x^2 f(x) dx + \int_0^\infty x^2 f(x) dx - \mu^2$$

$$= 0 + \int_0^\infty x^2 (\lambda^2 x e^{-\lambda x}) dx - \mu^2$$

$$= \lambda^2 \int_0^\infty x^3 e^{-\lambda x} dx - \frac{4}{\lambda^2}$$

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$$= \lambda^2 \left\{ \left[ x^3\left(\frac{e^{-\lambda x}}{-\lambda}\right)_0^\infty - 3x^2\left(\frac{e^{-\lambda x}}{\lambda^2}\right)_0^\infty + 6x\left(\frac{e^{-\lambda x}}{-\lambda^3}\right)_0^\infty - 6\left(\frac{e^{-\lambda x}}{\lambda^4}\right)_0^\infty \right] - \left(\frac{4}{\lambda^2}\right) \right\}$$

$$= \lambda^2 \left[ 0 - \frac{6}{\lambda^4}(0-1) - \frac{4}{\lambda^2} \right]$$

$$V(X) = \frac{6}{\lambda^2} - \frac{4}{\lambda^2} = \frac{2}{\lambda^2}$$

Q: A continuous RV 'X' is defined by

$$f(x) = \begin{cases} \frac{1}{16}(3+x)^2, & \text{if } -3 \le x < -1 \\ \frac{1}{16}(6-2x^2), & \text{if } -1 \le x < 1 \\ \frac{1}{16}(3-x)^2, & \text{if } 1 \le x \le 3 \\ 0, & \text{elsewhere} \end{cases}$$

Verify that $f(x)$ is a density function and find also the mean of 'X'.

Sol i) We have to show that, $\int_{-\infty}^{\infty} f(x)\,dx = 1$

Then $\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{-3} f(x)\,dx + \int_{-3}^{-1} f(x)\,dx + \int_{-1}^{1} f(x)\,dx + \int_{1}^{3} f(x)\,dx + \int_{3}^{\infty} f(x)\,dx$

$$= 0 + \int_{-3}^{-1} \frac{1}{16}(3+x)^2\,dx + \int_{-1}^{1} \frac{1}{16}(6-2x^2)\,dx + \int_{1}^{3} \frac{1}{16}(3-x)^2\,dx + 0$$

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$$= \frac{1}{16}\int_{-3}^{-1} \cancel{3+}(3+x)^2 dx + \frac{1}{16}\int_{-1}^{1} 6-2x^2 dx$$

$$+ \frac{1}{16}\int_{1}^{3} (3-x)^2 dx.$$

$$= \frac{1}{16}\left\{\int_{-3}^{-1} 9+x^2+6x\, dx + \int_{-1}^{1} 6-2x^2\, dx + \int_{1}^{3} 9+x^2-6x\, dx\right\}$$

$$= \frac{1}{16}\left\{9(-1+3)+\frac{1}{3}(-1+27)+\frac{6}{2}(1-9)+6(1+1)- \frac{2}{3}(1+1)+9(2)+\frac{1}{3}(27-1) -\frac{6}{2}(9-1)\right\}$$

$$= \frac{1}{16}\left\{18+\frac{26}{3}-24+12-\frac{4}{3}+18+\frac{26}{3}-24\right\}$$

$$= \frac{1}{16}\{16\}$$

$$= 1 = \text{RHS}$$

Hence It is a density function

(ii) Mean $= \int_{-\infty}^{\infty} x f(x)\, dx$

$$= \int_{-3}^{-1} x\frac{1}{16}(3+x)^2 dx + \int_{-1}^{1} x\frac{1}{16}(6-2x^2)\, dx + \int_{1}^{3} x\frac{1}{16}(3-x)^2 dx$$

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$$= \frac{1}{16}\left\{\int_{-3}^{-1} 9x+x^3+6x^2\,dx + \int_{-1}^{1} 6x-2x^3\,dx + \int_{1}^{3} 9x+x^3-6x^2\,dx\right\}$$

$$= \frac{1}{16}\left\{\frac{9}{2}(1-9)+\frac{1}{4}(1-81)+\frac{6}{3}(-1+27)+\frac{6}{2}(1-1)-\frac{2}{4}(1-1)+\right.$$
$$\left.\frac{9}{2}(9-1)+\frac{1}{4}(81-1)-\frac{6}{3}(27-1)\right\}$$

$$= \frac{1}{16}\left\{-\cancel{36}-20+\cancel{\frac{6\times 26}{3}}+\cancel{36}+\frac{80}{4}-\cancel{\frac{26\times 6}{3}}\right\}$$

Mean $= 0$

Q: A continous RV 'x' is defined by $f(x)$ and the measurements of 'x' b/w 0 & 1. with a probability functn

$$f(x) = \begin{cases} 12x^3 - 21x^2 + 10x & 0 \le x \le 1 \\ 0 & \text{elsewhere} \end{cases}$$

i) find $P(X \le 1/2)$ & $P(X > 1/2)$

ii) find a number 'k' such that $P(X \le k) = \frac{1}{2}$

Sol Given $f(x) = \begin{cases} 12x^3 - 21x^2 + 10x; & 0 \le x \le 1 \\ 0 & ; \text{elsewhere} \end{cases}$

i) Let $P(X \le 1/2) = \int_0^{1/2} f(x)\,dx$

$$= \int_0^{1/2} 12x^3 - 21x^2 + 10x\,dx$$

$$= \frac{\cancel{12}^{3}}{\cancel{4}}\left(\frac{1}{16}-0\right) - \frac{21}{3}\left(\frac{1}{8}-0\right) + \frac{10}{2}\left(\frac{1}{4}-0\right)$$

$$P(X \le \tfrac{1}{2}) = \frac{3}{16} - \frac{21}{24} + \frac{5}{4} = 9/16$$

ii) $P(X > 1/2) = 1 - P(X \leq 1/2)$

$$= 1 - \frac{9}{16}$$

$$P(X > 1/2) = 7/16$$

iii) $P(X \leq k) = 1/2$

$$\int_0^k f(x)\,dx = 1/2$$

$$\int_0^k (12x^3 - 21x^2 + 10x)\,dx = 1/2$$

$$= \frac{12}{4}(k^4) - \frac{21}{3}(k^3 - 0) + \frac{10}{2}(k^2) = \frac{1}{2}$$

$$3k^4 - 7k^3 + 5k^2 = 1/2$$

$$k^2(3k^2 - 7k + 5) = 1/2$$

$$k^2 = 1/2$$

$$k = \pm\frac{1}{\sqrt{2}}$$

$$\boxed{k = 0.707}$$

$$k(3k - 7) = -4$$

$$3k^2 - 7k + 4 = 0$$

$$3k^2 - 4k - 3k + 4 = 0$$

$$k(3k - 4) - 1(3k - 4)$$

$$k = 1, \quad k = \frac{4}{3}$$

Q: for the continuous RV 'X' whose probability density functn is given by $f(x) = \begin{cases} cx(2-x) & 0 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases}$

'c' is constant

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i) $c$
ii) Mean iii) variance

Sol i) $\int_0^2 f(x)\,dx = 1$ since, it is a probability density function

$$\int_0^2 cx(2-x)\,dx = 1$$

$$c\int_0^2 2x - x^2\,dx = 1$$

$$c\left[\frac{\cancel{2}}{\cancel{2}}(4) - \frac{1}{3}(8-0)\right] = 1$$

$$c\left(\frac{4}{3}\right) = 1$$

$$\boxed{c = 3/4}$$

ii) $\text{Mean} = \int_{-\infty}^{\infty} x f(x)\,dx$

$$= \int_{-\infty}^{0} x f(x)\,dx + \int_0^2 x f(x)\,dx + \int_2^{\infty} x f(x)\,dx$$

0 0

$$= \int_0^2 x \cdot cx(2-x)\,dx$$

$$= c\left[\int_0^2 2x^2 - x^3\,dx\right]$$

$$= c\left[\frac{2}{3}(8-0) - \frac{1}{4}(16-0)\right]$$

$$= c\left[\frac{16}{3} - 4\right] = \left(\frac{3}{4}\right)\left(\frac{4}{3}\right)$$

Mean $= 1$

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iii) Variance $= \int_{-\infty}^{\infty} x^2 f(x)\,dx - \mu^2$

$$= \int_0^2 x^2\, c\, x(2-x)\,dx - \mu^2$$

$$= c\int_0^2 2x^3 - x^4\,dx - \mu^2$$

$$= \left(\frac{3}{4}\right)\left[\frac{2}{4}(16) - \frac{1}{5}(32)\right] - 1$$

$$= \frac{3}{4}\left[8 - \frac{32}{5}\right] - 1$$

$$= 6 - \frac{24}{5} - 1$$

$$= 5 - \frac{24}{5}$$

$(a-b)(a^2-2ab+b^2)$
$a^3 - 2a^2b + ab^2 - a^2 + 2ab^2 - b^3$

Variance $= 1/5$

**Q:** find the standard deviation of the probability density functn

$$f(x) = \begin{cases} x^3, & 0 \le x \le 1 \\ (2-x)^3, & 1 \le x \le 2 \end{cases}$$

$\boxed{\text{Sol: } \frac{1}{\sqrt{15}}}$

Sol: first, Mean $= \int_{-\infty}^{\infty} x f(x)\,dx$

$$= \int_{-\infty}^{0} x f(x)\,dx + \int_0^1 x f(x)\,dx + \int_1^2 x f(x)\,dx + \int_2^{\infty} x f(x)\,dx$$

$$= 0 + \int_0^1 x\, x^3\,dx + \int_1^2 x(8 - x^3 - 12x + 6x^2) + 0$$

$$= \frac{1}{4}(1-0) + \frac{8}{2}(4-1) - \frac{1}{5}(32-1) - \frac{12}{2}(4-1) + \frac{6}{3}(7$$

$$= \frac{1}{4} + 12 - \frac{31}{5} - 18 + \frac{42}{3}$$

$$= -6 \cancel{\;\;} + \frac{42}{3} + \frac{5-124}{20}$$

$$\begin{array}{r} 168 \\ -114 \\ \hline 54 \end{array}$$

$$= \cancel{\frac{\;\;42\;\;164}{12}} \; \cancel{= \frac{\;\;}{12}}$$

$$\begin{array}{r} 42 \\ -18 \\ \hline 24 \end{array}$$

$$= \cancel{\frac{27}{6}}$$

$$\begin{array}{r} 119 \\ \times 3 \\ \hline 357 \end{array} \qquad \begin{array}{r} 480 \\ -357 \\ \hline 123 \end{array}$$

$$= \frac{24}{3} - \frac{119}{20}$$

$$= \frac{480-357}{60} = \frac{123}{60}$$

$$= \frac{41}{20}$$

$$\text{Variance} = \int_0^1 x^2 x^3\,dx + \int_1^2 x^2(8-x^3-12x+6x^2)\,dx - \mu^2$$

$$= \frac{1}{6}(1-0) + \frac{8}{3}(7) - \frac{1}{6}(63) - \frac{12}{4}(15) + \frac{6}{5}(31) - \mu^2$$

$$= \frac{1}{6} + \frac{56}{3} - \frac{21}{2} - 45 + \frac{186}{5} -$$

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Q: If 'X' is a continuous 'RV' and 'k' is a constant then prove that

i) $var(X+k) = var(X)$

ii) $var(kX) = k^2 V(X)$

Sol: By the defi, we have

$$var(X) = \sigma^2 = \int_{-\infty}^{\infty} x^2 f(x)\,dx - \mu^2$$

$$var(X) = \int_{-\infty}^{\infty} x^2 f(x)\,dx - \left[\int_{-\infty}^{\infty} x f(x)\,dx\right]^2$$

Consider $var(X+k) = \int_{-\infty}^{\infty} (x+k)^2 f(x)\,dx - \left[\int_{-\infty}^{\infty} (x+k) f(x)\,dx\right]^2$

$$= \int_{-\infty}^{\infty} (x^2+k^2+2xk) f(x)\,dx - \left[\int_{-\infty}^{\infty} x f(x)\,dx + k\int_{-\infty}^{\infty} f(x)\,dx\right]^2$$

$$= \int_{-\infty}^{\infty} x^2 f(x)\,dx + k^2\int_{-\infty}^{\infty} f(x)\,dx + 2k\int_{-\infty}^{\infty} x f(x)\,dx - \left[E(X)+k\right]^2$$

$$= E(X^2) + k^2 + 2k\,E(X) - \left[(E(X))^2 + k^2 + 2k\,E(X)\right]$$

$$= E(X^2) + \cancel{k^2} + \cancel{2k\,E(X)} - (E(X))^2 - \cancel{k^2} - \cancel{2k\,E(X)}$$

$$= E(X^2) - (E(X))^2$$

$var(X+k) = V(X)$

Hence proved

ii) Consider $var(kX) = \int_{-\infty}^{\infty} (kx)^2 f(x)\,dx - \left[\int_{-\infty}^{\infty} kx f(x)\,dx\right]^2$

$$= k^2 \int_{-\infty}^{\infty} x^2 f(x)\,dx - k^2\left(\int_{-\infty}^{\infty} x f(x)\,dx\right)^2$$

$$= k^2 E(X^2) - k^2 [E(X)]^2$$

$$= k^2\left[E(X^2) - (E(X))^2\right]$$

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$= k^2 V(X)$ $\qquad V(aX+b) = a^2 V(X)$, $V(k)$

$$V(kX) = k^2 V(X)$$

Hence proved

## Bits :

1) Maximum value of probability is $\underline{1}$

2) If 'k' is a constant then variance of 'k' $= V(X)$

3) If $f(x) = Ax^2$ in $0 \le x \le 1$ is a probability function then $A = \underline{3}$

4) If $X_1, X_2$ are Random variables and a, b are constants then
$$E(aX_1 + bX_2) = aE(X_1) + bE(X_2)$$

5) If $f(x) = \frac{1}{\sqrt{2\pi}} \cdot e^{-x^2/2}$ then Mean of the distribution = ________

Standard deviation = ________

$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x e^{-x^2/2} dx$

$x^2 = t$

$2x\,dx = dt$

$\frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t/2} dt$

$\frac{1}{2\sqrt{2\pi}} \left( \frac{e^{-t/2}}{-1/2} \right)_{-\infty}^{\infty}$

$\frac{1}{\sqrt{2\pi}} [0 - \cdot$

$\frac{1}{\sqrt{2\pi}} 2 \int_0^{\infty} x^2 e^{-x^2/2} dx$

$\frac{2}{\sqrt{2\pi}} \left[ x^2 \frac{e^{-x^2/2}}{\frac{-2x}{2}} - 2x \cdot e^{-x^2/2} \right.$

$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} t e^{-t/2} dx - 0$

$\left( t \frac{e^{-t/2}}{-1/2} \right)_{-\infty}^{\infty} - \int \frac{e^{-t/2}}{+1/2}$

$0 + 4 (e^{-t/2})_{-\infty}^{\infty}$

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⑥ If 'x' is a Random variable and $V(x) = 2$ then $V(2x+3) = 2^2 V(x)$ $\quad_{ax+b}$

$= 4(2) = 8$

⑦ If the density function of the Random variable 'x' is given by

$$f(x) = \begin{cases} \frac{K}{\sqrt{x}}, & 4 \le x \le 9 \\ 0 & \text{otherwise} \end{cases} \quad \text{then } K = ?$$

Sol:

$$\int_{-\infty}^{\infty} f(x)\, dx = 1$$

$$\int_{4}^{9} \frac{K}{\sqrt{x}}\, dx = 1$$

$$K\left[2(\sqrt{x})_4^9\right] = 1$$

$$2K[3-2] = 1$$

$$\boxed{K = \frac{1}{2}}$$

<u>Note</u>: Cumulative distribution function:

The cumulative distribution functn (or) simply the distribution functn of a continuous RV 'x' is denoted by $F(x)$ and is defined as $F(x) = P(X \le x) = \int_{-\infty}^{x} f(x)\, dx$

thus $F(x)$ gives the probability that the value of the variable 'x' will be $\le x$

Properties of $F(x)$ :-

1) $0 \leq F(x) \leq 1$ ; $-\infty < x < \infty$

2) $F'(x) = f(x) \geq 0$, so that $F(x)$ is a non-decreasing function.

3) $P(a \leq x \leq b) = \int_a^b f(x)\,dx = F(b) - F(a)$

4) Since $F'(x) = f(x)$ we have $f(x) = \frac{d}{dx}[F(x)]$

This is known as probability differential of $X'$.

---

Q: A continuous RV $X'$ has the distribution function

$$F(x) = \begin{cases} 0 & \text{if } x \leq 1 \\ k(x-1)^4 & \text{if } 1 \leq x \leq 3 \\ 1 & \text{if } x > 3 \end{cases}$$

Detemine i) $f(x)$ ii) $k$ iii) Mean.

Sol

i) We know that

$$f(x) = \frac{d}{dx}[F(x)]$$

$$f(x) = \begin{cases} 0 & ; x \leq 1 \\ 4k(x-1)^3 & ; 1 \leq x \leq 3 \\ 0 & ; x > 3 \end{cases}$$

ii) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\underbrace{\int_{-\infty}^{1} f(x)\,dx}_{0} + \int_1^3 4k(x-1)^3\,dx + \underbrace{\int_3^{\infty} f(x)\,dx}_{0} = 1$$

$$\Rightarrow \cancel{4}k\left[\frac{(x-1)^4}{\cancel{4}}\right]_1^3 = 1$$

$$k(16-0) = 1$$

$$\boxed{K = \frac{1}{16}}$$

iii) Mean $= \int_{-\infty}^{\infty} x f(x)\, dx$

$$= \int_{-\infty}^{1} f(x)\,dx + \int_{1}^{3} x f(x)\,dx + \int_{3}^{\infty} x f(x)\,dx.$$

$$= \int_{1}^{3} x\, 4k\,(x-1)^3\, dx.$$

$$= 4k \int_{1}^{3} x(x^3 - 1 - 3x^2 + 3x)\, dx$$

$$= 4k \left[\frac{1}{5}(x^5) - \frac{x^2}{2} - \frac{3}{4}x^4 + 3\frac{x^3}{3}\right]_1^3$$

$81$
$\frac{3}{243}$

$$= 4\left(\frac{1}{16}\right)\left[\frac{1}{5}(242) - \frac{1}{2}(8) - \frac{3}{4}(80) + 26\right]$$

$4$ $20$

$12$
$26$
$38$

$$= \frac{1}{4}\left[\frac{242}{5} - 38\right]$$

$38$
$5$
$190$

$$= \frac{242}{20} - \frac{38}{4}$$

$$= \frac{242 - 190}{20} = \frac{52}{20}$$

$$\text{Mean} = \frac{13}{5}$$

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