Inverse Extended Kalman Filter — Part I: Fundamentals

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Abstract-Recent advances in counter-adversarial systems have garnered significant research attention to inverse filtering from a Bayesian perspective. For example, interest in estimating the adversary's Kalman filter tracked estimate with the purpose of predicting the adversary's future steps has led to recent formulations of inverse Kalman filter (I-KF). In this context of inverse filtering, we address the key challenges of non-linear process dynamics and unknown input to the forward filter by proposing an inverse extended Kalman filter (I-EKF). The purpose of this paper and the companion paper (Part II) is to develop the theory of I-EKF in detail. In this paper, we assume perfect system model information and derive I-EKF with and without an unknown input when both forward and inverse state-space models are non-linear. In the process, I-KF-with-unknown-input is also obtained. We then provide theoretical stability guarantees using both bounded non-linearity and unknown matrix approaches. Numerical experiments validate our methods for various proposed inverse filters using the recursive Cramér-Rao lower bound as a benchmark. In the companion paper (Part II), we further generalize these formulations to highly non-linear models and propose reproducing kernel Hilbert space-based EKF to handle incomplete system model information.

Index Terms—Bayesian filtering, counter-adversarial systems, extended Kalman filter, inverse filtering, non-linear processes.

I. INTRODUCTION

In many engineering applications, it is desired to infer the parameters of a filtering system by observing its output. This inverse filtering is useful in applications such as system identification, fault detection, image deblurring, and signal deconvolution [1, 2]. Conventional inverse filtering is limited to non-dynamic systems. However, applications such as cognitive and counter-adversarial systems [3-5] have recently been shown to require designing the inverse of classical stochastic filters such as hidden Markov model (HMM) filter [6] and Kalman filter (KF) [7]. The cognitive systems are intelligent units that sense the environment, learn relevant information about it, and then adapt themselves in real-time to optimally enhance their performance. For example, a cognitive radar [8] adapts both transmitter and receiver processing in order to achieve desired goals such as improved target detection [9] and tracking [10]. In this context, [11] recently introduced inverse cognition, in the form of inverse stochastic filters, to detect cognitive sensor and further estimate the information that the same sensor may have learnt. In this two-part paper, we focus on inverse stochastic filtering for such inverse cognition applications.

At the heart of inverse cognition are two agents: 'defender' (e.g., an intelligent target) and an 'adversary' (e.g., a sensor or radar) equipped with a Bayesian tracker. The adversary infers an estimate of the defender's kinematic state and cognitively adapts its actions based on this estimate. The defender observes adversary's actions with the goal to predict its future actions in a Bayesian sense. In particular, [12] developed stochastic revealed preferences-based algorithms to ascertain if the adversary's actions are consistent with optimizing a utility function; and if so, estimate that function.

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If the defender aims to guard against the adversary's future actions, it requires an estimate of the adversary's inference. This is precisely the objective of inverse Bayesian filtering. In (forward) Bayesian filtering, given noisy observations, a posterior distribution of the underlying state is obtained. An example is the KF, which provides optimal estimates of the underlying state in linear system dynamics with Gaussian measurement and process noises. The inverse filtering problem, on the other hand, is concerned with estimating this posterior distribution of a Bayesian filter given the noisy measurements of the posterior. An example of such a system is the recently introduced inverse Kalman filter (I-KF) [11]. Note that, historically, the Wiener filter - a special case of KF when the process is stationary - has long been used for frequency-domain inverse filtering for deblurring in image processing [13]. Further, some early works [14] have investigated the inverse problem of finding cost criterion for a control policy.

1

Although KF and its continuous-time variant Kalman-Bucy filter [15] are highly effective in many practical applications, they are optimal for only linear and Gaussian models. In practice, many engineering problems involve non-linear processes [16, 17]. In these cases, a *linearized KF* is used, wherein the states of a linear system represent the deviations from a nominal trajectory of a non-linear system. The KF estimates the deviations from the nominal trajectory and obtains an estimate of the states of the non-linear system. The linearized KF is extended to directly estimate the states of a non-linear system in the extended KF (EKF) [18]. The linearization is locally at the state estimates through Taylor series expansion. This is very similar to the Volterra series filters [19] that are non-linear counterparts of adaptive linear filters.

While inverse non-linear filters have been studied for adaptive systems in some previous works [20, 21], the inverse of non-linear stochastic filters such as EKF remain unexamined so far. To address the aforementioned non-linear inverse cognition scenarios, contrary to prior works which focus on only linear I-KF [11], our goal is to derive and analyze inverse EKF (I-EKF). Note that the I-EKF is different from the *inversion of EKF* [22], which may not take the same form as EKF, is employed on the adversary's side, and is unrelated to our inverse cognition problem. Similarly, the non-linear extended information filter (EIF) proposed in [23] used inverse of covariance matrix and was compared with KF for estimation of the same states. Our inverse EKF is a different formulation that is focused on estimating the inference of an adversary who is also using an EKF to estimate the defender's state.

Preliminary results of this work appeared in our conference publication [24], where only I-EKF without any unknown inputs was formulated. In this paper, we present inverses of many other EKF formulations for systems with unknown inputs and provide their stability analyses. The companion paper (Part II) [25] further develops the I-EKF theory to highly non-linear systems where first-order EKF does not sufficiently address the linear approximation. Our main contributions in this paper (Part I) are:

1) I-KF and I-EKF with unknown inputs. In the inverse cognition scenario, the target may introduce additional motion or jamming that is known to the target but not to the adversarial cognitive sensor. In this context, while deriving I-EKF, we consider a more general nonlinear system model with unknown input. Unknown inputs refer to exogenous excitations to the system which affect the state transition and observations but are not known to the agent employing the

stochastic filter. In the process, we also obtain I-KF-with-unknown-input that was not examined in the I-KF developed in [11]. Here, similar to the inverse cognition frameworks investigated in [5, 11], we assume that the adversary's filter is known to the defender. In the companion paper (Part II) [25], we consider the case when no prior information about the adversary's filter is available.

2) Augmented states for I-EKF. For systems with unknown inputs, the adversary's state estimate depends on its estimate of the unknown input. As a result, the adversary's forward filters vary with system models. We overcome this challenge by considering augmented states in the inverse filter so that the unknown input estimation is performed jointly with state estimation, including for KF with direct feed-through. For different inverse filters, separate augmented states are considered depending on the state transitions for the inverse filter.

3) Stability of I-EKF. The treatment of linear filters includes filter stability and model error sensitivity. But, in general, stability and convergence results for non-linear KFs, and more so for their inverses, are difficult to obtain. In this work, we show the stability of I-EKF using two techniques. The first approach is based on bounded nonlinearities, which has been earlier employed for proving stochastic stability of discrete-time [26] and continuous-time [27] EKFs. Here, the estimation error was shown to be exponentially bounded in the mean-squared sense. The second method relaxes the bound on the initial estimation error by introducing unknown matrices to model the linearization errors [28]. Besides providing the sufficient conditions for error boundedness, this approach also rigorously justifies the enlarging of the noise covariance matrices to stabilize the filter [29]. Since the I-EKF's error dynamics depends on the forward filter's recursive updates, the derivations of these theoretical guarantees are not straightforward. We validate the estimation errors of all inverse filters through extensive numerical experiments with recursive Cramér-Rao lower bound (RCRLB) [30] as the performance metric.

The rest of the paper is organized as follows. In the next section, we provide the background of inverse cognition model. The inverse EKF with unknown input is then derived in Section III for the case of the forward EKF with and without direct feed-through. Here, we also obtain the standard I-EKF in the absence of unknown input. Then, similar cases are considered for inverse KF with unknown input in Section IV. We then derive the stability conditions in Section V. In Section VI, we corroborate our results with numerical experiments before concluding in Section VII.

Throughout the paper, we reserve boldface lowercase and uppercase letters for vectors (column vectors) and matrices, respectively. The transpose operation and l_2 norm (for a vector) are denoted by $(\cdot)^T$ and $||\cdot||_2$, respectively. The notation $\operatorname{Tr}(\mathbf{A})$, $\operatorname{rank}(\mathbf{A})$, and $||\mathbf{A}||$, respectively, denote the trace, rank, and spectral norm of \mathbf{A} . For matrices \mathbf{A} and \mathbf{B} , the inequality $\mathbf{A} \preceq \mathbf{B}$ means that $\mathbf{B} - \mathbf{A}$ is a positive semidefinite (p.s.d.) matrix. For a function $f: \mathbb{R}^n \to \mathbb{R}^m$, ∇f denotes the $\mathbb{R}^{m \times n}$ Jacobian matrix. Similarly, for a function $f: \mathbb{R}^n \to \mathbb{R}$, ∇f denote the gradient vector $(\mathbb{R}^{n \times 1})$. A $n \times n$ identity matrix is denoted by \mathbf{I}_n and a $n \times m$ all zero matrix is denoted by $\mathbf{0}_{n \times m}$. The notation $\{a_i\}_{i_1 \leq i \leq i_2}$ denotes a set of elements indexed by integer i. The notation $\mathbf{x} \sim \mathcal{N}(\boldsymbol{\mu}, \mathbf{Q})$ and $x \sim \mathcal{U}[u_l, u_u]$, respectively, represent a random variable drawn from a normal distribution with mean $\boldsymbol{\mu}$ and covariance matrix \mathbf{Q} , and the uniform distribution over $[u_l, u_u]$.

II. DESIDERATA FOR INVERSE COGNITION

Consider a discrete-time stochastic dynamical system as the defender's state evolution process $\{\mathbf{x}_k\}_{k\geq 0}$, where $\mathbf{x}_k\in\mathbb{R}^{n\times 1}$ is the state at the k-th time instant. The defender perfectly knows its current state \mathbf{x}_k . The control input $\mathbf{u}_k\in\mathbb{R}^{m\times 1}$ is known to the defender but not to the adversary. In a linear state-space model, we denote

the state-transition and control input matrices by $\mathbf{F} \in \mathbb{R}^{n \times n}$ and $\mathbf{B} \in \mathbb{R}^{n \times m}$, respectively. The defender's state evolves as

$$\mathbf{x}_{k+1} = \mathbf{F}\mathbf{x}_k + \mathbf{B}\mathbf{u}_k + \mathbf{w}_k,\tag{1}$$

where $\mathbf{w}_k \sim \mathcal{N}(\mathbf{0}_{n\times 1}, \mathbf{Q})$ is the process noise with covariance matrix $\mathbf{Q} \in \mathbb{R}^{n\times n}$. At the adversary, the observation and control input matrices are given by $\mathbf{H} \in \mathbb{R}^{p\times n}$ and $\mathbf{D} \in \mathbb{R}^{p\times m}$, respectively. The adversary makes a noisy observation $\mathbf{y}_k \in \mathbb{R}^{p\times 1}$ at time k as

$$\mathbf{y}_k = \mathbf{H}\mathbf{x}_k + \mathbf{D}\mathbf{u}_k + \mathbf{v}_k,\tag{2}$$

where $\mathbf{v}_k \sim \mathcal{N}(\mathbf{0}_{p \times 1}, \mathbf{R})$ is the adversary's measurement noise with covariance matrix $\mathbf{R} \in \mathbb{R}^{p \times p}$.

The adversary uses $\{\mathbf{y}_j\}_{1 \leq j \leq k}$ to compute the estimate $\hat{\mathbf{x}}_k$ of the defender's state \mathbf{x}_k using a (forward) stochastic filter. The adversary then uses this estimate to administer an action matrix $\mathbf{G} \in \mathbb{R}^{n_a \times n}$ on $\hat{\mathbf{x}}_k$. The defender makes noisy observations of this action as

$$\mathbf{a}_k = \mathbf{G}\hat{\mathbf{x}}_k + \boldsymbol{\epsilon}_k \in \mathbb{R}^{n_a \times 1}, \tag{3}$$

where $\epsilon_k \sim \mathcal{N}(\mathbf{0}_{n_a \times 1}, \mathbf{\Sigma}_{\epsilon})$ is the defender's measurement noise with covariance matrix $\mathbf{\Sigma}_{\epsilon} \in \mathbb{R}^{n_a \times n_a}$. Finally, the defender uses $\{\mathbf{a}_j, \mathbf{x}_j, \mathbf{u}_j\}_{1 \leq j \leq k}$ to compute the estimate $\hat{\mathbf{x}}_k \in \mathbb{R}^{n \times 1}$ of $\hat{\mathbf{x}}_k$ in the (inverse) stochastic filter. Define $\hat{\mathbf{u}}_k$ to be the estimate of \mathbf{u}_k as computed in the adversary's forward filter, while $\hat{\mathbf{u}}_k$ is an estimate of $\hat{\mathbf{u}}_k$ as computed by the defender's inverse filter. The noise processes $\{\mathbf{w}_k\}_{k \geq 0}, \{\mathbf{v}_k\}_{k \geq 1}$ and $\{\epsilon_k\}_{k \geq 1}$ are mutually independent and i.i.d. across time. These noise distributions are known to the defender as well as the adversary. When the unknown input is absent, either $\mathbf{B} = \mathbf{0}_{n \times m}$ or $\mathbf{D} = \mathbf{0}_{p \times m}$ or both vanish. Throughout the paper, we assume that both parties (adversary and defender) have perfect knowledge of the system model and parameters. The companion paper (Part II) [25] considers the case when the perfect knowledge is not available.

When the system dynamics are non-linear, then the matrix pairs $\{\mathbf{F}, \mathbf{B}\}$, $\{\mathbf{H}, \mathbf{D}\}$, and the matrix \mathbf{G} are replaced by non-linear functions $f(\cdot, \cdot)$, $h(\cdot, \cdot)$, and $q(\cdot)$, respectively, as

$$\mathbf{x}_{k+1} = f(\mathbf{x}_k, \mathbf{u}_k) + \mathbf{w}_k,\tag{4}$$

$$\mathbf{y}_k = h(\mathbf{x}_k, \mathbf{u}_k) + \mathbf{v}_k,\tag{5}$$

$$\mathbf{a}_k = g(\hat{\mathbf{x}}_k) + \boldsymbol{\epsilon}_k. \tag{6}$$

This is a *direct feed-through* (DF) model, wherein y_k depends on the unknown input. Without DF, observations (5) becomes

$$\mathbf{y}_k = h(\mathbf{x}_k) + \mathbf{v}_k. \tag{7}$$

We show in the following Section III, the presence or absence of the unknown input leads to different solution approaches towards forward and inverse filters. For simplicity, the presence of known exogenous inputs is also ignored in state evolution and observations. However, it is trivial to extend the inverse filters developed in this paper for these modifications in the system model. Throughout the paper, we focus on discrete-time models.

III. I-EKF WITH UNKNOWN INPUT

One of the earliest approach to treat the unknown input was to model the inputs as a stochastic process with known evolution dynamics and jointly estimate the state and inputs. Relaxing the known input dynamics assumption, [31–34] developed and analyzed unbiased minimum variance linear filters with unknown inputs. Recently, [35, 36] have also considered non-persistent and norm-constrained unknown input estimation in linear systems. Various EKF variants to handle unknown inputs in non-linear systems have also been proposed [37–41]. We consider a more general EKF with unknown inputs in case of both without [38] and with [37] DF. We do not make any other assumption on the inputs.

The EKF linearizes the model about the nominal values of the state vector and control input. It is similar to the iterated least squares (ILS) method except that the former is for dynamical systems and the latter is not [42]. Note that the optimal forward EKFs with and without DF are conceptually different. In the latter case, while the observation y_k is unaffected by the unknown input u_k , it is still dependent on \mathbf{u}_{k-1} through \mathbf{x}_k ; this induces a one-step delay in the adversary's estimate of \mathbf{u}_k . On the other hand, with DF, there is no such delay in estimating \mathbf{u}_k . We now show that this difference results in different inverse filters for these two cases.

A. I-EKF-without-DF unknown input

Consider the non-linear system without DF given by (4) and (7). Linearize the model functions as $\mathbf{F}_k \doteq \nabla_{\mathbf{x}} f(\mathbf{x}, \hat{\mathbf{u}}_{k-1})|_{\mathbf{x} = \hat{\mathbf{x}}_k}, \mathbf{B}_k \doteq$ $\nabla_{\mathbf{u}} f(\hat{\mathbf{x}}_k, \mathbf{u})|_{\mathbf{u} = \hat{\mathbf{u}}_{k-1}}$ and $\mathbf{H}_{k+1} \doteq \nabla_{\mathbf{x}} h(\mathbf{x})|_{\mathbf{x} = \hat{\mathbf{x}}_{k+1|k}}$.

1) Forward filter: The forward filter's recursive state estimation procedure first obtains the prediction $\hat{\mathbf{x}}_{k+1|k}$ of the current state using the previous state and input estimates, with $\Sigma_{k+1|k}^x$ as the associated state prediction error covariance matrix of $\hat{\mathbf{x}}_{k+1|k}$. Then, the state and input gain matrices \mathbf{K}_{k+1}^x and \mathbf{K}_k^u , respectively, are computed along with the input estimation (with delay) covariance matrix Σ_k^u . Finally, the state $\hat{\mathbf{x}}_{k+1}$, input $\hat{\mathbf{u}}_k$, and covariance matrix $\mathbf{\Sigma}_{k+1}^x$ are updated using current observation y_{k+1} , and gain matrices K_{k+1}^x and \mathbf{K}_{k}^{u} . Note that the current observation \mathbf{y}_{k+1} provides an estimate $\hat{\mathbf{u}}_{k}$ of the input \mathbf{u}_k at the previous time step. The adversary's forward EKF's recursions are [38]:

Prediction:
$$\hat{\mathbf{x}}_{k+1|k} = f(\hat{\mathbf{x}}_k, \hat{\mathbf{u}}_{k-1}),$$
 (8)

Gain computation: $\Sigma_{k+1|k}^x = \mathbf{F}_k \Sigma_k^x \mathbf{F}_k^T + \mathbf{Q}$,

$$\begin{split} \mathbf{K}_{k+1}^{x} &= \mathbf{\Sigma}_{k+1|k}^{x} \mathbf{H}_{k+1}^{T} \left(\mathbf{H}_{k+1} \mathbf{\Sigma}_{k+1|k}^{x} \mathbf{H}_{k+1}^{T} + \mathbf{R} \right)^{-1}, \\ \mathbf{\Sigma}_{k}^{u} &= \left(\mathbf{B}_{k}^{T} \mathbf{H}_{k+1}^{T} \mathbf{R}^{-1} (\mathbf{I}_{p \times p} - \mathbf{H}_{k+1} \mathbf{K}_{k+1}^{x}) \mathbf{H}_{k+1} \mathbf{B}_{k} \right)^{-1}, \\ \mathbf{K}_{k}^{u} &= \mathbf{\Sigma}_{k}^{u} \mathbf{B}_{k}^{T} \mathbf{H}_{k+1}^{T} \mathbf{R}^{-1} (\mathbf{I}_{p \times p} - \mathbf{H}_{k+1} \mathbf{K}_{k+1}^{x}), \\ Update: \hat{\mathbf{x}}_{k+1} &= \hat{\mathbf{x}}_{k+1|k} + \mathbf{K}_{k+1}^{x} (\mathbf{y}_{k+1} - h(\hat{\mathbf{x}}_{k+1|k})), \end{split}$$
(9)

$$\hat{\mathbf{u}}_k = \mathbf{K}_k^u (\mathbf{y}_{k+1} - h(\hat{\mathbf{x}}_{k+1|k}) + \mathbf{H}_{k+1} \mathbf{B}_k \hat{\mathbf{u}}_{k-1}), \tag{10}$$

Covariance matrix update: Σ_{k+1}^x

$$= \left(\mathbf{I}_{n \times n} - \mathbf{K}_{k+1}^{x} \mathbf{H}_{k+1}\right) \left(\mathbf{\Sigma}_{k+1|k}^{x} + \mathbf{B}_{k} \mathbf{\Sigma}_{k}^{u} \mathbf{B}_{k}^{T} \left(\mathbf{I}_{n \times n} - \mathbf{K}_{k+1}^{x} \mathbf{H}_{k+1}\right)^{T}\right)$$

Forward filter exists if rank(Σ_k^u) = m, for all $k \ge 0$, and $p \ge m$ [38].

2) Inverse filter: Consider an augmented state vector \mathbf{z}_k = $\begin{bmatrix} \hat{\mathbf{x}}_k^T & \hat{\mathbf{u}}_{k-2}^T \end{bmatrix}^T$. The defender's observation \mathbf{a}_k in (6) is the first observation that contains the information about unknown input estimate $\hat{\mathbf{u}}_{k-2}$, because of the delay in forward filter input estimate. Hence, the delayed estimate $\hat{\mathbf{u}}_{k-2}$ is considered in the augmented state \mathbf{z}_k . Define $\phi_k(\hat{\mathbf{x}}_k, \hat{\mathbf{u}}_{k-1}, \mathbf{x}_{k+1}, \mathbf{v}_{k+1}) = f(\hat{\mathbf{x}}_k, \hat{\mathbf{u}}_{k-1}) \mathbf{K}_{k+1}^{x}h(f(\hat{\mathbf{x}}_{k},\hat{\mathbf{u}}_{k-1})) + \mathbf{K}_{k+1}^{x}h(\mathbf{x}_{k+1}) + \mathbf{K}_{k+1}^{x}\mathbf{v}_{k+1}$. From (7)-(10), state transition equations of augmented state vector are $\hat{\mathbf{x}}_{k+1} = f_k(\hat{\mathbf{x}}_k, \hat{\mathbf{u}}_{k-2}, \hat{\mathbf{x}}_{k-1}, \mathbf{x}_k, \mathbf{x}_{k+1}, \mathbf{v}_k, \mathbf{v}_{k+1})$ and $\hat{\mathbf{u}}_{k-1} =$ $h_k(\hat{\mathbf{u}}_{k-2}, \hat{\mathbf{x}}_{k-1}, \mathbf{x}_k, \mathbf{v}_k)$, where

$$\tilde{h}_{k}(\hat{\mathbf{u}}_{k-2}, \hat{\mathbf{x}}_{k-1}, \mathbf{x}_{k}, \mathbf{v}_{k})
= \mathbf{K}_{k-1}^{u}(\mathbf{H}_{k}\mathbf{B}_{k-1}\hat{\mathbf{u}}_{k-2} - h(f(\hat{\mathbf{x}}_{k-1}, \hat{\mathbf{u}}_{k-2})) + h(\mathbf{x}_{k}) + \mathbf{v}_{k}),$$
(11)

$$\widetilde{f}_{k}(\widehat{\mathbf{x}}_{k}, \widehat{\mathbf{u}}_{k-2}, \widehat{\mathbf{x}}_{k-1}, \mathbf{x}_{k}, \mathbf{x}_{k+1}, \mathbf{v}_{k}, \mathbf{v}_{k+1})
= \widetilde{\phi}_{k}(\widehat{\mathbf{x}}_{k}, \widetilde{h}_{k}(\widehat{\mathbf{u}}_{k-2}, \widehat{\mathbf{x}}_{k-1}, \mathbf{x}_{k}, \mathbf{v}_{k}), \mathbf{x}_{k+1}, \mathbf{v}_{k+1}).$$
(12)

In these state transition equations, the actual states x_k and x_{k+1} are perfectly known to the defender and henceforth treated as known exogenous inputs. Note that, unlike the forward filter, the process noise terms \mathbf{v}_k and \mathbf{v}_{k+1} are non-additive because the filter gains \mathbf{K}_{k+1}^x and \mathbf{K}_{k-1}^u depend on the previous estimates (through the Jacobians).

Denote $\hat{\mathbf{z}}_{k+1} \doteq \begin{bmatrix} \hat{\mathbf{x}}_{k+1}^T & \hat{\mathbf{u}}_{k-1}^T \end{bmatrix}^T$. The state transition of the augmented state \mathbf{z}_{k+1} depends on the estimate $\hat{\mathbf{x}}_{k-1}$ which the defender approximates by its previous estimate $\hat{\mathbf{x}}_{k-1}$. With this approximation, $\hat{\mathbf{x}}_{k-1}$ is treated as a known exogenous input for the inverse filter while the augmented process noise vector is

$$\begin{bmatrix} \mathbf{v}_k^T & \mathbf{v}_{k+1}^T \end{bmatrix}^T. \text{ Define the Jacobians } \widetilde{\mathbf{F}}_k^z \stackrel{\cdot}{=} \begin{bmatrix} \nabla_{\hat{\mathbf{x}}_k} \widetilde{f}_k & \nabla_{\hat{\mathbf{u}}_{k-2}} \widetilde{f}_k \\ \mathbf{0}_{m \times n} & \nabla_{\hat{\mathbf{u}}_{k-2}} \widetilde{h}_k \end{bmatrix},$$

and $\mathbf{G}_{k+1} \doteq \begin{bmatrix} \nabla_{\hat{\mathbf{x}}_{k+1|k}} g & \mathbf{0}_{n_a \times m} \end{bmatrix}$ with respect to the augmented state; Jacobian $\widetilde{\mathbf{F}}_{k}^{v} \doteq \begin{bmatrix} \nabla_{\mathbf{v}_{k}} \widetilde{f}_{k} & \nabla_{\mathbf{v}_{k+1}} \widetilde{f}_{k} \\ \nabla_{\mathbf{v}_{k}} \widetilde{h}_{k} & \mathbf{0}_{m \times p} \end{bmatrix}$ with respect to the augmented process noise vector; and $\overline{\mathbf{Q}}_{k} = \widetilde{\mathbf{F}}_{k}^{v} \begin{bmatrix} \mathbf{R} & \mathbf{0}_{p \times p} \\ \mathbf{0}_{p \times p} & \mathbf{R} \end{bmatrix} (\widetilde{\mathbf{F}}_{k}^{v})^{T}$.

Then, the I-EKF-without-DF's recursions yield the estimate $\hat{\mathbf{z}}_k$ of the augmented state and the associated covariance matrix $\overline{\Sigma}_k$ as:

$$\begin{aligned} &\textit{Prediction: } \hat{\hat{\mathbf{x}}}_{k+1|k} = \widetilde{f}_{k}(\hat{\hat{\mathbf{x}}}_{k}, \hat{\hat{\mathbf{u}}}_{k-2}, \hat{\hat{\mathbf{x}}}_{k-1}, \mathbf{x}_{k}, \mathbf{x}_{k+1}, \mathbf{0}_{p \times 1}, \mathbf{0}_{p \times 1}), \\ \hat{\hat{\mathbf{u}}}_{k-1|k} = \widetilde{h}_{k}(\hat{\hat{\mathbf{u}}}_{k-2}, \hat{\hat{\mathbf{x}}}_{k-1}, \mathbf{x}_{k}, \mathbf{0}_{p \times 1}), \end{aligned}$$

$$\hat{\mathbf{z}}_{k+1|k} = \begin{bmatrix} \hat{\mathbf{x}}_{k+1|k}^T & \hat{\mathbf{u}}_{k-1|k}^T \end{bmatrix}^T,$$

$$\overline{\mathbf{\Sigma}}_{k+1|k} = \widetilde{\mathbf{F}}_{k}^{z} \overline{\mathbf{\Sigma}}_{k} (\widetilde{\mathbf{F}}_{k}^{z})^{T} + \overline{\mathbf{Q}}_{k}, \tag{13}$$

Update:
$$\overline{\mathbf{S}}_{k+1} = \mathbf{G}_{k+1} \overline{\boldsymbol{\Sigma}}_{k+1|k} \mathbf{G}_{k+1}^T + \boldsymbol{\Sigma}_{\epsilon},$$
 (14)

$$\hat{\mathbf{z}}_{k+1} = \hat{\mathbf{z}}_{k+1|k} + \overline{\mathbf{\Sigma}}_{k+1|k} \mathbf{G}_{k+1}^T \overline{\mathbf{S}}_{k+1}^{-1} \left(\mathbf{a}_{k+1} - g(\hat{\hat{\mathbf{x}}}_{k+1|k}) \right), \quad (15)$$

$$\overline{\Sigma}_{k+1} = \overline{\Sigma}_{k+1|k} - \overline{\Sigma}_{k+1|k} \mathbf{G}_{k+1}^T \overline{\mathbf{S}}_{k+1}^{-1} \mathbf{G}_{k+1} \overline{\Sigma}_{k+1|k}. \tag{16}$$

The I-EKF-without-DF's recursions take the same form as that of the standard EKF [43] but with modified system matrices. In particular, the former employs an augmented state such that the Jacobian of the state transition function with respect to the state is computed as $\tilde{\mathbf{F}}_k^z$ while for the latter, it is simply $\mathbf{F}_k \doteq \nabla_{\mathbf{x}} f(\mathbf{x})|_{\mathbf{x} = \hat{\mathbf{x}}_k}$. Further, unlike standard KF or EKF, the noise terms, i.e., \mathbf{v}_k and \mathbf{v}_{k+1} in (11) and (12) are non-additive such that linearization \mathbf{F}_k^v of the state transition function with respect to the noise terms yields the process noise covariance matrix approximation $\overline{\mathbf{Q}}_k$.

The forward filter gains \mathbf{K}_{k+1}^x and \mathbf{K}_{k-1}^u are treated as timevarying parameters of the state transition equation and not as a function of the state and input estimates $(\hat{\mathbf{x}}_k \text{ and } \hat{\mathbf{u}}_{k-1})$ in the inverse filter. The inverse filter approximates them by evaluating their values at its own estimates $(\hat{\mathbf{x}}_k \text{ and } \hat{\mathbf{u}}_{k-1})$ recursively in the similar manner as the forward filter evaluates them using its own estimates. On the contrary, in I-KF formulation introduced in [11], the forward Kalman gain \mathbf{K}_{k+1} is deterministic, fully determined by the model parameters for a given initial covariance estimate Σ_0 , and computed offline independent of the current I-KF's estimate.

B. I-EKF-with-DF unknown input

Consider the non-linear system with DF given by (4) and (5). Linearize the functions as $\mathbf{F}_k \doteq \nabla_{\mathbf{x}} f(\mathbf{x}, \hat{\mathbf{u}}_k)|_{\mathbf{x} = \hat{\mathbf{x}}_k}, \ \mathbf{H}_{k+1} \doteq$ $\nabla_{\mathbf{x}} h(\mathbf{x}, \hat{\mathbf{u}}_k)|_{\mathbf{x} = \hat{\mathbf{x}}_{k+1|k}}$ and $\mathbf{D}_k \doteq \nabla_{\mathbf{u}} h(\hat{\mathbf{x}}_{k+1|k}, \mathbf{u})|_{\mathbf{u} = \hat{\mathbf{u}}_k}$.

1) Forward filter: Denote the state and input estimation covariance and gain matrices identical to Section III-A. Here, the current observation y_{k+1} depends on the current unknown input u_{k+1} such that the forward filter infers $\hat{\mathbf{u}}_{k+1}$ without any delay. For input estimation covariance without delay, we use Σ_{k+1}^u . Then, the forward EKF-with-DF's recursions are [37]

Prediction:
$$\hat{\mathbf{x}}_{k+1|k} = f(\hat{\mathbf{x}}_{k}, \hat{\mathbf{u}}_{k}), \ \Sigma_{k+1|k}^{x} = \mathbf{F}_{k} \Sigma_{k}^{x} \mathbf{F}_{k}^{T} + \mathbf{Q},$$
 (17)

$$\mathbf{K}_{k+1}^{x} = \Sigma_{k+1|k}^{x} \mathbf{H}_{k+1}^{T} (\mathbf{H}_{k+1} \Sigma_{k+1|k}^{x} \mathbf{H}_{k+1}^{T} + \mathbf{R})^{-1},$$

$$\Sigma_{k+1}^{u} = \left(\mathbf{D}_{k}^{T} \mathbf{R}^{-1} (\mathbf{I}_{p \times p} - \mathbf{H}_{k+1} \mathbf{K}_{k+1}^{x}) \mathbf{D}_{k}\right)^{-1},$$

$$\mathbf{K}_{k+1}^{u} = \Sigma_{k+1}^{u} \mathbf{D}_{k}^{T} \mathbf{R}^{-1} (\mathbf{I}_{p \times p} - \mathbf{H}_{k+1} \mathbf{K}_{k+1}^{x}),$$

$$Update: \hat{\mathbf{u}}_{k+1} = \mathbf{K}_{k+1}^{u} \left(\mathbf{y}_{k+1} - h(\hat{\mathbf{x}}_{k+1|k}, \hat{\mathbf{u}}_{k}) + \mathbf{D}_{k} \hat{\mathbf{u}}_{k}\right),$$

$$\hat{\mathbf{x}}_{k+1} = \hat{\mathbf{x}}_{k+1|k} + \mathbf{K}_{k+1}^{x} \left(\mathbf{y}_{k+1} - h(\hat{\mathbf{x}}_{k+1|k}, \hat{\mathbf{u}}_{k}) - \mathbf{D}_{k} (\hat{\mathbf{u}}_{k+1} - \hat{\mathbf{u}}_{k})\right),$$
(18)

Covariance matrix update: Σ_{k+1}^x

$$= (\mathbf{I}_{n \times n} + \mathbf{K}_{k+1}^x \mathbf{D}_k \mathbf{\Sigma}_{k+1}^u \mathbf{D}_k^T \mathbf{R}^{-1} \mathbf{H}_{k+1}) (\mathbf{I}_{n \times n} - \mathbf{K}_{k+1}^x \mathbf{H}_{k+1}) \mathbf{\Sigma}_{k+1|k}^x.$$

The forward filter exists if $rank(\mathbf{D}_k) = m$ for all $k \geq 0$, which implies $p \geq m$ [37].

2) Inverse filter: Consider an augmented state vector $\mathbf{z}_k = \begin{bmatrix} \hat{\mathbf{x}}_k^T & \hat{\mathbf{u}}_k^T \end{bmatrix}^T$ (note the absence of delay in the input estimate). Define $\widetilde{\phi}_k(\hat{\mathbf{x}}_k, \hat{\mathbf{u}}_k, \hat{\mathbf{u}}_{k+1}, \mathbf{x}_{k+1}, \mathbf{u}_{k+1}, \mathbf{v}_{k+1}) = f(\hat{\mathbf{x}}_k, \hat{\mathbf{u}}_k) - \mathbf{K}_{k+1}^x h(f(\hat{\mathbf{x}}_k, \hat{\mathbf{u}}_k), \hat{\mathbf{u}}_k) - \mathbf{K}_{k+1}^x \mathbf{D}_k(\hat{\mathbf{u}}_{k+1} - \hat{\mathbf{u}}_k) + \mathbf{K}_{k+1}^x h(\mathbf{x}_{k+1}, \mathbf{u}_{k+1}) + \mathbf{K}_{k+1}^x \mathbf{v}_{k+1}.$ From (5) and (17)-(19), state transitions for inverse filter are $\hat{\mathbf{x}}_{k+1} = \widetilde{f}_k(\hat{\mathbf{x}}_k, \hat{\mathbf{u}}_k, \mathbf{x}_{k+1}, \mathbf{u}_{k+1}, \mathbf{v}_{k+1})$ and $\hat{\mathbf{u}}_{k+1} = \widetilde{h}_k(\hat{\mathbf{x}}_k, \hat{\mathbf{u}}_k, \mathbf{x}_{k+1}, \mathbf{u}_{k+1}, \mathbf{v}_{k+1})$, where

$$\widetilde{h}_{k}(\hat{\mathbf{x}}_{k}, \hat{\mathbf{u}}_{k}, \mathbf{x}_{k+1}, \mathbf{u}_{k+1}, \mathbf{v}_{k+1})
= \mathbf{K}_{k+1}^{u}(h(\mathbf{x}_{k+1}, \mathbf{u}_{k+1}) + \mathbf{v}_{k+1} - h(f(\hat{\mathbf{x}}_{k}, \hat{\mathbf{u}}_{k}), \hat{\mathbf{u}}_{k}) + \mathbf{D}_{k}\hat{\mathbf{u}}_{k})
\widetilde{f}_{k}(\hat{\mathbf{x}}_{k}, \hat{\mathbf{u}}_{k}, \mathbf{x}_{k+1}, \mathbf{u}_{k+1}, \mathbf{v}_{k+1})
= \widetilde{\phi}_{k}(\hat{\mathbf{x}}_{k}, \hat{\mathbf{u}}_{k}, \widetilde{h}_{k}(\hat{\mathbf{x}}_{k}, \hat{\mathbf{u}}_{k}, \mathbf{x}_{k+1}, \mathbf{u}_{k+1}, \mathbf{v}_{k+1}), \mathbf{x}_{k+1}, \mathbf{u}_{k+1}, \mathbf{v}_{k+1}).$$
(20)

Then, ceteris paribus, following similar steps as in I-EKF-without-DF, the I-EKF-with-DF estimate $\hat{\mathbf{z}}_k = \begin{bmatrix} \hat{\mathbf{x}}_k^T & \hat{\mathbf{u}}_k^T \end{bmatrix}^T$ from observations (6) is computed recursively. The predicted augmented state is $\hat{\mathbf{z}}_{k+1|k} = \begin{bmatrix} \hat{\mathbf{x}}_{k+1|k}^T & \hat{\mathbf{u}}_{k+1|k}^T \end{bmatrix}^T$, where $\hat{\mathbf{x}}_{k+1|k} = \widetilde{h}_k(\hat{\hat{\mathbf{x}}}_k, \hat{\hat{\mathbf{u}}}_k, \mathbf{x}_{k+1}, \mathbf{u}_{k+1}, \mathbf{0}_{p\times 1})$ and $\hat{\mathbf{u}}_{k+1|k} = \widetilde{h}_k(\hat{\hat{\mathbf{x}}}_k, \hat{\hat{\mathbf{u}}}_k, \mathbf{x}_{k+1}, \mathbf{u}_{k+1}, \mathbf{0}_{p\times 1})$. Hereafter, the remaining steps are as in (13)-(16). For I-EKF-with-DF, the Jacobians with

respect to the augmented state are
$$\widetilde{\mathbf{F}}_{k}^{z} \doteq \begin{bmatrix} \nabla_{\hat{\mathbf{x}}_{k}} \widetilde{f}_{k} & \nabla_{\hat{\mathbf{u}}_{k}} \widetilde{f}_{k} \\ \nabla_{\hat{\mathbf{x}}_{k}} \widetilde{h}_{k} & \nabla_{\hat{\mathbf{u}}_{k}} \widetilde{h}_{k} \end{bmatrix}$$
 and $\mathbf{G}_{k+1} \doteq \begin{bmatrix} \nabla_{\hat{\mathbf{x}}_{k+1}|k}} g & \mathbf{0}_{n_{a} \times m} \end{bmatrix}$; the Jacobian with respect to the

process noise term is $\widetilde{\mathbf{F}}_{k}^{v} \doteq \begin{bmatrix} \nabla_{\mathbf{v}_{k+1}} \widetilde{f}_{k} \\ \nabla_{\mathbf{v}_{k+1}} \widetilde{h}_{k} \end{bmatrix}$; and $\overline{\mathbf{Q}}_{k} = \widetilde{\mathbf{F}}_{k}^{v} \mathbf{R} (\widetilde{\mathbf{F}}_{k}^{v})^{T}$. Here, unlike I-EKF-without-DF, the inverse filter's prediction dispenses with any approximation of $\hat{\mathbf{x}}_{k-1}$. The absence of delay in input estimation also results in a simplified process point term.

in input estimation also results in a simplified process noise term \mathbf{v}_{k+1} , in place of I-EKF-without-DF's augmented noise vector. Examples of EKF with unknown inputs include fault detection with unknown excitations [37] and missile-target interception with

with unknown excitations [37] and missile-target interception with unknown target acceleration [38]. The inverse cognition in these applications would then resort to the I-EKFs described until now.

C. I-EKF without any unknown inputs

Consider a non-linear system model without unknown inputs in the system equations (4) and (7), i.e.,

$$\mathbf{x}_{k+1} = f(\mathbf{x}_k) + \mathbf{w}_k. \tag{21}$$

Linearize the functions as $\mathbf{F}_k \doteq \nabla_{\mathbf{x}} f(\mathbf{x})|_{\mathbf{x}=\hat{\mathbf{x}}_k}$ and $\mathbf{H}_{k+1} \doteq \nabla_{\mathbf{x}} h(\mathbf{x})|_{\mathbf{x}=\hat{\mathbf{x}}_{k+1|k}}$. Then, *ceteris paribus*, setting $\mathbf{B}_k = \mathbf{0}_{n \times p}$ and neglecting computation of $\mathbf{\Sigma}_k^u$, \mathbf{K}_k^u and $\hat{\mathbf{u}}_k$ in forward EKF-without-DF yields forward EKF-without-unknown-input whose state prediction and updates are

$$\hat{\mathbf{x}}_{k+1|k} = f(\hat{\mathbf{x}}_k),\tag{22}$$

$$\hat{\mathbf{x}}_{k+1} = \hat{\mathbf{x}}_{k+1|k} + \mathbf{K}_{k+1}(\mathbf{y}_{k+1} - h(\hat{\mathbf{x}}_{k+1|k})), \tag{23}$$

with $\mathbf{K}_{k+1} = \mathbf{\Sigma}_{k+1|k} \mathbf{H}_{k+1}^T \left(\mathbf{H}_{k+1} \mathbf{\Sigma}_{k+1|k} \mathbf{H}_{k+1}^T + \mathbf{R} \right)^{-1}$. Here, we have dropped the superscript in the covariance matrix $\mathbf{\Sigma}_{k+1|k}^x$ and gain \mathbf{K}_{k+1}^x to replace with $\mathbf{\Sigma}_{k+1|k}$ and \mathbf{K}_{k+1} , respectively (because only the state estimation covariances and gains are computed here). Thence, the I-EKF-without-DF's state transition equations and recursions yield I-EKF-without-unknown-input. Dropping the input estimate term in the augmented state \mathbf{z}_k , the state transition equations become

$$\hat{\mathbf{x}}_{k+1} = \widetilde{f}_k(\hat{\mathbf{x}}_k, \mathbf{x}_{k+1}, \mathbf{v}_{k+1}) = f(\hat{\mathbf{x}}_k) - \mathbf{K}_{k+1}h(f(\hat{\mathbf{x}}_k)) + \mathbf{K}_{k+1}h(\mathbf{x}_{k+1}) + \mathbf{K}_{k+1}\mathbf{v}_{k+1}.$$
(24)

Denote $\widetilde{\mathbf{F}}_k^x \doteq \nabla_{\mathbf{x}}\widetilde{f}_k(\mathbf{x},\mathbf{x}_{k+1},\mathbf{0}_{p\times 1})|_{\mathbf{x}=\hat{\mathbf{x}}_k}$, $\mathbf{G}_{k+1} \doteq \nabla_{\mathbf{x}}g(\mathbf{x})_{\mathbf{x}=\hat{\mathbf{x}}_{k+1}|k}$, $\widetilde{\mathbf{F}}_k^v \doteq \nabla_{\mathbf{v}}\widetilde{f}_k(\hat{\mathbf{x}}_k,\mathbf{x}_{k+1},\mathbf{v})|_{\mathbf{v}=\mathbf{0}_{p\times 1}}$, and $\overline{\mathbf{Q}}_k = \widetilde{\mathbf{F}}_k^v\mathbf{R}(\widetilde{\mathbf{F}}_k^v)^T$. Then, the I-EKF's recursions are similar to I-EKF-without-DF except that the I-EKF's predicted state estimate and the associated prediction covariance matrix are computed, respectively, as $\hat{\mathbf{x}}_{k+1|k} = \widetilde{f}_k(\hat{\mathbf{x}}_k,\mathbf{x}_{k+1},\mathbf{0}_{p\times 1})$ and $\overline{\mathbf{\Sigma}}_{k+1|k} = \widetilde{\mathbf{F}}_k^x\overline{\mathbf{\Sigma}}_k(\widetilde{\mathbf{F}}_k^x)^T + \overline{\mathbf{Q}}_k$, followed by the update procedure in (14)-(16).

Unlike I-KF [11], the I-EKF approximates the forward gain \mathbf{K}_{k+1} online at its own estimates recursively and is sensitive to the initial estimate of forward EKF's initial covariance matrix. I-EKF could be applied in various non-linear target tracking applications, where EKF is a popular forward filter [44].

The two-step prediction-update formulation (as discussed for EKF and I-EKF so far) infers an estimate of the current state. However, often for stability analyses, the one-step prediction formulation is analytically more useful. In this formulation, the estimate $\hat{\mathbf{x}}_k$ is the one-step prediction estimate, i.e., an estimate of state \mathbf{x}_k at k-th instant given the observations $\{\mathbf{y}_j\}_{1\leq j\leq k-1}$ up to time instant k-1 with Σ_k as the corresponding prediction covariance matrix. The forward one-step prediction EKF formulation [26] for the same system but with $\mathbf{F}_k \doteq \nabla_{\mathbf{x}} f(\mathbf{x})|_{\mathbf{x} = \hat{\mathbf{x}}_k}$ and $\mathbf{H}_k \doteq \nabla_{\mathbf{x}} h(\mathbf{x})|_{\mathbf{x} = \hat{\mathbf{x}}_k}$ is

$$\mathbf{K}_{k} = \mathbf{F}_{k} \mathbf{\Sigma}_{k} \mathbf{H}_{k}^{T} (\mathbf{H}_{k} \mathbf{\Sigma}_{k} \mathbf{H}_{k}^{T} + \mathbf{R})^{-1}, \tag{25}$$

$$\hat{\mathbf{x}}_{k+1} = f(\hat{\mathbf{x}}_k) + \mathbf{K}_k(\mathbf{y}_k - h(\hat{\mathbf{x}}_k)), \tag{26}$$

$$\Sigma_{k+1} = \mathbf{F}_k \Sigma_k \mathbf{F}_k^T + \mathbf{Q} - \mathbf{K}_k (\mathbf{H}_k \Sigma_k \mathbf{H}_k^T + \mathbf{R}) \mathbf{K}_k^T.$$
 (27)

From (7) and (26), the state transition equation for one-step formulation of I-EKF is $\hat{\mathbf{x}}_{k+1} = \tilde{f}_k(\hat{\mathbf{x}}_k, \mathbf{x}_k, \mathbf{v}_k) \doteq f(\hat{\mathbf{x}}_k) - \mathbf{K}_k h(\hat{\mathbf{x}}_k) + \mathbf{K}_k h(\hat{\mathbf{x}}_k) + \mathbf{K}_k \mathbf{v}_k$. With this state transition, the I-EKF one-step prediction formulation treating \mathbf{a}_k as the observation with the Jacobians with respect to state estimate $\tilde{\mathbf{F}}_k^x = \nabla_{\mathbf{x}} \tilde{f}_k(\mathbf{x}, \mathbf{x}_k, \mathbf{0})|_{\mathbf{x} = \hat{\mathbf{x}}_k} = \mathbf{F}_k - \mathbf{K}_k \mathbf{H}_k$ and $\mathbf{G}_k = \nabla_{\mathbf{x}} g(\mathbf{x})|_{\mathbf{x} = \hat{\hat{\mathbf{x}}}_k}$, and the process noise covariance matrix $\overline{\mathbf{Q}}_k = \mathbf{K}_k \mathbf{R} \mathbf{K}_k^T$.

IV. INVERSE KF WITH UNKNOWN INPUT

For linear Gaussian state-space models, our methods developed in the previous section are useful in extending the I-KF mentioned in [11] to unknown input. Again, the forward KFs employed by the adversary with and without DF are conceptually different [33] because of the delay involved in input estimation. The forward KFs with unknown input provide unbiased minimum variance state and input estimates.

A. I-KF-without-DF

Consider the system in (1) and (2) with $\mathbf{D} = \mathbf{0}_{p \times m}$.

1) Forward filter: Unlike EKF-without-DF, the forward KF-without-DF considers an intermediate state update step using the estimated unknown input before the final state updates. In this step, the unknown input is first estimated (with one-step delay) using the current observation \mathbf{y}_{k+1} and input estimation gain matrix \mathbf{M}_{k+1} . In the update step, the current state estimate $\hat{\mathbf{x}}_{k+1}$ is computed by again considering the current observation \mathbf{y}_{k+1} as [32]

Prediction:
$$\hat{\mathbf{x}}_{k+1|k} = \mathbf{F}\hat{\mathbf{x}}_k, \ \Sigma_{k+1|k} = \mathbf{F}\Sigma_k\mathbf{F}^T + \mathbf{Q},$$
 (28)

Unknown input estimation:
$$\mathbf{S}_{k+1} = \mathbf{H} \mathbf{\Sigma}_{k+1|k} \mathbf{H}^T + \mathbf{R},$$
 (29)

$$M_{k+1} = (\mathbf{B}^T \mathbf{H}^T \mathbf{S}_{k+1}^{-1} \mathbf{H} \mathbf{B})^{-1} \mathbf{B}^T \mathbf{H}^T \mathbf{S}_{k+1}^{-1}, \tag{30}$$

$$\hat{\mathbf{u}}_k = \mathbf{M}_{k+1} (\mathbf{y}_{k+1} - \mathbf{H} \hat{\mathbf{x}}_{k+1|k}), \tag{31}$$

$$\widetilde{\mathbf{x}}_{k+1|k+1} = \widehat{\mathbf{x}}_{k+1|k} + \mathbf{B}\widehat{\mathbf{u}}_k, \tag{32}$$

$$\widetilde{\boldsymbol{\Sigma}}_{k+1|k+1} = (\mathbf{I}_{n\times n} - \mathbf{B}\mathbf{M}_{k+1}\mathbf{H})\boldsymbol{\Sigma}_{k+1|k}(\mathbf{I}_{n\times n} - \mathbf{B}\mathbf{M}_{k+1}\mathbf{H})^T$$

$$+\mathbf{B}\mathbf{M}_{k+1}\mathbf{R}\mathbf{M}_{k+1}^T\mathbf{B}^T,\tag{33}$$

Update:
$$\mathbf{K}_{k+1} = \mathbf{\Sigma}_{k+1|k} \mathbf{H}^T \mathbf{S}_{k+1}^{-1},$$
 (34)

$$\hat{\mathbf{x}}_{k+1} = \tilde{\mathbf{x}}_{k+1|k+1} + \mathbf{K}_{k+1}(\mathbf{y}_{k+1} - \mathbf{H}\tilde{\mathbf{x}}_{k+1|k+1}), \tag{35}$$

$$\Sigma_{k+1} = \widetilde{\Sigma}_{k+1|k+1} - \mathbf{K}_{k+1} (\widetilde{\Sigma}_{k+1|k+1} \mathbf{H}^T - \mathbf{B} \mathbf{M}_{k+1} \mathbf{R})^T.$$
 (36)

The forward filter exists if $rank(\mathbf{HB}) = rank(\mathbf{B}) = m$ which implies $n \ge m$ and $p \ge m$ [32]. Here, unlike I-EKFs, the gain matrices \mathbf{K}_{k+1} and \mathbf{M}_{k+1} , are deterministic and completely determined by the model parameters and the initial covariance matrix similar to I-KF [11].

2) Inverse filter: Denote $\tilde{\mathbf{F}}_k = (\mathbf{I}_{n \times n} - \mathbf{K}_{k+1}\mathbf{H})(\mathbf{I}_{n \times n} - \mathbf{B}\mathbf{M}_{k+1}\mathbf{H})\mathbf{F}$ and $\mathbf{E}_k = \mathbf{B}\mathbf{M}_{k+1} - \mathbf{K}_{k+1}\mathbf{H}\mathbf{B}\mathbf{M}_{k+1} + \mathbf{K}_{k+1}$. From (2) with $\mathbf{D} = \mathbf{0}_{p \times m}$, and (28)-(35), the state transition equation for I-KF-without-DF is

$$\hat{\mathbf{x}}_{k+1} = \widetilde{\mathbf{F}}_k \hat{\mathbf{x}}_k + \mathbf{E}_k \mathbf{H} \mathbf{x}_{k+1} + \mathbf{E}_k \mathbf{v}_{k+1}. \tag{37}$$

Unlike the state transition (11) and (12) of I-EKF-without-DF, the state transition for I-KF-without-DF is not an explicit function of the forward filter input estimate and hence, an augmented state is not needed. The difference arises from the forward EKF-without-DF, where the current input estimate explicitly depends on the previous input estimates as observed in (10), which is not the case in KF-without-DF. The I-KF-without-DF's recursions with observation (3) are:

Prediction:
$$\hat{\mathbf{x}}_{k+1|k} = \widetilde{\mathbf{F}}_k \hat{\mathbf{x}}_k + \mathbf{E}_k \mathbf{H} \mathbf{x}_{k+1},$$
 (38)

$$\overline{\Sigma}_{k+1|k} = \widetilde{\mathbf{F}}_k \overline{\Sigma}_k \widetilde{\mathbf{F}}_k^T + \overline{\mathbf{Q}}_k, \tag{39}$$

Update:
$$\overline{\mathbf{S}}_{k+1} = \mathbf{G}\overline{\boldsymbol{\Sigma}}_{k+1|k}\mathbf{G}^T + \boldsymbol{\Sigma}_{\epsilon},$$
 (40)

$$\hat{\mathbf{x}}_{k+1} = \hat{\mathbf{x}}_{k+1|k} + \overline{\mathbf{\Sigma}}_{k+1|k} \mathbf{G}^T \overline{\mathbf{S}}_{k+1}^{-1} (\mathbf{a}_{k+1} - \mathbf{G} \hat{\mathbf{x}}_{k+1|k}), \tag{41}$$

$$\overline{\Sigma}_{k+1} = \overline{\Sigma}_{k+1|k} - \overline{\Sigma}_{k+1|k} \mathbf{G}^T \overline{\mathbf{S}}_{k+1}^{-1} \mathbf{G} \overline{\Sigma}_{k+1|k}, \tag{42}$$

where (inverse) process noise covariance matrix $\overline{\mathbf{Q}}_k = \mathbf{E}_k \mathbf{R} \mathbf{E}_k^T$.

B. I-KF-with-DF

Consider the linear system model with DF given by (1) and (2).

1) Forward filter: Denote the state estimation covariance, input estimation (without delay) covariance, and cross-covariance of state and input estimates by Σ_k^x , Σ_k^u and Σ_k^{xu} , respectively. The forward KF-with-DF is [33]:

Prediction:
$$\hat{\mathbf{x}}_{k+1|k} = \mathbf{F}\hat{\mathbf{x}}_k + \mathbf{B}\hat{\mathbf{u}}_k,$$
 (43)
$$\boldsymbol{\Sigma}_{k+1|k}^x = \begin{bmatrix} \mathbf{F} & \mathbf{B} \end{bmatrix} \begin{bmatrix} \boldsymbol{\Sigma}_k^x & \boldsymbol{\Sigma}_k^{xu} \\ \boldsymbol{\Sigma}_k^{xx} & \boldsymbol{\Sigma}_k^{x} \end{bmatrix} \begin{bmatrix} \mathbf{F}^T \\ \mathbf{B}^T \end{bmatrix} + \mathbf{Q},$$
Gain computation: $\mathbf{S}_{k+1} = \mathbf{H}\boldsymbol{\Sigma}_{k+1|k}^x \mathbf{H}^T + \mathbf{R},$

$$\mathbf{M}_{k+1} = (\mathbf{D}^T \mathbf{S}_{k+1}^{-1} \mathbf{D})^{-1} \mathbf{D}^T \mathbf{S}_{k+1}^{-1}, \quad \mathbf{K}_{k+1} = \mathbf{\Sigma}_{k+1|k}^x \mathbf{H}^T \mathbf{S}_{k+1}^{-1},$$

$$Update: \hat{\mathbf{u}}_{k+1} = \mathbf{M}_{k+1} (\mathbf{y}_{k+1} - \mathbf{H} \hat{\mathbf{x}}_{k+1|k}), \tag{44}$$

$$\hat{\mathbf{x}}_{k+1} = \hat{\mathbf{x}}_{k+1|k} + \mathbf{K}_{k+1}(\mathbf{y}_{k+1} - \mathbf{H}\hat{\mathbf{x}}_{k+1|k}),$$

$$\hat{\mathbf{x}}_{k+1} = \hat{\mathbf{x}}_{k+1|k} + \mathbf{K}_{k+1}(\mathbf{y}_{k+1} - \mathbf{H}\hat{\mathbf{x}}_{k+1|k} - \mathbf{D}\hat{\mathbf{u}}_{k+1}),$$
(45)

Covariance updates:
$$\mathbf{\Sigma}_{k+1}^u = (\mathbf{D}^T \mathbf{S}_{k+1}^{-1} \mathbf{D})^{-1}$$
,

$$\boldsymbol{\Sigma}_{k+1}^{x} = \boldsymbol{\Sigma}_{k+1|k}^{x} - \mathbf{K}_{k+1}(\mathbf{S}_{k+1} - \mathbf{D}\boldsymbol{\Sigma}_{k+1}^{u}\mathbf{D}^{T})\mathbf{K}_{k+1}^{T},$$

$$\mathbf{\Sigma}_{k+1}^{xu} = (\mathbf{\Sigma}_{k+1}^{ux})^T = -\mathbf{K}_{k+1}\mathbf{D}\mathbf{\Sigma}_{k+1}^{u}.$$

The forward filter exists if $rank(\mathbf{D}) = m$ (which implies $p \ge m$).

2) Inverse filter: Consider an augmented state vector $\mathbf{z}_k = \begin{bmatrix} \hat{\mathbf{x}}_k^T & \hat{\mathbf{u}}_k^T \end{bmatrix}^T$. Denote $\tilde{\mathbf{F}}_k = (\mathbf{I}_{n \times n} - \mathbf{K}_{k+1}\mathbf{H} + \mathbf{K}_{k+1}\mathbf{D}\mathbf{M}_{k+1}\mathbf{H})\mathbf{F}$, $\tilde{\mathbf{B}}_k = (\mathbf{I}_{n \times n} - \mathbf{K}_{k+1}\mathbf{H} + \mathbf{K}_{k+1}\mathbf{D}\mathbf{M}_{k+1}\mathbf{H})\mathbf{B}$, $\mathbf{E}_k = \mathbf{K}_{k+1}(\mathbf{I}_{p \times p} - \mathbf{D}\mathbf{M}_{k+1})$, $\tilde{\mathbf{H}}_k = -\mathbf{M}_{k+1}\mathbf{H}\mathbf{F}$ and $\tilde{\mathbf{D}}_k = -\mathbf{M}_{k+1}\mathbf{H}\mathbf{B}$. From (2), and (43)-(45), the state transition equations for I-KF-with-DF are

$$\hat{\mathbf{x}}_{k+1} = \widetilde{\mathbf{F}}_k \hat{\mathbf{x}}_k + \widetilde{\mathbf{B}}_k \hat{\mathbf{u}}_k + \mathbf{E}_k \mathbf{H} \mathbf{x}_{k+1} + \mathbf{E}_k \mathbf{D} \mathbf{u}_{k+1} + \mathbf{E}_k \mathbf{v}_{k+1},$$

and

$$egin{aligned} \hat{\mathbf{u}}_{k+1} \ &= \widetilde{\mathbf{H}}_k \hat{\mathbf{x}}_k + \widetilde{\mathbf{D}}_k \hat{\mathbf{u}}_k + \mathbf{M}_{k+1} \mathbf{H} \mathbf{x}_{k+1} + \mathbf{M}_{k+1} \mathbf{D} \mathbf{u}_{k+1} + \mathbf{M}_{k+1} \mathbf{v}_{k+1}. \end{aligned}$$

Also, $\left[\left(\mathbf{E}_{k} \mathbf{v}_{k+1} \right)^{T} \quad \left(\mathbf{M}_{k+1} \mathbf{v}_{k+1} \right)^{T} \right]^{T}$ is the augmented noise vector involved in this state transition with noise covariance matrix $\overline{\mathbf{Q}}_{k} = \begin{bmatrix} \mathbf{E}_{k} \mathbf{R} \mathbf{E}_{k}^{T} & \mathbf{E}_{k} \mathbf{R} \mathbf{M}_{k+1}^{T} \\ \mathbf{M}_{k+1} \mathbf{R} \mathbf{E}_{k}^{T} & \mathbf{M}_{k+1} \mathbf{R} \mathbf{M}_{k+1}^{T} \end{bmatrix}$. Then, *ceteris paribus*, following similar steps as in I-KF-without-DF, the I-KF-with-DF computes the estimate $\hat{\mathbf{z}}_{k} = \begin{bmatrix} \hat{\mathbf{x}}_{k}^{T} & \hat{\mathbf{u}}_{k}^{T} \end{bmatrix}^{T}$ of the augmented state vector using the observation \mathbf{a}_{k} given by (3). The system matrices for the augmented state are $\widetilde{\mathbf{F}}_{k}^{z} = \begin{bmatrix} \widetilde{\mathbf{F}}_{k} & \widetilde{\mathbf{B}}_{k} \\ \widetilde{\mathbf{H}}_{k} & \widetilde{\mathbf{D}}_{k} \end{bmatrix}$ and $\overline{\mathbf{G}} = \begin{bmatrix} \mathbf{G} & \mathbf{0}_{n_{a} \times m} \end{bmatrix}$. The I-KF-with-DF predicts the augmented state as

$$\begin{split} \hat{\mathbf{x}}_{k+1|k} &= \widetilde{\mathbf{F}}_k \hat{\mathbf{x}}_k + \widetilde{\mathbf{B}}_k \hat{\mathbf{u}}_k + \mathbf{E}_k \mathbf{H} \mathbf{x}_{k+1} + \mathbf{E}_k \mathbf{D} \mathbf{u}_{k+1}, \\ \hat{\mathbf{u}}_{k+1|k} &= \widetilde{\mathbf{H}}_k \hat{\mathbf{x}}_k + \widetilde{\mathbf{D}}_k \hat{\mathbf{u}}_k + \mathbf{M}_{k+1} \mathbf{H} \mathbf{x}_{k+1} + \mathbf{M}_{k+1} \mathbf{D} \mathbf{u}_{k+1}, \\ \hat{\mathbf{z}}_{k+1|k} &= \begin{bmatrix} \hat{\mathbf{x}}_{k+1|k}^T & \hat{\mathbf{u}}_{k+1|k}^T \end{bmatrix}^T, \ \overline{\mathbf{\Sigma}}_{k+1|k} = \widetilde{\mathbf{F}}_k^z \overline{\mathbf{\Sigma}}_k (\widetilde{\mathbf{F}}_k^z)^T + \overline{\mathbf{Q}}_k, \end{split}$$

followed by the update procedure (40)-(42) with \mathbf{G} and $\hat{\mathbf{x}}_{k+1}$ replaced by $\overline{\mathbf{G}}$ and $\hat{\mathbf{z}}_{k+1}$, respectively.

Since the observation y_k explicitly depends on the unknown input u_k for a system with DF, I-KF-with-DF and I-EKF-with-DF require perfect knowledge of the current input u_k as a known exogenous input to obtain their state and input estimates, which is not the case in I-KF-without-DF and I-EKF-without-DF.

V. STABILITY ANALYSES

For continuous-time non-linear Kalman filtering, some convergence results were mentioned in [45]. In case of EKF, sufficient conditions for stability of non-linear systems with linear output map were described in [46]. Recently, the stability of deterministic EKF was studied based on contraction theory in [47]. The asymptotic convergence of EKF for a special class of systems, where EKF is applied for joint state and parameter estimation of linear stochastic systems, was studied in [48, 49]. If the non-linearities have known bounds, then the Riccati equation is slightly modified to guarantee stability for the continuous-time EKF [50].

To derive the sufficient conditions for stochastic stability of nonlinear filters, one of the common approaches is to introduce unknown instrumental matrices to account for the linearization errors [28]. It does not assume any bound on the estimation error, but its sufficient conditions for stability, especially the bounds assumed on the unknown matrices, are difficult to verify for practical systems.

Alternatively, [26] considers the one-step prediction formulation of the filter and provides sufficient conditions under which the state prediction error is *exponentially bounded in mean-squared* sense. We restate some definitions and a useful Lemma from [26].

Definition 1 (Exponential mean-squared boundedness [26]). A stochastic process $\{\zeta_k\}_{k\geq 0}$ is defined to be exponentially bounded in mean-squared sense if there are real numbers $\eta, \nu > 0$ and $0 < \lambda < 1$ such that $\mathbb{E}\left[\|\zeta_k\|_2^2\right] \leq \eta \mathbb{E}\left[\|\zeta_0\|_2^2\right] \lambda^k + \nu$ holds for every $k \geq 0$.

Definition 2 (Boundedness with probability one [26]). A stochastic process $\{\zeta_k\}_{k\geq 0}$ is defined to be bounded with probability one if $\sup_{k\geq 0} \|\zeta_k\|_2 < \infty$ holds with probability one.

Lemma 1 (Boundedness of stochastic process [26, Lemma 2.1]). Consider a function $V_k(\zeta_k)$ of the stochastic process ζ_k and real numbers v_{\min} , v_{\max} , $\mu > 0$, and $0 < \lambda \le 1$ such that for all $k \ge 0$

$$|v_{min}||\zeta_k||_2^2 \le V_k(\zeta_k) \le v_{max}||\zeta_k||_2^2$$

and

$$\mathbb{E}\left[V_{k+1}(\zeta_{k+1})|\zeta_k\right] - V_k(\zeta_k) \le \mu - \lambda V_k(\zeta_k).$$

Then, the stochastic process $\{\zeta_k\}_{k\geq 0}$ is exponentially bounded in mean-squared sense, i.e.,

$$\mathbb{E}\left[\|\zeta_{k}\|_{2}^{2}\right] \leq \frac{v_{max}}{v_{min}} \mathbb{E}\left[\|\zeta_{0}\|_{2}^{2}\right] (1-\lambda)^{k} + \frac{\mu}{v_{min}} \sum_{i=1}^{k-1} (1-\lambda)^{i},$$

for every $k \geq 0$. Further, $\{\zeta_k\}_{k\geq 0}$ is also bounded with probability one.

In the bounded mean-squared sense, [26, Sec. III] showed that, while the two-step prediction and update recursion (described in previous sections) and one-step formulation of (forward) filters may differ in their performance and transient behaviour, they have similar convergence properties. However, the conditions of Lemma 1 were proved to hold when the error remained within suitable bounds; the guarantees fail if the error exceeds this bound at any instant. However, it was numerically shown [26, Sec. V] that the bound on the error was only of theoretical interest and, in practice, the filter remained stable for much larger estimation errors.

In the following, we first derive stability conditions for I-KF-without-DF in which we rely on the stability of the forward KF-without-DF as proved in [51]. The procedure is similar for the stability of I-KF-with-DF and I-KF-without-unknown-input [11] and hence, we omit the details for these filters. For I-EKF stability, we employ both unknown matrix and bounded non-linearity approaches. In the process, we also derive the forward EKF stability conditions using unknown matrix approach; note that the same was obtained using bounded non-linearity method in [26].

A. I-KF-with-unknown-input

Consider I-KF-without-DF of Section IV-A, where the forward filter is asymptotically stable under the sufficient conditions provided by [51]. The following Theorem 1 states conditions for stability of the inverse filter.

Theorem 1 (Stability of I-KF-without-DF). Consider an asymptotically stable forward KF-without-DF (28)-(36) such that the gain matrices $\overline{\mathbf{M}}_k$ and $\overline{\mathbf{K}}_k$ asymptotically approach to limiting gain matrices $\overline{\mathbf{M}}$ and $\overline{\mathbf{K}}$, respectively. The measurement noise covariance matrix Σ_{ϵ} is positive definite (p.d.). Denote the limiting matrices $\overline{\mathbf{F}} = (\mathbf{I} - \overline{\mathbf{K}}\mathbf{H})(\mathbf{I} - \mathbf{B}\overline{\mathbf{M}}\mathbf{H})\mathbf{F}$ and $\overline{\mathbf{Q}} = \overline{\mathbf{E}}\mathbf{R}\overline{\mathbf{E}}^T$, where

 $\overline{\mathbf{E}} = \mathbf{B}\overline{\mathbf{M}} - \overline{\mathbf{K}}\mathbf{H}\mathbf{B}\overline{\mathbf{M}} + \overline{\mathbf{K}}$. Then, the I-KF-without-DF (38)-(42) is asymptotically stable under the assumption that pair $(\overline{\mathbf{F}}, \mathbf{G})$ is observable and the pair $(\overline{\mathbf{F}}, \mathbf{C})$ is controllable for the system given by (3) and (37), where \mathbf{C} is such that $\overline{\mathbf{Q}} = \mathbf{C}^T \mathbf{C}$.

Note that, for I-KF-with-DF's stability, the stability conditions of basic KF need to hold for the augmented state considered in inverse filter formulation of Section IV-B. For forward KF-with-DF's stability conditions, we refer the reader to [51].

B. I-EKF-without-unknown-input: Unknown matrix approach

Consider the I-EKF's two-step prediction and update formulation of Section III-C, with forward filter as EKF-without-unknown-input.

1) Forward EKF stability: Denote the forward EKF's state prediction, state estimation and measurement prediction errors by $\tilde{\mathbf{x}}_{k+1|k} \doteq \mathbf{x}_{k+1} - \hat{\mathbf{x}}_{k+1|k}$, $\tilde{\mathbf{x}}_k \doteq \mathbf{x}_k - \hat{\mathbf{x}}_k$ and $\tilde{\mathbf{y}}_k \doteq \mathbf{y}_k - \hat{\mathbf{y}}_k$, with $\hat{\mathbf{y}}_k = h(\hat{\mathbf{x}}_{k|k-1})$, respectively. Using (21), (22) and the Taylor series expansion of $f(\cdot)$ at $\hat{\mathbf{x}}_k$, we get

$$\widetilde{\mathbf{x}}_{k+1|k} = \mathbf{F}_k(\mathbf{x}_k - \hat{\mathbf{x}}_k) + \mathbf{w}_k + \mathcal{O}(\|\mathbf{x}_k - \hat{\mathbf{x}}_k\|_2^2) \approx \mathbf{F}_k \widetilde{\mathbf{x}}_k + \mathbf{w}_k.$$

We consider the general case of time-varying process and measurement noise covariances and denote \mathbf{Q} , \mathbf{R} and $\mathbf{\Sigma}_{\epsilon}$ by \mathbf{Q}_{k} , \mathbf{R}_{k} and $\overline{\mathbf{R}}_{k}$, respectively. To account for the residuals and obtain an exact equality, we introduce an unknown instrumental diagonal matrix $\mathbf{U}_{k}^{x} \in \mathbb{R}^{n \times n}$ [28, 52] as

$$\widetilde{\mathbf{x}}_{k+1|k} = \mathbf{U}_k^x \mathbf{F}_k \widetilde{\mathbf{x}}_k + \mathbf{w}_k. \tag{46}$$

However, using (23), we have $\widetilde{\mathbf{x}}_k = \widetilde{\mathbf{x}}_{k|k-1} - \mathbf{K}_k \widetilde{\mathbf{y}}_k$, which when substituted in (46) yields $\widetilde{\mathbf{x}}_{k+1|k} = \mathbf{U}_k^x \mathbf{F}_k \widetilde{\mathbf{x}}_{k|k-1} - \mathbf{U}_k^x \mathbf{F}_k \mathbf{K}_k \widetilde{\mathbf{y}}_k + \mathbf{w}_k$. Similarly, using Taylor series expansion of $h(\cdot)$ at $\widehat{\mathbf{x}}_{k+1|k}$ in (7) and introducing an unknown diagonal matrix $\mathbf{U}_{k+1}^y \in \mathbb{R}^{p \times p}$ gives $\widetilde{\mathbf{y}}_{k+1} = \mathbf{U}_{k+1}^y \mathbf{H}_{k+1} \widetilde{\mathbf{x}}_{k+1|k} + \mathbf{v}_{k+1}$. The prediction error dynamics of the forward EKF becomes

$$\widetilde{\mathbf{x}}_{k+1|k} = \mathbf{U}_k^x \mathbf{F}_k (\mathbf{I} - \mathbf{K}_k \mathbf{U}_k^y \mathbf{H}_k) \widetilde{\mathbf{x}}_{k|k-1} - \mathbf{U}_k^x \mathbf{F}_k \mathbf{K}_k \mathbf{v}_k + \mathbf{w}_k.$$
(47)

Denote the true prediction covariance by $P_{k+1|k}$ $\mathbb{E}\left[\widetilde{\mathbf{x}}_{k+1|k}\widetilde{\mathbf{x}}_{k+1|k}^{T}\right].$ Define $\delta \mathbf{P}_{k+1|k}$ the difference as prediction of estimated $\Sigma_{k+1|k}$ covariance and prediction covariance $\mathbf{P}_{k+1|k}$ while $\Delta \mathbf{P}_{k+1|k}$ as the error in the approximation of the expectation $\mathbb{E}\left[\mathbf{U}_{k}^{x}\mathbf{F}_{k}(\mathbf{I}-\mathbf{K}_{k}\mathbf{U}_{k}^{y}\mathbf{H}_{k})\widetilde{\mathbf{x}}_{k|k-1}\widetilde{\mathbf{x}}_{k|k-1}^{T}(\mathbf{I}-\mathbf{K}_{k}\mathbf{U}_{k}^{y}\mathbf{H}_{k})^{T}\mathbf{F}_{k}^{T}\mathbf{U}_{k}^{x}\right]$ by $\mathbf{U}_k^x \mathbf{F}_k (\mathbf{I} - \mathbf{K}_k \mathbf{U}_k^y \mathbf{H}_k) \mathbf{\Sigma}_{k|k-1} (\mathbf{I} - \mathbf{K}_k \mathbf{U}_k^y \mathbf{H}_k)^T \mathbf{F}_k^T \mathbf{U}_k^x$. Denoting $\hat{\mathbf{Q}}_k = \mathbf{Q}_k + \mathbf{U}_k^x \mathbf{F}_k \mathbf{K}_k \mathbf{R}_k \mathbf{K}_k^T \mathbf{F}_k^T \mathbf{U}_k^x + \delta \mathbf{P}_{k+1|k} + \Delta \mathbf{P}_{k+1|k}$ and following similar steps as in [28, 52], we have

$$\Sigma_{k+1|k} = \mathbf{U}_{k}^{y} \mathbf{F}_{k} (\mathbf{I} - \mathbf{K}_{k} \mathbf{U}_{k}^{y} \mathbf{H}_{k}) \mathbf{\Sigma}_{k|k-1} (\mathbf{I} - \mathbf{K}_{k} \mathbf{U}_{k}^{y} \mathbf{H}_{k})^{T} \mathbf{F}_{k}^{T} \mathbf{U}_{k}^{x} + \hat{\mathbf{Q}}_{k}.$$

Similarly, denoting the true measurement prediction covariance and true cross-covariance by \mathbf{P}_{k+1}^{yy} and \mathbf{P}_{k+1}^{xy} , respectively, we obtain

$$\begin{split} \mathbf{S}_{k+1} &= \mathbf{U}_{k+1}^{y} \mathbf{H}_{k+1} \mathbf{\Sigma}_{k+1|k} \mathbf{H}_{k+1}^{T} \mathbf{U}_{k+1}^{y} + \hat{\mathbf{R}}_{k+1}, \\ \mathbf{\Sigma}_{k+1}^{xy} &= \begin{cases} \mathbf{\Sigma}_{k+1|k} \mathbf{U}_{k+1}^{xy} \mathbf{H}_{k+1}^{T} \mathbf{U}_{k+1}^{y}, & n \geq p \\ \mathbf{\Sigma}_{k+1|k} \mathbf{H}_{k+1}^{T} \mathbf{U}_{k+1}^{y} \mathbf{U}_{k+1}^{xy}, & n$$

where $\hat{\mathbf{R}}_{k+1} = \mathbf{R}_{k+1} + \Delta \mathbf{P}_{k+1}^{yy} + \delta \mathbf{P}_{k+1}^{yy}$ and \mathbf{U}_{k+1}^{xy} is an unknown instrumental matrix introduced to account for errors in the estimated cross-covariance Σ_{k+1}^{xy} [53].

The following Theorem 2 provides stability conditions for the forward EKF using the unknown matrices \mathbf{U}_k^x , \mathbf{U}_k^y and \mathbf{U}_k^{xy} .

Theorem 2 (Stochastic stability of forward EKF). Consider the non-linear stochastic system in (21) and (7). The two-step forward EKF

formulation is as in Section III-C. Let the following assumptions hold true:

1) There exist positive real numbers \overline{f} , \overline{h} , $\overline{\alpha}$, $\overline{\beta}$, $\overline{\gamma}$, $\underline{\sigma}$, $\overline{\sigma}$, \overline{q} , \overline{r} , \hat{q} and \hat{r} such that the following bounds are fulfilled for all $k \geq 0$.

$$\begin{split} &\|\mathbf{F}_k\| \leq \overline{f}, \quad \|\mathbf{H}_k\| \leq \overline{h}, \quad \|\mathbf{U}_k^x\| \leq \overline{\alpha}, \quad \|\mathbf{U}_k^y\| \leq \overline{\beta}, \\ &\|\mathbf{U}_k^{xy}\| \leq \overline{\gamma}, \quad \mathbf{Q}_k \preceq \overline{q}\mathbf{I}, \quad \mathbf{R}_k \preceq \overline{r}\mathbf{I}, \quad \hat{q}\mathbf{I} \preceq \hat{\mathbf{Q}}_k, \\ &\hat{r}\mathbf{I} \preceq \hat{\mathbf{R}}_k, \quad \underline{\sigma}\mathbf{I} \preceq \mathbf{\Sigma}_{k|k-1} \preceq \overline{\sigma}\mathbf{I}. \end{split}$$

2) \mathbf{U}_k^x and \mathbf{F}_k are non-singular for every $k \geq 0$.

Then, the prediction error $\widetilde{\mathbf{x}}_{k|k-1}$ and the estimation error $\widetilde{\mathbf{x}}_k$ of the forward EKF are exponentially bounded in mean-squared sense and bounded with probability one provided that the constants satisfy the inequality

$$\overline{\sigma\gamma}\overline{h}^2\overline{\beta}^2 < \hat{r}.\tag{48}$$

Proof: See Appendix B.

2) Inverse EKF stability: For a stable forward EKF in the previous subsection, we prove the stochastic stability of the I-EKF as an extension of Theorem 2. Similar to the forward EKF, we introduce unknown matrices $\overline{\mathbf{U}}_k^x$ and $\overline{\mathbf{U}}_k^a$ to account for the errors in the linearization of functions $\widetilde{f}_k(\cdot)$ and $g(\cdot)$, respectively, and $\overline{\mathbf{U}}_k^{xa}$ for the errors in cross-covariance matrix estimation. Similarly, denote $\widehat{\mathbf{Q}}_k$ and $\widehat{\mathbf{R}}_k$ as the counterparts of $\widehat{\mathbf{Q}}_k$ and $\widehat{\mathbf{R}}_k$, respectively, in the I-EKF dynamics. The following Theorem 3 states the stability criteria for I-EKF. Note that, when compared to Theorem 2, the following result requires an additional condition $\underline{r}\mathbf{I} \preceq \mathbf{R}_k$ for all $k \geq 0$ for some $\underline{r} > 0$.

Theorem 3 (Stochastic stability of I-EKF). Consider the adversary's forward EKF that is stable as per Theorem 2. Additionally, assume that the following hold true for all $k \ge 0$.

$$\begin{split} & \underline{r}\mathbf{I} \preceq \mathbf{R}_k, \quad \|\mathbf{G}_k\| \leq \overline{g}, \qquad \|\overline{\mathbf{U}}_k^a\| \leq \overline{c}, \qquad \|\overline{\mathbf{U}}_k^{xa}\| \leq \overline{d}, \\ & \overline{\mathbf{R}}_k \preceq \overline{\epsilon}\mathbf{I}, \qquad \hat{c}\mathbf{I} \preceq \hat{\overline{\mathbf{Q}}}_k, \qquad \hat{d}\mathbf{I} \preceq \hat{\overline{\mathbf{R}}}_k, \qquad p\mathbf{I} \preceq \overline{\boldsymbol{\Sigma}}_{k|k-1} \preceq \overline{p}\mathbf{I}, \end{split}$$

for some real positive constants $\underline{r}, \overline{g}, \overline{c}, \overline{d}, \overline{\epsilon}, \hat{c}, \hat{d}, \underline{p}, \overline{p}$. Then, the state estimation error of I-EKF is exponentially bounded in mean-squared sense and bounded with probability one provided that the constants satisfy the inequality $\overline{p}d\overline{g}^2\overline{c}^2 < \hat{d}$.

Note that Theorem 2 requires both $\hat{\mathbf{Q}}_k$ and $\hat{\mathbf{R}}_k$ to be p.d. In general, the difference matrices $\Delta \mathbf{P}_{k+1|k}$, $\delta \mathbf{P}_{k+1|k}$, $\Delta \mathbf{P}_{k+1}^{yy}$ and $\delta \mathbf{P}_{k+1}^{yy}$ may not be p.d. One could enhance the stability of EKF by enlarging the noise covariance matrices by adding sufficiently large $\Delta \mathbf{Q}_k$ and $\Delta \mathbf{R}_k$ to \mathbf{Q}_k and \mathbf{R}_k , respectively [28, 53]. The same argument also holds true for I-EKF noise covariance matrices.

C. I-EKF-without-unknown-input: Bounded non-linearity method

Consider the forward EKF's one step prediction formulation (25)-(27). Using Taylor series expansion around the estimate $\hat{\mathbf{x}}_k$, we have

$$f(\mathbf{x}_k) - f(\hat{\mathbf{x}}_k) = \mathbf{F}_k(\mathbf{x}_k - \hat{\mathbf{x}}_k) + \phi(\mathbf{x}_k, \hat{\mathbf{x}}_k),$$

$$h(\mathbf{x}_k) - h(\hat{\mathbf{x}}_k) = \mathbf{H}_k(\mathbf{x}_k - \hat{\mathbf{x}}_k) + \chi(\mathbf{x}_k, \hat{\mathbf{x}}_k),$$

where $\phi(\cdot)$ and $\chi(\cdot)$ are suitable non-linear functions to account for the higher-order terms of the expansions. Denoting the estimation error by $\mathbf{e}_k \doteq \mathbf{x}_k - \hat{\mathbf{x}}_k$, the error dynamics of the forward filter is

$$\mathbf{e}_{k+1} = (\mathbf{F}_k - \mathbf{K}_k \mathbf{H}_k) \mathbf{e}_k + \mathbf{r}_k + \mathbf{s}_k, \tag{49}$$

where $\mathbf{r}_k = \phi(\mathbf{x}_k, \hat{\mathbf{x}}_k) - \mathbf{K}_k \chi(\mathbf{x}_k, \hat{\mathbf{x}}_k)$ and $\mathbf{s}_k = \mathbf{w}_k - \mathbf{K}_k \mathbf{v}_k$.

The following Theorem 4 (reproduced from [26]) provides sufficient conditions for forward EKF's stochastic stability. **Theorem 4** (Exponential boundedness of forward EKF's error [26]). Consider a non-linear stochastic system defined by (21) and (7), and the one-step prediction formulation of forward EKF (25)-(27). Let the following assumptions hold true.

1) There exist positive real numbers $\overline{f}, \overline{h}, \sigma, \overline{\sigma}, \underline{q}, \underline{r}$, δ such that the following bounds are fulfilled for all $k \geq 0$.

$$\underline{\sigma}\mathbf{I} \preceq \mathbf{\Sigma}_k \preceq \overline{\sigma}\mathbf{I}, \qquad \underline{q}\mathbf{I} \preceq \mathbf{Q}_k \preceq \delta\mathbf{I},
r\mathbf{I} \preceq \mathbf{R}_k \preceq \delta\mathbf{I}, \qquad \|\mathbf{F}_k\| \leq \overline{f}, \quad \|\mathbf{H}_k\| \leq \overline{h}.$$

- 2) \mathbf{F}_k is non singular for every k > 0.
- 3) There exist positive real numbers κ_{ϕ} , ϵ_{ϕ} , ϵ_{χ} , ϵ_{χ} such that the non-linear functions $\phi(\cdot)$ and $\chi(\cdot)$ satisfy

$$\begin{aligned} &\|\phi(\mathbf{x}, \hat{\mathbf{x}})\|_2 \le \kappa_\phi \|\mathbf{x} - \hat{\mathbf{x}}\|_2^2 & for & \|\mathbf{x} - \hat{\mathbf{x}}\|_2 \le \epsilon_\phi, \\ &\|\chi(\mathbf{x}, \hat{\mathbf{x}})\|_2 \le \kappa_\chi \|\mathbf{x} - \hat{\mathbf{x}}\|_2^2 & for & \|\mathbf{x} - \hat{\mathbf{x}}\|_2 \le \epsilon_\chi. \end{aligned}$$

Then the estimation error given by (49) is exponentially bounded in mean-squared sense and bounded with probability one provided that the estimation error is bounded by suitable constant $\epsilon > 0$.

Theorem 4 guarantees that the estimation error remains exponentially bounded in mean-squared sense as long as the error is within suitable ϵ bounds. Further, the mean drift $\mathbb{E}[V_{k+1}(\mathbf{e}_{k+1})|\mathbf{e}_k] - V_k(\mathbf{e}_k)$ for a suitably defined $V_k(\cdot)$ (for application of Lemma 1) is negative when $\widetilde{\epsilon} \leq \|\mathbf{e}_k\|_2 \leq \epsilon$, which drives the system towards zero error in an expected sense. However, with some finite probability, the estimation error at some time-steps may be outside the ϵ bound. In this case, we cannot guarantee with probability one that the error will be within ϵ bound again at some future time-steps. As mentioned earlier, bounded non-linearity approach may not provide theoretical guarantees for the filter to be stable for all time-steps but, practically, the filter remains stable even if the estimation error is outside the ϵ bound provided that the assumed bounds on the system model are satisfied.

For the inverse filter observations (6), the Taylor series expansion of $g(\cdot)$ at estimate $\hat{\hat{\mathbf{x}}}_k$ of I-EKF's one step prediction formulation of Section III-C, considering suitable non-linear function $\overline{\chi}(\cdot)$ is

$$g(\hat{\mathbf{x}}_k) - g(\hat{\mathbf{x}}_k) = \mathbf{G}_k(\hat{\mathbf{x}}_k - \hat{\mathbf{x}}_k) + \overline{\chi}(\hat{\mathbf{x}}_k, \hat{\mathbf{x}}_k).$$

Finally, the error dynamics of the inverse filter, with the estimation error denoted by $\overline{\mathbf{e}}_k \doteq \hat{\mathbf{x}}_k - \hat{\hat{\mathbf{x}}}_k$ and the inverse filter's Kalman gain and estimation error covariance matrix by $\overline{\mathbf{K}}_k$ and $\overline{\mathbf{\Sigma}}_k$, respectively, is

$$\overline{\mathbf{e}}_{k+1} = (\widetilde{\mathbf{F}}_k^x - \overline{\mathbf{K}}_k \mathbf{G}_k) \overline{\mathbf{e}}_k + \overline{\mathbf{r}}_k + \overline{\mathbf{s}}_k, \tag{50}$$

where $\overline{\mathbf{r}}_k = \overline{\phi}_k(\hat{x}_k, \hat{\hat{\mathbf{x}}}_k) - \overline{\mathbf{K}}_k \overline{\chi}(\hat{\mathbf{x}}_k, \hat{\hat{\mathbf{x}}}_k)$ and $\overline{\mathbf{s}}_k = \mathbf{K}_k \mathbf{v}_k - \overline{\mathbf{K}}_k \boldsymbol{\epsilon}_k$ with $\overline{\phi}_k(\hat{\mathbf{x}}_k, \hat{\hat{\mathbf{x}}}_k) = \phi(\hat{\mathbf{x}}_k, \hat{\hat{\mathbf{x}}}_k) - \mathbf{K}_k \chi(\hat{\mathbf{x}}_k, \hat{\hat{\mathbf{x}}}_k)$.

The following Theorem 5 guarantees the stability of I-EKF. Note the additional assumption of \mathbf{H}_k to be full column rank for all $k \geq 0$, which implies $p \geq n$.

Theorem 5 (Exponential boundedness of I-EKF's error). *Consider the adversary's forward one-step prediction EKF that is stable as per Theorem 4. Additionally, assume that the following hold true.*

1) There exist positive real numbers \overline{g} , \underline{m} , \overline{m} , $\underline{\epsilon}$, $\underline{\epsilon}$ $\overline{\delta}$ such that the following bounds are fulfilled for all $k \geq 0$.

$$\|\mathbf{G}_k\| \leq \overline{g}, \ \underline{m}\mathbf{I} \preceq \overline{\Sigma}_k \preceq \overline{m}\mathbf{I}, \ \underline{\epsilon}\mathbf{I} \preceq \overline{\mathbf{R}}_k \preceq \overline{\delta}\mathbf{I}.$$

- 2) \mathbf{H}_k is full column rank for every $k \geq 0$.
- 3) There exist positive real numbers $\kappa_{\bar{\chi}}$ and $\epsilon_{\bar{\chi}}$ such that the non-linear function $\bar{\chi}(\cdot)$ satisfies

$$\|\overline{\chi}(\hat{\mathbf{x}},\hat{\hat{\mathbf{x}}})\|_2 \leq \kappa_{\bar{\chi}} \|\hat{\mathbf{x}} - \hat{\hat{\mathbf{x}}}\|_2^2 \quad for \quad \|\hat{\mathbf{x}} - \hat{\hat{\mathbf{x}}}\|_2 \leq \epsilon_{\bar{\chi}}.$$

Then, the estimation error for I-EKF given by (50) is exponentially bounded in mean-squared sense and bounded with probability one provided that the estimation error is bounded by suitable constant $\bar{\epsilon} > 0$.

VI. NUMERICAL EXPERIMENTS

We illustrate the performance of the proposed inverse filters for different example systems. The efficacy of the inverse filters is demonstrated by comparing the estimation error with RCRLB. The CRLB provides a lower bound on mean-squared error (MSE) and is widely used to assess the performance of an estimator. For the discrete-time non-linear filtering, we employ the RCRLB as $\mathbb{E}\left[(\mathbf{x}_k - \hat{\mathbf{x}}_k)(\mathbf{x}_k - \hat{\mathbf{x}}_k)^T\right] \succeq \mathbf{J}_k^{-1} \text{ where } \mathbf{J}_k = \mathbb{E}\left[-\frac{\partial^2 \ln p(Y^k, X^k)}{\partial \mathbf{x}_k^2}\right]$ is the Fisher information matrix [30]. Here, $X^k = \{\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_k\}$ is the state vector series while $Y^k = \{\mathbf{y}_0, \mathbf{y}_1, \dots, \mathbf{y}_k\}$ are the noisy observations. Also, $p(Y^k, X^k)$ is the joint probability density of pair (Y^k, X^k) and $\hat{\mathbf{x}}_k$ (a function of Y^k) is an estimate of \mathbf{x}_k with $\frac{\partial^2(\cdot)}{\partial \mathbf{x}^2}$ denoting the Hessian with second order partial derivatives. The information matrix \mathbf{J}_k can be computed recursively as [30]

$$\mathbf{J}_{k} = \mathbf{D}_{k}^{22} - \mathbf{D}_{k}^{21} (\mathbf{J}_{k-1} + \mathbf{D}_{k}^{11})^{-1} \mathbf{D}_{k}^{12}, \qquad (51)$$
where
$$\mathbf{D}_{k}^{11} = \mathbb{E} \left[-\frac{\partial^{2} \ln p(\mathbf{x}_{k} | \mathbf{x}_{k-1})}{\partial \mathbf{x}_{k-1}^{2}} \right],$$

$$\mathbf{D}_{k}^{12} = \mathbb{E} \left[-\frac{\partial^{2} \ln p(\mathbf{x}_{k} | \mathbf{x}_{k-1})}{\partial \mathbf{x}_{k} \partial \mathbf{x}_{k-1}} \right] = (\mathbf{D}_{k}^{21})^{T},$$

$$\mathbf{D}_{k}^{22} = \mathbb{E} \left[-\frac{\partial^{2} \ln p(\mathbf{x}_{k} | \mathbf{x}_{k-1})}{\partial \mathbf{x}_{k}^{2}} \right] + \mathbb{E} \left[-\frac{\partial^{2} \ln p(\mathbf{y}_{k} | \mathbf{x}_{k})}{\partial \mathbf{x}_{k}^{2}} \right].$$

For the non-linear system given by (21) and (7), the forward information matrices $\{J_k\}$ recursions reduces to [28]

$$\mathbf{J}_{k+1} = \mathbf{Q}_{k}^{-1} + \mathbf{H}_{k+1}^{T} \mathbf{R}_{k+1}^{-1} \mathbf{H}_{k+1} - \mathbf{Q}_{k}^{-1} \mathbf{F}_{k} (\mathbf{J}_{k} + \mathbf{F}_{k}^{T} \mathbf{Q}_{k}^{-1} \mathbf{F}_{k})^{-1} \mathbf{F}_{k}^{T} \mathbf{Q}_{k}^{-1},$$
(52)

where $\mathbf{F}_k = \nabla_{\mathbf{x}} f(\mathbf{x})|_{\mathbf{x} = \mathbf{x}_k}$ and $\mathbf{H}_k = \nabla_{\mathbf{x}} h(\mathbf{x})|_{\mathbf{x} = \mathbf{x}_k}$. Note that, for the information matrices recursion, the Jacobians \mathbf{F}_k and \mathbf{H}_k are evaluated at the true state \mathbf{x}_k while for forward EKF recursions, these are evaluated at the estimates of the state. These recursions can be trivially extended to other system models considered in this paper and to compute the information matrix $\overline{\mathbf{J}}_k$ for inverse filter's estimate $\hat{\mathbf{x}}_k$.

Throughout all experiments, 100 time-steps (indexed by k) were considered. The initial information matrices \mathbf{J}_0 and $\overline{\mathbf{J}}_0$ were set to $\mathbf{\Sigma}_0^{-1}$ and $\overline{\mathbf{\Sigma}}_0^{-1}$, respectively, unless mentioned otherwise. Note that these initial estimates only affect the RCRLB in the transient phase. The steady state RCRLB is independent of the initialization.

A. Inverse KF with unknown inputs

Consider a discrete-time linear system without DF [54],

$$\mathbf{x}_{k+1} = \begin{bmatrix} 0.1 & 0.5 & 0.08 \\ 0.6 & 0.01 & 0.04 \\ 0.1 & 0.7 & 0.05 \end{bmatrix} \mathbf{x}_k + \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} u_k + \mathbf{w}_k,$$
$$\mathbf{y}_k = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \mathbf{x}_k + \mathbf{v}_k, \quad a_k = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \hat{\mathbf{x}}_k + \epsilon_k,$$

with $\mathbf{w}_k \sim \mathcal{N}(\mathbf{0}, \mathbf{I}_3)$, $\mathbf{v}_k \sim \mathcal{N}(\mathbf{0}, 2\mathbf{I}_2)$ and $\epsilon_k \sim \mathcal{N}(0, 5)$. The unknown input u_k was set to 50 for $1 \leq k \leq 50$ and -50 thereafter. The initial state was $\mathbf{x}_0 = [1, 1, 1]^T$. For the forward filter, the initial state estimate was set to $[0, 0, 0]^T$ with initial covariance $\mathbf{\Sigma}_0 = \mathbf{I}_3$. For the inverse filter, the initial state estimate was set to \mathbf{x}_0 (known to the defender) itself with initial covariance $\overline{\mathbf{\Sigma}}_0 = 5\mathbf{I}_3$.

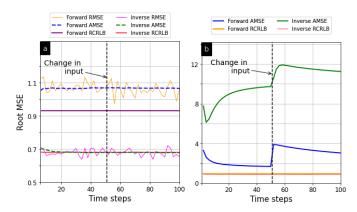


Fig. 1. RMSE, AMSE and RCRLB for forward and inverse filters (a) KF-without-DF: (b) KF-with-DF.

For KF-with-DF, we modify the forward filter's observations as [55]:

$$\mathbf{y}_k = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \mathbf{x}_k + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_k + \mathbf{v}_k.$$

Here, the initial input estimate was set to 10 with initial input estimate covariance $\Sigma_0^u = 10$ and initial cross-covariance $\Sigma_0^{xu} = [0,0,0]^T$. The inverse filter's initial augmented state estimate \mathbf{z}_0 was set to $[1,1,1,50]^T$ with initial covariance $\overline{\Sigma}_0 = 5\mathbf{I}_4$.

Fig. 1 shows the time-averaged RMSE (AMSE) = $\sqrt{(\sum_{i=1}^k \|\mathbf{x}_i - \hat{\mathbf{x}}_i\|_2^2)/nk}$ at k-th time step for n-dimensional actual state \mathbf{x}_i and its estimate $\hat{\mathbf{x}}_i$, and RCRLB for state estimation for both forward and inverse filters in the two cases, respectively, averaged over 200 runs. For KF-without-DF, we plot the root MSE (RMSE) = $\sqrt{(\|\mathbf{x}_k - \hat{\mathbf{x}}_k\|_2^2)/n}$ for comparison here but omit it for later plots for clarity. Note that in Fig. 1a, the I-KF-without-DF's RMSE fluctuates about the RCRLB because of a finite number of sample paths; see also similar phenomena in [28, 56, 57]. The RCRLB value for state estimation is $\sqrt{\text{Tr}(\mathbf{J}^{-1})}$ with \mathbf{J} denoting the associated information matrix.

Fig. 1 shows that the effect of change in unknown input after 50 time-steps is negligible for KF-without-DF in both forward and inverse filters. However, for KF-with-DF, the sudden change in unknown input leads to an increase in state estimation error of the forward filter and, consequently, of the inverse filter. The estimation error of I-KF-without-DF is less than the corresponding forward filter while for KF-with-DF, the inverse filter has a higher estimation error than the forward filter. Only I-KF-without-DF efficiently achieves the RCRLB bound on the estimation error. Note that in this and the following numerical experiments, the forward and inverse filters are compared only to highlight the relative estimation accuracy.

B. Inverse EKF without unknown inputs

Consider the discrete-time non-linear system model of FM demodulator without unknown inputs [43, Sec. 8.2]

$$\mathbf{x}_{k+1} \doteq \begin{bmatrix} \lambda_{k+1} \\ \theta_{k+1} \end{bmatrix} = \begin{bmatrix} \exp\left(-T/\beta\right) & 0 \\ -\beta \exp\left(-T/\beta\right) - 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_k \\ \theta_k \end{bmatrix} + \begin{bmatrix} 1 \\ -\beta \end{bmatrix} w_k,$$

$$\mathbf{y}_k = \sqrt{2} \begin{bmatrix} \sin \theta_k \\ \cos \theta_k \end{bmatrix} + \mathbf{v}_k, \quad a_k = \hat{\lambda}_k^2 + \epsilon_k,$$

with $w_k \sim \mathcal{N}(0, 0.01)$, $\mathbf{v}_k \sim \mathcal{N}(\mathbf{0}, \mathbf{I}_2)$, $\epsilon_k \sim \mathcal{N}(0, 5)$, $T = 2\pi/16$ and $\beta = 100$. Here, the observation function $g(\cdot)$ for the inverse filter is quadratic. Also, $\hat{\lambda}_k$ is the forward EKF's estimate of λ_k .

The initial state $\mathbf{x}_0 \doteq [\lambda_0, \theta_0]^T$ was set randomly with $\lambda_0 \sim \mathcal{N}(0, 1)$ and $\theta_0 \sim \mathcal{U}[-\pi, \pi]$. The initial state estimates of forward

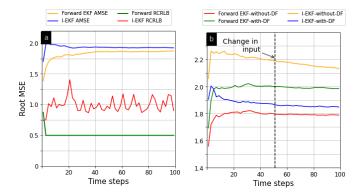


Fig. 2. (a) AMSE and RCRLB for forward and inverse EKF; (b) Time-averaged RMSE for forward and inverse EKF with and without DF, averaged over 200 runs.

and inverse EKF were also similarly drawn at random. The initial covariances were set to $\Sigma_0=10\mathbf{I}_2$ and $\overline{\Sigma}_0=5\mathbf{I}_2$ for forward and inverse EKF, respectively. The phase term of the state θ and its estimates $\hat{\theta}$ and $\hat{\theta}$ (for both prediction and measurement updates) were considered to be modulo 2π [43]. Note that the process covariance \mathbf{Q} is a singular matrix. For numerical stability and to facilitate computation of \mathbf{Q}^{-1} for evaluating information matrices \mathbf{J}_k , we used an enlarged covariance matrix by adding $10^{-10}\mathbf{I}_2$ to \mathbf{Q} in the forward filters. Similarly, we added $10^{-10}\mathbf{I}_2$ to $\overline{\mathbf{Q}}_k$ in the inverse filter because $\overline{\mathbf{Q}}_k$ is time-varying and may be ill-conditioned. The initial $\overline{\mathbf{J}}_0$ was taken close to the inverse of the steady state estimation covariance matrix of the forward filter. The initial $\overline{\mathbf{J}}_0$ only affects the RCRLB calculated for initial few time-steps. The RCRLB after these initial time-steps (around 20 for the considered system) shows same behaviour irrespective of the initial $\overline{\mathbf{J}}_0$.

Fig. 2a shows the AMSE and RCRLB for forward and inverse EKF averaged over 200 runs. The I-EKF's estimation error is comparable to that of forward EKF with I-EKF's average error being slightly higher than that of forward EKF. However, the difference between AMSE and RCRLB for I-EKF is less than that for forward EKF. Hence, we conclude that I-EKF is more efficient here. The I-EKF assumes initial covariance Σ_0 as $5\mathbf{I}_2$ (the true Σ_0 of forward EKF is $10\mathbf{I}_2$) and a random initial state for these recursions. In spite of this difference in the initial estimates, I-EKF's error performance is comparable to that of the forward EKF.

C. Inverse EKF with unknown inputs

For inverse EKF with unknown input, we modified the non-linear system model of Section VI-B to include an unknown input u_k as

$$\mathbf{x}_{k+1} = \begin{bmatrix} \exp\left(-T/\beta\right) & 0 \\ -\beta \exp\left(-T/\beta\right) - 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_k \\ \theta_k \end{bmatrix} + \begin{bmatrix} 0.001 \\ 1 \end{bmatrix} u_k + \begin{bmatrix} 1 \\ -\beta \end{bmatrix} w_k,$$

where u_k was set to $\pi/4$ for $1 \le k \le 50$ and $-\pi/4$ thereafter. The observation \mathbf{y}_k of the forward EKF-without-DF was same as in Section VI-B. Consider a linear measurement a_k for the inverse filter as $a_k = \hat{\lambda}_k + \epsilon_k$. For the forward filter, the initial input estimate was set to 0 while the inverse filter initial augmented state estimate consisted of the true state \mathbf{x}_0 and true input u_0 (known to the defender) with initial covariance estimate $\overline{\Sigma}_0 = 15\mathbf{I}_3$.

Similarly, for system with DF, we again considered the same non-linear system (without any unknown input in \mathbf{x}_k state transition) but with a modified forward filter's observation $\mathbf{y}_k = \sqrt{2} \begin{bmatrix} \sin{(\theta_k + u_k)} \\ \cos{(\theta_k + u_k)} \end{bmatrix} + \mathbf{v}_k$. The input estimates \hat{u} and \hat{u} were also, as before, modulo 2π . The Gaussian noise terms in the inverse filter

state transitions ((12) and (20)) are transformed through non-linear functions such that (52) is not applicable. The RCRLB in this case is derived using the general \mathbf{J}_k recursions given by (51), which is omitted here. Fig. 2b shows that for both EKF with and without DF, the change in unknown input after 50 time-steps does not increase the estimation error (as for KF-with-DF in Fig. 1b). The estimation error of I-EKF-without-DF (I-EKF-with-DF) is higher (lower) than that of the corresponding forward filter. Any change in unknown input affects the inverse filter's performance only when a significant change occurs in the forward filter's performance.

VII. SUMMARY

We studied the inverse filtering problem for non-linear systems with and without unknown inputs in the context of counter-adversarial applications. For systems with unknown inputs, the adversary's observations may or may not be affected by the unknown input known to the defender but not the adversary. The stochastic stability of a forward filter with certain additional system assumptions is also sufficient for the stability of the inverse filter. Our experiments suggested that the impact of the unknown input on inverse filter's performance strongly depends on its impact on the forward filter. For certain systems, the inverse filter may perform more efficiently than the forward filter. In the companion paper (Part II) [25], we develop I-EKF for second-order, Gaussian sum, and dithered EKFs and consider the case of uncertain information about the forward filter.

APPENDIX A PROOF OF THEOREM 1

Under the stability assumption of the forward filter, $\widetilde{\mathbf{F}}_k$ and \mathbf{E}_k converge to $\overline{\mathbf{F}}$ and $\overline{\mathbf{E}}$, respectively, where $\overline{\mathbf{F}}=(\mathbf{I}-\overline{\mathbf{K}}\mathbf{H})(\mathbf{I}-\mathbf{B}\overline{\mathbf{M}}\mathbf{H})\mathbf{F}$ and $\overline{\mathbf{E}}=\mathbf{B}\overline{\mathbf{M}}-\overline{\mathbf{K}}\mathbf{H}\mathbf{B}\overline{\mathbf{M}}+\overline{\mathbf{K}}$, obtained by replacing \mathbf{K}_{k+1} and \mathbf{M}_{k+1} by the limiting matrices $\overline{\mathbf{K}}$ and $\overline{\mathbf{M}}$, respectively, in $\widetilde{\mathbf{F}}_k$ and \mathbf{E}_k . In this limiting case, the state transition equation (37) becomes $\hat{\mathbf{x}}_{k+1}=\overline{\mathbf{F}}\hat{\mathbf{x}}_k+\overline{\mathbf{E}}\mathbf{H}\mathbf{x}_{k+1}+\overline{\mathbf{E}}\mathbf{v}_{k+1}$. From (39), (40), and (42) and substituting the limiting matrices, the Riccati equation $\overline{\mathbf{\Sigma}}_{k+1|k}=\overline{\mathbf{F}}\left[\overline{\mathbf{\Sigma}}_{k|k-1}-\overline{\mathbf{\Sigma}}_{k|k-1}\mathbf{G}^T(\mathbf{G}\overline{\mathbf{\Sigma}}_{k|k-1}\mathbf{G}^T+\overline{\mathbf{R}})^{-1}\mathbf{G}\overline{\mathbf{\Sigma}}_{k|k-1}\right]\overline{\mathbf{F}}^T+\overline{\mathbf{Q}}$ is obtained, where $\overline{\mathbf{Q}}=\overline{\mathbf{E}}\mathbf{R}\overline{\mathbf{E}}^T$. For the forward filter to be stable, covariance \mathbf{R} needs to be p.d. [51] and hence, $\overline{\mathbf{Q}}$ is a p.s.d. matrix. With $\overline{\mathbf{R}}$ being p.d. and the observability and controllability assumptions, $\overline{\mathbf{\Sigma}}_{k|k-1}$ tends to a unique p.d. matrix $\overline{\mathbf{\Sigma}}$ satisfying $\overline{\mathbf{\Sigma}}=\overline{\mathbf{F}}[\overline{\mathbf{\Sigma}}-\overline{\mathbf{\Sigma}}\mathbf{G}^T(\mathbf{G}\overline{\mathbf{\Sigma}}\mathbf{G}^T+\overline{\mathbf{R}})^{-1}\mathbf{G}\overline{\mathbf{\Sigma}}]\overline{\mathbf{F}}^T+\overline{\mathbf{Q}}$, and $\overline{\mathbf{F}}-\overline{\mathbf{F}}\overline{\mathbf{\Sigma}}\mathbf{G}^T(\mathbf{G}\overline{\mathbf{\Sigma}}\mathbf{G}^T+\overline{\mathbf{R}})^{-1}\mathbf{G}$ has eigenvalues strictly within the unit circle. These results follow directly from the application of [58, Proposition 4.1, Sec. 4.1] similar to the stability and convergence results for the standard KF for linear systems [58, Appendix E.4].

In this limiting case, the inverse filter prediction and update equations take the following asymptotic form

$$\begin{split} \hat{\mathbf{x}}_{k+1|k} &= \overline{\mathbf{F}} \hat{\mathbf{x}}_k + \overline{\mathbf{E}} \mathbf{H} \mathbf{x}_{k+1}, \\ \hat{\mathbf{x}}_{k+1} &= \hat{\mathbf{x}}_{k+1|k} + \overline{\mathbf{\Sigma}} \mathbf{G}^T (\mathbf{G} \overline{\mathbf{\Sigma}} \mathbf{G}^T + \overline{\mathbf{R}})^{-1} (\mathbf{a}_{k+1} - \mathbf{G} \hat{\mathbf{x}}_{k+1|k}). \end{split}$$

Denoting the inverse filter's one-step prediction error as $\overline{\mathbf{e}}_{k+1|k} \doteq \hat{\mathbf{x}}_{k+1} - \hat{\hat{\mathbf{x}}}_{k+1|k}$, the error dynamics for the inverse filter is obtained from this asymptotic form using (3) as

$$\begin{split} \overline{\mathbf{e}}_{k+1|k} &= \left(\overline{\mathbf{F}} - \overline{\mathbf{F}} \overline{\boldsymbol{\Sigma}} \mathbf{G}^T (\mathbf{G} \overline{\boldsymbol{\Sigma}} \mathbf{G}^T + \overline{\mathbf{R}})^{-1} \mathbf{G}\right) \overline{\mathbf{e}}_{k|k-1} \\ &- \overline{\mathbf{F}} \overline{\boldsymbol{\Sigma}} \mathbf{G}^T (\mathbf{G} \overline{\boldsymbol{\Sigma}} \mathbf{G}^T + \overline{\mathbf{R}})^{-1} \boldsymbol{\epsilon}_k + \overline{\mathbf{E}} \mathbf{v}_{k+1}. \end{split}$$

Since $\overline{\mathbf{F}} - \overline{\mathbf{F}} \overline{\mathbf{\Sigma}} \mathbf{G}^T (\mathbf{G} \overline{\mathbf{\Sigma}} \mathbf{G}^T + \overline{\mathbf{R}})^{-1} \mathbf{G}$ has eigenvalues strictly within the unit circle, this error dynamics is asymptotically stable.

APPENDIX B PROOF OF THEOREM 2

For simplicity, we consider the case of $n \geq p$ with $\mathbf{U}_{k+1}^{xy} \in \mathbb{R}^{n \times n}$. It is trivial to show that the proof remains valid for n < p as well. Using the expressions for $\mathbf{\Sigma}_{k+1}^{xy}$ and \mathbf{S}_{k+1} , we have

$$\begin{split} \mathbf{K}_{k+1} &= \mathbf{\Sigma}_{k+1|k} \mathbf{U}_{k+1}^{xy} \mathbf{H}_{k+1}^{T} \mathbf{U}_{k+1}^{y} \\ &\qquad \qquad \times \left(\mathbf{U}_{k+1}^{y} \mathbf{H}_{k+1} \mathbf{\Sigma}_{k+1|k} \mathbf{H}_{k+1}^{T} \mathbf{U}_{k+1}^{y} + \hat{\mathbf{R}}_{k+1} \right)^{-1}, \\ \mathbf{\Sigma}_{k+1} &= \mathbf{\Sigma}_{k+1|k} - \mathbf{\Sigma}_{k+1|k} \mathbf{U}_{k+1}^{xy} \mathbf{H}_{k+1}^{T} \mathbf{U}_{k+1}^{y} \\ &\qquad \qquad \times \left(\mathbf{U}_{k+1}^{y} \mathbf{H}_{k+1} \mathbf{\Sigma}_{k+1|k} \mathbf{H}_{k+1}^{T} \mathbf{U}_{k+1}^{y} + \hat{\mathbf{R}}_{k+1} \right)^{-1} \\ &\qquad \qquad \times \mathbf{U}_{k+1}^{y} \mathbf{H}_{k+1} (\mathbf{U}_{k+1}^{xy})^{T} \mathbf{\Sigma}_{k+1|k}. \end{split}$$

Define $V_k(\widetilde{\mathbf{x}}_{k|k-1}) = \widetilde{\mathbf{x}}_{k|k-1}^T \mathbf{\Sigma}_{k|k-1}^{-1} \widetilde{\mathbf{x}}_{k|k-1}$. Using the bounds assumed on $\mathbf{\Sigma}_{k|k-1}$, we have for all $k \geq 0$

$$\frac{1}{\overline{\sigma}}\|\widetilde{\mathbf{x}}_{k|k-1}\|_2^2 \leq V_k(\widetilde{\mathbf{x}}_{k|k-1}) \leq \frac{1}{\underline{\sigma}}\|\widetilde{\mathbf{x}}_{k|k-1}\|_2^2.$$

Hence, the first condition of Lemma 1 is satisfied with $v_{\min}=1/\overline{\sigma}$ and $v_{\max}=1/\underline{\sigma}$.

Using (47) and the independence of noise terms, we have

$$\mathbb{E}\left[V_{k+1}(\widetilde{\mathbf{x}}_{k+1|k})|\widetilde{\mathbf{x}}_{k|k-1}\right]$$

$$=\widetilde{\mathbf{x}}_{k|k-1}^{T}(\mathbf{U}_{k}^{T}\mathbf{F}_{k}(\mathbf{I}-\mathbf{K}_{k}\mathbf{U}_{k}^{y}\mathbf{H}_{k}))^{T}$$

$$\times \mathbf{\Sigma}_{k+1|k}^{-1}(\mathbf{U}_{k}^{T}\mathbf{F}_{k}(\mathbf{I}-\mathbf{K}_{k}\mathbf{U}_{k}^{y}\mathbf{H}_{k}))\widetilde{\mathbf{x}}_{k|k-1}$$

$$+\mathbb{E}\left[\mathbf{v}_{k}^{T}(\mathbf{U}_{k}^{T}\mathbf{F}_{k}\mathbf{K}_{k})^{T}\mathbf{\Sigma}_{k+1|k}^{-1}(\mathbf{U}_{k}^{T}\mathbf{F}_{k}\mathbf{K}_{k})\mathbf{v}_{k}|\widetilde{\mathbf{x}}_{k|k-1}\right]$$

$$+\mathbb{E}\left[\mathbf{w}_{k}^{T}\mathbf{\Sigma}_{k+1|k}^{-1}\mathbf{w}_{k}|\widetilde{\mathbf{x}}_{k|k-1}\right]. \tag{53}$$

The difference of two matrices $\mathbf{A} - \mathbf{B}$ is invertible if maximum singular value of \mathbf{B} is strictly less than the minimum singular value of \mathbf{A} . Using the assumed bounds, we have $\|\mathbf{K}_k\| \leq \overline{k} = (\overline{\sigma\gamma}\overline{h}\overline{\beta})/\hat{r}$. Hence, maximum singular value of $\mathbf{K}_k\mathbf{U}_k^y\mathbf{H}_k$ is upper-bounded by $(\overline{\sigma\gamma}\overline{h}^2\overline{\beta}^2)/\hat{r}$ and the inequality (48) guarantees that $\mathbf{I} - \mathbf{K}_k\mathbf{U}_k^y\mathbf{H}_k$ is invertible (singular value of \mathbf{I} is 1) such that

$$\begin{split} & \boldsymbol{\Sigma}_{k+1|k} \\ & = \mathbf{U}_k^x \mathbf{F}_k (\mathbf{I} - \mathbf{K}_k \mathbf{U}_k^y \mathbf{H}_k) (\boldsymbol{\Sigma}_{k|k-1} + (\mathbf{U}_k^x \mathbf{F}_k (\mathbf{I} - \mathbf{K}_k \mathbf{U}_k^y \mathbf{H}_k))^{-1} \\ & \times \hat{\mathbf{Q}}_k ((\mathbf{U}_k^x \mathbf{F}_k (\mathbf{I} - \mathbf{K}_k \mathbf{U}_k^y \mathbf{H}_k))^{-1})^T) (\mathbf{I} - \mathbf{K}_k \mathbf{U}_k^y \mathbf{H}_k)^T \mathbf{F}_k^T \mathbf{U}_k^x, \end{split}$$

because \mathbf{U}_k^x and \mathbf{F}_k are also assumed to be invertible. Again with the assumed bounds, we have $\|\mathbf{U}_k^x\mathbf{F}_k(\mathbf{I} - \mathbf{K}_k\mathbf{U}_k^y\mathbf{H}_k)\| \leq \overline{\alpha}\overline{f}(1 + \overline{k\beta h})$ which implies

$$\begin{aligned} (\mathbf{U}_k^x \mathbf{F}_k (\mathbf{I} - \mathbf{K}_k \mathbf{U}_k^y \mathbf{H}_k))^{-1} \hat{\mathbf{Q}}_k ((\mathbf{U}_k^x \mathbf{F}_k (\mathbf{I} - \mathbf{K}_k \mathbf{U}_k^y \mathbf{H}_k))^{-1})^T \\ & \succeq \frac{\hat{q}}{(\overline{\alpha} \overline{f} (1 + \overline{k\beta h}))^2} \mathbf{I}. \end{aligned}$$

Using this bound in the expression of $\Sigma_{k+1|k}$ as in [52], we have

$$(\mathbf{U}_k^x \mathbf{F}_k (\mathbf{I} - \mathbf{K}_k \mathbf{U}_k^y \mathbf{H}_k))^T \mathbf{\Sigma}_{k+1|k}^{-1} (\mathbf{U}_k^x \mathbf{F}_k (\mathbf{I} - \mathbf{K}_k \mathbf{U}_k^y \mathbf{H}_k))$$

$$\leq (1 - \lambda) \mathbf{\Sigma}_{k|k-1}^{-1},$$

where $1-\lambda=\left(1+\frac{\hat{q}}{\overline{\sigma}(\overline{\alpha}\overline{f}(1+\overline{k}\beta\overline{h}))^2}\right)^{-1}$ with $0<\lambda<1$. The last two expectation terms in (53) can be bounded by $\mu=(\overline{r}p\overline{\alpha}^2\overline{f}^2\overline{k}^2/\underline{\sigma})+(\overline{q}n/\underline{\sigma})>0$ following similar steps as in [52] such that

$$\mathbb{E}\left[V_{k+1}(\widetilde{\mathbf{x}}_{k+1|k})|\widetilde{\mathbf{x}}_{k|k-1}\right] - V_{k}(\widetilde{\mathbf{x}}_{k|k-1}) \leq -\lambda V_{k}(\widetilde{\mathbf{x}}_{k|k-1}) + \mu.$$

Hence, the second condition of Lemma 1 is also satisfied and the prediction error $\tilde{\mathbf{x}}_{k|k-1}$ is exponentially bounded in mean-squared sense and bounded with probability one.

Furthermore, with the bounds assumed on various matrices, it is straightforward to show that

$$\mathbb{E}\left[\|\widetilde{\mathbf{x}}_k\|_2^2\right] \leq (1 + \overline{k\beta h})^2 \mathbb{E}\left[\|\widetilde{\mathbf{x}}_{k|k-1}\|_2^2\right] + \overline{k}^2 \overline{r} p.$$

Finally, the exponential boundedness of $\widetilde{\mathbf{x}}_{k|k-1}$ leads to $\widetilde{\mathbf{x}}_k$ also being exponentially bounded in mean-squared sense as well as bounded with probability one.

APPENDIX C PROOF OF THEOREM 3

We will show that the I-EKF's dynamics also satisfies the assumptions of Theorem 2. For this, the following conditions C1-C13 need to hold true for all $k \geq 0$ for some real positive constants $\overline{a}, \overline{g}, \overline{b}, \overline{c}, \overline{d}, \hat{q}, \overline{\epsilon}, \hat{c}, \hat{d}, p, \overline{p}$.

- C1 $\|\widetilde{\mathbf{F}}_{k}^{x}\| \leq \overline{a};$
- C2 $\|\overline{\mathbf{U}}_{k}^{x}\| \leq \overline{b};$
- C3 $\overline{\mathbf{U}}_{k}^{x}$ is non-singular;
- **C4** $\widetilde{\mathbf{F}}_{k}^{x}$ is non-singular;
- C5 $\overline{\mathbf{Q}}_k \preceq \widetilde{q}\mathbf{I}$;
- C6 $\|\underline{\mathbf{G}}_{k}\| \leq \overline{g};$
- C7 $\|\underline{\overline{\mathbf{U}}}_{k}^{a}\| \leq \overline{c};$
- C8 $\|\overline{\mathbf{U}}_{k}^{xa}\| \leq \overline{d};$
- **C9** $\overline{\mathbf{R}}_k \leq \underline{\overline{\epsilon}}\mathbf{I};$
- C10 $\hat{c}\mathbf{I} \preceq \overline{\mathbf{Q}}_k$;
- C11 $\hat{d}\mathbf{I} \preceq \overline{\mathbf{R}}_k$;
- C12 $p\mathbf{I} \preceq \overline{\Sigma}_{k|k-1} \preceq \overline{p}\mathbf{I}$; and
- C13 the constants satisfy the inequality $\bar{p}d\bar{q}^2\bar{c}^2 < \hat{d}$.

Next, we prove that under the assumptions of Theorem 3, C1-C13 are satisfied. From the I-EKF's state transition (24), the Jacobians $\widetilde{\mathbf{F}}_k^x = \mathbf{F}_k - \mathbf{K}_{k+1}\mathbf{H}_{k+1}\mathbf{F}_k$ and $\widetilde{\mathbf{F}}_k^v = \mathbf{K}_{k+1}$ such that $\overline{\mathbf{Q}}_k = \mathbf{K}_{k+1}\mathbf{R}_{k+1}\mathbf{K}_{k+1}^T$.

For C1, using $\|\mathbf{K}_{k+1}\| \leq \overline{k}$ (as proved in Theorem 2) and the bounds on \mathbf{F}_k and \mathbf{H}_{k+1} from the assumptions of Theorem 2, it is trivial to show that $\|\widetilde{\mathbf{F}}_k^x\| = \|\mathbf{F}_k - \mathbf{K}_{k+1}\mathbf{H}_{k+1}\mathbf{F}_k\| \leq \overline{f} + \overline{khf}$. Hence, C1 is satisfied with $\overline{a} = \overline{f} + \overline{khf}$.

For C2-C4, consider the unknown matrix $\overline{\mathbf{U}}_k^x$ introduced to account for the residuals in linearization of $\widetilde{f}_k(\cdot)$. Let $\hat{\widetilde{\mathbf{x}}}_{k+1|k}$ and $\hat{\widetilde{\mathbf{x}}}_k$ denote the state prediction error and state estimation error of I-EKF. Similar to forward EKF with the introduction of the unknown matrix, we have

$$\hat{\widetilde{\mathbf{x}}}_{k+1|k} = \overline{\mathbf{U}}_{k}^{x} (\mathbf{F}_{k} - \mathbf{K}_{k+1} \mathbf{H}_{k+1} \mathbf{F}_{k}) \hat{\widetilde{\mathbf{x}}}_{k} + \mathbf{K}_{k+1} \mathbf{v}_{k+1}.$$
 (54)

Also, $\hat{\hat{\mathbf{x}}}_{k+1|k} = f(\hat{\mathbf{x}}_k) - f(\hat{\hat{\mathbf{x}}}_k) - \mathbf{K}_{k+1}(h(f(\hat{\mathbf{x}}_k)) - h(f(\hat{\hat{\mathbf{x}}}_k))) + \mathbf{K}_{k+1}\mathbf{v}_{k+1}$. Using the unknown matrices \mathbf{U}_k^x and \mathbf{U}_k^y introduced in the linearization of $f(\cdot)$ and $h(\cdot)$, respectively, we have

$$\hat{\widetilde{\mathbf{x}}}_{k+1|k} = (\mathbf{U}_k^x \mathbf{F}_k - \mathbf{K}_{k+1} \mathbf{U}_{k+1}^y \mathbf{H}_{k+1} \mathbf{U}_k^x \mathbf{F}_k) \hat{\widetilde{\mathbf{x}}}_k + \mathbf{K}_{k+1} \mathbf{v}_{k+1}.$$

Comparing with (54), we have

$$\overline{\mathbf{U}}_{k}^{x}(\mathbf{I} - \mathbf{K}_{k+1}\mathbf{H}_{k+1})\mathbf{F}_{k} = (\mathbf{I} - \mathbf{K}_{k+1}\mathbf{U}_{k+1}^{y}\mathbf{H}_{k+1})\mathbf{U}_{k}^{x}\mathbf{F}_{k}.$$
 (55)

With the additional assumption of $\underline{r}\mathbf{I} \leq \mathbf{R}_k$ and using matrix inversion lemma as in proof of [26, Lemma 3.1], we have

$$(\mathbf{I} - \mathbf{K}_{k+1} \mathbf{H}_{k+1}) \mathbf{\Sigma}_{k+1|k} = \left(\mathbf{\Sigma}_{k+1|k}^{-1} + \mathbf{H}_{k+1}^T \mathbf{R}_{k+1}^{-1} \mathbf{H}_{k+1} \right)^{-1}.$$

Since $\Sigma_{k+1|k}$ is invertible by the assumptions of Theorem 2, $\mathbf{I} - \mathbf{K}_{k+1}\mathbf{H}_{k+1}$ is invertible for all $k \geq 0$ and

$$(\mathbf{I} - \mathbf{K}_{k+1} \mathbf{H}_{k+1})^{-1} = \mathbf{I} + \mathbf{\Sigma}_{k+1|k} \mathbf{H}_{k+1}^T \mathbf{R}_{k+1}^{-1} \mathbf{H}_{k+1}.$$

With the bounds assumed on various matrices, we have $\|(\mathbf{I} - \mathbf{K}_{k+1}\mathbf{H}_{k+1})^{-1}\| \le 1 + \frac{\overline{\sigma_h}^2}{\underline{\Gamma}}$. Furthermore, using this bound and the invertibility of $\mathbf{I} - \mathbf{K}_{k+1}\mathbf{H}_{k+1}$ in (55), it is straightforward to show that $\overline{\mathbf{U}}_k^x = (\mathbf{I} - \mathbf{K}_{k+1}\mathbf{U}_{k+1}^y\mathbf{H}_{k+1})\mathbf{U}_k^x(\mathbf{I} - \mathbf{K}_{k+1}\mathbf{H}_{k+1})^{-1}$ is non-singular (both \mathbf{U}_k^x and $\mathbf{I} - \mathbf{K}_{k+1}\mathbf{U}_{k+1}^y\mathbf{H}_{k+1}$ are invertible under the assumptions of Theorem 2) and satisfies $\|\overline{\mathbf{U}}_k^x\| \le \overline{\alpha}(1 + \mathbf{K}_{k+1}\mathbf{U}_{k+1}^y\mathbf{H}_{k+1})$

 $\overline{k\beta h})(1+(\overline{\sigma}\overline{h}^2)/\underline{r})$. Also, since both $\mathbf{I}-\mathbf{K}_{k+1}\mathbf{H}_{k+1}$ and \mathbf{F}_k are invertible, $\widetilde{\mathbf{F}}_{k}^{x} = \mathbf{F}_{k}(\mathbf{I} - \mathbf{K}_{k+1}\mathbf{H}_{k+1})$ is non-singular. Hence, **C2-C4** are also satisfied with $\overline{b} = \overline{\alpha}(1 + \overline{k\beta h})(1 + (\overline{\sigma}\overline{h}^2)/r)$.

For C5, using the upper bound on \mathbb{R}_k from assumptions of Theorem 2, we have $\overline{\mathbf{Q}}_k \leq \overline{r} \mathbf{K}_{k+1} \mathbf{K}_{k+1}^T$. Since, $\|\mathbf{K}_{k+1}\| \leq \overline{k}$, the maximum eigenvalue of $\mathbf{K}_{k+1} \mathbf{K}_{k+1}^T$ is bounded by \overline{k}^2 such that $\overline{\mathbf{Q}}_k \preceq \overline{k}^2 \overline{r} \mathbf{I}$. Hence, C5 is satisfied with $= \overline{k}^2 \overline{r}$.

The conditions C6-C13 are assumed to hold true in Theorem 3. Hence, all the conditions hold true for the I-EKF's error dynamics and Theorem 2 is applicable for the I-EKF as well, i.e., the estimation error is exponentially bounded in mean-squared sense and bounded with probability one.

APPENDIX D PROOF OF THEOREM 5

We will show that the error dynamics of the I-EKF given by (50) satisfies the following conditions for all $k \geq 0$ for some real positive constants $\underline{c}, \kappa_{\bar{\phi}}, \epsilon_{\bar{\phi}}$.

C1 $\underline{c}\mathbf{I} \preceq \overline{\mathbf{Q}}_k$.

C2
$$\hat{\mathbf{F}}_{k}^{x}$$
 is non-singular matrix for all $k \geq 0$.
C3 $\|\bar{\phi}_{k}(\hat{\mathbf{x}}, \hat{\hat{\mathbf{x}}})\|_{2} \leq \kappa_{\bar{\phi}} \|\hat{\mathbf{x}} - \hat{\hat{\mathbf{x}}}\|_{2}^{2}$ for all $\|\hat{\mathbf{x}} - \hat{\hat{\mathbf{x}}}\|_{2} \leq \epsilon_{\bar{\phi}}$ for some $\kappa_{\bar{\phi}} > 0$ and $\epsilon_{\bar{\phi}} > 0$.

All other conditions of Theorem 4 can be proved to hold true for the I-EKF's error dynamics under the assumptions of Theorem 5 following similar approach as in proof of Theorem 3, such that the estimation error given by (50) is exponentially bounded in meansquared sense and bounded with probability one provided that the estimation error is bounded with $\bar{\epsilon} > 0$ where $\bar{\epsilon}$ depends on the various bounds in the same manner as ϵ depends in the forward filter case.

For C1, using the bound on \mathbf{R}_k from one of the assumptions of Theorem 4, we have $\overline{\mathbf{Q}}_k = \mathbf{K}_k \mathbf{R}_k \mathbf{K}_k^T \succeq \underline{r} \mathbf{K}_k \mathbf{K}_k^T$. Substituting for \mathbf{K}_k , we have

$$\mathbf{K}_{k}\mathbf{K}_{k}^{T} = \mathbf{F}_{k}\mathbf{\Sigma}_{k}\mathbf{H}_{k}^{T}(\mathbf{H}_{k}\mathbf{\Sigma}_{k}\mathbf{H}_{k}^{T} + \mathbf{R}_{k})^{-2}\mathbf{H}_{k}\mathbf{\Sigma}_{k}\mathbf{F}_{k}^{T}.$$

With the assumption that \mathbf{H}_k is full column rank, $\mathbf{K}_k \mathbf{K}_k^T$ is p.d. as \mathbf{F}_k is assumed to be non-singular in Theorem 4. Hence, there exists a constant $\tilde{q} > 0$ which is the minimum eigenvalue of $\mathbf{K}_k \mathbf{K}_k^T$ such that $\mathbf{K}_k \mathbf{K}_k^T \succeq \widetilde{q} \mathbf{I}$ and $\overline{\mathbf{Q}}_k \succeq \underline{r} \widetilde{q} \mathbf{I}$. Hence, C1 is satisfied with $\underline{c} = \underline{r} \widetilde{q}$. For C2, $\widetilde{\mathbf{F}}_{k}^{x} = \mathbf{F}_{k} - \mathbf{K}_{k} \mathbf{H}_{k}$ is proved to be invertible for all $k \geq 0$ as an intermediate result in the proof of Theorem 4 in [26, Lemma

For C3, using $\|\mathbf{K}_k\| \leq (\overline{f}\overline{\sigma}\overline{h}/\underline{r})$ (proved in [26, Lemma 3.1]) and the bounds on functions $\phi(\cdot)$ and $\chi(\cdot)$ from the assumptions of Theorem 4, we have $\|\overline{\phi}_k(\hat{\mathbf{x}},\hat{\hat{\mathbf{x}}})\|_2 \leq \|\phi(\hat{\mathbf{x}},\hat{\hat{\mathbf{x}}})\|_2 + \frac{\overline{f}\overline{\sigma}\overline{h}}{\underline{r}}\|\chi(\hat{\mathbf{x}},\hat{\hat{\mathbf{x}}})\|_2 \leq$ $\left(\kappa_{\phi} + \frac{\overline{f}\overline{\sigma}\overline{h}}{r}\kappa_{\chi}\right)\|\hat{\mathbf{x}} - \hat{\bar{\mathbf{x}}}\|_{2}^{2}, \text{ for } \|\hat{\mathbf{x}} - \hat{\bar{\mathbf{x}}}\|_{2} \leq \min(\epsilon_{\phi}, \epsilon_{\chi}). \text{ Hence, C3}$ is satisfied with $\kappa_{\bar{\phi}} = \kappa_{\phi} + (\overline{f}\overline{\sigma}\overline{h}/\underline{r})\kappa_{\chi}$ and $\epsilon_{\bar{\phi}} = \min(\epsilon_{\phi}, \epsilon_{\chi}).$

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