# Summer Of Science
# Financial Mathematics

End Term Report 2023

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**Abstract**

The project provides an in-depth understanding of financial derivatives and their applications in modern financial markets. It covers a broad range of topics, including options, futures, forwards, swaps, and other derivative instruments, as well as the underlying theories and concepts behind them. I have begun by introducing the basics of derivatives and their role in risk management and then dove into the mechanics and valuation of options and futures contracts, discussing various pricing models. Furthermore, the report covers the application of derivatives in real-world scenarios, such as trading strategies, exotic options, and credit derivatives.
The aim is to explores topics such as the Black-Scholes-Merton model and volatility.

References:

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2. Stochastic Calculus and Finance, STEVEN SHREVE
3. https://en.wikipedia.org/wiki/Mathematicalfinance , WIKIPEDIA
4. The Pricing of Options and Corporate Liabilities , F. BLACK and M. SCHOLES

# Contents

# 1 Introduction To Financial Mathematics

## 1.1 Types Of Traders

Three broad categories of traders can be identified: hedgers, speculators, and arbitrageurs. Hedgers use derivatives to reduce the risk that they face from potential future movements in a market variable. Speculators use them to bet on the future direction of a market variable. Arbitrageurs take offsetting positions in two or more instruments to lock in a profit.

## 1.2 Derivatives

A derivative is a financial instrument whose value is derived from the value of the underlying variables. For example, a stock option derives its value from the price of a stock which in turn derives its value from the value of the traded company.

## 1.3 Forward Contracts

A forward contract is a type of derivative contract that obligates two parties to buy or sell an asset at a predetermined price (the forward price) on a specified future date. It is a private agreement between the two parties, typically conducted over-the-counter (OTC), rather than being traded on a centralized exchange.

One of the parties to a forward contract assumes a long position and agrees to buy the underlying asset on a certain specified future date for a certain specified price. The other party assumes a short position and agrees to sell the asset on the same date for the same price. Generally, K is the delivery price and ST is the spot price (price at maturity). Profits and losses will be |K-ST| if any.

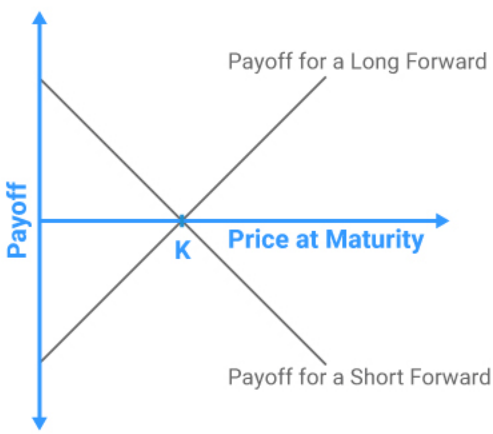


Fig. 1: (Profits in long and short forward)

### 1.4 Future Contracts

A futures contract is basically a forward contract that is traded on an exchange.

### 1.5 Future Contracts

Options are traded both on exchanges and in the over-the-counter market. There are two types of option. A call option gives the holder the right to buy the underlying asset by a certain date for a certain price. A put option gives the holder the right to sell the underlying asset by a certain date for a certain price. The price in the contract is known as the exercise price or strike price; the date in the contract is known as the expiration date or maturity. Buyers are referred to as having long positions; sellers are referred to as having short positions. Selling an option is also known as writing the option.

There are four types of participants in options markets:
1. Buyers of calls
2. Sellers of calls
3. Buyers of puts
4. Sellers of puts.

# 2 Future Markets

### 2.1 Specification Of A Future Contract

As a general rule, it is the party with the short position (the party that has agreed to sell the asset) that chooses what will happen when alternatives are specified by the exchange. When the party with the short position is ready to deliver, it files a notice of intention to deliver with the exchange.

The following are a part of the contracts:
1. The Asset
2. Delivery Arrangements
3. Price Quotes
4. Price Limits and Position Limits

The above terms are self explanatory.

### 2.2 Convergence Of Future Price To Spot Price

As the delivery period for a futures contract is approached, the futures price converges to the spot price of the underlying asset.

Traders then have a clear arbitrage opportunity:
1. Sell (i.e., short) a futures contract
2. Buy the asset
3. Make delivery

The futures price is very close to the spot price during the delivery period.

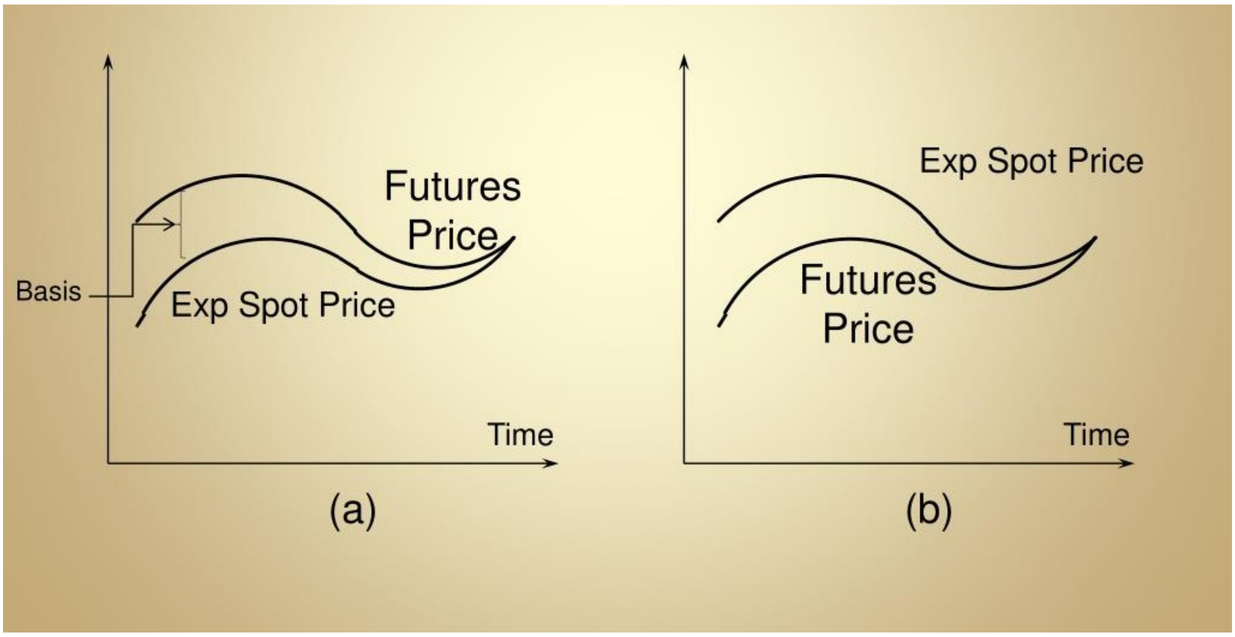


Fig. 2: Relationship between futures price and spot price as the delivery period is approached: (a) Futures price above spot price; (b) futures price below spot price

## 2.3 Orders

• Limit order specifies a particular price. The order can be executed only at this price or below. Thus, if the limit price is $30 for an investor to buy, the order will be executed only at a price of $30 or less.

• Stop order also specifies a particular price. The order is executed at the best available price once a bid or offer is made at that particular price or a less favourable price. It becomes an order to sell when and if the price falls below the specified price. It limits the loss that can be incurred.

• Stop-limit order is a combination of a stop order and a limit order. Two prices must be specified in a stop and limit order. Suppose the market price is $35, a stop limit-order to buy is issued with a stop price of $40 and a limit price of $41.

# 3 Capital Asset Pricing Model

There are two aspects to the risk associated with an asset's return. Systematic risk is an unavoidable risk that is connected to the return from the market as a whole. Non-systematic risk is a type of risk that is specific to an asset and can be mitigated by building a sizable portfolio of various assets.
Expected return on asset $= R_f + \beta(R_m - R_f)$
where $R_m$ is the return on the portfolio of all available investments, $R_f$ is the return on a risk-free investment, and $\beta$ (the Greek letter beta) is a parameter measuring systematic risk.

The return from the portfolio of all available investments, $R_m$, is referred to as the return on the market and is usually approximated as the return on a well-diversified stock index such as the SP 500. The $\beta$ of an asset is a measure of the sensitivity of its returns to returns from the market.

# 4 Interest Rates

An interest rate in a particular situation defines the amount of money a borrower promises to pay the lender. The interest rate applicable depends on the credit risk. This is the risk that there will be a default by the borrower of funds, so that the interest rate and principal are not paid to the lender as promised.

## 4.1 The Fed Funds Rate

financial institutions are required to maintain a certain amount of cash (known as reserves) with the Federal Reserve. At the end of a day, some financial institutions typically have surplus funds in their accounts with the Federal Reserve while others have requirement for funds.This leads to borrowing and lending overnight. This overnight rate is called federal funds rate.

## 4.2 Repo Rates

Repo rates are secured borrowing rates. In a repo, a financial institution that owns securities agrees to sell the securities for a certain price and buy them back at a higher price. The interest it pays if the difference the price at which the securities are sold and rebought. The interest rate is referred to as the repo rate. A repo involves very little credit risk as the lender has the right to keep the collateral/asset if the borrower defaults. Because a repo involves very little risk, repo rate is generally slightly below the corresponding fed funds rate.

## 4.3 The "Risk - Free" Rate

Derivatives are usually valued by setting up a riskless portfolio and arguing that the return on the portfolio should be the risk-free interest rate. The risk-free interest rate therefore plays a key role in the valuation of derivatives.

## 4.4 Measuring Interest Rates

The compound frequency defines the units in which an interest rate is measured. A rate expressed in one compounding frequency can be converted into an equivalent rate with a different compounding frequency. Suppose that an amount A is invested for n years at an interest rate of R per annum.
If the rate is compounded once times per annum, the terminal value of the investment is $A(1+R)^n$
If the rate is compounded m times per annum, the terminal value of the investment is $A(1+R/m)^n m$
When m=1, the rate is sometimes referred to as the equivalent annual interest rate.

## 4.5 Continuous Compounding

The limit as the compounding frequency (m) tends to infinity is known as continuous compounding. With continuous compounding, it can be shown that an amount A invested for n years at rate R grows to $Ae^{nR}$ . Where e is approximately 2.71828 .

Suppose that $R_c$ is a rate of interest with continuous compounding and $R_m$ is the equivalent rate with compounding m times per annum. So, we have
$Ae^{R_c n} = A(1 + Rm/m)^{mn}$
$e^{R_c} = (1 + Rm/m)^m$
$R_c = m(e^{R_c/m} - 1)$

## 4.6 Liquidity

In addition to creating problems in the way that has been described, a portfolio where maturities are mismatched can lead to liquidity problems. Consider a financial institution that funds 5-year fixed rate loans with wholesale deposits that last only 3 months. It might recognize its exposure to rising interest rates and hedge its interest rate risk. Wholesale depositors may, for some reason, lose confidence in the financial institution and refuse to continue to provide the financial institution with short-term funding. The financial institution, even if it has adequate equity capital, will then experience a severe liquidity problem that could lead to its downfall.

# 5 Options Markets

## 5.1 Strike Prices

The exchange normally chooses the strike prices at which options can be written so that they are spaced \$2.50, \$5, or \$10 apart. When a new expiration date is introduced, the two or three strike prices closest to the current stock price are usually selected by the exchange. Options are referred to as in the money, at the money, or out of the money. If S is the stock price and K is the strike price, a call option is in the money when S > K, at the money when S = K, and out of the money when S < K. A put option is in the money when S < K, at the money when S = K, and out of the money when S > K.

## 5.2 Stock Options

Options trade on several thousand different stocks. Stock options in the United States are on a January, February, or March cycle. If the expiration date for the current month has not yet been reached, options trade with expiration dates in the current month, the following month, and the next two months in the cycle.

## 5.3 Factors Affecting the Price of Stock Options

As we expect the price of the option to reflect the price of the underlying asset, it seems logical that a factor that affects the price of the underlying asset would also affect the price of the option. So for stock options we consider the factors that affect stock prices. There are six major factors. These are :

1. The current price the stock is trading at, $S_0$,
2. The strike price, $K$,
3. The time until expiration, $T$,
4. The volatility of the price of the stock, $\sigma$,
5. The risk-free interest rate, $r$,
6. The dividends that are expected to be paid, $q$.

Here the risk-free interest rate is the theoretical rate of return on a completely risk free investment. The volatility represents how the price varies over time, which we will give a more rigorous definition of this later. How each of these factors affects the price of European and American puts and calls is given in Table 1. We will discuss how each of these factors affects the option. Note that in Table 1 we use + to mean the factor increases the price of the option, a $-$ for a decrease and a ? when the relationship is unknown.

| Variable | European Call | European Put | American Call | American Put |
|---|---|---|---|---|
| Current stock price | + | - | + | - |
| Strike Price | - | + | - | + |
| Time to expiration | ? | ? | + | + |
| Volatility | + | + | + | + |
| Risk-free rate | + | - | + | - |
| Amount of future dividends | - | + | - | + |

Tab. 1: The effect different factors have on the price of different options

Most of the ways each factor affects each type of option is intuitivee. If the price goes up, the price of a call goes up as we are likely to see greater increases in the price of the stock and the price of a put goes down as the decrease will not be as fast as the increase. This is due to the fact that, as we will see later, the path the stock takes is the initial price multiplied by some variables. Hence increases tend to happen faster than decreases.

The risk free interest rate is a more complex idea. Within the economy, as interest rates increase, investors expect more return from the option, however the value of any money earnt in the future decreases, due to these interest rates, termed inflation. This increases the price of stocks slightly resulting in a higher chance of calls paying off and less of puts paying off.

With relation to the dividends, as the ex-dividend date approaches, that is the date at which entitlement to dividends changes, the stock price decreases. Hence, due to the relationships for calls and puts, the price of calls decreases and the price of a put increases.

The most anomalous observations are those of the time to expiration and the volatility. The time to expiration for American options increases for both puts and calls. This is due to that, for two options, if the only difference is that one has a longer time to expiration, then the owner of the option with a longer time has all of the same opportunities to exercise and more. This increases the value of the option. For European options, this is not necessarily the case. If we have two options that straddle the ex-dividend date, the one with the shorter life would be worth more.

Finally we consider volatility, we have not defined volatility until this point, we will discuss definition in later sections. For now we merely remark that volatility is a measure of how uncertain we are of the stocks future price. If volatility increases, the chance the stock will do very well or very poorly increases. As our maximum loss from an option contract is the price but the profit is either infinite or large this benefits the owner hugely.

## 5.4 Put - call parity

### 5.4.1 Assumptions and Notation

1. There are no transactions costs.
2. All trading profits are subject to the same tax rate.
3. Borrowing and lending are possible at the risk-free interest rate.

$S_0$ : Current stock price
K: Strike price of option
T : Time to expiration of option
$S_T$ : Stock price on the expiration date
r : Continuously compounded risk-free rate of interest for an investment maturing in time T
c : Value of call option to buy one share
p : Value of put option to sell one share

### 5.4.2 Upper Bound for Stock Options

No matter what happens, the option can never be worth more than the stock. Hence, the stock price is an upper bound to the option price: c $\leq S_0$
At maturity the option cannot be worth more than K. It follows that it cannot be worth more than the present value of K today: p $\leq Ke^{-rT}$

### 5.4.3 Lower Bound for Calls on Non-Dividend-Paying Stocks

A lower bound for the price of a call option on a non-dividend-paying stock is $S_0 - Ke^{-rT}$
Portfolio A: one call option plus a zero-coupon bond that provides a payoff of K at time T
Portfolio B: one share of the stock.
If $S_T >$ K, the call option is exercised at maturity and is worth $S_T$ .
If $S_T <$ K, the call option expires worthless and the portfolio is worth K. Hence, at time T, portfolio A is worth : max($S_0$,K)
Portfolio B is worth $S_T$ at time T

$S_0 - Ke^{-rT} \leq c$
$max(S_0 - Ke^{-rT}, 0) \leq c$

### 5.4.4 Lower Bound for Puts

Portfolio C: one put option plus one share
Portfolio D: a zero-coupon bond paying off K at time T.
If $S_T <$ K, then the option in portfolio C is exercised at option maturity and the portfolio becomes worth K.
If $S_T > K$ , then the put option expires worthless and the portfolio is worth $S_T$ at this time. Hence, portfolio C is worth $max(S_T, K)$
Portfolio D is worth K in time T. Hence, portfolio C is always worth as much as, and can sometimes be worth more than, portfolio D in time T. Hence, $Ke^{-rT} - S_0 \leq p$
$max(Ke^{-rT} - S_0, 0) \leq p$

### 5.4.5 Put - Call Parity

The components of portfolio A are worth c and $Ke^{-rT}$ today, and the components of portfolio C are worth p and $S_0$ today. Hence, $c + Ke^{-rT} = p + S_0$
This relationship is known as put - call parity.

## 5.5 Black-Scholes Model

Firstly, we assume that the percentage return's variability over a small time interval, $\Delta t$, is constant and independent of stock price. This means that a buyer is as uncertain of the return (as a percentage) when the stock costs \$1 as when the stock costs \$1000. This leads to the fact that the the stock price should be proportional to the standard deviation over a small period of time $\Delta t$. This leads to the following final model,

$$\mathrm{d}S = \mu S \mathrm{d}t + \sigma S \mathrm{d}W, \tag{1}$$

where the variable $\sigma$ is the volatility of the stock per year and $\mu$ is the expected rate of return on the stock per year. This is the most widely used model for stock behavior. Now using (1) and applying Itô's lemma we obtain, for a function $G(S, t)$ we have the process $G$ follows is given by,

$$\mathrm{d}G = \left( \frac{\partial G}{\partial S} + \frac{\partial G}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 G}{\partial S^2} \right) \mathrm{d}t + \frac{\partial G}{\partial S} \sigma S \mathrm{d}W. \tag{2}$$

We see from (1) that the volatility is a measure of how unsure we are about the path the stock will take. This is because it is multiplying the random component. It can also be viewed as the standard deviation of the lognormal distribution of $S_T$, as we will see in the next section.

## 5.6 The Black-Scholes Differential Equation

Given that $f$ is an option subject to $S$, then $f$ must be some function of $S$ and $t$. Hence, from (2),

$$\mathrm{d}f = \left( \frac{\partial f}{\partial S} + \frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} \right) \mathrm{d}t + \frac{\partial f}{\partial S}\sigma S \mathrm{d}W.$$

The equations (1) and the above have discretized versions,

$$\Delta S = \mu S \Delta t + \sigma S \Delta W. \tag{3}$$

$$\Delta f = \left( \frac{\partial f}{\partial S} + \frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} \right) \Delta t + \frac{\partial f}{\partial S}\sigma S \Delta W, \tag{4}$$

over a time interval $\Delta t$. As the Wiener processes contained in $\Delta f$ and $\Delta S$ are the same, it follows that we may construct a portfolio to eliminate this. Such a portfolio should sell an option and buy $\frac{\partial f}{\partial S}$ shares. Then by definition our portfolio, $\Pi$, is,

$$\Pi = -f + \frac{\partial f}{\partial S} S. \tag{5}$$

The change in this over $\Delta t$ is,

$$\Delta \Pi = -\Delta f + \frac{\partial f}{\partial S}\Delta S.$$

By substituting in (3) and (4) we obtain,

$$\Delta \Pi = \left( \frac{\partial f}{\partial t} - \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2 \right) \Delta t \tag{6}$$

Over the time period $\Delta t$ we have eliminated $\Delta W$, so the portfolio must be riskless in this time period and must therefore make the riskfree interest rate. Thus,

$$\Delta \Pi = r \Pi \Delta t$$

substituting (5) and (6) into the above, we yield,

$$\left( \frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} \right) \Delta t = r \left( f - \frac{\partial f}{\partial S} S \right) \Delta t$$

so that,

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf.$$

The above is known as the Black-Scholes differential equation. It is solvable for some boundary conditions and unsolvable analytically for others.