ME 793 Project # 2

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1 Introduction

In this paper, we present solutions to the Sod Shock Tube problem, originally proposed by Gary Sod in 1978. The problem involves solving the one dimensional Euler equations with a sharp discontinuity in pressure and density at the midsection of a 10m tube. Solutions are generated using the Lax Friedrichs scheme, Lax Wendroff 2 step scheme, and the Lax Wendroff scheme with artificial viscosity. Full code is available at `https://github.com/vanillaBrooks/finite-volume`

2 Methods

The one dimensional Euler equations are specified by equations 1 and 2.

$$\frac{\partial \boldsymbol{q}}{\partial t} + \frac{\partial \boldsymbol{F}}{\partial t} = 0 \tag{1}$$

$$\boldsymbol{q} = \begin{bmatrix} \rho \\ \rho u \\ \rho \left(e + \frac{1}{2}u^2\right) \end{bmatrix} \quad \boldsymbol{F} = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ u \left(p + \rho \left(e + \frac{1}{2}u^2\right)\right) \end{bmatrix} \tag{2}$$

The Lax-Friedrichs (LF) scheme is give by equation 3. A grid resolution study was performed using the LF scheme and is plotted in figure 2. A higher number of grid points results in more refined peaks in the plots. Using the results of the grid resolution study, a final value of $n = 1501$ was used for plotting purposes in combination with a conservative CFL number of 0.1.

$$q_j^{n+1} = \frac{1}{2}\left(q_{j+1}^n + q_{j-1}^n\right) - \frac{\Delta t}{2h}\left(F_{j+1}^n - F_{j-1}^n\right) \tag{3}$$

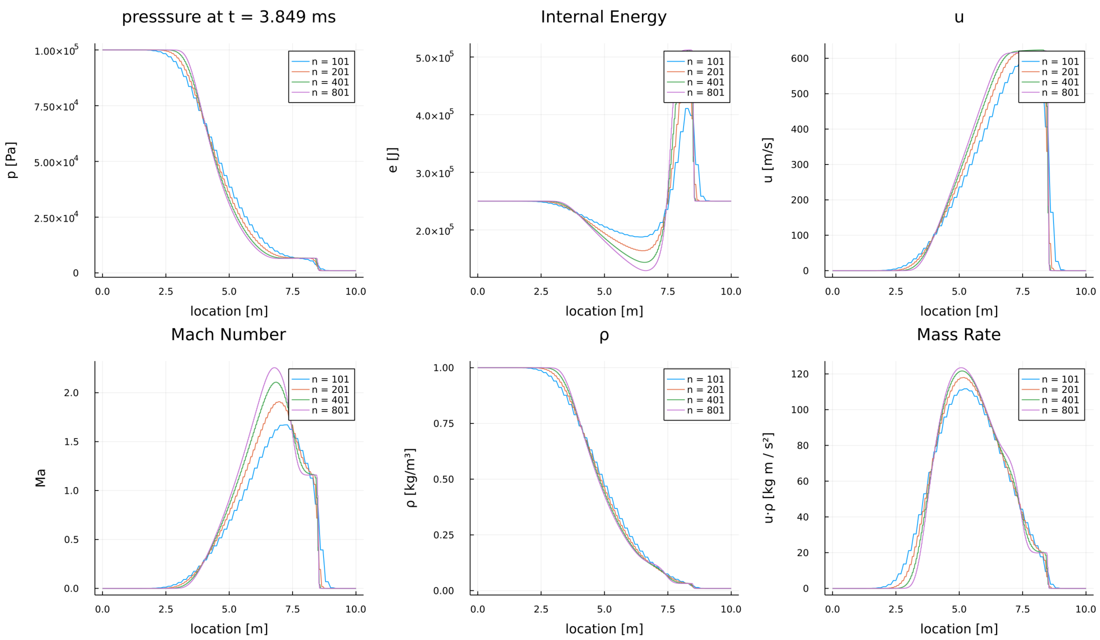

Figure 1: A grid resolution study performed using the LF method

The Lax-Wendroff 2-step scheme (LW-II) is given by equations 4 and 5. Since the LW-II scheme is subject to oscillations resembling Gibbs phenomena, an artificial flux may be added to the half step F using equation 6.

$$q^*_{j+1/2} = \frac{1}{2}\left(q^n_j + q^n_{j+1}\right) + \frac{\Delta t}{2h}\left(F^n_{j+1} + F^n_j\right) \tag{4}$$

$$q^{n+1}_j = q^n_j + \frac{\Delta t}{2}\left(F^*_{j+1/2} - F^*_{j-1/2}\right) \tag{5}$$

$$F^* = F^* - \alpha h^2 \rho \text{ abs}\left(\frac{\partial u}{\partial x}\right)\frac{\partial u}{\partial x}\begin{bmatrix}0\\1\\u\end{bmatrix} \tag{6}$$

3 Results

Figure 3 provides an overview of the solutions plotted against each other. While the LF scheme provides a good overview of the shape of the solution, the LW-II schemes better capture the sharp discontinuities in the solution. The tradeoff in the LW-II schemes is the introductions of oscillations near discontinuities. Artificial viscosity introduction (a la equatin 6) does not show a significant improvement in the shape of the solutions as compared to the regular LW-II schemes ($\alpha = 0$).

A comparison of different α values in the flux adjustment is located in figure 3. Large differences in α values fail to produce significant changes in the solution.

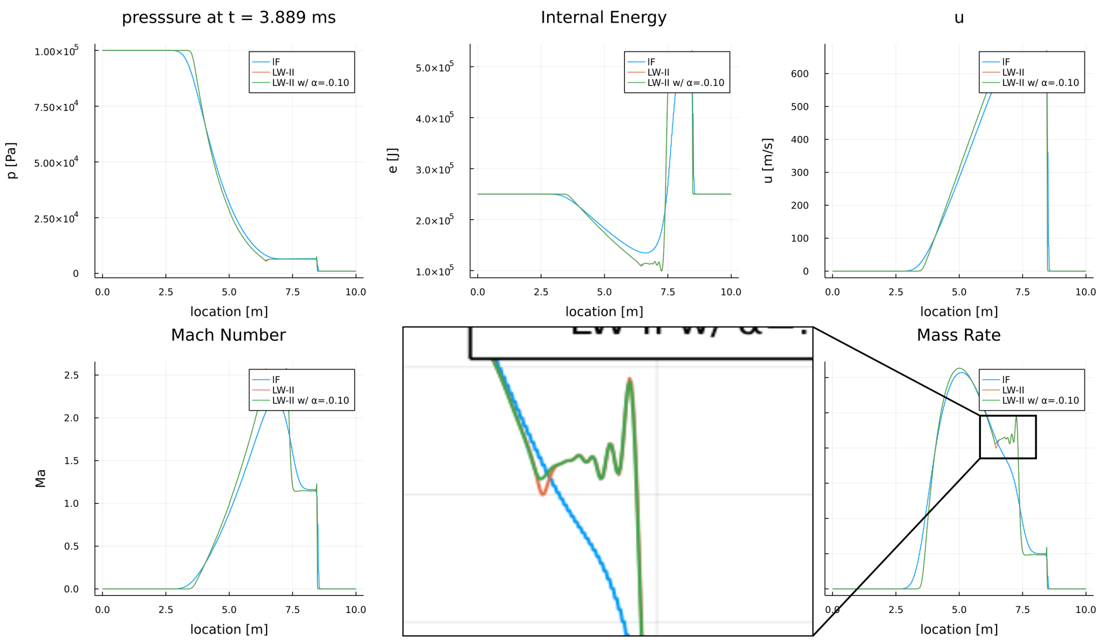

Figure 2: A qualitative comparison of the three methods. The artificial flux added to the LW-II scheme does not produce significant changes in the solution.

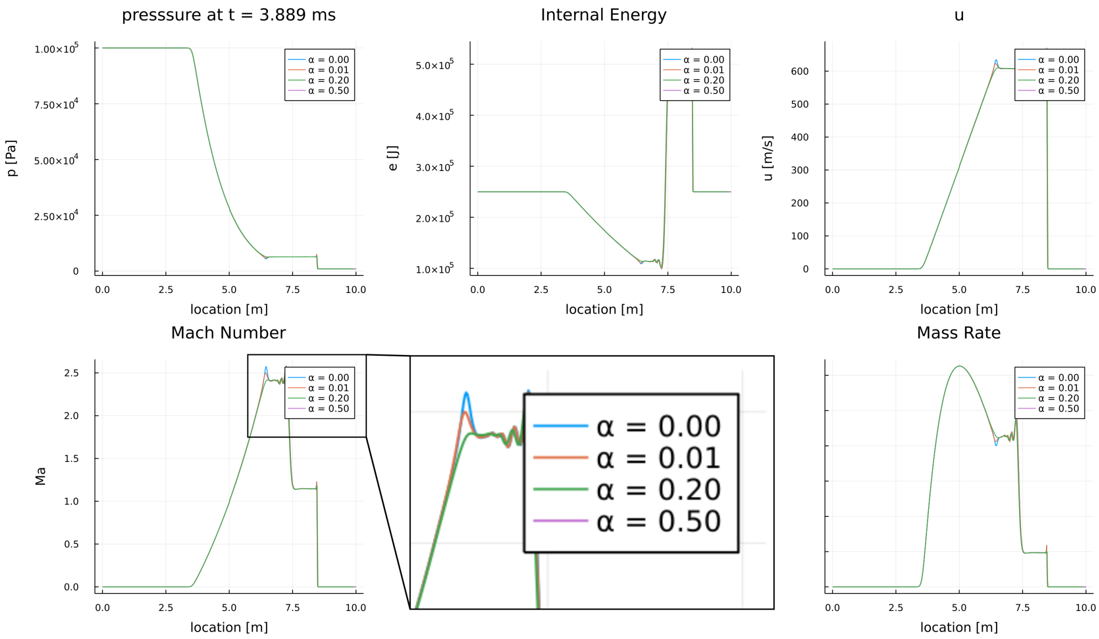

Figure 3: A comparison of the α value used in artificial flux construction on the shape of the solution. The choice of α value does not produce significant differences in the solution.