UNIT-III

FLEXURAL STRESSES

When the beam is loaded with some external loads, the bending moment and shearing forces are set up at all sections of the beam. As a matter of fact, the bending moment at a section tends to bend or deflect the beam and the internal stresses resist its bending. The process of bending stops, when every cross section sets up full resistance to the bending moment. The resistance offered by the internal stresses to the bending is called bending stress and the relevant theory is called the theory of simple bending.

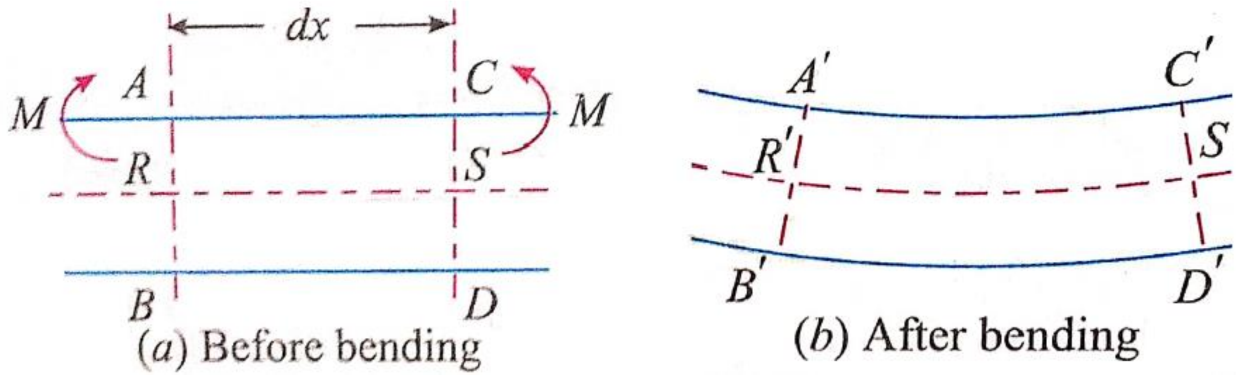
Assumptions in the theory of Simple Bending: The following assumptions are made in the theory of simple bending.

1. The material of the beam is perfectly homogeneous (same kind throughout) and isotropic (equal elastic properties in all directions).
2. The beam material is stressed within its elastic limit and thus, obeys Hook’s law.
3. The transverse sections, which were plane before bending, remains plane after bending also.
4. Each layer of the beam is free to expand or contract, independently, of the layer above or below it.
5. The value of E (Young’s Modulus of elasticity) is the same in tension and compression.
6. The beam is in equilibrium i.e., there is no resultant pull or push in the beam section.

Theory of Simple Bending: Consider a small length of a simply supported beam subjected to a bending moment as shown fig. Now consider two sections AB and CD, which are normal to the axis of the beam RS. Due to action of the bending moment, the beam as a whole will bend as shown in fig.

Since we are considering a small length of dx of the beam, therefore the curvature of the beam in this length is taken to be circular. When the beam is subjected to a load, the top layer of the beam has suffered compression and reduced to A’C’. As we proceed towards the lower layers of the beam, we find that the layers have no doubt suffered compression, but to lesser degree; until we come across the layers RS, which has suffered no change in its length. Though bent into R’S’. If we further proceed towards the lower layers, we find the layers have suffered tension, as a result of which the layers are stretched. The amount of extension increases as we proceed lower until we come across the lowermost layer BD which has been stretched to B’D’.

Now we see that the layers above have been compressed and those below RS have been stretched. The amount by which layer is compressed or stretched, depends upon the position of the layer with reference to RS. This layer RS, which is neither compressed nor stretched, is known as neutral plane or neutral layer. This theory of bending is called theory of simple bending.



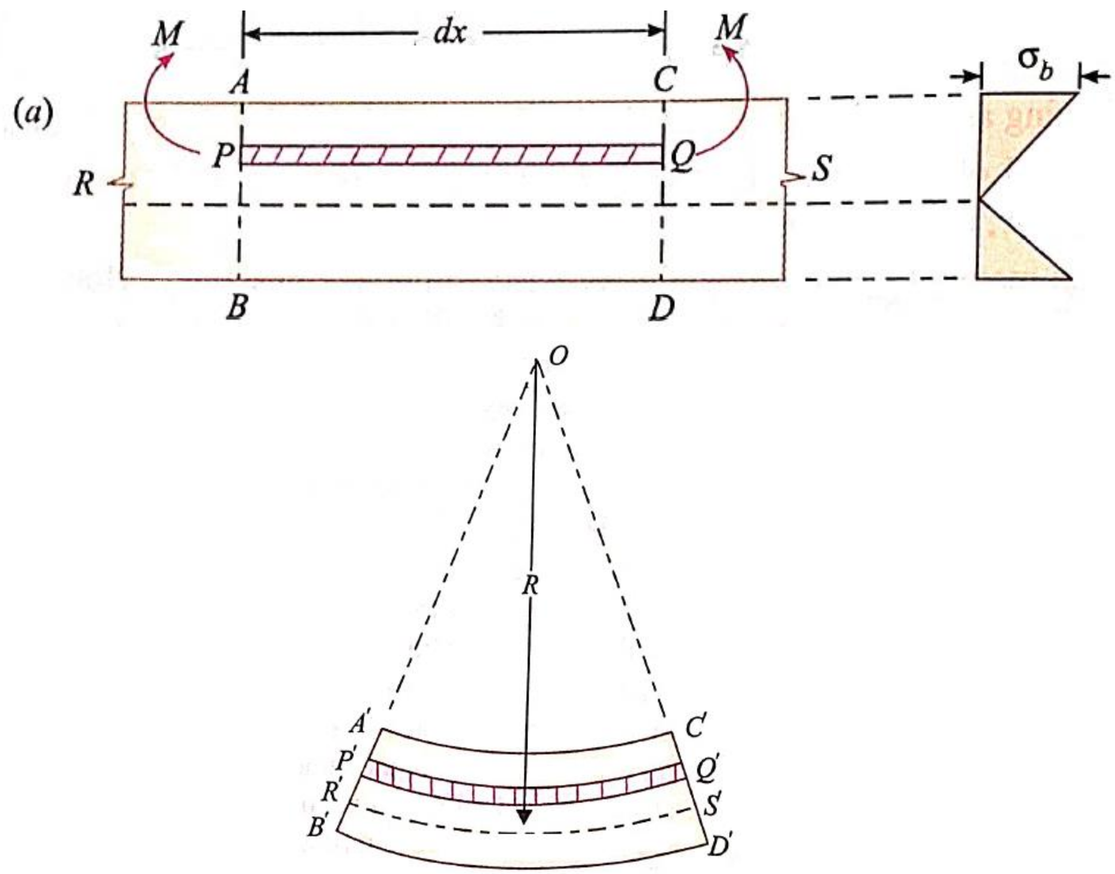
(a) Before bending (b) After bending

Bending Stress: Consider a small length dx of a beam subjected to a bending moment as shown in fig. As a result of this moment, let this small length of beam bend into an arc of a circle with O as centre as shown in fig.

Let M = Moment acting on the beam

θ = Angle subtended at the centre by the arc and

R = Radius of curvature of the beam.



(b) Bending stress

Now consider a layer PQ at a distance y from RS the neutral axis of the beam. Let this layer be compressed to $P'Q'$ after bending as shown in Fig. 14.2 (b).

We know that decrease in length of this layer,

$$\delta l = PQ - P'Q'$$

$$\therefore \quad \text{Strain } \varepsilon = \frac{\delta l}{\text{Original length}} = \frac{PQ - P'Q'}{PQ}$$

Now from the geometry of the curved beam, we find that the two sections $OP'Q'$ and $OR'S'$ are similar.

$$\therefore \quad \frac{P'Q'}{R'S'} = \frac{R-y}{R}$$

or

$$1 - \frac{P'Q'}{R'S'} = 1 - \frac{R-y}{R}$$

or

$$\frac{R'S' - P'Q'}{PQ} = \frac{y}{R}$$

$$\frac{PQ - P'Q'}{PQ} = \frac{y}{R} \qquad (PQ = R'S' = \text{Neutral axis})$$

$$\varepsilon = \frac{y}{R} \qquad \ldots\left(\because \varepsilon = \frac{PQ - P'Q'}{PQ}\right)$$

It is thus obvious, that the strain (ε) of a layer is proportional to its distance from the neutral axis. We also know that the bending stress,

$$\sigma_b = \text{Strain} \times \text{Elasticity} = \varepsilon \times E$$

$$= \frac{y}{R} \times E = y \times \frac{E}{R} \qquad \ldots\left(\because \varepsilon = \frac{y}{R}\right)$$

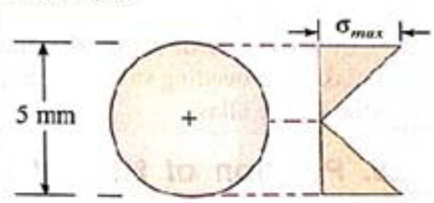
Since E and R are constants in this expression, therefore the stress at any point is directly proportional to y, *i.e.*, the distance of the point from the neutral axis. The above expression may also be written as,

$$\frac{\sigma_b}{y} = \frac{E}{R} \quad \text{or} \quad \sigma_b = \frac{E}{R} \times y$$

NOTE. Since the bending stress is inversely proportional to the radius (R), therefore for maximum stress the radius should be minimum and vice versa.

EXAMPLE *A steel wire of 5 mm diameter is bent into a circular shape of 5 m radius. Determine the maximum stress induced in the wire. Take E = 200 GPa.*

SOLUTION. Given : Diameter of steel wire (d) = 5 mm ; Radius of circular shape (R) = 5 m = 5×10^3 mm and modulus of elasticity (E) = 200 GPa = 200×10^3 N/mm^2.



We know that distance between the neutral axis of the wire and its extreme fibre,

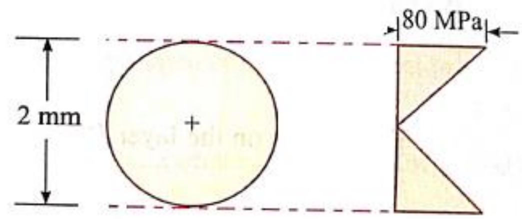
$$y = \frac{d}{2} = \frac{5}{2} = 2.5 \text{ mm}$$

and maximum bending stress induced in the wire,

$$\sigma_{b\,(max)} = \frac{E}{R} \times y = \frac{200 \times 10^3}{5 \times 10^3} \times 2.5 = 100 \text{ N/mm}^2 = 100 \text{ MPa} \qquad \textbf{Ans.}$$

Example *A copper wire of 2 mm diameter is required to be wound around a drum. Find the minimum radius of the drum, if the stress in the wire is not to exceed 80 MPa. Take modulus of elasticity for the copper as 100 GPa.*

Solution. Given : Diameter of wire (d) = 2 mm ; Maximum bending stress $\sigma_{b\,(max)} = 80$ MPa $= 80$ N/mm^2 and modulus of elasticity (E) = 100 GPa = 100×10^3 N/mm^2.



We know that distance between the neutral axis of the wire and its extreme fibre

$$y = \frac{2}{2} = 1 \text{ mm}$$

$\therefore$ Minimum radius of the drum

$$R = \frac{y}{\sigma_{b(\max)}} \times E = \frac{1}{80} \times 100 \times 10^3 \qquad \ldots\left(\because \frac{\sigma_b}{y} = \frac{E}{R}\right)$$

$$= 1.25 \times 10^3 \text{ mm} = 1.25 \text{ m} \qquad \textbf{Ans.}$$

Example *A metallic rod of 10 mm diameter is bent into a circular form of radius 6 m. If the maximum bending stress developed in the rod is 125 MPa, find the value of Young's modulus for the rod material.*

Solution. Given : Diameter of rod (d) = 10 mm ; Radius (R) = 6 m = 6×10^3 mm and maximum bending stress $\sigma_{b\,(max)} = 125$ MPa $= 125$ N/mm^2.

We know that distance between the neutral axis of the rod and its extreme fibre,

$$y = \frac{10}{2} = 5$$

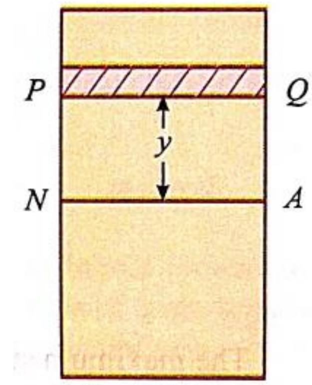
$\therefore$ Value of Young's modulus for the rod material,

$$E = \frac{\sigma_{b(\max)}}{y} \times R = \frac{125}{5} \times (6 \times 10^3) \text{ N/mm}^2 \qquad \ldots\left(\because \frac{\sigma_b}{y} = \frac{E}{R}\right)$$

$$= 150 \times 10^3 \text{ N/mm}^2 = 150 \text{ GPa} \qquad \textbf{Ans.}$$

1. A copper rod 20 mm diameter is bent into a circular arc of 8 m radius. Determine the intensity of maximum bending stress induced in the metal. Take $E = 100$ GPa. [**Ans.** 125 MPa]
2. A steel wire of 3 mm diameter is to be wound around a circular component. If the bending stress in the wire is limited to 80 MPa, find the radius of the component. Take Young's modulus for the steel as 200 GPa. [**Ans.** 3.75 m]
3. An alloy wire of 5 mm diameter is wound around a circular drum of 3 m diameter. If the maximum bending stress in the wire is not to exceed 200 MPa, find the value of Young's modulus for the alloy. [**Ans.** 120 GPa]

Moment of Resistance: We know that, on one side of the neutral axis there are compressive stresses and on the other side of the neutral axis are tensile stresses. These stresses form couple, whose moment must be equal to the external moment (M). the moment of the couple which resists the external bending moment is known as moment of resistance.



Moment of resistance

Consider a section of the beam as shown in fig.
Let NA be the neutral axis of the section. Now consider a small layer PQ of the beam section at a distance y from the neutral axis .

Let δa = Area of the layer PQ

Then, the intensity of stress in the layer PQ, $\sigma = y \; x \, \frac{E}{R}$

Total stress in the layer PQ = $y \;\; x \;\; \frac{E}{R} \;\; x \;\; \delta a$

and moment of this total stress about the neutral axis

$$= y \; X \; \frac{E}{R} \; X \; \delta a \; X \; y \; = \frac{E}{R} y2 \,.\, \delta a$$

The algebraic sum of all such moments about the neutral axis must be equal to M.

Therefore $\qquad M = \Sigma \frac{E}{R} y2 \,.\, \delta a \; = \; \frac{E}{R} \Sigma y2 \,.\, \delta a$

The expression $\Sigma y2.\delta a$ represents the moment of inertia of the area of the whole section about the neutral axis. Therefore

$$M = \frac{E}{R} X \; I$$

$$\frac{M}{I} = \frac{E}{R}$$

We also know that, $\frac{\sigma}{y} = \frac{E}{R}$ then, $\frac{M}{I} = \frac{\sigma}{y} = \frac{E}{R}$

It is the most important equation in the theory of simple bending, which gives us relation between various characteristics of a beam.

Distribution of bending stress across the section: There is no bending stress at the neutral axis. In simply supported beam, there is a compressive stress above the neutral axis and a tensile stress below it. In cantilever beam, there is a tensile stress above the neutral axis and a compressive stress below it.

The stress at a point is directly proportional to its distance from the neutral axis. The maximum stress (either compressive or tensile) takes place at the outer most layer. Or in other words, while obtaining maximum bending stress at a section , the value of y is taken as maximum.

Modulus of section: The relation for finding out the bending stress on the extreme fibre of a section, i.e.,

$$\frac{M}{I} = \frac{\sigma}{Y} \qquad \text{or} \qquad M = \sigma \, x \, \frac{I}{y}$$

From this relation, we find that the stress in a fibre is proportional to its distance from the centre of gravity. If y_{max} is the distance between the c.g. of the section and the extreme fibre of the stress, then

$$M = \sigma\text{max} \; x \frac{I}{ymax} = \sigma\text{max} \; x \, Z$$

Where $Z = \frac{I}{y}$, The term 'Z' is known as modulus of section or section modulus. The general practice of writing the above equation is M = σ x Z, where σ denotes the maximum stress, tensile or compressive in nature.

We shall consider the Modulus of section for following sections:

1. **Rectangular section.** We know that moment of inertia of a rectangular section about an axis through its centre of gravity is $I = \frac{bd3}{12}$

   ∴ Modulus section $Z = \frac{I}{y} = \frac{bd2}{6}$

2. **Circular section.** Moment of inertia for a circular section about an axis through its c.g.

$$I = \frac{\pi}{64}(d)4$$

   ∴ Modulus section $Z = \frac{I}{y} = \frac{\pi}{32}(d)3$

If the given section is hallow, then the corresponding values for external and internal dimensions should be taken.

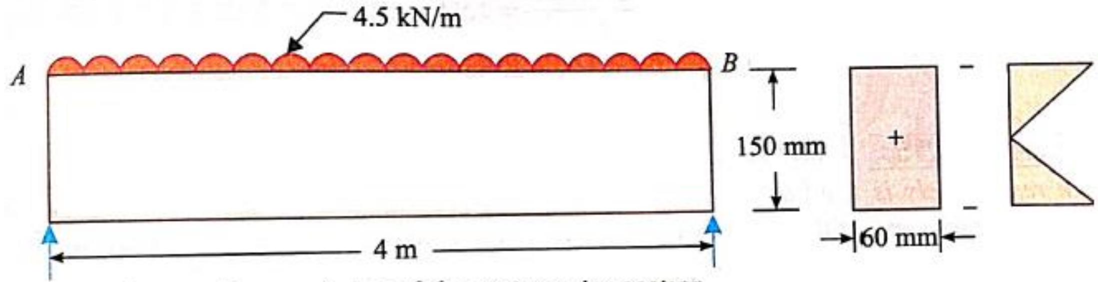
Problem-1: A rectangular beam 60mm wide and 150mm deep is simply supported over span of 6 m. If the beam is subjected to central point load of 12 kN, find the maximum bending stress induced in the beam.[**Ans:** 80 MPa.]

Problem-2: A rectangular beam 300 mm deep is simply supported over a span of 4 meters, What uniformly distributed load the beam may carry, if the bending stress is not to exceed 120 MPa. Take I = 225 x 10^6 mm^4. [**Ans:** 90 kN/m.]

Problem-3: A cantilever beam is rectangular in section having 80 mm width and 120 mm depth. If the cantilever is subjected to a point load of 60 kN at the free end and the bending stress is not to exceed 40 MPa, find the span of the cantilever beam. [**Ans:** 1.28 mm.]

Problem-4: A rectangular beam 60mm wide and 150mm deep is simply supported over span of 4 m. If the beam is subjected to a uniformly distributed load of 4.5 kN/m, find the maximum bending stress induced in the beam.

SOLUTION. Given : Width (b) = 60 mm ; Depth (d) = 150 mm ; Span (l) = 4 m = 4×10^3 mm and uniformly distributed load (w) = 4.5 kN/m = 4.5 N/mm.



We know that section modulus of the rectangular section,

$$Z = \frac{bd^2}{6} = \frac{60 \times (150)^2}{6} = 225 \times 10^3 \text{ mm}^3$$

and maxmum bending moment at the centre of a simply supported beam subjected to a uniformly distributed load,

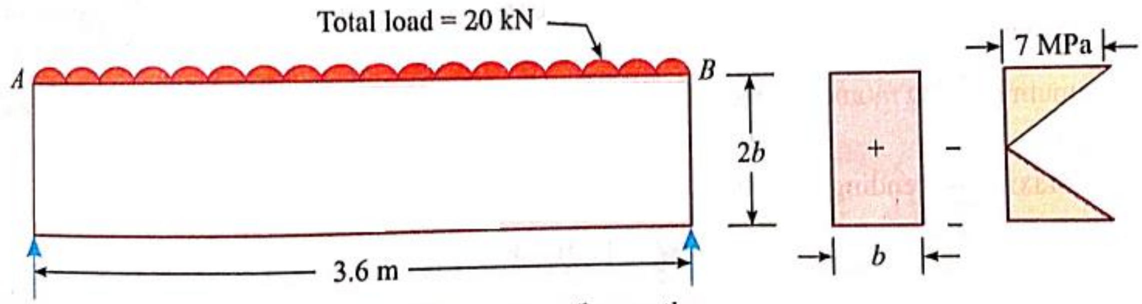
$$M = \frac{wl^2}{8} = \frac{4.5 \times (4 \times 10^3)^2}{8} = 9 \times 10^6 \text{ N-mm}$$

∴ Maximum bending stress,

$$\sigma_{max} = \frac{M}{Z} = \frac{9 \times 10^6}{225 \times 10^3} = 40 \text{ N/mm}^2 = 40 \text{ MPa} \qquad \textbf{Ans.}$$

Problem-5: A timber beam of rectangular cross section supports a load of 20 kN uniformly distributed over a span of 3.6 m. If depth of the beam section is twice the width and maximum stress is not to exceed 7 MPa, find the dimensions of the beam section.

SOLUTION. Given : Total load $(W) = 20$ kN $= 20 \times 10^3$ N ; Span $(l) = 3.6 \times 10^3$ mm ; Depth of beam section $(d) = 2b$ and $(\sigma_{max}) = 7$ MPa $= 7$ N/mm^2.



We know that section modulus of the rectangular section,

$$Z = \frac{bd^2}{6} = \frac{b \times (2b)^2}{6} = \frac{2b^3}{3}$$

and maximum bending moment at the centre of a smiply supported beam subject to a uniformly distributed load,

$$M = \frac{wl^2}{8} = \frac{Wl}{8} = \frac{(20 \times 10^3) \times (3.6 \times 10^3)}{8} = 9 \times 10^6 \text{ N-mm}$$

$\therefore$ Maximum bending stress (σ_{max}),

$$7 = \frac{M}{Z} = \frac{9 \times 10^6}{\frac{2b^2}{3}} = \frac{13.5 \times 10^6}{b^3}$$

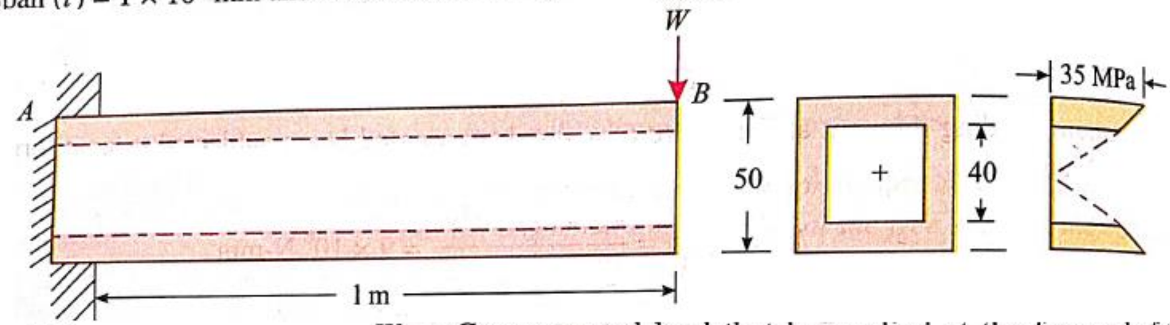
or $$b^3 = \frac{(13.5 \times 10^6)}{7} = 1.93 \times 10^6$$

$\therefore$ $$b = 1.25 \times 10^2 = 125 \text{ mm}$$ **Ans.**

and $$d = 2b = 2 \times 125 = 250 \text{ mm}$$ **Ans.**

Problem-6: A hollow square section with outer and inner dimensions of 50 mm and 40 mm respectively is used as a cantilever of span 1 m. How much concentrated load can be applied at the free end of the cantilever, if the maximum bending stress is not to exceed 35 MPa.

SOLUTION. Given : Outer width (or depth) $(B) = 50$ mm ; Inner width (or depth) $= (b) = 40$ mm; Span $(l) = 1 \times 10^3$ mm and maximum bending stress $\sigma_{b\,(max)} = 35$ MPa $= 35$ N/mm^2.



Let W = Concentreated load that be applied at the free end of the cantilever.

We know that moment of inertia of the hollow square section,

$$I = \frac{BD^3}{12} - \frac{bd^3}{12} = \frac{BB^3}{12} - \frac{bb^3}{12} = \frac{B^4}{12} - \frac{b^4}{12} = \frac{(50)^4}{12} - \frac{(40)^4}{12}\ \text{mm}^4$$
$$= 307.5 \times 10^3\ \text{mm}^4$$

$\therefore$ Modulus of section,
$$Z = \frac{I}{y} = \frac{307.5 \times 10^3}{25} = 12300\ \text{mm}^3$$

and maximum bending moment at the fixed end of the cantilever subjected to a point load at the free end

$$M = Wl = W \times (1 \times 10^3) = 1 \times 10^3\ W$$

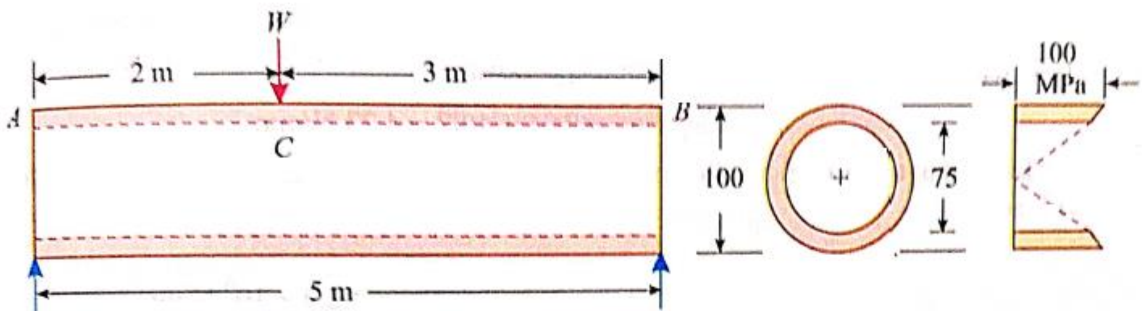
$\therefore$ Maximum bending stress (σ_{max}),

$$35 = \frac{M}{Z} = \frac{1 \times 10^3\, W}{12300}$$

or
$$W = \frac{35 \times 12300}{1 \times 10^3} = 430.5\ \text{N} \qquad \textbf{Ans.}$$

Problem-7: A hollow steel tube having external and internal diameter of 100 mm and 75 mm respectively is simply supported over a span of 5 m. The tube carries a concentrated load of W at a distance of 2 m from one of the supports. What is the value of W, if the maximum bending stress is not to exceed 100 MPa.

Solution: Given: External diameter (D) = 100 mm, Internal diameter (d) = 75 mm, Span (l) = 5 m = 5 x 10^3 mm, Distance AC (a) = 2 m = 2 x 10^3 mm, Distance BC = 5-2=3 m = 3 x 10^3 mm and maximum bending stress (σ_{max}) = 100 MPa = 100 N/mm^2.



We know that maximum bending moment over a simply supported beam subjected to an eccentric load,

$$M = \frac{Wab}{l} = \frac{W \times (2 \times 10^3) \times (3 \times 10^3)}{5 \times 10^3} = 1.2 \times 10^3\ W$$

and section modulus of a hollow circular section,

$$Z = \frac{\pi}{32 \times D} \times \left[D^4 - d^4\right] = \frac{\pi}{32 \times 100} \times \left[(100)^4 - (75)^4\right] \text{mm}^3$$

$$= 67.1 \times 10^3\ \text{mm}^3$$

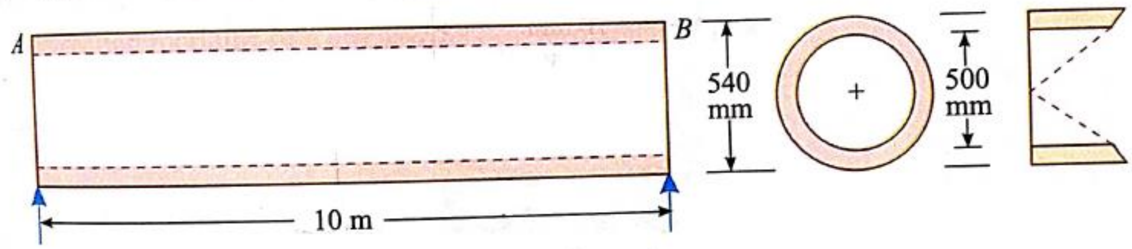
We also know that maximum bending stress $[\sigma_{b\,(max)}]$,

$$100 = \frac{M}{Z} = \frac{1.2 \times 10^3\ W}{67.1 \times 10^3} = 0.018\ W$$

$$W = \frac{100}{0.018} = 5.6 \times 10^3\ \text{N} = 5.6\ \text{kN} \qquad \textbf{Ans.}$$

Problem-8: A cast iron water pipe of 500 mm inside diameter and 20 mm thickness is supported over a span of 10 meters. Find the maximum stress in the pipe metal, when the pipe is running full. Take density of cast iron as 70.6 kn/m^3 and that of water as 9.8 kN/m^3.

SOLUTION. Given : Inside diameter (d) = 500 mm ; Thickness (t) = 20 mm or outside diameter (D) = $d + 2t = 500 + (2 \times 20) = 540$ mm ; Span (l) = 10 m = 10×10^3 mm ; density of cast iron = 70.6 kN/m^3 = 70.6×10^{-6} N/mm^2 and density of water = 9.8 kN/m^3 = 9.8×10^{-6} N/mm^2.



We know that cross-sectional area of the cast iron pipe,

$$= \frac{\pi}{4} \times \left[D^2 - d^2\right] = \frac{\pi}{4} \times \left[(540)^2 - (500)^2\right] = 32.67 \times 10^3 \text{ mm}^2$$

and its weight (w_1) = $(70.6 \times 10^{-6}) \times (32.67 \times 10^3) = 2.31$ N/mm

We also know that cross-sectional area of the water section

$$= \frac{\pi}{4} \times (d)^2 = \frac{\pi}{4} \times (500)^2 = 196.35 \times 10^3 \text{ mm}^2$$

and its weight (w_2) = $(9.8 \times 10^{-6}) \times (196.35 \times 10^3) = 1.92$ N/mm

∴ Total weight of the cast iron pipe and water section

$$w = w_1 + w_2 = 2.31 + 1.92 = 4.23 \text{ N/mm}$$

We also know that maximum bending moment at the centre of the beam subjected to a uniformly distributed load,

$$M = \frac{wl^2}{8} = \frac{4.23 \times (10 \times 10^3)^2}{8} = 52.9 \times 10^6 \text{ N-mm}$$

and section modulus of a hollow circular section,

$$Z = \frac{\pi}{32D} \times \left[D^2 - d^4\right] = \frac{\pi}{32 \times 540} \times \left[(540)^4 - (500)^4\right] \text{mm}^3$$

$$= 4.096 \times 10^6 \text{ mm}^3$$

∴ Maximum bending stress,

$$\sigma_{b\,(max)} = \frac{M}{Z} = \frac{52.9 \times 10^6}{4.096 \times 10^6} = 12.9 \text{ N/mm}^2 = 12.9 \text{ MPa} \qquad \textbf{Ans.}$$

Problem-9: A beam 3 m long has rectangular section of 80 mm width and 120 mm depth. If the beam is carrying a uniformly distributed load of 10 kN/m, find the maximum bending stress induced in the beam. [**Ans:** 58.6 MPa.]

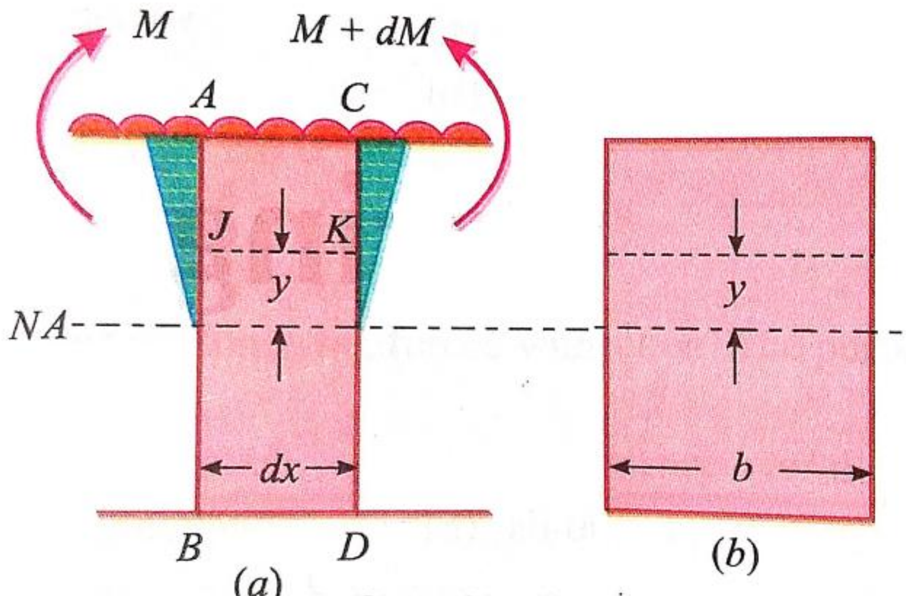
Problem-10: A rectangular beam 200 mm deep is simply supported over a span of 2 m.Find the uniformly distributed load, the beam can carry if the bending stress is not to exceed 30 MPa. Take I for the beam as $8\text{x}10^6$ mm^4. [**Ans:** 4.8 N/mm]

Problem-11: A rectangular beam, simply supported over a span of 4 m.is carrying uniformly distributed load of 50 kN/m. Find the dimensions of the beam, if depth of the beam section is 2.5 times its width. Take maximum bending stress in the beam section is 60 MPa. [**Ans:** 125 mm x 300 mm]

SHEAR STRESSES

In actual practice, when a beam is loaded, the shear force at a section always comes into play, alongwith the bending moment. It has been observed that the effect of shearing stress, as compared to the bending stress, is quite negligible, and is not of much importance. But sometimes the shear stresses at a section assumes much importance in the design criteria.

Shear stress at a section in a loaded beam: Consider a small portion ABCD of length dx of a beam loaded with uniformly distributed load as shown in fig.



Shearing stress

We know that when a beam is loaded with a uniformly distributed load, the shear force and bending moment vary at every point along the length of the beam.

Let M = Bending moment at AB,

M + dM = Bending moment at CD,

F = Shear force at AB,

F + dF = Shear force at CD, and

I = Moment of Inertia of the section about its neutral axis.

Now consider an elementary strip a distance y from the neutral axis as shown in fig.

Now let σ = Intensity of bending stress across AB at distance y from the neutral axis and
a = Cross-sectional area of the strip

We have already discussed that

$$\frac{M}{I} = \frac{\sigma}{y} \quad \text{or} \quad \sigma = \frac{M}{I} \times y$$

Similarly, $$\sigma + d\sigma = \frac{M + dM}{I} \times y$$

here $\sigma + d\sigma$ = Intensity of bending stress across CD.

We know that the force acting across AB

$$= \text{Stress} \times \text{Area} = \sigma \times a = \frac{M}{I} \times y \times a \qquad \text{...(i)}$$

Similarly, force acting across CD

$$= (\sigma + d\sigma) \times a = \frac{M + dM}{I} \times y \times a \qquad \text{...(ii)}$$

∴ Net unbalanced force on the strip

$$= \frac{M + dM}{I} \times y \times a - \frac{M}{I} \times y \times a = \frac{dM}{I} \times y \times a$$

The total *unbalanced force (F) above the neutral axis may be found out by integrating the above equation between 0 and $d/2$.

or $$= \int_0^{\frac{d}{2}} \frac{dM}{I} a \cdot y \cdot dy = \frac{dM}{I} \int_0^{\frac{d}{2}} a \cdot y \cdot dy = \frac{dM}{I} A\bar{y} \qquad \text{...(iii)}$$

where A = Area of the beam above neutral axis, and $\bar{y}$ = Distance between the centre of gravity of the area and the neutral axis.

We know that the intensity of the shear stress,

$$\tau = \frac{\text{Total force}}{\text{Area}} = \frac{\frac{dM}{I} \cdot A\bar{y}}{dx \cdot b} \qquad \text{...(Where } b \text{ is the width of beam)}$$

$$= \frac{dM}{dx} \times \frac{A \cdot \bar{y}}{Ib}$$

$$= F \times \frac{A\bar{y}}{Ib} \qquad \left(\text{Substituting } \frac{dM}{dx} = F = \text{Shear force}\right)$$

Distribution of Shearing stress: We have obtained a relation, which helps us in determining the value of shear stress at any section on a beam. Now we shall study the distribution of the shear stress along the depth of a beam. For doing so, we shall calculate the intensity of shear stress at important sections of a beam and then sketch a shear stress diagram. Such a diagram helps us in obtaining the value of shear at any section along the depth of the beam.We shall discuss the the distribution of shear stress over the following sections.

1. Rectangular sections,
2. Triangular sections,
3. Circular sections,
4. I-sections,
5. T-sections and
6. Miscellaneous sections.

Distribution of Shearing Stress over a Rectangular Sections: Consider a beam of rectangular section ABCD of width and depth as shown in fig. We know that the shear stress on a layer JK of beam, at a distance y from the neutral axis.

$$\tau = F \times \frac{A\bar{y}}{Ib}$$

Where F = Shear force at the section,

A = Area of section above y (i.e., shaded area AJKD),

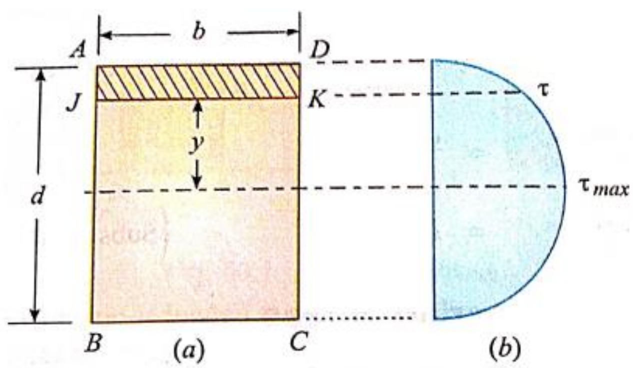
y = Distance of the shaded area from the neutral axis,

A y = Moment of the shaded area about the neutral axis,

I = Moment of inertia of the whole section about its neutral axis,

b = Width of the section

.



Rectangular section

We know that area of the shaded portion *AJKD*,

$$A = b\left(\frac{d}{2} - y\right) \qquad \text{...}(ii)$$

$$\therefore \quad \bar{y} = y + \frac{1}{2}\left(\frac{d}{2} - y\right) = y + \frac{d}{4} - \frac{y}{2}$$

$$= \frac{y}{2} + \frac{d}{4} = \frac{1}{2}\left(y + \frac{d}{2}\right) \qquad \text{...}(iii)$$

Substituting the above values of A and $\bar{y}$ in equation (i),

$$\tau = F \times \frac{A\bar{y}}{Ib} = F \times \frac{b\left(\frac{d}{2} - y\right) \times \frac{1}{2}\left(y + \frac{d}{2}\right)}{Ib}$$

$$= \frac{F}{2I}\left(\frac{d^2}{4} - y^2\right) \qquad \text{...}(iv)$$

We see, from the above equation, that τ increase as y decreases. At a point, where $y = d/2$, $\tau = 0$; and where y is zero, τ is maximum. We also see that the variation of τ with respect to y is a parabola.

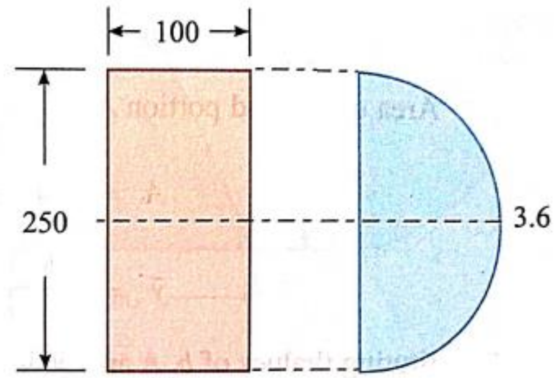
at neutral axis, the value of τ is maximum. Thus substituting $y = 0$ and $I = \dfrac{bd^3}{12}$ in the above equation,

$$\tau_{max} = \frac{F}{2 \times \frac{ba^3}{12}}\left(\frac{d^2}{4}\right) = \frac{3F}{2bd} = 1.5\,\tau_{av} \qquad \text{...}\left(\because \ \tau_{av} = \frac{F}{\text{Area}} = \frac{F}{bd}\right)$$

Now draw the shear stress distribution diagram as shown in fig.b.

Problem-1: A wooden beam 100 mm wide, 250 mm deep and 3 m long is carrying a uniformly distributed load of 40 kN/m. Determine the maximum shear stress and sketch the variation of shear stress along the depth of the beam.

SOLUTION. Given: Width (b) = 100 mm ; Depth (d) = 250 mm ; Span (l) = 3 m = 10^3 mm and uniformly distributed load (w) = 40 kN/m = 40 N/mm.



(a) (b)

We know that shear force at one end of the beam,

$$F = \frac{wl}{2} = \frac{40 \times (3 \times 10^3)}{2} \text{ N}$$
$$= 60 \times 10^3 \text{ N}$$

and area of beam section,

$$A = b \cdot d = 100 \times 250 = 25\,000 \text{ mm}^2$$

$\therefore$ Average shear stress across the section,

$$\tau_{av} = \frac{F}{A} = \frac{60 \times 10^3}{25000} = 2.4 \text{ N/mm}^2 = 2.4 \text{ MPa}$$

and maximum shear stress,

$$\tau_{max} = 1.5 \times \tau = 1.5 \times 2.4 = 3.6 \text{ MPa} \qquad \textbf{Ans.}$$

The diagram showing the variation of shear along the depth of the beam is shown in Fig.(b).

Problem-2: A rectangular beam 80 mm wide and 150 mm deep is subjected to a shearing force of 30 kN. Calculate the maximum shear stress and draw the distribution diagram for the shear stress. [**Ans:** 3.75 MPa.]

Problem-3: A rectangular beam 100 mm wide is subjected to a shearing force of 50 kN. Calculate the depth of the beam, if maximum shear stress is 3 MPa. [**Ans:** 250 mm.]

Distribution of Shearing Stress over a Triangular Section:

Consider a beam of triangular cross-section ABC of base b and height h as shown in Fig. (a).

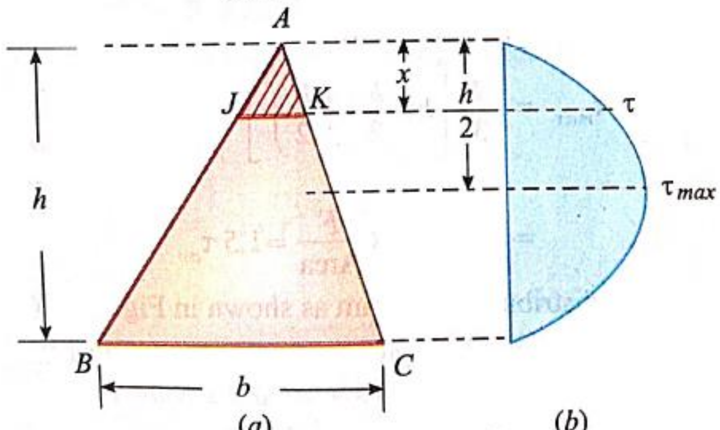
We know that the shear stress on a layer JK at a distance y from the neutral axis,

$$\tau = F \times \frac{A\,\overline{y}}{Ib} \qquad \text{...}(i)$$

F = Shear force at the section,

$A\,\overline{y}$ = Moment of the shaded area about the neutral axis and

I = Moment of inertia of the triangular section about its neutral axis.



Triangular section.

We know that width of the strip JK,

$$b = \frac{bx}{h}$$

$\therefore$ Area of the shaded portion AJK,

$$A = \frac{1}{2} JK \times x = \frac{1}{2}\left(\frac{bx}{h} \times x\right) = \frac{bx^2}{2h}$$

and

$$\bar{y} = \frac{2h}{3} - \frac{2x}{3} = \frac{2}{3}(h - x)$$

Substituting the values of b, A and $\bar{y}$ in equation (i),

$$\tau = F \times \frac{\left(\frac{bx^2}{2h}\right) \times \frac{2}{3}(h-x)}{I \times \frac{bx}{h}} = \frac{F}{3I} \times [x(h-x)]$$

$$= \frac{F}{3I} \times \left[hx - x^2\right] \qquad ...(ii)$$

Thus we see that the variation of τ with respect to x is parabola. We also see that as a point where $x = 0$ or $x = h$, $\tau = 0$. At neutral axis, where $x = \frac{2h}{3}$,

$$\tau = \frac{F}{3I}\left[h \times \frac{2h}{3} - \left(\frac{2h}{3}\right)^2\right] = \frac{F}{3I} \times \frac{2h^2}{9} = \frac{2Fh^2}{27I}$$

$$= \frac{2Fh^2}{27 \times \frac{bh^3}{36}} = \frac{8F}{3bh} \qquad ... \left(\because I = \frac{bh^3}{36}\right)$$

$$= \frac{4}{3} \times \frac{F}{\text{Area}} = 1.33\, \tau_{av} \qquad ... \left(\because \text{Area} = \frac{bh}{2}\right)$$

Now for maximum intensity, differentiating the equation (ii) and equating to zero,

$$\frac{d\tau}{dx}\left[\frac{F}{3I}(hx - x^2)\right] = 0$$

$$\therefore \qquad h - 2x = 0 \quad \text{or} \quad x = \frac{h}{2}$$

Now substituting this value of x in equation (ii),

$$\tau_{max} = \frac{F}{3I}\left[h \times \frac{h}{2} - \left(\frac{h}{2}\right)^2\right] = \frac{Fh^2}{12I} = \frac{Fh^2}{12 \times \frac{bh^3}{36}} \qquad ... \left(\because I = \frac{bh^3}{36}\right)$$

$$= \frac{3F}{bh} = \frac{3}{2} \times \frac{F}{\text{Area}} = 1.5\, \tau_{av}$$

Now draw the shear stress distribution diagram as shown in Fig. (b).

Problem-4: A beam of triangular cross section having base width of 100 mm and height of 150 mm is subjected to a shear force of 13.5 kN. Find the value of maximum shear stress and sketch the shear stress distribution along the depth of beam.

Solution: Given: Base width (b) = 100 mm; Height (h) = 150 mm; and shear force (F) = 13.5 kN.

We know that area of beam section,

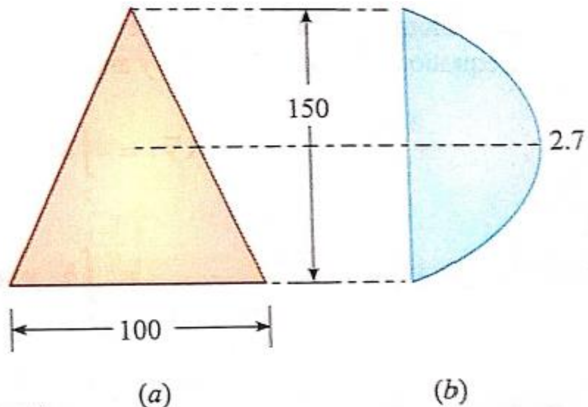
$$A = \frac{b \cdot h}{2} = \frac{100 \times 150}{2} \text{ mm}^2$$
$$= 7500 \text{ mm}^2$$

$\therefore$ Average shear stress across the section,

$$\tau_{av} = \frac{F}{A} = \frac{13.5 \times 10^3}{7500} \text{ N/mm}^2$$
$$= 1.8 \text{ N/mm}^2 = 1.8 \text{ MPa}$$

and maximum shear stress,

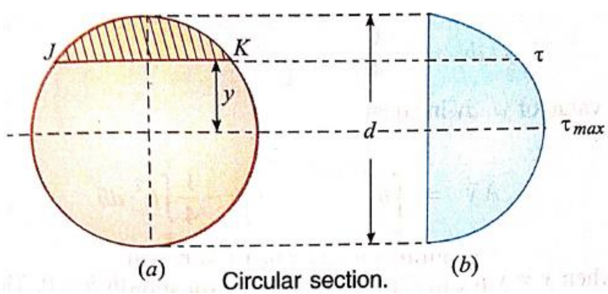
$$\tau_{av} = 1.5 \times \tau_{av} = 1.5 \times 1.8 = \mathbf{2.7\ MPa} \quad \textbf{Ans.}$$



The diagram showing the variation of shear stress along the depth of the beam is shown in Fig. (*b*).

Problem-5: A triangular beam of base width 80 mm and height 100 mm is subjected to a shear force of 12 kN. What is the value of maximum shear stress? Also draw the shear stress distribution diagram over the beam section. [**Ans:** 4.5 MPa.]

Distribution of Shearing Stress over a Circular Section:



Circular section.

Consider a circular section of diameter d as shown in Fig. 16.6 (*a*). We know that the shear stress on a layer JK at a distance y from the neutral axis,

$$\tau = F \times \frac{A\bar{y}}{Ib}$$

F = Shear force at the section,

$A\bar{y}$ = Moment of the shaded area about the neutral axis,

r = Radius of the circular section,

I = Moment of inertia of the circular section and

b = Width of the strip JK.

we know that in a circular section,

width of the strip JK, $\quad b = 2\sqrt{r^2 - y^2}$

and area of the shaded strip

$$A = 2\sqrt{r^2 - y^2} \cdot dy$$

$\therefore$ Moment of this area about the neutral axis

$$= 2y\sqrt{r^2 - y^2} \cdot dy$$

Now moment of the whole shaded area about the neutral axis may be found out by integrating the above equation between the limits y and r, *i.e.*,

$$A\bar{y} = \int_y^r 2y\sqrt{r^2 - y^2} \cdot dy$$

$$= \int_y^r b \cdot y \cdot dy \qquad \ldots (\because b = 2\sqrt{r^2 - y^2}) \quad \ldots (ii)$$

We know that width of the strip JK,

$$b = 2\sqrt{r^2 - y^2}$$

or

$$b^2 = 4\sqrt{r^2 - y^2} \qquad \ldots \text{(Squaring both sides)}$$

Differentiating both sides of the above equation,

$$2b \cdot db = 4(-2y)\, dy = -8y \cdot dy$$

or

$$y \cdot dy = -\frac{1}{4} b \cdot db$$

Substituting the value of $y \cdot dy$ in equation (*ii*),

$$A\bar{y} = \int_y^r b\left(-\frac{1}{4} b \cdot db\right) = -\frac{1}{4}\int_y^r b^2 \cdot db \qquad \ldots (iii)$$

We know that when y = y, width b = b and when y = r, width b = o. Therefore, the limits of integration may be changed from y to r, from b to zero in equation (iii),

or

$$A\bar{y} = -\frac{1}{4}\int_b^0 b^2 \cdot db$$

$$= \frac{1}{4}\int_0^b b^2 \cdot db \qquad \ldots \text{(Eliminating –ve sign)}$$

$$= -\frac{1}{4}\left[\frac{b^3}{3}\right]_0^b = \frac{b^3}{12}$$

Now substituting this value of $A\bar{y}$ in our original formula for the shear stress,

$$\tau = F \times \frac{A\bar{y}}{Ib} = F \times \frac{\frac{b^3}{12}}{Ib} = F \times \frac{b^2}{12I}$$

$$= F \times \left[\frac{(2\sqrt{r^2 - y^2})^2}{12I}\right] \qquad \ldots (\because b = 2\sqrt{r^2 - y^2})$$

$$= F \times \frac{r^2 - y^2}{3I}$$

Thus we again see that τ increases as y decreases. At a point, where $y = r$, $\tau = 0$, $= 0$ and where y is zero, τ is maximum. We also see that the variation of τ with respect to y is a parabolic curve. We see that at neutral axis τ is maximum.

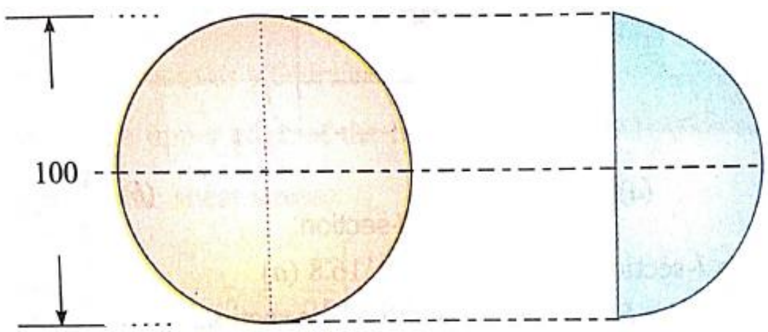
Substituting $y = 0$ and $I = \frac{\pi}{64} \times d^4$ in the above equation,

$$\tau_{max} = F \times \frac{r^2}{3 \times I} = F \times \frac{\left(\frac{d}{2}\right)^2}{3 \times \frac{\pi}{64} \times d^4} = \frac{4F}{3 \times \frac{\pi}{4} \times d^2} = 1.33\, \tau_{av}$$

Now draw the shear stress distribution diagram as shown in Fig. (b).

Problem-6: A circular beam of 100 mm diameter is subjected to a shear force of 30 kN. Calculate the value of maximum shear stress and sketch the variation of shear stress along the depth of the beam.

Solution: Given: Diameter (d) = 100 mm and shear force (F) = 30 kN = 30 x 10^3 N



(a) (b)

We know that area of the beam section,

$$A = \frac{\pi}{4}(d)^2 = \frac{\pi}{4}(100)^2 \text{ mm}^2 = 7854 \text{ mm}^2$$

$\therefore$ Average shear stress across the section

$$\tau_{av} = \frac{F}{A} = \frac{30 \times 10^3}{7854} = 3.82 \text{ N/mm}^2 = 3.82 \text{ MPa}$$

and maximum shear stress

$$\tau_{max} = 1.33 \times \tau_{av} = 1.33 \times 3.82 = \mathbf{5.08\ MPa} \qquad \textbf{Ans.}$$

The diagram showing the variation of shear stress along the depth of the beam is shown in Fig.

Problem-7: A circular beam of diameter 150 mm is subjected to a shear force of 70 kN. Find the value of maximum shear stress and sketch the shear stress distribution diagram over the beam section. [**Ans:** 5.27 MPa]