Holographic Quantum Error Correction

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Abstract

This article presents a review examining the relationship between holography and quantum information theory. This is done by using tools from quantum error correction and tensor networks to build toy models called holographic quantum error correction codes, which replicate some of the important properties of AdS/CFT conjecture. Specifically, we describe the puzzle of radial locality in AdS/CFT and then try to resolve that by the construction of three qutrit code. We then use this three qutrit model as an inspiration to build a more complex model which has more structure to it, we describe the construction and properties of HaPPY code in detail. And then we finish the paper by shedding light on some of the possible extension to the HaPPY code and recent explorations of other properties of these models.

I. Introduction

The discovery of *holographic principle* [1] more specifically *AdS/CFT correspondence* [2] changed the way we tackle the problem of quantum gravity. The extensive ongoing research in holographic duality has led to a bold claim, which is that the geometry of spacetime itself emerges from quantum entanglement in a lower dimensional theory.

The origins of holographic principle could be traced back to 1970's when Hawking and Bekenstein realized that black holes are thermodynamical objects with temperature and entropy [3, 4], they described quantities like *Hawking temperature* and *Bekenstein-Hawking entropy* of a black hole [1]. When the formula for entropy expressed in natural units it takes the form $S_{BH} = A_{\text{hor}} /4G$, where, A_{hor} is the area of event horizon of the black hole and G is gravitational constant. This equation suggests that the entropy of the black hole increases with its surface area rather than its volume, which was counter intuitive at the time. This led to very bold suggestion that information of black hole's microstates are holographically encoded on its horizon. Based on this observation, Susskind and 't Hooft concluded that that a theory of quantum gravity would need to follow a holographic principle, meaning the theory of quantum gravity in $(D + 1)$ dimension can be reduced to an effective D dimensional theory without gravity [1, 5]. Even though promising holographic principle was still vague, it was until Maldacena in 1997 showed that this principle can be realized in the string theory setup [2]. He conjectured that there is a correspondence between observales in *anti-de sitter* space and operators in *conformal field theory* (a quantum field theory without gravity), this is what we know today as *AdS/CFT correspondence*. This correspondence had a huge impact on the field of high energy physics and completely changed string theory.

Now, AdS/CFT correspondence is a connection, a duality between the AdS space and a CFT living on its boundary. For our use throughout the article we are going to refer AdS space as the *bulk* and CFT as the *boundary.* The AdS/CFT mapping relates the bulk fields ϕ to the boundary operators O, as you would see in a little while. Even though it is a conjecture, it still has been very successfully in finding application in high energy physics and surprisingly in condensed matter physics.

In the past few years amazing connections have been found between holography and quantum information theory, which has had a huge impact in tackling some of the problems in holography. One such tool that has made a surprising connection is quantum error correction, originally used to make quantum computers fault tolerant [6]. The encoding of logical information in physical qubits in order to protect the information from erasures and also not violating the no cloning theorem [7] has turned out to be suitable framework for bulk reconstruction in AdS/CFT[2] as we would see in section three.

Another surprising tool that has made connection with AdS/CFT and holography is *tensor networks.* Tensor networks are used in theoretical and computational study of quantum many-body systems. These are used to capture entanglement structure of quantum systems. Vidal in [8] formulated a class of tensor networks called the *multi-scale entanglement renormalization ansatz* (MERA), which is made out of two types of tensors called directional isometries (triangles) and unitary disentangler (square), arranged in a tree like structure. These tensor networks capture more complicated entanglement structure more effectively. MERA networks also have a special property

[1]The explicit formula can be given by $T_H = \frac{\hbar c^3}{8\pi k_B GM}$ and $S_{BH} = \frac{4GM^2}{\hbar c} = \frac{c^3 A_{hor}}{4\hbar G}$

[2]logical state/qubit → bulk degrees of freedom
physical state/qubit → boundary degress of freedom

called entanglement renormalization [9] which converts a fine grained state to a coarse grain state and vice versa. Later Swingle in [10] established a direct relation between holography and MERA using entanglement renormalization as the basis. We will see in section four that how these tensor networks are built using perfect tensors and would be used as a mapping between the bulk and the boundary.

This article covers a new class of toy models called *holographic quantum error correction codes* (HECC). These are used to understand and resolve some of the key features of AdS/CFT correspondence from a quantum information lens. The holographic codes like the *three qutrit code* [11] act as minimal model of holography in the form of quantum error correction, showcasing how the bulk operators are encoded as boundary operators in a way it mimics local error correction (protect bulk from the local boundary erasures). Similarly taking the idea one step forward we get holographic codes like *HaPPY code* [12] which uses tensor networks as a map from the bulk to the boundary. These models have establish a deep connection between geometry and entanglement[10]. Then there are generalization based on the choice of tensors, tiling's and geometry of the space. These toy models could help us better tackle problems in quantum gravity like the emergence of spacetime and black-hole physics, not only that these codes could also be used build better quantum error correction protocols for fault tolerant quantum computation.

II. AdS/CFT Puzzle

The AdS/CFT correspondence [2] has been one of the best example to study the problem of quantum gravity. It has also resolved many of the puzzles of quantum gravity and is of great practical importance for studying strongly interacting QFT's.

Though one of the key problem in AdS/CFT which is still somewhat mysterious is the emergence of bulk locality. In the AdS/CFT Dictionary we have a simple relation [13, 14]

$$\lim_{r\to\infty} r^{\Delta}\phi(r,x) = \mathcal{O}(x) \tag{1}$$

between the bulk field ϕ and the CFT operator O. Here $\phi(r,x)$ is the bulk field in AdS space, where r is the radial coordinate and as r increases (moving radially outward) you get closer to the boundary and decreasing r means moving to the center of the bulk, x is the boundary coordinate describing locations on the boundary of AdS space, this is where the CFT lives. $O(x)$ is the local operator on the boundary CFT which represents some physical observable. The limit $r \to \infty$ is very crucial as this establishes the link between the bulk and the boundary, as r goes to infinity we are approaching the boundary. The scaling factor r^{Δ} makes sure that when we take the limit, the relationship between the bulk field and boundary operator is scaled correctly [11]. This equation is important because it expresses a holographic duality between bulk and boundary.

Normally our standard day to day physics is local in nature, meaning fields in one region of space interact only with fields that are nearby, thus preserving locality. Locality in ordinary QFT means operators which are spacelike separated[3] should commute, this is a very standard feature of local QFT. But this is more complicated in AdS/CFT as there is violation of radial locality. To put this problem in a more mathematical way as mentioned in [11], let's say we have bulk operator $\phi(x)$ in the middle of bulk and a boundary operator $O(X)$ at the boundary in the same time slice. Then can we say

$$[\phi(x), O(X)] = 0? \tag{2}$$

In the bulk of AdS space, since $\phi(x)$ and $O(X)$ are spacelike separated, then according to principle of locality the commutator must indeed vanish and locality must be preserved. The problem occurs when we start looking at things from the CFT perspective. If a bulk operator $\phi(x)$ commutes with all the boundary operators $O(X)$, then according to the *time-slice axiom*, in quantum field theory if an operator commutes with all other local operators $O(x)$ then in same time-slice that commuting operator is proportional to identity. This means if Eq 2 is indeed zero, then $\phi(x)$ should be identity across that timeslice[4]. But now this creates a problem, how can $\phi(x)$ be something trivial (identity), if it is supposed to represent something meaningful in the bulk like a particle or field excitation. So this creates a serious problem, how these bulk operators represent something meaningful if they commute with every operator on the boundary?

Well according to AdS-Rindler reconstruction, we consider an operator $\phi(x)$ in the center of the bulk and we can split the boundary into three regions R_1, R_2 and R_3. We can see this in figure below

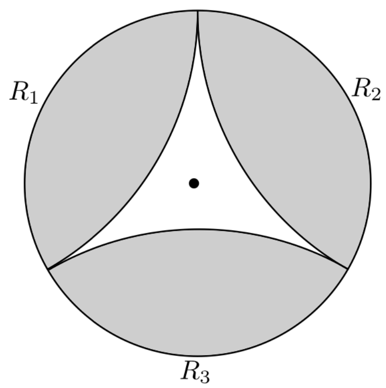

Figure 1: Redundant encoding of the bulk field using two boundary region. (Taken from [15])

Here the operator does not lie in $C[R_1], C[R_2]$, or $C[R_3]$, but it does lie in $C[R1 \cup R2], C[R1 \cup R3]$, and $C[R2 \cup$

[3] meaning causally disconnected
[4] Remember we are talking from CFT (boundary theory perspective)

$R3$]. Therefore, according to AdS-Rindler reconstruction the operator in the bulk can be represented via any two boundary regions [11, 15]. In order to solve these puzzles of locality it is believed that the condition of Eq 2 holds only for a subspace of states in the full CFT Hilbert space. This is where quantum error correction comes in play, which tries to proves the previous statement. We can see this explicitly in the next section.

III. Three Qutrit Code

This code was proposed as the first toy model of bulk/boundary correspondence in the language of quantum error correction[11]. At first it may seem very odd as QEC and AdS/CFT are two completely different fields of study with different applications. One is a conjecture tackling the problem of quantum gravity[2] whereas other is used to make fault tolerant quantum computers[16]. But as you would see with this example, this framework of QEC turned out to be quite valuable as it gave a fresh perspective on AdS/CFT and on the problem of bulk reconstruction[11].

Now, let's discuss this example in detail to see why methods from quantum information turn out to be a good tool in resolving some of the issues mentioned in the previous section [11] and also serve as a good motivation for construction of HaPPY code [12], which is a better generalization of HECC.

We will use our two friends Alice and Bob to demonstrate this error correction scheme, Alice wants to send a message to Bob using a qutrit state[5]

$$|\psi\rangle = \sum_{i=0}^{2} C_i |i\rangle \tag{3}$$

But Alice is concerned that along the way it can be lost or erased and the message might not reach Bob. If this were a classical state, we could have used a repetition code, where we send multiple copies of the same classical state. But in quantum mechanics, the no cloning theorem [7, 17] prevents us from creating copies of a quantum state. So, we need to find some way to send the information without violating the laws of quantum mechanics. Instead of sending $|\Psi\rangle$, Alice can send the message by encoding the single qutrit into three qutrits and we can write the encoded state as

$$|\widetilde{\psi}\rangle = \sum_{i=0}^{2} C_i |\widetilde{i}\rangle \tag{4}$$

where,

$$\begin{aligned} |\widetilde{0}\rangle &= \frac{1}{\sqrt{3}}(|000\rangle + |111\rangle + |222\rangle) \\ |\widetilde{1}\rangle &= \frac{1}{\sqrt{3}}(|012\rangle + |120\rangle + |201\rangle) \\ |\widetilde{2}\rangle &= \frac{1}{\sqrt{3}}(|021\rangle + |102\rangle + |210\rangle). \end{aligned} \tag{5}$$

Here, we are encoding the *logical* qutrit $|\widetilde{\Psi}\rangle$ into a larger system of three *physical* qutrits. $|\widetilde{0}\rangle, |\widetilde{1}\rangle$ and $|\widetilde{2}\rangle$ are the encoded/ logical basis states mapped to a superposition of three physical qutrits. The subspace spanned by Eq 5 is called the *code subspace.*

This code subspace is symmetric under cyclic permutation of physical qutrits and each basis state his highly entangled. The structure of these superpositions are very carefully chosen [11], such that the information about the original qutrit state is not localized in any single physical qutrit. Instead, it is spread out evenly among all three physical qutrits, this property of the code ensures robustness against the loss or corruption of any one qutrit. To show this let's analyze logical basis state $|\widetilde{1}\rangle$

$$|\widetilde{1}\rangle = \frac{1}{\sqrt{3}}(|012\rangle + |120\rangle + |201\rangle. \tag{6}$$

This state in Eq(6) represents the superposition of cyclic permutation of $|0\rangle, |1\rangle$ and $|2\rangle$, meaning if first qutrit is in $|0\rangle$ then second and third qutrit have to be in $|1\rangle$ and $|2\rangle$ respectively (shown in the first term of (6)). Same goes of the second and third term. This encoded state distributes the information in the three physical qutrits in cyclic manner.

Now, the claim to fame of this code is that it has an encoding map which acts only on the first two qutrit[6]. We can demonstrate that the error correcting properties of this code arises because each basis state can be represented as [15]

$$|\widetilde{i}\rangle = U_{12}\left(|i\rangle_1 \otimes |\chi\rangle_{23}\right), \tag{7}$$

here $|i\rangle_1$ is the first qutrit which can be in $|0\rangle, |1\rangle$ or $|2\rangle$ and $|\chi\rangle_{23}$ is a maximally entangled state of second and third qutrit given by

$$|\chi\rangle \equiv \frac{1}{\sqrt{3}}(|00\rangle + |11\rangle + |22\rangle), \tag{8}$$

and $|i\rangle_1 \otimes |\chi\rangle_{23}$ is a product state, where the first qutrit is not entangled with second and third. However, U_{12} is an unitary transformation acting on first and second qutrit as

$$\begin{array}{lll} |00\rangle \to |00\rangle & |11\rangle \to |20\rangle & |22\rangle \to |10\rangle \\ |01\rangle \to |11\rangle & |12\rangle \to |01\rangle & |20\rangle \to |21\rangle \\ |02\rangle \to |22\rangle & |10\rangle \to |12\rangle & |21\rangle \to |02\rangle \end{array}. \tag{9}$$

[5]'Qutrit' is a three level quantum system, like a spin-1 particle.

[6]The actual encoding map is tensor $U_{12} \otimes I_3$, where I is identity acting on the third qutrit, we have not written this in Eq 7, as to make the notation easier, but in a lot of literature it is implied (though some mathematicians might get offended as it messes up the mapping), but throughout this paper when we say $U_{12} \to U_{12} \otimes I_3$

Once we apply U_{12} to first and second qutrit as given in Eq 7, the product state gets transformed into fully entangled state of three qutrits, which is $|\widetilde{i}\rangle$. Given that we are able to do this for basis state, we can also unitarily encode the full logical state $|\widetilde{\Psi}\rangle$ as

$$|\widetilde{\Psi}\rangle = U_{12}\left(|\Psi\rangle_1 \otimes |\chi\rangle_{23}\right). \tag{10}$$

Preparing the encoded state this way ensures that the no cloning theorem is not violated[15]. Moreover, because of the cyclic symmetry of the code subspace we can also have encoding unitaries U_{23} and U_{13} with support on second and third qutrit or first and third qutrit repectively as

$$\begin{aligned}|\widetilde{\psi}\rangle &= U_{23}\left(|\psi\rangle_2 \otimes |\chi\rangle_{13}\right)\\ |\widetilde{\psi}\rangle &= U_{31}\left(|\psi\rangle_3 \otimes |\chi\rangle_{12}\right).\end{aligned} \tag{11}$$

Now say Alice instead of sending state $|\Psi\rangle$ (spin-one particle), but she sends three spin-one particles, which are prepared in advance in the state $|\widetilde{\Psi}\rangle$ by acting U_{12} as in Eq 10 in order to protect information. And that Bob receives only two out the three particles, say the first two qutrits[7]. Then Bob can take those two qutrits and use a quantum computer to act on them with $U_{12}^{\dagger}$, upon which he would then have

$$U_{12}^{\dagger}|\widetilde{\psi}\rangle = |\psi\rangle_1 \otimes |\chi\rangle_{23}, \tag{12}$$

which is a product state, with our original state $|\Psi\rangle$, which is what Alice wanted to send in the first place. Also, the same could be done using U_{23} or U_{31} if Bob had received the second and third or first and third qutrit respectively, this works again because of the cyclic symmetry of code subspace. Thus, Bob can correct the loss of any one qutrits. This is described as a quantum error correction code[8] which can protect against single qutrit erasures[18].

In our discussion of bulk reconstruction from the previous section we were interested in operators and their actions, instead of recovery of states[11]. We can rewrite this error correcting scheme in the language of operators. Already this error correction is somewhat similar to the idea of bulk (operator) reconstruction on any two of the three boundary subregion as described in the previous section. We can make this idea more concrete by introducing the idea of *logical operators*. We can describe logical operator as a linear map on the logical qutrit state as

$$O|i\rangle = \sum_j (O)_{ji}|j\rangle. \tag{13}$$

For any such operator O, there can be an encoded logical operator $\widetilde{O}$ which implements the same transformation on the code subspace [9] as

$$\begin{aligned}\widetilde{O}|\widetilde{i}\rangle &= \sum_j (O)_{ji}|\widetilde{j}\rangle\\ \widetilde{O}^{\dagger}|\widetilde{i}\rangle &= \sum_j (O)^*_{ij}|\widetilde{j}\rangle,\end{aligned} \tag{14}$$

however the action of $\widetilde{O}$ on the part of Hilbert space that is outside the code subspace[10] is not relevant to us [15]. For a general code subspace, $\widetilde{O}$ would have non-trivial support on all three qutrits, but for the subspace spanned by the Eq 5, we can construct an operator that has the same action on the code subspace that O has on the Hilbert space, but has non trivial support only on the first two qutrits [19]. We can say

$$O_{12} \equiv U_{12}^{\dagger} O U_{12}. \tag{15}$$

We can show the action of O_{12} on code subspace as

$$\begin{aligned}O_{12}|\tilde{i}\rangle &= U_{12}^{\dagger} O U_{12}|\tilde{i}\rangle\\ &= U_{12}^{\dagger} O|i\rangle \otimes \frac{1}{\sqrt{3}}(|00\rangle + |11\rangle + |22\rangle)\\ &= \sum_j (O)_{ji} U_{12}^{\dagger}|j\rangle \otimes \frac{1}{\sqrt{3}}(|00\rangle + |11\rangle + |22\rangle)\\ &= \sum_j (O)_{ji}|\tilde{j}\rangle.\end{aligned} \tag{16}$$

Therefore we have O_{12} as an logical operator which has support only on first two qutrits. We can also write this as

$$\begin{aligned}O_{12}|\widetilde{\psi}\rangle &= \widetilde{O}|\widetilde{\psi}\rangle\\ O_{12}^{\dagger}|\widetilde{\psi}\rangle &= \widetilde{O}^{\dagger}|\widetilde{\psi}\rangle.\end{aligned} \tag{17}$$

We can analogously create operators O_{13} and O_{23} which have support on two of the three qutrits but have the same action on the code subspace. This idea is reminiscent of the overlapping wedges from AdS-Rindler reconstruction.

Now we can make our connection between bulk reconstruction and this error correcting code more explicit. We can make the following statements [11, 15]:

- The logical qutrit in the center lattice site represents the bulk degree of freedom, we can say that logical operators are interpreted as bulk fields.
- The three lattice sites on the boundary are the three physical qutrits which are the local degrees of freedom of a boundary CFT.

[7] Here we know which qutrit was lost, infact it is important to know, as by this code we are correcting the *erasure channel* (in quantum information language.), which is a specific type of error.

[8] It is also very important to emphasize that the redundancy of the three qutrit code depends crucially on the entanglement in state $|\chi\rangle$[15].

[9] Simply, operator O acting on logical qutrit is translated to encoded operator $\widetilde{O}$ which acts on the physical qutrits. The encoded operators still leads to a transformation of the logical state, but now it does so in the larger space of physical qutrits.

[10] Physical/Hilbert space is 27 dimensional space of the 3 physical qutrits. The code subspace is a 3 dimensional subspace of the physical space, where the the encoded logical qutrit lives.

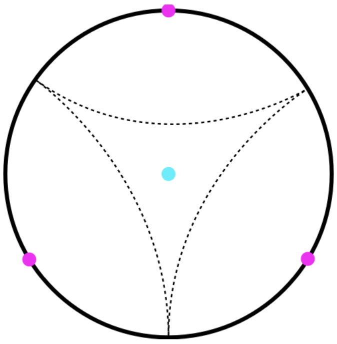

Figure 2: This image could essential be called the "three qutrit model of holography", where in the center you have the logical (bulk) qutrit encoded into three physical (boundary CFT) qutrits. (Taken from [15]).

The above statements can visually realized in figure 2. Stemming from our discussion of bulk reconstruction in the the previous section, where the we can say that the bulk qutrit is lying in the casual wedge of any two boundary qutrits and we can interpret the construction of O_{12}, O_{23} and O_{31} as the AdS-Rindler reconstruction. The main argument of the paper [11] is that this is more than an analogy, this is exactly how AdS/CFT is producing the bulk. The bulk logical operators encoded on the code subspace becomes better protected from the local boundary errors as we move radially inwards (towards the bulk) [11]. We can see this idea visually in figure below.

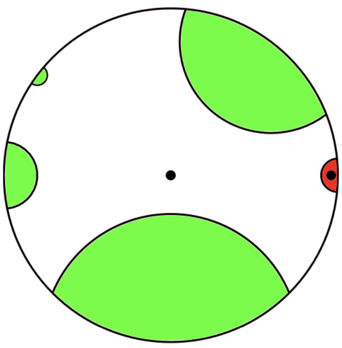

Figure 3: Bulk information in the center is better protected in CFT against the boundary erasure in any one of the green regions, but near the boundary in the red region the information is completely lost by a boundary erasure. (Taken from [11]).

One of the issues that this reformulation of AdS/CFT in terms QEC addresses is that the radial locality seems to hold in the following way: Let's consider an encoded logical operator $\widetilde{O}$ and an operator σ_3 acting on the third physical qutrit. Now for any two states $|\widetilde{\psi}\rangle$ and $|\widetilde{\phi}\rangle$ in the subspace, we have

$$\langle\widetilde{\phi}|\left[\widetilde{O},\sigma_3\right]|\widetilde{\psi}\rangle = \langle\widetilde{\phi}|\left[O_{12},\sigma_3\right]|\widetilde{\psi}\rangle = 0. \tag{18}$$

Same can be true for any other operator σ_2 or σ_1 acting on second and first qutrit respectively, as we can replace $\widetilde{O}$ with O_{31} or O_{23}. It is important to emphasize that these non trivial bulk operators can commute all the local operators in the boundary, but this is happening only between states in the *code subspace*, where there is no issue with the time-slice axiom.

According to the authors of [11], they have made compelling attempt at interpreting AdS-Rindler reconstruction as quantum error correction. But in order to do so they have relied on bulk physics and the assumption that the certain operators in CFT obey the bulk equation of motions on a subspace. Though this assumption is possible due to the large-N structure of the CFT [20], but it would still be nice if we could explicitly describe the structure of quantum error correcting code in the CFT. A good starting point in resolving this issue is looking at MERA tensor network construction which models a discrete version of AdS/CFT [8, 10]. As you would see in the next section it is possible to confirm the structure of quantum error correction using MERA networks in a controlled setting [12]. Three qutrit model of holography does fairly well as a minimal model. It is useful to probe such exciting ideas in simple setting, but it would nice to have a model with a bit more structure of AdS/CFT.

IV. HaPPY Code

AdS/CFT duality in recent times has shed light on a remarkable relationship between entanglement and geometry. This idea was first suggested by Maldcena in [21], where two entangled CFT's have a bulk dual connecting them through a wormhole. This notion was then quantified by Ryu and Takayanagi [22, 23], where they proposed that the entanglement entropy in the CFT can be computed by the area of a minimal surface in the bulk geometry. This Ryu-Takayanagi formula has led to better efforts in establishing more concrete and quantifiable connections between entanglement and geometry [24, 25, 26].

The idea of holography and particularly AdS/CFT from the get-go is described in a continuous setting. One of the motivation behind the development of holographic codes is to a obtain discrete formulation of AdS/CFT (at least some of the important properties of the conjecture). Before the advent of these HECC other discrete version of holography were studied. It was Swingle [10] who pointed out that some of the elements of AdS/CFT can be modelled by a MERA like tensor network and that these tensor networks provide a good blueprint for discretized holographic duality. This tensor network representation of the duality came with certain similarities such as RT formula and entanglement entropy of tensor networks more or less take the same form, but this representation did not reproduce many other holographic properties. It was Harlow and Almheiri [11] who pointed out that other properties such as emergence of bulk locality, more specifically causal wedge reconstruction can be interpreted as a quantum error correcting code. Interpreting bulk operators as logical operators on a certain subspace of states in the CFT, protects theses operators from boundary erasures, which is what we did in the previous section. It was further suggested by the authors that there could be a MERA like tensor networks which should be able to model these ideas inspired from [10, 27].

HaPPY code was therefore inspired by the [11] and was proposed as a family of exactly solvable toy models of the AdS/CFT correspondence based on novel tensor network construction of quantum error correction codes [12]. There have been many authors who have used ideas from holography [28, 29] and tensor networks [30, 31] to build quantum error correction codes, but all of them converge to the family of codes described by the HaPPY code [12]. In this section we will describe the construction of holographic states and codes, then will discuss the holographic properties of these states mainly the RT formula and AdS-Rindler reconstruction. We will close this section by discussing the error correction properties of these holographic codes.

IV.1. Construction of holographic states and codes

IV.1.1. Perfect tensors

An isometry from H_A to H_B is a linear map $T : H_A \to H_B$ with the property that it preserves the inner product. One of the main condition is T can exist only if their dimensionalities $\dim(A)$ and $\dim(B)$ obey $\dim(A) \leq \dim(B)$. If $\dim(A) = \dim(B)$, then T is an unitary. If T is an isometry then we can represent the map T as a tensor as

$$T : |a\rangle \mapsto \sum_b |b\rangle T_{ba} \tag{19}$$

where, $\{|a\rangle\}$ and $\{|b\rangle\}$ are orthonormal basis for for H_A and H_B respectively. given the above the equation, we can say an *isometric tensor* is the one which follows the condition

$$\sum_b T^\dagger_{a'b} T_{ba} = \delta_{a'a} \tag{20}$$

Now, we can define the *perfect tensor* (taken directly from [12]) as a $2n$-index tensor $T_{a1a2\ldots a2n}$ for any bipartition of its indices into a set A and complementary set A_c with $|A| \leq |A_c|$, T is proportional to an isometric tensor from A to A_c.

Perfect tensor can describe a pure state of $2n$ spins, it has a special property that any set of spin n is maximally entangled with the complimentary set of n spins, these are called AME states (absolutely maximal states) [12]. A perfect tensor can be regarded as the encoding map of a quantum error correction code, if it is a linear map from one spin to $2n-1$ spin which encodes single logical spin in a $2n-1$ spin block. Here logical spin spin is protected against the erasure of $n-1$ physical spins. Because n is more than half of the physical spins, therefore it is compatible with no cloning theorem and provides best protection against erasure. This is how perfect tensors could be used as an encoding map for quantum error correction.

IV.1.2. Holographic states and codes

These holographic states and codes are built by placing *perfect tensors* such that the tensor network is laid on a uniform tiling of hyperbolic space (*hyperbolic tessellation*). The reason behind this choice is if the tiling is extend to an infinite system, then the tensor network has no inherent directionality and all points in the bulk can be treated equally. Also these tilings have desirable symmetries for constructing a toy model for AdS/CFT correspondence[11] Now we can explicitly define Holographic states and codes as below:

- **Definition 1. Holographic State:** The state interpretation of a tensor network made up of perfect tensors which covers a geometric manifold with a boundary. All the interior legs are contracted. All the uncontracted legs are located at the boundary of the manifold.

In order to illustrate the idea of holographic state, the authors in [12] used hexagons on a uniform tiling of hyperbolic space, where at each vertex there are four hexagons as show in Fig 4. A six legged perfect tensor is placed at each hexagon and the legs on the interior are contracted with neighbouring tensors (at the shared edges of hexagons). All the uncontracted open legs are present at the boundary and we can associate physical spin to them. This tensor network represents a pure state of these boundary physical spins which we call a holographic state.

- **Definition 2. Holographic Codes:** It is a tensor network made up perfect tensors which cover some geometric manifold with boundaries. This tensor network is a holographic code if there exists an isometric map from uncontracted bulk legs to uncontracted boundary legs.

In the example of holographic codes we again take a uniform tiling of a hyperbolic space but this time with pentagons. Again there are four pentagons at each vertex and six legged perfect tensor is placed at each pentagon so that each tensor has one additional uncontracted open leg. These additional legs can be interpreted as the bulk /logical input of the tensor network. We can see these in Fig 5, where the red dots represent the bulk indices. This tensor network can be viewed as a big tensor with logical legs in the bulk and physical legs on the boundary.

We can explicitly say that this *pentagon tiling network is an isometric tensor from the bulk to the boundary and is called holographic pentagon code* [12]. This can be demonstrated by observing that each tensor has a maximum

[11] Particularly they are discretely scale invariant and also there exist graph isomorphisms that bring any point in the graph to the center while preserving the local structure of the tiling [12].

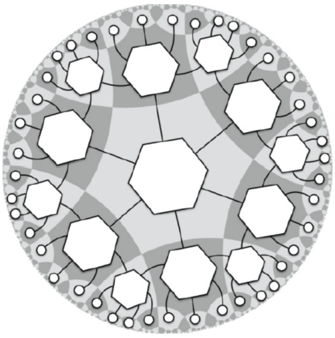

Figure 4: Representation of holographic state using hexagons, the white dots on the boundary are uncontracted open legs of the tensor network. (Taken from [12])

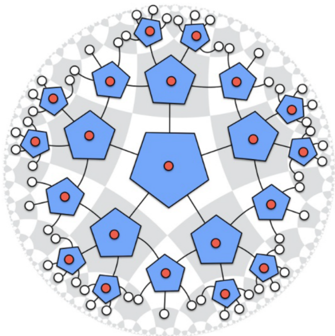

Figure 5: This is the holographic pentagon code, where the red dots are the bulk (logical) indicies and white dots are the boundary (physical) indicies. (Taken from [12])

of two legs contracted with the tensors at the preceding layer if the tensors are arranged into layers represented by increasing the graph's distance from the centre (this property is a consequence of the "negative curvature" of the graph). Consequently, we may consider each tensor as an isometry from input legs to output legs since, even if we consider the bulk logical index of the pentagon to be an input leg, there are only a maximum of three input legs overall and therefore we obtain an isometry mapping all of the logical indices in the bulk to the physical indices by layer-by-layer application of the perfect tensors and keeping in mind that the product of isometries is an isometry. This is how we create holographic codes, now lets move onto discuss some properties these codes.

IV.2. Holographic properties

IV.2.1. Ryu-Takayanagi Formula

Having defined holographic states and codes, we now want to know what holographic properties are contained in this holographic description and how much of these distinct attributes hold. We will first discuss the idea of how the RT-formula holds for the entropy of the tensor legs, this would be because these codes have MERA like structure. For an arbitrary tensor network with a cut c separating it into two tensors P and Q as shown in Fig 6, an explicit description of the reduced density matrix of a subsystem of uncontracted legs is achieved by tracing out the complementary subsystem.

$$|\psi\rangle = \sum_i |P_i\rangle_A \otimes |Q_i\rangle_{A^c} \tag{21}$$

$$\rho_A = \operatorname{tr}_{A^c} |\psi\rangle\langle\psi| = \sum_{i,j} \langle Q_j \mid Q_i\rangle |P_i\rangle \langle P_j| \tag{22}$$

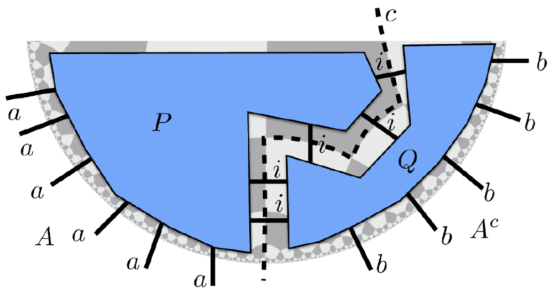

Figure 6: A tensor network divided by a cut c into two tensors P and Q. (Taken from [12])

We can bound the entropy of ρ_A as follows, if the tensor network is the one of a holographic state and such a network is made of contractions of a unique perfect tensor T with all of its legs of dimension v, such as the code illustrated in Fig 4. Then the entropy could be classified as [32]

$$S(\rho_A) = -\operatorname{tr}(\rho_A \log \rho_A) \leqslant \min \left\{ \sum_{b \in \tilde{\gamma}_A} \log v \right\} \tag{23}$$

where $\tilde{\gamma}_A$ is any cut separating A from A^C. Because every tensor leg has the same contribution to the entropy, the cut that minimizes this quantity is the *geodesic curve* γ_A. This is defined as the curve that separates A from A^c by cutting the minimum number of tensor legs. So this geometric object will bound the entropy of the subsystem as

$$S_A := S(\rho_A) \leq |\gamma_A| \log v \tag{24}$$

In order to saturate this bound the tensors P and Q must be isometries from the inside to the boundaries. The entropy bound would achieve equality because the implication of the given case is that $\{|P_i\rangle\}_i$ and $\{|Q_j\rangle\}_j$ are orthogonal. This bound's saturation holds when the tensor under consideration is a holographic state (over a non-positively curved manifold) [12]. Though is not easily apparent but by using the max-flow min-cut theorem, one can demonstrate this phenomenon. The satisfied equality

$$S_A = |\gamma_A| \log(\nu) \tag{25}$$

is thus a discrete version of RT formula. It relates a geometrical quantity $|\gamma_A|$, which is the number of tensor legs crossed by a minimal cut to the entropy of boundary subregion A.

There will also be uncontracted tensor legs in the bulk when a geodesic is used to cut a holographic code. Then, even though we ask holographic codes to be isometries from the bulk to the boundary, we are providing extra legs as an input while cutting the tensor network, so the remaining tensors P and Q, without any further constraint, won't necessarily be isometries. Therefore, we cannot be certain that the remaining map will be an isometry once more. The authors in [12] introduce the use of greedy algorithm to quantify the extent to which a holographic code's tensors P and Q break as isometries. The goal is to find the largest subset network that exists within P or Q, which is in fact an isometry.

We can now define the *greedy algorithm* [12], it is an algorithm that is used to get a maximal cut $\gamma_A^\star$ from a boundary region A, called the *greedy geodesic*, such that the tensors between the boundary A and the cut $\gamma_A^\star$ is an isometry. The steps of the algorithm (taken from [33]) are the following:

1. Set $\gamma_0 = A$, the cut that trivially cuts all tensors legs of A. Set the tensor P_0 as the identity map from A to A.
2. Identify if there is a perfect tensor T_i next to P_i such that it has half of its legs or more contracted with P_i. If not, go to step 4.
3. Set P_{i+1} as the contraction of P_i with T_i and define γ_{i+1} as the cut obtained by pushing γ_i through T_i as shown in Fig 6. Then, repeat step 2.
4. Return the greedy geodesic $\gamma_A^\star = \gamma_N$ where N is the last cycle of the algorithm.

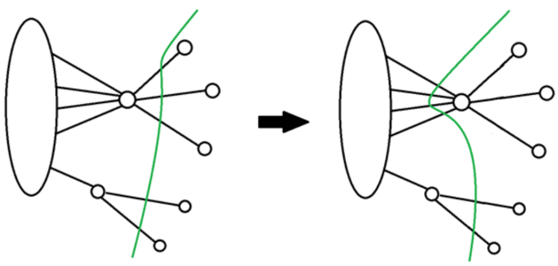

Figure 7: One iteration of the greedy algorithm passing though an admissible isometry. The green curve is the cut that will eventually converge to the greedy geodesic.

The greedy geodesic is well-defined, as clearly shown in [12], and the algorithm's convergence is not affected by the sequence in which the new perfect tensors on step 2 are selected. We can measure the degree to which the RT formula fails in holographic codes thanks to this process. When A is an arbitrary subregion of the boundary of holographic state or code. Then, there is a lower bound on the entropy of A

$$S_A \geq |\gamma_A^\star \cap \gamma_{A^c}^\star| \log(\nu) \tag{26}$$

where $|\gamma_A^\star \cap \gamma_{A^c}^\star|$ is the number of tensor legs cut by greedy geodesic of both A and A^c. We refer to the tensor between the two cuts as the "bipartite residual region", and the discrete RT equation also applies when it is empty. The good thing about this version of RT formula is that using discrete formulation we are able to recover properties that are valid for the continuous AdS/CFT, which make quantum error correction a good candidate for discretized holography.

IV.2.2. AdS-Rindler (Causal-Wedge) Reconstruction

The holographic states and codes can showcase another very important property of AdS/CFT correspondence, which is *causal wedge reconstruction.*

The set of bulk tensor legs that are between A and greedy geodesic $\gamma_A^\star$ is the causal wedge $C[A]$ of the connected boundary region A [33]. Similarly we can define the causal wedge for an arbitrary boundary region B such that $B = \bigcup_i B_i$ and B_i is connected, then $C[B] = \bigcup_i C[B_i]$.

According to the definition of greedy algorithm, we can construct any bulk local operator $\phi(x)$ that is present in the causal wedge of a connected boundary region A as a boundary operator on A. Let $P_{\gamma_A^\star}$ be an isometry which is enclosed by the greed geodesic $\gamma_A^\star$ and the boundary region A. Then the boundary operator that does the reconstruction of bulk operator $\phi(x)$ is $P_{\gamma_A^\star}[\phi(x)]$.

The quantum information perspective of AdS/CFT also take an attempt at the *entanglement wedge* conjecture, which says sometimes we can also reconstruct operators that are beyond the causal wedge of a boundary region. We can see this idea in Fig 8 , entanglement wedge is the region of A, where the minimal surface encloses a larger region when the size of A is more than half the boundary space.

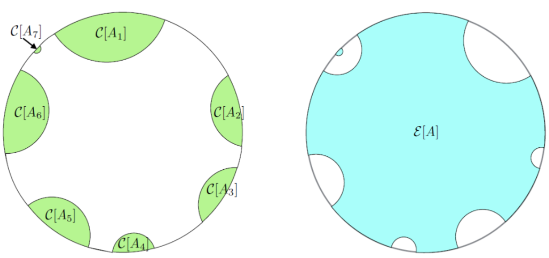

Figure 8: Causal wedge (green) $C[A]$ and entanglement wedge (blue) $E[A]$. [12]

We can describe this in terms of holographic codes. The bulk region enclosed between boundary A and greedy geodesic $\gamma_A^\star$, which we get when we simultaneously apply greedy algorithm to every connected component of boundary region A_i, this is defined as the entanglement wedge

$E[A]$ of boundary region $A = \bigcup_i A_i$. The information of operator on the entanglement wedge (outside causal wedge) would be encoded on the entanglement of A_i but on the individual A_i.

V. Black Holes and HECC

Bulk operators in holographic codes are only reconstructed on a subspace of the boundary Hilbert space. This can appear concerning because all conceivable states on the boundary should be assigned a bulk interpretation by the holographic correspondence. A solution to this puzzle was put out in [11]: for the majority of boundary states, a certain bulk operator may not always be reconstructable due to its deep location inside a black hole. In fact, if we include black holes in our models in the way we describe below, we can observe this immediately.

To demonstrate this idea we can take look at the pentagon code but without the central tensor. The free bulk index of the central tensor is replaced by five bulk indices, these were the ones which were contracted with the missing pentagon's legs. Now the tensor network transforms into an isometry which maps the five indices along with the remaining bulk to the boundary. Therefore the code subspace is larger for this boundary Hilbert space compared to the conventional pentagon code. This larger code subspace represent the configurations of the bulk with a black hole in the center. The microstate of this black hole is determined by the input to the new bulk legs. Larger black hole constructions are possible by removing more central layers and this construction can be viewed in Fig 9.

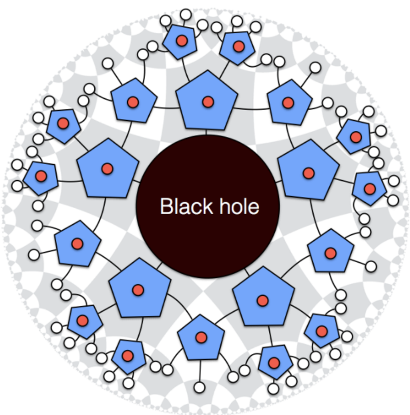

Figure 9: Black holes placed at the center of a holographic code. [12]

It is evident that the entropy of the black hole would scale proportional to the horizon area, this is what was predicted by Hawking and Bekenstein [3, 4]. From [12] we can say that the entropy of black hole is the logarithm of dimension of Hilbert space of microstates of black holes. There is decrease in the number of bulk legs outside the black hole as it grows, which leads to fewer bulk operator reconstruction. Eventual the whole tensor network would be swallowed by the black hole and the isometry would become unitary (trivial).

From the above discussion we can say that these models do assign bulk interpretation to all boundary states, which what's required by AdS/CFT, therefore most boundary states correspond to a large black hole in the bulk. We can also describe the some configurations of a two sided by wormhole by simply preparing tensor network with equal size central black holes and maximally entangle the bulk legs at their horizon[12]. This can be seen in Fig 10.

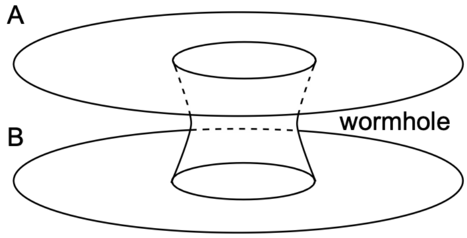

Figure 10: Wormhole geometry built using holographic code.

VI. Extension of HaPPY Code: Further Approaches

The HaPPY model is already quite versatile in itself and it captures a variety of holographic properties but it is not the only way to create holographic codes. There are two distinct possible directions for constructing more general codes. First is to consider tensors which are more general encoding isometries. Second would be to consider tilings which are more complicated in comparison to the ones used in the paper. Recently one approach has been found which combine both these direction [34]. This approach generalizes the quantum error correction code by incorporating gauge degrees of freedom and this is done by choosing tensors such that they represent a *Bacon–Shor code*[32]. The authors use a alternating hyperbolic tiling of squares and hexagons to embed the code. The logical qubits are encoded on a squares and hexagons have perfect tensors without bulk indices, this construction can be seen in Fig 11. This model has some of the same problems as HaPPY model such as residual bulk regions[13], but this allows for a deformation to a skewed code which has approximate error correction, effects of this on entanglement wedges is similar to gravitational back-reaction of massive bulk deformation.

Another approach to constructing the holographic code is that we can consider these codes as mappings between local Hamiltonians, this ideas was developed in [35]. Using

[12]Though this is highly speculative and would require incorporating a lot of dynamcis into these models
[13]

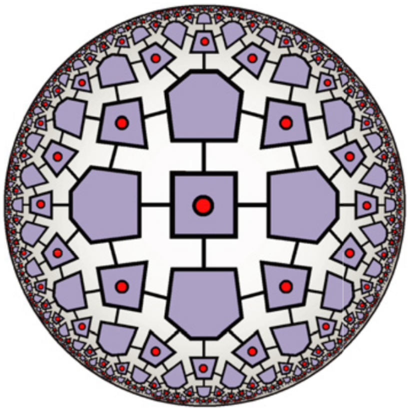

Figure 11: Generalization of HaPPY code with different tiling and bacon-shor code as the logical degrees of freedom. (taken from [32])

something called *permutation gadgets*[14], a mapping is possible for a discretized hyperbolic bulk of three or more spatial dimension[15]. These models preserve locality both in thee bulk and the boundary makes it possible for a generic time evolution which mimics certain aspect of black hole formation. But since these models are in higher dimensions it would require a generalization of regular tilling to a tessellation of polytopes.

Instead of first searching for holographic tensor network models with suitable geometry and then choosing suitable tensors, we can also generate geometry of tensor networks dynamically using boundary entanglement. This process is called *entanglement distillation* and is applied iteratively to the boundary state [36]. Through this approach we get tensor networks that satisfy holographic entanglement entropy and and entanglement purification (in case of geometries other than tree structure) of CFT states and this happens due inherent structure of the construction, and these tensor networks are suitable for representing different properties of continuum AdS/CFT and not act only as a toy models.

In the pursuit of discretizing AdS/CFT correspondence, people have made remarkable connection between very different subject matters, for example the proposal in [37] establishes a connection between p-adic number (number theory) and AdS/CFT. In this paper the author tries describe how classical dynamics on an infinite tree graph can be dual to a conformal field theory defined over the p-adic numbers. A more recent connection is between quantum error correction and celestial holography has been established [38], in here they initiate a study of quantum error correction in celestial CFT.

[14] Permutation gadget is tool used in Hamiltonian simulation theory.
[15] This is different from HaPPY code as it is modelled in two dimension

VII. Outlook

We have presented an introduction to a relatively new and rapidly growing field of research which is at the interface of quantum information theory and high energy physics. The broader aim of this body of work is to reproduce some aspects of holography in a discretized picture. In order to do so there are two main ingredients, on one hand we have quantum error correction, which originally arose in the context of handling noise in quantum computers, but has been closely intertwined with the ideas from holography. On the other hand we have tensor networks which meaningfully and quantitatively capture the entanglement structure of quantum states. We have discussed in detail the three qurit model of holography and how reconstruction bulk logical qutrit is possible using any two physical boundary qutrits, this serves as a minimal model of AdS/CFT using quantum error correction. Next we saw how this idea could be used as an inspiration to build the HaPPY code, which uses tensor networks as the encoding map for the quantum error correction, and serves as a toy model for AdS/CFT with properties like RT formula and AdS-Rindler Reconstruction. We also discussed few other approaches to tensor network models of holographic quantum error correction and discretized holography.

All the methods mentioned in this paper are just start of what is an exciting and rich field. There are still many questions that remain open, like the study of *dynamics* in these holographic models is just starting to take place [39, 40]. Another question of the *continuum limit* of tensor networks which relates the discerte and continuous model of holography is being pursued actively [41, 42, 43, 44]. This has has interesting potential for describing regimes of interacting quantum fields [45, 46, 47]. Due to *complexity equal volume* [48] and *complexity equal action* [49] conjectures, the notion of circuit and state complexity [50] have been found crucial in discussions of holography [51, 52, 53, 54]. The holographic tensor network models can be used to create practical stabilizer codes for actual error corrections in quantum computers [55]. There are many more practical and theoretical applications of these family of models which are being extensively studied.

It is the hope that beyond the brief overview, this article can act as motivation for further exploration of intricate connections between holography, tensor networks and quantum error correction.

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