Candidate Code- jtk011

Decoding Factorials: Surprising Paths to Approximate Uncommon Inputs

Introduction

During my math's class we were doing the topic of induction where we came across a question using the concept of factorial. Learning and understanding the concept of factorial, my teacher said that the factorial of 0 is 1. I had asked him why the factorial of 0 is 1, as it had not made any sense to me, but he had asked me to explore the question for myself and find out why. I later tried to solve more and more questions which involved the factorial. The symbol itself is very fascinating. Who would ever thing that an exclamation mark would be used in mathematics. Solving proving by induction questions with the factorial function had defined my interest and I had decided that I wanted to do my Internal Assessment on something relevant to the factorial function. While I was researching for the topics which I could explore for my math's exploration, on google there was a "People also search for" section where I found a question, "Why does the factorial not work for fractional powers and negative integers?" This questions curiosity began to grow larger and larger, fuelling my research deeper and deeper into the cave that is the factorial function. Now we arrive at today, where I, armed with functions to extend the factorial, am ready to take on all the challenges that the factorial function has in store for me.

Aim

I want to have calculated fractional and negative factorials by the end of this internal assessment and utilize the data to develop straightforward models of the factorial function. The aim of this investigation is to investigate the application of these results in approximating other fractional or negative factorials. This study will delve into mathematical techniques, algorithms, and practical approaches to compute factorials for non-integer values, with a focus on understanding the underlying principles and implications for broader factorial approximations. My research question for this exploration is: **How can you calculate fractional and negative factorials, and how can you utilize these results to approximatively calculate other fractional or negative factorials?**

Background Research

The Gamma Function:

$$(z-1)! = \Gamma(Z) = \int_0^\infty t^{Z-1} e^{-t} dt$$ [1]

The gamma function, as you've seen above, is essentially an extended version of the factorial function. It takes on various forms, each with its own unique purpose and characteristics. When I use these different versions of the gamma function, I'll define and explain why I'm using a particular form and what sets it apart from the traditional definition of the gamma function. In the function, the variable "z" represents the factorial of "z - 1". This notation of the gamma function can be attributed to Legendre, who made modifications to Euler's original integral method for expanding the factorial function. In case of the Gamma function, "z" takes on values from the realm of complex numbers.

The gamma function is still called a "function" because it associates each input with a specific output, following the basic concept of a mathematical function. Even though it fails the horizontal line test, which is a test for one-to-one functions, the gamma function is defined for its domain and provides a result for each valid input. A function is essentially a rule that assigns exactly one output to each input (one to one), and the gamma function adheres to this definition within its specific context, even if its behaviour is more intricate due to its complex nature.

[1] "Gamma Function." From Wolfram MathWorld, mathworld.wolfram.com/GammaFunction.html. Accessed 17 Dec. 2023.

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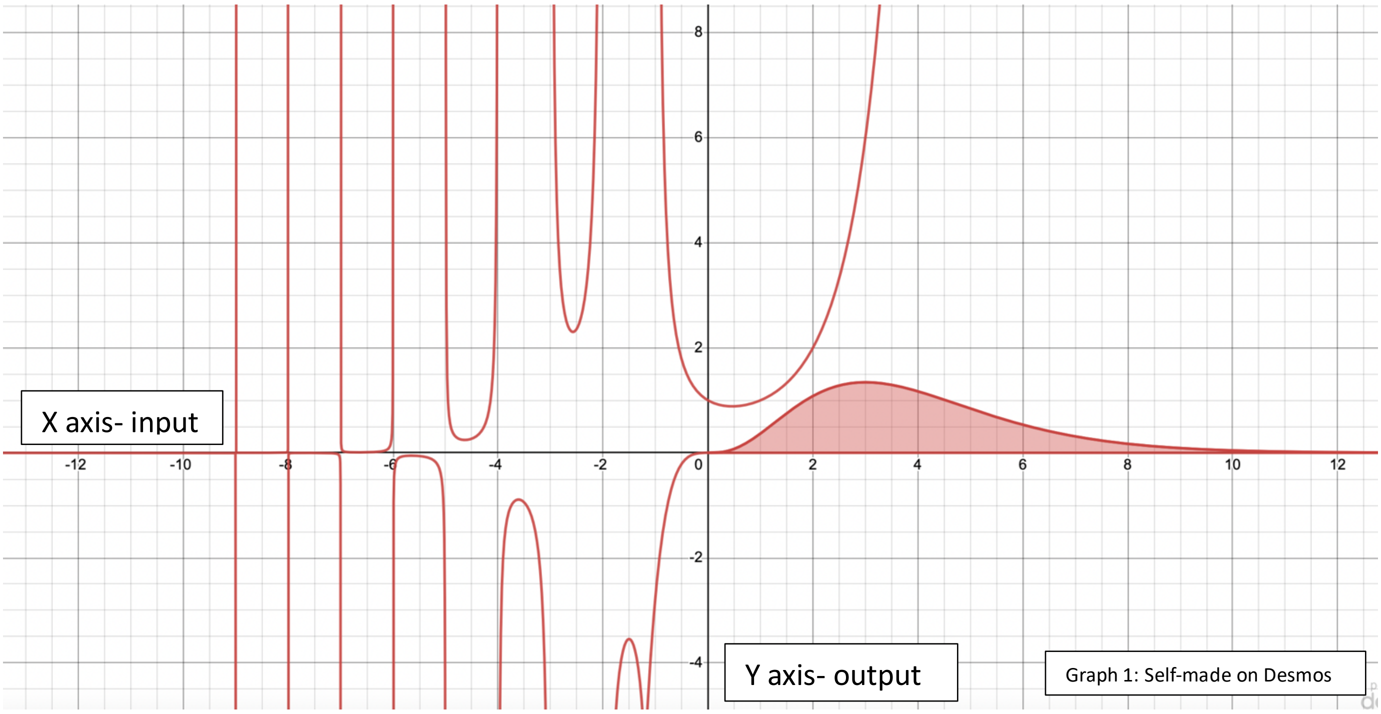

Graph 1: A graph which represents the Gamma function on Desmos.

When we plot the Gamma function, we observe that in the negative range of the function, there are vertical asymptotes at every integer value. This means that this form of the gamma function isn't suitable for calculating the factorials of negative integers. To address this limitation and calculate factorials for negative integers, I'll be using a function introduced by Steven Roman.

$$n! = \frac{(-1)^{(-n-1)}}{(-n-1)!} \; for \; n < 0$$

With this formula, I'll be able to calculate factorials for negative integers in my upcoming investigation. However, there's a challenge with these functions since they tend to be quite complex and not very user-friendly. To overcome this complexity, I aim to employ these functions to create approximations for factorials that are more straightforward and easier to work with. I'll be using methods like quadratic regression, exponential regression, and tangent lines to approximate values using simpler functions.

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Improper integrals broaden the concept of definite integrals to handle functions with infinite domains or unbounded intervals. They involve integrating over intervals extending to infinity or incorporating discontinuities. To evaluate them, limits are applied, ensuring convergence, and addressing the challenges posed by infinite or unbounded regions in calculus. To solve the gamma function, I will be required to calculate a few equations suing improper integrals.

An example of solving an improper integral is:

$$\int_0^{\infty} e^{-3t} dt$$

$$= \lim_{b \to \infty} \left[-\frac{1}{3} e^{-3t} \right]_0^b$$

$$= \lim_{b \to \infty} \left[-\frac{1}{3} e^{-3b} + \frac{1}{3} e^{-3(0)} \right] as\ b\ will\ tend\ to\ \infty, -\frac{1}{3} e^{-3t}\ will\ tend\ to\ 0$$

$$= -\frac{1}{3}(0) + \frac{1}{3}(1)$$

$$= \frac{1}{3}$$

Gaussian Integral- Special integrals: - $\int_{-\infty}^{\infty} e^{-x^2}\ dx = \sqrt{\pi}$

The Gaussian integral works with the gamma function in mathematical contexts. Using the Gaussian integral's properties in connection with the gamma function's integral representation, simplifies the handling of complex integrals involving non-integer parameters. This will help us in simplifying out equations.

Throughout this investigation, my hypothesis is that I'll discover a consistent pattern of growth or decay for factorials with unconventional inputs, such as rational and negative numbers. I expect to find a pattern that can be modelled and used to approximate factorial values with a high degree of accuracy, typically to three significant digits.

I initially thought that it's very important to understand and prove that the Gamma function could be used. Before we prove for the gamma function it's imperative that we prove $\Gamma(n) = (n-1)!$

Let us assume n=k

$$\Gamma(\mathrm{k}) = \int_0^\infty t^{\mathrm{k}-1} e^{-t} dt$$

First, we need the derivative of e^{-t}

$$de^{-t} = -e^{-t} dt$$

$$-de^{-t} = e^{-t} dt$$

$$\int_0^\infty t^{\mathrm{k}-1} (-de^{-t}) = -\int_0^\infty t^{\mathrm{k}-1} de^{-t}$$

Integrating by parts:

$$-t^{k-1} \cdot e^{-t} \Big|_0^\infty + \int_0^\infty e^{-t} \, (t^{k-1})$$

$$\lim_{t\to\infty} t^{k-1} e^{-t} = \lim_{t\to\infty} \frac{t^{k-1}}{e^t} = \lim_{t\to\infty} \frac{(k-1)t^{k-2}}{e^t} = \lim_{t\to\infty} \frac{(k-1)(k-2)t^{k-3}}{e^t}$$

$$= \lim_{t\to\infty} \frac{(k-1)(k-2)\cdots 2\cdot 1}{e^t} = \lim_{t\to\infty} \frac{(k-1)!}{e^t} = 0$$

$$\Longrightarrow -t^{k-1} \cdot e^{-t} \Big|_0^\infty = 0$$

$$= \int_0^\infty e^{-t} \, (t^{k-1}) = \int_0^\infty e^{-t} \, (k-1)(t^{k-2}) \cdot dt$$

$$= (k-1) \int_0^\infty e^{-t} \, (t^{k-2}) \cdot dt$$

Since $\Gamma(\mathrm{k}) = \int_0^\infty t^{\mathrm{k}-1} e^{-t} dt$

$$\Gamma(\mathrm{k}-1) = \int_0^\infty t^{\mathrm{k}-2} e^{-t} dt$$

$$\Rightarrow (k-1)\Gamma(\mathrm{k}-1)$$

$$\Gamma(\mathrm{k}) = (k-1)\Gamma(\mathrm{k}-1)$$

$$(k-1) = \frac{\Gamma(\mathrm{k})}{\Gamma(\mathrm{k}-1)}$$

Imputing a few values to establish a pattern.

When k=2

$$\frac{\Gamma(2)}{\Gamma(1)} = 1$$

When k=3

$$\frac{\Gamma(3)}{\Gamma(2)} = 2$$

When k=n-1

$$\frac{\Gamma(\mathrm{n}-1)}{\Gamma(\mathrm{n}-1)} = n-2$$

When k=n

$$\frac{\Gamma(\mathrm{n})}{\Gamma(\mathrm{n}-1)} = n-1$$

With the following chain could be established

$$\frac{\Gamma(2)}{\Gamma(1)} \cdot \frac{\Gamma(3)}{\Gamma(2)} \cdot \frac{\Gamma(4)}{\Gamma(3)} \cdots \frac{\Gamma(\mathrm{n}-1)}{\Gamma(\mathrm{n}-2)} \cdot \frac{\Gamma(\mathrm{n})}{\Gamma(3\mathrm{n}-1)} = 1 \cdot 2 \cdot 3 \cdot 4 \cdots (n-2)(n-1)$$

After cancelling everything we get

$$\frac{\Gamma(\mathrm{n})}{\Gamma(1)} = 1 \cdot 2 \cdot 3 \cdot 4 \cdots (n-2)(n-1)$$

When we input $\Gamma(1)$ in the Gamma function, we get:

$$\Gamma(1) = \int_0^\infty e^{-t} \cdot dt \Rightarrow 1$$

Hence, we have proved $\Gamma(n) = (n-1)!$

Proof the gamma function by induction,

$$a)\ Proving\ for\ n = 1$$

$$\Gamma(1) = \int_0^\infty t^{1-1} e^{-t} dt$$

$$\Gamma(\mathrm{k}) = \int_0^\infty e^{-t} dt$$

$$\Rightarrow -e^{-t} \Big|_0^\infty$$

$$\Rightarrow (1-1)! = 1$$

$$b)\ Asuming\ this\ function\ is\ true\ for\ n = k$$

$$\Gamma(\mathrm{k}) = \int_0^\infty t^{\mathrm{k}-1}e^{-t}dt \equiv (k-1)!$$

$$c)\ Proving\ this\ function\ is\ true\ for\ n = k+1$$

$$To\ prove\colon \int_0^\infty t^{(\mathrm{k}+1)-1}e^{-t}dt$$

$$= \int_0^\infty t^{\mathrm{k}}e^{-t}dt$$

Using integration by parts to prove this integral.

$$u = t^k$$

$$du = kt^{k-1}dx$$

$$dv = e^{-t}dt$$

$$v = -e^{-t}$$

$$\therefore -t^ke^{-t} - \int_0^\infty -e^{-t}kt^{k-1}\,dt$$

$$-t^ke^{-t} = 0\ (proved\ in\ step\ a)$$

$$\Rightarrow 0 + \int_0^\infty e^{-t}kt^{k-1}\,dt$$

$$\Rightarrow k\int_0^\infty e^{-t}t^{k-1}\,dt$$

$$using\ the\ assumption\ made\ in\ step\ b\ \int_0^\infty e^{-t}t^{k-1}\,dt = (k-1)!$$

$$\therefore k(k-1)! = k!$$

$$\Rightarrow \int_0^\infty e^{-t}t^{(k+1)-1}\,dt = k!$$

$$\Rightarrow \int_0^\infty e^{-t}t^{(k+1)-1}\,dt = ((k+1)-1)!$$

$$\Rightarrow \int_0^\infty e^{-t}t^{(k+1)-1}\,dt = (n-1)!$$

Hence by Principal of Mathematical induction, the function holds true $\forall\ n \in \mathbb{R}$

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Non integer positive numbers

To calculate non-integer factorials, my first step is to select specific non-integer values that I'll be evaluating using the Gamma Function. For this investigation, I've chosen the variables "$\frac{1}{2}$," "$\frac{1}{5}$," "$\frac{1}{7}$," "$\frac{1}{9}$," and "$\frac{1}{11}$." I opted for these values because they allow me to observe patterns involving ascending denominators up to "$\frac{1}{11}$" I'll demonstrate the manual calculations for the factorial. Subsequently, I'll provide a table of values for the other factorials, computed using Wolfram Alpha with precision up to one significant figure. Example of solving the Factorial of $\frac{1}{2}$

$$\Gamma\left(1+\frac{1}{2}\right) = \int_0^\infty t^{1+\frac{1}{2}-1}\, e^{-1} dt$$

applying the gamma function

$$\Gamma\left(\frac{3}{2}\right) = \int_0^\infty t^{\frac{1}{2}} e^{-t} dt$$ -Simplification

$$Let\ u = t^{\frac{1}{2}}\ therefore\ t = u^2\ \ and\ dt = 2u\ du$$ -assigning variables

$$\Gamma\left(\frac{3}{2}\right) = \int_0^\infty u e^{-u^2} 2u du$$

$$\Gamma\left(\frac{3}{2}\right) = 2\int_0^\infty u \times u e^{-u^2} du$$

Let r=u and $dv = ue^{-u^2}du$ therefore dr = du and $v = -\frac{1}{2}e^{-u^2}$

$$\Gamma\left(\frac{3}{2}\right) = 2(\left(u \times -\frac{1}{2}e^{-u^2}\right)]\,{}_0^\infty - \int_0^\infty -\frac{1}{2}e^{-u^2}\, du$$

$$\Gamma\left(\frac{3}{2}\right) = -ue^{-u^2}]\,{}_0^\infty - \int_0^\infty e^{-u^2}\, du$$

$\lim_{u\to\infty} -ue^{-u^2} = \infty\ \times\ \infty$ which is the intermediate form

$$\lim_{u\to\infty} \frac{-1}{2ue^{u^2}} = 0$$

$$\Gamma\left(\frac{3}{2}\right) = (0-0) + \int_0^\infty e^{-u^2}\, du\, \Gamma\left(\frac{3}{2}\right) = \int_0^\infty e^{-u^2}\, du$$

Because e^{-x^2} is an even function it is symmetric across the y-axis, therefore the vaule of $\Gamma\left(\frac{3}{2}\right)$ should be equal to $\frac{1}{2}$ of the Gaussian Integral.

$$\Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}$$

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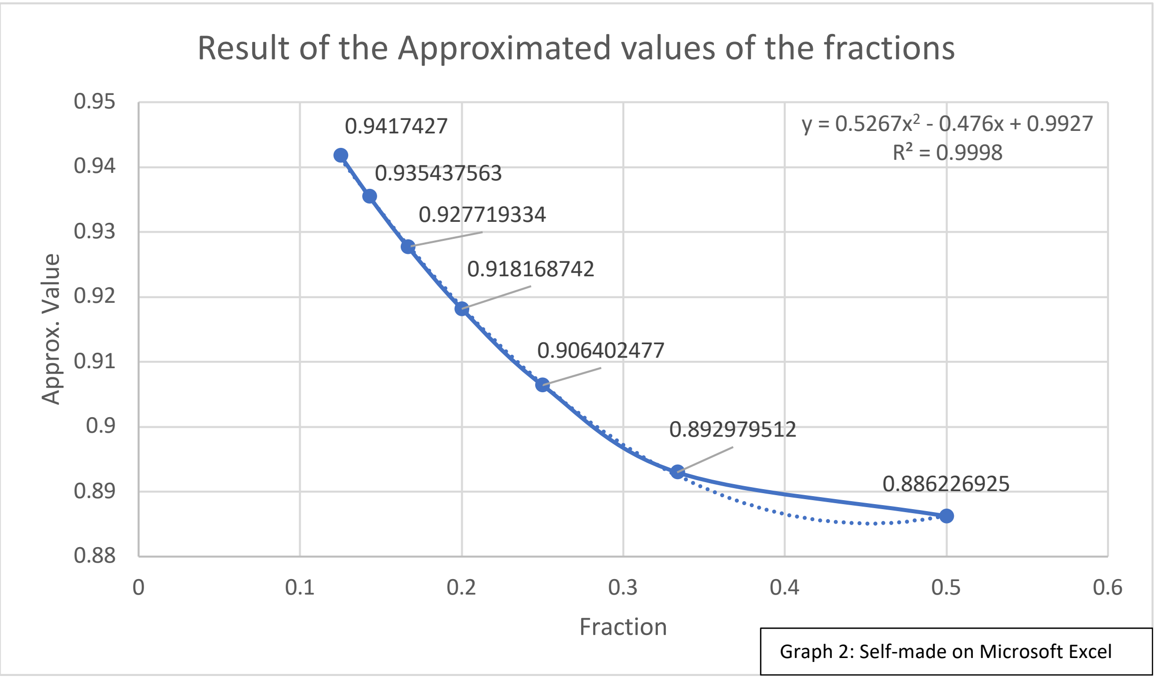
Input Value	Approximated Factorial Value
$\frac{1}{2}$	≈ 0.886226925452
$\frac{1}{3}$	≈ 0.892979511569
$\frac{1}{4}$	≈ 0.906402477055
$\frac{1}{5}$	≈ 0.918168742399
$\frac{1}{6}$	≈ 0.927719333630
$\frac{1}{7}$	≈ 0.935437562892
$\frac{1}{8}$	≈ 0.941742699849

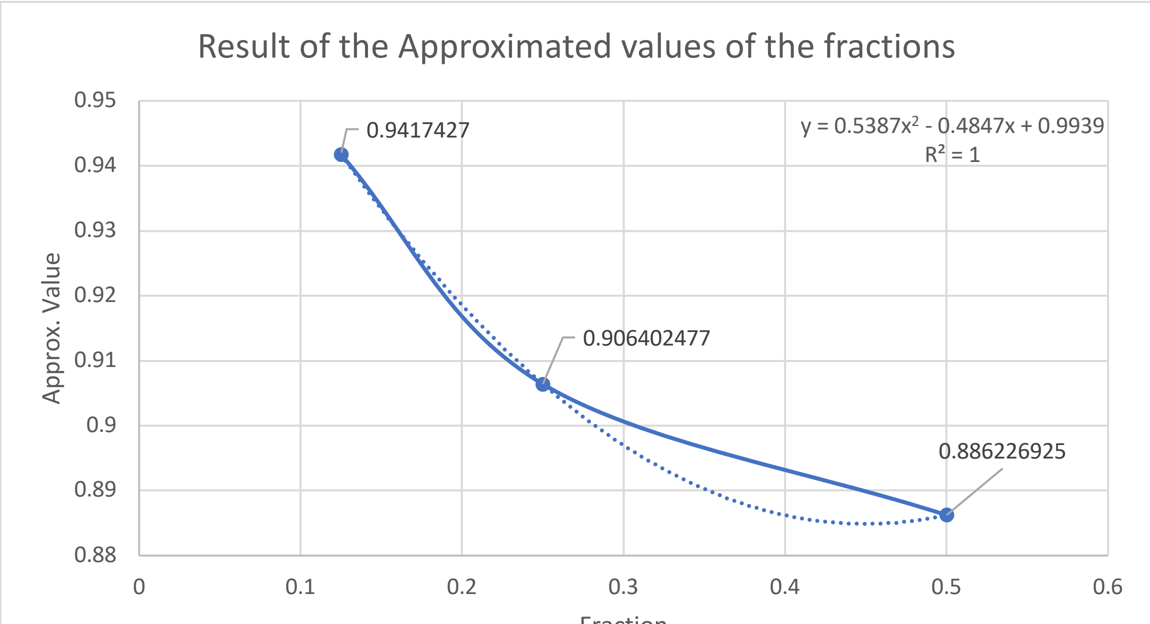
Table 1: A table with all the results of the fractional values which were tested.



Graph 2: Self-made on Microsoft Excel

Graph 2: A graph which represents the results of the fractional values.

After plotting the data points and employing various regression equations, I observed a strong quadratic relationship within the dataset of $y = 0.5267x^2 - 0.476x + 0.9927$, evidenced by a R-squared value of 0.9998. On the X axis I have the fractions mapped out while on the y axis I have the approximate value mapped out. Using the quadratic relationship obtained, I attempted to extrapolate to calculate the factorial value of $\frac{1}{9}$ and $\frac{1}{10}$. The calculated value of the factorial value of the fraction $\frac{1}{9}$ comes to about 0.9463135802 however the actual value is 0.9469653488, which is accurate to 3 significant figures. The calculated value of the factorial value of the fraction $\frac{1}{10}$ comes to about 0.9518675442 however the actual value is 0. 0.9513507698, which again is accurate to only 3 significant figures. While graphing the values of $\frac{1}{2^n}$, I found out that the R-squared value was coming out to be 1, so I graphed the points of $\frac{1}{2}$, $\frac{1}{4}$ and $\frac{1}{8}$ so that I can explore and extrapolate more points in the graph of $\frac{1}{2^n}$, with the quadratic expression $y = 0.5387x^2 - 0.4847x + 0.9939$. As shown in the graph below.



Graph 3: Self-made on Microsoft Excel

Graph 3: A graph which represents the trend of $\frac{1}{2^n}$.

I tried to find the value of $\frac{1}{16}$ and $\frac{1}{32}$ using the same quadratic formula. With the quadratic expression which I got for $\frac{1}{16}!$ was 0.9675105469 but the actual value was 0.9675800675. This value is accurate to only 3 significant figures again. Similarly with $\frac{1}{32}!$, the value was accurate to only 3 significant figures.

Non positive numbers

Now I will be investigating the factorial values of negative integers, specifically focusing on -1, -2, -3, -4, -5, -6, -7 and -8. I've opted for Roman's formula to carry out these calculations. This formula simplifies so many calculations and instead of calculating the values computationally, I will be using this formula.

Value of (-1)! using the Roman formula

$$(-1)! = \frac{(-1)^{(-(-1)-1)}}{(-(-1)-1)!} = \frac{(-1)^0}{0!} = 1$$

Value of (-2)! using the Roman formula

$$(-1)! = \frac{(-1)^{(-(-2)-1)}}{(-(-2)-1)!} = \frac{(-1)^1}{1!} = -1$$

Value of (-3)! using the Roman formula

$$(-1)! = \frac{(-1)^{(-(-3)-1)}}{(-(-3)-1)!} = \frac{(-1)^2}{2!} = \frac{1}{2}$$

Value of (-4)! using the Roman formula

$$(-1)! = \frac{(-1)^{(-(-4)-1)}}{(-(-4)-1)!} = \frac{(-1)^3}{3!} = -\frac{1}{3}$$

Value of (-5)! using the Roman formula

$$(-1)! = \frac{(-1)^{(-(-5)-1)}}{(-(-5)-1)!} = \frac{(-1)^4}{4!} = \frac{1}{24}$$

Value of (-6)! using the Roman formula

$$(-1)! = \frac{(-1)^{(-(-6)-1)}}{(-(-6)-1)!} = \frac{(-1)^5}{5!} = -\frac{1}{120}$$

Value of (-7)! using the Roman formula

$$(-1)! = \frac{(-1)^{(-(-7)-1)}}{(-(-7)-1)!} = \frac{(-1)^6}{6!} = \frac{1}{720}$$

Value of (-8)! using the Roman formula

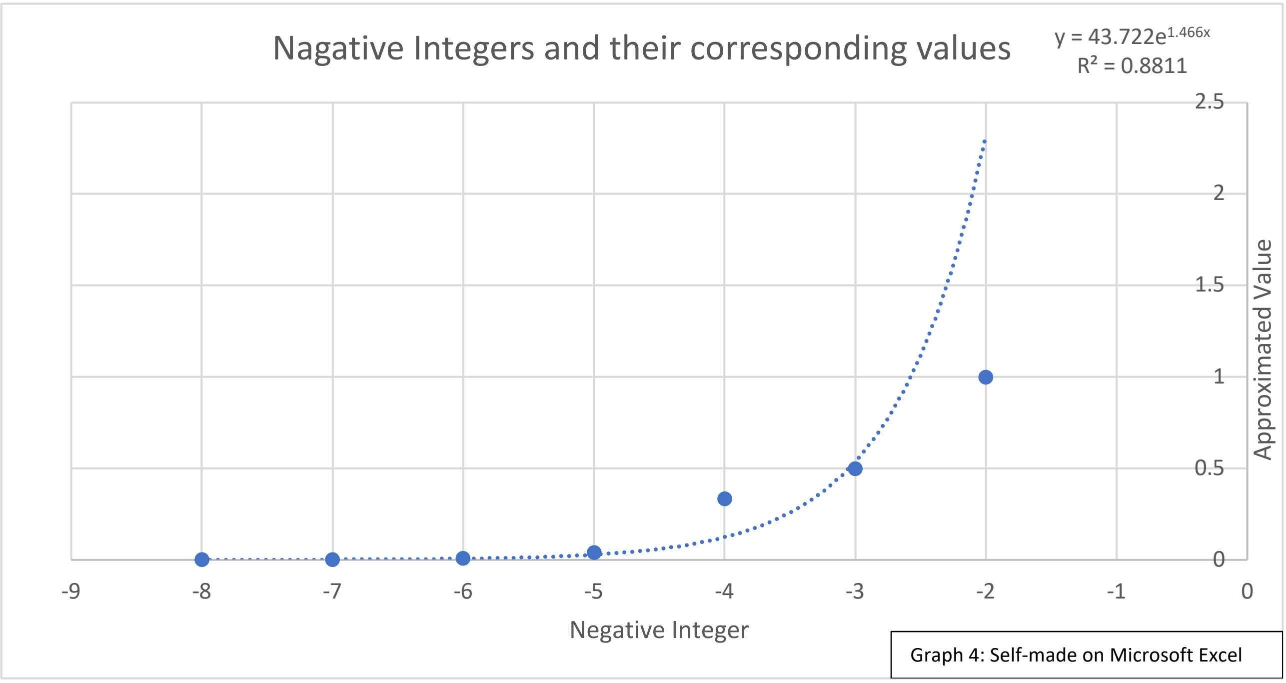
$$(-1)! = \frac{(-1)^{(-(-1)-1)}}{(-(-1)-1)!} = \frac{(-1)^7}{7!} = -\frac{1}{5040}$$

Input Value	Factorial Value
-1	1
-2	-1
-3	$\frac{1}{2}$
-4	$-\frac{1}{3}$
-5	$\frac{1}{24}$
-6	$-\frac{1}{120}$
-7	$\frac{1}{720}$
-8	$-\frac{1}{5040}$

Table 2: A table with all the results of the negative values which were tested.

Upon collecting these factorial values, an intriguing pattern emerged: there's an alternating nature to the results. Notably, even for negative integer inputs, the factorial values turn out negative, while odd negative integer inputs yield positive values. This characteristic stems from the $(-1)^{(-n-1)}$ component in Roman's formula. When attempting to graph these values, I faced challenges implementing a regression model initially, owing to the alternating nature of the results. To address this, I opted to take the absolute value of all factorial values. Additionally, I excluded the -1 input from the relationship due to the repeated factorial values of 1 for both -1 and -2, which could significantly impact any correlation equation.

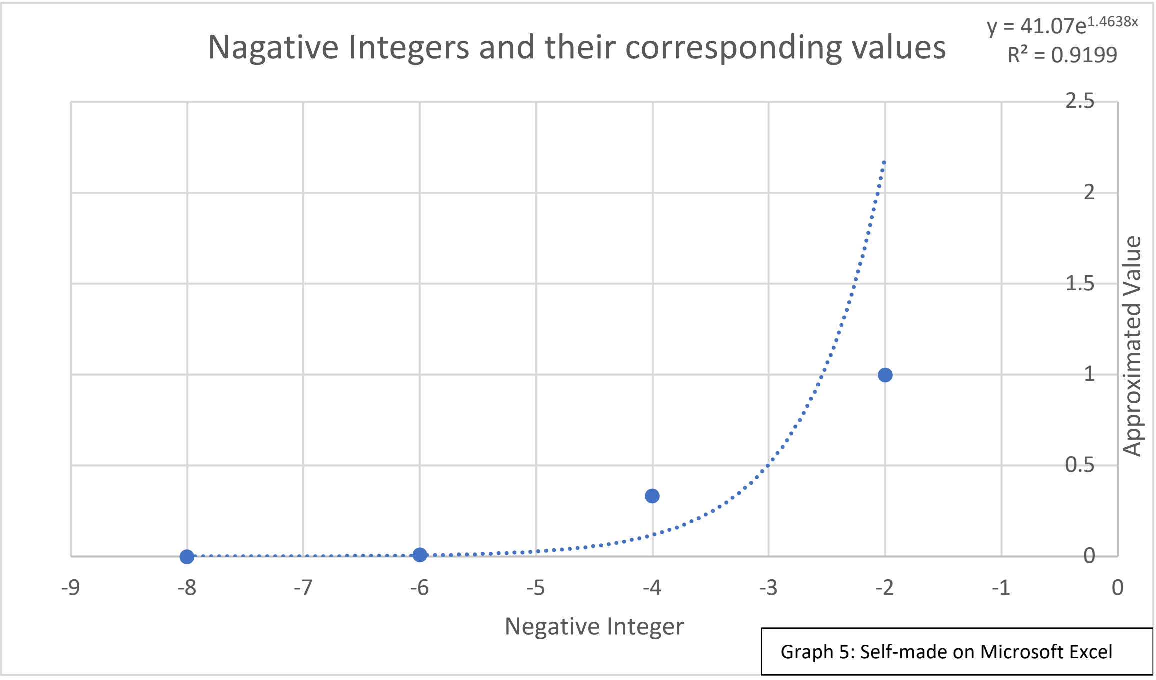
Subsequently, I plotted the values and applied an exponential regression rather than a quadratic one. The reason behind this choice lies in the expectation that the negative integer factorial values should approach zero as the input decreases. A quadratic regression, which suggests an eventual increase in factorial values with a decrease in input, doesn't align with the nature of Roman's formula and was deemed inappropriate for this dataset.



Graph 4: A graph which represents the results of the negative integers.

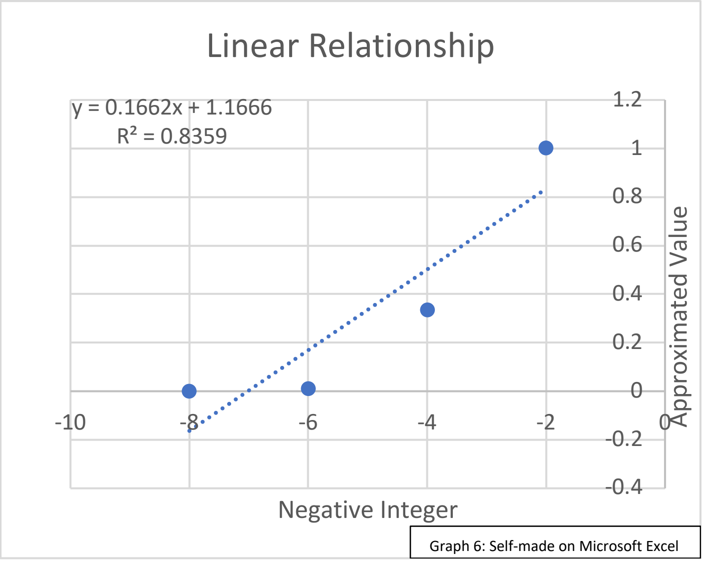
The depicted graph reveals a notably weaker correlation in the relationship compared to the positive non-integer factorials, contrary to my initial expectations. Although employing a quadratic regression yields a higher R-squared value, it is an inappropriate fit for this dataset. To address this, I opted to extrapolate the factorial value for -9, deeming it a more suitable approximation given its alignment with the original dataset. Utilizing the regression equation, I computed -9! to be 0.0000258333, whereas the actual value stands at 1/40320 or 0.00002480158. Unexpectedly, the regression equation exhibited accuracy only to one significant digit. Despite selecting a value within the dataset range, the lower R-squared value justifies this outcome.

In contrast to non-integer factorials, this regression equation tended to overestimate, potentially influenced by the choice of regression types. Subsequently, when graphing exclusively for values -2, -4, -6 and -8, an exponential regression equation proved to offer a markedly improved correlation.

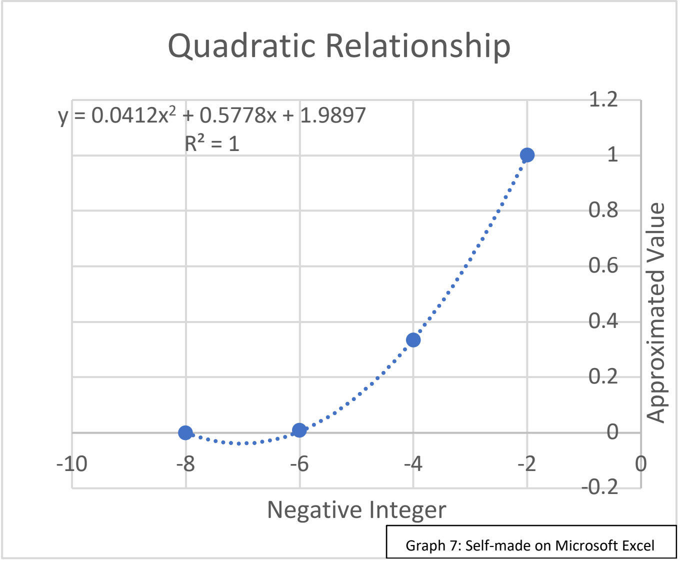


Graph 5: A graph which represents the results of only even negative values.

Subsequently, I plotted the values and applied an multiple regression to see which regression would be the best for the model. I will be comparing the R^2 values of the various regression models and then jump to a conclusion on which regression model would best suit my exploration. The R^2 or coefficient of determination value serves as an indicator of the accuracy of fit for a regression model. Ranging from 0 to 1, where 1 signifies a perfect fit, R^2 gauges the proportion of variability in the dependent variable explained by the independent variable. A higher R^2 value implies a more effective model.



Graph 6: A graph which shows the linear relationship of the points.



Graph 7: A graph which shows the quadratic relationship of the points.

With an R-squared value of 1, the regression equation exhibits a notably superior fit to the data when compared to the calculation based on the complete dataset mentioned earlier. The problem with the quadratic regression is that it has already surpassed its vertex point so that the curve would be heading toward positive infinite in the Y axis, while the value is expected to get smaller and smaller. The exponential regression's outcome was predicted, given the reduced number of data points, facilitating the development of a regression equation with improved accuracy. Using this regression equation to determine the factorial of -16, I obtained a value of $2.76694 \text{ x} 10^{-9}$ after adjusting for the even input. In contrast, the actual value for -16! is $-7.647163732 \text{ x } 10^{-13}$. The substantial disparity in values, differing by orders of magnitude, was unexpected. However, it can be rationalized by recognizing that, with non-integer factorials, extrapolating from -2 to -8 represents a much smaller interval than extrapolating from -8 to -16.

I was still not satisfied with an R^2 value of 0.9199 so I decided to linearize the graph to get a value using semi-logs. Linearizing data puts the data in a simpler form. It helps make sense of relationships by turning curves into straight lines, making it easier to use common equation for future trend analysis. Linear models are straightforward and let us understand trends, estimate stuff, and predict things more clearly, giving us a better handle on the data.

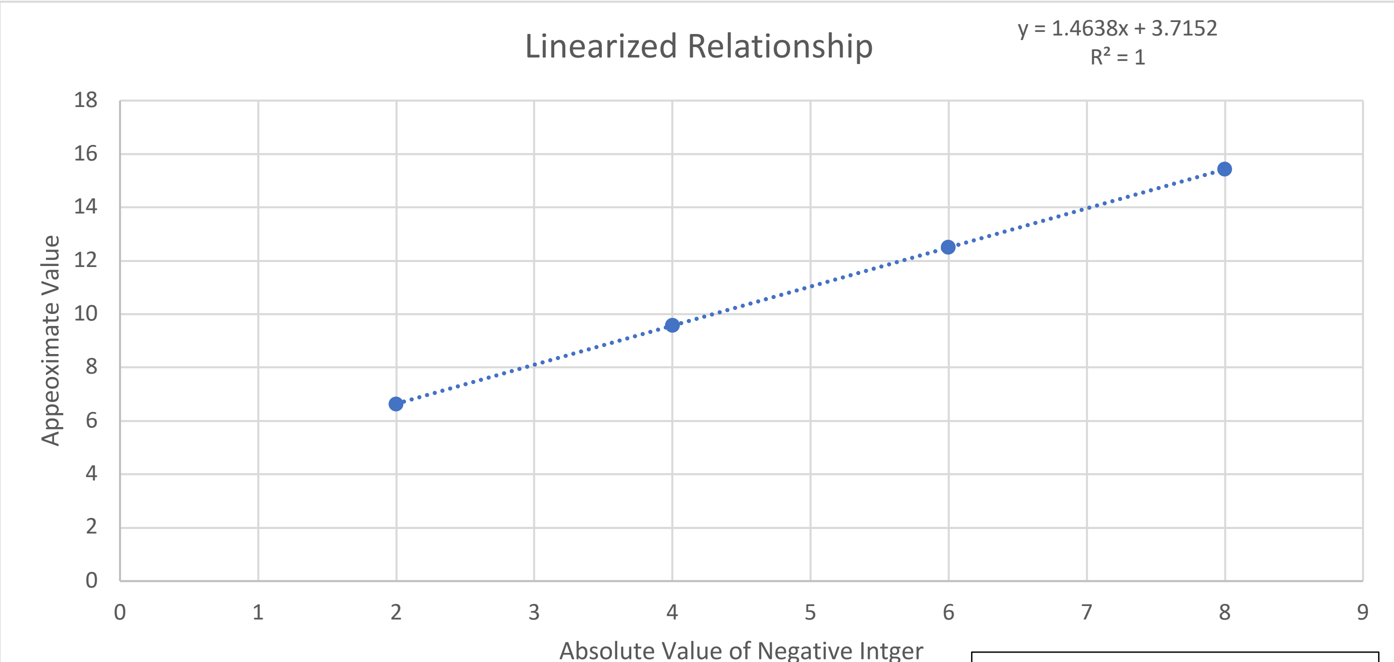
For applying the semi-logs method, I started off by applying natural logarithms to both sides. The equation I got was: $y = 41.07e^{14638x}$

$$\ln(y) = \ln(41.07) + 1.4638x$$

$$let\ \ln(y) = Y$$

$$Y = 3.7152 + 1.4638x$$

The graph which I obtained from this linearizing was as follows:



Graph 8: Self-made on Microsoft Excel

Graph 8: A graph which shows the linear relationship of the points.

Now that I have linearized the graph, I will try to find the value of $-16!$ With the trendline and with Roman's Formula of factorial.

Using the trendline the value comes to:

$$Y = 3.7152 + 1.4638(-16)$$

$$Y = -19.7056$$

$$Since\ Y = \ln(y)$$

$$y = e^{-19.7056}$$

$$y = 2.76672922 \times 10^{-9}$$

Since we have use the absolute value for determining a trend, the actual value would be about - $2.76672922 \times 10^{-9}$. Although using Roman's Formula, the value should have come to $-7.647163732 \text{ x } 10^{-13}$. Which again is not accurate at all, proving that my hypothesis is incorrect and there is no relationship and correlation in the value of negative factorial and the trends which lead up to these values.

Conclusion

Throughout this investigation, my hypothesis was proven incorrect. Initially, I believed I could model the factorial function within small intervals, using those models to approximate other factorial values with a precision of 3 significant figures. However, none of the models I employed met this level of accuracy. As I extended my approximations beyond the initial dataset, the accuracy significantly declined. Reflecting on this challenge, I arrived at a realization: the factorial function's rapid growth outpaces the capabilities of second-order polynomials or exponential functions to accurately model it. Underestimating the immense disparity in function growth led me to overestimate the effectiveness of these models.

Despite falling short of successfully modelling the factorial function, this exploration was immensely educational. It provided deep insights into the nature of this remarkable function, offering a satisfying journey of discovery.

Nevertheless, this investigation harbour's limitations. In hindsight, exploring higher-order polynomial regressions could have been beneficial. If I were to repeat this study, I would likely make the regression function's order the independent variable, aiming to assess the proximity of approximation achievable. Additionally, expanding the dataset by calculating more factorial values might have enhanced the accuracy of regression equations. However, I recognize that regardless of the number of data points, the factorial function's complexity exceeds the capabilities of regression models to faithfully represent it.

During my exploration, I decided to take a new look at factorials. I played around with different math tools and visualization techniques, trying to make my research question more clear. This exploration has not just expanded my grasp of factorials; it has also sharpened my problem-solving abilities and honed my mathematical intuition. It was noticed that there was some what a corelation in the various graph, even though the coefficient of determination was not exactly 1, it was always close to one displaying a relationship between the various x axis values along with the y axis values.

The gamma function has quite a few advantages and applications such as the integral representation makes it easier to simplify complex math problems and integrals making complicated equations easier to handle. The functions versatility makes it applicable in many fields such as number theory, differential equations and special functions. The function can come in handy while studying density functions as it is useful for modelling waiting times and random variables in statistical analysis. Similarly the Roman Factorial function has many applications. It is very handy in the simplification of double factorial. It makes the factorial simpler and easier to simplify. It is also used in many different combinatorial problems. It also comes up in the representation of special series and sequences providing a convenient way to express certain patterns.

In summary, delving into the realm of factorials has been a personal journey fuelled by curiosity, challenges, and a genuine admiration for the beauty of mathematics. Throughout this exploration, I've not only unravelled the mysteries of factorials but also forged a more profound connection with the vast landscape of combinatorial mathematics.

Further Scope of Research

I could explore higher order polynomial regressions. They may provide better fits for my models and graphs. Moreover I could expand my dataset instead of keeping it limited to 1-8 this could help in better approximations. I can also expand and include Mathematical Python Libraries which would help me in exploring different regression techniques and help to analyse larger datasets more efficiently. The reasons why I couldn't include all these mathematical concepts in this exploration was due to the lack of space. There was also a lot of deviation and variations which made me focus more on the main goal more than expanding this exploration. Using higher order polynomial requires other mathematical concept which was challenging for me to learn.

Bibliography

BlackTshirtMathProfessor. "The Gamma Function." YouTube, 4 May 2021, www.youtube.com/watch?v=-GXHHg6kFZM

"Graphing Calculator." Desmos, www.desmos.com/calculator Accessed 1 Jan. 2024.

"Gamma Function." From Wolfram MathWorld, mathworld.wolfram.com/GammaFunction.html Accessed 5 Jan. 2024.

Gamma Function Definition, Properties & Examples | Study.Com, study.com/academy/lesson/gamma-function-properties-examples.html Accessed 4 Jan. 2024.

National Library of Medicine. Ashwani K Thukral, 6 Nov. 2014, www.ncbi.nlm.nih.gov/pmc/articles/PMC4247832. Accessed 6 Jan. 2024.

"Roman Factorial." From Wolfram MathWorld, mathworld.wolfram.com/RomanFactorial.html Accessed 6 Jan. 2024.

Tom Rocks Maths. "What Is the Gamma Function?" YouTube, 10 Oct. 2017, www.youtube.com/watch?v=7y-XTrfNvCs

Wiki, Contributors to Googology. "Roman Factorial." Googology Wiki, googology.fandom.com/wiki/Roman_factorial Accessed 7 Jan. 2024.

"Wolfram: Alpha: Making the World's Knowledge Computable." WolframAlpha Computational Knowledge AI, www.wolframalpha.com/ Accessed 6 Jan. 2024.