Existence and Uniqueness of Solutions

(for first order differential equation)

1. Introduction

Equations which have one dependent variable and its derivatives with respect to one or more than one independent variable are called differential equations. However, an ordinary differential equation is one in which there is only one independent variable.

Main problem with this ODE is that only few types of differential equation can be solved in terms of known elementary functions. for example, first order linear ODE, separable equation, or second order linear ODE whose solution can be expressed in form of power series. However, most of the equations are outside this category.

For this general first order equation,

$$y' = f(x, y). \tag{1}$$

Where f is a continuous function. One of the special cases for solving this in explicit way is if it's a linear equation of form:

$$y' = g(x)y + h(x). \tag{2}$$

Where g, h are continuous on some interval I. Then its solution would be of form

$$\phi(x) = e^{\int_{x_0}^{x} g(x)\,dx} \int_{x_0}^{x} e^{-\int_{x_0}^{x} g(x)\,dx} h(t)\; dt + ce^{\int_{x_0}^{x} g(x)\,dx}.$$

x_0 is in interval I, and c is arbitrary constant.

For this kind of differential equation, we get many solutions, but for initial value problems we only get unique solutions. Also, these unique solutions are mostly solutions that exist around the interval of initial point. The equation mostly does not give clue as for how far a solution will exist. Also, we can solve the equation with variable separable of form,

$$f(x, y) = \frac{g(x)}{h(y)}.$$

This kind of variable separable equation can be easily solved using integration method, also if it is initial value problem then by that we can find value of arbitrary constant we got from integration for particular solution.

Another type of solving is using the exactness of equation. Suppose the first order equation $y' = f(x,y)$ is written as below,

$$y' = \frac{-M(x,y)}{N(x,y)}$$

$$\therefore \mathrm{M(x,y)dx + N(x,y)dy = 0.}$$

So, if $F(x,y)$ is a solution of given equation then,

$$\frac{\partial \mathrm{F}}{\partial \mathrm{x}} = \mathrm{M} \quad \text{and} \quad \frac{\partial \mathrm{F}}{\partial \mathrm{y}} = \mathrm{N}.$$

For being exact this equation should satisfy,

$$\frac{\partial \mathrm{M}}{\partial \mathrm{y}} = \frac{\partial \mathrm{N}}{\partial \mathrm{x}}\ .$$

And then we can solve this equation by partial differentiation of F, if given equation is not exact then we can find integrating factor to make it exact by doing necessary calculations, but this also helps us for solving only some of the equation.

Another method of solving,

The method of successive approximation

This method is used for finding solutions for general equations (1). Where f is any continuous real valued function defined on some rectangle R satisfying following condition,

$$|x - x_0| \leq a\,, \qquad |y - y_0| \leq b, \qquad (a, b > 0).$$

Our main motive is to prove that on some interval I containing x_0, there exist a solution ϕ which satisfies,

$$\phi(x_0) = y_0.$$

In this way we mean that ϕ is a function such that the point $\big(x, \phi(x)\big)$ are in R for x in I. And

$$\phi'(x) = f\big(x, \phi(x)\big).$$

Such ϕ is said to be solution of initial value problem,

$$y' = f(x,y),\ \ y(x_0) = y_0. \tag{3}$$

Theorem: A solution ϕ is a solution of the IVP (3) on an interval I, if and only if it is solution of this integral equation,

$$y = y_0 + \int_{x_0}^{x} f(t,y)\, dt. \tag{4}$$

Proof:

Suppose ϕ is a solution of (3) on an interval I. Then we got

$$\phi'(t) = f\big(t, \phi(t)\big) \text{ And } \phi(x_0) = y_0. \tag{5}$$

Where $\phi(t)$ is continuous on I, So now integrating (4) from x_0 to x, we got

$$\phi(\mathrm{x}) = \phi(x_0) + \int_{x_0}^{x} f(t, \phi(t))\, dt\, .$$

So $\phi(x)$ is a solution of (3) as $\phi(x_0) = y_0$. (because (5))

Similarly, we now have to check for the solution of (4). Let's assume ϕ satisfy (4) on I, by differentiating (4) we got $\phi'(t) = f\big(t, \phi(t)\big)$.

Also, as it is a solution of (4), we have $\phi(x_0) = y_0$, so we can clearly say that ϕ is a solution of (3). Hence proved.

So now we will solve (4), and whatever solution we get will be the solution of (3), according to this theorem. As a rough approximation we can consider $\phi_0(x) = y_0$ as our solution. But this function does not satisfy (4), only satisfies initial value condition. But for better approximation, we got,

$$\phi_1(x) = y_0 + \int_{x_0}^{x} f\big(t, \phi_0(t)\big) dt.$$

By ϕ_1 we can now get more closer approximation, by doing this step successively we can get most specific approximation.

$$\phi_{(k+1)}(x) = y_0 + \textstyle\int_{x_0}^{x} f\big(t, \phi_k(t)\big) dt. \tag{6}$$

As k tends to infinite, we would obtain $\phi_k(x) \to \phi(x)$. Where $\phi(x)$ would satisfy (4). So ϕ would be our solution.

This is the method of successive approximation.

Remember that this ϕ_k is a continuous function on interval I which follows following properties: (*)

$$|x - x_0| \le \alpha = min\,\{a, b/M\},$$

And in this interval,

$$|\phi_k(x) - y_0| \le M|x - x_0|.$$

Which can easily be interpreted using geometric interpretation.

Lipschitz condition:

If a function defined on a set S containing (x, y) satisfies following condition on S for some $\mathrm{K} > 0$ for all $(\mathrm{x}, \mathrm{y}_0), (\mathrm{x}, \mathrm{y}_1)$ in S. Then this function satisfies Lipschitz condition, and K is said to be Lipschitz condition.

$$|f(x, y_0) - f(x, y_1)| \le K|y_0 - y_1|.$$

$$or$$

$$\left|\frac{\partial \mathrm{f}(\mathrm{x}, \mathrm{y})}{\partial \mathrm{y}}\right| \le \mathrm{K}$$

2. Picard's theorem

If $f(x, y)$ and $\partial f/\partial y$ are continuous function on a closed rectangle R, which have following properties,

$$|x - x_0| \leq a, |y - y_0| \leq b \qquad (a, b > 0),$$

And let (x_0, y_0) be any point from R, then there exist a number $h > 0$ such that IVP,

$$y' = f(x, y) \ \leq M, \ \ y(x_0) = y_0 \tag{3}$$

Has one and only one solution $\phi(x)$ on interval

$$|x - x_0| \leq \mathrm{h} = min\{a, b/M\}$$

Proof:

We know that the solution of IVP (3) is a continuous solution of integral equation,

$$y = y_0 + \int_{x_0}^{x} f(t, y)\, dt \tag{4}$$

That means we have to prove that (4) has a unique solution continuous solution for that interval. We first have to check for successive approximations, that they converge to $\phi(x)$. this simple means that given series.

$$\phi_0(x) + \Sigma_{k=1}^{\infty}\left[\phi_k(x) - \phi_{(k-1)}(x)\right] \tag{7}$$

Converges to $\phi(x)$ as $\phi_k(x)$ is partial sum of this series.

Now as we know f (x, y) and $\partial f/\partial y$ are continuous function on a closed rectangle R, but as R is bounded and closed; we can compute that there exist such M and K, for which

$$f(x, y) \leq M, \qquad \frac{\partial f}{\partial y} \leq K \tag{8}$$

Also $$|f(x, y_1) - f(x, y_0)| \leq K|y_1 - y_0|. \tag{9}$$

We can clearly see that all solutions of (4) are bounded by these lines,

$$y - y_0 = M(x - x_0), \qquad y - y_0 = -M(x - x_0)$$

$$and$$

$$x - x_0 = h, \qquad x - x_0 = -h$$

So, by graphical interpretation of these can get $\mathrm{h} = min\{a, b/M\}$. Let this interval be R'. Also, by this we have $Kh < 1$.

Now that these things are clear, we will focus on interval R' only, so we will start working on series convergence. For proving convergence of series, we can simply prove convergence of series given below due to comparison test:

$$|\phi_0(x)| + \Sigma_{k=1}^{\infty}\left[\left|\phi_k(x) - \phi_{(k-1)}(x)\right|\right] \tag{S}$$

Also, we know that $\phi_k(x)$ lies in R' hence in R.

Putting $\phi(x)$ in (4), we got:

$$|\phi_1(x) - \phi_0| = \left|\int_{x_0}^{x} f\big(t, \phi_0(t)\big)dt\right| \leq Mh$$

Similarly,

$$|\phi_k(x) - \phi_0| = \left|\int_{x_0}^{x} f\big(t, \phi_{k-1}(t)\big)dt\right| \leq Mh$$

Also, we have $|\phi_1(x) - \phi_0| \ \leq b$, because $\phi_k(x)$ lies in R.

So, we will calculate next terms of series,

$$|\phi_2(x) - \phi_1(x)| = \left|\int_{x_0}^{x} f\big(t, \phi_1(x)\big) - f\big(t, \phi_0(x)\big)dt\right| \leq Kbh$$

Similarly,

$$|\phi_k(x) - \phi_{k-1}(x)| = \left|\int_{x_0}^{x} f\big(t, \phi_{k-1}(x)\big) - f\big(t, \phi_{k-2}(x)\big)dt\right| \leq b(Kh)^{(k-1)}$$

So, our series is now less than or equal to this new series,

$$|y_0| + b + b(Kh) + b(Kh)^2 + \cdots$$

Where $Kh < 1$, so we proved that this series converges, hence series (7) converges to $\phi(x)$, as all $\phi_k(x)$ lies in R, hence $\phi(x)$ follows same property.

Now we will prove that $\phi(x)$ is continuous solution of (4). As we take larger k, $\phi_k(x)$ nearly became $\phi(x)$. More precisely we can say that, for some $\epsilon > \ 0$, there exist a positive number n_0 such that $\left|\phi(x) - \phi_{n(x)}\right| < \epsilon$ for $n \geq n_0$, in given interval. Also, as $\phi_n(x)$ is continuous, due to this uniformity of $\phi(x)$, we got that $\phi(x)$ is continuous.

For $\phi(x)$ being a solution of (4),

$$\phi(\mathrm{x}) - \mathrm{y}_0 - \int_{x_0}^{x} f\big(t, \phi(t)\big)\, dt = 0 \tag{10}$$

Given that $\phi_n(x)$ is a solution of (4), so we have,

$$\phi_{\mathrm{n(x)}} - \mathrm{y}_0 - \int_{x_0}^{x} f\left(t, \phi_{(\mathrm{n}-1)}(t)\right) dt = 0 \tag{11}$$

By comparing (10) and the subtraction of (10) and (11),

$$\phi(\mathrm{x}) - \mathrm{y}_0 - \int_{x_0}^{x} f\big(t, \phi(t)\big)\, dt = \phi(\mathrm{x}) - \int_{x_0}^{x} f\big(t, \phi(t)\big)\, dt - \phi_{\mathrm{n(x)}} + \int_{x_0}^{x} f\left(t, \phi_{(\mathrm{n}-1)}(t)\right) dt$$

More specifically,

$$\left|\phi(x) - y_0 - \int_{x_0}^{x} f(t, \phi(t))\, dt\right| \leq |\phi(x) - \phi_{n(x)}| + \left|\int_{x_0}^{x} f(t, \phi(t)) - f\left(t, \phi_{(n-1)}(t)\right) dt\right|$$

As we discussed before, by taking sufficient large number n, right side can be rounded off to 0. So, we can get (10). So, we proved that (x) is continuous solution of (4). Let's check for uniqueness of solution.

Let $\overline{\phi}(x)$ and ϕ(x) both be solution of (4). for uniqueness we have to prove $\overline{\phi}(x) = \phi(x)$. First, we will prove that $\overline{\phi}(x)$ is in interval R'. Let's assume that $\overline{\phi}(x)$ leaves R'. So there exist x_1 such that

$$\frac{|\overline{\phi}(x_1) - y_0|}{|x_1 - x_0|} > \text{M}$$

But as given in theorem $y' = f(x, y) \leq M$, so

$$\frac{|\overline{\phi}(x_1) - y_0|}{|x_1 - x_0|} \leq M$$

Hence this is contradiction, we proved that $\overline{\phi}(x)$ lies in R'. Now let's take $\overline{\phi}(x)$ and ϕ(x) both as solution of (4). So, we have,

$$|\phi(x) - \overline{\phi}(x)| = \left|\int_{x_0}^{x} f(t, \phi(x)) - f\left(t, \overline{\phi}(x)\right) dt\right|$$

$$\therefore |\phi(x) - \overline{\phi}(x)| \leq Kh\, max|\phi(x) - \overline{\phi}(x)|$$

But we have $Kh < 1$, so $\phi(x) - \overline{\phi}(x) = 0$,

$$\therefore \phi(x) = \overline{\phi}(x)$$

So, the property of uniqueness for this theorem is also proved.

Therefore, Picard's theorem is proven.

Remark:

1) This theorem takes two conditions first of continuity of derivative and of partial derivative of that derivative. However partial derivative's condition is only used once, so we can replace that condition with $|f(x, y_1) - f(x, y_0)| \leq K|y_1 - y_0|$, which can strengthen our theorem by weakening its hypotheses. This is Lipschitz condition for function f.
2) If we drop the part of boundness of partial derivative or Lipschitz condition, then we can prove converges of series and continuity of solution but can't prove uniqueness. So, we get the existence theorem, which is also called Peano's theorem.
3) Picard's theorem is also known as local existence theorem as it only checks for solution for small intervals around initial point, but this restriction can be removed for some of the cases. There is an theorem for removing this constraint which is given below

   Statement:

   If a function $f(x, y)$ is continuous function satisfies Lipschitz condition with Lipschitz constant K, on a strip which follows condition given below,

$$a \le x \le b, \qquad -\infty < y < \infty$$

If some point (x_0, y_0) is contained by this strip, then the initial value problem (3) has an unique solution $y(x)$ on interval $a \le x \le b$.

Proof:

The existence of the solution can also be proved in similar way we have done for Picard theorem (By making series and convergence of it). So first we will define three variables $M_0,\ M_1,\ M$. Were,

$$M_0 = y_0, \qquad M_1 = max|y_1(x)|, \qquad M = M_0 + M_1$$

By this we found that $|y_0(x)| \le M$ and difference of $y_1(x)$ and $y_0(x)$ is bounded by M. So we have to find the general term of series for finding convergence. So for $x_0 \le x \le b$,

$$|y_2(x) - y_1(x)| = \left|\int_{x_0}^{x} \big(f(t, y_1(t)) - f(t, y_0(t))\big)\, dt\right|$$

$$\le K \int_{x_0}^{x} |y_1(t) - y_0(t)| dt$$

$$\le KM(x - x_0) \qquad \text{(because lipschitz condition)}$$

Similar calculation led us to this equation,

$$|y_n(x) - y_{n-1}(x)| \le K^{n-1} M \frac{(x - x_0)^{n-1}}{(n-1)!}$$

Similarly, we can do for another interval $a \le x \le x_0$. And finally we will get,

$$|y_n(x) - y_{n-1}(x)| \le K^{n-1} M \frac{(b - a)^{n-1}}{(n-1)!}$$

So we can clearly see that series S

$$|y_0(x)| + \Sigma_{n=1}^{\infty}\big[|y_n(x) - y_{(n-1)}(x)|\big]$$

Converges, and converges uniformly on the interval $a \le x \le b$. So as previously explained our solution is on continuous on given interval. So now we must prove the uniqueness of the solution. So, let's assume $\overline{y}(x)$ is also solution along with $y(x)$. Our main point to prove is that $\overline{y}(x) = y(x)$, on given interval of x.

As $\overline{y}(x)$ is a solution, we have,

$$\overline{y}(x) = y_0 + \int_{x_0}^{x} f(t, \overline{y}(t))\, dt$$

Let assume $\mathrm{A} = \max|\overline{\mathrm{y}}(\mathrm{x}) - \mathrm{y}_0|$, then for interval $x_0 \le x \le b$,

$$|\overline{y}(x) - y_1(x)| = \left|\int_{x_0}^{x} \big(f(t, \overline{y}(t)) - f(t, y_0(t))\big)\, dt\right|$$

$$\le KA(x - x_0)$$

In general, as we done before, for whole interval we get,

$$|\overline{y}(x) - y_n(x)| \leq K^n A \frac{(b-a)^n}{(n)!}$$

Right side of this equation can approach zero as we take sufficient large n, so, we found that $\overline{y}(x) = y(x)$. Hence the proof is completed.

Problem practice:

1. Find all real solution of the following equations,

   (a) $\frac{dy}{dx} = \frac{e^{(x-y)}}{1+e^x}$

$$\therefore e^y dy = \frac{e^x}{1+e^x} dx$$

$$\therefore \int e^y dy = \int \frac{e^x}{1+e^x} dx$$

$$\therefore e^y = \int \frac{du}{1+u} \qquad (u = e^x)$$

$$\therefore e^y = ln(1+u) + c.$$

   Where c is arbitrary constant.

   (b) $y' = x^2y^2 - 4x^2$

$$\therefore \frac{dy}{y^2-4} = x^2 dx$$

$$\therefore \frac{1}{4}\int \frac{4}{(y-2)(y+2)} dy = \int x^2 dx$$

$$\therefore \frac{1}{4}\int \frac{(y+2)-(y-2)}{(y-2)(y+2)} dy = \int x^2 dx$$

$$\therefore \frac{1}{4}\int \frac{1}{y-2} - \frac{1}{y+2} dy = \int x^2 dx$$

$$\therefore 3\left(\ln(y-2) - \ln(y+2)\right) = 4x^2 + c$$

$$\therefore 3ln\left(\frac{y-2}{y+2}\right) = 4x^2 + c.$$

   Where c is arbitrary constant.

2. If a function satisfies,

$$f(tx, ty) = t^k f(x, y).$$

   Then, this is called homogeneous function. To solve such equation, we take $y = ux$. Show that following equation is homogeneous and then find real values solution for it.

$$y' = \frac{x+y}{x-y}.$$

   Ans.

We have,

$$f(x,y) = \frac{x+y}{x-y}$$

And,

$$f(tx,ty) = \frac{x+y}{x-y}.$$

So, $f(tx,ty) = f(x,y)$. So, we can say this is a homogeneous equation.
Now, $y = ux$ then $y' = u + xu'$.

$$\therefore u + xu' = \frac{1+u}{1-u}$$

$$\therefore xu' = \frac{1+u^2}{1-u}.$$

Then we can solve it as we solve separable equations, using integration. Also, there are some questions in which we can convert equations to homogeneous. Which we can do by $x = X + h, and\ y = Y + k$.

3. Check whether this equation is exact or not. If it is exact, then find real values solution of it.
(a) $2xydx + (x^2 + 3y^2)dy = 0$.

Here, $M = 2xy$ and $N = x^2 + 3y^2$.

$$\frac{\partial M}{\partial y} = 2x,$$

$$\frac{\partial N}{\partial x} = 2x.$$

Hence

$$\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}.$$

Therefore, this is an exact equation. Now,

$$\frac{\partial f}{\partial x} = M.$$

$$\therefore f(x,y) = \int M dx + g(y)$$

$$\therefore f(x,y) = x^2y + g(y)$$

$$\therefore \frac{\partial f}{\partial y} = x^2 + g'(y)$$

$$\therefore N = x^2 + g'(y)$$

$$\therefore g'(y) = 3y^2$$

So, solution of this equation $x^2y + y^3 = c$.
(note: if given equation is not exact, then you have to first find integrating factor for equation.)

4. It is important to see how Picard's successive approximation theorem works with just a choice of constant function $y_0(x) = y_0$. Check successive method for this initial value problem, $y' = x + y,\ y(0) = 1$. Apply this method for this IVP using,
(a) $y_0(x) = e^x$;
(b) $y_0(x) = 1 + x$;

Ans.
The equivalent integral equation is,

$$y(x) = 1 + \int_0^x [t + y(t)]dt,$$

With $y_0(x) = y_0$, doing successive method yields

$$y_1(x) = 1 + \int_0^x (t+1)dt = 1 + x + \frac{x^2}{2!},$$

$$y_2(x) = 1 + \int_0^x \left(1 + 2t + \frac{t^2}{2!}\right) dt = 1 + x + x^2 + \frac{x^3}{3!},$$

$$y_3(x) = 1 + \int_0^x \left(1 + 2t + t^2 + \frac{t^3}{3!}\right) dt = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{4!},$$

So, in general,

$$y_n(x) = 1 + x + 2\left(\frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}\right) + \frac{x^{(n+1)}}{(n+1)!},$$

This converges to,

$$= 2e^x - x - 1,$$

Which is similar to exact solution.

(a) Now we will check for $y_0(x) = e^x$.

$$y_1(x) = 1 + \int_0^x (x + e^t)dt = \frac{x^2}{2} + e^x,$$

$$y_2(x) = 1 + \int_0^x \left(t + \frac{t^2}{2} + e^t\right) dt = \frac{x^2}{2} + \frac{x^3}{3!} + e^x,$$

$$y_3(x) = 1 + \int_0^x \left(1 + t + \frac{t^2}{2} + e^t\right) dt = \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + e^x,$$

So basically,

$$y_n(x) = 1 + x + 2(e^x - x - 1).$$

(b) Now, it's turn to check for $y_0(x) = 1 + x$,

$$y_1(x) = 1 + \int_0^x (1 + 2t)dt = 1 + x + x^2,$$
$$y_2(x) = 1 + \int_0^x (1 + t + t^2)dt = 1 + x + x^2 + \frac{x^3}{3},$$
$$y_3(x) = 1 + \int_0^x \left(1 + t + t^2 + \frac{t^3}{3}\right)dt = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{12},$$

Generally,

$$\mathrm{y_n(x) = 2e^2 - x - 1.}$$

We can say that we can get an exact solution by any initial approximation, but n should be large enough.

5. Show that $f(x, y) = xy^2$
(a) Satisfies a Lipschitz condition on rectangle $a \le x \le b$ and $c \le y \le d$.
(b) Doesn't satisfy that condition on $a \le x \le b$ and $-\infty \le y \le \infty$.

Ans.

For satisfying Lipschitz condition this inequality should be bounded,

$$\frac{|xy_1^2 - xy_2^2|}{|y_1 - y_2|}$$
$$= |x(y_1 + y_2)|. \tag{10}$$

(a) For satisfying Lipschitz condition, equation (10) should be bounded. We are already given that $a \le x \le b$ and $c \le y \le d$.

$$\therefore |x(y_1 + y_2)| \le 2|ad|$$

So, this equation is bounded, hence Lipschitz condition is satisfied for this interval.

(b) Now for interval $a \le x \le b$ and $-\infty \le y \le \infty$, equation (10) is not bounded. Because $|y_1 + y_2|$ is not bounded. Hence Lipschitz condition doesn't satisfy in this interval.

6. Show that $f(x, y) = xy$,
(a) Satisfies Lipschitz condition on rectangle $a \le x \le b$ and $c \le y \le d$.
(b) Satisfies that condition on strip $a \le x \le b$ and $-\infty \le y \le \infty$.
(c) Doesn't satisfy that condition on entire plane.

Ans.

For satisfying Lipschitz condition there should exist K>0, for which

$$\frac{|f(x, y_1) - f(x, y_2)|}{|y_1 - y_2|} \le K.$$

$$\therefore \frac{|xy_1 - xy_2|}{|y_1 - y_2|} \le K.$$

$$\therefore |x| \leq K.$$

(a) For given interval $a \leq x \leq b$ and $c \leq y \leq d$, We have $K = b$. So, for this interval Lipschitz condition is satisfied.

(b) Similarly for this strip $a \leq x \leq b$ and $-\infty \leq y \leq \infty$, $K = b$. So, for this strip also Lipschitz condition is satisfied.

(c) For entire plane, x is not bounded. So, inequality is not bound. Hence for entire plane Lipschitz continuity is not satisfied,

7. For what points (x_0, y_0) does Picard's theorem imply that IVP

$$y' = y|y|, \qquad y(x_0) = y_0.$$

Has a unique solution on the interval $|x - x_0| \leq h$?

Ans.

$f(x,y) = y|y|$ and

$$\frac{\partial f}{\partial y} = 2y, when\, y > 0,$$

$$\frac{\partial f}{\partial y} = -2y, when\, y < 0.$$

So $f(x,y)$ is continuous everywhere, and $\frac{\partial f}{\partial y}$ is continuous everywhere. Therefore, we have unique solutions for all point (x_0, y_0) in interval $|x - x_0| \leq h$.

3. Peano's Theorem:

Now, as we have completed Picard's theorem, we will now focus on Peano's theorem, which we discussed before in weakening the hypothesis.
For learning this, we must learn these things first, 1) Cauchy Euler theorem, 2) Ascoli lemma.

Theorem (Cauchy Euler Theorem):
Let f be a continuous function on rectangle R satisfying following condition:

$$|x - x_0| \leq a, \qquad |y - y_0| \leq b.$$

For given $\epsilon > 0$, there exists an ϵ-approximate solution ϕ of IVP,

$$y' = f(x, y), \;\; y(x_0) = y_0. \tag{3}$$

on interval $|x - x_0| \leq \alpha$.

Proof:
As given, f is continuous on rectangle R, hence f is bounded. So there exist a $M > 0$ for which,

$$f(x, y) \leq M.$$

First, we must see what is an ϵ-approximate solution is,

Let f be continuous on domain D on (x, y) plane. If a continuous ϕ is ϵ-approximate solution of IVP (3) on interval I. Then it should satisfy following conditions:

1) $(x, \phi(x)) \in D$ for $x \in I$.
2) $\phi \in C^1$ on I. Except some points where ϕ' may be discontinuous (which point lies in set S)
3) $|\phi'(x) - f(x, y(x)| \leq \epsilon, \quad (t \in I - S)$.

First, we will construct ϵ-approximate solution on interval $(x_0, x_0 + \alpha)$, similar construction will do for $(x_0 - \alpha, x_0)$. Also given that f is uniform continuous. So, for given ϵ there exist $\delta_\epsilon > 0$, such that,

$$|f(x, y) - f(\overline{x}, \overline{y})| \leq \epsilon. \tag{1.1}$$

If $|x - \bar{x}| \leq \delta_\epsilon$ and $|y - \bar{y}| \leq \delta_\epsilon$. For (x, y) and $(\bar{x}, \bar{y})$, in R.

Now we will start with dividing $(x_0, x_0 + \alpha)$ into n parts, such that.

$$x_0 = t_0 < t_1 < t_2 < \cdots < t_n = x_0 + \alpha,$$

Such that, $\max\left|t_k - t_{(k-1)}\right| \leq \min\left(\delta_\epsilon, \frac{\delta_\epsilon}{M}\right)$. (1.2)

Now, from (x_0, y_0), we construct a line segment with slope $f(x_0, y_0)$ which ends at (t_1, y_1), which should be Triangular region T made by lines,

$$x = x_0 + \alpha, \qquad y = y_0 + M(x - x_0), \qquad y = y_0 - M(x - x_0).$$

Similarly, construct line segment from (t_1, y_1) with slope $f(t_1, y_1)$, which ends at (t_2, y_2). Continue this process till this construction meet line $x = x_0 + \alpha$.

Overall, this whole construction lies in T, because $f(x, y) \leq M$.
This constructed ϕ is ϵ-approximate solution. Which can be expressed as following,

$$\phi(x) = \phi(t_{(k-1)}) + f\left(t_{(k-1)}, \phi(t_{(k-1)})\right)(x - t_{(k-1)}), k = 1, \ldots, \mathrm{n}. \tag{1.3}$$

From construction, it's clear that $\phi \in C_p^1$ on given interval. And,

$$|\phi(x) - \phi(\overline{x})| \leq M|x - \bar{x}| \tag{1.4}$$

Where $x, \bar{x}$ are in interval $[x_0, x_0 + \alpha]$.

So, for some x, $t_{(k-1)} < x < t_k$, then by (1.2) and (1.4), we got,

$$\left|\phi(x) - \phi\left(t_{(k-1)}\right)\right| \leq \delta_\epsilon.$$

And by (1.1) and (1.3), we got,

$$\left|f\left(x, \phi(x)\right) - f\left(t_{(k-1)}, \phi\left(t_{(k-1)}\right)\right)\right| \leq \epsilon.$$

Hence, its proved that, ϕ is an ϵ-approximate solution of given IVP.

Lemma (Ascoli):

Let K={k} be an infinite, Equi continuous and uniformly bounded set of functions on bounded Interval I. Then F contains $\{f_n\}$ sequence where n belongs to N, which is(sequence) uniformly convergent on I.

Theorem (Peano's Existence Theorem / Cauchy-Peano's Existence Theorem):

If f is continuous on the rectangle R, which satisfies following conditions,

$$|x - x_0| \leq a, \qquad |y - y_0| \leq b.$$

Then there exists a solution $\phi \in C'$ of IVP (3), on $|x - x_0| \leq \alpha$.

Proof:

Let $\{\epsilon_n\}$ where $n \in N$, be a monotonically decreasing sequence of real numbers such that as we take sufficient large n ($n \to \infty$), $\epsilon_n \to \infty$.

Since by Cauchy Euler's theorem for each $\epsilon_n > 0$ there exist ϵ_n-approximate solution ϕ_n of IVP (3), on given interval.

As we choose ϕ_n as such solution for ϵ_n. As we know from Cauchy Euler's theorem,

$$|\phi_n(x) - \phi_n(t)| \leq M|x - t|. \tag{11}$$

As we take $t = x_0$, we got

$$|\phi_n(x) - y_0| \leq M|x - x_0|$$

$$\therefore |x - x_0| \leq \frac{b}{M}$$

$$|\phi_n(x)| \leq b + |y_0|.$$

So, $\{\phi_n(x)\}$ is uniformly bounded. Also, by (11) we can say it is Equi continuous.

By Ascoli lemma, we know that there exists a subsequence $\{\phi_{n_k}\}$of main sequence of ϵ_n-approximate solution ϕ_n. Which is uniformly convergent to a limit function on given interval. Which will be continuous ϕ because ϵ_n-approximate solution ϕ_n is continuous.

As we know that ϕ_n is an ϵ_n-approximate solution, so we can say,

$$\phi_n'(x) = f\left(x, \phi_n(x)\right) + \Delta_n(x)$$

$$|\Delta_n(x)| = \left|\phi_n'(x) - f\left(x, \phi_n(x)\right)\right| < \epsilon_n. \tag{12}$$

Also,

$$\phi_n(x) = y_0 + \int_{x_0}^{x} [f(t, \phi_n(t)) + \Delta_n(t)]dt.$$

Due to (12), we have $\phi_{nk}(x) \to \phi(x)$ and $\Delta_n(t) \to 0$ on given interval $|x - x_0| \le \alpha$ as $k \to \infty$. (13)

Put $n \to nk$,

$$\phi_{nk}(x) = y_0 + \int_{x_0}^{x} [f(t, \phi_{nk}(t)) + \Delta_{nk}(t)]dt.$$

But as $k \to \infty$,

$$\phi_n(x) = y_0 + \int_{x_0}^{x} [f(t, \phi_n(t))]dt. \qquad (\because (13))$$

So, we know that if ϕ is solution of integral equation (13), then it is a solution of IVP (13). Hence, we proved Peano's existence theorem.

Problem Practice:

1. For given function f, where $x \in R, y \in R$. Find the biggest interval where solution for given initial value problem exist.
   (a) $y' = \sqrt{(x-3)},\ y(2) = 4.$
   (b) $y' = \frac{1}{x},\ y(-3) = 5.$
   (c) $y' = x + y^2,\ y(1) = 1.$

   Ans.

   (a) $f(x, y) = \sqrt{x-3}$, this function is continuous for every $x > 3$.

   But given IVP, says that $y(2) = 4$. As y' is not continuous for $x = 2$, these does not exist any solution for given IVP.

   (b) $f(x, y) = \frac{1}{x}$, this function is clearly continuous for given interval,
   $$x \in (-\infty, 0) \cup (0, \infty).$$

   Given IVP, , $y(-3) = 5$ need interval which involve , $x = -3$. So biggest interval where solution of given IVP exist is,
   $$x \in (-\infty, 0).$$

   (c) $f(x, y) = x + y^2$, clearly this is continuous for every $x \in R, y \in R$.

   So there exist solution for this IVP, on whole $x \in R$.

2. For given functions, find the interval where solution exists, also check on which interval it has unique solutions.
   (a) $y' = |y|,\ \ y(x_0) = y_0.$
   (b) $y' = tany,\ y(x_0) = y_0.$

Ans.

(a) $f(x, y) = |y|$. Which is continuous all over $x \in R, y \in R$.
$\frac{\partial f}{\partial y} = +1$, when $y \geq 0$, or $\frac{\partial f}{\partial y} = -1$.

Hence $\frac{\partial f}{\partial y}$ is not continuous at $y = 0$. Hence for this IVP, there exist solution every where because y' is continuous everywhere. ($\because$ Peano's theorem)

But hence $\frac{\partial f}{\partial y}$ is not continuous at $y = 0$, solution will not be unique $y = 0$. ($\because$ Picard's theorem)

(b) $f(x, y) = tany$. Which is continuous all over for $y \in R - \{\frac{(2n+1)\pi}{2} : n \in N\}$.
$\frac{\partial f}{\partial y} = sec^2 y$, which is continuous all over for $y \in R - \{n\pi : n \in N\}$.

Hence for this IVP, there exist solution everywhere on $y \in R - \{\frac{(2n+1)\pi}{2} : n \in N\}$ because y' is continuous everywhere on that interval. ($\because$ Peano's theorem)

But hence $\frac{\partial f}{\partial y}$ is continuous on interval $y \in R - \{n\pi : n \in N\}$ hence, solution will be unique on this interval. ($\because$ Picard's theorem)

4. Continuation of solutions:

Cauchy Convergent criterion: A sequence of function f_n is said to be uniformly convergent if it satisfies following condition,
For every $\epsilon > 0$, there is an N such that if $m, n > N$, then $|f_n(x) - f_m(x)| < \epsilon$, for every x.

Theorem 3: Let f be a continuous function on a domain D in (x, y) plane and let (x_0, y_0) is a point inside D. then there exist a solution on some t interval containing x_0 in its interior.

Proof: We are not given that D is bounded or not, so we will assume that D is open. So there exist $r > 0$, such that all points having distance less then r from point (x_0, y_0) are contained in D. Let R be a rectangle containing point (x_0, y_0), and inside the circle of radius r. Then by Piano's theorem, there exist a solution passing through this point in given rectangle. Hence proved.

Coming on to the topic,

Theorem:

Let f be continuous function on domain D of (x, y) plane, and suppose f is bounded in D. If ϕ is solution IVP (3), on interval (a, b), then limits $\phi(a + 0)$ and $\phi(b - 0)$ do exist. If $(a, \phi(a + 0))$ is in D, then solution can be continued to the left of a. (Similarly continued for right of b using $\phi(b - 0)$)

Proof:

Suppose $f(x, y)$ is continuous on some rectangle R, of the (x, y) plane and that IVP has a solution $\phi(x)$ on the interval (a, b) passing through initial point $(x_0, y_0) \in R$.

Where $|f| \leq M$ on R.
Also,

$$\phi(a+0) = \lim_{t \to a+0} \phi(t), \quad \phi(b-0) = \lim_{t \to b-0} \phi(t).$$

If $a < x_1 < x_2 < b$,

$$|\phi(x_1) - \phi(x_2)| \leq \int_{x_1}^{x_2} \left| f\big(s, \phi(s)\big) \right| ds \leq M|x_1 - x_2|.$$

As x_1 and x_2 approaches $a + 0$, $|\phi(x_1) - \phi(x_2)|$ approaches 0, hence by Cauchy criterion for convergence, we say that $\phi(a + 0)$ exists, similarly we can check for $\phi(b - 0)$.

Now, let $\bar{\phi}$ be a function is a function which have following properties,

$$\bar{\phi}(x) = \phi(x), when \;\; a < t < b,$$

$$= \phi(b-0), when \;\; t = b.$$

So, now $\bar{\phi}$ is continuous on (a, b].

Also, let assume $\bar{\phi}$ is in D, we have,

$$\bar{\phi}(x) = y_0 + \int_{x_0}^{x} f\left(s, \bar{\phi}(s)\right) ds$$

Hence, Left hand derivative of $\bar{\phi}$ at b exists,

$$\bar{\phi}(b^-) = \bar{\phi}(b-0) = f\left(b, \bar{\phi}(b)\right).$$

This function $\bar{\phi}$ is said to be continuation of the solution ϕ on (a, b].

Example: The function $y' = y^2$ have a solution $y = -1/x$, which passes through $(-1,1)$, which exist on $(-1,0)$. Can it be continued on $(-1,0]$.

Ans.

No, it can't be continued on $(-1,0]$. Because solution y is not bounded on given interval.

We can also carry this process further, by theorem 3 we know that there exists a function $\omega(x)$ (which is continuous and derivative also continuous) passing through $\big(b, \phi(b-0)\big)$. which exist on some interval $[b, b+\beta]$, $\beta > 0$. Let $\hat{\phi}(x)$ be a function defined as,

$$\hat{\phi}(x) = \bar{\phi}(x) \qquad (x \in (a, b]),$$

$$\hat{\phi}(x) = \omega(x) \qquad (x \in [b, b+\beta]).$$

Then $\hat{\phi}(x)$ is solution of given IVP on $(a, b+\beta]$ of class C'. For this we only need to check at the point b for existence and continuity of derivative of $\hat{\phi}'(x)$.

It will be clear that is we show that,

$$\hat{\phi}(x) = y_0 + \int_{x_0}^{x} f\left(s, \hat{\phi}(s)\right) ds. \tag{*}$$

This for $a < x \leq b$. For $x > b$ it follows this,

$$\hat{\phi}(x) = \phi(b-0) + \int_{b}^{x} f\left(s, \hat{\phi}(s)\right) ds.$$

But we know that,

$$\phi(b-0) = y_0 + \int_{x_0}^{b} f\left(s, \hat{\phi}(s)\right) ds.$$

Hence, we showed (*). So, we have proved from (*), that $\hat{\phi}(x)$ is continuous and its derivative is also continuous on interval $(a, b+\beta]$.

This $\hat{\phi}(x)$ is said to be continuation of $\phi(x)$ on $(a, b+\beta]$. Hence proved.

5. System of Differential equations:

Let n be a positive number and $f_1, f_2, \dots, f_n$ are n continuous function defined on domain D of the real space $(x, y_1, y_2, \dots, y_n)$.
So, we must find n functions defined on real x interval I, such that it follows following conditions:

1) $(x, \phi_1, \phi_2, \dots, \phi_n) \in D, \quad x \in I.$
2) $\phi_i'(x) = f_i\big(x, \phi_1(x), \phi_2(x), \dots, \phi_n(x)\big), \quad where\ i = 1,2, \dots, n.$

This is called a system of n ordinary differential equation of first order denoted by,

$$y_i' = f_i(x, y_1, y_2, \dots, y_n), \quad where\ (i = 1,2, \dots, n). \tag{15}$$

Such an I(interval), and functions $\phi_1(x), \phi_2(x), \dots, \phi_n(x)$ are said to be solution of this DE (15), on I.

The IVP for (15)is based on finding such $\phi_1(x), \phi_2(x), \dots, \phi_n(x)$, such that $\phi_i(x_0) = y_{0_i}$.

Let X denote the Euclidean n dimensional space with points y having coordinate $(y, y_1, y_2, \dots, y_n)$. Then the functions f_i defined on the $(x, y_1, y_2, \dots, y_n)$ space have functions $\overline{f}_i$ on the (x, y) space defined by,

$$\overline{f}_i(x, y) = f_i(x, y_1, y_2, \dots, y_n).$$

Any point y in the y space is represented as,

$$y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix},$$

Is known as vector associated with y. Clearly,

$$\hat{f}_{\mathrm{i}}(x,\hat{y}) = \overline{f}_{\iota}(x,y).$$

$\hat{f}_i(x,\hat{y})$ is represented as,

$$\hat{f}(x,\hat{y}) = \begin{pmatrix} \hat{f}_1(x,\hat{y}) \\ \hat{f}_2(x,\hat{y}) \\ \vdots \\ \hat{f}_n(x,\hat{y}) \end{pmatrix},$$

Here y_i is a function of x, then $\widehat{y'}$ is defined by,

$$\widehat{y'} = \begin{pmatrix} y_1' \\ y_2' \\ \vdots \\ y_n' \end{pmatrix},$$

Then it can be written as,

$$\widehat{y'} = \hat{f}(x,\hat{y}).$$

But for not getting confused in $y, \bar{y}, \hat{y}$ we will just simply use

$$y' = f(x,y).$$

Where f is a vector of real x and $y \in Y$.

Now, let $\phi(x)$ be a solution of given differential equation with components $\phi_1(x), \phi_2(x), \dots, \phi_n(x)$ defined on I. then it should satisfy,

$$\big(x.\phi(x)\big) \in D, \qquad for\ x \in I$$

And

$$\phi'(\mathrm{x}) = \mathrm{f}\big(\mathrm{x}, \phi(\mathrm{x})\big), \qquad for\ x \in I.$$

Magnitude of vector y, are defined by,

$$|y| = \Sigma_1^n |y_i|$$

And Euclidean length,

$$\big||y|\big| = (\Sigma_1^n |y_i|^2)^{(1/2)}.$$

If distance between y and x which are in Y, is defined by $|y - x|$ then following are true,

1) $|y - x| = |x - y|$
2) $|y - x| \geq 0, \quad |y - x| = 0 \Rightarrow y = x.$
3) $|y - x| \leq |y - z| + |x - z|.$

Hence, Y containing all this vector is matrix space.

Let $\{y_k\}$ be sequence of vectors of Y. So, this sequence is said to be convergent if it is convergent with respect to distance function. For that $\{x_{k_i}\}$ should be convergent where $i = 1,2, \dots, n$.

Suppose a function f (vector having finite components) is defined by (x, y) on domain D, if there exists a k such that for every $(x, y), (x, \bar{y}) \in D$.

$$|\mathrm{f(x, y) - f(x, \bar{y})}| \leq \mathrm{k|y - \bar{y}|}.$$

Hence f is said to be Lipschitz continuous on given D.

So now, we can prove all the theorems we have done till now just by some simple conversion as like changing x and f by vector x and f. and magnitude of vector as we have done before. This will be doing for following equation,

$$y_i' = f_i(x, y_1, y_2, \dots, y_n) \qquad where\ (i = 1,2, \dots, n).$$

An interesting system is linear system,

$$y_i' = \Sigma_{j=1}^{n} a_{(ij)}(x) y_j$$

If f is the vector with components f_i then

$$f_i(x, y) = \Sigma_{j=1}^{n} a_{(ij)}(x) y_j \tag{L}$$

Where $a_{(ij)}$ are continuous functions on some close interval $[a, b)$.

Therefore, f satisfies Lipschitz condition on region,

$$D: \qquad a \leq x \leq b, \qquad |y| < \infty.$$

In fact, there exist a k for which following conditions hold where (x, y) and $(x, \bar{y})$ are in D.

$$|f(x, y) - f(x, \bar{y})| \leq k|y - \bar{y}|.$$

And $k = max\ \Sigma_{i=1}^{n} a_{(ij)}(y)$.

Theorem (a):
Suppose a function f is Lipschitz continuous on domain D with Lipschitz constant k. Let ϕ_1 and ϕ_2 be two different epsilon approximate solution (ϵ_1, ϵ_2) of IVP of class C_p ^1 on some interval (a, b), for which,

$$|\phi_1(x_0) - \phi_2(x_0)| \leq \delta.$$

Where δ is positive constant. Let ϵ be sum of both ϵ_1, ϵ_2 then, for every x in given interval.

$$|\phi_1(x) - \phi_2(x)| \leq \delta e^{k|x - x_0|} + \frac{\epsilon}{k}\left(e^{k|x - x_0|} - 1\right).$$

Theorem:

For (L), where $a_{(ij)}$ is continuous functions on interval $[a, b]$. Then there exists unique solution $\phi(x)$ of (L) on $[a, b]$ passing through $(x_0, y_0) \in D$.

Proof:

We know that f is Lipschitz continuous on D. So, existence and uniqueness of solution ω (L) through (x_0, y_0), on some subinterval $[c, d]$ of $[a, b]$. Also, by continuation theorem, we can expand the continuation interval of ϕ $[c, d]$ to $[a, b]$.

Hence proved.

Using similar argument, unique solution can be continued to any unbounded interval I, too.

6. nth order Equation:

Let f be a real and continuous function in some domain D, of space $(x, y_1, y_2, \dots, y_n)$,then nth order equation of this will be,

$$y^{(n)} = f\left(x, y, y', y'', \dots, y^{(n-1)}\right), \qquad (E)$$

Our problem of this kind of equation will be to find a function on real interval I, for which these are satisfied,

1) $\left(x, \phi(x), \phi'(x), \dots, \phi^{(n-1)}(x)\right) \in D, \quad (for\ x \in I)$
2) $\phi^{(n)}(x) = f\left(x, \phi'(x), \phi''(x), \dots, \phi^{(n-1)}(x)\right), \quad (for\ x \in I)$

Such ϕ is said to be solution of (E) on interval I. Here y, f, ϕ are not vectors.

Initial value problem of nth order equation consists of finding ϕ for some x_0 in interval I, such that $\phi(x_0) = y_0, \phi'(x_0) = y_{0_1}, \dots, \phi^{(n-1)}(x_0) = y_{0_{(n-1)}}$. Where $\left(x_0, y_0, y_{0_1}, \dots, y_{0_{(n-1)}}\right) \in D$.

This theory of finding solution of nth order differential equations can be reduced to theory of finding solution of system of equations. Here is how we can do it using (E),

$$\begin{aligned} y_1 &= y \\ y_2 &= y_1' \\ y_3 &= y_2' \\ &. \\ &. \\ &. \\ y_n' &= f(x, y_1, y_2, \dots, y_n) \end{aligned} \qquad (\bar{E})$$

If the vector $\bar{\phi}$ having components $(\phi_1, \phi_2, \phi_3, \dots, \phi_n)$ is a solution of $(\bar{E})$ then we have,

$$\phi_2 = \phi_1',\ \phi_3 = \phi_1'', \dots,\ \phi_n = \phi_1^{(n-1)}.$$

And,

$$\phi_1^{(n)}(x) = f\left(x, \phi_1(x), \phi_2(x), \dots, \phi_n(x)\right) = f\left(x, \phi_1(x), \phi_1'(x), \dots, \phi_1^{(n-1)}(x)\right).$$

So, $\bar{\phi}$ is a solution of system of equation $(\bar{E})$ and it's first component ϕ_1 is a solution of (E). Such system $(\bar{E})$ is called *system associated with the nth order equation (E).* So, all statements which we proved for system of equation are applicable in nth order equation.

Hence,

If f is a continuous function of space $(x, y_1, y_2, \dots, y_n)$ in a domain D, and if P is a point of this domain D. Then, there exists a solution $\phi \in C^n$ of (E) on some interval , which passes through P.

And if there exist $k > 0$, such that,

$$|f(x, y_1, y_2, \dots, y_n) - f(x, \bar{y}_1, \bar{y}_2, \dots, \bar{y}_n)| \leq k\Sigma_{i=1}^{n}|y_i - \bar{y}_i|.$$

If this condition holds, that means f is Lipschitz continuous, and therefore ϕ will be unique through P.

7. Solution's dependence on initial conditions and parameters

A solution of differential equation on interval I is not only a function of x but also of coordinate of initial value problem's initial point. Hence, ϕ is a function of (x, x_0, y_0). Now we will check this for general case of system of equations.
Let, D be a domain of space $(n + 1)$ dimension and suppose f is Lipschitz continuous in D. Let ϕ be a solution of following equation,

$$y' = f(x, y).$$

$$\therefore \left(x, \phi(x)\right) \in D, \qquad for\ x \in I.$$

Also, from existence theorem we know that this equation has a unique solution through (x_0, y_0). But through existence theorem, we only got solution for very small interval. To solve this, we can show that the solution exists on whole interval I.

Theorem:
Let f be continuous and Lipschitz continuous in a domain D of the (n+1) dimensional (x, y) space. Let assume $\omega(x)$ be a solution of $y' = f(x, y)$ on interval $[a, b]$. there exist a $\delta > 0$ such that initial point (x_0, y_0) is inside U, where

$$U: \qquad a < x_0 < b \qquad |y_0 - \omega(x_0)| < \delta$$

Then there exists a unique solution ϕ of $y' = f(x, y)$ on I with initial condition $\phi(x_0, x_0, y_0) = y_0$. Also, ϕ is continuous on (n+2) dimensional set,

$$V: \qquad a < x < b \qquad (x_0, y_0) \in U$$

Proof:

First choose a region U_1 in (x, y),

$$U_1 : x \in I, |y - \omega(x)| \leq \delta_1.$$

Where $\delta_1 \geq 0$, such that U_1 is inside D.
Then choose δ such that $\delta < e^{(-k|b-a|)}\delta_1$, where k is Lipschitz constant. So, now we can define a region U as in statement of theorem.

Also there exists a solution locally, such that,

$$\phi(x, x_0, y_0) = y_0 + \int_{x_0}^{x} f\left(t, \phi(t, x_0, y_0)\right) dt.$$

And for $x \in I$,

$$\omega(x) = \omega(x_0) + \int_{x_0}^{x} f\left(t, \omega(t)\right) dt.$$

As both are exact solution of f, according to inequality of theorem (a),

$$
\begin{aligned}
|\phi(x, x_0, y_0) - \omega(x)| &\leq |y_0 - \omega(x_0)| e^{(k|x-x_0|)} \\
&\leq (max|y - \omega(x)|) e^{(k|x - x_0)} \\
&< \delta_1.
\end{aligned}
$$

Hence proved that ϕ can't leave U_1. And can be continuous on whole I, using continuation theorem.

Now, once we have existence and continuity of ϕ as a function (x, x_0, y_0). It is also important to give reasonable condition for existence and continuity of $\frac{\partial \phi}{\partial x_0}, \frac{\partial \phi}{\partial y_{0_1}}, \frac{\partial \phi}{\partial y_{0_2}}$ ..., where $\{y_{0_j}\}, j = 1,2,3, \ldots, n$ are components of y_0.

A condition which is sufficient for all this, is continuity and existence of $, \frac{\partial f}{\partial y_j}$ on D.

Now let f_x be a matrix with element $\frac{\partial f_i}{\partial y_j}$ on ith row and jth column, also let ϕ_{y_0} be a matrix with element $\frac{\partial \phi_i}{\partial y_{0_j}}$ on ith row and jth column. A matrix is said to be continuous if all its elements are continuous.
In square matrix A, $det\ A$ is notation of determinant, $tr\ A$ is notation of trace of it. The e^u is shown as $exp\ u$.
Theorem:
Let M be a n sized square matrix which is continuous on interval $I = [a, b]$ and let be ϕ is a matrix of a function on I satisfying

$$\phi'(x) = M(x)\phi(x)$$

Then for x and x_0 in I,

$$det\ \phi(x) = det\ \phi(x_0)\ exp \int_{x_0}^{x} tr\ M(t) dt$$

Theorem:
Let f be a continuous function which also satisfy Lipschitz condition, then we know $\phi \in C'$ on V, by previous theorem and suppose f_x exists and f_x is continuous on D. Then

$$\det \phi_{y_0} (\mathrm{x}, x_0, y_0) = \exp \int_{x_0}^{x} \mathrm{tr} \left(f_x\big(\mathrm{t}, \phi(\mathrm{t}, x_0, y_0)\big) \right) \ \mathrm{dt}$$

Proof:
First, we need to prove existence of ϕ_{y_0}, it is sufficient to prove existence of $\frac{\partial \phi}{\partial y_{0_1}}$ to prove whole existence of ϕ_{y_0}, where $y_0 = \left(y_{0_1}, y_{0_2}, \ldots, y_{0_n}\right)$, let $h = (h_1, 0,0, \ldots, 0)$. $\overline{y_0} = y_0 + h$, let (x_0, y_0) and $(x_0, \overline{y_0})$ in U. If we define a function ω for $(x, x_0, y_0) \in V$. Then we need to prove existence of

$$\lim_{h \to 0} \omega(x, x_0, y_0, h), \qquad where\ \omega(x, x_0, y_0, h) = \frac{\phi(x, x_0, y_0) - \phi(x, x_0, y_0)}{h_1}$$

We know that ϕ is a solution of $y' = f(x, y)$,

$$\therefore \phi'(x, x_0, y_0) = f\big(x, \phi(x, x_0, y_0)\big)$$

$$\therefore \left(\frac{\partial \phi}{\partial y_{0_1}}\right)' = f_x\big(x, \phi(x, x_0, y_0)\big)\frac{\partial \phi}{\partial y_{0_1}}(x, x_0, y_0)$$

From this we can easily say $\frac{\partial \phi}{\partial y_{0_1}}$ is a solution of a linear differential equation. Let

$$\theta(x, x_0, y_0, h) = \phi(x, x_0, \overline{y_0}) - \phi(x, x_0, y_0)$$

Therefore,

$$\phi(x, x_0, y_0, h) \leq \phi(x_0, x_0, y_0, h)e^{(k|x-x_0|)}$$

Because both are exact solution of $y' = f(x, y)$, we have $\epsilon = 0$, therefore,

$$\theta(x, x_0, y_0, h) \leq |h_1|e^{(k|x-x_0|)} \tag{7.1}$$

As $h_1 \to 0, \theta \to 0$ for $(x, x_0, y_0) \in V$.

$$\theta'(x, x_0, y_0, h) = f\big(x, \phi(x, x_0, \overline{y_0})\big) - f\big(x, \phi(x, x_0, y_0)\big)$$

Hence there exists a Matrix T such,

$$\therefore \theta'(x, x_0, y_0, h) = \Big(f_x\big(x, , \phi(x, x_0, y_0)\big) + T\Big)\theta(x, x_0, y_0, h)$$

Now $|T| = \Sigma_{i,j=1}^{n}\left|T_{(ij)}\right| \leq \epsilon_1$ if $|\theta| < \delta_1$ for $(x, x_0, y_0) \in V$, and we already knew that as $h_1 \to 0, \theta \to 0$, therefore, as $h_1 \to 0, |T| \to 0$.

Now,

$$\frac{\theta'(x, x_0, y_0, h)}{h_1} = \frac{f_x(x, \phi(x, x_0, y_0))\theta(x, x_0, y_0, h) + T\theta}{h_1}$$

$$\therefore \omega(\mathrm{x}, \mathrm{x}_0, \mathrm{y}_0, \mathrm{h}) = \mathrm{f_x}\big(\mathrm{x}, \phi(\mathrm{x}, \mathrm{x}_0, \mathrm{y}_0)\big)\omega(\mathrm{x}, \mathrm{x}_0, \mathrm{y}_0, \mathrm{h}) + \beta$$

Where

$$\beta = |\mathrm{T}|\left|\frac{\theta}{\mathrm{h}_1}\right|$$

$$\therefore \beta \leq |T|e^{(k|x-x_0|)} \tag{$\because$ 7.1}$$

Hence, $\beta \to 0$, as $h_1 \to 0$ uniformly on V, for any $\epsilon > 0$, there exist $\delta_\epsilon > 0$ such that $|\beta| < \epsilon$ if $|h_1| \leq \delta_{(\epsilon)}$, thus ω became ϵ-approximate solution of linear equation

$$y' = f_x\big(x, \phi(x, x_0, y_0)\big)y \tag{7.2}$$

And the initial value $\omega(x_0, x_0, y_0, h)$ is $e_1 = (1,0,0, \dots ,0)$.
Also, we know that there exists a solution $\bar{\phi}$ of (7.2) with initial condition e_1 at $x = x_0$. Then it can be continued on whole $[a, b]$. Now we have,

$$\left|\omega(x, x_0, y_0, h) - \bar{\phi}(x, x_0, y_0)\right| \leq \frac{\epsilon}{k}\left(e^{(|k(b-a)|)} - 1\right)$$

Hence, we can clearly see that

$$\lim_{h\to 0}\omega(x, x_0, y_0, h) = \bar{\phi}(x, x_0, y_0)$$

This proves the existence of $\frac{\partial \phi}{\partial y_{0_1}}$ and that $\frac{\partial \phi}{\partial y_{0_1}}$ is a solution of (7.2), By assuming initial value e, at $x = x_0$, Also it implies continuity $\frac{\partial \phi}{\partial y_{0_1}}$ on V.

Also, by following same method, we can prove existence and continuity of $\frac{\partial \phi}{\partial y_{0_1}}$, $j = 1,2,3 \dots$ on V. Also e_j is vector with all zeroes and one only at jth place.

$$\frac{\partial\phi}{\partial y_{0_1}}(x_0, x_0, y_0) = e_j$$

Hence $\frac{\partial\phi}{\partial y_{0_1}}$ or ϕ_{y_0} is solution of (7.2). we have $\phi_{y_0}(x_0, y_0, z_0) = E = \begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix}$

Using same method to show that $\frac{\partial\phi}{\partial x_0}$ is also solution of linear system (7.2), we got its initial condition as following,

$$\frac{\partial\phi}{\partial x_0}(x_0, x_0, y_0) = f(x_0, y_0).$$

Then by using previous theorem,

$$\det \phi_{y_0}(x, x_0, y_0) = \exp \int_{x_0}^{x} \operatorname{tr}\left(f_x\big(t, \phi(t, x_0, y_0)\big)\right) dt.$$

Hence proved.

8. Complex system

Till now, we have assumed that equation $y' = f(x, y)$ have x, y, f all real. If we take f as a continuous complex function on an open set D in (x, y) space, where x is real and y is complex n dimensional vector, which means real 2n dimensional vector. Our equation will be following,

$$y' = f(x, y). \qquad (Ec)$$

Basic problem related to this equation would be finding some interval I on some real x line and a complex function ϕ which is also differentiable on I. Such that it satisfies following,

1) $(x, \phi(x)) \in D,\ for\ x \in I$
2) $\phi'(x) = f\big(x, \phi(x)\big),\ \ for\ x \in I.$

We can clearly see that all the theorems which we have proved till now, for existence and uniqueness, continuation of solution and many other, are valid for given equation (Ec). Which can be done by defining $|y|$ of a complex vector $y = (y_1, y_2, \dots, y_n)$ as following,

$$|y| = \Sigma_{i=1}^{n} |y_i|.$$

Where $|y_i| = \left(\big(r(y_i)\big)^2 + \big(i(y_i)\big)^2\right)^{(1/2)}$, where $r(x), i(x)$ are real and imaginary part of complex number x, respectively. Usually, functions defined on a set of complex numbers which occur in differential equations are mostly analytic. Let f be function defined on D on y vector space. Then of some neighborhood of some point $y_0 \in D$, $|y - y_0| < \delta$ and $\delta > 0$. Each component of f, f_j is continuous in

$$y = (y_1, y_2, \dots, y_n)$$

And is analytic in each y_i where all other y_j are held fixed. What we meant to say can be explained in another way too. Like, each f_j can be represented by convergent power series.

$$f_j(y_1, y_2, \dots, y_n) = \Sigma_{m_1=0}^{\infty} \dots \Sigma_{m_n=0}^{\infty} A_{(m_1,\dots,m_n)} \big(y_1 - y_{0_1}\big)^{(m_1)} \dots \big(y_n - y_{0_n}\big)^{(m_n)}$$

The $A_{(m_1,\ldots,m_n)}$ is complex constant. Also, a function f is said to be analytic in whole domain if it is analytical at each point of domain. We know that analytic functions do have derivatives of all order on given domain. If a sequence of analytic function converges uniformly on a domain D, then limit function is also analytic on that domain.

We can extend this problem further by taking x as a complex number.

If f is an analytic complex valued vector function on domain D of (x, y) space, x is one complex dimensional, and y has n dimensional space.

Then for equation

$$y' = f(x, y) \qquad (E_2)$$

Our challenge would be finding domain H in the complex x plane such that any solution ϕ of equation (E_2) satisfies following,

1) $\big(x, \phi(x)\big) \in D, \quad for\ x \in H$
2) $\phi'(x) = f\big(x, \phi(x)\big), \quad for\ x \in H$

The existence and uniqueness of solution of (E_2) can be found by successive approximation, as we done in Picard's theorem. Suppose f has components $f_1, f_2, f_3, \ldots, f_n$ and similarly y has components $y_1, y_2, \ldots, y_n$ and f is analytic on given rectangular (n+1) complex dimensional rectangle,

$$R_2: \quad |x - x_0| < a, \qquad |y - y_0| < b$$

Where $(a, b > 0)$.

Theorem:

f is analytic and bounded on open rectangle R_2 and suppose,

$$M = \sup_{(x,y)\in R} |f(x, y)|$$

And

$$\alpha = min\,(a, b/M)$$

Then there exist unique solution of (E_2), ϕ such that $\phi(x_0) = y_0$ on $|x - x_0| < \alpha$ interval.

Proof:

The matric f_w having $\frac{\partial f_i}{\partial y_j}$ on ith row and jth column as components, will be bounded on every closed rectangle $R_3 \subset R_2$, hence f passes Lipschitz condition with some $k > 0$ as Lipschitz constant on R_3.

Now, we will construct successive approximation for it with initial rough approximation,

$$\phi_0(x) = y_0$$

And

$$\phi_{(k+1)}(x) = y_0 + \int_{x_0}^{x} f\big(t, \phi_k(t)\big)dt \qquad (*)$$

Where $k = 0,1,2 \ldots$

Applying same argument we have done in Picard's theorem; we can see existence of unique solution ϕ on the circle $|x - x_0| < \alpha$ with passing through (x_0, y_0).

We know that $\phi_0(x)$ will be analytic function as it is constant function, hence $f_0 = f(x, \phi_0)$ would also be analytic on given interval $|x - x_0| < \alpha$.

Using equation $(*)$ we can surely say that ϕ_1 would also be analytic, hence further all ϕ_k approximations would be analytic. Since the solution ϕ is uniform convergent of $\{\phi_k\}$ series, therefore it would also be analytic on $|x - x_0| < \alpha$.

Hence proved.

Similarly all the previous theorems, can be applied to complex systems by doing some necessary changes only.

9. Conclusion:

In this we learned theorems related to existence and uniqueness of solutions of ODEs and system of ODEs. Also, by doing this we also solved some initial value problems and some real-life problems. Also concepts such as continuation of solution on open interval, system of equation, nth order equation and many things were covered during this work.
After completion of this work, we can answer questions such as, does a solution exist? Is it unique or not? And many others.

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