Calculus II – Homework

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Group members

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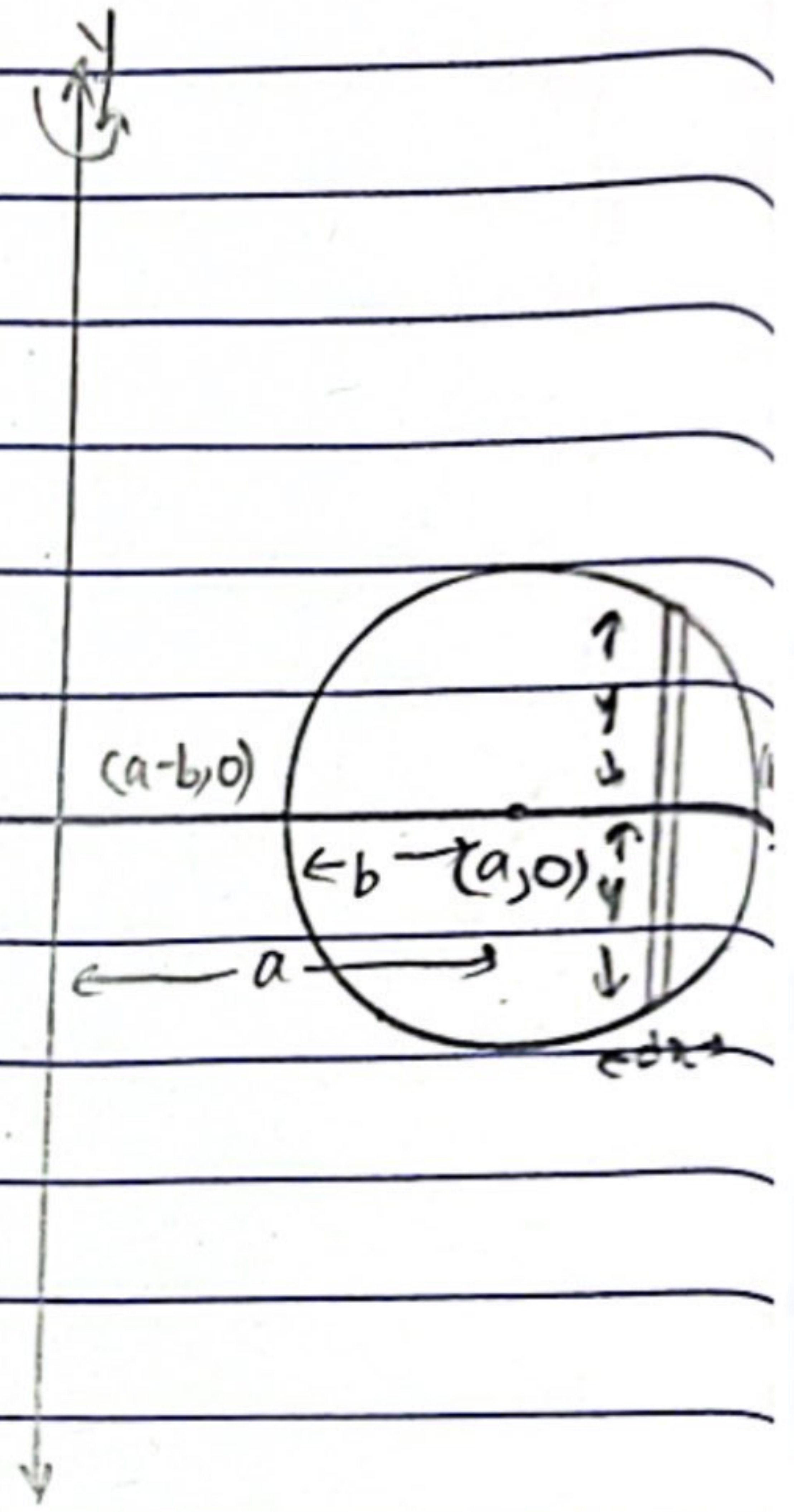
Q1) a) Eqn of circle given,

$$(x-a)^2+y^2=b^2.$$

$\therefore$ Center $=(a,0)$

radius $=b$.



To find volume using cylindrical shell method,

$$h(x)=2y.$$

$$(x-a)^2+y^2=b^2.$$

$$\therefore y^2=b^2-(x-a)^2$$

$$y=\sqrt{b^2-(x-a)^2}.$$

$$\therefore h(x)=2\sqrt{b^2-(x-a)^2}.$$

$$r(x)=x.$$

$$\therefore V=\int_{a-b}^{a+b}2\pi r(x)h(x)\,dx.$$

$$=2\pi\int_{a-b}^{a+b}x\left(2\sqrt{b^2-(x-a)^2}\right)dx.$$

$$=4\pi\int_{a-b}^{a+b}x\sqrt{b^2-(x-a)^2}\,dx.$$

$$=4\pi\int_{a-b}^{a+b}\left[(x-a)+a\right]\sqrt{b^2-(x-a)^2}\,dx.$$

$$=4\pi\left[\int_{a-b}^{a+b}(x-a)\sqrt{b^2-(x-a)^2}\,dx+\int_{a-b}^{a+b}a\sqrt{b^2-(x-a)^2}\,dx\right]$$

$$=4\pi\left[I_1+I_2\right]$$

$$I_1 = \int_{a-b}^{a+b} (x-a)\sqrt{b^2-(x-a)^2}\, dx.$$

Let $b^2-(x-a)^2 = u.$

$$-2(x-a)\,dx = du.$$

$$(x-a)\,dx = \frac{du}{-2}$$

$\therefore \cancel{I_1} \rightarrow \int (x-a)\sqrt{b^2-(x-a)^2}\, dx.$ [Ignoring limits].

$$= -\frac{1}{2}\int \sqrt{u}\, du.$$

$$= -\frac{1}{\cancel{2}}\left(\frac{\cancel{2}}{3}u^{3/2}\right) + C = -\frac{(b^2-(x-a)^2)^{3/2}}{3} + C.$$

$$\therefore I_1 = -\frac{(b^2-(x-a)^2)^{3/2}}{3}\Bigg|_{a-b}^{a+b}.$$

$$= -\frac{(b^2-(\cancel{a}+b-\cancel{a})^2)^{3/2}}{3} + \frac{(b^2-(\cancel{a}-b-\cancel{a})^2)^{3/2}}{3}$$

$$= -\frac{(\cancel{b^2}-\cancel{b^2})^{3/2}}{3} + \frac{(\cancel{b^2}-\cancel{b^2})^{3/2}}{3} = 0$$

$$I_2 = a\int_{a-b}^{a+b} \sqrt{b^2-(x-a)^2}\, dx.$$

Let $x-a = u.$ $\therefore dx = du.$ | $\therefore$ when $x = a-b,$ $u = a-b-a = -b$ | when $x = a+b,$ $u = a+b-a = b.$

$$\therefore I_2 = a\int_{-b}^{+b} \sqrt{b^2-u^2}\, du.$$

Let $u = b\sin\theta \Rightarrow \theta = \sin^{-1}\left(\frac{u}{b}\right).$

$$\frac{du}{d\theta} = b\cos\theta$$

$$\therefore du = b\cos\theta\, d\theta$$

$\therefore I_1 : \int \sqrt{b^2 - u^2}\, du$. [Ignoring limits].

$= \int \sqrt{b^2 - b^2\sin^2\theta}\, (b\cos\theta\, d\theta)$.

$= \int b\cos\theta\, (b\cos\theta)\, d\theta$

$= b^2 \int \cos^2\theta\, d\theta$

$= \frac{b^2}{2} \int (1 + \cos 2\theta)\, d\theta$

$= \frac{b^2}{2}\left(\theta + \frac{\sin 2\theta}{2}\right) + c = \frac{b^2}{2}\left(\sin^{-1}\left(\frac{u}{b}\right) + \frac{\sin 2\theta}{2}\right)$

$\therefore \sin\theta = \frac{u}{b}$.

$\cos\theta = \sqrt{1 - \sin^2\theta} = \sqrt{1 - \frac{u^2}{b^2}}$.

$\sin 2\theta = 2\sin\theta\cos\theta$

$= 2\left(\frac{u}{b}\right)\left(\sqrt{1 - \frac{u^2}{b^2}}\right)$

$= 2\left(\frac{u}{b^2}\right)\sqrt{b^2 - u^2}$

$\therefore \frac{b^2}{2}\left(\sin^{-1}\left(\frac{u}{b}\right) + \frac{\sin 2\theta}{2}\right)$

$= \frac{b^2}{2}\left(\sin^{-1}\left(\frac{u}{b}\right) + \left(\frac{u}{b^2}\right)\sqrt{b^2 - u^2}\right) = \frac{1}{2}\left(b^2\sin^{-1}\left(\frac{u}{b}\right) + u\sqrt{b^2 - u^2}\right)$

$\therefore I_2 = \frac{a}{2}\left(b^2\sin^{-1}\left(\frac{u}{b}\right) + u\sqrt{b^2 - u^2}\right)\Big|_{-b}^{+b}$

$= \frac{a}{2}\left(b^2\sin^{-1}(1) + u\cancel{\sqrt{b^2 - b^2}}^{\,0} - b^2\sin^{-1}(-1) - u\cancel{\sqrt{b^2 - b^2}}^{\,0}\right)$

$= \frac{a}{2}\left(b^2\frac{\pi}{2} + b^2\frac{\pi}{2}\right) = \frac{a}{2} b^2\pi = \frac{1}{2} ab^2\pi$

$\therefore V = 4\pi\,[I_1 + I_2]$.

$= 4\pi\left(0 + \frac{1}{2}ab^2\pi\right)$.

$\boxed{V = 2\pi^2 ab^2}$

(b) $V = 2\pi^2\,(6\,cm)\,(3.5\,cm)^2$

$= 147\,\pi^2\,cm^3$.

$= \boxed{1450.832\ cm^3.}$

(c) ~~SH~~ ~~let~~ $(x-a)^2 + y^2 = b^2$.

$(x-a)^2 = b^2 - y^2$.

$x - a = \sqrt{b^2 - y^2}$.

$x = \pm\sqrt{b^2 - y^2} + a$.

$\frac{dx}{dy} = \frac{1}{\cancel{2}\sqrt{b^2-y^2}}(-\cancel{2}y)$

$= \frac{-y}{\sqrt{b^2-y^2}}$.

$\left(\frac{dx}{dy}\right)^2 + 1 = \left(\frac{-y}{\sqrt{b^2-y^2}}\right)^2 + 1 = \frac{y^2}{b^2-y^2} + 1 = \frac{\cancel{y^2} + b^2 - \cancel{y^2}}{b^2-y^2} = \frac{b^2}{b^2-y^2}$

c) $\left(\frac{dx}{dy}\right)^2 + 1 = \frac{b^2}{b^2-y^2}$.

Surface area ~~Surface area~~ $= 2\pi \int_{-b}^{b} x \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy$. $+ \cancel{2\pi \int_{-b}^{b} x}$

$A_1 + A_2 = 4\pi \int_0^b \left(\sqrt{b^2-y^2}+a\right) \sqrt{\frac{b^2}{b^2-y^2}}\, dy$ [symmetry]. $+ 4\pi \int_0^b \left(-\sqrt{b^2-y^2}+a\right)\sqrt{\frac{b^2}{b^2-y^2}}\, dy$

$\therefore A_1 = 4\pi \int_0^b \frac{b\sqrt{b^2-y^2}}{\sqrt{b^2-y^2}} + \frac{ab}{\sqrt{b^2-y^2}}\, dy.$

$= 4\pi \int_0^b b + \frac{ab}{\sqrt{b^2-y^2}}\, dy.$

Let $y = b\sin\theta$. ~~when y = 0~~, $\theta = \sin^{-1}\left(\frac{y}{b}\right)$ | when $y=0$, $\theta = \sin^{-1}(0) = \underline{\underline{0}}$ | when $y = b$, ~~sin~~ $\theta = \sin^{-1}(1) = \underline{\underline{\pi/2}}$

$dy = b\cos\theta\, d\theta$

$\therefore 4\pi \int_0^b b + \frac{ab}{\sqrt{b^2-y^2}}\, dy.$

$= 4\pi \left[\int_0^b b\, dy + \int_0^{\frac{\pi}{2}} \frac{ab}{\sqrt{b^2-b^2\sin^2\theta}} \cdot b\cos\theta\, d\theta\right].$

$= 4\pi \left[by\Big|_0^b + \int_0^{\frac{\pi}{2}} \frac{ab^2 \cancel{\cos\theta}\, d\theta}{\cancel{b\cos\theta}} \right].$

$= 4\pi \left[by\Big|_0^b + ab\theta\Big|_0^{\pi/2} \right]$

$= 4\pi \left[b^2 + \frac{ab\pi}{2} \right] = \underline{4\pi b^2 + 2ab\pi^2}$

$$A_2 = 4\pi \int_{a}^{b} \left(-\sqrt{b^2-y^2}+a\right) \sqrt{\frac{b^2}{b^2-y^2}}\, dy.$$

$$= 4\pi \int_{0}^{b} -b + \frac{ab}{b^2-y^2}\, dy.$$

$$= 4\pi \left[-by \Big|_0^b + ab\frac{\pi}{2} \right]$$

$$= 4\pi\left[-b^2 + ab\frac{\pi}{2}\right] = -4\pi b^2 + 2\pi^2 ab.$$

$$\therefore A = A_1 + A_2 = \cancel{4\pi b^2} + 2\pi^2 ab + 2\pi^2 ab - \cancel{4\pi b^2}$$

$$= \boxed{4\pi^2 ab}$$

when $a = 6$ cm and $b = 3.5$ cm,

$$A = 4\pi^2(6)(3.5)$$

$$= \boxed{829.047 \text{ cm}^2.}$$

) $A = 4\pi^2 ab$.

$V = 2\pi^2 ab^2$.

V should stay constant, therefore V is considered a const, $V = 1450.832$ cm³.

$\therefore V = 2\pi^2 ab^2$.

$$a = \frac{V}{2\pi^2 b^2}$$

ab in A,

$$A = 4\pi^2 b \left(\frac{V}{2\pi^2 b^2}\right) = \frac{2V}{b}.$$

~~For A to be~~ $\therefore A \propto \frac{1}{b}$

$\therefore$ For A to be min. b should be max.

Since $b \geq a$.

b can at max be $b = a$.

$\therefore 1450.832 = 2\pi^2 a^3$.

$$a^3 = \frac{1450.832}{2\pi^2} = 73.5$$

$a = \sqrt[3]{73.5} \approx 4.1889$ cm.

$b \approx 4.1889$ cm

2a) $\sum_{n=3}^{\infty} n e^{-n/2}$

Product rule:

$y' = 1e^{-n/2} + (-\frac{1}{2} n^{-n/2}) =$ (a negative number) $\to$ function decreases

$\boxed{1e^{-n/2} - \frac{n}{2} e^{-n/2}}$

$\int x e^{-x/2}\, dx$

~~$u = e^{-\frac{x}{2}}$ $dv = x$~~

~~$du = -\frac{1}{2} e^{-\frac{x}{2}}$ $v = \frac{x^2}{2}$~~

	D	I
+	x	$e^{-x/2}$
−	1	$\frac{e^{-x/2}}{-\frac{1}{2}}$
+	0	$\frac{e^{-x/2}}{\frac{1}{4}}$

$x\left(\frac{e^{-x/2}}{-\frac{1}{2}}\right) - \frac{1e^{-x/2}}{1/4}$

$-2xe^{-x/2} - 4e^{-x/2}$

$\lim_{n\to\infty} -2e^{-x/2}(x-2)$

$\lim_{n\to\infty} \frac{-2(n-2)}{e^{n/2}}$

L.H Rule

$\frac{-2}{\frac{1}{2}e^{n/2}} = \frac{-2}{\infty} = 0$

(b) $\sum_{n=3}^{\infty} \frac{ne^{n/2}}{1+e^n}$

$a_n = \frac{ne^{n/2}}{1+e^n}$ $\quad$ $b_n = \frac{e^{n/2}}{e^n}$

$b_n = \frac{1}{e^{\frac{1}{2}n}} = 0$

$a_n = 0$

Both converge.

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35) ~~to~~ We can find the arc length ~~by~~ to find the cable length.

$$L = \int_0^{0.2} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\ \cancel{dy}\ dx.$$

$$y = 1 + 2x^{7/2}.$$

$$\frac{dy}{dx} = \left(\frac{7}{2}\right)(2)\left(x^{5/2}\right)$$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + 49x^5$$

$$\therefore L = \int_0^{0.2} \left(1 + 49x^5\right)^{1/2} dx.$$

Using binomial expansion,

$$\left(1+49x^5\right)^{1/2} = 1 + \frac{1}{2}\left(49x^5\right) + \frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)\left(49x^5\right)^2}{2!} + \frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\left(49x^5\right)^3}{3!} + \dots$$

$$= 1 + \frac{49}{2}x^5 - \frac{49^2}{8}x^{10} + \frac{49^3}{16}x^{15} + \dots$$

$$\therefore L = \int_0^{0.2} 1 + \frac{1}{2}49x^5 - \frac{49^2}{8}x^{10} + \frac{49^3}{16}x^{15} + \dots\ dx.$$

$$= x\Big|_0^{0.2} + \frac{49}{2}\left(\frac{x^6}{6}\right)\Big|_0^{0.2} - \frac{49^2}{8}\cdot\frac{x^{11}}{11}\Big|_0^{0.2} + \frac{49^3}{16}\cdot\frac{x^{16}}{16}\Big|_0^{0.2} + \dots$$

$$= 0.2 + 2.613\times10^{-4} - 5.567\times10^{-7} + 3.011\times10^{-9} - \dots$$

$$3.011\times10^{-9} < 10^{-7}\ \text{(error)}.$$

$\therefore$ Neglecting everything after.

$$= 0.2 + 2.613\times10^{-4} - 5.567\times10^{-7}$$

$$= 0.2002607913 \text{ units.}$$