AnisotropicElasticity

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0.1 Isotropic Linear Elasticity stiffness

The static isotropic linear elasticity problem can be written as $\nabla \cdot \boldsymbol{\sigma} = \mathbf{f}$, where the stress field $\boldsymbol{\sigma}(\mathbf{x})$ is related to the linearized strain field $\boldsymbol{\epsilon}(\mathbf{x}) = 1/2(\nabla \mathbf{u} + (\nabla \mathbf{u})^T)$, $\mathbf{u}(\mathbf{x}) \in \mathbb{R}^d$, $\mathbf{x} \in \Omega \subset \mathbb{R}^d$ through the constitutive equation $\boldsymbol{\sigma} = \lambda(\mathrm{Tr}\,\boldsymbol{\epsilon})\mathbb{I} + 2\mu\boldsymbol{\epsilon}$. We consider both essential $\mathbf{u} = \mathbf{u}_0$ on Γ_D and natural $\sigma\mathbf{n} = \mathbf{g}$ on Γ_N boundary conditions. Its associated weak form is:

$$\left[\begin{array}{l} a(\mathbf{u},\mathbf{v}) = \frac{1}{2}\int_\Omega \boldsymbol{\sigma} : (\nabla \mathbf{v} + (\nabla \mathbf{v})^T)\ \mathrm{d}\mathbf{x} = \int_\Omega \mathbf{f}\cdot\mathbf{v} + \int_{\Gamma_N} \mathbf{g}\cdot\mathbf{v} \\ \forall \mathbf{v} \in H_0^1(\Omega) \quad \mathbf{v} = \mathbf{0} \ on\ \Gamma_D \\ \mathbf{u} \in H^1(\Omega) \quad \mathbf{u} = \mathbf{u}_0 \ on\ \Gamma_D \end{array}\right.$$

where $\mathbf{A} : \mathbf{B} = \mathrm{Tr}(\mathbf{A}^T\mathbf{B})$. Following a Bubnov Galerkin approach, the discrete displacement field $\mathbf{u}_h$ and test function $\mathbf{v}_h$ are defined on a high order simplicial element mesh $\Omega_h = \bigcup_i K_i$ such that $\mathbf{u}_h, \mathbf{v}_h \in V_h = \{w_h \in \mathcal{C}(\Omega_h) | \quad w_h \restriction_K \in \widehat{P}, \forall K \in \Omega_h\}$.

Proposition 1 *The local symmetric stiffness matrix of size $(dN_d)^2$ defined on K associated with the isotropic linear elastic problem can be written as*

$$K_{\mathbf{pq}} = \int_{S_n} \left(\lambda \mathbf{D_p}\mathbf{D_q}^T + \mu \mathbf{D_q}\mathbf{D_p}^T + \mu(\mathbf{D_p}\cdot\mathbf{D_q})\mathbb{I}\right)\ J(\boldsymbol{\tau})\mathrm{d}\boldsymbol{\tau} \tag{1}$$

Proof: From the definition of the displacement field $\mathbf{u}(\boldsymbol{\tau}) = \sum_{|\mathbf{p}|=n} \mathbf{u_p} N_\mathbf{p}(\boldsymbol{\tau})$, the gradient of the displacement is written as the sum of tensor products between shape vectors and nodal displacements : $\nabla \mathbf{u}(\boldsymbol{\tau}) = \sum_{|\mathbf{p}|=n} \mathbf{u_p}\mathbf{D_p}^T$. Thus the strain becomes $\boldsymbol{\epsilon} = \sum_{|\mathbf{p}|=n} \frac{1}{2}(\mathbf{u_p}\mathbf{D_p}^T + \mathbf{D_p}\mathbf{u_p}^T)$, its trace as $\mathrm{Tr}(\boldsymbol{\epsilon}) = \sum_{|\mathbf{p}|=n}(\mathbf{u_p}\cdot\mathbf{D_p})$ and the stress $\boldsymbol{\sigma} = \sum_{|\mathbf{p}|=n}\left(\lambda(\mathbf{u_p}\cdot\mathbf{D_p})\mathbb{I} + \mu(\mathbf{u_p}\mathbf{D_p}^T + \mathbf{D_p}\mathbf{u_p}^T)\right)$. The stiffness matrix is built from the bilinear form $\frac{1}{2}\int_\Omega \boldsymbol{\sigma} : (\nabla\mathbf{v} + (\nabla\mathbf{v})^T)\ \mathrm{d}\mathbf{x}$ with $\mathbf{v}(\boldsymbol{\tau}) = \mathbf{v_q}N_\mathbf{q}$:

$$\begin{aligned} \boldsymbol{\sigma} : (\nabla\mathbf{v} + (\nabla\mathbf{v})^T) &= \sum_{|\mathbf{p}|=n}\left(2\lambda(\mathbf{u_p}\cdot\mathbf{D_p})(\mathbf{v_q}\cdot\mathbf{D_q}) + \mu(\mathbf{u_p}\mathbf{D_p}^T + \mathbf{D_p}\mathbf{u_p}^T) : (\mathbf{v_q}\mathbf{D_q}^T + \mathbf{D_q}\mathbf{v_q}^T)\right) \\ &= \sum_{|\mathbf{p}|=n}\left(2\lambda\mathbf{u_p}^T\mathbf{D_p}\mathbf{D_q}^T\mathbf{v_q} + 2\mu((\mathbf{u_p}\cdot\mathbf{v_q})(\mathbf{D_p}\cdot\mathbf{D_q}) + (\mathbf{D_p}\cdot\mathbf{v_q})(\mathbf{u_p}\cdot\mathbf{D_q}))\right) \\ &= \sum_{|\mathbf{p}|=n} 2\mathbf{u_p}^T\left(\lambda\mathbf{D_p}\mathbf{D_q}^T + \mu(\mathbf{D_p}\cdot\mathbf{D_q})\mathbb{I} + \mu\mathbf{D_q}\mathbf{D_p}^T\right)\mathbf{v_q} = 2\sum_{|\mathbf{p}|=n}\mathbf{u_p}^T\bar{K}_\mathbf{pq}\mathbf{v_q} \end{aligned}$$

where we have used the following relation $(\mathbf{ab}^T + \mathbf{ba}^T) : (\mathbf{cd}^T + \mathbf{dc}^T) = 2((\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) + (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c}))$. Therefore the weak form written on K writes as $a(\mathbf{u}_h, \mathbf{v}_h) = \sum_{|\mathbf{p}|=n} \mathbf{u}_{\mathbf{p}}^T (\int_K \bar{K}_{\mathbf{pq}} \, \mathrm{d}\mathbf{x}) \mathbf{v}_{\mathbf{q}}$. ■

The stiffness matrix can be integrated exactly on a linear tetrahedron leading to

$$\tilde{K}_{ij} = \mathcal{V}_P(K) \left(\lambda \widetilde{\mathbf{D}}_{\mathbf{e}_i} \widetilde{\mathbf{D}}_{\mathbf{e}_j}^T + \mu \widetilde{\mathbf{D}}_{\mathbf{e}_j} \widetilde{\mathbf{D}}_{\mathbf{e}_i}^T + \mu (\widetilde{\mathbf{D}}_{\mathbf{e}_i} \cdot \widetilde{\mathbf{D}}_{\mathbf{e}_j}) \mathbb{I} \right)$$

0.2 Anisotropic Linear Elasticity stiffness

The prior expression based on isotropic elasticity also generalizes to the anisotropic case. Indeed, the linear relationship between strain and stress tensors is captured by the existence of a left, right minor and major symmetric fourth order elasticity tensor $\mathbb{C}$ such that $\boldsymbol{\sigma} = \mathbb{C}\boldsymbol{\epsilon}$. This fourth order tensor may also be written in terms of $d \times d$ symmetric matrices :

Proposition 2 *In a linear elastic material, there exists $r_d = \binom{d+1}{2}$ elastic constants $\alpha_k \in \mathbb{R}$ and symmetric matrix $\mathbf{A}_k$ of size $d \times d$ such that*

$$\boldsymbol{\sigma} = \sum_{k=1}^{r_d} \alpha_k \mathbf{A}_k (\mathbf{A}_k : \boldsymbol{\epsilon}) \tag{2}$$

Proof: The Kelvin notation[1] of fourth order tensors allows to transform $\boldsymbol{\sigma} = \mathbb{C}\boldsymbol{\epsilon}$ into a linear system of equations : $\bar{\boldsymbol{\sigma}} = \mathbf{C}\bar{\boldsymbol{\epsilon}}$ where $\bar{\boldsymbol{\sigma}}$ and $\bar{\boldsymbol{\epsilon}}$ are vectors of size $\binom{d+1}{2}$ and $\mathbf{C}$ is a symmetric matrix of size $\binom{d+1}{2}^2$. This symmetric real matrix can be diagonalized as $\mathbf{C} = \sum_{k=1}^{r_d} \alpha_k \, \mathbf{a}_k \mathbf{a}_k^T$ where α_k and $\mathbf{a}_k$ are the eigenvalues and eigenvectors of the matrix $\mathbf{C}$[**?**]. Therefore, the application of the elasticity tensor on $\bar{\boldsymbol{\epsilon}}$ can be written as $\mathbf{C}\bar{\boldsymbol{\epsilon}} = \sum_{k=1}^{r_d} \alpha_k \, \mathbf{a}_k (\mathbf{a}_k \cdot \bar{\boldsymbol{\epsilon}})$. The eigenvectors $\mathbf{a}_k$ can then be written as $d \times d$ symmetric matrices $\mathbf{A}_k$ such that $\mathbf{a}_k \cdot \bar{\boldsymbol{\epsilon}} = \mathbf{A}_k : \boldsymbol{\epsilon}$ following the property of Kelvin notation. ■

This alternative way of decomposing the elasticity tensor has the advantage of being non-ambiguous as opposed to the $\binom{d+1}{2}^2$ matrix $\mathbf{C}$ for which one has to state whether it is written in Voigt or Kelvin notations. Furthermore, it allows to write the density of strain energy as $w = \sum_k \alpha_k (\mathbf{A}_k : \boldsymbol{\epsilon})^2$.

Following the spectral decomposition of elasticity tensors[3, 2], the isotropic elasticity corresponds to the following decomposition for $d = 3$: $\alpha_1 = 3\lambda + 2\mu$, $\mathbf{A}_1 = \frac{\mathbb{I}}{\sqrt{3}}$, $\alpha_i = 2\mu$ for $2 \leq i \leq 6$, $\mathbf{A}_2 = \frac{2\,\mathbf{e}_1\,\mathbf{e}_1^T - \mathbf{e}_2\,\mathbf{e}_2^T - \mathbf{e}_3\,\mathbf{e}_3^T}{\sqrt{6}}$, $\mathbf{A}_3 = \frac{\mathbf{e}_2\,\mathbf{e}_2^T - \mathbf{e}_3\,\mathbf{e}_3^T}{\sqrt{2}}$, and $\mathbf{A}_{k+3} = \frac{\mathbf{e}_i\,\mathbf{e}_j^T + \mathbf{e}_j\,\mathbf{e}_i^T}{\sqrt{2}}$ for $\{i, j, k\} = \{1, 2, 3\}$. In this specific case, expression Eq. 2 leads to $\boldsymbol{\sigma} = \lambda(\mathrm{Tr}\,\boldsymbol{\epsilon})\mathbb{I} + 2\mu\boldsymbol{\epsilon}$. In the case of cubic anisotropy for instance, the same matrices $\mathbf{A}_k$ and coefficients α_k are considered than in the isotropic case with the exception that $\alpha_i = 2\mu^0$ for $i \geq 3$. For other symmetries of the elasticity tensor, the expressions of $\mathbf{A}_k$ and α_k can be derived as function of dilation, isochoric extension and pure shearing modes [3, 2].

Proposition 3 *The local stiffness matrix defined on K associated with the anisotropic linear elastic problem can be written as*

$$K_{\mathbf{pq}} = \int_{S_n} \left(\sum_{k=1}^{r_d} \alpha_k \mathbf{A}_k \mathbf{D}_{\mathbf{q}} \mathbf{D}_{\mathbf{p}}^T \mathbf{A}_k \right) \; J(\boldsymbol{\tau}) \mathrm{d}\boldsymbol{\tau} \tag{3}$$

Proof: The Frobenius dot product of the strain matrix with each of the eigen-stiffness $\mathbf{A}_k$ can be simplified as $\boldsymbol{\epsilon} : \mathbf{A}_k = \sum_{|\mathbf{p}|=n} \frac{1}{2}(\mathbf{u}_{\mathbf{p}} \mathbf{D}_{\mathbf{p}}^T + \mathbf{D}_{\mathbf{p}} \mathbf{u}_{\mathbf{p}}^T) : \mathbf{A}_k = \sum_{|\mathbf{p}|=n} \mathbf{u}_{\mathbf{p}}^T \mathbf{A}_k \mathbf{D}_{\mathbf{p}}$ thus leading to the stress tensor $\boldsymbol{\sigma} = \sum_{k=1}^{r_d} \alpha_k \sum_{|\mathbf{p}|=n} \mathbf{u}_{\mathbf{p}}^T \mathbf{A}_k \mathbf{D}_{\mathbf{p}} \mathbf{A}_k$

$$\begin{aligned}
\boldsymbol{\sigma} : (\nabla \mathbf{v} + (\nabla \mathbf{v})^T) &= \sum_{k=1}^{r_d} \alpha_k \sum_{|\mathbf{p}|=n} \big((\mathbf{u}_{\mathbf{p}}^T \mathbf{A}_k \mathbf{D}_{\mathbf{p}})(\mathbf{A}_k : (\mathbf{v}_{\mathbf{q}} \mathbf{D}_{\mathbf{q}}^T + \mathbf{D}_{\mathbf{q}} \mathbf{v}_{\mathbf{q}}^T) \big) \\
&= \sum_{k=1}^{r_d} 2\alpha_k \sum_{|\mathbf{p}|=n} \big((\mathbf{u}_{\mathbf{p}}^T \mathbf{A}_k \mathbf{D}_{\mathbf{p}})(\mathbf{D}_{\mathbf{q}}^T \mathbf{A}_k \mathbf{v}_{\mathbf{q}}) \big) = 2 \sum_{|\mathbf{p}|=n} \mathbf{u}_{\mathbf{p}}^T \bar{K}_{\mathbf{pq}} \mathbf{v}_{\mathbf{q}}
\end{aligned}$$

where we use the following identity : $(\mathbf{a}\mathbf{b}^T + \mathbf{b}\mathbf{a}^T) : \mathbf{H} = 2\mathbf{a}^T \mathbf{H} \mathbf{b} = 2\mathbf{b}^T \mathbf{H} \mathbf{a}$ for any symmetric matrix $\mathbf{H}$.■

In the case of a linear tetrahedron, the stiffness matrix can be written in closed form as in Eq.**??**:

$$\tilde{K}_{ij} = \mathcal{V}_P(K) \left(\sum_{k=1}^{r_d} \alpha_k \mathbf{A}_k \mathbf{D}_{\mathbf{e}_j} \mathbf{D}_{\mathbf{e}_i}^T \mathbf{A}_k \right)$$

References

[1] J. Dellinger, D. Vasicek, and C. Sondergeld. Kelvin notation for stabilizing elastic-constant inversion. *Revue de L?institut Francais du Petrole*, 53:709–719, 1998.

[2] Sandrine Germain. *On Inverse Form Finding for Anisotropic Materials in the Logarithmic Strain Space.* PhD thesis, University of Erlangen-Nuremberg, 2013.

[3] Rolf Mahnken. Anisotropy in geometrically non-linear elasticity with generalized Seth–Hill strain tensors projected to invariant subspaces. *Communications in Numerical Methods in Engineering*, 21(8):405–418, 2005.