INDIAN INSTITUTE OF TECHNOLOGY MANDI
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IC-115 (ODE)
Assignment-1
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1. (a) Show that by applying the substitution $y = uv$ to the homogeneous equation $y'' + P(x)y' + Q(x)y = 0$ it is possible to obtain a homogeneous second order linear equation for v with no v' term present. Find u and the equation for v in terms of the original coefficients $P(x)$ and $Q(x)$.

   (b) Use the method of part (a) to find the general solution of $y'' + 2xy' + (1 + x^2)y = 0$.

2. It is clear that $\sin x$, $\cos x$ and $\sin x$, $\sin x - \cos x$ are two distinct pairs of linearly independent solutions of $y'' + y = 0$. Thus, if y_1 and y_2 are linearly independent solutions of the homogeneous equation

$$y'' + P(x)y' + Q(x)y = 0,$$

   we see that y_1 and y_2 are not uniquely determined by the equation.

   (a) Show that

$$P(x) = -\frac{y_1 y_2'' - y_2 y_1''}{W(y_1, y_2)}$$

   and

$$Q(x) = -\frac{y_1' y_2'' - y_2' y_1''}{W(y_1, y_2)}$$

   so that the equation is uniquely determined by any given pair of linearly independent solutions.

   (b) Use (a) to reconstruct the equation $y'' + y = 0$ from each of the two pairs of linearly independent solutions mentioned above.

3. A radioactive substance decays at a rate proportional to the amount present, and half the original quantity Q_0 is left after 1500 years. In how many years would the original amount be reduced to $3Q_0/4$? How much will be left after 2000 years?

4. Assume an object weighing 2 lb stretches a spring 6 in. Find the equation of motion if the spring is released from the equilibrium position with an upward velocity of 16 ft/sec. What is the period of the motion?

5. Verify that the following equations are exact or not. Find an integrating factor for non-exact equations and solve.

   (a) $(x^3 + xy^3)\,dx + 3y^2\,dy = 0$.

   (b) $x\,dy + y\,dx = \sqrt{xy}\,dy$.

   (c) $(\sin x \tan y + 1)\,dx + \cos x \sec^2 y\,dy = 0$.

6. Solve the following differential equations:

   (a) $\dfrac{4y^2 - 2x^2}{4xy^2 - x^3}dx + \dfrac{8y^2 - x^2}{4y^3 - x^2 y}dy = 0$;

   (b) $dy + \dfrac{y}{x}dx = \sin x dx$;

   (c) $x\dfrac{dy}{dx} + y = x^4 y^3$;

   (d) $x\dfrac{dy}{dx} = 2x^2 y + y \log y$.

7. A spherical raindrop, starting from rest, falls under the influence of gravity. If it gathers in water vapor (assumed at rest) at a rate proportional to its surface, and if its initial radius is 0, show that it falls with constant acceleration $g/4$.

8. A man is pushing a loaded sled across a level field of ice at the constant speed of 10 ft/sec. When the man is halfway across the ice field, he stops pushing and lets the loaded sled continue on. The combined weight of the sled and its load is 80 lb; the air resistance (in pounds) is numerically equal to $\frac{3}{4}v$, where v is the velocity of the sled (in feet per second); and the coefficient of friction of the runners on the ice is 0.04. How far will the sled continue to move after the man stops pushing?

9. A boat weighing 150 lb with a single rider weighing 170 lb is being towed in a certain direction at the rate of 20 mph. At time $t = 0$ the tow rope is suddenly cast off and the rider begins to row in the same direction, exerting a force equivalent to a constant force of 12 lb in this direction. The resistance (in pounds) is numerically equal to twice the velocity (in feet per second).

   (a) Find the velocity of the boat 15 sec after the tow rope was cast off.

   (b) How many seconds after the tow rope is cast off will the velocity be one-half that at which the boat was being towed?