PXT992 Project Plan
Holographic Error Correcting Codes with Tensor Networks

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Compiled: 10 May 2024

1 Introduction

1.1 Overview

The discovery of Holographic Principle [1] and AdS/CFT correspondence [2] has entirely changed our outlook towards the ideas of quantum gravity in last few decades. These developments have lead to a bold statement that spacetime geometry essentially emerges from quantum entanglement in a lower dimensional theory. This statement provides a resolution for long standing problem of black hole information paradox. Over last two decades, a new player has emerged central to this new paradigm, which is quantum information theory. These new information theoretic concepts like that entanglement entropy and quantum error correction (QEC) have made surprising connections with holography [3, 4] and in today's time are playing a key role in describing quantum gravity.

Particularly, a new class of quantum codes, called holographic quantum error correction codes (HECC) have emerged as a framework to understand key aspects of AdS/CFT correspondence from a different lens. These HECC like *Three Qutrit Code* [5] tries to model how bulk states (qubits) are encoded in boundary space in a way that makes it mimics local error correction. Similarly, HECC based on tensor networks such as *HaPPY Code* [6] have established a deep connection between geometry and entanglement. Despite significant progress, there are many open question which could be better understood from a deeper understanding of design and functionality of HECC. These could shed light on the problems in quantum gravity like emergence of spacetime and black hole physics, but can also help in making better QEC protocols for fault tolerant quantum computers.

1.2 Research Questions

This research project aims to gain insight into the workings of HECC by addressing the following questions:

1. **How does foundational concepts such as quantum error correction, tensor networks, and AdS/CFT correspondence combine and lead to creation of HECC?**
2. **How does tensor network models capture key holographic properties?**
3. **What implications do these holographic codes have on fundamental physics and QEC?**

1.3 Aims and Objectives

Aim 1: To understand the basic formalism of Quantum Error Correction and Tensor Networks.

Objective 1.1 By the end of week 2 (01/07/2024) – Using [7, 8] Create an operational understanding of stabilizer codes, specifically five qubit code and summarise key properties.

Objective 1.2 By the end of week 2 (01/07/2024) – Construct a comprehensive mathematical framework for tensor network models, focusing on tensor operations and MERA (Multiscale Entanglement Renomarlization Ansatz) networks as discussed in references [9, 10].

Objective 1.3 By the end of week 3 (08/07/2024) – Document the similar properties between MERA networks and AdS/CFT correspondence [3], as these shed light on how tensor networks captures holographic properties.

Aim 2: To acquire an understanding of design and functionality of HECCs.

Objective 2.1 By the end of week 4 (15/07/2024) – Rederive and summarise results from [5], which motivate and demonstrate the first connection between quantum error correction and holography.

Objective 2.2 By the end of week 6 (29/07/2024) – Using tools and methods learnt while achieving objective 1.1, 1.2 and 1.3, rederive results from [6] and construct the HaPPY code from scratch.

Objective 2.3 By the end of week 7 (05/08/2024) – Write a Python code to simulate a segment of HaPPY code (a Pentagon tile using five tensor nodes) using tensor network library and check whether it is feasible to simulate the whole structure.

Aim 3: To explore implications of these codes for quantum error correction and fundamental physics.

Objective 3.1 By the end of week 8 (12/08/2024), Using the HECC, address the idea of emergent spacetime [11]. Make a case of why it could be true and does the mathematics support it?

Objective 3.2 By the end of week 8 (12/08/2024), Assess the potential of HECC for actual error correction and scalability in quantum computers [12], based on fidelity and standard threshold measures.

Objective 3.3 By the end of week 8 (12/08/2024) – Do a comparative analysis of different types of holographic codes based on their structure, error correction capabilities and efficiency.

2 Literature Review

Before getting into actual development of HECC, it is important to briefly touch upon various fields of research, as HECC is an amalgmation of many different topics which in itself are huge research fields. So I want to start this review by briefly discussing basic ideas of holographic principle, quantum error correction and tensor networks in section 2.1, 2.2 and 2.3, respectively. Section 2.4 is on HECC, focusing three qutrit code and HaPPY code.

2.1 Holographic Principle

The origins of holographic principle can be traced back to the 1970's, where physicist started looking at black holes from a completely new perspective. It was through the works of Hawking and Bekenstein, it was realized that black holes are also thermodynamical objects with well defined temperature and entropy[13, 14]. For a black hole with mass M, Theses are given by

$$T_H = \frac{\hbar c^3}{8\pi k_B G M} \qquad S_{BH} = \frac{4GM^2}{\hbar c} = \frac{c^3 A_{hor}}{4\hbar G} \tag{1}$$

and are called the *Hawking temperature* and *Bekenstein-Hawking entropy* of a black hole, respectively [15]. When the equation of entropy expressed in natural units, can be written as $S_{BH} = A_{hor}/4G$. where, A_{hor} is the area of event horizon of the black hole and G is gravitational constant. This equation contains a surprising insight, the entropy of the black hole increases with its surface area rather than its volume unlike a conventional thermodynamic system. This insight led to a bold suggestion that information of black hole's microstates are *holographically* encoded on its horizon. This encoding is happening on the order of the Plank scale, and we can see this when we write Eq(1) in the terms of Planck length l_p as $4\hbar G/c^3 = 4{l_p}^2$ which would make $S_{BH} = A_{hor}/4{l_p}^2$.

Susskind and 't Hooft were the ones who concluded based on this observation that a consistent theory of quantum gravity would need to follow a *holographic principle*, which would mean that the dynamics of gravity in $(3+1)$ dimensional spacetime would have to be reducible to an effective $(2+1)$ dimensional description [1, 16]. There have been other instances, where entropy scaling in terms of area appeared in gravitational settings other than black holes [17], but holographic principle was not clear. It did not specify which theory of quantum gravity would produce such a mapping between systems in different dimensions and how it would be implemented.

It was Maldacena in 1997, despite holographic principle's vagueness, he showed that the realization of this principle is possible in string theory setup [2]. He conjectured that there is a correspondence between observables in *Anti-de Sitter space*, a type of spacetime with gravity and operators in a *conformal field theory*, a quantum field theory without gravity. This correspondence is called *AdS/CFT correspondence*. This fundamentally changed the field of string theory and has met with a huge amount of research activity in wide range of areas.

2.1.1 AdS/CFT Setup

The two key component of this correspondence, as given by the name are *Anti-de Sitter* spacetime and *conformal field theory*. In this section we briefly discuss these components and some key properties of the conjecture.

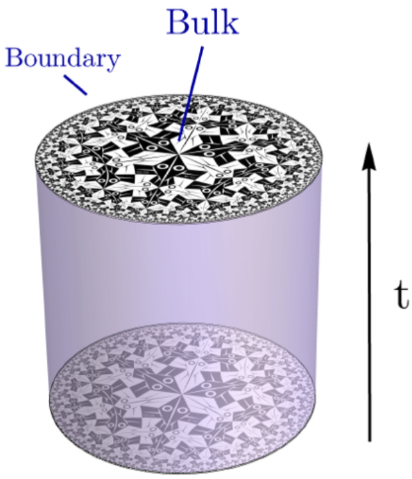


Figure 1: The bulk is AdS_3, which is a Poincare disk moving through time. The bulk refers to gravitational theory and boundary refers to a $(1+1)$dimensional CFT [15].

- **Anti-de Sitter (AdS) Spacetime**: AdS spacetime is a maximally symmetric spacetime with negative curvature. It is a solution of Einstein field equations. To be more precise it is maximally symmetric Lorentzian manifold with negative scalar curvature. It is a differentiable n-dimensional manifold which has a non-degenerate symmetric metric tensor. The metric has a signature of $(n-1, 1)$, these types of manifolds arise from Einstein's field equations. These manifolds have different curvatures, if it has positive curvature, it is a called *de Sitter space*. If it has negative curvature, it is called AdS space. A famous example used through out literature is of AdS_3, it is a $(2+1)$ spacetime which is traced by a Poincare disk as shown in Figure(1). But for generalization of the theory we are mostly going use AdS_{d+1} so that we can describe the CFT in d-dimension.
- **Conformal Field Theory:** It is a quantum field theory which is invariant under conformal transformation. Conformal transformation are special because they lead to scale invariance. Being an element of conformal group, which is a group of all possible transformation from space

to itself that preserves angles between vectors. This group contains transformations from the Poincare group (translations and Lorentz transformations), but still contain larger sets of maps, like scale transformations $x^\mu \to \lambda x^\mu$, and more. A QFT is said to be invariant under conformal transformation if the n point function remain same before and after transformation.

$$\langle \phi_1(x_1) \cdots \phi_n(x_n) \rangle = \langle \pi_f \phi_1(x_1) \cdots \pi_f \phi_n(x_n) \rangle \tag{2}$$

Where in Eq(2), f is an element of conformal group and π_f is the group representation. While it is hard to analytically solve higher dimensional CFT's, in $(1 + 1)$ dimensions there are many examples which are exactly solavable [18]. That is why taking AdS_3 make sense because the corresponding CFT is in $(1 + 1)$ dimensions.

Now coming to the AdS/CFT correspondence [2], It is a connection between AdS space and CFT that is living on it's boundary. The space is referred to as the *bulk* and the latter is called the *boundary*. The AdS/CFT mapping usually relates fields ϕ in AdS space i.e in the bulk, to the operators $\mathcal{O}$ in the boundary CFT. The dynamics for both the theories coincide with each other in the language of partition function, which was first proposed by Witten [19].

$$Z_{bulk}[\phi] = Z_{boundary}[\mathcal{O}] \tag{3}$$

While this correspondence is still a conjecture, it has been a highly successful theory with applications from high energy to condensed matter physics. In past decades, a lot of amazing connections have been unearthed between quantum information and holography [20], which had a huge impact on quantum information theory. One such connection is development of *HECC*, which apart from being a new type QEC code, used for error correction in quantum computers. It also aims reproduce some key aspect of AdS/CFT through a different lens.

2.2 Quantum Error Correction

Any type of system that aims to store, transmit or compute information needs error correction. In case of classical systems, these errors can occur due to the corruption of bits on the storage devices or due to presence of noise while transmitting the information. As errors are bound to happen and we can't make a system without errors, it means there is a need for encoding of information such that with small errors, the recovery of logical data can take place. The recovery of logical data can simply be done by storing and transmitting multiple copies of data, via what came to be known as *repetition codes*. These classical codes are often denoted as $[n, k, d]$, which means encoding of k logical bits in n physical ones, with d Hamming distance. Well, classical codes required for modern systems are much more complicated than these. Classical coding theory is a huge field and is not our focus here, our aim is to explore error correction in quantum systems.

The manifestation of errors in quantum systems is fundamentally different from classical systems. Quantum systems when interacting with it's environment are prone to a phenomena, called *decoherence*. This decoherence ruins the special properties these system, specifically superposition and entanglement. These two properties are a necessary resource in quantum computation.

The usual approach of duplicating information, as used in classical error correction is not useful. We can't duplicate quantum state because of *no-cloning theorem* [21, 22], which states that there is no unitary operator (no physical time evolution) that can make an exact copy of quantum states. This raises a lot of problems in using repetition approach, in order to counter this, people created different approaches to QEC. We are going to use one of the most famous approach first developed by Daniel Gottesman in his PhD thesis [7], called *stabilizer codes*. This was based on the earlier works of Shor and Steane [23, 24].

The basic idea of the stabilizer codes is that the encoding of information can be represented in the ground state of the Hamiltonian. These are given by sum of operators S_i, often called *generators* of stabilizer.

$$H_S = -\sum_{i=1}^{m} S_i \tag{4}$$

$$S = \{S_1, S_2, \dots, S_m\} \tag{5}$$

These generators commute with each other. For a qubit, these are chosen as a tensor product of Pauli $(\sigma_x, \sigma_y, \sigma_z)$ and Identity (I) operators. We can also use these operators as a basis set to represent local errors. The notation of stabilizer codes is inspired from classical codes. We denote them as $[[n, k, d]]$, where n is the physical qubit, k is the logical qubit, however d is not exactly the hamming distance, it has a more nuanced meaning [25].

One of the stabilizers codes that would be useful in our work is called the "five-qubit code" or $[[5, 1, 3]]$ code [15, 7]. This code can correct a Pauli type error on one logical qubit. This condition is put up by *quantum hamming bound* [25], which says that for one logical qubit there is requirement of 5 or more physical qubits. This code is built from stabilizers

$$S_5 = \{\sigma_x\sigma_z\sigma_z\sigma_x I, \quad I\sigma_x\sigma_z\sigma_z\sigma_x, \quad \sigma_x I\sigma_x\sigma_z\sigma_z, \quad \sigma_z\sigma_x I\sigma_x\sigma_z\} \tag{6}$$

we can see that these generators are cyclic permutation of one another, and multiplying all will give the fifth missing generator $\sigma_z\sigma_z\sigma_x I\sigma_x$. This code ideal as it exactly meet the criteria for quantum hamming bound [26] and quantum singleton bound [7]. It follows from reconstructability conditions after erasure [27] and is given by

$$n \geq 2(d-1) + k \tag{7}$$

for an $[[n, k, d]]$ code. There are many other classes of stabilizer codes such as *Calderbank–Shor–Steane* (CSS)[24, 28], these were the first codes to be realized experimentally. Stabilizer based topological codes like *surface code*[29] and *color code* [30], have gained much traction in recent years for error correction in large systems. For holographic realization of quantum error correction we will stick to the *five qubit code*, as this is the one which would be crucial for the study of HAPPY code.

2.3 Tensor Networks

The field of Tensor Networks (TNs) has established many connections in different areas of physics, one common area is use of TN in theoretical and computational study of quantum many body systems. Th other slightly surprising connection it drew is it's relation to holographic principle and AdS/CFT correspondence [3]. I aim to describe the basic structure of tensor networks and briefly mention which TNs capture holographic properties.

Hilbert spaces suffers through a curse of dimensionality. A lot of problems on the classical side are solved with efficient analytical and numerical methods, because these problems could be reduced to some small parameter space. In case of quantum mechanical problems, that is not the case. Beyond certain perturbative methods, not a lot is exactly solvable.

We can look at simple quantum system to see how this Hilbert space becomes a problem. Let's assume a system with N degree of freedom, each degree corresponding to M state. For such a system we can define a pure quantum state as

$$|\psi\rangle = \sum_{k_1,k_2,\dots,k_N}^{M} T_{k_1,k_2,\dots,k_N} |k_1, k_2, \dots, k_N\rangle \tag{8}$$

where the state $|\psi\rangle$ is expressed by M^N amplitudes, $T_{k_1,k_2,\dots,k_N} \in C$. Now, let say our system is spin chain($M = 2$) of 40 sites, then we would need exponential memory to store 2^{40} complex numbers. This is roughly on scale of how much data LHC produces every year (definitely less!). Now even if our operations in algorithms are linearly scaling, computationally tacking a simple system of spin chain with just 40 sites becomes a large resource drain. We can avoid this problem by simply ignoring the states in the Hilbert space which are not relevant to our problem. Let's say we have identified a set of states with properties, which are of interest to us. How can we restrict the Hilbert space? We need a description that captures the entanglement between our set of states and discard the long range entanglement? Well, this description is effectively captured by the framework of TN, which is an ansatz for the state amplitudes $T_{k_1,k_2,\dots,k_N}$ in terms of contraction of multiple tensors[15].

As the name goes, a tensor network can be represented as a graph, with nodes as tensors and edges as indices. We can define a *tensor*, as a multidimensional array of complex numbers. Well, they are not

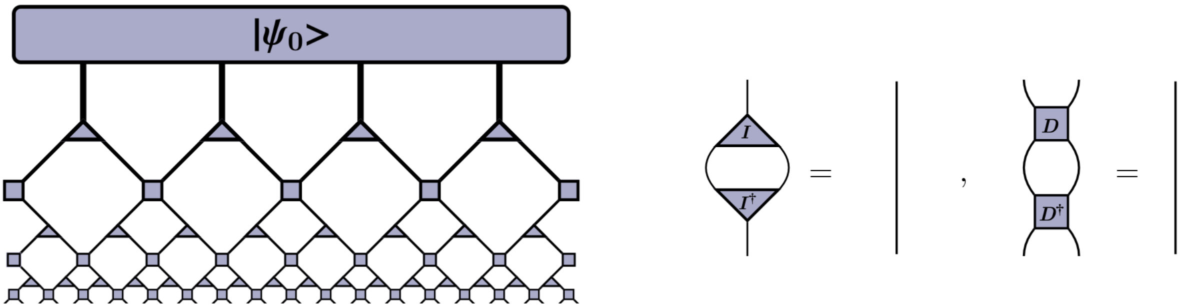
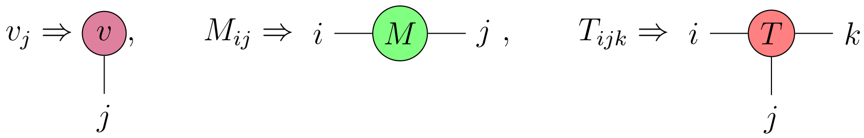


Figure 2: MERA network composed of isometries and disentanglers. [15]

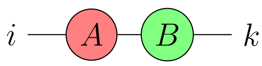
just that, they have a much deeper meaning, but for simplicity we'll say they are an array. The number of indices is the *rank* of the tensor. Below we represent tensors of various ranks in TN notation[10].



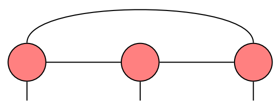
(9)

One of the key property to keep in mind while dealing TNs is *index contraction*, which is sum of all possible values of the repeated indices of a set of tensors, as given in Eq(10). Using our above example and setting value of $N = 3$, we would get T_{k_1,k_2,k_3} and it could be decomposed as a sum of three different rank 3 tensors. We can make a TN notation of the same as shown in Eq.(11).

$$C_{ik} = \sum_{j=1}^{n} A_{ij} B_{jk} \Rightarrow \tag{10}$$



$$T_{k_1,k_2,k_3} = \sum_{j_1,j_2,j_3}^{\chi_j} U_{k_1,j_3,j_1} V_{k_2,j_1,j_2} W_{k_3,j_2,j_3} \quad \Rightarrow \tag{11}$$



let χ_k and χ_j be dimension of k and j respectively (which is basically 2). Right now we can see that use of TN for $N = 3$, is not very useful as the number of coefficients expressed on on the right side (χ_k^3) are of the same dimension as on the left$(3\chi_k\chi_j^2\ )$. So there is no reduction in number of coefficient to describe a state. But as soon as we generalize the the Eq.(11) to N sites, we can immediately see a drastic reduction in the number coefficient $(\chi_k^N \to N\chi_k\chi_j^2)$ and through this generalization we get a new class of TN called *Matrix Product State* (MPS). These MPS are mainly used to to describe ground state of gapped one-dimensional hamiltonians [31] and many other applications where the complexity of entanglement is much simpler.

Similarly, what sort of TN geometries would we need, if we wish to capture more complicated entanglement? The answer was provided by Vidal [32], who formulated another class of TNs called the *multi-scale entanglement renormalization ansatz* (MERA), as seen in Figure(2) it is made out of two types of tensors called directional isometries (triangles) and unitary disentangler (square), arranged in a tree like structure. These tensors have special properties which makes it useful to capture complicated entanglement much effectively. MERA networks also have a special property called *entanglement renormalization* [33] which coverts a fine grained state to a coarse grain state and vice versa. Each layer in this network therefore has an energy/length scale which creates entanglement in the desired output state. We'll see in our project that MERA some properties which are very similar to that of AdS/CFT correspondence [3]. There is deep connection between hyperbolic geometries and MERA, both featuring in AdS/CFT like properties in terms of causal structure [34].The main motivation of this section has really been to briefly describe the usefulness of tensor networks as tool in studying and modelling properties of AdS/CFT, which what a good portion of our project would focus on.

2.4 Holographic Error Correction Codes (HECC)

These HECC are a new class quantum error correcting codes, which captures the properties of holographic principle, specifically AdS/CFT correspondence. These models, in the language of QEC were developed less than a decade ago. They are a new, state of the art framework for realizing the emergence of bulk phenomena from the boundary phenomena. The two types of codes I wish to focus in this section are the *Three Qutrit Code* [5] and the *HaPPY code*[6].

2.4.1 Three Qutrit Code

The qutrit code developed by Almheiri and Harlow [5], is a minimal model of holography in terms of error correcting code. To define it more succinctly, qutrit code is a quantum error correction code that protects any single "logical" qutrit state by encoding it into a system three "physical" qutrits. Any logical state can be written into three dimensional code subspace of physical states, as shown in Eq(12). The basis states $|\widetilde{i}\rangle$ are given by Eq(13). We can see that the subspace is in cyclic permutation of physical qutrits.

$$|\psi\rangle = \sum_{i=0}^{2} C_i|i\rangle \quad \Rightarrow \quad |\widetilde{\psi}\rangle = \sum_{i=0}^{2} C_i|\widetilde{i}\rangle \tag{12}$$

$$\begin{aligned} |\widetilde{0}\rangle &= \frac{1}{\sqrt{3}}(|000\rangle + |111\rangle + |222\rangle) \\ |\widetilde{1}\rangle &= \frac{1}{\sqrt{3}}(|012\rangle + |120\rangle + |201\rangle) \\ |\widetilde{2}\rangle &= \frac{1}{\sqrt{3}}(|021\rangle + |102\rangle + |210\rangle) \end{aligned} \tag{13}$$

The error correcting properties comes from Eq(14), for now we will take this as a fact and not go into too much detail [35]. From this we can say that for any logical state $|\psi\rangle$, we can unitarily encode it via Eq(14) without violating the no cloning theorem.

$$|\widetilde{i}\rangle = U_{12}\left(|i\rangle_1 \otimes |\chi\rangle_{23}\right) \quad \Rightarrow \quad |\widetilde{\psi}\rangle = U_{12}\left(|\psi\rangle_1 \otimes |\chi\rangle_{23}\right) \tag{14}$$

$$U_{12}^{\dagger}|\widetilde{\psi}\rangle = |\psi\rangle_1 \otimes |\chi\rangle_{23} \tag{15}$$

Now if I want to send you a qutrit state $|\psi\rangle$, I can do that by sending you one particle (spin 1 particle) in that state, but it may get lost. But if I were to send you three particles in $|\widetilde{\psi}\rangle$ by acting U_{12}. But somehow you only receive two of the three particles, you can still take those two qutrits and apply $U_{12}^{\dagger}$ as given in Eq(15) and you can find your first qutrit state, which was sent to you. This is how we can recover quantum state even after loss of a qutrit.

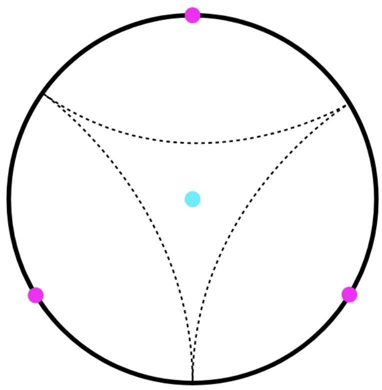


Figure 3: One bulk qutrit is encoded into 3 boundary qutrits [35].

The holographic interpretation of this comes, if we say that logical qutrit state is identified as bulk and 3 physical qutrit states identify as boundary regions. This could be shown visually in Figure(3). Similarly, you can reconstruct a bulk field (blue), with any two boundary regions(pink). This was one of the first discrete toy model for holographic codes proposed in [5]. This work was further extended in [36]. This particular holographic code became an inspiration for the HaPPY code [6].

2.4.2 HaPPY Code

The HaPPY code was proposed by its authors (Harlow, Pastawski, Preshkill and Yoshida) in 2015 [6], the code is based on their names. Alternatively, it's called hyperbolic pentagon code (HyPeC). It is a type of holographic tensor network code, which is constructed by combining MERA tensor network [32] with the [[5,1,3]] (five qubit code) [7]. The main idea of this code is to capture properties of bulk/boundary correspondence. One such property it capture's is called Ryu-Takayanagi formula for entanglement entropy [4, 37].

The key component in construction of HaPPY code is the perfect tensors, these are special tensors that form ismometries from any subset of indices to the complement subset (condition: that number of indices in the subset does not exceed half the total indices). The five qubit code we discussed in section 2.2 can be represented as a six-leg perfect tensor, where there is one logical (bulk) index and five physical (boundary) index [6]. These tensors are arranged in a pentagon tiling of the hyperbolic disk. The structure resembles the MERA tensor network geometry.

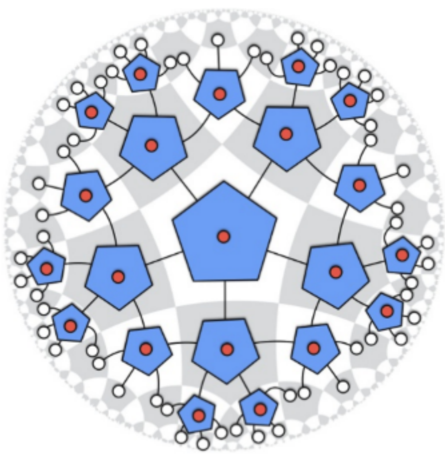


Figure 4: Hyperbolic pentagon tiling of perfect tensors corresponding to encoding isometry of [[5, 1, 3]] quantum error-correcting code [6].

The bulk geometry corresponds to the tensor network structure, where logical (bulk) indices corresponds to the bulk degree of freedom. Similarly, boundary regions corresponds open tensor indices and there physical (boundary) indices corresponds to boundary degree of freedoms. The entanglement between the bulk indices via tensor network give rise to encoding map between from bulk to boundary. This tensor network could be thought of as a MERA coarse graining structure [32] that maps bulk states to boundary states. These codes exhibits Ryu-Takayanagi's entanglement entropy [4], which is a generalization of Eq(1) and states that the entanglement entropy of boundary region is proportional to the area of the minimal surface γ_A in the bulk, given by Eq(16). This captures essential feature of holographic entanglement entropy in AdS/CFT.

$$S_A = \frac{|\gamma_A|}{4G}, \tag{16}$$

While the HaPPY code is a simplified toy model, it represents a significant development in the understanding of holographic duality from the perspective of quantum information theory.It provides a concrete realization of key holographic properties using well understood quantum error correction codes and tensor networks. Along with subsequent development these codes will turn out to be an absolute tool to gauge fundamental physics.

3 Methodology and Implementation Details

3.1 Logistics

Any numerical simulation within this project will use Python. Most likely it would be simulate tensor networks, in order to do so we would be using packages like 'TensorNetwork' and 'TenPy'. 'TenPy' is used for the simulation of quantum systems with tensor networks and 'TensorNetwork' is python library for easy and efficient contraction of tensor networks, supporting multiple powerful backends.

Creating and maintaining a detailed project dairy would be very essential. To have a single document with all the details of project, its progress and further plans, ensures project is structured and organized. Organization is one critical thing that is very important due to analytical nature of the project. While writing the dissertation, a huge amount of information would need to be accumulated, thus keeping careful notes of the progress will ease the process at the end.

3.2 Timeline

This project officially begins on 17th June 2023, All the aspects of the project must be completed in a sequence, as every objective builds a base for the later objectives. Due to a tight constraint on time, wherever possible we have arranged the timeline in such a way that the multiple objectives could be achieved simultaneously. We have established a provision to drop objective 2.3 and spend that time on objective 2.2, if necessary. Also, if there are any delays in accomplishing any previous objectives, we may have to choose either objective 3.1 or 3.2, in order to timely finish the project.

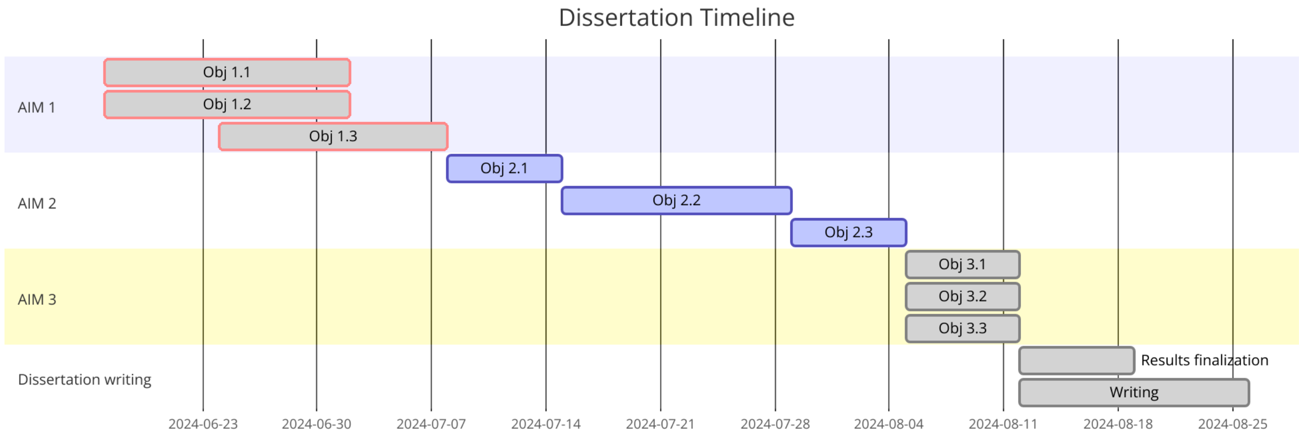


Figure 5: Gantt chart detailing the project timeline - each objective starts on Monday and ends on Sunday night.

3.3 Methodology

3.3.1 Aim 1

The goal of AIM 1 is to get acquainted to the background knowledge required to undertake this project, the aim is to develop operational efficiency in QEC and tensor network methods. Three week's time has been allotted for this, from week 1–3 (17/06/2024 – 08/07/2024).

- **Objective 1.1:** Creating an operational efficiency in QEC would first require revision of basics of quantum information, this would be done through review of chapter 1, 2 and 3 of Preshkill's lecture notes [38] or chapter 2, 8 and 9 of Nielsen's book [39], whichever treatment of material suits better. Once basic principles of quantum information are reviewed then a good place to start for QEC would be chapter 10 (typically section 10.1 to 10.5) of Nielsen's book [39], in which section 10.5 covers stabilizer formalism. If more time is needed on general formalism then could refer section 7 of [8]. If not, then we can straight move to *five qubit code* [7], which would later be used in **Objective 2.2** to construct HaPPY code. This objective 1 would end with creating summaries of the material learned, which would later help in writing the dissertation. **Objective 1.1** would typically go in parallel with **Objective 1.2** and needs to be completed by week 2.

- **Objective 1.2 and 1.3:** Using references [9, 10], we learn and create mathematical framework, which would include of tensors and tensor operation such as trace, index contraction and ideas about entanglement entropy and how they are captured by tensor networks. Then we will describe MPS tensor networks, through which we would get the idea of tensor contraction on an relatively simpler tensor network. Then our main focus would be on MERA networks, and their construction which would be achieved via study of Vidal's seminal paper [32], as these are the ones which captures the holographic properties mentioned in [3]. The **Objective 1.2** would naturally lead to **Objective 1.3**, where we would look into relationship between entanglement and geometry via this MERA network, and then connect those relations to AdS/CFT correspondence by the approach mentioned in [3]. These MERA networks along with five qubit code would be used in **Objective 2.2** to recreate the HaPPY code. **Objective 1.3** has been assigned week 3 for it's completion.

3.3.2 Aim 2

Aim 2 will gain insight into design and functionality of HECC, this would be done by re-deriving results from [5] and creating the HaPPY code as mentioned in paper [6]. Three weeks are set aside for this part of the project from week 4–7 (08/07/2024 – 05/08/2024)

- **Objective 2.1:** the general methods developed in **Objective 1.1** will be used to produce an error correction scheme applied to a qutrit (3–level system). This would be done by encoding a single logical qutrit state into 3 physical qutrit state. This recovery of logical state from the physical state after erasure should resemble the recovery of bulk information from the boundary information. It should take one week to reproduce, summarise and document these results. This objective spans over the week 4 of the project.

- **Objective 2.2:** Finally the skills developed in **Objective 1.1**, **Objective 1.2** and **Objective 1.3** will be used, all together. This objective deals with recreating the HaPPY code [6] from scratch. This would be done by encoding logical (bulk) qubits into physical (boundary) qubits, the encoding will be done through a complicated tensor. Now this complicated tensor would be simplified using the tensor network, this network would be MERA with perfect tensors in it. To build HaPPY code, perfect tensors for five qubit code are ideal. The HaPPY code encoding map would then be constructed by process of hyperbolic tessellation. This code is said to capture general properties of emergent bulk phenomena. We will be writing a general process of creating such a code. Week 5–6 are allotted to this section project. Though depending on the progress we can also utilize some days of week 7 as well.

- **Objective 2.3** We will be writing a code using Python and TenPy library to simulate a section (pentagonal tile of five tensor nodes) and check whether it is feasible to simulate the whole structure. Though this is an optional objective and can be dropped if there are time constraints. Week 7 has been kept for this but can be used for **Objective 2.2**.

3.3.3 Aim 3

After understanding two important model of HECC, our third aim is to find out applications of HECC in quantum error correction and fundamental physics. This would be done by reviewing literature related to these fields, in case of fundamental physics the search would focus on topic of emergent spacetime. On error correction front the search would focus on error correcting capabilities of the HECC. Week 8 (05/08/2024 – 12/08/2024) has been assigned. Though work for this aim could be started from week 7, if objective 2.3 is dropped.

- **Objective 3.1 and 3.2 :** To address the idea of emergent spacetime using HECC, we would do this by looking through the literature and using the techniques learnt in the process of achieving **Objective 2.2** to make an mathematical argument as to why this could be true. Similarly for **Objective 3.2**, we will look at the error correcting capabilities these codes by analysing the fidelity rates and threshold measures and documenting the limitations of such codes. But it would depend on the time, as we only have 11 week to finish the project we may have to choose between **Objective 3.1** and **Objective 3.2**.

- **Objective 3.3:** We achieve this objective by writing down comparison between different types of HECC and drawing distinction based on their design, error correcting capabilities and usefulness as a toy model. This could also be done after week 8, as this documentation could directly become a part of the dissertation.

3.4 Risk Identification and Management

The main resource to manage throughout the project is time. Even though the aims and objectives have been established and put in a sequential manner, being a completely theoretical project it is extremely difficult to estimate time for each step and process. In order to address this risk, we have planned several objectives in parallel. In cases where this is not possible the, lack of time management would lead to delay in individual tasks and overall project. As such, managing time would be crucial and therefore amount of time spent would need to be carefully planned.

The primary risk associated with aim 1 is the complexity of abstract concepts of QEC and TNs, these may require significant effort. This could lead to delays in our timeline. To mitigate this risk, during the weeks assigned, I plan to allocate sufficient time to deeply go through the cited sources. Also weekly meetings will be organised to keep the progress in check and any issues identified are addressed immediately.

For aim 2, the main risk is difficulty in deriving results from [5, 6]. These papers are technically very dense and may require significant amount of time and effort to reproduce the results. To mitigate this risk, I'll collaborate with the supervisor closely and break the papers into smaller sections and go through each section step by step. Also seek help from other experts to ensure all the difficulties are solved. Another risk associated with objective 2.3 is the computational complexity for simulating HaPPY code, even for a single section. To counter this, we can create smaller structures first and then scale the complexity gradually.

The main risk for aim 3 is misinterpretation or over-interpretation of results. This is a state of the art, very new framework so it is important to remain grounded in the capabilities of these codes. To manage this risk, we will carefully validate the all the claims of the theory with available evidence and seek guidance from the experts in the field. We will also acknowledge any assumptions, limitations before making any assertion in regards to the capabilities of these codes. The other risk that there is potential for comparative analysis to be too broad and superficial. To counter this we will select a set of established codes focusing on properties and variations of interest, making sure each code is analyzed based on well defined metrics.

The nature of this project is highly exploratory, because the field itself is roughly a decade old, the resources are fairly limited. So there is an inherent risk in the entire project specially with aim 3, that our arguments for objective 3.1, 3.2 and 3.3 may simply not workout or may not be in agreement with current literature. In that case, this project would largely lead to literature review, connecting and summarising what is known in the field. Though trying to implement mathematically motivated arguments to a tough problem, still contains some value, even if it is wrong. By identifying and actively managing the risks, we aim for a successful completion of the project.

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