Differential Equation

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Abstract

113 微分方程式

Contents

1 Introduction

1.1 Some Mathematical Models

In order to motivate this course, we begin with some examples from [BD22, Che16]. The most simplest and important example which can be modeled by ordinary differential equations (ODE) is a relaxation process, i.e. the system starts from state and eventual reaches an equilibrium state.

Example 1.1.1. *A falling object*
We now consider an object with mass m falling from height y_0 at time $t = 0$.
Let $v(t)$ be its velocity at time t.
According to physics law, we know that the acceleration of the object at time t is the rate of change of velocity $v(t)$, that is, $a(t) = \frac{d}{dt}v(t) = v'(t)$.
According to Newton's second law, the net force F exerted on the on the object is expressed by the equation $F(t) = ma(t) = mv'(t)$.
Next, we consider the forces that act on the object as it falls.
The gravity exerts a force equal to the weight of the object given by mg, where g is the acceleration due to the gravity.
The drag force due to air resistance has the magnitude $\gamma v(t)$, where γ is a constant called the drag coefficient.
Therefore the net force is given by $F(t) = mg - \gamma v(t)$.
Combining $F(t) = ma(t) = mv'(t)$ and $F(t) = mg - \gamma v(t)$, we reach the ODE $mv'(t) = mg - \gamma v(t)$, for $t > 0$.

Example 1.1.2. *Heating (or cooling) of an object*
We now consider an object, with initial temperature T_0 at time $t = 0$, which is taken out of refrigerator to defrost.
Let $T(t)$ be its temperature at time t.
Suppose that the room temperature is given by K.
The Newton's law of cooling/heating says that the rate change of temperature $T(t)$ is proportional to the difference between $T(t)$ and K,
more precisely, $T'(t) = -\alpha(T(t) - K)$, where $\alpha > 0$ is a conductivity coefficient.

Example 1.1.3. *Population growth model*
We first describe the population model proposed by Malthus ($1766 \sim 1834$).
Let $y(t)$ be the population (in a large area) at time t.
He built a model based on the following hypothesis :
$y'(t) =$ births $-$deaths $+$migration, and he assume that the births and the deaths are proportion to the current population $y(t)$, i.e., births $-$deaths $= ry(t)$, where the constant $r \in \mathbb{R}$ is called the net growth rate.
If there is no migration at all, then the model reads $y'(t) = ry(t)$, which is called

the simple population growth model.
Suppose that the initial population is y_0 at time $t = 0$.
In fact, the unique solution is $y(t) = y_0 e^{rt}$, which is not make sense, since the environment limitation is not taken into account.
With this consideration, we should expect that $\exists$ an environment carrying capacity $K \ni y'(t) > 0$ when $y(t) < K, y'(t) < 0$ when $y(t) > K$ due to a competition of resource.
Verhulst (1804 $\sim$ 1849) *proposed another model which take the limit of environment into consideration (i.e. in a small area) : $y'(t) = ry(1 - \frac{y(t)}{K})$, which is called the logistic population model.*
Note the above simple population growth model formally corresponds to the case when $K = +\infty$.
See also [BD22, Section 2.5] for further explainations.

1.2 Classification of ODE

We say that the system of equations of the form $F(t, u(t), u''(t), \ldots, u^{(m)}(t)) = 0$, or in equation form $\begin{cases} F_1(t, u(t), u''(t), \ldots, u(m)(t)) = 0 \\ \vdots \\ F_\ell(t, u(t), u''(t), \ldots, u^{(m)}(t)) = 0 \end{cases}$, the ordinary differential equation (ODE) of order m, where we write $u(k) := (u_1^{(k)}, \ldots, u_n^{(k)})$ the k^{th}-order derivative of $u = (u_1, \ldots, u_n)$.

Example 1.2.1.
$u''' + 2e^t u'' + uu' = t^4$ is a third order ODE.

In many cases, we only consider the system of ODE of the form (with $n = \ell$) $u^{(m)}(t) = f(t, u(t), u''(t), \cdots, u^{(m-1)}(t))$.
Otherwise, for example, the equation $(u')2 + tu' + 4u = 0$ leads to two equations $u' = \frac{-t+\sqrt{t^2-16u}}{2}, u' = \frac{-t-\sqrt{t^2-16u}}{2}$.

Definition 1.2.1.
An ODE is said to be linear if it takes the form $a_0(t)u^{(n)}+a_1(t)u^{(n`1)}+\cdots+a_n(t)u = g(t)$, for some matrices $a_0, \ldots, a_n$, otherwise we say that the ODE is nonlinear.
If $g(t) \equiv 0$, then we say that the ODE is homogeneous, otherwise inhomogeneous.

2 First order nonlinear ODE

Assume f is known.
This chapter deals with first order nonlinear ODE of the form $u'(t) = f(t, u)$ with $u(t_0) = u_0$, for some vector $u_0 \in \mathbb{R}^n$.
u_0 是給定的初始狀態
We again remind the readers that we use the notation $u(t) = (u_1(t), \ldots, u_n(t))$, and $f(t) = (f_1(t), \ldots, f_n(t))$, and we also use the notation $|u(t)| = \max\{|u_1(t)|, \ldots, |u_n(t)|\}$.
Question：什麼樣的 $f \ni u$ 存在？

2.1 Well-posedness of ODE

If $f \equiv 0$, then $u'(t) = f(t, u)$ with $u(t_0) = u_0$ reads $u'(t) = 0, u(t_0) = u_0$, and one easily sees that the constant function $u = u_0$ is a solution which is valid $\forall t \in \mathbb{R}$. We first state the fundamental existence theorem when $f \not\equiv 0$.
先來看看非零函數時的存在性

Theorem 2.1.1. 證明存在性 (*The Picard-Lindeloef theorem*)
Let $a > 0$ and $b > 0$.
If $f = f(t, y)$ is a (real-valued) continuous function on a closed cylinder $\mathcal{R} = \{(t, y) \in \mathbb{R} \times \mathbb{R}^n; |t - t_0| \leq a, |y - u_0| \leq b\} \ni M := \max\limits_{(t,y)\in\mathcal{R}} |f(t, y)| > 0$, then $\exists$ a function $u \in C^1((t_0 - \alpha, t_0 + \alpha); \mathbb{R}^n)$ with $\alpha = \min\{a, \frac{b}{M}\}$ satisfying $u'(t) = f(t, u)$ and $u(t_0) = u_0$ in $(t_0 - \alpha, t_0 + \alpha)$.

However, the uniqueness does not hold true in general without further assumption on f. We demonstrate this in the following few examples.
然而存在並不保證函數解的唯一性
Question：What condition makes the uniqueness hold？

Example 2.1.1.
Check that $\begin{cases} f(t, y) = ty^{\frac{1}{5}} \text{ and } t_0 = 3 \\ u'(t) = f(t, u(t)), \forall t \in \mathbb{R}, u(t_0) = 0 \end{cases}$.
⟨Solution⟩
We define the function $u(t) := \begin{cases} 0 & if\ t \leq 3 \\ \left(\frac{2}{5}(t^2 - 9)\right)^{\frac{5}{4}} & if\ t > 3 \end{cases}$.
By using left and right limits, first check that $u \in C(\mathbb{R})$.
Since chain rule and product rule only holds on open set.
Then $u'(t) = \left(\frac{2}{5}\right)^{\frac{1}{4}} t(t^2 - 9)^{\frac{1}{4}}$, for $t > 3$, and $u'(t) = 0$, for $t < 3$.
Let's watch $t = 3$.

Since $\lim\limits_{h\to 0^+} \frac{u(3+h)-u(3)}{h} = \lim\limits_{h\to 0^+} \frac{1}{h}\big(\frac{2}{5}\big((3+h)^2-9\big)\big)^{\frac{5}{4}} = \lim\limits_{h\to 0^+} \frac{1}{h}\big(\frac{2}{5}h(h+6)\big)^{\frac{5}{4}}$
$= \lim\limits_{h\to 0^+} h^{\frac{1}{4}}\big(\frac{2}{5}(h+6)\big)^{\frac{5}{4}} = 0$ *and* $\lim\limits_{h\to 0^-} \frac{u(3+h)-u(3)}{h} = \lim\limits_{h\to 0^-} \frac{0}{h} = 0.$
Then limit exists, $u'(3) := \lim\limits_{h\to 0} \frac{u(3+h)-u(3)}{h} = 0.$
Hence, $u \in C(\mathbb{R})$*.*
By $u' \in C(\mathbb{R})$*, we can check* $u \in C^1(\mathbb{R})$*.*
Since $\lim\limits_{t\to 3^+} u'(t) = \lim\limits_{t\to 3^+} \big(\frac{2}{5}\big)^{\frac{1}{4}} t(t^2-9)^{\frac{1}{4}} = 0$ *and* $\lim\limits_{t\to 3^-} u'(t) = \lim\limits_{t\to 3^-} 0 = 0.$
Then $u \in C^1(\mathbb{R})$*.*

Note :
f is continuous in $\mathbb{R}\times\mathbb{R}$, and hence the assumptions in Theorem above satisfy.
Since $u \equiv 0$ is also another solution of $\begin{cases} f(t,y) = ty^{\frac{1}{5}} \text{ and } t_0 = 3 \\ u'(t) = f(t,u(t)), \forall t \in \mathbb{R}, u(t_0) = 0 \end{cases}$, one sees that the solution of initial value problem $\begin{cases} f(t,y) = ty^{\frac{1}{5}} \text{ and } t_0 = 3 \\ u'(t) = f(t,u(t)), \forall t \in \mathbb{R}, u(t_0) = 0 \end{cases}$ is not unique.
e.g. Since $y = 0$ or $y = 1$, then solution in equations not unique.

Exercise 2.1.
Verify that the initial-value problem $u'(t) = \big(u(t)\big)^{\frac{1}{3}}, \forall t \in \mathbb{R}, u(t_0) = 0$ *has at least two non-trivial* $C^1(R)$*-solutions:*
$u(t) = \begin{cases} 0 & if\ t \le t_0 \\ \big(\frac{2}{3}(t-t_0)\big)^{\frac{3}{2}} & if\ t > t_0 \end{cases}$ *and* $u(t) = \begin{cases} 0 & if\ t \le t_0 \\ -\big(\frac{2}{3}(t-t_0)\big)^{\frac{3}{2}} & if\ t > t_0 \end{cases}$.
⟨*Solution*⟩
Goal 1 : check $u(t) = \begin{cases} 0 & if\ t \le t_0 \\ \big(\frac{2}{3}(t-t_0)\big)^{\frac{3}{2}} & if\ t > t_0 \end{cases}$
Define $u(t) := \begin{cases} 0 & if\ t \le t_0 \\ \big(\frac{2}{3}(t-t_0)\big)^{\frac{3}{2}} & if\ t > t_0 \end{cases}$.
claim : $u \in C(\mathbb{R})$*.*
By using left and right limits,
$\lim\limits_{t\to {t_0}^+} u(t) = \lim\limits_{t\to {t_0}^+} (\frac{2}{3}(t-t_0))^{\frac{3}{2}} = 0 = u(t_0)$ *and* $\lim\limits_{t\to {t_0}^-} u(t) = \lim\limits_{t\to {t_0}^-} 0 = 0 = u(t_0)$*.*
claim : u is differentiable.
Since chain rule and product rule only holds on open set.
Then $u'(t) = \frac{d}{dx}u(t) = \frac{d}{dx}(\frac{2}{3}(t-t_0))^{\frac{3}{2}} = \frac{3}{2}((\frac{2}{3}(t-t_0))^{\frac{1}{2}})\frac{2}{3} = (\frac{2}{3}(t-t_0))^{\frac{1}{2}}$*, for* $t > t_0$*, and* $u'(t) = 0$*, for* $t < t_0$*.*
Watch $t = t_0$*.*
Since $\lim\limits_{h\to 0^+} \frac{u(t_0+h)-u(t_0)}{h} = \lim\limits_{h\to 0^+} \frac{1}{h}\big(\frac{2}{3}\big(h^{\frac{3}{2}}-0\big)\big) = \lim\limits_{h\to 0^+} \frac{2}{3}h^{\frac{1}{2}} = 0$

and $\lim\limits_{h\to 0^-}\frac{u(t_0+h)-u(t_0)}{h}=\lim\limits_{h\to 0^-}\frac{0-0}{h}=0.$
Then limit exists, $u'(t_0):=\lim\limits_{h\to t_0}\frac{u(t_0+h)-u(t_0)}{h}=0.$
claim : $u\in C^1(\mathbb{R})$.
By $u'\in C(\mathbb{R})$, *we can check* $u\in C^1(\mathbb{R})$.
Since $\lim\limits_{t\to {t_0}^+}u'(t)=\lim\limits_{t\to {t_0}^+}(\frac{2}{3}(t-t_0))^{\frac{1}{2}}=0=u'(t_0)$
and $\lim\limits_{t\to {t_0}^-}u'(t)=\lim\limits_{t\to {t_0}^-}0=0=u'(t_0)$.
Then $u\in C^1(\mathbb{R})$, *and* $u'(t)=\big(u(t)\big)^{\frac{1}{3}},\forall t\in\mathbb{R}$.
Similarly for the other one.

Goal 2 : check $u(t)=\begin{cases}0 & if\ t\le t_0\\ -\big(\frac{2}{3}(t-t_0)\big)^{\frac{3}{2}} & if\ t>t_0\end{cases}$

Define $u(t):=\begin{cases}0 & if\ t\le t_0\\ -\big(\frac{2}{3}(t-t_0)\big)^{\frac{3}{2}} & if\ t>t_0\end{cases}.$

claim : $u\in C(\mathbb{R})$.
By using left and right limits,
$\lim\limits_{t\to {t_0}^+}u(t)=\lim\limits_{t\to {t_0}^+}-(\frac{2}{3}(t-t_0))^{\frac{3}{2}}=0=u(t_0)$ *and* $\lim\limits_{t\to {t_0}^-}u(t)=\lim\limits_{t\to {t_0}^-}0=0=u(t_0)$.
claim : u is differentiable.
Since chain rule and product rule only holds on open set.
Then $u'(t)=\frac{d}{dx}u(t)=\frac{d}{dx}(-(\frac{2}{3}(t-t_0))^{\frac{3}{2}})=\frac{3}{2}(\frac{-2}{3}(t-t_0))^{\frac{1}{2}})\frac{-2}{3}=(\frac{2}{3}(t-t_0))^{\frac{1}{2}}$, *for* $t>t_0$, *and* $u'(t)=0$, *for* $t<t_0$.
Watch $t=t_0$.
Since $\lim\limits_{h\to 0^+}\frac{u(t_0+h)-u(t_0)}{h}=\lim\limits_{h\to 0^+}\frac{1}{h}\big(\frac{-2}{3}\big(h^{\frac{3}{2}}-0\big)\big)=\lim\limits_{h\to 0^+}\frac{-2}{3}h^{\frac{1}{2}}=0$
and $\lim\limits_{h\to 0^-}\frac{u(t_0+h)-u(t_0)}{h}=\lim\limits_{h\to 0^-}\frac{0-0}{h}=0.$
Then limit exists, $u'(t_0):=\lim\limits_{h\to t_0}\frac{u(t_0+h)-u(t_0)}{h}=0.$
claim : $u\in C^1(\mathbb{R})$.
By $u'\in C(\mathbb{R})$, *we can check* $u\in C^1(\mathbb{R})$.
Since $\lim\limits_{t\to {t_0}^+}u'(t)=\lim\limits_{t\to {t_0}^+}(\frac{-2}{3}(t-t_0))^{\frac{1}{2}}=0=u'(t_0)$
and $\lim\limits_{t\to {t_0}^-}u'(t)=\lim\limits_{t\to {t_0}^-}0=0=u'(t_0)$.
Then $u\in C^1(\mathbb{R})$, *and* $u'(t)=\big(u(t)\big)^{\frac{1}{3}},\forall t\in\mathbb{R}$.
Therefore, the initial-value problem $u'(t)=\big(u(t)\big)^{\frac{1}{3}},\forall t\in\mathbb{R},u(t_0)=0$ *has at least two non-trivial* $C^1(R)$*-solutions.*

Note :
f is continuous in $\mathbb{R} \times \mathbb{R}$, and hence the assumptions in Theorem 2.1.1 satisfy.
Since $u \equiv 0$ is also another solution of $\begin{cases} f(t,y) = ty^{\frac{1}{5}} \text{ and } t_0 = 3 \\ u'(t) = f(t,u(t)), \forall t \in \mathbb{R}, u(t_0) = 0 \end{cases}$, one sees that the solution of initial value problem $\begin{cases} f(t,y) = ty^{\frac{1}{5}} \text{ and } t_0 = 3 \\ u'(t) = f(t,u(t)), \forall t \in \mathbb{R}, u(t_0) = 0 \end{cases}$ is not unique.

Exercise 2.2.
Verify that the initial-value problem $u'(t) = \sqrt{|u(t)|}, \forall t \in \mathbb{R}, u(t_0) = 0$ has at least one non-trivial $C^1(R)$-solutions:
$u(t) = \begin{cases} -\frac{1}{4}(t-t_0)^2 & if\ t \le t_0 \\ \frac{1}{4}(t-t_0)^2 & if\ t > t_0 \end{cases}$.
⟨Solution⟩
goal : check $u(t) = \begin{cases} -\frac{1}{4}(t-t_0)^2 & if\ t \le t_0 \\ \frac{1}{4}(t-t_0)^2 & if\ t > t_0 \end{cases}$
Define $u(t) := \begin{cases} -\frac{1}{4}(t-t_0)^2 & if\ t \le t_0 \\ \frac{1}{4}(t-t_0)^2 & if\ t > t_0 \end{cases}$.
claim : $u \in C(\mathbb{R})$.
By using left and right limits,
$\lim\limits_{t \to t_0^+} u(t) = \lim\limits_{t \to t_0^+} \frac{1}{4}(t-t_0)^2 = 0 = u(t_0)$ *and* $\lim\limits_{t \to t_0^-} u(t) = \lim\limits_{t \to t_0^-} \frac{-1}{4}(t-t_0)^2 = 0 = u(t_0)$, *then* $u \in C(\mathbb{R})$.
claim : u is differentiable.
Since chain rule and product rule only holds on open set.
Then $u'(t) = \frac{d}{dx}u(t) = \frac{d}{dx}(\frac{1}{4}(t-t_0)^2) = 2(\frac{1}{4}(t-t_0)) = \frac{1}{2}(t-t_0)$, *for* $t > t_0$,
and $u'(t) = \frac{d}{dx}u(t) = \frac{d}{dx}(\frac{-1}{4}(t-t_0)^2) = 2(\frac{-1}{4}(t-t_0)) = \frac{-1}{2}(t-t_0)$, *for* $t < t_0$.
Watch $t = t_0$.
Since $\lim\limits_{h \to 0^+} \frac{u(t_0+h)-u(t_0)}{h} = \lim\limits_{h \to 0^+} \frac{\frac{1}{4}h^2 - 0}{h} = \lim\limits_{h \to 0^+} \frac{1}{4}h = 0$
and $\lim\limits_{h \to 0^-} \frac{u(t_0+h)-u(t_0)}{h} = \lim\limits_{h \to 0^-} \frac{\frac{-1}{4}h^2 - 0}{h} = \lim\limits_{h \to 0^+} \frac{-1}{4}h = 0$.
Then limit exists, $u'(t_0) := \lim\limits_{h \to t_0} \frac{u(t_0+h)-u(t_0)}{h} = 0$.
claim : $u \in C^1(\mathbb{R})$.
By $u' \in C(\mathbb{R})$, *we can check* $u \in C^1(\mathbb{R})$.
Since $\lim\limits_{t \to t_0^+} u'(t) = \lim\limits_{t \to t_0^+} \frac{1}{2}(t-t_0) = 0 = u'(t_0)$
and $\lim\limits_{t \to t_0^-} u'(t) = \lim\limits_{t \to t_0^-} \frac{1}{2}(t-t_0) = 0 = u'(t_0)$.
Then $u \in C^1(\mathbb{R})$, *and* $u'(t) = \sqrt{|u(t)|}, \forall t \in \mathbb{R}$.
Therefore, the initial-value problem $u'(t) = \sqrt{|u(t)|}, \forall t \in \mathbb{R}, u(t_0) = 0$ *has at least*

one non-trivial $C^1(R)$-solutions.

We now state a sufficient condition to guarantee also the uniqueness of the solution.
給一個充分條件使得解有唯一性

Theorem 2.1.2. *Fundamental theorem of ODE*
Suppose that all assumptions in Theorem 2.1.1 hold.
Additionally, assume that $|f(t,y_1) - f(t,y_2)| \leq L|y_1 - y_2|$ whenever (t,y_1) and (t,y_2) are in $\mathcal{R}$.
Then the solution described in Theorem 2.1.1 is the unique $C^1((t_0-\alpha, t_0+\alpha);\mathbb{R}^n)$ solution.

Remark : Watch Theorem 2.1.3 below.

Example 2.1.2.
Verify that the ODEs in Example 2.1.2, Exercise 2.1.3 and Exercise 2.1.4 do not satisfy the Lipschitz condition $|f(t,y_1) - f(t,y_2)| \leq L|y_1 - y_2|$.

Example 2.1.3.
Under the assumptions of Theorem 2.1.2, show that the initial value problem $u'(t) = f(t,u)$ with $u(t_0) = u_0$, for some vector $u_0 \in \mathbb{R}^n$ is equivalent to the integral equation
$u(t) = u_0 + \int_{t_0}^t f(s,u(s))ds, \forall t \in (t_0 - \alpha, t_0 + \alpha)$.

By using Example 2.1.5, under the assumptions of Theorem 2.1.2,
if $u_1 \in C^1((t_0-\alpha, t_0+\alpha))$ and $u_2 \in C^1((t_0-\alpha, t_0+\alpha))$ are the unique solution of $u'(t) = f(t,u)$ with $u(t_0) = u_0$, for some vector $u_0 \in \mathbb{R}^n$ corresponding to initial data u_0^1 and u_0^2 respectively,
then one sees that $u_1(t) - u_2(t) = u_0^1 - u_0^2 + \int_{t_0}^t ((f(s,u_1(s)) - (f(s,u_2(s)))ds$.

By using the Lipschitz condition $|f(t,y_1) - f(t,y_2)| \leq L|y_1 - y_2|$, one sees that
$|u_1(t) - u_2(t)|$
$\leq |u_0^1 - u_0^2| + \int_{t_0}^t |(f(s,u_1(s)) - (f(s,u_2(s))|ds$
$\leq |u_0^1 - u_0^2| + L\int_{t_0}^t |u_1(s) - u_2(s)|ds, \forall t \in (t_0-\alpha, t_0+\alpha)$.
If $t \leq t_0$, then $|u_1(t) - u_2(t)| \leq |u_0^1 - u_0^2|$.
If $t \geq t_0$, then we will use the following useful lemma.

Lemma 2.1.3. 解何時會穩定？
If $g \in C([t_0, t_1])$ *satisfies the inequality* $0 \le g(t) \le K + L\int_{t_0}^{t} g(s)ds, \forall t \in [t_0, t_1]$, *then* $0 \le g(t) \le Ke^{L(t-t_0)}, \forall t \in [t_0, t_1]$.

Proof.
Set $v(t) := \int_{t_0}^{t} g(s)ds$.
From $0 \le g(t) \le K + L\int_{t_0}^{t} g(s)ds, \forall t \in [t_0, t_1]$, we have
$\frac{dv}{dt} \le K + Lv(t), \forall t \in (t_0, t_1)$ with $v(t_0) = 0$.
Use the method of integrating factors (i.e., $\times e^{-L(t-t_0)}$),
$\frac{d}{dt}(e^{-L(t-t_0)}v(t)) = e^{-L(t-t_0)}\frac{dv}{dt} - Le^{-L(t-t_0)}v(t) = e^{-L(t-t_0)}(\frac{dv}{dt} - Lv(t))$
$\le Ke^{-L(t-t_0)}, \forall t \in (t_0, t_1)$.
By FTC (i.e., integrating the above inequality from t_0 to $\tau \in (t_0, t_1)$),
$e^{-L(\tau-t_0)}v(\tau) \le K\int_{t}^{\tau} e^{-L(t-t_0)}dt = \frac{K}{L}(1 - e^{-L(\tau-t_0)}), \forall \tau \in (t_0, t_1))$
$\Rightarrow \int_{t_0}^{t} g(s)ds = v(t) = \frac{K}{L}(e^{L(t-t_0)} - 1)$.
By condition $0 \le g(t) \le K + L\int_{t_0}^{t} g(s)ds, \forall t \in [t_0, t_1]$,
$0 \le g(t) \le K + L\int_{t_0}^{t} g(s)ds \le K + K(e^{L(t-t_0)} - 1) = Ke^{L(t-t_0)}, \forall t \in (t_0, t_1)$.
By $g \in C([t_0, t_1])$, $0 \le g(t) \le Ke^{L(t-t_0)}, \forall t \in [t_0, t_1]$. □

Note : Condition in Lemma have some reasons -
1. To use FTC, so g is continuous.
2. Avoid $\frac{1}{x}$ exists, the domain is $[t_0, t_1]$.
3. K and L are independent to t, so open interval can take limit on both side $\ni$ the result on $[t_0, t_1]$.

In view of $|u_1(t) - u_2(t)| \le |u_0^1 - u_0^2|, \forall t \le t_0$ and $|u_1(t) - u_2(t)| \le |u_0^1 - u_0^2| + L\int_{t_0}^{t} |u_1(s) - u_2(s)|ds, \forall t \in (t_0 - \alpha, t_0 + \alpha)$,
we now choose any $0 < t_1 < t + \alpha$ and $g(t) = |u_1(t) - u_2(t)|$ as well as $K = |u_0^1 - u_0^2|$ in Lemma above to see that:

Theorem 2.1.4. *(Dependence on data)*
If all assumptions in Theorem 2.1.5 hold, then the stability estimate $|u_1(t) - u_2(t)| \le |u_0^1 - u_0^2|e^{\alpha L}, \forall t \in (t_0 - \alpha, t_0 + \alpha)$ *hold, where* $u^1 \in C^1((t_0 - \alpha, t_0 + \alpha))$ *and* $u^2 \in C^1((t_0 - \alpha, t_0 + \alpha))$ *are the unique solution of* $u'(t) = f(t, u)$ *and* $u(t_0) = u_0$ *corresponding to initial data* u_0^1 *and* u_0^2 *respectively.*

Remark : We refer to [HS99, Chapter II] for further generalizations of Theorem above.

Exercise 2.3.

Show that the initial value problem $\begin{cases} u'(t) = \dfrac{1}{(3-(t-1)^2)(9-(t-5)^2)}, \forall t \in (1-\sqrt{2}, 1+\sqrt{2}) \\ u(1) = 5 \end{cases}$

has a unique $C^1((1-\sqrt{2}, 1+\sqrt{2}))$*-solution.*

⟨Solution⟩

Let $f(t,u) = u'(t)$.

As $t = 2 \in (1-\sqrt{2}, 1+\sqrt{2})$, *the dominator* $= 0$, *and the function value of* $f(t,u)$ *is not exists.*

Then $\nexists$ u *has a unique* $C^1((1-\sqrt{2}, 1+\sqrt{2}))$*-solution..*

Exercise 2.4.

Show that the initial value problem $\begin{cases} u'(t) = \dfrac{1}{(1+(t-4)^2)(5+(t-3)^2)}, \forall t \in \mathbb{R} \\ u(4) = 3 \end{cases}$

has a unique $C^1(\mathbb{R})$*-solution.*

⟨Solution⟩

Let $f(t,u) = u'(t)$.

Since $1 + (t-4)^2 > 0$ *and* $5 + (t-3)^2 > 0, \forall t \in \mathbb{R}$.

Then the rational function $f(t,u) = u'(t) \in C(\mathbb{R})$.

Since $u'(t) = \frac{1}{(1+(t-4)^2)(5+(t-3)^2)}, \forall t \in \mathbb{R}$, *then* $f(t,u)$ *is independent of* u.

$\forall u_1, u_2 \in \mathbb{R}$, *and* $t \in \mathbb{R}$,

$|f(t,u_1) - f(t,u_2)| = 0 \leq L|u_1 - u_2|$, *for any* $L \geq 0$.

By Fundamental theorem of ODE with the Lipschitz condition holds on $L = 0$,

the initial value problem $\begin{cases} u'(t) = \dfrac{1}{(1+(t-4)^2)(5+(t-3)^2)}, \forall t \in \mathbb{R} \\ u(4) = 3 \end{cases}$

has a unique $C^1(\mathbb{R})$*-solution.*

2.2 Some Techniques for Solving the Equation

Now consider a single equation of ODE: $u'(t) = f(t, u(t)), u(t_0) = u_0$.
Unfortunately, there is no universally applicable method for solving solution(s) $u \in C^1$ for the equation $u'(t) = f(t, u(t)), u(t_0) = u_0$.
We now exhibit some methods which can help to solve some certain class of ODE.
現在不存在一個通解去解決 $u \in C^1$ 的問題.

Definition 2.2.1.
The ODE $u'(t) = f(t, u(t)), u(t_0) = u_0$ is said to be separable if can be expressed in the form of $M(t) + N(u(t))u'(t) = 0$, for some continuous functions M and N. Sometimes we abuse the notation by writing $M(t)dt + N(u)du = 0$.

For $u \in C^1$, since integral of continuous function N is C^1, we use chain rule to see that $\frac{d}{dt}\big(\int_{u_0}^{u(t)} N(z)dz\big) = N(u(t))u'(t)$, then rewrite separable form $M(t) + N(u(t))u'(t) = 0$ as $\frac{d}{dt}\big(\int_{u_0}^{u(t)} N(z)dz\big) = -M(t)$.
Use FTC integral both sides with respect to the variable t from t_0 to τ,
$\int_{t_0}^{\tau} -M(t)dt = \int_{t_0}^{\tau} \left(\frac{d}{dt}\big(\int_{u_0}^{u(t)} N(z)dz\big)\right)dt = \int_{u_0}^{u(t)} N(z)dz$,
which solves the ODE implicitly.

Example 2.2.1.
Find the general solution of $\frac{du}{dt} = \frac{t^2}{1-u^2}$ and the initial condition $u(t_0) = u_0 \neq 1$.
⟨Solution⟩
Since $\frac{du}{dt} = \frac{t^2}{1-u^2} \Rightarrow -t^2 + (1-u^2)\frac{du}{dt} = 0$, then $\frac{du}{dt} = \frac{t^2}{1-u^2}$ is separable.
Since $u_0 \neq 1$ and the continuity of $u \Rightarrow u(t) \neq 1, \forall t$ near t_0.
Then the conditions in FTODE hold, $\exists!$ solution $u \in C^1$ near t_0.
By chain rule, $t^2 = (1-u^2)\frac{du}{dt} = \frac{d}{dt}\big(u(t) - \frac{1}{3}(u(t))^3\big)$.
Use FTC integral both sides with respect to the variable t from t_0 to τ,
$\big(u(\tau) - \frac{1}{3}(u(\tau))^3\big) - \big(u_0 - \frac{1}{3}{u_0}^3\big) = \int_{t_0}^{\tau} \big(\frac{d}{dt}\big(u(t) - \frac{1}{3}(u(t))^3\big)\big)dt = \int_{t_0}^{\tau} t^2 dt = \frac{1}{3}\tau^3 - \frac{1}{3}{t_0}^3$.
Then $\forall t$ near t_0, $u(\tau) - \frac{1}{3}(u(\tau))^3 - u_0 + \frac{1}{3}{u_0}^3\big|_{\tau=t} = \frac{1}{3}\tau^3 - \frac{1}{3}{t_0}^3\big|_{\tau=t}$
$\Rightarrow -t^3 + 3u(t) - (u(t))^3 = -{t_0}^3 + 3u_0 - {u_0}^3$.

Exercise 2.5.
Find the general solution of $\frac{du}{dt} = \frac{3t^2+4t+2}{2(u-1)}$.
⟨Solution⟩
Since $u = 1$ such that dominator $= 0$, then we impose the initial condition $u(t_0) = u_0 \neq 1$.
Since $-(3t^2 + 4t + 2) + 2(u-1)\frac{du}{dt} = 0$ and $\forall t$ near t_0, $u(t)$ is continue, then the ODE is separable.
Since for every closed interval, exists maximum and minimum value, we can choose

the maximum slope of tangent line for constant L.
Then satisfies Lipschilz condition in FTODE such that $\exists!$ solution $u \in C^1$ near t_0.
By chain rule, $3t^2 + 4t + 2 = \frac{d}{dt}\big((u(t))^2 - 2u(t)\big)$.
Use FTC integral both sides with respect to the variable t from t_0 to τ,
$\int_{t_0}^{\tau}\big(3t^2 + 4t + 2\big)dt = \int_{t_0}^{\tau}\big(\frac{d}{dt}\big((u(t))^2 - 2u(t)\big)\big)dt$
$\Rightarrow \big(\tau^3 + 2\tau^2 + 2\tau\big) - \big({t_0}^3 + 2{t_0}^2 + 2t_0\big) = \big((u(\tau))^2 - 2u(\tau)\big) - \big({u_0}^2 - 2u_0\big)$.
Hence, $\forall t$ *near* t_0, $-(u(t))^2 + 2u(t) + t^3 + 2t^2 + 2t = -{u_0}^2 + 2u_0 + {t_0}^3 + 2{t_0}^2 + 2t_0$.

Exercise 2.6.
Find the general solution of $\frac{du}{dt} = \frac{4t - t^3}{4 + u^3}$.
⟨Solution⟩
Since $u = -4^{\frac{1}{3}}$ *such that dominator* $= 0$, *then we impose the initial condition* $u(t_0) = u_0 \neq -4^{\frac{1}{3}}$.
Since $(t^3 - 4t) + (u^3 + 4)\frac{du}{dt} = 0$ *and* $\forall t$ *near* t_0, $u(t)$ *is continue, then the ODE is separable.*
Since for every closed interval, exists maximum and minimum value, we can choose the maximum slope of tangent line for constant L, i.e., $\max|u'(t)|$.
Then satisfies Lipschilz condition in FTODE such that $\exists!$ solution $u \in C^1$ near t_0.
By chain rule, $3t^2 - 4 = -\frac{d}{dt}\big(\frac{1}{4}(u(t))^4 + 4u(t)\big)$.
Use FTC integral both sides with respect to the variable t from t_0 to τ,
$\int_{t_0}^{\tau}\big(3t^2 - 4\big)dt = \int_{t_0}^{\tau}\big(-\frac{d}{dt}\big(\frac{1}{4}(u(t))^4 + 4u(t)\big)\big)dt$
$\Rightarrow \big(\tau^3 - 2\tau^2\big) - \big({t_0}^3 - 2{t_0}^2\big) = -\big(\frac{1}{4}(u(\tau))^4 + 4u(\tau)\big) + \big(\frac{1}{4}{u_0}^2 + 4u_0\big)$.
Hence, $\forall t$ *near* t_0, $\frac{1}{4}(u(t))^4 + 4u(t) + t^3 - 2t^2 = \frac{1}{4}{u_0}^2 + 4u_0 + {t_0}^3 - 2{t_0}^2$.

Question：什麼樣的合理假設才會使得解存在？

We now consider the following ODE: $M(t, u(t)) + N(t, u(t))u'(t) = 0, u(t_0) = u_0$.
Note that $u'(t) = f(t, u(t)), u(t_0) = u_0$ and $M(t) + N(u(t))u'(t) = 0$ are both special case of $M(t, u(t)) + N(t, u(t))u'(t) = 0, u(t_0) = u_0$.
We now want to solve $M(t, u(t)) + N(t, u(t))u'(t) = 0, u(t_0) = u_0$ under some sufficient conditions.

Assume that $M = M(t, y), N = N(t, y), \partial_y M$ and $\partial_t N$ are continuous in an open rectangle $(t_1, t_2) \times (y_1, y_2)$ and $u \in C^1((t_1, t_2))$ satisfies $y_1 < u(t) < y_2, \forall t \in (y_1, y_2)$.
For each $\psi \in C^1((t_1, t_2) \times (y_1, y_2))$, by using chain rule,
$\frac{d}{dt}\psi(t, u(t)) = \partial_t\psi(t, u(t)) + \partial_y\psi(t, u(t))u'(t)$.
期望$M = \partial_t\psi(t, u(t)), N = \partial_y\psi(t, u(t))u'(t)$.

Comparing this equality with the ODE $M(t, u(t)) + N(t, u(t))u'(t) = 0, u(t_0) = u_0$, it is natural to find $\psi \ni \partial_y\psi = N$ in $(t_1, t_2) \times (y_1, y_2)$, which can be achieved by choosing $\psi(t, y) := \int_{u_0}^{y} N(t, z)dz, \forall (t, y) \in (t_1, t_2) \times (y_1, y_2)$.

The following lemma for further computations.
(one way to prove this is to utilize the Lebesgue dominated convergence theorem).

Lemma 2.2.1.
Suppose that $f(t,y)$ and $\partial_t f(t,y)$ are continuous in the closed rectangle $[s_1,s_2]\times[z_1,z_2]$, then $\frac{d}{dt}\big(\int_{z_1}^{z_2} f(t,y)dy\big) = \int_{z_1}^{z_2}\partial_t f(t,y)dy, \forall t\in[s_1,s_2]$.
先積分再對第一位置的變數作微分 = 對第一位置的變數先作偏微分再積分

Note : the above lemma guarantees that
$\partial_t\psi(t,y) = \int_{u_0}^{y}\partial_t N(t,z)dz, \forall (t,y)\in(t_1,t_2)\times(y_1,y_2)$.

Now if $\partial_t N = \partial_y M$ in $(t_1,t_2)\times(y_1,y_2)$, then we reach
$\partial_t\psi(t,y) = \int_{u_0}^{y}\partial_z M(t,z)dz = M(t,y) - M(t,u_0), \forall (t,y)\in(t_1,t_2)\times(y_1,y_2)$.

From $\frac{d}{dt}\psi(t,u(t)) = \partial_t\psi(t,u(t)) + \partial_y\psi(t,u(t))u'(t)$, and consequently by $M(t) + N(u(t))u'(t) = 0$, we see that
$\frac{d}{dt}\psi(t,u(t)) = M(t,u(t)) - M(t,u_0) + N(t,u(t))u'(t) = -M(t,u_0)$.
Hence, $-\int_{t_0}^{\tau} M(t,u_0)dt = \int_{t_0}^{\tau}\frac{d}{dt}(\psi(t,u(t)))dt = \psi(\tau,u(\tau)) - \psi(t_0,u(t_0))$, which solves u implicitly.

Definition 2.2.2.
The ODE $M(t,u(t)) + N(t,u(t))u'(t) = 0, u(t_0) = u_0$ is said to be exact if $\partial_t N = \partial_y M$ holds.

Remark : If the ODE is separable, then it also exact.
In this case, $\psi(\tau,u(\tau)) - \psi(t_0,u(t_0))$ reduces to $\int_{t_0}^{\tau} -M(t)dt = \int_{u_0}^{u(t)} N(z)dz$.

Example 2.2.2.
Solve the ODE $(u(t)\cos t + 2te^{u(t)}) + (\sin t + t^2e^{u(t)} - 1)u'(t) = 0$ with suitable initial condition $u(t_0) = u_0$.
⟨Solution⟩

Now we want to deal with the ODE $M(t,u(t)) + N(t,u(t))u'(t) = 0, u(t_0) = u_0$ which is not necessarily to be exact in the sense of Definition 2.2.2.
處理不像exact形式的exact

Idea : multiply an integrating factor $\mu(t,y)$ so that
$\tilde{M}(t,u(t)) + \tilde{N}(t,u(t))u'(t) = 0, u(t_0) = u_0$ is exact,
where $\tilde{M}(t,y) = \mu(t,y)M(t,u(t))$ and $\tilde{N}(t,y) = \mu(t,y)N(t,u(t))$.
Now, $\partial_t N = \partial_y M$ in $(t_1,t_2) \times (y_1,y_2)$ reads
$\partial_t \mu N + \mu \partial_t N = \partial_t \tilde{N} = \partial_y \tilde{M} = \partial_y \mu M + \mu \partial_y M$,
i.e., $M\partial_y \mu - N\partial_t \mu + (\partial_y M - \partial_t N)\mu = 0$.
This is a transport equation, which will be discussed in Section 2.3 below.

Remark : one can directly check that if $k := \frac{\partial_y M - \partial_t N}{N}$ is a function of t only, the integrating factor μ is also a function of t only (which is independent of y), and it satisfies the linear ODE $\mu'(t) = k(t)\mu(t)$.
For each K with $K' = k$, then $\frac{d}{dt}\big(e^{-K(t)}\mu(t)\big) = -k(t)e^{-K(t)}\mu(t) + e^{-K(t)}\mu'(t) = 0$.
This can be achieved by choosing $\mu(t) := e^{K(t)}$.

Example 2.2.3.
Solve the ODE $(3tu+u^2)+(t^2+tu)u' = 0$ *with suitable initial condition* $u(t_0) = u_0$.
$\langle$*Solution*$\rangle$

2.3 From ODE to PDE

Give an application of ODE in the theory of partial differential equations (PDE). We begin our discussions from a simple model.

2.3.1 Linear equations

Example 2.3.1. *Linear equations*
Given a horizontal pipe of fixed cross section in the (positive) x-direction.
Suppose that there is a fluid flowing at a constant rate c ($c = 0$ means the fluid is stationary; $c > 0$ means flowing toward right, otherwise towards left).
We now assume that there is a substance is suspended in the water.
Fix a point at the pipe, and we set the point as the origin 0, and let $u(t, x)$ be the concentration of such substance.
The amount of pollutant in the interval $[0, y]$ at time t is given by $\int_0^y u(t, x)dx$.
At the later time $t + \tau$, the same molecules of pollutant moved by the displacement $c\tau$, and this means $\int_0^y u(t, x)dx = \int_{c\tau}^{y+c\tau} u(t + \tau, x)dx$.
If u is continuous, by using the FTC, by differentiating the above equation with respect to y, one sees that $u(t, y) = u(t + \tau, y + c\tau), \forall y \in \mathbb{R}$.
If we further assume $u \in C^1$, then differentiating the equation above with respect to τ, we reach the following transport equation :
$0 = u(t + \tau, y + c\tau)|_{\tau=0} = \partial_t u(t, x) + c\partial_x u(t, x), \forall (t, x) \in \mathbb{R} \times \mathbb{R}.$

Question：PDE 的解有∃!性？
Consider the transport equation with variable coecient equation of the form
$\partial_t u + c(t, x)\partial_x u = 0, u(0, x) = f(x)$, where $f \in C^1(\mathbb{R})$ and $c = c(t, x)$ satisfies all assumption in FTODE.
Given any $s \in \mathbb{R}$.
Consider a curve $x = \gamma_s(t)$, where γ solves the ODE $\gamma_s'(t) = c(t, \gamma_s(t)), \gamma_s(0) = s$.
We now restrict u on a curve $x = \gamma_s(t)$, then
$\partial_t(u\big|_{\gamma_s}(t)) = \partial_t(u(t, \gamma_s(t))) = (\partial_t u + \gamma'(t)\partial_x u)\big|_{x=\gamma_s(t)} = (\partial_t u + c(t, x)\partial_x u)\big|_{x=\gamma_s(t)} = 0$
This means that u is constant along the characteristic curve γ_s.
Hence, $u(t, \gamma_s(t)) = u(0, \gamma_s(0)) = f(\gamma_s(0)) = f(s)$.
Write $x = \gamma(t, s) = \gamma_s(t)$ for later convenience.
Fix $x \in \mathbb{R}$ and now we want to solve the equation $x = \gamma(t, s)$.
From $\gamma(0, x) = x$, and since $\partial_s\gamma(0, x) = (\partial_s\gamma_s(0))\big|_{s=x} = 1 \neq 0$, then we can apply the implicit function theorem to guarantee that
$\exists$ an open neighborhood $U(0, x) \subset \mathbb{R}$ and $g_x \in C^1(U_x) \ni g_x(0) = x$ and $x = \gamma(t, s)\big|_{s=g_x(t)}, \forall t \in U_x$.

i.e., we found a solution $s = g_x(t) \equiv g(x,t)$ of the equation $x = \gamma(t,s)$ in U_x.
Plugging this solution into $u(t,\gamma_s(t)) = u(0,\gamma_s(0)) = f(\gamma_s(0)) = f(s)$, we conclude $u(t,x) = f(g(x,t)), \forall x \in \mathbb{R}$ and $t \in U_x$.
This completes the local existence proof.
Since u is constant along the characteristic curve γ, i.e., under $C^1(\mathbb{R})$ this condition, the uniqueness holds.

Example 2.3.2.
Given any $f \in C^1(\mathbb{R})$.
Consider $\partial_t u + c(t,x)\partial_x u = 0, u(0,x) = f(x)$ *with* $c =$ *constant.*
In this case, $\gamma_s'(t) = c(t,\gamma_s(t)), \gamma_s(0) = s$ *reads* $\gamma'(t) = c$.
For each $s \in \mathbb{R}$*, it is easy to see that the solution of* ${\gamma_s}'(t) = c$ *with* $\gamma_s(0) = s$ *is* $\gamma(t,s) \equiv \gamma_s(t) = ct + s$.
For each $x \in \mathbb{R}$*, the solution of* $x = \gamma(t,s)$ *is clearly given by* $s = g(t,x) \equiv x - ct$*, and thus from* $u(t,x) = f(g(x,t)), \forall x \in \mathbb{R}$ *and* $t \in U_x$ *we conclude that*
$u(t,x) = f(x - ct)$*, and the solution is valid* $\forall x \in \mathbb{R}$ *and* $t \in \mathbb{R}$.

Example 2.3.3.
Given any $f \in C^1(\mathbb{R})$.
We now want to solve $\partial_t u + x\partial_x u = 0$ *with* $u(0,x) = f(x), \forall x \in \mathbb{R}$.
Write $c(t,x) = x$*, and for each* $s \in \mathbb{R}$ *we consider the ODE* ${\gamma_s}'(t) = c(t,\gamma_s(t)) \equiv \gamma_s(t), \gamma_s(0) = s$.
By using the integrating factor, one can easily see that the solution of the ODE is$\gamma_s(t) = e^{ts}$.
For each $x \in \mathbb{R}$*, the solution of* $x = \gamma_s(t)$ *is given by* $s = g(x,t) \equiv e^{-tx}$.
Thus, from $u(t,x) = f(g(x,t)), \forall x \in \mathbb{R}$ *and* $t \in U_x$ *we conclude that*
$u(t,x) = f(g(x,t)) = f(e^{-t}x)$*, and the solution is valid* $\forall x \in \mathbb{R}$ *and* $t \in \mathbb{R}$.

Example 2.3.4.
Given any $f \in C^1(\mathbb{R})$.
We want to solve $\partial_t u + 2tx^2\partial_x u = 0$ *with* $u(0,x) = f(x), \forall x \in \mathbb{R}$.
Write $c(t,x) = 2tx^2$.
For each $s \neq 0$ *we consider the ODE* ${\gamma_s}'(t) = 2t(\gamma_s(t))^2, \gamma_s(0) = s^{-1}$.
By using the method of separation of variables, one can easily see that the solution of the ODE is $\gamma_s(t) = (s - t^2)^{-1}$*, which is valid* $\begin{cases} \forall t \in \mathbb{R}, & when\ s < 0 \\ \forall t^2 < s & when\ s > 0 \end{cases}$*, but the ODE is not solvable when* $s = 0$.
When $s \neq 0$*, the solution of* $x = \gamma_s(t)$ *is given by* $s = t^2 + \frac{1}{x}$*, and thus from (2.3.5) we conclude that* $u(t,x) = f(s^{-1}) = f(\frac{x}{1+t^2x}), \forall x \neq -t^{-2}$.
Here we emphasize that the uniqueness only holds true in the region
$\{(t,x) : x > 0\} \cup \{(t,x) : x < 0, t < |x|^{-\frac{1}{2}}\}$.

Summarize the above ideas in the following algorithm :

Algorithm 1 Solving $\partial_t u + c(t,x)\partial_x u + d(t,x)u = F(t,x)$ with $u(0,x) = f(x)$

1. Solve the ODE ${\gamma_s}'(t) = c(t, \gamma_s(t))$ with given $\gamma_s(0)$ for any suitable parameter s.
2. Compute $\partial_t(u(t, \gamma_s(t)))$.
3. Rewrite the identity $x = \gamma_s(t)$ in the form of $s = g(x,t)$.
4. Identify the domain for which $u(t,x) = f(g(x,t))$ solves $\partial_t u + c(t,x)\partial_x u = 0$.

Exercise 2.7.
Given any $f \in C^1(\mathbb{R})$, solve the equation $(1+t^2)\partial_t u + \partial_x u = 0$ with $u(0,x) = f(x)$ and identify the range of x.
⟨Solution⟩
Since $(1+t^2)\partial_t u + \partial_x u = 0$, then rewrite the formula becomes $\partial_t u + \frac{1}{1+t^2}\partial_x u = 0$.
Write $c(t,x) = \frac{1}{1+t^2}$.
Given $s \in \mathbb{R}$.
Consider $x = \gamma(t,s) = \gamma_s(t)$, where γ solves the ODE $\begin{cases} \gamma_s{}'(t) = c(t, \gamma_s(t)) = \dfrac{1}{1+t^2}, \\ \gamma_s(t_0) = s_0 \end{cases}$
and t_0 and $s_0 \in \mathbb{R}$.
Restrict u on $x = \gamma_s(t)$, then we observe that
$\partial_t u\big|_{x=\gamma_s(t)} = \partial_t u(t, \gamma_s(t)) = \partial_t u + \partial_x u \cdot \gamma_s{}'(t)\big|_{x=\gamma_s(t)} = \partial_t u + \partial_x u \cdot \frac{1}{1+t^2} = 0.$
claim : find the initial value $\gamma_s(t)$.
By FTC, $\gamma_s(\tau) = \int_{t_0}^{\tau} \frac{1}{1+t^2} dt = \tan^{-1}(t)\big|_{t_0}^{\tau} = \tan^{-1}(\tau) - \tan^{-1}(t_0)$.
Since $u(0,x) = f(x)$, then choose $t_0 = 0$, then $\gamma_s(t_0) = \tan^{-1} 0 = 0 = s_0$.
Hence, we can know that the domain of γ_s is $t \in \mathbb{R}$.
claim : find $u(t,x)$, i.e., rewrite the identity $x = \gamma_s(t)$ in the form of $s = g(x,t)$.
Since the solution of $x = \gamma_s(t)$ is given by $s = g(x,t) \equiv \tan^{-1} x$, then from the form $u(t,x) = f(g(t,x)), \forall x \in \mathbb{R}, t \in U_x$, we can get $u(t,x) = f(g(t,x)) = f(\tan^{-1} x)$.
Hence, $x \in (\frac{-\pi}{2} + k\pi, \frac{\pi}{2} + k\pi), k \in \mathbb{N}$.

Exercise 2.8.
Given any $f \in C^1(\mathbb{R})$, solve the equation $t\partial_t u + x\partial_x u = 0$ with $u(0,x) = f(x)$ and identify the range of x.
⟨Solution⟩
Since $t\partial_t u + x\partial_x u = 0$, then rewrite the formula becomes $\partial_t u + \frac{x}{t}\partial_x u = 0$.
Let $c(t,x) = \frac{x}{t}$.
Given $s \in \mathbb{R}$.
Consider $x = \gamma(t,s) = \gamma_s(t)$, where γ is the solution of ODE $\begin{cases} \gamma_s{}'(t) = c(t, \gamma_s(t)) = \dfrac{x}{t}, \\ \gamma_s(t_0) = s_0 \end{cases}$
and t_0 and $s_0 \in \mathbb{R}$.
Since the initial point from initial condition $u(0,x) = f(x)$, then $\frac{x}{t}$ does not continuous at $t_0 = t = 0$.
claim : $u(t,x)$ has no solution.
Suppose $u(t,x)$ is a solution. Then
$\gamma_s{}'(t) = \frac{\gamma_s(t)}{t}$
$\Rightarrow \frac{\gamma_s{}'(t)}{\gamma_s(t)} = \frac{1}{t}$
$\Rightarrow \int_{t_0}^{\tau} \frac{\gamma_s{}'(t)}{\gamma_s(t)} dt = \int_{t_0}^{\tau} \frac{1}{t} dt$
$\Rightarrow \int_{t_0}^{\tau} \frac{1}{\gamma_s(t)} dt = \int_{t_0}^{\tau} \frac{1}{t} dt$

$\Rightarrow \ln|\gamma_s(t)|\Big|_{t_0}^{\tau} = \ln|t|\Big|_{t_0}^{\tau}$
$\Rightarrow \ln|\gamma_s(\tau)| = \ln|\tau| - \ln|t_0| + \ln|s_0|, \forall t > 0$
Then $|\gamma_s(t)| = |\frac{t}{t_0}s_0|$.
The solution of $s = g(t,x) \equiv |\frac{t}{t_0}s_0|$.
From the form $u(t,x) = f(g(t,x)), \forall x \in \mathbb{R}, t \in U_x$,
we can get $u(t,x) = f(g(t,x)) = f(|\frac{t}{t_0}x|), \forall x > 0$.
But this contradiction to $(0,x)$ *is defined.*

Exercise 2.9.
Solve the equation $x\partial_t u + t\partial_x u = 0$ *with* $u(0,x) = e^{-x^2}$.
⟨*Solution*⟩
$x\partial_t u + t\partial_x u = 0 \Rightarrow \partial_t u + \frac{t}{x}\partial_x u = 0$.
Let $c(t,x) = \frac{t}{x}$.
Given $s \in \mathbb{R}$.
Consider $x = \gamma_s(t) = \gamma(t,s)$, *where* γ *is the solution of ODE* $\begin{cases} {\gamma_s}'(t) = c(t,\gamma_s(t)) = \dfrac{t}{x} \\ \gamma_s(t_0) = s_0 \end{cases}$,
and t_0 *and* $s_0 \in \mathbb{R}$.
claim : find the initial value $\gamma_s(t)$.
$\int_{t_0}^{\tau} {\gamma_s}'(t)dt = \int_{t_0}^{\tau} \frac{t}{x}dt = \int_{t_0}^{\tau} \frac{t}{\gamma_s(t)}dt$
$\Rightarrow \int_{t_0}^{\tau} \gamma_s(t)d\gamma_s(t) = \int_{t_0}^{\tau} tdt$
$\Rightarrow \gamma_s(\tau)^2 - \gamma_s(t_0)^2 = \tau^2 - {t_0}^2$
$\Rightarrow \gamma_s(t) = \pm\sqrt{t^2 - {t_0}^2 + {s_0}^2}, \forall {t_0}^2 < t^2 + {s_0}^2$.
Since $u(0,x) = e^{-x^2}$, *then choose* $t_0 = 0$, *then* $\gamma_s(0) = \pm s_0$.
Since the solution of $s = g(t,x) \equiv \sqrt{x^2 + {s_0}^2}$, *then from the form*
$u(t,x) = f(g(t,x)), \forall x \in \mathbb{R}, t \in U_x$, *we can get*
$e^{-x^2} = u(t,x) = f(g(t,x)) = f(\sqrt{x^2 + {s_0}^2}), \forall x \in \mathbb{R}$.
If $s = -x$, *then we get the same conclusion.*

2.3.2 Quasilinear equations

The ideas in previous subsection can be extend for quasilinear equation of the form $a(x,y,u)\partial_x u + b(x,y,u)\partial_y u = c(x,y,u)$.
Here we follow the approach in [Joh78, Sections 1.41.6].
Rewrite the form as $(a,b,c)\cdot(\partial_x u, \partial_y u, -1) = 0$.
We represent the the function u by a surface $z = u(x,y)$ in $\mathbb{R}^3$, and we write $(a,b,c)\cdot(\frac{dz}{dx}, \frac{dz}{dy}, -1) = 0$.
Note that $(\frac{dz}{dx}, \frac{dz}{dy}, -1)$ is the normal vector of the surface, thus (a,b,c) is a tangent vector.
Consider a regular curve $(x(t), y(t), z(t))$ in that surface, and now we see that $(x'(t), y'(t), z'(t))$ is a tangent vector at the point $(x(t), y(t), z(t))$.
This suggests us to consider the characteristic ODE: $\begin{cases} x'(t) = a(x(t), y(t), z(t)) \\ y'(t) = b(x(t), y(t), z(t)) \\ z'(t) = c(x(t), y(t), z(t)) \end{cases}$,
which is a special case of the ODE $u'(t) = 0, u(t_0) = u_0$.
Here, the system is even autonomous,
i.e. the coefficients are independent of variable t does not appear explicitly.
If we assume that $a, b, c \in C^1$, then one can apply Theorem 2.1.1 to ensure the existence of characteristic curve $(x(t), y(t), z(t))$ which is C^1.
We now prove that the above choice of the characteristic ODE really describes the surface $z = u(x,y)$.

Lemma 2.3.1.
Assume that $a, b, c \in C^1$ near $(x_0, y_0, z_0) \in S$, where S is the surface described by $z = u(x,y)$.
If γ is a C^1 curve described by $(x(t), y(t), z(t))$ with $(x(t_0), y(t_0), z(t_0)) = (x_0, y_0, z_0)$, then γ lies completely on S.

Proof.
For convenience, we write $U(t) := z(t) - u(x(t), y(t))$ so that $U(t_0) = 0$ since $(x_0, y_0, z_0) \in S$.
Using chain rule and from $\begin{cases} x'(t) = a(x(t), y(t), z(t)) \\ y'(t) = b(x(t), y(t), z(t)) \\ z'(t) = c(x(t), y(t), z(t)) \end{cases}$ one sees that

$$\begin{aligned} U'(t) &= z'(t) - (\partial_x u)x'(t) - (\partial_y u)y'(t) \\ &= c(x,y,z) - \partial_x u(x,y)a(x,y,z) - \partial_y u(x,y)b(x,y,z) \\ &= c(x,y,U+u(x,y)) - \partial_x u(x,y)a(x,y,U+u(x,y)) - \partial_y u(x,y)b(x,y,U+u(x,y)) \end{aligned}.$$

From $a(x,y,u)\partial_x u + b(x,y,u)\partial_y u = c(x,y,u)$, we see that $U \equiv 0$ is a solution of the ODE $U'(t)$ above.

By using the FTODE (Theorem 2.1.2), we see that $U \equiv 0$ is the unique solution of the ODE $U'(t)$ above, which concludes our lemma. $\square$

Goal : Solve the Cauchy problem for $a(x,y,u)\partial_x u + b(x,y,u)\partial_y u = c(x,y,u)$ with the Cauchy data $h(s) = u(f(s), g(s))$ for some $f, g, h \in C^1$ near s_0.
i.e., 一個點附近有解
Note that the initial value problem we previous considered is simply the special case when $f(s) \equiv x_0$ and $g(s) = s$.

Now the characteristic ODE $\begin{cases} x'(t) = a(x(t), y(t), z(t)) \\ y'(t) = b(x(t), y(t), z(t)) \\ z'(t) = c(x(t), y(t), z(t)) \end{cases}$ (with suitable parameterization) reads $\begin{cases} \partial_t X(s,t) = a(X(s,t), Y(s,t), Z(s,t)) \\ \partial_t Y(s,t) = b(X(s,t), Y(s,t), Z(s,t)) \\ \partial_t Z(s,t) = c(X(s,t), Y(s,t), Z(s,t)) \\ \text{with initial conditions :} \\ X(s,0) = f(s), Y(s,0) = g(s), Z(s,0) = h(s) \end{cases}$.

If $a, b, c \in C^1$ near $(f(s_0), g(s_0), h(s_0))$, thus the FTODE (Theorem 2.1.5) guarantees that there exists a unique solution of equation above :
$(X(s,t), Y(s,t), Z(s,t))$ which is C^1 for (s,t) near $(s_0, 0)$.
If $\det \begin{pmatrix} f'(s_0) & g'(s_0) \\ a(x_0, y_0, z_0) & b(x_0, y_0, z_0) \end{pmatrix} = \det \begin{pmatrix} \partial_s X(s_0, 0) & \partial_s Y(s_0, 0) \\ \partial_t X(s_0, 0) & \partial_t Y(s_0, 0) \end{pmatrix} \neq 0$, then we can use implicit function theorem [Apo74, Theorem 13.7] to guarantee that
$\exists$ a unique solution $(s,t) = (S(x,y), T(x,y))$ of $x = X(S(x,y), T(x,y)), y = Y(S(x,y), T(x,y))$ of class C^1 is a neighborhood of (x_0, y_0) and satisfying $S(x_0, y_0) = s_0, T(x_0, y_0) = 0$.
So that we finally we conclude that the local solution of the Cauchy problem for $a(x,y,u)\partial_x u + b(x,y,u)\partial_y u = c(x,y,u)$ with the Cauchy data $h(s) = u(f(s), g(s))$ for some $f, g, h \in C^1$ near s_0 is given by $u(x,y) = Z(S(x,y), T(x,y))$.

Theorem 2.3.2. *Extend for higher dimensional case*
Consider the Cauchy problem $\sum_{i=1}^n a_i(x_1, \ldots, x_n, u) u_{x_i} = c(x_1, \ldots, x_n, u)$
with Cauchy data $h(s_1, \ldots, s_{n-1}) = u(f_1(s_1, \ldots, s_{n-1}), \ldots, f_n(s_1, \ldots, s_{n-1}))$, *for some* $f_1, \ldots, f_n, h \in C^1$ *near* $(s_1^0, \ldots, s_{n-1}^0)$.
If $a_1, \ldots, a_n, c \in C^1$ *near* $(f_1(s_0), \ldots, f_n(s_0), h(s_0)) \ni$

$$\det \begin{pmatrix} \partial_{s_1} f_1(s_1^0, \ldots, s_{n-1}^0) & \cdots & \partial_{s_1} f_n(s_1^0, \ldots, s_{n-1}^0) \\ \vdots & \ddots & \vdots \\ \partial_{s_n} f_1(s_1^0, \ldots, s_{n-1}^0) & \cdots & \partial_{s_n} f_n(s_1^0, \ldots, s_{n-1}^0) \\ a_1(x_1^0, \ldots, x_n^0, z^0) & \cdots & a_n(x_1^0, \ldots, x_n^0, z^0) \end{pmatrix} \neq 0, \text{ where}$$

$x_i^0 = f_i(s_1^0, \ldots, s_{n-1}^0), \forall 1 \leq i \leq n$ *and* $z^0 = h(s_1^0, \ldots, s_{n-1}^0)$, *then* $\exists$ *a unique solution* $u = (x_1, \ldots, x_n) \in C^1$ *near* $(x_1^0, \ldots, x_n^0, z^0)$.

Remark :
The corresponding characteristic ODE is

$$\begin{cases} \partial_t x_i(s_1, \ldots, s_{n-1}, t) = a_i(x_1, \ldots, x_n, z), \text{ for } 1 \le i \le n \\ \partial_t z(s_1, \ldots, s_{n-1}, t) = c(x_1, \ldots, x_n, z) \\ \text{with initial conditions :} \\ x_i(s_1, \ldots, s_{n-1}, 0) = f_i(s_1, \ldots, s_{n-1}), \text{ for } 1 \le i \le n \\ z(s_1, \ldots, s_{n-1}, t) = h(x_1, \ldots, x_n, z) \end{cases}$$

Example 2.3.5.
Solve the initial value problem $u\partial_x u + \partial_y u = 0, u(x,0) = h(x)$.
The characteristic ODE is $\partial_t x(s,t) = z, \partial_t y(s,t) = 1, \partial_t z(s,t) = 0$ *with initial condition* $x(s,0) = s, y(s,0) = 0, z(s,0) = h(s)$.
⟨Solution⟩
Solving the ODE yields $x = s + zt, y = t, z = h(s)$.
Eliminating s, t *yields the implicit equation* $u(x,y) = h(x - u(x,y)y)$.
It is interesting to see that $\partial_x u = h'(x - u(x,y)y)(1 - y\partial_x u)$, *then*
$\partial_x u + yh'(x - u(x,y)y)\partial_x u = h'(x - u(x,y)y)$
$\Rightarrow \partial_x u(x,y) = \frac{h'(x-u(x,y)y)}{1+yh'(x-u(x,y)y)}$.

Take $h(z) = -z$ to above equation for example, $\partial_x u(x,y) = \frac{-1}{1-y}$.
This quantity will blow up at $y = 1$, which means that there cannot exist a strict solution $u \in C^1$ beyond $y = 1$.
This type of behavior is typical for a nonlinear partial differential equation.
In general, we need to consider weak solution to study the PDE, but we will not going to go too far beyond this point.

Exercise 2.10.
Solve the initial value problem $xu\partial_x u - \partial_y u = 0, u(x,0) = x$.
⟨Solution⟩

The characteristic ODE : $\begin{cases} \partial_t x(s,t) = xz \\ \partial_t y(s,t) = -1 \\ \partial_t z(s,t) = 0 \end{cases}$, *with* $\begin{cases} x(s,0) = s \\ y(s,0) = 0 \\ z(s,0) = s \end{cases}$.

Solving the ODE $\Rightarrow \begin{cases} x = xzt + s \\ y = -t \\ z = x \end{cases}$.

Elimination $s, t \Rightarrow$ *the implicit equation*
$u(x,y) = x^2 u(x,y) + y$
$\Rightarrow u(x,y) = \frac{y}{1-x^2} = y(1-x^2)^{-1}$

$\Rightarrow \partial_x u(x,y) = -y(1-x^2)^{-2}(-2x)$.
Similarly, $\partial_y u(x,y) = (1-x^2)^{-1}$.

Exercise 2.11.
Solve the initial value problem $-y\partial_x u + x\partial_y u = u, u(x,0) = h(x)$,
provide that $h \in C^1(\mathbb{R})$.
$\langle$*Solution*$\rangle$

The characteristic ODE : $\begin{cases} \partial_t x(s,t) = x \\ \partial_t y(s,t) = -y \\ \partial_t z(s,t) = -z \end{cases}$, *with* $\begin{cases} x(s,0) = s \\ y(s,0) = 0 \\ z(s,0) = h(s) \end{cases}$.

Solving the ODE $\Rightarrow \begin{cases} x = xt + s \\ y = -yt \\ z = -th(s) + h(s) = (1-t)h(s) \end{cases}$.

Elimination $s, t \Rightarrow$ *the implicit equation*
$u(x,y) = (1-y)h(x(y-1))$
$\Rightarrow \partial_x u(x,y) = (1-y)h'(x(1-y))(y-1) = -(y-1)^2 h'(x(1-y))$.
Similarly, $\partial_y u(x,y) = -h(x(y-1)) + (1-y)h'(x(y-1))x$.

Remark :
For the general first order equation, we refer [Joh78, Sections 1.7] for details. Here we will not going to discuss here.

3 Linear ODE

Study the structure of solutions of a linear system $y'(t) = A(t)y + b(t)$, where the entries of the matrix $A(t) \in M_{n\times n}(\mathbb{C})$ are continuous functions of a independent variable $t \in \mathbb{R}$, and $b(t)$ is a complex-valued continuous function.
Well-posedness of the ODE can be guaranteed by the FTODE (Theorem 2.1.2).
We will follow the approach in some parts of [HS99, Chapter IV].

3.1 Homogeneous ODE with constant coefficients

Goal :
Show that the unique matrix-valued solution $Y = Y(t)$ of $\begin{cases} Y'(t) = AY(t), \forall t \in \mathbb{R} \\ Y(0) = I \end{cases}$,
which is called the fundamental matrix solution, where I is the $n \times n$ identity matrix.
Guaranteed by the FTODE in Theorem 2.1.2, we can takes the form
$Y(t) = \exp(tA), \forall t \in \mathbb{R}$.
Note :
Before, we may write $e_1 = (1, 0, \cdots), e_2 = (0, 1, 0, \cdots), \cdots$, now we can write it in short that e_j be the j^{th} column of I.

If this is the case, then the unique solution of $y'(t) = Ay(t), y(t_0) = p$ is exactly $y(t) = exp((t-t_0)A)p, \forall t \in \mathbb{R}$.
If $n = 1$, the above discussions are trivial, we are interested in the case when $n \geq 2$.

Let $M_{n\times n}(\mathbb{C})$ denote the set of all $n \times n$ matrices whose entries are complex numbers.
For any $A \in M_{n\times n}(\mathbb{C})$, we denote $A^{(jk)} \in M_{n-1\times n-1}(\mathbb{C})$ be the matrix obtained from A by crossing out j^{th} row and k^{th} column.

Definition 3.1.1.
For $A \in M_{1\times 1}(\mathbb{C}) \cong \mathbb{C}$, we simply define $\det(A) := A$.
For each $A \in M_{n\times n}(\mathbb{C})$, we define $\det(A) := \sum_{k=1}^{n} (-1)^{1+k} A_{1k} \det(A^{(1k)})$.

Definition 3.1.2.
The collection $GL(n, \mathbb{C})$ is the set of all invertible matrix with entries in $\mathbb{C}$, which can be characterized by $GL(n, \mathbb{C}) = \{A \in M_{n\times n}(\mathbb{C}); \det A \neq 0\}$.
The collection $GL(n, \mathbb{C})$ is known as the general linear group of order n.

Definition 3.1.3.
For any $A \in M_{n\times n}(\mathbb{C})$, we define the Hilbert-Schmidt norm $\|A\| := \left(\sum_{j,k=1}^{n} |A_{jk}|^2\right)^{\frac{1}{2}}$, where A_{jk} is the element of A on the j^{th} row and the k^{th} column.

Definition 3.1.4.
The trace of $A \in M_{n\times n}(\mathbb{C})$ *is defined by* $tr(A) := \sum_{j=1}^{n} A_{jj}$.

Exercise 3.1.
Let $A, B \in M_{n\times n}(\mathbb{C})(= \mathbb{C}^{n\times n})$, *show that*
a° $\|A\| = (tr(A^*A))^{\frac{1}{2}}$
b° $\|A+B\| \le \|A\| + \|B\|$
c° $\|AB\| \le \|A\|\|B\|$
$\langle$*Solution*$\rangle$
a°

$$\begin{aligned}(tr(A^*A))^{\frac{1}{2}} &= (\sum_{j=1}^{n}(A^*A)_{jj})^{\frac{1}{2}}\\ &= (\sum_{j=1}^{n}(\sum_{k=1}^{n} {A^*}_{jk}A_{kj}))^{\frac{1}{2}}\\ &= (\sum_{j=1}^{n}\sum_{k=1}^{n}|A_{kj}|^2)^{\frac{1}{2}} = \|A\|\end{aligned}$$

b°

$$\begin{aligned}\|A+B\| &= \Big(\sum_{j=1}^{n}\sum_{k=1}^{n}|(A+B)_{jk}|^2\Big)^{\frac{1}{2}} = \Big(\sum_{j=1}^{n}\sum_{k=1}^{n}|A_{jk}+B_{jk}|^2\Big)^{\frac{1}{2}}\\ &\le \Big(\sum_{j=1}^{n}\sum_{k=1}^{n}|A_{jk}|^2+|B_{jk}|^2\Big)^{\frac{1}{2}}\\ &\le \Big(\sum_{j=1}^{n}\sum_{k=1}^{n}|A_{jk}|^2\Big)^{\frac{1}{2}} + \Big(\sum_{j=1}^{n}\sum_{k=1}^{n}|B_{jk}|^2\Big)^{\frac{1}{2}} = \|A\|+\|B\|\end{aligned}$$

c°

$$\begin{aligned}\|AB\| &= \Bigg(\sum_{j=1}^{n}\sum_{k=1}^{n}|(AB)_{jk}|\Bigg)^{\frac{1}{2}} = \Bigg(\sum_{j=1}^{n}\sum_{k=1}^{n}|\sum_{m=1}^{n}A_{jm}B_{mk}|^2\Bigg)^{\frac{1}{2}}\\ &\le \Bigg(\sum_{j=1}^{n}\sum_{k=1}^{n}(|\sum_{m=1}^{n}A_{jm}|^2|\sum_{m=1}^{n}B_{mk}|^2)\Bigg)^{\frac{1}{2}}\\ &\le \Bigg((\sum_{j=1}^{n}|\sum_{m=1}^{n}A_{jm}|^2)(\sum_{k=1}^{n}|\sum_{m=1}^{n}B_{mk}|^2)\Bigg)^{\frac{1}{2}}\\ &= (\sum_{j=1}^{n}|\sum_{m=1}^{n}A_{jm}|^2)^{\frac{1}{2}}(\sum_{k=1}^{n}|\sum_{m=1}^{n}B_{mk}|^2)^{\frac{1}{2}} = \|A\|\|B\|\end{aligned}$$

Definition 3.1.5.
Let $\{A_m\}$ *be a sequence of complex matrices in* $M_{n\times n}(\mathbb{C})$.
We say that A_m *converges to matrix* A *if* $\lim_{m\to\infty}(A_m)_{jk} = A_{jk}, \forall 1 \le j,k \le n$.

Exercise 3.2.
Show that A_m *converges to* A *iff* $\lim\limits_{m\to\infty} \|A_m - A\| = 0$.
⟨*Solution*⟩
Since $\dim A_m = n^2 < \infty$, *then* $\sum_{j=1}^n A_{jk}$ *and* $\sum_{k=1}^n A_{jk}$ *converge absolutely.*
Hence, double summation can exchange.
Since in finite dimension that converge point-wisely is equivalent to converge uniformly, then summation and limit can exchange.
Therefore, we can prove it.
A_m *converges to* A.
iff $\lim\limits_{m\to\infty} (A_m)_{jk} = A_{jk} = \lim\limits_{m\to\infty} A_{jk}, \forall 1 \le j,k \le n$
iff $\lim\limits_{m\to\infty} ((A_m)_{jk} - A_{jk}) = 0, \forall 1 \le j,k \le n$
iff $\lim\limits_{m\to\infty} ((A_m - A)_{jk}) = 0, \forall 1 \le j,k \le n$.
iff $|\lim\limits_{m\to\infty} ((A_m - A)_{jk})|^2 = 0, \forall 1 \le j,k \le n$.
iff $\lim\limits_{m\to\infty} |(A_m - A)_{jk}|^2 = 0, \forall 1 \le j,k \le n$.
iff $\sum_{j=1}^n \sum_{k=1}^n \lim\limits_{m\to\infty} |(A_m - A)_{jk}|^2 = 0$
iff $\lim\limits_{m\to\infty} \sum_{j=1}^n \sum_{k=1}^n |(A_m - A)_{jk}|^2 = 0$
iff $\lim\limits_{m\to\infty} |\sum_{j=1}^n \sum_{k=1}^n (A_m - A)_{jk}|^2 = 0$
iff $(\lim\limits_{m\to\infty} |\sum_{j=1}^n \sum_{k=1}^n (A_m - A)_{jk}|^2)^{\frac{1}{2}} = 0$
iff $\lim\limits_{m\to\infty} (|\sum_{j=1}^n \sum_{k=1}^n (A_m - A)_{jk}|^2)^{\frac{1}{2}} = 0$
iff $\lim\limits_{m\to\infty} \|A_m - A\| = 0$.

Lemma 3.1.1.
For each $A \in M_{n\times n}(\mathbb{C})$, *we define* A_m *be the repeated matrix product of* X *with itself and* $A_0 = I$.
Then the series $\exp(A) := \sum_{m=0}^\infty \frac{A^m}{m!}$ *converges absolutely.*
In addition, the function $A \in M_{n\times n}(\mathbb{C}) \mapsto e^A \in M_{n\times n}(\mathbb{C})$ *is a continuous function* (*with respect to the Hilbert-Schmidt norm* $\|A\|_{HS} = \|A\| := \big(\sum_{j,k=1}^n |A_{j,k}|^2\big)^{\frac{1}{2}}$).

Proof.
Since $\|AB\| \le \|A\|\|B\|$, for $A, B \in M_{n\times n}(\mathbb{C})$, then $\|A^m\| \le \|A\|^m, \forall m \in \mathbb{N}$.
Hence, $\sum_{m=0}^\infty \|\frac{A^m}{m!}\| \le \|I\| + \sum_{m=0}^\infty \frac{\|A\|^m}{m!} = e^{\|A\|} < \infty$
$\Rightarrow \exp(A) = \sum_{m=0}^\infty \frac{A^m}{m!}$ converge absolutely.
On the other hand, we see that
$\sup\limits_{\|A\|\le R} \|\exp(A) - \sum_{m=0}^N \frac{A^m}{m!}\| = \sup\limits_{\|A\|\le R} \|\sum_{m=N+1}^\infty \frac{A^m}{m!}\| \le \sum_{m=N+1}^\infty \frac{R^m}{m!}$.
Hence, $\sup\limits_{\|A\|\le R} \|\exp(A) - \sum_{m=0}^N \frac{A^m}{m!}\| \to 0$ as $N \to \infty$
$\Rightarrow \exp(A)$ converges uniformly on the closed ball $\{A \in M_{n\times n}(\mathbb{C}); \|A\| \le R\}$, and

$A \mapsto e^A$ is continuous on the open ball $\{A \in M_{n\times n}(\mathbb{C}); \|A\| < R\}$.
Since this holds true $\forall R > 0$, hence we conclude our result. $\square$

Remark : Lemma 3.1.1 can be rephrase
The radius of convergence of the power series $\exp(A) := \sum_{m=0}^{\infty} \frac{A^m}{m!}$ is $+\infty$.
級數的收斂半徑 $= \infty$

觀察一下有哪些性質
It is easy to see that

$$\begin{cases} \exp(0) = I, (exp(A))^* = exp(A^*) \\ \exp(BAB^{-1}) = B\exp(A)B^{-1}, \forall B \in GL(n, \mathbb{C}) \end{cases}$$

The following lemma is crucial (see also [Hal15, Theorem 5.1] for a generalization).

Lemma 3.1.2.
If $A, B \in M_{n\times n}(\mathbb{C})$ are commute (i.e. $AB = BA$), then
$\exp(A+B) = \exp(A)\exp(B) = \exp(B)\exp(A)$.

Proof.
Since both series converge absolutely by Lemma 3.1.1, then simply multiply the two power series $\exp(A)$ and $\exp(B)$ term by term.
i.e., $\exp(A)\exp(B) = \sum_{m=0}^{\infty}\sum_{k=0}^{m} \frac{A^k}{k!}\frac{B^{m-k}}{(m-k)!}$
Since A and B are commute, then we able to rearrange the terms , so we can collect terms where the power of A plus the power of B equals to m.
i.e., $\sum_{m=0}^{\infty} \frac{1}{m!}\sum_{k=0}^{m} \frac{m!}{k!(m-k)!}A^kB^{m-k} = \sum_{m=0}^{\infty} \frac{(A+B)^m}{m!} = \exp(A+B)$

$$\begin{aligned} \exp(A)\exp(B) &= \sum_{m=0}^{\infty}\sum_{k=0}^{m} \frac{A^k}{k!}\frac{B^{m-k}}{(m-k)!} \\ &= \sum_{m=0}^{\infty}\frac{1}{m!}\sum_{k=0}^{m}\frac{m!}{k!(m-k)!}A^kB^{m-k}. \\ &= \sum_{m=0}^{\infty}\frac{(A+B)^m}{m!} = \exp(A+B) \end{aligned}$$

Hence, we are done. $\square$

Example 3.1.1. *Use Jordom form to construct counterexample*
In general, the identity $\exp(A+B) = \exp(A)\exp(B) = \exp(B)\exp(A)$ does not hold true for those $A \in M_{n\times n}(\mathbb{C})$ and $B \in M_{n\times n}(\mathbb{C})$ which are not commute.
For example, we choose $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.
Then $AB = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = BA$.

Since $A^2 = O$ *and* $B^2 = O$*, then we see that*
$\exp(A) = I + A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$
$\exp(B) = I + B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$
Hence, $\exp(A)\exp(B) = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \neq \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = \exp(B)\exp(A).$
Since $(A+B)^2 = I$*, then we see that*

$$\begin{aligned} \exp(A+B) &= \sum_{m \text{ is even}} \frac{1}{m!} I + \sum_{m \text{ is odd}} \frac{1}{m!} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &= \sum_{m=0}^{\infty} \frac{1}{(2m)!} I + \sum_{m=0}^{\infty} \frac{1}{(2m+1)!} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &= \cosh(1) I + \sinh(1) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \cosh(1) & \sinh(1) \\ \sinh(1) & \cosh(1) \end{pmatrix}, \end{aligned}$$

where $\cosh x = \frac{e^{-x}+e^{x}}{2}$ *and* $\sinh x = \frac{e^{-x}-e^{x}}{2}$.
Hence, $\exp(A+B), \exp(A)\exp(B), \exp(B)\exp(A)$ *are not equal.*

Remark :
It is also interesting to compare Lma 3.1.2 with the Lie product formula (Thm 3.1.51) below.

The following are immediate consequences of Lemma 3.1.2

Corollary 3.1.3.
Given any $A \in M_{n\times n}(\mathbb{C})$.
Then $\exp(\alpha A)\exp(\beta A) = \exp((\alpha+\beta)A), \forall \alpha, \beta \in \mathbb{C}$.

Corollary 3.1.4.
Given any $A \in M_{n\times n}(\mathbb{C})$.
Then $\exp(A) \in GL(n, \mathbb{C})$ *with* $(\exp(A))^{-1} = \exp(-A)$.

Proof.
$\exp(A)\exp(-A) = \exp(A-A) = I = \exp(-A+A) = \exp(-A)\exp(A)$ □

Theorem 3.1.5.
Each $A \in GL(n, \mathbb{C})$ *can be expressed as* $\exp(B)$ *for some* $B \in M_{n\times n}(\mathbb{C})$.
In other words, the mapping $B \in M_{n\times n}(\mathbb{C}) \to \exp(B) \in GL(n, \mathbb{C})$ *is surjective.*
Note that the mapping is not injective, and it also not invertible.

Goal : Verify $Y(t) = exp(tA), \forall t \in \mathbb{R}$ solves $Y'(t) = AY(t), \forall t \in \mathbb{R}, Y(0) = I$.

Theorem 3.1.6.
Let $A \in M_{n\times n}(\mathbb{C})$.
Then $t \to \exp(tA)$ is a smooth curve in $M_{n\times n}(\mathbb{C})$, i.e., $\in C^\infty$, and by communicate gets $\frac{d}{dt}\exp(tA) = A\exp(tA) = \exp(tA)A$.

Proof.
It is well-known that one can differentiate a power series term by term within its radius of convergence (see e.g. [Pug15, Theorem 12 in Chapter 4]).
In view of Remark above that the radius of convergence of the power series $\exp(A) := \sum_{m=0}^{\infty} \frac{A^m}{m!}$ is $+\infty$, our theorem immediate follows by differentiating the power series $\exp(tA)$. $\square$

Take A with $\|A\| = 1$, we see that Thm 3.1.6 is nothing by just a directional derivative.
For each fixed $1 \le j_0, k_0 \le n$, if we choose $A = \begin{cases} 1, & j = j_0 \text{ and } k = k_0 \\ 0, & \text{otherwise} \end{cases}$, then the derivative in Thm 3.1.6 is simply a partial derivative.
In fact, the matrix exponential map is (total) differentiable:

Theorem 3.1.7.
The matrix exponential map $\exp : M_{n\times n}(\mathbb{C}) \cong \mathbb{R}^{2n^2} \to GL(n, \mathbb{C})$ is an infinitely differentiable map, i.e., $\in C^\infty$.

Proof.
Fix any $A \in M_{n\times n}(\mathbb{C})$.
Note that for each j and k, the quantity $(A_m)_{jk}$ is a homogeneous polynomial of degree m in the entries of A.
Thus, the series for the function $(A_m)_{jk}$ has the form of a multivariable power series on $M_{n\times n}(\mathbb{C} \sim= \mathbb{R}^{2n^2}$.
Since the series converges on all $\mathbb{R}^{2n^2}$ (more precisely, the radius of convergence $= \infty$), it is permissible to differentiate the series term by term as many times as we wish (see e.g. [Pug15, Theorem 12 in Chapter 4]). $\square$

Note : Thm 3.1.7 is false on $M_{n\times n}(\mathbb{R})$

3.1.1 Computations of the exponential

Theorem 3.1.8. (*Cofactor expansion*)
For each $A \in M_{n\times n}(\mathbb{C})$ and for each fixed $j = 1, 2, \cdots, n$, we have

$\det(A) = \sum_{k=1}^{n} (-1)^{j+k} A_{jk} \det(A^{(jk)})$, *i.e., the determinant is independent of* j. *In addition, we have* $\det(A) = \sum_{j=1}^{n} (-1)^{j+k} A_{jk} \det(A^{(jk)})$, *i.e., the determinant is independent of* k.

In real world, calculate the det by definition need a lot of resource in computer, we use some technique from Algebra.

Definition 3.1.6.
S_n be the set of permutation on the indices $\{1, \dots, n\}$,
that is, $S_n = \{$bijective function $\sigma : \{1, \dots, n\} \to \{1, \dots, n\}\}$.

Theorem 3.1.9.
The determinant of $A \in M_{n\times n}(\mathbb{C})$ can be computed by
$\det(A) = \sum_{\sigma\in S_n} c_\sigma A_{\sigma(1)1} A_{\sigma(2)2} A_{\sigma(3)3} \cdots A_{\sigma(n)n}$, *for some constant* $c_\sigma \in \{-1, 1\}$.

Remark : for those who familiar with abstract algebra
c_σ is exact the sign of the permutation $\sigma \in S_n$, denoted by $sign(\sigma)$.
In addition, it is also related to determinant via the formula $sign(\sigma) = \det(e_{\sigma(1)} \cdots e_{\sigma(n)})$, where e_j is the j^{th} column of I.

Theorem 3.1.10.
Given any $A \in M_{n\times n}(\mathbb{C})$ and $B \in M_{n\times n}(\mathbb{C})$. Then
$\det(A) = \det(A^T), \det(AB) = \det(A)\det(B)$, *and* $\det(\bar{A}) = \det(A^*) = \overline{\det(A)}$.

Definition 3.1.7.
A matrix $A \in M_{n\times n}(\mathbb{C})$ is said to be diagonal if $a_{jk} = 0, \forall j \neq k$.
Denote $diag(\lambda_1, \lambda_2, \cdots, \lambda_n)$ is the diagonal matrix with entries $\lambda_1, \lambda_2, \dots, \lambda_n$ on the main diagonal.

Definition 3.1.8.
A matrix $A \in M_{n\times n}(\mathbb{C})$ is said to be diagonalizable
if there exists a matrix $P \in GL(n, \mathbb{C}) \ni P^{-1}AP$ is diagonal.

If $P^{-1}AP = diag(\lambda_1, \lambda_2, \cdots, \lambda_n)$, then by $\exp(BAB^{-1}) = B\exp(A)B^{-1}, \forall B \in GL(n, \mathbb{C})$, we can compute its exponential of a diagonalizable matrix A as

$$\begin{aligned}\exp(A) &= \exp(P diag(\lambda_1, \lambda_2, \cdots, \lambda_n) P^{-1}) \\ &= P \exp(diag(\lambda_1, \lambda_2, \cdots, \lambda_n)) P^{-1} \\ &= P diag(e^{\lambda_1}, e^{\lambda_2}, \cdots, e^{\lambda_n}) P^{-1}\end{aligned}$$

Definition 3.1.9.
A vector $p \neq 0$ is said to be an eigenvector of A with eigenvalue λ if $Ap = \lambda p$.

Since λ is an eigenvalue of A iff $\det(A-\lambda I)=0$, then consider the characteristic polynomial $p(z)=\det(zI-A)=z^n+\sum_{j=0}^{n-1}c_jz^j$, for some $c_0,\dots,c_j\in\mathbb{C}$.
By using the fundamental theorem of algebra, $\exists$ exactly n complex roots.
Define $p(B):=B^n+\sum_{j=0}^{n-1}c_jB^j, \forall B\in M_{n\times n}(\mathbb{C})$.

Theorem 3.1.11. *Cayley-Hamilton*
If $A\in M_{n\times n}(\mathbb{C})$, then $p(A)=0$.

Definition 3.1.10.
A complex number λ is called a root of p if $p(\lambda)=0$.
The multiplicity of this root is called the algebraic multiplicity of the eigenvalue λ.
There is another notion of multiplicity of an eigenvalue: the dimension of the eigenspace $\ker(\lambda I-A):=\{p\in\mathbb{R}^n;(\lambda I-A)p=0\}$ *is called the geometric multiplicity of the eigenvalue λ.*

Lemma 3.1.12.
Let $A\in M_{n\times n}(\mathbb{C})$.
Then A is diagonalizible iff for each eigenvalue λ the dimension of the eigenspace $\ker(A-\lambda I)$ *coincides with its algebraic multiplicity.*

Example 3.1.2. (*A nondiagonalizable matrix*)
Consider the matrix $A=\begin{bmatrix}1&1\\0&0\end{bmatrix}$.
Its characteristic polynomial is
$p(z)=\det(zI-A)=\det\left(\begin{bmatrix}z-1&-1\\0&z-z\end{bmatrix}\right)=(z-1)^2$,
so A has an eigenvalue 1 of algebraic multiplicity 2.
One see that $\dim\big(zI-A\big)=\det\left(\ker\left(\begin{bmatrix}0&-1\\0&0\end{bmatrix}\right)\right)=1$, *which shows that A is not diagonalizable.*

Definition 3.1.11.
A set of vectors $\{p_1,\dots,p_k\}$ is said to be linear independent if
$\sum_{i=1}^k c_ip_i=0$ *iff* $c_i=0,\forall 1\le i\le k$.

Lemma 3.1.13.
A matrix $A\in M_{n\times n}(\mathbb{C})$ is diagonalizable iff A has n linearly independent eigenvectors $p_1,\dots,p_n$.

Corollary 3.1.14.
If $A\in M_{n\times n}(\mathbb{C})$ has n distinct eigenvalues, then A is diagonalizable.

Exercise 3.3.
Show that the set of diagonalizable $n \times n$ *matrix is a proper subset* (*i.e. a subset which is not equal*) *of* $M_{n\times n}(\mathbb{C}), \forall n \geq 2$.
(*Hint : modify the ideas in Example 3.1.2*)

Proof.
Given $\mathscr{M} = \{A \in M_{n\times n}(\mathbb{C}); A \text{ is diagonal}, n \geq 2\}$.
claim : $\mathscr{M} \subset M_{n\times n}(\mathbb{C})$.
Suppose $\mathscr{M} = M_{n\times n}(\mathbb{C})$.
But we can find that $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \in M_{2\times 2}(\mathbb{C})$ is not diagonal.
Then by contradiction, we conclude that $\mathscr{M} \subset M_{n\times n}(\mathbb{C})$. $\square$

However, it is not easy to check whether a matrix A is diagonalizable or not.
There are some sufficient conditions which are relatively easy to check.
檢查是否對角化其實並不容易，現在有一些充分條件更容易去檢驗。

Definition 3.1.12.
A matrix $U \in M_{n\times n}(\mathbb{C})$ *is called unitary if* $U^*U = I$.
The unitary group is the set of unitary matrix is denoted by $U(n, \mathbb{C})$.

Lemma 3.1.15.
If $U \in M_{n\times n}(\mathbb{C})$ *is unitary, then* $U \in GL(n, \mathbb{C})$ *with* $U^{-1} = U^*$*, which is unitary.*
i.e., $U(n, \mathbb{C}) \subset GL(n, \mathbb{C})$.
In particular, if $U \in M_{n\times n}(\mathbb{R})$ *is unitary, then* $\det(U)^2 = 1 \Rightarrow \det(U) = 1$ *is rotation or* $\det(U) = -1$ *is the projection, i.e., is the rigid motion.*

定義matrix的長度

Definition 3.1.13.
A matrix $A \in M_{n\times n}(\mathbb{C})$ *is called normal if* $A^*A = AA^*$.
A matrix $A \in M_{n\times n}(\mathbb{C})$ *is called Hermitian (or self-adjoint) if* $A = A^*$.
We write $(u, v)_{M_{n\times n}(\mathbb{C})} := v^*u, \forall u, v \in M_{n\times n}(\mathbb{C})$.

Note :
$\overline{(v, u)_{M_{n\times n}(\mathbb{C})}} = ((v, u)_{M_{n\times n}(\mathbb{C})})^* = (u^*v)^* = v^*u = (u, v)_{M_{n\times n}(\mathbb{C})}, \forall u, v \in M_{n\times n}(\mathbb{C})$.

Lemma 3.1.16.
A is Hermitian iff $(Au, v)_{M_{n\times n}(\mathbb{C})} = (u, Av)_{M_{n\times n}(\mathbb{C})}, \forall u, v \in M_{n\times n}(\mathbb{C})$.

Proof.
$(u, Av)_{M_{n\times n}(\mathbb{C})} = (Av)^*u = v^*A^*u = v^*Au = (Au, v)_{M_{n\times n}(\mathbb{C})}$,
$\forall A \in M_{n\times n}(\mathbb{C}), u, v \in \mathbb{C}^n$ $\square$

Theorem 3.1.17. *All normal matrices are unitary diagonalizable*
If $A \in M_{n\times n}(\mathbb{C})$ is normal, then $\exists$ a unitary matrix $U \in M_{n\times n}(\mathbb{C}) \ni D := U^*AU$ is diagonal.
If $A \in M_{n\times n}(\mathbb{C})$ is Hermitian, then D is real-valued.

Definition 3.1.14.
A matrix $A \in M_{n\times n}(\mathbb{C})$ is said to be upper triangular if $a_{jk} = 0$, for $j > k$.

Lemma 3.1.18. (*Schur representation*)
For each $A \in M_{n\times n}(\mathbb{C})$, there exists a unitary matrix $U \in M_{n\times n}(\mathbb{C}) \ni U^*AU$ is upper triangular.
If $A \in M_{n\times n}(\mathbb{R})$ and all its eigenvalues are real, then we can choose $U \in M_{n\times n}(\mathbb{C})$.

Lemma 3.1.19.
The set of diagonalizable $n \times n$ matrix is dense in $M_{n\times n}(\mathbb{C})$.
(i.e., A_m converges to A iff $\lim\limits_{m\to\infty} \|A_m - A\| = 0$).
In other words, given any $A \in M_{n\times n}(\mathbb{C})$, there exists a sequence of diagonalizable matrix B_k which is converges to A.

Proof.
By Schur representation, $\exists P \in GL(n, \mathbb{C}) \ni B = P^{-1}AP$ is upper triangular.
WLOG, if $\exists$ a sequence of diagonalizable matrix $\tilde{B_k}$ which is converges to B, then from Exercise 3.1.1 and Exercise 3.1.3 we have
$\limsup\limits_{k\to\infty} \|P\tilde{B_k}P^{-1} - A\| =$ $\square$

Exercise 3.4.
For each $A \in M_{n\times n}(\mathbb{C})$*, show that* $\det(\exp(A)) = e^{tr(A)}$.
Use to show that $tr(A) = \lambda_1 + \cdots + \lambda_n$*, where* $\lambda_j \in \mathbb{C}$ *are eigenvalues (may identical) of* A.

Proof.
Given $A \in M_{n\times n}(\mathbb{C})$.
By Shur representation, for each $A \in M_{n\times n}(\mathbb{C})$,
$\exists$ a unitary $U \in M_{n\times n}(\mathbb{C}) \ni P = U^*AU$ is upper triangular.
Note that all the eigenvalue of upper triangular P is on $P_{jj}, \forall 1 \leq j \leq n$.
Since $U \in M_{n\times n}(\mathbb{C})$ is unitary, then $U \in GL(n, \mathbb{C})$ with $U^* = U^{-1}$, which is also unitary.
Since A and P are similar, then the eigenvalues of A and P are the same.
$\det(\exp(A)) = \det(\sum_{m=0}^{\infty} \frac{A^m}{m!})$
$= \det(\sum_{m=0}^{\infty} \frac{(UPU^*)^m}{m!})$
$= \det(\sum_{m=0}^{\infty} \frac{UP^mU^*}{m!})$
$= \det(\sum_{m=0}^{\infty} (U\frac{P^m}{m!}U^*))$
$= \det(U(\sum_{m=0}^{\infty} \frac{P^m}{m!})U^*)$
$= \det(U)\det(\sum_{m=0}^{\infty} \frac{P^m}{m!})\det(U^*)$
$= \det(\sum_{m=0}^{\infty} \frac{P^m}{m!})$
$= \det(\exp(P))$
$= e^{tr(P)}$
$= e^{tr(A)}$. $\square$

Question：目前的matrix跟diagonal差多少？
Answer：差nilpotent

Definition 3.1.15.
A matrix $N \in M_{n\times n}(\mathbb{C})$ is said to be nilpotent if $N^n = 0$.

Remark :
If N is nilpotent, then $\exp(N)$ is simply a finite sum.
One can directly verify that $\exp(N)$ is unipotent (i.e., $\exp(N) - I$ is nilpotent).

Lemma 3.1.20.
A matrix $N \in M_{n\times n}(\mathbb{C})$ is nilpotent iff all eigenvalues of N are zero.

Proof.
Let λ be an eigenvalue of a nilpotent matrix N with eigenfunction $p \not\equiv 0$, i.e. $Np = \lambda p$.
Then $0 = N^n p = \lambda^n p$, which implies $\lambda^n = 0$.
Since $|\lambda|^n = |\lambda^n| = 0$, then we conclude that $\lambda = 0$.
The converse can be easily verified as well. □

Lemma 3.1.21. *A relation between the set of diagonalizable matrix and $M_{n\times n}(\mathbb{C})$*
A matrix $N \in M_{n\times n}(\mathbb{C})$ (resp. $N \in M_{n\times n}(\mathbb{R})$) is nilpotent iff there exists a unitary matrix $U \in M_{n\times n}(\mathbb{C})$ (resp. $U \in M_{n\times n}(\mathbb{R})$) $\ni T = U^*NU$ is upper triangle with $T_{jj} = 0, \forall j = 1, \dots, n$.

Theorem 3.1.22. (*Jordan-Chevalley decomposition, another relation*)
Let $A \in M_{n\times n}(\mathbb{C})$.
Then there exist a diagonalizable matrix $D \in M_{n\times n}(\mathbb{C})$ and a nilpotent matrix $N \in M_{n\times n}(\mathbb{C}) \ni A = D + N$ and $DN = ND$ and the decomposition is unique.
If $A \in M_{n\times n}(\mathbb{R})$, then $D, N \in M_{n\times n}(\mathbb{R})$.

The general algorithm to compute the decomposition of D and N in Thm :

Algorithm 2 Computation of D and N in Thm

---
1. Input the matrix $A \in M_{n\times n}(\mathbb{C})$.
2. Compute the characteristic polynomial $p(z) = \det(zI - A)$
3. Decompose $\frac{1}{p(z)}$ into partial fractions $\frac{1}{p(z)} = \sum_{j=1}^{k} \frac{Q_j(z)}{(z-\lambda_j)^{m_j}}$,
   where for each j the quantity $Q_j(z)$ is a nonzero polynomial with $\deg(Q_j) \leq m_j - 1$, and $\lambda_1, \dots, \lambda_k$ are distinct zeros of p (i.e. eigenvalues of A)
4. For each j = 1, ・・・,n, we define $P_j(a) := Q_j(A) \prod_{s\neq j} (A - \lambda_j I)^{m_s}$.
5. Output $D = \lambda_1 P_1(A) + \cdots + \lambda_k P_k(A)$ and $N = A - D$.

Since the trace operator is linear and $tr(N) = 0$, we immediately reach the following corollary.

Corollary 3.1.23.
If $A \in M_{n\times n}(\mathbb{C})$ satisfies $tr(A) = 0$, then the diagonalizable matrix D in $A = D+N$ and $DN = ND$ satisfies $tr(D) = 0$.

Remark :
There always exists a unique decomposition in Step 3 of Algorithm 2. (by Complex Analysis)
This algorithm highlighted the proof of Theorem 3.1.35. Here we also highlight that $P_j(A)P_k(A) = 0, \forall j \neq k$ and $P_j(A)^2 = P_j(A), \forall j = 1, \ldots, n$.

Exercise 3.5.

Let $A_1 = \begin{bmatrix} 0 & -a \\ a & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix}, A_3 = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$.

Compute $\exp(A_1), \exp(A_2)$ *and* $\exp(A_3)$.

$\langle$*Solution*$\rangle$

Since ${A_1}^2 = \begin{bmatrix} 0 & -a \\ a & 0 \end{bmatrix}^2 = \begin{bmatrix} -a^2 & 0 \\ 0 & -a^2 \end{bmatrix} = -a^2 I$, *then*

$\exp(A_1) = \sum_{m=0}^{\infty} \frac{{A_1}^m}{m!} = \sum_{m=0}^{\infty} \frac{{A_1}^{2m}}{(2m)!} + \sum_{m=0}^{\infty} \frac{{A_1}^{2m+1}}{(2m+1)!} = \sum_{m=0}^{\infty} \frac{(-a^2)^m}{(2m)!} I + \sum_{m=0}^{\infty} \frac{(-a^2)^m}{(2m+1)!} A_1$

$= \cos(a) I + \frac{\sin(a)}{a} A_1$.

Note : $\cos(a) = \sum_{m=0}^{\infty} (-1)^m \frac{a^{2m}}{(2m)!}, \sin(a) = \sum_{m=0}^{\infty} (-1)^m \frac{a^{2m+1}}{(2m+1)!}$.

Since ${A_2}^2 = \begin{bmatrix} 0 & 0 & ac \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ *and* ${A_2}^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, *then* A_2 *is nilpotent.*

Hence, $\exp(A_2) = \sum_{m=0}^{\infty} \frac{{A_2}^m}{m!} = I + A_2 + \frac{1}{2!} \begin{bmatrix} 0 & 0 & ac \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & a & b + \frac{ac}{2} \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$.

Since ${A_3}^n = \begin{bmatrix} a^n & na^{n-1}b \\ 0 & a^n \end{bmatrix}$, *then*

$$\exp(A_3) = \sum_{m=0}^{\infty} \frac{{A_3}^m}{m!} = I + \sum_{m=1}^{\infty} \frac{1}{m!} \begin{bmatrix} a^m & ma^{m-1}b \\ 0 & a^m \end{bmatrix}$$

$$= \begin{bmatrix} \sum_{m=0}^{\infty} \frac{a^m}{m!} & \sum_{m=1}^{\infty} \frac{ma^{m-1}b}{m!} \\ 0 & \sum_{m=0}^{\infty} \frac{a^m}{m!} \end{bmatrix} = \begin{bmatrix} e^a & b \sum_{m=0}^{\infty} \frac{a^m}{m!} \\ 0 & e^a \end{bmatrix}$$

$$= \begin{bmatrix} e^a & be^a \\ 0 & e^a \end{bmatrix}.$$

Exercise 3.6.
Show that for any $a, b, d \in \mathbb{C}, \exp \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = \begin{pmatrix} e^a & b\frac{e^a-e^d}{a-d} \\ 0 & e^d \end{pmatrix}$.
Since $\lim\limits_{a \to d} \frac{e^a-e^d}{a-d} = e^a$, *we simply interpret* $\frac{e^a-e^d}{a-d}$ *as* e^a *when* $d = a$.
(*Hint : Show that* $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}^m = \begin{pmatrix} a^m & b\frac{a^m-d^m}{a-d} \\ 0 & d^m \end{pmatrix}, \forall m \in \mathbb{N}$ *and* $a \neq d$)

Proof.
For $m = 1$, $\begin{pmatrix} a^1 & b\frac{a^1-d^1}{a-d} \\ 0 & d^1 \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}^1$.
Suppose $m = k-1$ is hold that $\begin{pmatrix} a^{k-1} & b\frac{a^{k-1}-d^{k-1}}{a-d} \\ 0 & d^{k-1} \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}^{k-1}$.
For $m = k$,

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}^k = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}^{k-1} \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$$
$$= \begin{pmatrix} a^{k-1} & b\frac{a^{k-1}-d^{k-1}}{a-d} \\ 0 & d^{k-1} \end{pmatrix} \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$$
$$= \begin{pmatrix} a^k & a^{k-1}b + bd\frac{a^{k-1}-d^{k-1}}{a-d} \\ 0 & d^k \end{pmatrix}$$
$$= \begin{pmatrix} a^k & \frac{a^kb-a^{k-1}bd}{a-d} + \frac{a^{k-1}bd-bd^k}{a-d} \\ 0 & d^k \end{pmatrix} = \begin{pmatrix} a^k & b\frac{a^k-d^k}{a-d} \\ 0 & d^k \end{pmatrix}.$$

Hence, by math induction, $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}^m = \begin{pmatrix} a^m & b\frac{a^m-d^m}{a-d} \\ 0 & d^m \end{pmatrix}, \forall m \in \mathbb{N}$ and $a \neq d$.

$$\exp \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = \sum_{m=0}^{\infty} \frac{1}{m!} \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}^m = \sum_{m=0}^{\infty} \frac{1}{m!} \begin{pmatrix} a^m & b\frac{a^m-d^m}{a-d} \\ 0 & d^m \end{pmatrix}$$
$$= \begin{pmatrix} \sum\limits_{m=0}^{\infty} \frac{1}{m!}a^m & \sum\limits_{m=0}^{\infty} \frac{1}{m!}b\frac{a^m-d^m}{a-d} \\ 0 & \sum\limits_{m=0}^{\infty} \frac{1}{m!}d^m \end{pmatrix} = \begin{pmatrix} e^a & b\frac{\sum\limits_{m=0}^{\infty} \frac{a^m}{m!} - \sum\limits_{m=0}^{\infty} \frac{d^m}{m!}}{a-d} \\ 0 & e^d \end{pmatrix} = \begin{pmatrix} e^a & b\frac{e^a-e^d}{a-d} \\ 0 & e^d \end{pmatrix} \qquad \square$$

Combining Lemma 3.1.2 and Theorem 3.1.32, we reach
$\exp(A) = \exp(D)\exp(N)$
$= \exp(\lambda_1 P_1(A)) \cdots \exp(\lambda_k P_k(A)) \exp(N)$
$= \exp(N)\exp(D)$
$= \exp(N)\exp(\lambda_1 P_1(A)) \cdots \exp(\lambda_k P_k(A))$.

Example 3.1.3.

The characteristic polynomial of the matrix $A = \begin{pmatrix} 252 & 498 & 4134 & 698 \\ -234 & -465 & -3885 & -656 \\ 15 & 30 & 252 & 42 \\ -10 & -20 & -166 & -25 \end{pmatrix}$
is $p(z) = (z-4)^2(z-3)^2$.
One can compute $\frac{1}{p(z)} = \frac{1}{(z-4)^2} - \frac{2}{z-4} + \frac{1}{(z-3)^2} + \frac{2}{z-3} = \frac{1-2(z-4)}{(z-4)^2} + \frac{1+2(z-3)}{(z-3)^2}$.
Accordingly, we set
$P_1(z) := (1-2(z-4))(z-3)^2, \quad \lambda_1 = 4,$
$P_2(z) := (1+2(z-3))(z-4)^2, \quad \lambda_2 = 3,$
and we compute $P_1(A) = \begin{pmatrix} -1 & -2 & 134 & 198 \\ 1 & 2 & -125 & -186 \\ 0 & 0 & 9 & 12 \\ 0 & 0 & -6 & -8 \end{pmatrix}$, $P_2(A) = \begin{pmatrix} 2 & 2 & -134 & -198 \\ -1 & -1 & 125 & 186 \\ 0 & 0 & -8 & -12 \\ 0 & 0 & 6 & 9 \end{pmatrix}$.
Therefore, $S = \lambda_1 P_1(A) + \lambda_2 P_2(A) = \begin{pmatrix} 2 & -2 & 134 & 198 \\ 1 & 5 & -125 & -186 \\ 0 & 0 & 12 & 12 \\ 0 & 0 & -6 & -5 \end{pmatrix}$,
and $N = A - S = \begin{pmatrix} 250 & 500 & 4000 & 500 \\ -235 & -470 & -3760 & 470 \\ 15 & 30 & 240 & 30 \\ -10 & -20 & -160 & -20 \end{pmatrix}$.

Exercise 3.7.
Decompose $A = \begin{pmatrix} 3 & 4 & 3 \\ 2 & 7 & 4 \\ -4 & 8 & 3 \end{pmatrix}$ *by using Algorithm 2.*

Exercise 3.8. *Bochner's subordination*
Let $0 < s < 1$*, by using the integration by parts on* $\Gamma(1-s)$*, where* Γ *is the gamma function, show that* $\lambda^s = \frac{1}{|\Gamma(-s)|}\int_0^\infty (1-e^{t\lambda})t^{-1-s}dt, \forall \lambda > 0$.
Note : gamma function $\Gamma(s) = \int_0^\infty e^{-t}t^{s-1}dt$.

Proof.
claim : the property for $\Gamma(1-s)$

$$\Gamma(1-s) = \int_0^\infty e^{-t}t^{-s}dt = \lim_{k\to\infty}\int_0^k e^{-t}t^{-s}dt$$

$$\text{let } u = e^{-t}, dv = t^{-s} \Rightarrow du = -e^{-t}, v = \frac{t^{1-s}}{1-s}$$

$$= \lim_{k\to\infty}\left(\frac{e^{-t}t^{1-s}}{1-s}\bigg|_{t=0}^{k} + \int_0^k \frac{1}{1-s}t^{1-s}e^{-t}dt\right)$$

$$= \lim_{k\to\infty}\int_0^k \frac{1}{1-s}t^{1-s}e^{-t}dt$$

$$= \frac{1}{1-s}\Gamma(2-s)$$

$$\int_0^\infty (1-e^{-\lambda t})t^{-1-s}dt = \int_0^\infty t^{-1-s}dt - \int_0^\infty t^{-1-s}dt$$

$$\text{let } \lambda t = v$$

$$= \lim_{k\to\infty}\frac{t^{-s}}{-s}\bigg|_{t=0}^{k} - \lim_{k\to\infty}\int_0^k e^{-v}\left(\frac{v}{\lambda}\right)^{-1-s} d\left(\frac{v}{\lambda}\right)$$

$$= 0 - \lambda^s\Gamma(-s)$$

$\Rightarrow \lambda^s = \frac{1}{|\Gamma(-s)|}\int_0^\infty (1-e^{t\lambda})t^{-1-s}dt, \forall\lambda > 0$ $\square$

Definition 3.1.16.
A Hermitian matrix $A \in M_{n\times n}(\mathbb{C})$ is said to positive definite, denoted by $A \succ 0$, if $p^*Ap > 0, \forall p \in \mathbb{C}^n \setminus \{0\}$.

By using Thm 3.1.17, we see that $A \succ 0$ iff $A = Udiag(\lambda_1, \cdots, \lambda_n)U^*$, for some $\lambda_1, \ldots, \lambda_n > 0$ and unitary $U \in M_{n\times n}(\mathbb{C})$, and accordingly we define
$A^s := Udiag({\lambda_1}^s, \cdots, {\lambda_n}^s)U^*$.

Using the Bochner's subordination and $Pdiag(e^{\lambda_1}, e^{\lambda_2}, \cdots, e^{\lambda_n})P^{-1}$, we can compute A^s via the formula $A^s = \frac{1}{|\Gamma(-s)|}\int_0^\infty (1-e^{tA})t^{-1-s}dt, \forall A \succ 0$, which gives an application of the matrix fundamental solution for $Y(t) = \exp(tA), \forall t \in \mathbb{R}$.

Remark：分數次方的矩陣
The Fourier transform suggests us to (formally) replace A by $-\Delta$ in $A^s = \frac{1}{|\Gamma(-s)|}\int_0^\infty (1-e^{tA})t^{-1-s}dt$, and we reach the fractional Laplacian $(-\Delta)^s := \frac{1}{|\Gamma(-s)|}\int_0^\infty (1-e^{-t\Delta})t^{-1-s}dt$.

3.1.2 The Matrix Logarithm

Goal：Define a matrix logarithm, which should be an inverse function to the matrix exponential.

One simplest way to dene the matrix logarithm is by a power series.
We recall the following fact concerning the principal branch of complex logarithm.

Lemma 3.1.24.
The radius of convergence of the complex power series $\log z := \sum_{m=1}^{\infty}(-1)^{m+1}\frac{(z-1)^m}{m}$ is 1. i.e., the series is defined and holomorphic in a circle of radius 1 about $z = 1$.
In addition, we have $e^{\log z} = z, \forall z$ with $|z-1| < 1$.
Moreover, we have $|e^u - 1| < 1$ and $\log e^u = u, \forall u$ with $|u| < \log 2$.

Based on the above lemma, we now can define the matrix logarithm by the following theorem.

Theorem 3.1.25.
The matrix logarithm $\log(A) := \sum_{m=1}^{\infty}(-1)^{m+1}\frac{(A-I)^m}{m}$ is defined and continuous on the set of all matrices $A \in M_{n\times n}(\mathbb{C})$ with $\|A - I\| < 1$.
In addition, $\exp(\log(A)) = A, \forall A \in M_{n\times n}(\mathbb{C})$ with $\|A - I\| < 1$.
Moreover, $\|\exp(B) - I\| < 1$ and $\log(\exp(B)) = B, \forall B \in M_{n\times n}(\mathbb{C})$ with $\|B\| < \log 2$.

Proof.
By $\|AB\| \le \|A\|\|B\|$, we have $\|(A-I)^m\| \le \|A-I\|^m$.
By using the similar arguments in Lma 3.1.4, one can show that $\log(A)$ is defined and continuous on the set of all matrices $A \in M_{n\times n}(\mathbb{C})$ with $\|A-I\| < 1$.
(We left the details for readers as an exercise)
Let $A \in M_{n\times n}(\mathbb{C})$ with $\|A-I\| < 1$.
By using Lma 3.1.19 (i.e, $\forall A \in M_{n\times n}(\mathbb{C}), \exists$ a sequence of diagonalizable matrix B_k, which is converges to A), one can find a sequence of diagonalizable matrix $A_k \in M_{n\times n}(\mathbb{C}) \ni \|A_k - A\| \to 0$ as $k \to \infty$.
Since $\|A_k - I\| < 1, \forall$ sufficiently large k, then we know that
$\log(A_k) = \sum_{m=1}^{\infty}(-1)^{m+1}\frac{(A_k-I)^m}{m}$ is defined and continuous, $\forall$ sufficiently large k.

We write $A_k = Q_k\, diag(\lambda_{k,1}, \cdots, \lambda_{k,n}) {Q_k}^{-1}$ and we see that
$(A_k - I)^m = Q_k\, diag((\lambda_{k,1}-1)^m, \cdots, (\lambda_{k,n}-1)^m)Q_k^{-1}$.
Hence, $\log(A_k) = Q_k\left(\sum\limits_{m=1}^{\infty}\frac{(-1)^{m+1}}{m} diag((\lambda_{k,1}-1)^m, \cdots, (\lambda_{k,n}-1)^m)\right)Q_k^{-1}$
$= Q_k diag(\log(\lambda_{k,1}), \cdots, \log(\lambda_{k,n})){Q_k}^{-1}$.
By using (3.1.5) and Lemma 3.1.45 we see that

$$\begin{aligned}\exp(\log(A_k)) &= Q_k \exp\, diag(\log(\lambda_{k,1}), \cdots, \log(\lambda_{k,n})){Q_k}^{-1}\\ &= Q_k\, diag(\exp\log(\lambda_{k,1}), \cdots, \exp\log(\lambda_{k,n})){Q_k}^{-1}\\ &= Q_k\, diag(\lambda_{k,1}, \cdots, \lambda_{k,n}){Q_k}^{-1} = A_k.\end{aligned}$$

Finally, by continuity of the mapping $A \mapsto \log(A), B \mapsto \exp(B)$, we conclude $\exp(\log(A)) = A$ by taking $k \to \infty$.
If $\|B\| < \log 2$, then using $\|A+B\| \le \|A\| + \|B\|$ we see that
$\|\exp(B) - I\| = \left\|\sum\limits_{m=1}^{\infty}\frac{B^m}{m!}\right\| \le \sum\limits_{m=1}^{\infty}\frac{\|B\|^m}{m!} = e^{\|B\|-1} < 2-1 = 1.$
Thus, $\log(\exp(B))$ is well-defined.
The proof of $\log(\exp(B)) = B$ is very similar to the proof of $\exp(\log(A)) = A$, therefore we left the details for readers as an exercise. $\square$

Remark :
If A is unipotent (i.e. $A-I$ is nilpotent), then $\log(A)$ is simply a finite sum, which can be defined without the assumption $\|A-I\| < 1$.
In this case, one can easily verify that $\log(A)$ is nilpotent.

Exercise 3.9.
Show that :
(a) If A is unipotent, then $\exp(\log(A)) = A$*.*
(b) If B is nilpotent, then $\log(\exp(B)) = B$*.*
(*Hint. Let $A(t) := I + t(A - I)$ and show that* $\exp(\log(A(t)))$ *depends polynomially on t and that* $\exp(\log(A(t))) = A(t), \forall$ *sufficiently small t*)

Proof.
(a) Let $A(t) := I + t(A - I)$.
Since A is unipotent, then $A - I$ is nilpotent $\Rightarrow \exists n \in \mathbb{N} \ni (A - I)^n = 0$.

$$\begin{aligned}
\lim_{h\to 0} A(t) &= \lim_{h\to 0} \frac{1}{h}\Big(\exp(\log(A(t+h))) - \exp(\log(A(t)))\Big) \\
&= \lim_{h\to 0} \frac{1}{h}\Big(\sum_{k=0}^{\infty} \frac{1}{k!}\big(\log(A(t+h)) - \log(A)\big)\Big) \\
&= \lim_{h\to 0} \frac{1}{h}\Big(\sum_{k=0}^{\infty} \frac{1}{k!}\big(\log(I - (t+h)(A - I)) - \log(I - t(A - I))\big)\Big) \\
&= \lim_{h\to 0} \frac{1}{h}\Big(\sum_{k=0}^{\infty} \frac{1}{k!}\big(\sum_{m=1}^{\infty}(-1)^{m+1}\big(\frac{(-(t+h)(A - I))^m}{m!} - \frac{(-t(A - I))^m}{m!}\big)\big)\Big) \\
&= \lim_{h\to 0} \frac{1}{h}\Big(\sum_{k=0}^{\infty} \frac{1}{k!}\big(\sum_{m=1}^{\infty}(-1)^{2m+1}\big(\frac{(t+h)^m(A - I)^m}{m!} - \frac{t^m(A - I)^m}{m!}\big)\big)\Big) \\
&= \sum_{k=0}^{\infty} \frac{1}{k!}\Big(\sum_{m=1}^{\infty} \big(\lim_{h\to 0} \frac{1}{h}\big(\frac{[-(t+h)^m + t^m](A - I)^m}{m!}\big)\big)\Big) \\
&= \sum_{k=0}^{\infty} \frac{1}{k!}\Big(\sum_{m=1}^{\infty} \frac{1}{m!}(-t^{m-1}(A - I)^m)\Big) \\
&= \sum_{k=0}^{\lceil \frac{n}{m} \rceil} \frac{1}{k!}\Big(\sum_{m=1}^{n} \frac{1}{m!}(-t^{m-1}(A - I)^m)\Big) = I, \forall \text{sufficient small } t
\end{aligned}$$

Hence, $\exp(\log(A)) = A$.
(b) Since B is nilpotent, then $\exists n \in \mathbb{N} \ni B^n = 0$.
Hence, $\exp(B) = \sum\limits_{m=0}^{\infty} \frac{B^m}{m!} = \sum\limits_{m=0}^{n-1} \frac{B^m}{m!} = I + B + \frac{B^2}{2!} + \frac{B^3}{3!} + \cdots + \frac{B^{n-1}}{(n-1)!}$.

$$
\begin{aligned}
\log(\exp(B)) &= \log(I + B + \frac{B^2}{2!} + \frac{B^3}{3!} + \cdots + \frac{B^{n-1}}{(n-1)!}) \\
&= \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \big(B + \frac{B^2}{2!} + \frac{B^3}{3!} + \cdots + \frac{B^{n-1}}{(n-1)!}\big)^k \\
&= \sum_{k=1}^{\lceil \frac{n}{m} \rceil} \frac{1}{k!} \big(B + \frac{B^2}{2!} + \frac{B^3}{3!} + \cdots + \frac{B^{n-1}}{(n-1)!}\big)^k
\end{aligned}
$$

Hence, $\log(\exp(B)) = B$. $\square$

Exercise 3.10.

Show that there exists a constant $c > 0 \ni \|\log(I + A) - A\| \leq c\|A\|^2$ holds true $\forall A \in M_{n\times n}(\mathbb{C})$ with $\|A\| \leq \frac{1}{2}$.

Proof.

$\log(I + A) = \sum\limits_{m=1}^{\infty} (-1)^{m+1} \frac{A^m}{m} = A - \frac{A^2}{2} + \frac{A^3}{3} - \frac{A^4}{4} + \cdots$

$\Rightarrow \log(I + A) - A = -\frac{A^2}{2} + \frac{A^3}{3} - \frac{A^4}{4} + \cdots$

$\Rightarrow \|\log(I + A) - A\| = \| - \frac{A^2}{2} + \frac{A^3}{3} - \frac{A^4}{4} + \cdots \|$

$\Rightarrow \|\log(I + A) - A\| \leq \sum\limits_{k=2}^{\infty} \frac{\|A\|^k}{k} = \|A\|^2 \sum\limits_{k=0}^{\infty} \frac{\|A\|^k}{k+2}$

$\Rightarrow \|\log(I + A) - A\| \leq \frac{1}{4} \sum\limits_{k=0}^{\infty} \frac{1}{2^k(k+2)}$ (since $\|A\| \leq \frac{1}{2}$). $\square$

Remark :

We may restate $\|\log(I+A)-A\| \leq c\|A\|^2$ by saying that $\log(I+A) = A+O(\|A\|^2)$, where $O(\|A\|^2)$ denotes a quantity of order $\|A\|^2$, i.e. a quantity that is bounded by a constant times $\|A\|^2, \forall$ sufficiently small values of $\|A\|$.

3.1.3 One parameter subgroup, Lie group and Lie algebra

We now exhibit a result involving the exponential of a matrix that will be important in the study of Lie algebras.
李代數中存在一些和矩陣指數相關的重要結果

Theorem 3.1.26. (*Lie product formula*)
For each $A, B \in M_{n\times n}(\mathbb{C})$*, we have* $\exp(A+B) = \lim_{m\to\infty}(\exp(\frac{A}{m})\exp(\frac{B}{m}))^m$.

Proof.
By multiplying the power series for $\exp(\frac{A}{m})$ and $\exp(\frac{B}{m})$, one sees that
$\exp(\frac{A}{m})\exp(\frac{B}{m}) = I + \frac{A}{m} + \frac{B}{m} + O(m^{-2})$.
Since $\exp(\frac{A}{m})\exp(\frac{B}{m}) \to I$ as $m \to \infty$, then $\log(\exp(\frac{A}{m})\exp(\frac{B}{m}))$ is well-defined for all sufficient large m.
By using $\log(I+A) - A\| \le c\|A\|^2$ in Exercise 3.10, we see that

$$\begin{aligned}\log(\exp(\frac{A}{m})\exp(\frac{B}{m})) &= \log\left(I + \frac{A}{m} + \frac{B}{m} + O(m^{-2})\right)\\ &= \frac{A}{m} + \frac{B}{m} + O(m^{-2}) + O\left(\left\|\frac{A}{m} + \frac{B}{m} + O(m^{-2})\right\|^2\right)\\ &= \frac{A}{m} + \frac{B}{m} + O(m^{-2}).\end{aligned}$$

Thm 3.1.25 guarantees that $\exp(\frac{A}{m})\exp(\frac{B}{m}) = \exp(\frac{A}{m} + \frac{B}{m} + O(m^{-2}))$, then
$\exp(\frac{A}{m})\exp(\frac{B}{m}) = \exp(A + B + O(m^{-1}))$.
By the continuity of the exponential, we conclude by taking $m \to \infty$. □

Remark :
There is a version of this result, known as the Trotter product formula, which holds for suitable unbounded operators on an infinite-dimensional Hilbert space. e.g. [Hal13, Theorem 20.1].

Definition 3.1.17.
$\{Y(t)\}_{t\in\mathbb{R}}$ *is a one parameter subgroup of* $GL(n,\mathbb{C})$ *if*
(a) $Y : \mathbb{R} \to GL(n,\mathbb{C})$ *is continuous*
(b) $Y(0) = I$
(c) $Y(t+s) = Y(t)Y(s), \forall t, s \in \mathbb{R}$.
Note :
$Y(t) \in M_{n\times n}(\mathbb{C})$*, and any matrix with* exp *function must belongs to* $GL(n,\mathbb{C})$.

Example 3.1.4.
The fundamental matrix solution $\{\exp(tA)\}_{t\in\mathbb{R}}$ *given in* $Y(t) = \exp(tA), \forall t \in \mathbb{R}$ *forms a one parameter subgroup, where*

(a) *is verified by Theorem 3.1.6*
(b) *can be found in the basic properties*
$\begin{cases} (1)\ \exp(0) = I, (exp(A))^* = exp(A^*), \\ (2)\ \exp(BAB^{-1}) = B\exp(A)B^{-1}, \forall B \in GL(n, \mathbb{C}) \end{cases}$
(c) *is a special case of Corollary 3.1.3.*

Lemma 3.1.27. 說明矩陣如何開根號
Let $0 < \varepsilon < \log 2$.
Let $B_{\frac{\varepsilon}{2}} := A \in M_{n\times n}(\mathbb{C}); \|A\| < \frac{\varepsilon}{2}$ *and* $\exp(B_{\frac{\varepsilon}{2}}) := \{\exp(A) \in M_{n\times n}(\mathbb{C}; \|A\| < \frac{\varepsilon}{2}\}$. *Given any* $A \in \exp(\exp(B_{\frac{\varepsilon}{2}}))$, $\exists! \sqrt{A} \in \exp(B_{\frac{\varepsilon}{2}}) \ni \sqrt{A}^2 = A$, *which is given by* $\sqrt{A} = \exp(\frac{1}{2}\log(A))$.

Proof.
Tips : By using Lma 3.1.2, $\sqrt{A}^2 = \exp(\frac{1}{2}\log(A) + \frac{1}{2}\log(A)) = \exp(\log(A)) = A$.
claim : check domain $\ni \varepsilon \in (\|2\log B\|, \log 2)$.
Suppose $B \in \exp(B_{\frac{\varepsilon}{2}})$ satisfies $B^2 = A$.
Using Lma 3.1.2,
$\exp(2\log(B)) = \exp(\log(B) + \log(B)) = \exp(\log(B))\exp(\log(B) = B^2 = A$.
Since $B \in \exp(B_{\frac{\varepsilon}{2}})$, then we can check that $\|2\log(B)\| < \varepsilon < \log 2$.
Use Thm 3.1.25, $\log(A) = \log(\exp(2\log(B))) = 2\log(B)$, then $\log(B) = \frac{1}{2}\log(A)$.
Hence, we conclude the uniqueness by using Thm 3.1.25 again,
$B = \exp(\log(B)) = \exp\left(\frac{1}{2}\log(A)\right) = \sqrt{A}$. ☐

Goal : show that Example 3.1.4 already exhibit all one parameter subgroup.

Theorem 3.1.28.
If $\{Y(t)\}_{t\in\mathbb{R}}$ *is a one parameter subgroup of* $GL(n, \mathbb{C})$, *then there exists a unique* $A \in M_{n\times n}(\mathbb{C}) \ni Y(t) = e^{tA}, \forall t \in \mathbb{R}$.

Proof.
claim : the uniqueness of such A.
Suppose that $e^{tA} = e^{tB}, \forall t \in \mathbb{R}$.
By Thm 3.1.6, we can differentiate the power series of matrix exponential term-by-term, $A = \frac{d}{dt}e^{tA}\Big|_{t=0} = \frac{d}{dt}e^{tB}\Big|_{t=0} = B$.
Since the function $\log : \exp(B_{\frac{\varepsilon}{2}}) \to B_{\frac{\varepsilon}{2}}$ is bijective and continuous, then we see that $\exp(B_{\frac{\varepsilon}{2}})$ is an open set in $GL(n, \mathbb{C})$.
Since $Y(0) = I \in \exp(B_{\frac{\varepsilon}{2}})$ and $t \to Y(t)$ is continuous, then
$\exists t_0 > 0 \ni Y(t) \in \exp(B_{\frac{\varepsilon}{2}}), \forall t \in [-t_0, t_0]$.
Define $A := \frac{1}{t_0}\log(Y(t_0))$, so that $t_0 A = \log(Y(t_0))$.

Since $t_0 A \in B_{\frac{\varepsilon}{2}}$, then Thm 3.1.25 allows us to apply matrix exponential on the identity $t_0 A = \log(Y(t_0))$, $\exp(t_0 A) = \exp(\log(Y(t_0))) = Y(t_0)$.
Since $Y(\frac{t_0}{2}) \in \exp(B_{\frac{\varepsilon}{2}})$ and by def'n 3.1.17(c), then we have $Y(\frac{t_0}{2})^2 = Y(t_0)$.
By using Lma 3.1.27, $Y(\frac{t_0}{2}) = \sqrt{Y(t_0)} = \exp\left(\frac{1}{2}\log(Y(t_0))\right) = \exp(\frac{t_0 A}{2})$.
Applying repeatedly, we conclude that $Y(\frac{t_0}{2^k}) = \exp(\frac{t_0 A}{2^k}), \forall k \in \mathbb{N}$.
By def'n 3.1.17(c) and Lma 3.1.2, $Y(\frac{mt_0}{2^k}) = Y(\frac{t_0}{2^k})^m = \exp(\frac{mt_0 A}{2^k}), \forall k \in \mathbb{N}$ and $m \in \mathbb{Z}$, i.e., $Y(t) = \exp(tA), \forall t \in \mathbb{R}$ of the form $t = \frac{m}{2^k}t_0$.
Since the set $\{t \in \mathbb{R} : t = \frac{m}{2^k}t_0$ for some $k \in \mathbb{N}$ and $m \in \mathbb{Z}\}$ is dense in $\mathbb{R}$, and by continuity of $t \to Y(t)$ and $t \to \exp(tA)$, then we conclude that
$Y(t) = \exp(tA), \forall t \in \mathbb{R}$. □

Thm 3.1.28 says that there is a one-to-one correspondence between $M_{n\times n}(\mathbb{C})$ and the collection of one parameter subgroups of $GL(n,\mathbb{C})$. This also suggests us to extend Definition 3.1.17 as follows:

Definition 3.1.18.
Let $\mathcal{G}$ be a matrix Lie group, i.e. a subgroup of $GL(n,\mathbb{C})$ with respect to matrix multiplication. In other words, $\mathcal{G}$ is a subset of $GL(n,\mathbb{C})$ satisfying :
(1) $AB \in \mathcal{G}, \forall A, B \in \mathcal{G}$
(2) $I \in \mathcal{G}$
(3) $A^{-1} \in \mathcal{G}, \forall A \in \mathcal{G}$.
We call $\{Y(t)\}_{t\in\mathbb{R}}$ a one parameter subgroup of $\mathcal{G}$ if
(1) $Y : \mathbb{R} \to \mathcal{G}$ *is continuous*
(2) $Y(0) = I$
(3) $Y(t+s) = Y(t)Y(s), \forall t, s \in \mathbb{R}$.

Example 3.1.5.
The trivial subgroup $\{Y(t)\}_{t\in\mathbb{R}}$ is a one parameter subgroup of $\mathcal{G}$, which means the existence of one parameter subgroup holds.

We now able to give some examples. By using Lemma 3.1.27, it is easy to check that the unitary group U(n,C) is a matrix Lie group.

Theorem 3.1.29.
If $\{Y(t)\}_{t\in\mathbb{R}}$ is a one parameter subgroup of $U(n,\mathbb{C})$, then there exists a unique $A \in M_{n\times n}(\mathbb{C})$ which is skew-Hermitian (i.e. $A^* = -A$) $\ni Y(t) = \exp(tA), \forall t \in \mathbb{R}$.

Proof.

$$\begin{aligned} exp(tA^*) = \exp((tA)^*) &= I + (tA)^* + \frac{((tA)^*)^2}{2!} + \cdots \\ &= \left(I + tA + \frac{(tA)^2}{2!} + \cdots\right)^* = (exp(tA))^*, \forall t \in \mathbb{R}. \end{aligned}$$

If $A \in M_{n\times n}(\mathbb{C})$ is skew-Hermition, then by Corollary 3.1.4,
$(\exp(tA))^* = \exp(tA^*) = \exp(-tA) = (\exp(tA))^{-1}, \forall t \in \mathbb{R}$.
By Theorem 3.1.25, we see that $tB^* = \log(\exp(tB^*)) = \log(\exp(-tB)) = -tB$, $\forall t$ with small $|t|$, which shows that B is skew-Hermitian. $\square$

The special linear group $SL(n, \mathbb{C}) := \{A \in M_{n\times n}(\mathbb{C}) : \det(A) = 1\}$ is a matrix Lie group.

Theorem 3.1.30.
If $\{Y(t)\}_{t\in\mathbb{R}}$ is a one parameter subgroup of $SL(n, \mathbb{C})$, then there exists a unique $A \in M_{n\times n}(\mathbb{C})$ with $tr(A) = 0 \ni Y(t) = \exp(tA), \forall t \in \mathbb{R}$.

Proof.
Let $A \in M_{n\times n}(\mathbb{C})$ with $tr(A) = 0$.
By using Exercise 3.4, check that $\det(\exp(tA)) = e^{tr(tA)} = e^0 = 1, \forall t \in \mathbb{R}$,
i.e., $\{\exp(tA)\}_{t\in\mathbb{R}}$ forms a one parameter subgroup of $SL(n, \mathbb{C})$.
Let $\{Y(t)\}_{t\in\mathbb{R}}$ be a one parameter subgroup of SL(n,C).
By Theorem 3.1.25, $\exists$ a matrix $A \in M_{n\times n}(\mathbb{C}) \ni Y(t) = \exp(tA), \forall t \in \mathbb{R}$.
By Exercise 3.4, $1 = \det(Y(1)) = e^{tr(A)}$, then $tr(A) = \log(e^{tr(A)}) = 0$,
which conclude our theorem. $\square$

Remark : See also Lma 3.1.36.
It is interesting to mention that, if $A \in M_{n\times n}(\mathbb{C})$ satisfies $tr(A) = 0$, then there exist matrices $X \in M_{n\times n}(\mathbb{C})$ and $Y \in M_{n\times n}(\mathbb{C})$ so that $A = XY - YX$, where X is Hermitian and $tr(Y) = 0$.

Note : check that the special unitary group $SU(n, \mathbb{C}) := U(n, \mathbb{C}) \cap SL(n, \mathbb{C})$ is also a matrix Lie group.

Corollary 3.1.31.
If $\{Y(t)\}_{t\in\mathbb{R}}$ is a one parameter subgroup of $SU(n, \mathbb{C})$, then there exists a unique skew-Hermitian matrix $A \in M_{n\times n}(\mathbb{C})$ with $tr(A) = 0 \ni Y(t) = \exp(tA), \forall t \in \mathbb{R}$.

Definition 3.1.19.
Define the following sets :
(1) $\mathfrak{gl}(n, \mathbb{C}) := M_{n\times n}(\mathbb{C})$
(2) $\mathfrak{u}(n, \mathbb{C}) := \{A \in M_{n\times n}(\mathbb{C}) : A^* = -A\}$
(3) $\mathfrak{sl}(n, \mathbb{C}) := \{A \in M_{n\times n}(\mathbb{C}) : tr(A) = 0\}$
(4) $\mathfrak{su}(n, \mathbb{C}) := \mathfrak{u}(n, \mathbb{C}) \cap \mathfrak{sl}(n, \mathbb{C})$

We point out that Theorem 3.1.28, Thm 3.1.29, Thm 3.1.30 and Cor 3.1.31 say

that one has the following one-to-one correspondence :
(1) $GL(n,\mathbb{C}) \leftrightarrow \mathfrak{gl}(n,\mathbb{C})$
(2) $U(n,\mathbb{C}) \leftrightarrow \mathfrak{u}(n,\mathbb{C})$
(3) $SL(n,\mathbb{C}) \leftrightarrow \mathfrak{sl}(n,\mathbb{C})$
(4) $SU(n,\mathbb{C}) \leftrightarrow \mathfrak{su}(n,\mathbb{C})$

Definition 3.1.20.
Let $\mathcal{G}$ be a matrix Lie group.
The Lie algebra of $\mathcal{G}$, denoted $\mathfrak{g}$, is the set of all matrices $A \ni e^{tA} \in \mathcal{G}, \forall t \in \mathbb{R}$.

Summarize the above examples :

matrix Lie Group G	Lie algebra $\mathfrak{g}$
$GL(n,\mathbb{C})$	$\mathfrak{gl}(n,\mathbb{C})$
$U(n,\mathbb{C})$	$\mathfrak{u}(n,\mathbb{C})$
$SL(n,\mathbb{C})$	$\mathfrak{sl}(n,\mathbb{C})$
$SU(n,\mathbb{C})$	$\mathfrak{su}(n,\mathbb{C})$

Definition 3.1.21.
Define the commutator by $[A,B] := AB - BA, \forall A,B \in M_{n\times n}(\mathbb{C})$.
Note : A and B are commute iff $[A,B] = 0$.

The following theorem exhibit some basic properties of Lie algebra.

Theorem 3.1.32.
Let G be a matrix Lie group with Lie algebra $\mathfrak{g}$. The following holds :
(a) $A\mathfrak{g}A^{-1} \subset \mathfrak{g}, \forall A \in G$, i.e., for each $B \in \mathfrak{g}$, $ABA^{-1} \in \mathfrak{g}, \forall A \in G$.
(b) $\mathfrak{g}$ is $\mathbb{R}$-linear, that is, $aA + bB \in \mathfrak{g}, \forall A,B \in \mathfrak{g}$ and $a,b \in \mathbb{R}$.
(c) $[A,B] \in \mathfrak{g}, \forall A,B \in \mathfrak{g}$.

Proof.
(a)
(b) claim 1 : $tA \in \mathfrak{g}, \forall t \in \mathbb{R}$.
For each $A \in \mathfrak{g}$, it is easy to check that $tA \in \mathfrak{g}, \forall t \in \mathbb{R}$.
claim 2 : $aA + bB \in \mathfrak{g}$.
For each $A,B \in G$ and for each $m \in \mathbb{N}$, $(\exp(\frac{tA}{m})\exp(\frac{tB}{m}))^m \in G$.
Using the Lie product formula, $\exp(t(A+B)) = \lim_{m\to\infty}(\exp(\frac{tA}{m})\exp(\frac{tB}{m}))^m \in G$.
(c) By using (a), one sees that $\exp(tA)B\exp(-tA) \in \mathfrak{g}, \forall t \in \mathbb{R}$.
By (b), $\frac{\exp(tA)B\exp(-tA)-B}{t} \in \mathfrak{g}, \forall t \in \mathbb{R}\setminus\{0\}$.
Hence, $\frac{d}{dt}\bigg(\exp(tA)B\exp(-tA)\bigg)\bigg|_{t=0} = \lim_{m\to\infty}\frac{\exp(tA)B\exp(-tA)-B}{t} \in \mathfrak{g}$.
Using product rule, $\frac{d}{dt}\big(\exp(tA)B\exp(-tA)\big)\big|_{t=0}$
$= A\exp(tA)B\exp(-tA) + \exp(tA)B(-A\exp(-tA))\big|_{t=0} = AB - BA = [A,B]$. $\square$

Definition 3.1.22.
Let G and H be matrix Lie groups.
A map $\Phi : G \to H$ is called a Lie group homomorphism if :
(a) $\Phi : G \to H$ is a group homomorphism, i.e., $\Phi(A)\Phi(B) = \Phi(AB), \forall A, B \in G$
(b) $\Phi : G \to H$ is continuous
In addition, if $\Phi : G \to H$ is bijective and its inverse $\Phi^{-1} : H \to G$ is continuous, then Φ is called a Lie group isomorphism.

The following theorem tells us that a Lie group homomorphism between two Lie groups gives rise in a natural way to a map between the corresponding Lie algebras.

Theorem 3.1.33.
Let G and H be matrix Lie groups, with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, respectively.
Suppose that $\Phi : G \to H$ is a Lie group homomorphism.
Then $\exists!\mathbb{R}$-linear map $\Phi : \mathfrak{g} \to \mathfrak{h} \ni \Phi(\exp(A)) = \exp(\Phi(A)), \forall A \in \mathfrak{g}$, which satisfies
(a) $\Phi(BAB^{-1}) = \Phi(B)\Phi(A)\Phi(B)^{-1}, \forall A \in \mathfrak{g}$ and $B \in G$
(b) $\Phi([A, B]) = [\Phi(A), \Phi(B)], \forall A, B \in \mathfrak{g}$
(c) $\Phi(A) = \frac{d}{dt}\Phi(\exp(tA))\Big|_{t=0}, \forall A \in \mathfrak{g}$.
In addition, if $\Phi : G \to H$ is a Lie group isomorphism, then such $\mathbb{R}$-linear map $\phi : \mathfrak{g} \to \mathfrak{h}$ is bijective.
We call such mapping $\phi : \mathfrak{g} \to \mathfrak{h}$ the associated Lie algebra homomorphism of the Lie group homomorphism $\Phi : G \to H$.

Definition 3.1.23.
Let G be a matrix Lie group.
A representation of a matrix Lie group G is a Lie group homomorphism
$\prod : G \to GL(n, \mathbb{C})$.
A representation of a Lie algebra $\mathfrak{g}$ is a Lie algebra homomorphism
$\pi : \mathfrak{g} \to \mathfrak{gl}(n, \mathbb{C})$.

In this note, we only deal with $M_{n\times n}(\mathbb{C})$.
The notations in this section can be extend to abstract vector spaces, and this is related to the group representation theory, see e.g. the monograph [Hal15] for further details.

3.2 Homogeneous ODE with Variable Coefficients

In this section, we explain the basic results concerning the structure of solutions of a homogeneous system of ODE given by $y'(t) = A(t)y(t), y(t_0) = p = (p_1, \ldots, p_n)$.

Assume that A is continuous near t_0.
By using the FTODE (Thm 2.1.2), there exist a unique C^1-solution $Y \ni$ $Y'(t) = A(t)Y(t)$ near t_0, $Y(t_0) = I$.
By using $\det(Y(t_0)) = 1$ and the continuity of $\det : M_{n\times n}(\mathbb{C}) \to \mathbb{C}$, one sees that $\det(Y(t)) \neq 0, \forall t$ near t_0, i.e. $Y(t) \in GL(n, \mathbb{C}), \forall t$ near t_0.

We call such $Y(t)$ a fundamental matrix solution near t_0.
Furthermore, the columns $y_1(t), \cdots, y_n(t)$ of $Y(t)$, which forms a linearly independent set in $\mathbb{C}^n$ near t_0, are called the fundamental set of n linearly independent solution near t_0.
We see that $Y(t)p = \sum_{j=1}^n p_j y_j$ is the unique solution of $y'(t) = A(t)y(t), y(t_0) = p = (p_1, \ldots, p_n)$ near t_0.
By arbitrariness of p, we can rephrase the above in the following theorem.

Theorem 3.2.1.
If A is continuous near t_0, then the set of all solutions of the ODE $y'(t) = A(t)y(t)$ near t_0 forms an n-dimensional vector space over $\mathbb{C}$.

Similarly, by using the FTODE, $\exists!$ C^1-solution $Y \ni Z'(t) = -Z(t)A(t)$ near t_0, $Z(t_0) = I$, which satisfies $Z(t) \in GL(n, \mathbb{C}), \forall t$ near t_0.
By using the chain rule,

$$\begin{cases} \dfrac{d}{dt}(Z(t)Y(t)) = Z'(t)Y(t) + Z(t)Y'(t) = -Z(t)A(t)Y(t) + Z(t)A(t)Y(t) = 0 \\ Z(t_0)Y(t_0) = I \end{cases}$$

Hence, $\dfrac{d}{dt}(Z(t)Y(t) - I) = 0, (Z(t)Y(t) - I)\big|_{t=t_0} = 0$.
By using (uniqueness part in) the FTODE, $Z(t)Y(t) - I = 0, \forall t$ near t_0.
Hence, $Y(t) = Z(t)^{-1} \in GL(n, \mathbb{C}), \forall t$ near t_0.
We can refer $Z'(t) = -Z(t)A(t)$ be the adjoint problem of $Y'(t) = A(t)Y(t)$.

We now want to compute $Y(t)$ in terms of $A(t)$.
In this case when $A(t) \in M_{1\times 1}(\mathbb{C}) \cong C$, i.e., the ODE (3.2.2) is scalar ($n = 1$), then by using the FTC, we can easily obtain $Y(t) = \exp(\int_{t_0}^t A(s)ds), \forall t$ near t_0.
When $n \geq 2$, the situation become tricky: By using the product rule,

$$
\begin{aligned}
Y'(t) &= \Big(\exp\big(\int_{t_0}^{t} A(s)ds \big) \Big)' \\
&= \Big(I + \big(\int_{t_0}^{t} A(s)ds \big) + \frac{1}{2!} \big(\int_{t_0}^{t} A(s)ds \big)^2 + \cdots \Big)' \\
&= A(t) + \frac{1}{2!} \Big(\big(\int_{t_0}^{t} A(s)ds \big)^2 \Big)' + \cdots \\
&= A(t) + \frac{1}{2!} \Big(A(t) \big(\int_{t_0}^{t} A(s)ds \big) + \big(\int_{t_0}^{t} A(s)ds \big) A(t) \Big) + \cdots .
\end{aligned}
$$

If $A(t)$ and $\int_{t_0}^{t} A(s)ds$ are commute, then

$$Y'(t) = A(t) + A(t)\big(\int_{t_0}^{t} A(s)ds \big) + \frac{1}{2!} A(t) \big(\int_{t_0}^{t} A(s)ds \big)^2 + \cdots = A(t)Y(t).$$

In addition, by using exercise 3.4, from $Y(t) = \exp(\int_{t_0}^{t} A(s)ds), \forall t$ near t_0, then

$$\det(Y(t)) = \exp\Big(\int_{t_0}^{t} tr(A(s))ds \Big), \forall t \text{ near } t_0.$$

Example 3.2.1.
$A(t) = diag(\lambda_1(t), \ldots, \lambda_n(t))$ for some scalar functions $\lambda_1, \ldots, \lambda_n$ which are continuous near t_0.

We remind the readers that the existence of unique fundamental matrix solution $Y(t)$ does not require the additional assumption that $A(t)$ and $\int_{t_0}^{t} A(s)ds$ are commute.
The requirement that $A(t)$ and $\int_{t_0}^{t} A(s)ds$ are commute is quite restrictive:
In general we do not know the explicit formula of the unique fundamental matrix solution $Y(t)$.
Due to this reason, it is worth to mention the following theorem.

Theorem 3.2.2. (*Abel's Formula*)
If A is continuous near t_0, then the fundamental matrix solution $Y(t)$ satisfying $Y'(t) = Y(t)A(t)$ satisfies $\det(Y(t)) = \exp(\int_{t_0}^{t} tr(A(s))ds)$.

Proof.
Using the product rule and Thm 3.1.9,

$$
\begin{aligned}
\frac{d}{dt}(\det(Y(t))) = \quad & \sum_{\sigma\in S_n} \operatorname{sign}(\sigma)\left(\frac{d}{dt}Y_{\sigma(1),1}(t)\right) Y_{\sigma(2),2}(t)\cdots Y_{\sigma(n),n}(t)+\cdots \\
& + \sum_{\sigma\in S_n} \operatorname{sign}(\sigma) Y_{\sigma(1),1}(t) Y_{\sigma(2),2}(t)\cdots\left(\frac{d}{dt}Y_{\sigma(n),n}(t)\right) \\
= \quad & \sum_{\sigma\in S_n} \operatorname{sign}(\sigma)\left(\frac{d}{dt}Y(t)\right)_{\sigma(1),1} Y_{\sigma(2),2}(t)\cdots Y_{\sigma(n),n}(t)+\cdots \\
& + \sum_{\sigma\in S_n} \operatorname{sign}(\sigma) Y_{\sigma(1),1}(t) Y_{\sigma(2),2}(t)\cdots\left(\frac{d}{dt}Y(t)\right)_{\sigma(n),n}
\end{aligned}
$$

Since $Y'(t) = A(t)Y(t)$, then

$$
\begin{aligned}
\frac{d}{dt}(\det(Y(t))) = \quad & \sum_{\sigma\in S_n} \operatorname{sign}(\sigma)(A(t)Y(t))_{\sigma(1),1} Y_{\sigma(2),2}(t)\cdots Y_{\sigma(n),n}(t)+\cdots \\
& + \sum_{\sigma\in S_n} \operatorname{sign}(\sigma) Y_{\sigma(1),1}(t) A_{\sigma(2),2}(t)\cdots(A(t)Y(t))_{\sigma(n),n} \\
= \quad & \sum_{\sigma\in S_n} \operatorname{sign}(\sigma)\big(a_{\sigma(1)}(t)y_1(t)\big)\big(e^T_{\sigma(2)}(t)y_2(t)\big)\cdots\big(e^T_{\sigma(n)}(t)y_n(t)\big)+\cdots \\
& + \sum_{\sigma\in S_n} \operatorname{sign}(\sigma)\big(e^T_{\sigma(1)}(t)y_1(t)\big)\cdots\big(e^T_{\sigma(n-1)}(t)y_{n-1}(t)\big)\big(a_{\sigma(n)}(t)y_n(t)\big). \\
= \quad & \det\left(\begin{pmatrix} a_1(t) \\ e_2^T(t) \\ \vdots \\ e_n^T(t)\end{pmatrix} Y(t)\right)+\cdots+\det\left(\begin{bmatrix} e_1^T(t) \\ \vdots \\ e_{n-1}^T(t) \\ a_n(t)\end{bmatrix} Y(t)\right),
\end{aligned}
$$

where a_j be the j^{th} row of A and y_j is the j^{th} column of Y.
Use Thm 3.1.10,

$$
\begin{aligned}
\frac{d}{dt}(\det(Y(t))) &= \det\left(\begin{bmatrix} a_1(t) \\ e_2^T(t) \\ \vdots \\ e_n^T(t)\end{bmatrix} \det(Y(t))\right)+\cdots+\det\left(\begin{bmatrix} e_1^T(t) \\ \vdots \\ e_{n-1}^T(t) \\ a_n(t)\end{bmatrix} \det(Y(t))\right) \\
&= a_{11}(t)\det(Y(t))+\cdots+a_{nn}(t)\det(Y(t)) \\
&= \operatorname{tr}(A(t))\det(Y(t)),
\end{aligned}
$$

which shows that $\det(Y(t))$ satisfies the scalar ODE of the form $Y'(t) = A(t)Y(t)$ near t_0, $Y(t_0) = I$.
Since $\det(Y(t_0)) = 1$, by using the FTODE, we conclude that the unique solution of the scalar ODE is given by $\det(Y(t)) = \exp(\int_{t0}^t tr(A(s))ds), \forall t$ near t_0. $\square$

Definition 3.2.1.
The quantity $W(t) := \det(Y(t))$ *is called the Wronskian and the Abel formula reads* $W(t) = \exp\big(\int_{t_0}^t tr(A(s))ds\big), \forall t$ *near* t_0.

3.3 Nonhomogeneous Equations

Goal : how to solve an initial-value problem $\begin{cases} y'(t) = A(t)y(t) + b(t) \\ y(t_0) = p \end{cases}$.

Assume that both A and b are continuous, $\forall t$ near t_0.
Let $Y(t) = Y(t;t_0) \in GL(n,\mathbb{C})$ be the fundamental matrix solution satisfying $\begin{cases} Y'(t) = A(t)Y(t) \\ Y(t_0) = I \end{cases}$, which was mentioned in previous subsections.

By plugging the anzats $y(t) = Y(t)z(t), \forall t$ near t_0, into $\begin{cases} y'(t) = A(t)y(t) + b(t) \\ y(t_0) = p \end{cases}$ and by using the product rule, we see that
(1) $y'(t) = Y'(t)z(t) + Y(t)z'(t) = A(t)Y(t)z(t) + Y(t)z'(t)$ and
(2) $y'(t) = A(t)y(t) + b(t) = A(t)Y(t)z(t) + b(t)$
$\Rightarrow Y(t)z'(t) = b(t)$.

Multiply the above equation by $(Y(t))^{-1} \in GL(n,\mathbb{C})$, we now see that $z'(t) = (Y(t))^{-1}b(t), z(t_0) = p$, and its unique solution (guaranteed by the FTODE (Thm 2.1.2)) is given by $z(t) = p + \int_{t_0}^{t} (Y(s))^{-1}b(s)ds, \forall t$ near t_0.

Now we see that the unique solution $Y(t)$ of $y'(t) = A(t)y(t) + b(t), y(t_0) = p$ is given by $y(t) = Y(t)z(t) = Y(t;t_0)\big(p + \int_{t_0}^{t} (Y(s;t_0))^{-1}b(s)ds\big), \forall t$ near t_0.

Remark :
The numerical computation of $(Y(t))^{-1}b(t)$ is quite fundamental, but keep in mind that one never compute the inverse of $(Y(t))^{-1}$ directly (the computation is both inaccurate and slow).
In practical, this is computed via iterative algorithm such as GMRES, conjugate gradient, etc. One can refer to the monograph [TB22] for a nice introduction of numerical linear algebra.
Here we recall that in fact $(Y(t))^{-1}$ is the fundamental matrix solution of the adjoint problem (3.2.3), which allows us to compute $(Y(t))^{-1}$ by solving an ODE, which is much better than compute $(Y(t))^{-1}b(t)$ via numerical method.

In fact, the above formula can be further simplified and we now label the subscript for clarification.
For each fixed s near t_0, we now see that
$$\begin{cases} \dfrac{d}{dt}(Y(t;t_0)(Y(s;t_0))^{-1}) = A(t)Y(t;t_0)(Y(s;t_0))^{-1} \\ Y(t;t_0)(Y(s;t_0))^{-1}\big|_{t=s} = I \end{cases},$$

and t the FTODE (Thm 2.1.2) says that $Y(t;s) = Y(t;t_0)(Y(s;t_0))^{-1}$.
Then the unique solution of $y'(t) = A(t)y(t) + b(t), y(t_0) = p$ is given by $y(t) = Y(t;t_0)p + \int_{t_0}^{t} Y(t;t_0)(Y(s;t_0))^{-1}b(s)ds = Y(t;t_0)p + \int_{t_0}^{t} Y(t;s)b(s)ds, \forall t$ near t_0.
It is worth to mention that the solution formula $y(t) = Y(t;t_0)p + \int_{t_0}^{t} Y(t;s)b(s)ds$ do not involve $Y(t)^{-1}$, as well as the adjoint problem $Z'(t) = -Z(t)A(t)$ near $t_0, Z(t_0) = I$, at all.

Remark :
When $A(t) \equiv A$ is a constant matrix, we see that $Y(t;s) = \exp((t-s)A), \forall t, s \in \mathbb{R}$, and now $y(t) = Y(t;t_0)p + \int_{t_0}^{t} Y(t;s)b(s)ds$ reads
$y(t) = exp((t\grave{}t0)A)p + \int_{t_0}^{t} \exp((t-s)A)b(s)ds$,
where $\int_{t_0}^{t} \exp((t-s)A)b(s)ds$ is the convolution.

Example 3.3.1.
Solve the initial value problem $\begin{cases} y'(t) = Ay(t) + b(t) \\ y(0) = p \end{cases}$ *with*
$A = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 0 \\ 3 & 2 & 1 \end{bmatrix}, b(t) = \begin{bmatrix} 2 \\ 0 \\ t \end{bmatrix}, p = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.
The fundamental matrix solution is given by
$Y(t) = \exp(tA) = \begin{bmatrix} e^{-2t} & te^{-2t} & 0 \\ 0 & e^{-2t} & 0 \\ e^{t} - e^{-2t} & e^{t} - (1+t)e^{-2t} & e^{t} \end{bmatrix}$.
Therefore, the solution is given by $y(t) = \begin{bmatrix} 1 + te^{-2t} \\ e^{-2t} \\ -4 - t + 5e^{t} - (1+t)e^{-2t} \end{bmatrix}$.

Exercise 3.11.

Prove that $Y(t) = \exp(tA) = \begin{bmatrix} e^{-2t} & te^{-2t} & 0 \\ 0 & e^{-2t} & 0 \\ e^t - e^{-2t} & e^t - (1+t)e^{-2t} & e^t \end{bmatrix}.$

Proof.
claim : check the matrix A can diagonalize or not.
Let λ is the eigenvalue that satisfies the characteristic function $\det(A - \lambda I) = 0$.

$$\det(A - \lambda I) = 0$$

$$\Rightarrow \det\left(\begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 0 \\ 3 & 2 & 1 \end{bmatrix} - \lambda I\right) = 0$$

$$\Rightarrow (-2-\lambda)^2(1-\lambda) = 0$$

$$\Rightarrow \lambda_1 = -2 \text{ or } \lambda_2 = 1$$

For $\lambda_1 = -2$,

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 3 & 2 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

For $\lambda_2 = 1$,

$$\begin{bmatrix} -3 & 1 & 0 \\ 0 & -3 & 0 \\ 3 & 2 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

For $\lambda_1 = -2$,
algebraic multiplicity $= 2 \neq 1 =$ geometric multiplicity, then A cannot diagonalize.
claim : find the Jordon form J from A, i.e., find the matrix $P \ni P^{-1}AP = J$
For $\lambda_1 = -2$, we need to find a vector that $(A+2I)^2 x = 0$ and $(A+2I)x \neq 0$.

$$(A+2I)^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 3 & 2 & 3 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 9 & 9 & 9 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 9 & 9 & 9 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}.$$

$$(A+2I)\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 3 & 2 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

Then $P = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ -1 & -2 & 1 \end{bmatrix} \Rightarrow P^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

Hence, the Jordon form

$$J = P^{-1}AP = \left(\begin{bmatrix} -1 & -1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}\right)^{-1}\begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 0 \\ 3 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ -1 & -2 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

claim : verify $\exp(tA)$
Since $\exp(tA) = P \cdot \exp(t \cdot P^{-1}AP) \cdot P^{-1} = P \cdot \exp(t \cdot J) \cdot P^{-1}$, then

$$
\begin{aligned}
\exp(tA) &= P \cdot \exp(t \cdot P^{-1}AP) \cdot P^{-1} = P \cdot \exp(t \cdot J) \cdot P^{-1} \\
&= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ -1 & -2 & 1 \end{bmatrix} \exp\left(t \begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \\
&= \begin{bmatrix} e^{-2t} & t \cdot e^{-2t} & 0 \\ 0 & e^{-2t} & 0 \\ -e^{-2t} & -2e^{-2t} - te^{-2t} & e^{t} \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \\
&= \begin{bmatrix} e^{-2t} & t \cdot e^{-2t} & 0 \\ 0 & e^{-2t} & 0 \\ e^{t} - e^{-2t} & e^{t} - (t+1)e^{-2t} & e^{t} \end{bmatrix}
\end{aligned}
$$

$\square$

3.4 Higher Order Linear ODE

Goal : how to solve the initial value problem of an nth order linear ODE

$$\begin{cases} u^{(n)} + a_1(t)u^{(n-1)} + \cdots + a_{n-1}(t)u' + a_n(t)u = b(t), \quad \forall t \text{ near } t_0, \\ u(t_0) = p_1,\, u'(t_0) = p_2,\, \cdots,\, u^{(n-1)}(t_0) = p_n \end{cases}$$

Theorem 3.4.1.
If the coefficients $a_1, \ldots, a_n, b$ are continuous near t_0, then there exists a unique C^n-solution u of ODE above near t_0.

Proof.
Goal 1 : Existence of solution

$$\text{Define } A(t) = \begin{bmatrix} 0_{n-1} & -a_n(t) \\ I_{n-1} & (-a_{n-1}(t), \cdots, -a_1(t)) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n(t) & -a_{n-1}(t) & -a_{n-2}(t) & \cdots & -a_1(t) \end{bmatrix}$$

where $0_{n-1} \in \mathbb{R}^{n-1}$ is the zero vector, $I_{n-1} \in \mathbb{R}^{(n-1)\times(n-1)}$ is the identity matrix,

$$\text{and } b(t) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ b(t) \end{bmatrix}, \quad p = \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{bmatrix}.$$

In previous section (Sec 3.3), we have showed that there exists a unique C^1-solution $y(t)$ of $y'(t) = A(t)y(t) + b(t), y(t_0) = p$.
For each t and s, which are close to t_0,

$$\text{let } Y(t;s) = \begin{bmatrix} y_1(t;s) \\ y_2(t;s) \\ \vdots \\ y_{n-1}(t;s) \\ y_n(t;s) \end{bmatrix} \text{ be the fundamental matrix solution satisfying } Y(s) = I$$

and

$$\begin{bmatrix} y_1'(t;s) \\ \vdots \\ y_{n-1}'(t;s) \\ y_n'(t;s) \end{bmatrix} = Y'(t;s) = A(t)Y(t;s)$$

$$= \begin{bmatrix} 0_{n-1} & I_{n-1} \\ -a_n(t) & (-a_{n-1}(t), \cdots, -a_1(t)) \end{bmatrix} Y(t;s)$$

$$= \begin{bmatrix} y_2(t;s) \\ \vdots \\ y_n(t;s) \\ -a_n(t)y_1(t;s) - \cdots - a_1(t)y_n(t;s) \end{bmatrix}$$

and we see that ${y_j}'(t;s) = y_{j+1}(t;s), \forall j = 1, \ldots, n-1$.

From $y(t) = Y(t;t_0)p + \displaystyle\int_{t_0}^t Y(t;s)b(s)ds$, we have $y(t) = Y(t;t_0)p + \displaystyle\int_{t_0}^t Y(t;s)b(s)ds$.

Define $u(t) := y_1(t) = y_1(t;t_0)p + \int_{t_0}^t y_1(t;s)b(s)ds \in C^1$ near t_0.

From ${y_j}'(t;s) = y_{j+1}(t;s)$ we see that $u \in C^n$ near t_0 and $y(t) = \begin{bmatrix} u(t) \\ u'(t) \\ \vdots \\ u^{(n-1)}(t) \end{bmatrix}$,

then we see that

$$A(t)y(t) + b(t) = \begin{bmatrix} 0_{n-1} & I_{n-1} \\ -a_n(t) & (-a_{n-1}(t), \cdots, -a_1(t)) \end{bmatrix} \begin{bmatrix} u(t) \\ u'(t) \\ \vdots \\ u^{(n-1)}(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ b(t) \end{bmatrix}$$

$$= \begin{bmatrix} u'(t) \\ u''(t) \\ \vdots \\ u^{(n-2)}(t) \\ -a_1(t)u^{(n-1)}(t) - a_2(t)u^{(n-2)}(t) - \cdots - a_n(t)u(t) + b(t) \end{bmatrix}$$

Hence, $y'(t) = A(t)y(t) + b(t), y(t_0) = p$ gives the ODE equation in title of this subsection.

Goal 2 : Uniqueness

If $u \in C^n$ is a solution of ODE equation near t_0, then the C^1-function y given in $y(t) = \begin{bmatrix} u(t) \\ u'(t) \\ \vdots \\ u^{(n-1)}(t) \end{bmatrix}$ satisfies $y'(t) = A(t)y(t) + b(t), y(t_0) = p$.

□

Recall : Definition 3.2.xx
The Wronskian is defined by $W(t) := \det(Y(t))$, and using the Abel's formula (Theorem 3.2.3), we see that $W(t) = \exp\big(\int_{t_0}^{t} tr(A(s))ds\big) = \exp\big(-\int_{t_0}^{t} a_1(s)ds\big)$.

Remark : 2-dimensional case
In practical computation, the most difficult part is to compute $y_1(t;s) = y_1(t)(y_1(s))^{-1}$.
When $n = 2$, the computation of $(y_1(s))^{-1}$ can be further simplfiied.
As mentioned in the proof, one has $y_1(t) = \begin{bmatrix} \phi_1(t) & \phi_2(t) \\ {\phi_1}'(t) & {\phi_2}'(t) \end{bmatrix}$,
where $\phi_1(t)$ and $\phi_2(t)$ are two linearly independent solutions of the associated homogeneous equation $v'' + a_1(t)v' + a_2(t)v = 0$.
We see that $(y_1(s;t0))^{-1} = \frac{1}{W(t)} \begin{bmatrix} \phi_1(t) & \phi_2(t) \\ {\phi_1}'(t) & {\phi_2}'(t) \end{bmatrix}$.
In this case, the Wronskian reads $W(t) = \phi_1(t){\phi_2}'(t) - {\phi_1}'(t)\phi_2(t)$.
Hence, from $u(t) := y_1(t) = y_1(t;t_0)p + \int_{t_0}^{t} y_1(t;s)b(s)ds, \forall t$ near t_0, we know that the unique solution u of $\begin{cases} u'' + a_1(t)u' + a_2(t)u = b(t), \forall t \text{ near } t_0 \\ u(t_0) = p_1, u'(t_0) = p_2 \end{cases}$ is given by

$$u(t) := p_1\phi_1(t) + p_2\phi_2(t) - \phi_1(t)\int_{t_0}^{t} \frac{\phi_2(s)}{W(s)}b(s)ds + \phi_2(t)\int_{t_0}^{t} \frac{\phi_1(s)}{W(s)}b(s)ds.$$

Write $W(t) = \phi_1(t){\phi_2}'(t) - {\phi_1}'(t)\phi_2(t)$ as $\left(\frac{\phi_2(t)}{\phi_1(t)}\right)' = \frac{{\phi_2}'(t)}{\phi_1(t)} - \frac{{\phi_1}'(t)}{(\phi_1(t))^2}\phi_2(t) = \frac{W(t)}{(\phi_1(t))^2}$.
Let $\tilde{W}$ be any function $\ni \tilde{W}'(t) = \frac{W(t)}{(\phi_1(t))^2}$, and one sees that $\phi_2(t) = \phi_1(t)\tilde{W}(t)$ satisfies the above ODE.
Using Cor 3.4.5 below. we conclude that the solution set of $v''+a_1(t)v'+a_2(t)v = 0$ is a 2-dimensional vector space with basis $\{\phi_1(t), \phi_1(t)\tilde{W}(t)\}$.

Remark : (3-dimensional case)
Write $Y(t) = \begin{bmatrix} \phi(t) & \phi_1(t) & \phi_2(t) \\ \phi'(t) & {\phi_1}'(t) & {\phi_2}'(t) \\ \phi''(t) & {\phi_1}''(t) & {\phi_2}''(t) \end{bmatrix}$, and we see that $\phi(t)$, $\phi_1(t)$ and $\phi_2(t)$ are linearly independent solutions of the homogeneous equation
$u'''(t) + a_1(t)u'' + a_2(t)u' + a_3(t)u = 0$.
By using cofactor expansion of determinant, the Wronskian reads

$$\begin{aligned} W(t) =& \phi''(t)\det\left(\begin{bmatrix} \phi_1(t) & \phi_2(t) \\ {\phi_1}'(t) & {\phi_2}'(t) \end{bmatrix}\right) \\ & - \phi'(t)\det\left(\begin{bmatrix} \phi_1(t) & \phi_2(t) \\ {\phi_1}''(t) & {\phi_2}''(t) \end{bmatrix}\right) + \phi(t)\det\left(\begin{bmatrix} {\phi_1}'(t) & {\phi_2}'(t) \\ {\phi_1}''(t) & {\phi_2}''(t) \end{bmatrix}\right) \end{aligned}$$

and we write $\phi''(t) + \frac{A_1(t)}{A_0(t)}\phi'(t) + \frac{A_2(t)}{A_0(t)}\phi(t) = \frac{W(t)}{A_0(t)}$, where

$A_0(t) = \det\left(\begin{bmatrix}\phi_1(t) & \phi_2(t)\\ {\phi_1}'(t) & {\phi_2}'(t)\end{bmatrix}\right), A_1(t) = \left(\begin{bmatrix}\phi_1(t) & \phi_2(t)\\ {\phi_1}''(t) & {\phi_2}''(t)\end{bmatrix}\right),$
$A_2(t) = \left(\begin{bmatrix}{\phi_1}'(t) & {\phi_2}'(t)\\ {\phi_1}''(t) & {\phi_2}''(t)\end{bmatrix}\right).$
i.e., the general solution of ϕ can be expressed in terms of ϕ_1, ϕ_2 and W.

The proof of Thm 3.4.1 itself gives an algorithm to compute the unique solution. Since $Y(t;s) \in GL(n, \mathbb{C})$, then we also have the following corollary.

Corollary 3.4.2.
If the coefficients $a_1, \dots, a_n$ are continuous near t_0, then the solution set $\{u \in \mathbb{C}^n \text{ near } t_0 : u^{(n)} + a_1(t)u^{(n-1)} + \cdots + a_{n-1}(t)u' + a_n(t)u = 0 \text{ near } t_0\}$ forms an n-dimensional vector space.

Example 3.4.1.
Let us solve the initial value problem
$$\begin{cases} u''' - 2u'' - 5u + 6u = 3t,\\ u(0) = 1, u'(0) = 2, u''(0) = 0.\end{cases}$$
⟨Solution⟩
The matrix $A(t)$ given in proof of existence in Thm 3.4.1 and the vectors b as well as p given in proof of existence in Thm 3.4.1 read
$$A = \begin{bmatrix} 0 & 1 & 0\\ 0 & 0 & 1\\ -6 & 5 & 2\end{bmatrix}, b(t) = \begin{bmatrix}0\\0\\3t\end{bmatrix}, p = \begin{bmatrix}1\\2\\0\end{bmatrix}.$$
We can compute
$$\begin{aligned} Y(t) &= \exp(tA)\\ &= -\frac{e^t}{6}\begin{pmatrix}-6 & -1 & 1\\ -6 & -1 & 1\\ -6 & -1 & 1\end{pmatrix} + \frac{e^{-2t}}{15}\begin{pmatrix}3 & -4 & 1\\ -6 & 8 & -2\\ 12 & 16 & 4\end{pmatrix} + \frac{e^{3t}}{10}\begin{pmatrix}-2 & 1 & 1\\ -6 & 3 & 3\\ -18 & 9 & 9\end{pmatrix}.\end{aligned}$$
By a direct but long computations we reach
$$\begin{aligned}\int_0^t Y(s)^{-1}b(s)\,ds = &-\frac{1}{2}\begin{pmatrix}1-(t+1)e^t\\ 1-(t+1)e^t\\ 1-(t+1)e^t\end{pmatrix}\\ &+ \frac{1}{20}\begin{pmatrix}1+(2t-1)e^{2t}\\ -2(1+(2t-1)e^{2t})\\ -4(1+(2t-1)e^{2t})\end{pmatrix} + \frac{1}{30}\begin{pmatrix}1+(3t+1)e^{3t}\\ 3(1+(3t+1)e^{3t})\\ 9(1+(3t+1)e^{3t})\end{pmatrix}.\end{aligned}$$
Hence, $y(t) = \dfrac{1}{60}\begin{pmatrix}30t + 25 - 17e^{-2t} + 50e^t + 2e^{3t}\\ 2(15 + 17e^{-2t} + 25e^{2t} + 2e^{3t})\\ 2(-34e^{-2t} + 25e^t + 9e^{3t})\end{pmatrix}.$
We finally conclude from $u(t) := y_1(t) = y_1(t;t_0)p + \int_{t_0}^t y_1(t;s)b(s)ds \in C^1$ near t_0 that $u(t) = \frac{1}{60}(30t + 25 - 17e^{-2t} + 50e^t + 2e^{3t})$ is the unique solution.

Exercise 3.12.
Proof $Y(t) = \exp(tA)$

$$= -\frac{e^t}{6}\begin{bmatrix} -6 & -1 & 1 \\ -6 & -1 & 1 \\ -6 & -1 & 1 \end{bmatrix} + \frac{e^{-2t}}{15}\begin{bmatrix} 3 & -4 & 1 \\ -6 & 8 & -2 \\ 12 & 16 & 4 \end{bmatrix} + \frac{e^{3t}}{10}\begin{bmatrix} -2 & 1 & 1 \\ -6 & 3 & 3 \\ -18 & 9 & 9 \end{bmatrix}$$

using Algorithm 2 with $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & 5 & 2 \end{bmatrix}$. (*Hint. See Exercise 3.1.40*)

Proof.
claim 1 : Compute the characteristic polynomial $p(z) = \det(zI - A)$.

$$\begin{aligned} p(z) = \det(zI - A) &= \det\left(\begin{bmatrix} z & -1 & 0 \\ 0 & z & -1 \\ 6 & -5 & z-2 \end{bmatrix}\right) \\ &= z\det\left(\begin{bmatrix} z & -1 \\ -5 & z-2 \end{bmatrix}\right) - (-1)\left(\begin{bmatrix} 0 & -1 \\ 6 & z-2 \end{bmatrix}\right) \\ &= z(z^2 - 2z - 5) + 6 = z^3 - 2z^2 - 5z + 6 \\ &= (z+2)(z-1)(z-3) \end{aligned}$$

claim 2 : compute the partial fraction decomposition of $\frac{1}{p(z)}$.

$$\frac{1}{p(z)} = \frac{Q_1(z)}{z-1} + \frac{Q_2(z)}{z-3} + \frac{Q_3(z)}{z+2}$$

$$\Rightarrow 1 = Q_1(z)(z-3)(z+2) + Q_2(z)(z-1)(z+2) + Q_3(z)(z-1)(z-3)$$

$$\Rightarrow Q_1(1) = \frac{-1}{6}, Q_2(3) = \frac{1}{10}, Q_3(-2) = \frac{1}{15}$$

claim 3 : Compute the projection polynomials $P_j(A), j = 1, 2, 3$.

$$P_1(A) = Q_1(A)\cdot(A - 3I)\cdot(A + 2I) = \frac{-1}{6}\begin{bmatrix} -6 & -1 & 1 \\ -6 & -1 & 1 \\ -6 & -1 & 1 \end{bmatrix}$$

$$P_2(A) = Q_2(A)\cdot(A - I)\cdot(A + 2I) = \frac{1}{10}\begin{bmatrix} -2 & 1 & 1 \\ -6 & 3 & 3 \\ -18 & 9 & 9 \end{bmatrix}$$

$$P_3(A) = Q_3(A)\cdot(A - I)\cdot(A - 3I) = \frac{1}{15}\begin{bmatrix} 3 & -4 & 1 \\ -6 & 8 & -2 \\ 12 & 16 & 4 \end{bmatrix}$$

claim 4 : Compute diagonal matrix D and nilpotent matrix N.
By Jordan-Chevalley decomposition, the diagonal matrix

$$D = 1P_1(A) + 3P_2(A) - 2P_3(A) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & 5 & 2 \end{bmatrix} \Rightarrow N = O_{3\times 3}$$

$$\begin{aligned}\exp(tA) &= \exp(tD) = \exp(t(1P_1(A) + 3P_2(A) - 2P_3(A))) \\ &= -\frac{e^t}{6}\begin{bmatrix} -6 & -1 & 1 \\ -6 & -1 & 1 \\ -6 & -1 & 1 \end{bmatrix} + \frac{e^{-2t}}{15}\begin{bmatrix} 3 & -4 & 1 \\ -6 & 8 & -2 \\ 12 & 16 & 4 \end{bmatrix} + \frac{e^{3t}}{10}\begin{bmatrix} -2 & 1 & 1 \\ -6 & 3 & 3 \\ -18 & 9 & 9 \end{bmatrix}\end{aligned}$$

□

Exercise 3.13.

Using Thm 3.4.8 solve the initial value problem

$$\begin{cases} u''' - 2u'' - 5u + 6u = 3t, \\ u(0) = 1, u'(0) = 2, u''(0) = 0. \end{cases}$$

⟨*Solution*⟩

Consider the homogeneous equation $u''' - 2u'' - 5u + 6u = 0$.

The characteristic equation $P(\lambda) = \lambda^3 - 2\lambda^2 - 5\lambda + 6 = 0$.

Then the roots of P *is* $-2, 1, 3$.

Hence, the homogeneous solution $u_h(t) = c_1e^t + c_2e^{-2t} + c_3e^{3t}$.

Since the non-homogeneous part of original equation is $3t$.

Then let the particular solution $u_p(t) = At + B$.

Put into the original function,

$0 - 2 \cdot 0 - 5 \cdot A + 6At + 6B = 3t \Rightarrow A = \frac{1}{2}, B = \frac{5}{12}$.

The general solution is the sum of homogeneous and non-homogeneous solution.

$$\begin{cases} u(t) = u_h(t) + u_p(t) = c_1e^t + c_2e^{-2t} + c_3e^{3t} + \frac{1}{2}t + \frac{5}{12}, \\ u(0) = 1, u'(0) = 2, u''(0) = 0. \end{cases}$$

$$\Rightarrow \begin{cases} c_1 + c_2 + c_3 + \frac{5}{12} = 1 \\ c_1 + 2c_2 + 3c_3 + \frac{1}{2} = 2 \\ c_1 + 4c_2 + 9c_3 = 0 \end{cases} \Rightarrow \begin{cases} c_1 = 1 \\ c_2 = \frac{17}{4} \\ c_3 = \frac{-29}{4} \end{cases}$$

Hence, $u(t) = e^t + \frac{17}{4}e^{-2t} + \frac{-29}{4}e^{3t} + \frac{1}{2}t + \frac{5}{12}$.

Let $u \in C^n$ be the unique solution of (3.4.1).
Let $v \in C^n$ be any function satisfying $v^{(n)} + a_1(t)v^{(n-1)} + \cdots + a_{n-1}(t)v' + a_n(t)v = b(t), \forall t$ near t_0, then one sees that the solution $w = u - v$ is the unique solution to
$$\begin{cases} w^{(n)} + a_1(t)w^{(n-1)} + \cdots + a_{n-1}w' + a_n(t)w = 0, \\ w(t_0) = p_1 - v(t_0), u'(t_0) = p_2 - v'(t_0), \ldots, u^{(n-1)}(t_0) = p_n - v^{(n-1)}(t_0). \end{cases}, \forall t \text{ near } t_0.$$
For the case when $a_1, \ldots, a_n$ are constants, there is an efficient way (based on Cor 3.4.2) to compute the solution w of equation above.
By plugging the anzats $w(t) = e^{\lambda t}$ into the equation
$w^{(n)} + a_1 w^{(n-1)} + \cdots + a_{n-1}w' + a_n w = 0$
$\Rightarrow P(\lambda) := \lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n = 0$
$\Rightarrow P\left(\dfrac{d}{dt}\right)w = 0.$
By FTA (fundamental theorem of algebra), $\exists$ distinct $\lambda_1, \ldots, \lambda_k \in \mathbb{C} \ni$
$P(\lambda) = (\lambda - \lambda_1)^{m_1} \cdots (\lambda - \lambda_k)^{m_k}$ with $m_1 + \cdots + m_k = n$.
For any differentiable function g, we see that $\left(\dfrac{d}{dt} - \lambda_j\right)(e^{\lambda_j t}g(t)) = e^{\lambda_j t}g'(t)$,
and inductively one can show $\left(\dfrac{d}{dt} - \lambda_j\right)^{m_j}(e^{\lambda_j t}g(t)) = e^{\lambda_j t}g^{m_j}(t)$.
For each $c_1, \ldots, c_{m_j} \in \mathbb{C}$,
we now choose $g(t) = c_1 + c_2 t + \cdots + c_{m_j}t^{m_j}$, then we see that $g^{(m_j)}(t) \equiv 0$.
$$\left(\frac{d}{dt} - \lambda_j\right)^{m_j}(c_1 e^{\lambda_j t} + c_2 t e^{\lambda_j t} + \cdots + c_{m_j}t^{m_j-1}e^{\lambda_j t}) = 0$$
$$\Rightarrow P\left(\frac{d}{dt}\right)(c_1 e^{\lambda_j t} + c_2 t e^{\lambda_j t} + \cdots + c_{m_j}t^{m_j-1}e^{\lambda_j t}) = 0.$$

Since the above argument works $\forall j = 1, \ldots, k$, combining with Cor 3.4.2, we reach the following theorem.

Theorem 3.4.3.
The solution set of $w^{(n)} + a_1 w^{(n-1)} + \cdots + a_{n-1}w' + a_n w = 0$ *is a* $\mathbb{C}$*-vector space with basis* $\bigcup_{j=1}^{k}\{e^{t\lambda_j}, te^{t\lambda_j}, \ldots, t^{m_j-1}e^{t\lambda_j}\}$.

Example 3.4.2.
Revisit Example 3.4.1, note that
$v(t) := \frac{1}{60}(30t + 25) = \frac{t}{2} + \frac{5}{12}$ *is a particular solution to* $v''' - 2v'' - 5v + 6v = 3t$.
We now see that $w = u - v$ *solves* $\begin{cases} w''' - 2w'' - 5w + 6w = 0, \\ w(0) = \dfrac{7}{12}, u'(0) = \dfrac{3}{12}, w''(0) = 0. \end{cases}$
Consider $P(\lambda) := \lambda^3 - 2\lambda^2 - 5\lambda + 6 = 0$.
One can check that the roots of P *are* $-2, 1, 3$.

Hence the general solution of w is $w(t) = c_1e^{-2t} + c_2e^t + c_3e^{3t}$.

Now we see that $\begin{cases} \dfrac{7}{12} = w(0) = c_1 + c_2 + c_3 \\ \dfrac{3}{2} = w'(0) = -2c_1 + c_2 + 3c_3, \\ 0 = w''(0) = 4c_1 + c_2 + 9c_3 \end{cases} \Rightarrow \begin{cases} c_1 = -\dfrac{17}{60} \\ c_2 = \dfrac{5}{6}, \\ c_3 = \dfrac{1}{30} \end{cases}$

$\Rightarrow w(t) = \dfrac{1}{60}(-17e^{-2t} + 50e^t + 2e^{3t})$.

Hence, we conclude that $u(t) = v(t) + w(t) = \frac{1}{60}(30t + 25 - 17e^{-2t} + 50e^t + 2e^{3t})$ *is the unique solution of* $\begin{cases} u''' - 2u'' - 5u + 6u = 3t, \\ u(0) = 1, u'(0) = 2, u''(0) = 0. \end{cases}$

Exercise 3.14.
Given any $b, c \in \mathbb{R}$, *find general solutions for the equation* $u'' + bu' + cu = 0$.
⟨Solution⟩
Assume $u(t) = e^{rt}$, *then we get the characteristic equation* $r^2 + br + c = 0$.
There are three cases.
Case 1 : $b^2 - 4c < 0 \Rightarrow$ *there have complex roots.*
$r_1 = \frac{-b}{2} + \frac{\sqrt{4c - b^2}}{2} i, r_2 = \frac{-b}{2} - \frac{\sqrt{4c - b^2}}{2} i$
Then the general solution is
$u(t) = c_1 e^{\frac{-b}{2}} \cos(\frac{\sqrt{4c - b^2}}{2} t) + c_2 e^{\frac{-b}{2}} \sin(\frac{\sqrt{4c - b^2}}{2} t), \forall c_1, c_2 \in \mathbb{R}\cdot$
Case 2 : $b^2 - 4c > 0 \Rightarrow$ *there have 2 different real roots.*
$r_1 = \frac{-b + \sqrt{b^2 - 4c}}{2}, r_2 = \frac{-b - \sqrt{b^2 - 4c}}{2}$
The general solution is $u(t) = c_1 e^{r_1 \cdot t} + c_2 r^{r_2 t}, \forall c_1, c_2 \in \mathbb{R}$.
Case 3 : $b^2 - 4c = 0 \Rightarrow$ *there has a repeat real root.*
$r = r_1 = r_2 = \frac{-b}{2}$
The general solution is $u(t) = (c_1 + c_2) e^{rt}, \forall c_1, c_2 \in \mathbb{R}$.

Exercise 3.15.
Fix $\alpha, \beta \in \mathbb{R}$.
Find general solutions for the Euler equation $t^2 u''(t) + \alpha t u'(t) + \beta u(t) = 0, \forall t > 0$.
⟨Solution⟩
Let $u(t) = t^r$. *Then* $u'(t) = r t^{r-1}, u''(t) = r(r-1) t^{r-2}$.
Hence, $(r^2 - r) t^r + \alpha r t^r + \beta t^r = 0 \Rightarrow t^r (r^2 - r + \alpha r + \beta) = 0$.
Since $t^r > 0, \forall t > 0$, *then* $r^2 - r + \alpha r + \beta = 0$.
$r_1 = \frac{(1 - \alpha) + \sqrt{(\alpha - 1)^2 - 4\beta}}{2}, r_2 = \frac{(1 - \alpha) - \sqrt{(\alpha - 1)^2 - 4\beta}}{2}$
Case 1 : $(\alpha - 1)^2 - 4\beta > 0 \Rightarrow$ *there have 2 different real roots.*
The general solution is $u(t) = c_1 t^{r_1} + c_2 t^{r_2}, \forall c_1, c_2 \in \mathbb{R}$.
Case 2 : $(\alpha - 1)^2 - 4\beta = 0 \Rightarrow$ *there has a repeat real root.*
The general solution is $u(t) = c_1 t^r + c_2 t^r \ln(t), \forall c_1, c_2 \in \mathbb{R}$.
Case 3 : $(\alpha - 1)^2 - 4\beta < 0 \Rightarrow$ *there have complex roots.*
The general solution is
$u(t) = t^{\frac{1-\alpha}{2}} c_1 \cos(\frac{\sqrt{4\beta - (\alpha - 1)^2}}{2} \ln(t)) + c_2 \sin(\frac{\sqrt{4\beta - (\alpha - 1)^2}}{2} \ln(t)), \forall c_1, c_2 \in \mathbb{R}\cdot$

3.5 Strum-Liouville Eigenvalue Problem

Let $a, b, \theta_1, \theta_2 \in \mathbb{R}$ with $a < b$, let $q \in C([a,b])$ is real-valued and let $p \in C^1(a,b) \cap C[a,b]$ is real-valued $\ni p(t) > 0, \forall t \in [a,b]$.
Define the linear operator $\mathscr{L} : C^2(a,b) \cap C^1[a,b] \to C(a,b)$ by
$(\mathscr{L}[u])(t) := \dfrac{d}{dt}\bigg(p(t)\dfrac{du}{dt} + q(t)u\bigg)$.
In this section, we consider the eigenvalue problem
$\mathscr{L}[u] = -\lambda u, \forall t \in (a,b)$ subject to the boundary conditions
$\begin{cases} u(a)\cos\theta_1 - p(a)u'(a)\sin\theta_1 = 0, \\ u(b)\cos\theta_2 - p(b)u'(b)\sin\theta_2 = 0. \end{cases}$

It is easy to see that the eigenvalue problem has a solution at $u \equiv 0$, which is called the trivial solution.
We see that the boundary conditions in eigenvalue problem is over-determined, and we expect that in general the nonexistence of nontrivial solution (i.e. $u \not\equiv 0$) without any further assumptions.
We are interested in the following object:

Definition 3.5.1.
If there exists $\lambda \in \mathbb{C}$ and a nontrivial solution $u \in C^2(a,b) \cap C^1[a,b]$ of the eigenvalue problem, then we say that such λ is a Strum-Liouville eigenvalue and such nontrivial solution u is called the corresponding Strum-Liouville eigenfunction.
The boundary value problem is called the Strum-Liouville eigenvalue problem.

Define $(u,v)_{L^2(a,b)} := \displaystyle\int_a^b u(t)\overline{v(t)}dt$, and $\|u\|_{L^2(a,b)} := (u,u)_{L^2(a,b)}{}^{\frac{1}{2}} = \bigg(\displaystyle\int_a^b |u(t)|^2 dt\bigg)^{\frac{1}{2}}$.
We remind the inner product is skew-Hermitian :
$(v,u)_{L^2(a,b)} = \displaystyle\int_a^b v(t)\overline{u(t)}dt = \overline{(u,v)_{L^2(a,b)}}$.
One can compare this with the the inner product $(\cdot,\cdot)_{M_{n\times n}(\mathbb{C})}$ given in (3.1.9).

Lemma 3.5.1.
The operator L : C2((a, b)) →C((a, b)) is Hermitian or self-adjoint in the sense of $(\mathscr{L}[u],v)_{L^2(a,b)} = (u,\mathscr{L}[u])_{L^2(a,b)}, \forall u,v \in C^2(a,b) \cap C^1[a,b]$ both satisfy the boundary condition $\begin{cases} u(a)\cos\theta_1 - p(a)u'(a)\sin\theta_1 = 0, \\ u(b)\cos\theta_2 - p(b)u'(b)\sin\theta_2 = 0. \end{cases}$
In addition, if λ is a Strum-Liouville eigenvalue, then $\lambda \in \mathbb{R}$.

Proof.
By using the integration by parts, we see that

□

Remark :
It is interesting to compare (3.5.2) with a characterization of Hermitian matrix in Lemma 3.1.28, and compare the second statement with the result for Hermitian matrix in Theorem 3.1.29.

Theorem 3.5.2.
Let $a, b, \theta_1, \theta_2 \in \mathbb{R}$ with $a < b$, let $q \in C[a,b]$ is real-valued and let $p \in C^1(a,b) \cap C[a,b]$ is real-valued $\ni p(t) > 0, \forall t \in [a,b]$. Then
(1) *there exists a countable sequence of eigenvalues $\lambda_1 < \lambda_2 < \lambda_3 < \cdots \to +\infty$ of the Strum-Liouville eigenvalue problem.*
(2) *Let λ_i and λ_j be two distinct eigenvalues, then the corresponding eigenfunctions u_i and u_j are orthogonal in $L^2(a,b)$, that is, $(u_i, u_j)_{L^2(a,b)} = 0$.*
(3) *For every $u \in C^2(a,b) \cap C^1[a,b]$ satisfying the boundary condition, the series $\sum_{j=1}^{\infty} (f, u_j)_{L^2(a,b)} u_j$ converges to u in $L^\infty(a,b)$, provided the eigenfunctions are normalized to $\|u_j\|_{L^2(a,b)} = 1$.*

Example 3.5.1.
For simplicity, we put $a = 0$ and $b = \pi$. We now choose $p(t) \equiv 1$ and $q(t) \equiv 0$, and now the Strum-Liouville problem reads $u''(t) = -\lambda u(t), \forall t \in (0, \pi)$ subject to the boundary conditions $\begin{cases} u(a)\cos\theta_1 - u'(a)\sin\theta_1 = 0, \\ u(b)\cos\theta_2 - u'(b)\sin\theta_2 = 0. \end{cases}$
By choosing $\theta_1 = \theta_2 = 0$, we reach an orthogonal sequence $\{\sin(nt)\}_{n\in\mathbb{N}}$ of eigenfunctions.
By choosing $\theta_1 = \theta_2 = \frac{\pi}{2}$, we reach an orthogonal sequence $\{\cos(nt)\}_{n=0}^{\infty}$ of eigenfunctions. This induces Fourier series.

4 Partial Differential Equation in Classical Sense

4.1 Preliminaries

We will use Lebesgue integral rather than Riemannian integral.
Throughout this lecture note, the abbreviation "a.e." means "almost everywhere," and we usually omit this if there is no ambiguity.

Notations :
1. $x := (x_1, x_2, \dots, x_n)$ is the column vector.
Let Ω be any open set in $\mathbb{R}^n$, and for each $1 \le p \le \infty$, we define
2-1. $1 \le p < \infty$, $L^p(\Omega) = \{f : \Omega \to \mathbb{R}^n \text{ with } \|f\|_{L^p(\Omega)} := (\int_\Omega |f(x)|^p dx)^{1/p} < \infty\}$
2-2. $p = \infty$, $L^\infty(\Omega) = \{f : \Omega \to \mathbb{R}^n \text{ with } \|f\|_{L^p(\Omega)} := \sup_{x \in \Omega} |f(x)| < \infty\}$

Note : For each $1 \le p < \infty$, one sees that $\||f|^p\|_{L^1(\Omega)} = \|f\|^p_{L^p(\Omega)}$, therefore in many cases it is suffice to consider L^1-functions.

Some properties of L^1-spaces.

Lemma 4.1.1. (*Monotone Convergence Theorem, MCT*)
Let Ω be an open set in $\mathbb{R}^n$.
If $\{f_k\}_{k\in\mathbb{N}}$ be a sequence of functions in $L^1(\Omega)$ satisfying $f_1(x) \le f_2(x) \le \cdots \le f_k(x) \le f_{k+1}(x) \le \cdots$ for a.e. $x \in \Omega$ and $\sup_{k\in\mathbb{N}} \int_\Omega f_k < \infty$, then $f_k(x)$ converges a.e. on Ω to a finite limit $f(x)$.
In addition, such limit function $f(x)$ belongs to $L^1(\Omega)$ and satisfies $f_k \to f$ in $L^1(\Omega)$, i.e., $\lim\limits_{k\to\infty} \|f_k - f\|_{L^1(\Omega)} = 0$.

Lemma 4.1.2. (*Lebesgue Dominated Convergence Theorem, LDCT*)
Let Ω be an open set in $\mathbb{R}^n$.
If $\{f_k\}_{k\in\mathbb{N}}$ be a sequence of functions in $L^1(\Omega)$ satisfying $f_k(x) \to f(x)$ a.e. in Ω and $\exists$ a function $g \in L^1(\Omega) \ni |f_k(x)| \le g(x)$ a.e. in $\Omega, \forall k \in \mathbb{N}$. Then $f \in L^1(\Omega)$ and $f_k \to f$ in $L^1(\Omega)$.

Lemma 4.1.3. (*Fatou's Lemma*)
Let Ω be an open set in $\mathbb{R}^n$.
If $\{f_k\}_{k\in\mathbb{N}}$ be a sequence of functions in $L^1(\Omega)$ satisfying
$f_k(x) \ge 0$ for $x \in \Omega$, and $\sup_{k\in\Omega} \int_\Omega f_k < \infty$.

Set $f(x) := \sup\limits_{m\in\mathbb{N}} \big(\inf\limits_{k\ge m} f_k(x)\big) = \lim\limits_{m\to\infty} \big(\inf\limits_{k\ge m} f_k(x)\big) = \liminf\limits_{k\to\infty} f_k(x) \le \infty$.
Then $f \in L(\Omega)$ and $\int_\Omega f(x)dx \le \liminf_{k\to\infty} \int_\Omega f_k(x)dx$.

Lemma 4.1.4. (*Fubini Thm*)
Let $\Omega_1 \subseteq \mathbb{R}^n$ and $\Omega_2 \subseteq \mathbb{R}^m$ be an open set, and let $F : \Omega_1 \times \Omega_2 \to \mathbb{R}$ be a (measurable) function.
(1) *If $F \geq 0$ a.e. in $\Omega_1 \times \Omega_2$, then* $\displaystyle\int_{\Omega_1}\left(\int_{\Omega_2} F(x,y)dy\right)dx = \int_{\Omega_2}\left(\int_{\Omega_1} F(x,y)dx\right)dy.$
(2) *If $F \in L^1(\Omega_1 \times \Omega_2)$, i.e.,* $\displaystyle\int_{\Omega_2}\left(\int_{\Omega_1} |F(x,y)|dx\right)dy < \infty$, *then*
$\displaystyle\int_{\Omega_1}\left(\int_{\Omega_2} F(x,y)dy\right)dx = \int_{\Omega_2}\left(\int_{\Omega_1} F(x,y)dx\right)dy.$

Some elementary properties of L^p spaces.

Theorem 4.1.5.
Assume $\Omega \subseteq \mathbb{R}^n$ be an open set.
Let $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$ with $1 \leq p \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$.
Then $fg \in L(\Omega)$ and the following Hoelder's inequality holds :
$\displaystyle\int_\Omega |f(x)g(x)|dx \leq \|f\|_{L^p(\Omega)}\|g\|_{L^q(\Omega)}.$
Moreover, $\displaystyle\int_\Omega |f(x)g(x)|dx = \|f\|_{L^p(\Omega)}\|g\|_{L^q(\Omega)}$ *holds when $\exists c \in \mathbb{R} \ni |g(x)| = c|f(x)|^{p-1}$ for a.e. $x \in \Omega$.*
In addition, the function $\|\cdot\|_{L^p(\Omega)} : L^p(\Omega) \to \mathbb{R}$ defines a norm, and $(L^p(\Omega), \|\cdot\|_{L^p(\Omega)})$ is a Banach space, $\forall 1 \leq p \leq \infty$. (i.e., well-known as Fischer-Riesz Thm)

Exercise 4.1.
For each $p, q \in (1, \infty)$ *and* $\frac{1}{p} + \frac{1}{q} = 1$.
Show that $ab \leq \dfrac{1}{p}a^p + \dfrac{1}{q}b^q, \forall a \geq 0$ *and* $b \geq 0$.

Proof.
Define the function $f : \mathbb{R}_{\geq 0} \to \mathbb{R}$ by $f(x) = \frac{a^p}{p} + \frac{x^q}{q} - ax, \forall a \geq 0$.
Then $f'(x) = x^{q-1} - a$.
We can get $f'(x) = 0$ as $x = a^{\frac{1}{q-1}}$.
Moreover, $f'(x) > 0$ as $x > a^{\frac{1}{q-1}}$, and $f'(x) < 0$ as $x < a^{\frac{1}{q-1}}$.
Hence, $x = a^{\frac{1}{q-1}}$ is global minimum.
Since $1 = \frac{1}{p} + \frac{1}{q} = \frac{p+q}{pq}$ and $\frac{1}{p} = \frac{q-1}{q}$ iff $\frac{1}{q-1} = \frac{p}{q}$, then
$f(x) \geq f(a^{\frac{1}{q-1}}) = f(a^{\frac{p}{q}}) = \dfrac{a^p}{p} + \dfrac{(a^{\frac{p}{q}})^q}{q} - a \cdot a^{\frac{p}{q}} = \dfrac{a^p}{p} + \dfrac{a^p}{p} - \left(a^{\frac{p+q}{pq}}\right)^p = 0.$ ∎

Exercise 4.2.
Let Ω be an open set in $\mathbb{R}^n$.
Assume that $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$ with $1 \leq p, q \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$.
Show that $fg \in L^1(\Omega)$ and the following Holder's inequality holds :
$$\int_\Omega |f(x)g(x)|dx \leq \|f\|_{L^p(\Omega)} \|g\|_{L^q(\Omega)}$$
and the equality holds iff $\exists c \in \mathbb{R} \ni |g(x)| = c|f(x)|^{p-1}$, for a.e. $x \in \Omega$.

Proof.
For $p = 1, q = \infty$,
$$\begin{aligned}\int_\Omega |f(x)g(x)|dx = & \int_\Omega \Big(|f(x)| \cdot |g(x)| \Big) \\ \leq & \int_\Omega \Big(|f(x)| \cdot \sup_{x \in \Omega} g(x) \Big) = \sup_{x\in\Omega} g(x) \cdot \int_\Omega |f(x)|dx \\ & = \|f\|_{L^1(\Omega)} \|g\|_{L^\infty(\Omega)}\end{aligned}$$
Same technique for $p = \infty, q = 1$.
For $1 < p, q < \infty$,
If $\left(\int_\Omega |f|^p \right)^{1/p} = 0$ or $\left(\int_\Omega |f|^q \right)^{1/q} = 0$, then $f = 0$ or $g = 0$ a.e. on Ω.
Hence, $fg = 0$ a.e. and the inequality holds.
Assume $f, g > 0$.
By Young's inequality,
$$\left| \frac{f}{(\int_\Omega |f|^p)^{1/p}} \cdot \frac{g}{(\int_\Omega |f|^q)^{1/q}} \right| \leq \frac{1}{p} \left(\frac{|f|}{(\int_\Omega |f|^p)^{1/p}} \right)^p + \frac{1}{q} \left(\frac{|g|}{(\int_\Omega |g|^q)^{1/q}} \right)^q \text{ a.e. on } \Omega.$$
$$\begin{aligned}\int_\Omega \frac{|fg|}{(\int_\Omega |f|^p)^{1/p} (\int_\Omega |g|^q)^{1/q}} \leq & \frac{1}{p} \cdot \frac{1}{\int_\Omega |f|^p} \int_\Omega |f|^p + \frac{1}{q} \cdot \frac{1}{\int_\Omega |g|^q} \int_\Omega |g|^q \\ = & \frac{\|f\|^p_{L^p(\Omega)}}{p \cdot \|f\|^p_{L^p(\Omega)}} + \frac{\|g\|^q_{L^q(\Omega)}}{q \cdot \|g\|^q_{L^q(\Omega)}} = \frac{1}{p} + \frac{1}{q} = 1\end{aligned}$$
Therefore, $\int_\Omega |f(x)g(x)|dx \leq \|f\|_{L^p(\Omega)} \|g\|_{L^q(\Omega)}$.
Suppose $\int_\Omega |f(x)g(x)|dx = \|f\|_{L^p(\Omega)} \|g\|_{L^q(\Omega)}$.
Then the equality in Young's inequality holds, i.e., $ab = \dfrac{a^p}{p} + \dfrac{b^q}{q}$ iff $a^p = b^q$.
Since $\dfrac{1}{p} + \dfrac{1}{q} = 1 \Rightarrow 1 + \dfrac{p}{q} = p$, then we can get that

$$\left(\frac{|f|}{\|f\|_{L^p(\Omega)}}\right)^p = \left(\frac{|g|}{\|g\|_{L^q(\Omega)}}\right)^q$$
$$\Rightarrow \left(\frac{|f|}{\|f\|_{L^p(\Omega)}}\right)^{p/q} = \frac{|g|}{\|g\|_{L^q(\Omega)}}$$
$$\Rightarrow |g| = \|g\|_{L^q(\Omega)}\left(\frac{|f|}{\|f\|_{L^p(\Omega)}}\right)^{p/q} = \left(\frac{\|g\|_{L^q(\Omega)}^{q/p}}{\|f\|_{L^p(\Omega)}}\right)^{p/q}|f|^{p/q} = \left(\frac{\|g\|_{L^q(\Omega)}^{q/p}}{\|f\|_{L^p(\Omega)}}\right)^{1-p}|f|^{1-p}$$

Hence, $\exists c = \left(\frac{\|g\|_{L^q(\Omega)}^{q/p}}{\|f\|_{L^p(\Omega)}}\right)^{1-p} \in \mathbb{R} \ni |g(x)| = c|f(x)|^{p-1}$, for a.e. $x \in \Omega$.

Suppose $\exists c \in \mathbb{R} \ni |g(x)| = c|f(x)|^{p-1}$, for a.e. $x \in \Omega$

$$\int_\Omega |fg| = \int_\Omega \big(|f||g|\big) = \int_\Omega \big(|f|c|f|^{p-1}\big) = c\int_\Omega |f|^p = c\|f\|_{L^p(\Omega)}^p$$

Since $\frac{1}{p} + \frac{1}{q} = 1$ iff $p + q = pq$, then

$$\begin{aligned}\|f\|_{L^p(\Omega)}\|g\|_{L^q(\Omega)} &= \|f\|_{L^p(\Omega)}\left(\int_\Omega |g|^q\right)^{1/q} = \|f\|_{L^p(\Omega)}\left(\int_\Omega c^q|f|^{(p-1)q}\right)^{1/q} \\ &= \|f\|_{L^p(\Omega)}\left(c^q\int_\Omega |f|^p\right)^{1/q} = \|f\|_{L^p(\Omega)}\cdot c\left[\left(\int_\Omega |f|^p\right)^{1/p}\right]^{p-1} \\ &= \|f\|_{L^p(\Omega)}\cdot c\cdot \|f\|_{L^p(\Omega)}^{p-1} = c\cdot\|f\|_{L^p(\Omega)}^p\end{aligned}$$

Therefore, the equality holds iff $\exists c \in \mathbb{R} \ni |g(x)| = c|f(x)|^{p-1}$, for a.e. $x \in \Omega$. $\square$

Exercise 4.3.
Let Ω be an open set in $\mathbb{R}^n$.
Show that $\|\cdot\|_{L^p(\Omega)}$ defines a norm for each $1 \le p \le \infty$.

Proof.
For $1 \le p < \infty$,
Given $f, g \in L^p(\Omega)$.
$\|f\|_{L^p(\Omega)} = \left(\int_\Omega |f|^p \right)^{1/p} \ge 0.$
If $f = 0$ a.e. on Ω, then $\left(\int_\Omega |f|^p \right)^{1/p} = \left(\int_\Omega 0^p \right)^{1/p} = 0^{1/p} = 0.$
Since $0 \le f \in L^p(\Omega)$ and $\int_\Omega f = 0$, then $f = 0$ a.e. on Ω.
Given $\alpha \in \mathbb{R}$.

$$\begin{aligned} \|\alpha f\|_{L^p(\Omega)} &= \left(\int_\Omega |\alpha f|^p \right)^{1/p} = \left(\int_\Omega |\alpha|^p |f|^p \right)^{1/p} = \left(|\alpha|^p \int_\Omega |f|^p \right)^{1/p} \\ &= |\alpha| \left(\int_\Omega |f|^p \right)^{1/p} = |\alpha| \cdot \|f\|_{L^p(\Omega)} \end{aligned}$$

$$\begin{aligned} \|f + g\|_{L^p(\Omega)} &= \left(\int_\Omega |f + g|^p \right)^{1/p} \\ &\le \left(\int_\Omega |f|^p \right)^{1/p} \left(\int_\Omega |g|^p \right)^{1/p} \text{ (by Minkowski inequality)} \end{aligned}$$

For $p = \infty$,
Given $f, g \in L^\infty(\Omega)$.
$\|f\|_{L^\infty(\Omega)} = \sup_{x \in \Omega} |f(x)| \ge 0.$
If $f = 0$ a.e. on Ω, then $\sup_{x \in \Omega} |0| = 0$.
Since $0 \le f \in L^\infty(\Omega)$ and $\sup_{x \in \Omega} |f| = 0$, then $f = 0$ on Ω.
Given $\alpha \in \mathbb{R}$.

$$\|\alpha f\|_{L^\infty(\Omega)} = \sup_{x \in \Omega} |\alpha f(x)| = \sup_{x \in \Omega} |\alpha||f(x)| = |\alpha| \sup_{x \in \Omega} |f(x)| = |\alpha| \cdot \|f\|_{L^\infty(\Omega)}$$

$$\begin{aligned} \|f + g\|_{L^\infty(\Omega)} &= \sup_{x \in \Omega} |f(x) + g(x)| \\ &\le \sup_{x \in \Omega} \big(|f(x)| + |g(x)| \big) = \sup_{x \in \Omega} |f(x)| + \sup_{x \in \Omega} |g(x)| \\ &= \|f\|_{L^\infty(\Omega)} + \|g\|_{L^\infty(\Omega)} \end{aligned}$$

Hence, $\|\cdot\|_{L^p(\Omega)}$ defines a norm for each $1 \le p \le \infty$. $\square$

Exercise 4.4.
Show that $\left(\int_{\Omega_2}\left|\int_{\Omega_1} F(x,y)dx\right|^p dy\right)^{1/p} \le \left(\int_{\Omega_1}\left|\int_{\Omega_1} F(x,y)dx\right|^p dy\right)^{1/p}$

Proof.
Let q be the conjugate number of p that satisfy $\frac{1}{p}+\frac{1}{q}=1$.
Then $\frac{1}{p}=\frac{q-1}{q}$ iff $p(q-1)=q$.
Let $H(y)=\int_{\Omega_1} F(x,y)dx$. Then

$$\begin{aligned}
\int_{\Omega_2}\left|\int_{\Omega_1} F(x,y)dx\right|^p dy &= \int_{\Omega_2}\left|H(y)\right|^p dy \\
&= \int_{\Omega_2}\left|H(y)\right|\cdot\left|H(y)\right|^{p-1} dy \\
&= \int_{\Omega_2}\left|\int_{\Omega_1} F(x,y)dx\right|\cdot\left|H(y)\right|^{p-1} dy \\
&= \int_{\Omega_2}\left|\int_{\Omega_1} F(x,y)\cdot H(y)^{p-1}dx\right|dy \\
&\le \int_{\Omega_2}\int_{\Omega_1}\left|F(x,y)\cdot H(y)^{p-1}\right|dxdy
\end{aligned}$$

Then we use Tonelli Thm and Hoelder inequality,

$$\begin{aligned}
\int_{\Omega_2}\left|H(y)\right|^p dy = &\int_{\Omega_2}\left|\int_{\Omega_1} F(x,y)dx\right|^p dy \\
\le &\int_{\Omega_2}\int_{\Omega_1}\left|F(x,y)\cdot H(y)^{p-1}\right|dxdy \\
= &\int_{\Omega_1}\int_{\Omega_2}\left|F(x,y)\cdot H(y)^{p-1}\right|dydx \text{ (by Tonelli Thm)} \\
\le &\int_{\Omega_1}\|F(x,y)\|_{L^p(\Omega_2)}\|H(y)^{p-1}\|_{L^q(\Omega_2)}dx \text{ (by Hoelder's Inequality)} \\
= &\int_{\Omega_1}\left[\int_{\Omega_2}\left|F(x,y)\right|^p dy\right]^{1/p}\left[\int_{\Omega_2}\left|H(y)\right|^{q(p-1)} dy\right]^{1/q} dx \\
= &\int_{\Omega_1}\left[\int_{\Omega_2}\left|F(x,y)\right|^p dy\right]^{1/p}\left[\int_{\Omega_2}\left|H(y)\right|^{p} dy\right]^{1/q} dx \\
= &\left[\int_{\Omega_2}\left|H(y)\right|^p dy\right]^{1/q}\cdot\int_{\Omega_1}\left[\int_{\Omega_2}\left|F(x,y)\right|^p dy\right]^{1/p} dx
\end{aligned}$$

Hence, we can get

$$\left[\int_{\Omega_2}\left|H(y)\right|^p dy\right]^{\frac{q-1}{q}} = \left[\int_{\Omega_2}\left|H(y)\right|^p dy\right]^p \le \int_{\Omega_1}\left[\int_{\Omega_2}\left|F(x,y)\right|^p dy\right]^{1/p} dx$$ ☐

Exercise 4.5.
Deduce that if $f \in L^p(\Omega) \cap L^q(\Omega)$ with $1 \le p, q \le \infty$, then $f \in L^r(\Omega), \forall r \in (p,q)$. More precisely, write $\dfrac{1}{r} = \dfrac{\alpha}{p} + \dfrac{1-\alpha}{q}$ with $0 \le \alpha \le 1$ and prove that
$\|f\|_{L^r(\Omega)} \le \|f\|^{\alpha}_{L^p(\Omega)} \|f\|^{1-\alpha}_{L^q(\Omega)}$.

Proof.
For $p = 1, q = \infty$.
Since $\dfrac{1}{r} = \dfrac{\alpha}{p} + \dfrac{1-\alpha}{q}$, then $r = \dfrac{p}{\alpha}$ iff $p = r\alpha$.

$$\begin{aligned}
\|f\|^r_{L^r(\Omega)} = & \int_\Omega |f|^r = \int_\Omega |f|^{r\alpha + r(1-\alpha)} \\
\le & \int_\Omega |f|^{r\alpha} \cdot \left(\sup_\Omega |f|\right)^{r(1-\alpha)} = \left(\sup_\Omega |f|\right)^{r(1-\alpha)} \int_\Omega |f|^{r\alpha} \\
= & \left(\sup_\Omega |f|\right)^{r(1-\alpha)} \left(\int_\Omega |f|^p\right)^{r\alpha/p} = \|f\|^{r\alpha}_{L^p(\Omega)} \|f\|^{r(1-\alpha)}_{L^\infty(\Omega)}
\end{aligned}$$

Hence, $\|f\|_{L^r(\Omega)} \le \|f\|^{\alpha}_{L^1(\Omega)} \|f\|^{1-\alpha}_{L^\infty(\Omega)}$.
For $1 < p, q < \infty$.
If $(\int_\Omega |f|^p)^{1/p} = 0$, then $f = 0$ a.e. on Ω.
Hence, the inequality holds.
Assume $f > 0$.
Since $\dfrac{1}{r} = \dfrac{\alpha}{p} + \dfrac{1-\alpha}{q}$, then $1 = \dfrac{r\alpha}{p} + \dfrac{r(1-\alpha)}{q}$ iff $1 = \dfrac{1}{\frac{p}{r\alpha}} + \dfrac{1}{\frac{q}{r(1-\alpha)}}$.
By Hoelder's inequality,

$$\begin{aligned}
\|f\|^r_{L^r(\Omega)} = & \int_\Omega |f|^r = \int_\Omega |f|^{r\alpha + r(1-\alpha)} = \int_\Omega |f|^{r\alpha} |f|^{r(1-\alpha)} \\
\le & \left[\int_\Omega \left(|f|^{r\alpha}\right)^{\frac{p}{r\alpha}}\right]^{\frac{r\alpha}{p}} \left[\int_\Omega \left(|f|^{r(1-\alpha)}\right)^{\frac{q}{r(1-\alpha)}}\right]^{\frac{r(1-\alpha)}{q}} \quad \text{(by Hoelder's)} \\
= & \left(\int_\Omega |f|^p\right)^{\frac{r\alpha}{p}} \left(\int_\Omega |f|^q\right)^{\frac{r(1-\alpha)}{q}} = \|f\|^{r\alpha}_{L^p(\Omega)} \|f\|^{r(1-\alpha)}_{L^q(\Omega)}
\end{aligned}$$

Hence, $\|f\|_{L^r(\Omega)} \le \|f\|^{\alpha}_{L^p(\Omega)} \|f\|^{1-\alpha}_{L^q(\Omega)}$.
Since $f \in L^p(\Omega) \cap L^q(\Omega)$, then $\|f\|^{\alpha}_{L^p(\Omega)} \|f\|^{1-\alpha}_{L^q(\Omega)} < \infty \Rightarrow \|f\|^r_{L^r(\Omega)} < \infty$. $\square$

Exercise 4.6.
Given any $f \in L^p(\Omega)$, show that
$$\|f\|_{L^p(\Omega)} = \sup_{\|g\|_{L^q(\Omega)}=1} \int_\Omega f(x)g(x)dx \text{ and } \|f\|_{L^p(\Omega)} = \sup_{\|g\|_{L^q(\Omega)}=1} \int_\Omega |f(x)g(x)|dx.$$

Proof.
Assume $g \in L^q(\Omega)$ with $\|g\|_{L^q(\Omega)} = 1$.
$$\begin{aligned}\int_\Omega fg \le & \int_\Omega |fg| \\ \le & \left(\int_\Omega |f|^p\right)^{1/p}\left(\int_\Omega |f|^q\right)^{1/q} \text{ (by Hoelder Inequality)} \\ &= \|f\|_{L^p(\Omega)}\|g\|_{L^q(\Omega)} = \|f\|_{L^p(\Omega)}\end{aligned}$$
Hence, $\|f\|_{L^p(\Omega)} \ge \sup_{\|g\|_{L^q(\Omega)}=1} \int_\Omega fg$ and $\|f\|_{L^p(\Omega)} \ge \sup_{\|g\|_{L^q(\Omega)}=1} \int_\Omega |fg|$.
If $f = 0$ a.e. on Ω and for any g with $\|g\|_{L^q(\Omega)} = 1$, then $\|f\|_{L^p(\Omega)} = 0$ and $\int_\Omega f(x)g(x)\, dx = 0$.
Hence, the supremum is 0, and equality holds.
Assume $0 \ne f \in L^p(\Omega)$, then $\|f\|_{L^p(\Omega)} > 0$.
Define $g(x) = \begin{cases} \dfrac{|f(x)|^{p-1} sgn(f(x))}{\|f\|^{p-1}_{L^p(\Omega)}}, & \text{if } f(x) \ne 0 \\ 0, & \text{if } f(x) = 0 \end{cases}$, where $sgn(f(x)) = \dfrac{f(x)}{|f(x)|}$.
claim : $g \in L^q(\Omega)$.
Since $\dfrac{1}{p} + \dfrac{1}{q} = 1$, then $\dfrac{p+q}{pq} = 1$ iff $p = pq - q$
$$\begin{aligned}\|g\|^q_{L^q(\Omega)} &= \int_\Omega |g|^q = \int_\Omega \left|\frac{|f|^{p-1}sgn(f)}{\|f\|^{p-1}_{L^p(\Omega)}}\right|^q = \frac{1}{\|f\|^{pq-q}_{L^p(\Omega)}} \int_\Omega \left(|f|^{pq-q} \cdot \frac{f^q}{|f|^q}\right) \\ &= \frac{1}{\|f\|^p_{L^p(\Omega)}} \int_\Omega \left(|f|^p \cdot \frac{f^q}{|f|^q}\right) = \frac{1}{\|f\|^p_{L^p(\Omega)}} \int_\Omega \left(|f|^{p/q} \cdot f^q\right) \\ &= \frac{1}{\|f\|^p_{L^p(\Omega)}} \cdot \|f\|^p_{L^p(\Omega)} = 1\end{aligned}$$
Hence, g is well-defined and in $L^q(\Omega)$.
$$\begin{aligned}\int_\Omega fg &= \int_\Omega \left(f \cdot \frac{|f|^{p-1}sgn(f)}{\|f\|^{p-1}_{L^p(\Omega)}}\right) = \frac{1}{\|f\|^{p-1}_{L^p(\Omega)}} \int_\Omega f \cdot sgn(f) \cdot |f|^{p-1} \\ &= \frac{1}{\|f\|^{p-1}_{L^p(\Omega)}} \int_\Omega |f| \cdot |f|^{p-1} = \frac{1}{\|f\|^{p-1}_{L^p(\Omega)}} \int_\Omega |f|^p = \frac{1}{\|f\|^{p-1}_{L^p(\Omega)}} \cdot \|f\|^p_{L^p(\Omega)} \\ &= \|f\|_{L^p(\Omega)}\end{aligned}$$
Hence, g achieves the value $\|f\|_{L^p(\Omega)}$, so $\|f\|_{L^p(\Omega)} \ge \sup_{\|g\|_{L^q(\Omega)}=1} \int_\Omega f(x)g(x)dx$.

$$\begin{aligned}\int_\Omega |fg| &= \int_\Omega \left| f \cdot \frac{|f|^{p-1} sgn(f)}{\|f\|_{L^p(\Omega)}^{p-1}} \right| = \frac{1}{\|f\|_{L^p(\Omega)}^{p-1}} \int_\Omega \left| f \cdot sgn(f) \cdot |f|^{p-1} \right| \\ &= \frac{1}{\|f\|_{L^p(\Omega)}^{p-1}} \int_\Omega |f| \cdot |f|^{p-1} = \frac{1}{\|f\|_{L^p(\Omega)}^{p-1}} \int_\Omega |f|^p = \frac{1}{\|f\|_{L^p(\Omega)}^{p-1}} \cdot \|f\|_{L^p(\Omega)}^p \\ &= \|f\|_{L^p(\Omega)}\end{aligned}$$

Hence, g achieves the value $\|f\|_{L^p(\Omega)}$, so $\|f\|_{L^p(\Omega)} \geq \sup_{\|g\|_{L^q(\Omega)}=1} \int_\Omega |f(x)g(x)|dx$. □

Exercise 4.7.
Let $f \in L^p(\Omega), g \in L^q(\Omega), h \in L^r(\Omega)$ for some $1 \le p, q, r \le \infty$ with $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$. Show that $fgh \in L^1(\Omega)$ and $\displaystyle\int_\Omega |f(x)g(x)h(x)|dx \le \|f\|_{L^p(\Omega)}\|g\|_{L^q(\Omega)}\|h\|_{L^r(\Omega)}$.

Proof.
If $(\displaystyle\int_\Omega |f|^p)^{1/p} = 0, (\int_\Omega |g|^q)^{1/q} = 0$ or $(\displaystyle\int_\Omega |h|^r)^{1/r} = 0$, then $f = 0, g = 0$ or $h = 0$ a.e. on Ω.
Hence, $fgh = 0$ a.e. and the inequality holds.
Assume $f, g, h > 0$.
Use Hoelder's inequlity twice.

$$\begin{aligned}\int_\Omega |f(x)g(x)h(x)|dx = & \int_\Omega |f(x)g(x)| \cdot |h(x)|dx \\ \le & \left(\int_\Omega |f(x)g(x)|dx\right)^{\frac{r-1}{r}} \left(\int_\Omega |h(x)|dx\right)^{\frac{1}{r}} \text{ (by Hoelder's)} \\ & = \|fg\|_{L^{\frac{r}{r-1}}(\Omega)} \cdot \|h\|_{L^r(\Omega)}\end{aligned}$$

Let $\dfrac{1}{s} = \dfrac{1}{p} + \dfrac{1}{q}$.
By $\dfrac{1}{p} + \dfrac{1}{q} + \dfrac{1}{r} = 1$, we get $\dfrac{1}{s} = 1 - \dfrac{1}{r} = \dfrac{r-1}{r} \Rightarrow s = \dfrac{r}{r-1}$.
Hence, s and r are conjugate exponent.

$$\begin{aligned}\int_\Omega |f(x)g(x)|^{\frac{r}{r-1}}dx = & \int_\Omega |f(x)|^{\frac{r}{r-1}}|g(x)|^{\frac{r}{r-1}}dx \\ \le & \left[\int_\Omega \left(|f(x)|^{\frac{r}{r-1}}\right)^s dx\right]^{1/s} \left[\int_\Omega \left(|f(x)|^{\frac{r}{r-1}}\right)^r dx\right]^{1/r} \\ & = \left(\int_\Omega |f(x)|^{\frac{r^2}{(r-1)^2}}dx\right)^{\frac{r-1}{r}} \left(\int_\Omega |g(x)|^{\frac{r^2}{r-1}}dx\right)^{\frac{1}{r}}\end{aligned}$$

$$\Rightarrow \left(\int_\Omega \left|f(x)g(x)\right|dx\right)^{\frac{r-1}{r}} \le \left(\int_\Omega |f(x)|^{\frac{r^2}{(r-1)^2}}dx\right)^{\frac{(r-1)^2}{r^2}} \left(\int_\Omega |g(x)|^{\frac{r^2}{r-1}}dx\right)^{\frac{r-1}{r^2}}$$

Choose $p = \frac{r^2}{(r-1)^2}$ and $q = \frac{r^2}{r-1}$, then

$$\begin{aligned}\int_\Omega \left|f(x)g(x)h(x)\right|dx \le & \left(\int_\Omega |f(x)|^p dx\right)^{\frac{1}{p}} \left(\int_\Omega |g(x)|^q dx\right)^{\frac{1}{q}} \|h\|_{L^r(\Omega)} \\ & = \|f\|_{L^p(\Omega)}\|g\|_{L^q(\Omega)}\|h\|_{L^r(\Omega)}\end{aligned}$$

Since $f \in L^p(\Omega), g \in L^q(\Omega), h \in L^r(\Omega)$, then $\|f\|_{L^p(\Omega)}\|g\|_{L^q(\Omega)}\|h\|_{L^r(\Omega)} < \infty$.
Hence, $fgh \in L^1(\Omega)$. $\square$

Exercise 4.8.
Show that $\|\cdot\|^p_{L^p(\Omega)}$ *defines a norm for each* $0 < p < 1$.
In addition, show that $\|\cdot\|_{L^p(\Omega)}$ *does not define a norm for each* $0 < p < 1$.

Proof.
Given $f, g \in L^p(\Omega)$, for $0 < p < 1$.
$\|f\|^p_{L^p(\Omega)} = \int_\Omega |f|^p \geq 0.$
If $f = 0$ a.e. on Ω, then $\int_\Omega |f|^p = \int_\Omega |0|^p = 0$.
If $\int_\Omega |f|^p = 0$ and $|f| \geq 0$, then $f = 0$ a.e. on Ω.
Given $\alpha \in \mathbb{R}$.

$$\|\alpha f\|^p_{L^p(\Omega)} = \int_\Omega |\alpha f|^p = \int_\Omega |\alpha|^p |f|^p = |\alpha|^p \int_\Omega |f|^p = |\alpha|^p \cdot \|f\|^p_{L^p(\Omega)}$$

$$\begin{aligned}\|f+g\|^p_{L^p(\Omega)} &= \int_\Omega |f+g|^p \\ &\leq \int_\Omega \left(|f|^p + |g|^p\right) = \int_\Omega |f|^p + \int_\Omega |g|^p = \|f\|^p_{L^p(\Omega)} + \|g\|^p_{L^p(\Omega)}\end{aligned}$$

Hence, $\|\cdot\|^p_{L^p(\Omega)}$ defines a norm for each $0 < p < 1$.
Let $f = \chi_{[0,\frac{1}{2})}, g = \chi_{[\frac{1}{2},1]}$, and $\Omega = [0,1]$.

$$\begin{aligned}1 = &\left(\int_{[0,1]} 1^p \ dx\right)^{1/p} = \|f+g\|_{L^p([0,1])} \\ \leq &\ \|f\|_{L^p([0,1])} + \|g\|_{L^p([0,1])} = \left(\int_{[0,1]} \chi_{[0,\frac{1}{2})}\right)^{1/p} + \left(\int_{[0,1]} \chi_{[\frac{1}{2},1]}\right)^{1/p} \\ &= \left(\int_{[0,\frac{1}{2})} 1^p\right)^{1/p} \left(\int_{[\frac{1}{2},1]} 1^p\right)^{1/p} = 2 \cdot \left(\frac{1}{2}\right)^{1/p} \rightarrow\leftarrow \ \text{for } 0 \leq p \leq 1\end{aligned}$$

Hence, $\|\cdot\|_{L^p(\Omega)}$ does not define a norm for each $0 < p < 1$. $\square$

Define a new operation !

Definition 4.1.1.
The convolution of two measurable functions $f, g : \mathbb{R}^n \to \mathbb{R}$ *is the function* $f * g : \mathbb{R}^n \to \mathbb{R}$ *given by* $(f * g)(x) = \int_{\mathbb{R}^n} f(t)g(x-t)dt$ *provided the integral exists a.e.*
Note : $f * g = g * f$

The convolution is well-defined in the following sense:

Lemma 4.1.6. (*Young's Inequality*)
Let $1 \leq p, q \leq \infty \ni \frac{1}{p} + \frac{1}{q} \geq 1$. *Set* $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$ *so that* $1 \leq r \leq \infty$.
If $x \in L^p(\mathbb{R}^n)$ *and* $y \in L^q(\mathbb{R}^n)$, *then*
$x * y \in L^r(\mathbb{R})$ *and* $\|y * x\|_{L^r(\mathbb{R}^\ltimes)} \leq \|x\|_{L^p(\mathbb{R}^\ltimes)}\|y\|_{L^q(\mathbb{R}^\ltimes)}$.

What is the differentiation ?

Definition 4.1.2.
Let $u \in C(\mathbb{R})$.
The derivative of f *is defined by* $u'(x) = \lim_{h \to 0} \dfrac{u(x+h) - u(x)}{h}$.

Definition 4.1.3. *multi-derivative differentiation*
Let $u \in C(\mathbb{R}^n)$.
Let e_j *be the* j^{th}*-column of the* $n \times n$ *identity matrix, and the partial derivatives are (formally) defined by* $\partial_j u(x) := \lim_{h \to 0} \dfrac{u(x + he_j) - u(x)}{h}$.

We need some condition to assure that the multi differentiation holds !

Lemma 4.1.7.
Let $u \in C(\mathbb{R}^n)$.
If both partial derivatives $\partial_{i_1} u$ *and* $\partial_{i_2} u$ *exist near a point* x *and if both* $\partial_{i_1} u\ \partial_{i_2} u$ *and* $\partial_{i_2} u\ \partial_{i_1} u$ *are continuous at* x, *then* $\partial_{i_1}\partial_{i_2} u(x) = \partial_{i_2}\partial_{i_1} u(x)$

We usually denote $\partial_{i_1}\partial_{i_2} = \partial_{i_1}\partial_{i_2} = \partial_{i_2}\partial_{i_1}$.
How about multi-index in higher order derivatives ?

Definition 4.1.4.
For each multi-index $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{Z}^n_{\geq 0}$, *we define*
$supp(\alpha) = \{j \in \{1, 2, \ldots, n\}; \alpha_j \neq 0\}$ *and* $|\alpha| := \sum_{j=1}^n \alpha_j$, *as well as*
$\partial^\alpha := \prod_{j \in \{1,2,\ldots,n\}} \partial_j^{\alpha_j}$ *with the convention* $\partial^{(0,\ldots,0)} := I$, *where* I *is identity matrix and the partial derivatives in* ∂^α *are pairwise commute.*

Definition 4.1.5.
Let $f : \mathbb{R}^n \to \mathbb{R}$.
The support of f is defined to be supp$(f) = \overline{\{x \in \mathbb{R}^n ; f(x) \neq 0\}}$.
i.e., the smallest closed set in $\mathbb{R}^n$ and outside of supp(f) is zero.
Note : f is compact support if supp(f) is compact.

In view of Lemma 4.1.7, for each open set Ω and $k \in \mathbb{Z}_{\geq 0}$, we define the spaces :
1. $C^k(\Omega) := \{u : \Omega \to \mathbb{C}; \partial^\alpha u \text{ is continuous}, \forall \alpha \text{ with } |\alpha| \leq k\}$
2. $C^k(\overline{\Omega}) := \{u|_{\overline{\Omega}} : u \in C^k(U) \text{ for some open set } U \supset \overline{\Omega}\}$
3. $C_c^k(\Omega) := \{u \in C^k(\Omega); supp(u) \subset \Omega \text{ is compact}\}$
4. $C_c^\infty(\Omega) := \{u \in C^\infty(\Omega); supp(u) \subset \Omega \text{ is compact}\}$

By considering the zero extension, one can also see that
1. $\mathbb{C}_c^k(\Omega) = \{: u \in C^k(\mathbb{R}^n); supp(u) \subset \Omega \text{ is compact}\}$.
2. $C_c^\infty(\Omega) = \{u \in C^\infty(\mathbb{R}^n); supp(u) \subset \Omega \text{ is compact}\}$.

Lemma 4.1.8.
Let $\Omega \subseteq \mathbb{R}^n$ be an open set.
Then $C_c^\infty(\Omega)$ is dense in $L^p(\Omega)$, for any $1 \leq p < \infty$, i.e., given any $f \in L^p(\Omega)$, $\exists$ a sequence of functions $\{f_k\}_{k\in\mathbb{N}}$ in $C_c^\infty(\Omega) \ni f_k \to f$ in $L^p(\Omega)$.

Let's think about the different FTC by Riemann integral and Lebesgue Integral.

For Riemann integral, if $f \in C^1(a, b)$, then $\begin{cases} \dfrac{d}{dx}\displaystyle\int_{x_0}^{x} f(t)dt = f(x) \\ \displaystyle\int_a^b f'(x)dx = f(b) - f(a) \end{cases}$

For Lebesgue integral, $\displaystyle\int_a^b f'(x)dx = f(b) - f(a)$ iff f is absolute continuous.

Proposition 4.1.1. (*Divergence theorem*)
Let Ω be a bounded domain in $\mathbb{R}^n$ with a piecewise-C^1 boundary $\partial\Omega$.
Let $\nu = (\nu_1, \ldots, \nu_n)$ be the unit outward normal vector on $\partial\Omega$, then $\displaystyle\int_\Omega \partial_i f dx = \int_{\partial\Omega} \nu_i f dS_x$, $\forall f \in C^1(\overline{\Omega})$, where dS_x is the surface element (can be characterized in terms of Hausdorff measure) on $\partial\Omega$.

Exercise 4.9.
Let Ω be a bounded domain in $\mathbb{R}^n$ with a piecewise-C^1 boundary $\partial\Omega$.
Let $\nu = (\nu_1, \ldots, \nu_n)$ be the unit outward normal vector on $\partial\Omega$.
Show that $\displaystyle\int_\Omega \nabla \cdot f \; dx = \int_{\partial\Omega} \nu \cdot f dS_x, \forall f = (f_1, \ldots, f_n) \in C^1(\overline{\Omega})$,
where $\nabla \cdot f(x) := \sum_{i=1}^n \partial_i f_i(x)$ is called the divergence of f.

Using the Divergence Thm and product rule, we can see that
$\displaystyle\int_{\partial\Omega} \nu_i f g \; dS_x = \int_{\Omega} \partial_i(fg) \; dx = \int_{\partial\Omega} \partial_i f(x) g(x) \; dx + \int_{\partial\Omega} f(x)\partial_i g(x) \; dx,$
then we can reach the following useful corollary :

Corollary 4.1.9. (*Integration by parts*)
Let Ω be a bounded domain in $\mathbb{R}^n$ with a piecewise-C^1 boundary $\partial\Omega$.
Let $\nu = (\nu_1, \ldots, \nu_n)$ be the unit outward normal vector on $\partial\Omega$.
Then $\displaystyle\int_{\Omega} \partial_i f(x) g(x) \; dS_x = \int_{\partial\Omega} \nu_i(x) f(x) g(x) \; dS_x - \int_{\Omega} f(X)\partial_i g(x) \; dx, \forall f, g \in C^1(\overline{\Omega}).$

Remark :
We now restrict ourselves when $n = 1$.
When $\Omega = (a, b)$, the above identity simply reads
$\displaystyle\int_a^b f'(x)g(x)dx = f(x)g(x)\Big|_{x=a}^{x=b} - \int_a^b f(x)g'(x)dx$
which is just the usual integration by parts.
In fact, each open set Ω can be written as union of countably many disjoint open intervals, and therefore the above formula can be extended for arbitrary open sets in $\mathbb{R}^1$.
The FTC is simply a special case of divergence theorem. Indeed Corollary 4.1.9 can be extended for general bounded Lipschitz domains and in weak sense (see Thm 5.2.10 below).

Exercise 4.10.
Suppose $n = 2$. *Verify that Green's theorem is a special case of divergence theorem. Let* Ω *be a bounded domain in* $\mathbb{R}^n$ *with a piecewise-*C^1 *boundary* $\partial\Omega$.
Let $\nu = (\nu_1, \ldots, \nu_n)$ *be the unit outward normal vector on* $\partial\Omega$.
The Green's theorem stated that
$\displaystyle\int_\Omega (\partial_x q(x, y) - \partial_y p(x, y))dxdy = \int_{\partial D} (pdx + qdy), \forall p, q \in C^1(\overline{\Omega}),$
where the right-hand-side is the line integral defined by
$\displaystyle\int_{\partial D} (pdx + qdy) := \int_{\partial D} (p(\gamma(s)), q(\gamma(s))) \cdot t(s)ds,$
where $\gamma(s)$ *is the arc-length parametrization of* ∂D *and* $t(s)$ *is the unit tangent vector field* (*usually chosen to be counterclockwise oriented*).

Proof.
Define a vector field $F = (F_1, F_2)$.
Since Green's theorem involves p and q in the line integral $p\ dx+q\ dy$, and $\partial_x q-\partial_y p$, then F is relate to (p, q).
Consider the unit normal vector ν in divergence theorem versus the tangent vector field t in Green's theorem, and the orientation of the boundary.
For a counterclockwise-oriented boundary $\partial\Omega$, the unit tangent $t = (t_1, t_2)$ and the outward normal $\nu = (\nu_1, \nu_2)$ are related.
If $t = \left(\frac{dx}{ds}, \frac{dy}{ds}\right)$ for a parametrization $\gamma(s) = (x(s), y(s))$, then the outward normal is obtained by rotating t 90 degrees clockwise (follows the right-hand rule) can get $\nu = \left(\frac{dy}{ds}, \frac{-dx}{ds}\right)$.
Hence, $t = (-\nu_2, \nu_1)$.
Let $F = (-q, p)$. Then $div(F) = \nabla \cdot F = \frac{\partial}{\partial x}(-q) + \frac{\partial}{\partial y}p$.
By divergence theorem, $\displaystyle\int_\Omega (-\partial_x q + \partial_y p)dx\ dy = \int_{\partial\Omega} F \cdot \nu\ ds$.
For a point on $\partial\Omega$ parametrized by arc length $\gamma(s)$ with
$\nu(s) = \left(\frac{dy}{ds}, \frac{-dx}{ds}\right) = (\nu_2, -\nu_1)$.
Then $F = (-q(\gamma(s)), p(\gamma(s)))$ and $F \cdot \nu = -q \cdot \frac{dy}{ds} + p \cdot \left(-\frac{dx}{ds}\right)$.
$\displaystyle\int_{\partial\Omega} F \cdot \nu\ ds = \int_{\partial\Omega} \left(-q\frac{dy}{ds} - p\frac{dx}{ds}\right)ds = -\int_{\partial\Omega} (p\ dx + q\ dy) = -\int_{\partial\Omega} (p, q) \cdot t\ ds.$
Hence, $\displaystyle -\int_\Omega (-\partial_x q + \partial_y p)dxdy = \int_{\partial\Omega} (p\ dx + q\ dy).$ $\square$

4.2 What Is Partial Differential Equations

In many cases, it is not convenient to write down the function explicitly.
For example, for the function $u(t) := \sin^{-1} t$ for $t \in (-1, 1)$, it is more convenient to write it as $\sin(u(t)) = t$.
Taking derivative on both sides of the above equation, or in some fancy words "performing implicit differentiation", we reach $\cos(u(t))u'(t)$.
If we write $F(t, u, u') := \cos(u(t))u'(t) - 1$, then we see that the above equation is simply a special case of the following first-order ordinary differential equation : $F(t, u, u') = 0$.
In general, for any $k \in \mathbb{N}$, where $\mathbb{N} = \{1, 2, 3, \cdots\}$, the most general k^{th}-order ordinary differential equation (ODE) takes the form $F(t, u, u', \ldots, u^{(k)}) = 0$, where $u(k)$ is the k^{th}-derivative of u.
The key defining property of a partial differential equation is that there is more than one independent variable $x_1, x_2, \cdots, x_n$ ($n \in \mathbb{N}$).
Similar as above, we now introduce the following definition.

Definition 4.2.1.
The general k^{th}-order partial differential equation (PDE) takes the form
$F(x, \{\partial^\alpha u\}_{|\alpha|\leq k}) \equiv F(x, \{\partial^\alpha u\}_{|\alpha|=1}, \ldots, \{\partial^\alpha u\}_{|\alpha|=k}) = 0.$
A solution of the function u that satisfies the equation identically in some region (open sets) in $\mathbb{R}^n$.

In view of Lma 4.1.7, the above definition is at least well-defined for C^k-solutions of function u. It is convenient to write the PDE in operator form: We write $\mathscr{L}u := F(x, \{\partial^\alpha u\}_{|\alpha|\leq k})$, where F is the function given in Def'n 4.2.1.

Definition 4.2.2.
Given any function g, and we consider a PDE $\mathscr{L}u = g$.
If $g \equiv 0$, then we say that the PDE is homogeneous.
We say that the PDE is :
(1) *linear when* $(\mathscr{L}u)(x) = \sum_{|\alpha|\leq k} c_\alpha(x)\partial^\alpha u(x)$ *for some functions* c_α
(2) *semilinear when* $(\mathscr{L}u)(x) = \sum_{|\alpha|=k} c_\alpha(x)\partial^\alpha u(x) + G(x, \{\partial^\alpha u\}_{|\alpha|<k})$
(3) *quasilinear when* $(\mathscr{L}u)(x) = \sum_{|\alpha|=k} c_\alpha(x, \{\partial^\alpha u\}_{|\alpha|<k})\partial^\alpha u(x)$
Note :
If $\mathscr{L}$ is semilinear, sometimes we refer $(\mathscr{L}_{prin}u)(x) = \sum_{|\alpha|=k} c_\alpha(x)\partial^\alpha u(x)$ is the principal part of $\mathscr{L}$.

Example 4.2.1.
1. *The transport equation* $\sum_{|\alpha|=1} \partial^\alpha u = 1$ *is a first order linear PDE.*
2. *The Laplace equation* $\partial_1^2 u + \cdots + \partial_n^2 u = 0$ *is a second order linear PDE. We often denote the Laplace operator (or Laplacian) by* $\Delta u = 0$.

3. *The diffusion/heat/caloric equation* $\Theta_t u - \Delta u = 0$ *is a second order linear PDE.*
4. *The Korteweg-deVries (KdV) equation* $\partial_t u + \partial_x^3 u + 6u\partial_x u = 0$ *is a third order semilinear PDE.*

4.3 First Order PDE

4.3.1 Transport equation

Example 4.3.1. *Transport equation*
Given a horizontal pipe of fixed cross section in the (positive) x-direction.
Suppose that there is a fluid flowing at a constant rate c ($c = 0$ means the fluid is stationary; $c > 0$ means flowing toward right, otherwise towards left).
We now assume that there is a substance is suspended in the water.
Fix a point at the pipe, and we set the point as the origin 0, and let $u(t, x)$ be the concentration of such substance.
The amount of pollutant in the interval $[0, y]$ at time t is given by $\int_0^y u(t, x)dx$.
At the later time $t+\tau$, the same molecules of pollutant moved by the displacement $c\tau$, and this means $\int_0^y u(t, x)dx = \int_{c\tau}^{y+c\tau} u(t+\tau, x)dx$.
If u is continuous, by using the FTC, by differentiating the above equation with respect to y, one sees that $u(t, y) = u(t+\tau, y+c\tau), \forall y \in \mathbb{R}$.
If we further assume $u \in C^1$, then differentiating the equation above with respect to τ, we reach the following transport equation :
$0 = u(t+\tau, y+c\tau)|_{\tau=0} = \partial_t u(t, x) + c\partial_x u(t, x), \forall (t, x) \in \mathbb{R} \times \mathbb{R}$.

4.3.2 The constant coefficient equation

The simplest possible PDE is $\partial_t u(t, x) = 0$.
Its general solution is $u(t, x) = f(x)$, where f is any function of one variables. Because the solutions are independent of t, they are constant on the lines $x =$ constant in the (t, x)-plane.
Let $0 \neq c$ is a constant.
Goal : solve the transport equation $\partial_t u + c\partial_x u = 0$ for a function $u = u(t, x) \in C^1(\mathbb{R} \times \mathbb{R})$.

4.3.3 The variable coefficient equation

4.4 Linear PDE of Second Order

We usually classify them by considering its principal part (i.e. the term with highest order derivatives).
For simplicity, let us consider the principal part of the constant coefficient case :
$A : \nabla^{\otimes 2} u := \sum_{i,j=1}^{n} a_{ij}\partial_i\partial_j u.$

Suppose that the matrix $A = (a_{ij})$ is real symmetric.
It is well-known that A is unitary diagonalizable, i.e. $\exists$ an invertible Q with $Q^{-1} = Q^T \ni A = QDQ^T$ for some diagonal matrix $D = diag(\lambda_1, \ldots, \lambda_n)$, where λ_j are called the eigenvalues of the matrix A.

Some classifications of the second order linear PDE :
1. 2^{nd} order linear PDE is elliptic if $\lambda_j > 0, \forall 1 \leq j \leq n$.
2. 2^{nd} order linear PDE is parabolic if $\lambda_{j_0} = 0$ and $\lambda_j > 0, \forall j \in \{1, \ldots, n\} \setminus \{j_0\}$
3. 2^{nd} order linear PDE is hyperbolic if $\lambda_{j_0} < 0$ and $\lambda_j > 0, \forall j \in \{1, \ldots, n\} \setminus \{j_0\}$
4. 2^{nd} order linear PDE is ultrahyperbolic if $n \geq 4$ and $\exists$ 4 different indices $j_0, j_0', j_1, j_1' \in \{1, \ldots, n\} \ni \lambda_{j_0} < 0, \lambda_{j_0'} < 0, \lambda_{j_1} > 0$ and $\lambda_{j_1'} > 0$.
Note : the above are not complete classifications of the second order linear PDE.

Exercise 4.11.
Let A be a real symmetric matrix.
Show that all its eigenvalue are positive iff $A\xi \cdot \xi \equiv \xi^T A\xi > 0, \forall \xi \in \mathbb{R}^n \setminus \{0\}$.

Proof.
Given $\xi \in \mathbb{R}^n \setminus \{0\}$.
Suppose λ is an eigenvalue of A.
Then $A\xi = \lambda\xi \Rightarrow \xi^T A\xi = \xi^T \lambda\xi = \lambda\xi^T\xi > 0$.
Suppose $\exists\ \lambda < 0$ is an eigenvalue of $A \ni A\xi = \lambda\xi$.
Then $A\xi = \lambda\xi \Rightarrow \xi^T A\xi = \xi^T \lambda\xi = \lambda\xi^T\xi < 0 \rightarrow\leftarrow$.
Hence, all the eigenvalue of A are positive iff $A\xi \cdot \xi \equiv \xi^T A\xi > 0, \forall \xi \in \mathbb{R}^n \setminus \{0\}$. $\square$

Remark :
If we consider the linear second order PDE with principal part ...

Exercise 4.12.
Let $n = 2$. Classify each of the equations :

Exercise 4.13.
Classify each of the equations :

Remark :
Many authors would prefer put a minus sign in front of the Laplacian Δ (or the elliptic operator (2.3.1b)) as indicated in Exercise 2.3.4, due to the maximum principle (Lemma 3.5.5), eigenvalue decomposition (Theorem 3.6.4) as well as Fourier transform (Exercise 4.2.13) below.
This minus sign is actually come from the integration by parts (Corollary 1.0.19).

4.5 Wave equation

4.5.1 1-dimensional Wave Equation on the Whole line $\mathbb{R}$

4.5.2 1-dimensional Wave Equation on the Half-line $(0, \infty)$

4.5.3 1-dimensional Wave Equation on the Finite Interval $(0, L)$

4.5.4 1-dimensional Wave with an External Source : Duhamel's Principle

4.5.5 n-dimensional wave equation in space time

5 Partial Differential Equations in Weak Sense

5.1 Weak Derivatives and Distribution Derivatives

In practical application, we should expect there are singularities in solution, for example :
1. the general solution of transport equation, which in general need not to be C^1.
2. the general solution of $1D$ wave equation, i.e., d'Alembert formula, which in general need not to be C^2.
3. the general solution of $3D$ wave equation, i.e., Kirchhoff's formula, which in general need not to be C^2.

One simplest way to interpret weak solutions is directly write down the explicit solution. However, this idea is difficult in general. Therefore we need some systematic way to interpret the "weak solutions".

Recall :
Let Ω be an open set in $\mathbb{R}^n$.
By using integration by parts formula, $\displaystyle\int_\Omega (\partial_j f)\varphi = -\int_\Omega f(\partial_j \varphi), \forall f, \varphi \in C_c^1(\Omega)$.

Exercise 5.1.
Show that $\displaystyle\int_\Omega (\partial^\alpha f)\varphi = (-1)^{|\alpha|}\int_\Omega f(\partial^\alpha \varphi), \forall \text{multi-indices } \alpha.$

Example 5.1.1.
Let $f(x) = \frac{1}{x}, \forall x \in \mathbb{R}$. *Then* $\displaystyle\int_{\mathbb{R}} \frac{1}{x}dx = \infty$.
Hence, we need to consider the domain locally

Therefore, it is quite natural to consider the following definition (and we can interpret the PDE using the following weak derivatives) :

Definition 5.1.1. (*Weak Derivatives*)
We define the locally-L^1 space by
$L^1_{loc}(\Omega) := \left\{ f \text{ is defined on } \Omega; \|f\|_K := \displaystyle\int_K |f(x)|dx < \infty, \forall \text{ compact set } K \subset \Omega \right\}.$
A function $g \in L^1_{loc}(\Omega)$ (if exists) is called a weak derivative of $f \in L^1_{loc}(\Omega)$ (of order α) if $\displaystyle\int_\Omega g\varphi = (-1)^{|\alpha|}\int_\Omega f(\partial^\alpha \varphi), \forall \varphi \in C_c^\infty(\Omega)$.
Note : we often denote g as $\partial^\alpha f$ (in a.e. sense).

Note : $L^1_{loc}(\Omega)$ is not Banach space.

現在還不確定弱導數定義是否合理
Q : Is it possible to find $g_1, g_2 \in L^1_{loc}(\Omega)$ are both weak derivative of $f \in L^1_{loc}(\Omega)$?
If yes, then $\displaystyle\int_\Omega (g_1 - g_2)\varphi = 0, \forall \varphi \in C_c^\infty(\Omega)$
The well-definedness of the weak derivatives (i.e., each function $g \in L^1_{loc}(\Omega)$ produced from $f \in L^1_{loc}(\Omega)$ must be unique) is guaranteed by the following lemma.

Lemma 5.1.1. (*Uniqueness of weak derivatives*)
If $g \in L^1_{loc}(\Omega)$ satisfying $\displaystyle\int_\Omega g\varphi = 0, \forall \varphi \in C_c^\infty(\Omega)$, then $g = 0$ a.e. in Ω.
i.e, $g \in L^1_{loc}(\Omega)$ satisfying $g = 0$ in $\mathscr{D}'(\Omega) \Rightarrow g = 0$ a.e. in Ω.

Remark : The converse of Lemma 5.1.1 is trivial.
i.e, $g \in L^1_{loc}(\Omega) \Rightarrow g = 0$ in $\mathscr{D}'(\Omega)$ iff $g = 0$ a.e. in Ω.
Here and after, we shall omit the notation "a.e." if there is no any ambiguity.
The above lemma only guarantee uniqueness, but not existence.

Example 5.1.2.
Consider the Heaviside function $H(x) := \begin{cases} 1, \forall x > 0 \\ 0, \forall x \leq 0 \end{cases}.$
It is easy to see that $H \in L^1_{loc}(\mathbb{R})$.
We define $f(x) := \begin{cases} x, \forall x > 0 \\ 0, \forall x \leq 0 \end{cases}$. *Then $f \in L^1_{loc}(\mathbb{R})$.*
$\forall \varphi \in C_c^\infty(\mathbb{R})$,

$$\begin{aligned} -\int_{\mathbb{R}} f(x)\varphi'(x)\ dx &= -\int_0^\infty x\varphi'(x)\ dx \\ &= -\lim_{k\to\infty} x\varphi(x)\Big|_{x=0}^{k} + \int_0^\infty \varphi(x)\ dx = \int_{\mathbb{R}} H(x)\varphi(x)\ dx \end{aligned}$$

Hence, the Heaviside function $H(x)$ is the weak derivative of f (of order 1),
i.e, $f'(x) = H(x)$ a.e. in $\mathbb{R}$.

Example 5.1.3. *Not all $L^1_{loc}(\Omega)$ function admits weak derivatives* (反證法)
claim : the weak derivative of the Heaviside function H of order 1 does not exist.
Suppose the contrary, that H has a weak derivative of order 1, says $g \in L^1_{loc}(\Omega)$.
By definition 5.1.1 and FTC,
$$\int_{-\infty}^{\infty} g(x)\varphi(x)\ dx = -\int_{-\infty}^{\infty} H(x)\varphi'(x)\ dx = -\int_{0}^{\infty} \varphi'(x)\ dx = \varphi(0), \forall \varphi \in C_c^\infty(\mathbb{R})$$
$$\Rightarrow \int_{-\infty}^{\infty} g(x)\varphi(x)\ dx = 0, \forall \varphi \in C_c^\infty(\mathbb{R}\setminus\{0\})\ (\textit{Restrict } \mathbb{R} \textit{ to } \mathbb{R}\setminus\{0\})$$
$$\Rightarrow g = 0 \textit{ a.e. in } \mathbb{R}\setminus\{0\}\ (\textit{by Lma 5.1.1})$$
$$\Rightarrow g = 0 \textit{ a.e. in } \mathbb{R}$$
$$\Rightarrow \varphi(0) = 0, \forall \varphi \in C_c^\infty(\mathbb{R}) \to\leftarrow$$

Since the weak derivatives may not exist, it is much more convenient to consider a generalization the weak derivatives, called the distribution derivatives.
To do so, we need to explain what is a distribution (or generalized functions).

Definition 5.1.2.
Fixing any compact set $K \subseteq \mathbb{R}^n$.
We denote $\mathscr{D}_K := \{\varphi \in C_c^\infty(\mathbb{R}^n) : supp(\varphi) \subset K\}$ is a distribution.

Definition 5.1.3.
Given a vector space X over a subfield F of the complex numbers $\mathbb{C}$, a norm on X is a function $\|\cdot\| : X \to \mathbb{R}$ with the following properties :
1. Positive definiteness : $\|x\| \geq 0, \forall x \in X$ and $\|x\| = 0$ iff $x = 0$.
2. Absolute homogeneity : $\|sx\| = |s|\|x\|, \forall x \in X$ and scalars $s \in F$.
3. Subadditivity / Triangle inequality : $\|x+y\| \leq \|x\| + \|y\|, \forall x, y \in X$.
In this case, we call the pair $(X, \|\cdot\|)$ is the normed space.
A sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ is Cauchy if $\forall \varepsilon > 0, \exists N \in \mathbb{N} \ni \forall n, m \geq N, \|x_n - x_m\| < \varepsilon$
The normed space $(X, \|\cdot\|)$ is complete or Banach if each Cauchy sequence converges in X, i.e., for each Cauchy sequence $\{x_n\}_{n\in\mathbb{N}} \subset X, \exists x \in X \ni \|x_n - x\| \to 0$.
complete 即是能夠將 *limit point* 表示出來

For each fixed $n \in \mathbb{Z}_{\geq 0}$ and any compact set $K \subseteq \mathbb{R}^n$., it is easy to see that $\|\varphi\|_{n,K} := \sum_{|\alpha|\leq n} \left\|\partial^\alpha \varphi\right\|_{L^\infty(K)}$ is a norm defined on $\mathscr{D}_K$.
However, the normed space $(\mathscr{D}_K, \|\cdot\|, K)$ is not complete, since the norm does not involve derivatives of order $> N$.
We can further generalize the notion "norm" in the following definition :

Definition 5.1.4.
Given a set M, a metric is a function $d : M \times M \to \mathbb{R}$ satisfies :
1. Positive definiteness : $d(x, y) \geq 0, \forall x, y \in M$ and $d(x, y) = 0$ iff $x = y$.
2. Symmetry : $d(x, y) = d(y, x), \forall x, y \in M$.
3. Subadditivity / Triangle inequality : $d(x, y) \leq d(x, z) + d(z, y), \forall x, y, z \in M$.
In this case, we call the pair (M, d) the metric space.
A sequence $\{x_n\}_{n\in\mathbb{N}} \subset M$ is Cauchy if $\forall \varepsilon > 0, \exists N \in \mathbb{N} \ni \forall n, m \geq N, d(x_n, x_m) < \varepsilon$
The normed space $M, d)$ is complete or Frechet if each Cauchy sequence converges in X, i.e., for each Cauchy sequence $\{x_n\}_{n\in\mathbb{N}} \subset M, \exists x \in M \ni d(x_n, x) \to 0$.

Exercise 5.2.
Show that each normed space $(X, \|\cdot\|)$ is also a metric space.

Proof.
Define the metric $d : X \times X \to [0, \infty)$ by $d(x, y) = \|x - y\|$.
claim : $d(x, y) \geq 0$, and $d(x, y) = 0$ iff $x = y$.
Since $\|\cdot\|$ is a norm, then $d(x, y) = \|x - y\| \geq 0, \forall x, y \in X$.
If $d(x, y) = 0$, then $\|x - y\| = d(x, y) = 0 \Rightarrow x - y = 0 \Rightarrow x = y$.
If $x = y$, then $x - y = 0 \Rightarrow d(x, y) = \|x - y\| = \|0\| = 0$.
claim : $d(x, y) = d(y, x), \forall x, y \in X$.
$d(x, y) = \|x - y\| = \|(-1)(y - x)\| = |-1|\|y - x\| = \|y - x\| = d(y, x), \forall x, y \in X$.
claim : $d(x, y) \leq d(x, z) + d(z, y), \forall x, y, z \in X$.

$$\begin{aligned} d(x, y) = \|x - y\| &= \|x - z + z - y\| \\ &\leq \|x - z\| + \|z - y\| = d(x, z) + d(z, y), \forall x, y, z \in X. \end{aligned}$$

Hence, each normed space $(X, \|\cdot\|)$ is also a metric space. $\square$

Lemma 5.1.2.
$\mathscr{D}_K$ is Frechet equipped with the metric $d_{\mathscr{D}_K}(\varphi, \psi) := \displaystyle\sum_{n=0}^{\infty} 2^{-n} \frac{\|\varphi - \psi\|_{n,K}}{1 + \|\varphi - \psi\|_{n,K}}$.

Remark :
Since $\mathscr{D}_K$ is a (Grothedieck) nuclear space, then it is not possible to find a norm which is complete, i.e., if we equipped $\mathscr{D}_K$ by any norm $\|\cdot\|$, then $(\mathscr{D}_K, \|\cdot\|)$ cannot be Banach.

Exercise 5.3.
Verify that $d_{\mathscr{D}_K}$ is a metric.

Proof.
claim : $d_{\mathscr{D}_K}(\varphi,\psi) \geq 0, \forall \varphi,\psi \in \mathscr{D}_K$.

$$d_{\mathscr{D}_K}(\varphi,\psi) = \sum_{n=0}^{\infty} 2^{-n} \frac{\|\varphi-\psi\|_{n,K}}{1+\|\varphi-\psi\|_{n,K}} = \sum_{n=0}^{\infty} 2^{-n} \frac{\sum_{|\alpha|\leq n} \left\|\partial^\alpha(\varphi-\psi)\right\|_{L^\infty(K)}}{1+\sum_{|\alpha|\leq n} \left\|\partial^\alpha(\varphi-\psi)\right\|_{L^\infty(K)}}$$
$$= \sum_{n=0}^{\infty} 2^{-n} \frac{\sum_{|\alpha|\leq n} \sup_{x\in K} \left|\partial^\alpha(\varphi(x)-\psi(x))\right|}{1+\sum_{|\alpha|\leq n} \sup_{x\in K} \left|\partial^\alpha(\varphi(x)-\psi(x))\right|} \geq 0$$

claim : $d_{\mathscr{D}_K}(\varphi,\psi) = 0$ iff $\varphi = \psi, \forall \varphi,\psi \in \mathscr{D}_K$.
Suppose $\varphi = \psi$. Then $\varphi - \psi = 0 \Rightarrow \partial^\alpha(\varphi-\psi), \forall \alpha$.
Hence, $\sum_{|\alpha|\leq n} \sup_{x\in K} \left|\partial^\alpha(\varphi(x)-\psi(x))\right| = 0 \Rightarrow d_{\mathscr{D}_K}(\varphi,\psi) = 0$.
Suppose $d_{\mathscr{D}_K}(\varphi,\psi) = 0$.
Then $\displaystyle\sum_{n=0}^{\infty} 2^{-n} \frac{\sum_{|\alpha|\leq n} \left\|\partial^\alpha(\varphi-\psi)\right\|_{L^\infty(K)}}{1+\sum_{|\alpha|\leq n} \left\|\partial^\alpha(\varphi-\psi)\right\|_{L^\infty(K)}} = 0.$
Since $2^{-n} > 0, \forall n \in \mathbb{N}$ and $\displaystyle\frac{\sum_{|\alpha|\leq n} \left\|\partial^\alpha(\varphi-\psi)\right\|_{L^\infty(K)}}{1+\sum_{|\alpha|\leq n} \left\|\partial^\alpha(\varphi-\psi)\right\|_{L^\infty(K)}} \geq 0$. Then

$$\frac{\sum_{|\alpha|\leq n} \left\|\partial^\alpha(\varphi-\psi)\right\|_{L^\infty(K)}}{1+\sum_{|\alpha|\leq n} \left\|\partial^\alpha(\varphi-\psi)\right\|_{L^\infty(K)}} = 0$$
$$\Rightarrow \sum_{|\alpha|\leq n} \left\|\partial^\alpha(\varphi-\psi)\right\|_{L^\infty(K)} = 0$$
$$\Rightarrow \sum_{|\alpha|\leq n} \sup_{x\in K} \left|\partial^\alpha(\varphi(x)-\psi(x))\right| = 0$$
$$\Rightarrow \varphi = \psi$$

Hence, $d_{\mathscr{D}_K}(\varphi,\psi) = 0$ iff $\varphi = \psi, \forall \varphi,\psi \in \mathscr{D}_K$.
claim : $d_{\mathscr{D}_K}(\varphi,\psi) = d_{\mathscr{D}_K}(\psi,\varphi), \forall \varphi,\psi \in \mathscr{D}_K$.

$$d_{\mathscr{D}_K}(\varphi,\psi) = \sum_{n=0}^{\infty} 2^{-n} \frac{\sum_{|\alpha|\leq n} \sup_{x\in K} \left|\partial^\alpha(\varphi(x)-\psi(x))\right|}{1+\sum_{|\alpha|\leq n} \sup_{x\in K} \left|\partial^\alpha(\varphi(x)-\psi(x))\right|}$$
$$= \sum_{n=0}^{\infty} 2^{-n} \frac{\sum_{|\alpha|\leq n} \sup_{x\in K} \left|\partial^\alpha(\psi(x)-\varphi(x))\right|}{1+\sum_{|\alpha|\leq n} \sup_{x\in K} \left|\partial^\alpha(\psi(x)-\varphi(x))\right|} = d_{\mathscr{D}_K}(\psi,\varphi)$$

claim : $d_{\mathscr{D}_K}(\varphi,\psi) \leq d_{\mathscr{D}_K}(\varphi,\phi) + d_{\mathscr{D}_K}(\phi,\psi), \forall \varphi,\psi,\phi \in \mathscr{D}_K$.

$$
\begin{aligned}
d_{\mathscr{D}_K}(\varphi,\psi) = & \sum_{n=0}^{\infty} 2^{-n} \frac{\sum_{|\alpha|\le n} \sup_{x\in K} \left|\partial^\alpha(\varphi(x)-\psi(x))\right|}{1+\sum_{|\alpha|\le n} \sup_{x\in K} \left|\partial^\alpha(\varphi(x)-\psi(x))\right|} \\
= & \sum_{n=0}^{\infty} 2^{-n} \left(1 - \frac{1}{1+\sum_{|\alpha|\le n} \sup_{x\in K} \left|\partial^\alpha(\varphi(x)-\psi(x))\right|}\right) \\
= & \sum_{n=0}^{\infty} 2^{-n} \left(1 - \frac{1}{1+\sum_{|\alpha|\le n} \sup_{x\in K} \left|\partial^\alpha(\varphi(x)-\phi(x)+\phi(x)-\psi(x))\right|}\right) \\
\le & \sum_{n=0}^{\infty} 2^{-n} \left(1 - \frac{1}{1+\sum_{|\alpha|\le n} \sup_{x\in K} \left|\partial^\alpha(\varphi(x)-\phi(x))+\partial^\alpha(\phi(x)-\psi(x))\right|}\right) \\
\le & \sum_{n=0}^{\infty} 2^{-n} \left(1 - \frac{1}{1+\sum_{|\alpha|\le n} \left(\sup_{x\in K} \left|\partial^\alpha(\varphi(x)-\phi(x))\right| + \sup_{x\in K} \left|\partial^\alpha(\phi(x)-\psi(x))\right|\right)}\right) \\
= & \sum_{n=0}^{\infty} 2^{-n} \frac{\sum_{|\alpha|\le n} \left(\sup_{x\in K} \left|\partial^\alpha(\varphi(x)-\phi(x))\right| + \sup_{x\in K} \left|\partial^\alpha(\phi(x)-\psi(x))\right|\right)}{1+\sum_{|\alpha|\le n} \left(\sup_{x\in K} \left|\partial^\alpha(\varphi(x)-\phi(x))\right| + \sup_{x\in K} \left|\partial^\alpha(\phi(x)-\psi(x))\right|\right)} \\
= & \sum_{n=0}^{\infty} 2^{-n} \frac{\sum_{|\alpha|\le n} \sup_{x\in K} \left|\partial^\alpha(\varphi(x)-\phi(x))\right|}{1+\sum_{|\alpha|\le n} \left(\sup_{x\in K} \left|\partial^\alpha(\varphi(x)-\phi(x))\right| + \sup_{x\in K} \left|\partial^\alpha(\phi(x)-\psi(x))\right|\right)} \\
& + \sum_{n=0}^{\infty} 2^{-n} \frac{\sum_{|\alpha|\le n} \sup_{x\in K} \left|\partial^\alpha(\phi(x)-\psi(x))\right|}{1+\sum_{|\alpha|\le n} \left(\sup_{x\in K} \left|\partial^\alpha(\varphi(x)-\phi(x))\right| + \sup_{x\in K} \left|\partial^\alpha(\phi(x)-\psi(x))\right|\right)} \\
\le & \sum_{n=0}^{\infty} 2^{-n} \frac{\sum_{|\alpha|\le n} \sup_{x\in K} \left|\partial^\alpha(\varphi(x)-\phi(x))\right|}{1+\sum_{|\alpha|\le n} \sup_{x\in K} \left|\partial^\alpha(\phi(x)-\psi(x))\right|} \\
& + \sum_{n=0}^{\infty} 2^{-n} \frac{\sum_{|\alpha|\le n} \sup_{x\in K} \left|\partial^\alpha(\phi(x)-\psi(x))\right|}{1+\sum_{|\alpha|\le n} \sup_{x\in K} \left|\partial^\alpha(\phi(x)-\psi(x))\right|} \\
= & d_{\mathscr{D}_K}(\varphi,\phi) + d_{\mathscr{D}_K}(\phi,\psi)
\end{aligned}
$$

Hence, $d_{\mathscr{D}_K}(\varphi,\psi) \le d_{\mathscr{D}_K}(\varphi,\phi) + d_{\mathscr{D}_K}(\phi,\psi), \forall \varphi,\psi,\phi \in \mathscr{D}_K$. □

Definition 5.1.5.
Let $\Omega \subset \mathbb{R}^n$ be an open set.
We now define the set of test functions by $\mathscr{D}(\Omega) := \bigcup_{K\subset\Omega \text{ is compact}} \mathscr{D}_K$.

If we view $\mathscr{D}(\Omega)$ as a set, then it is easy to see that $\mathscr{D}(\Omega) = C_c^\infty(\Omega)$.
Of course, one can equipped $\mathscr{D}(\Omega)$ by the metric

$$d_{\mathscr{D}(\Omega)} := \sum_{n=0}^{\infty} 2^{-n} \frac{\|\varphi-\psi\|_{n,\Omega}}{1+\|\varphi-\psi\|_{n,\Omega}}, \ \|\varphi\|_{n,\Omega} := \sum_{|\alpha|\le n} \|\partial^\alpha\varphi\|_{L^\infty(\Omega)}.$$

However, unlike Lemma 5.1.2, $(\mathscr{D}(\Omega), d_{\mathscr{D}(\Omega)})$ is not complete.

Exercise 5.4.
Take $n = 1$ and $\Omega = \mathbb{R}$. Let $\phi \in \mathscr{D}(\mathbb{R})$ with $supp(\phi) \subset [0, 1]$ and $\phi > 0$ in $(0, 1)$.
Define $\Psi_m(x) := \phi(x-1) + \dfrac{1}{2}\phi(x-2) + \cdots + \dfrac{1}{m}\phi(x-m)$.
Show that $\{\Psi_m\}$ s a Cauchy sequence in $(\mathscr{D}(\mathbb{R}), d_{\mathscr{D}(\Omega)})$, but the limit $\lim_{m\to\infty} \Psi_m(x)$ does not have compact support.

Proof.
claim : analysis the support of Ψ_m.
Since $supp(\phi) \subset [0, 1]$ and $\phi(x-k)$ is a shifted version of $\phi(x)$.
Then $supp(\phi(x-k)) = [k, k+1]$.
Since $\Psi_m(x) = \sum_{k=1}^m \frac{1}{k}\phi(x-k)$, then $supp(\Psi) = \bigcup_{k=1}^m [k, k+1] = [1, m+1]$.
claim : $\{\Psi_m\}$ is a Cauchy sequence.
Compute the distance $d_{\mathscr{D}(\mathbb{R})}(\Psi_m, \Psi_{m+p})$ for large m and $p \geq 1$.

$$\begin{aligned}
\|\Psi_{m+p} - \Psi_m\|_{n,\mathbb{R}} &= \sum_{|\alpha|\leq n} \left\| \partial^\alpha \big(\Psi_{m+p} - \Psi_m\big) \right\|_{L^\infty(\mathbb{R})} \\
&= \sum_{|\alpha|\leq n} \left\| \partial^\alpha \left(\sum_{k=m+1}^{m+p} \frac{1}{k}\phi(x-k) \right) \right\|_{L^\infty(\mathbb{R})} \\
&= \sum_{|\alpha|\leq n} \left\| \sum_{k=m+1}^{m+p} \frac{1}{k}\partial^\alpha \phi(x-k) \right\|_{L^\infty(\mathbb{R})} \\
&= \sum_{|\alpha|\leq n} \sup_{x\in[m+1,m+p+1]} \left| \sum_{k=m+1}^{m+p} \frac{1}{k}\partial^\alpha \phi(x-k) \right| \\
&= \sum_{|\alpha|\leq n} \sup_{k=m+1,\ldots,m+p} \left| \frac{1}{k}\partial^\alpha \phi(x-k) \right| \\
&= \sum_{|\alpha|\leq n} \left(\sup_{k=m+1,\ldots,m+p} \frac{1}{k} \right) \left\| \partial^\alpha \phi \right\|_{L^\infty(\mathbb{R})} = \frac{1}{m+1} \sum_{|\alpha|\leq n} \left\| \partial^\alpha \phi \right\|_{L^\infty(\mathbb{R})}
\end{aligned}$$

Let $C_n = \sum_{|\alpha|\leq n} \left\| \partial^\alpha \phi \right\|_{L^\infty(\mathbb{R})}$, which is a constant depending only on ϕ and n.
Then $\|\Psi_{m+p} - \Psi_m\|_{n,\mathbb{R}} = \frac{C_n}{m+1}$

$$\begin{aligned}
d_{\mathscr{D}(\mathbb{R})}(\Psi_m, \Psi_{m+p}) &= \sum_{n=0}^\infty 2^{-n} \frac{\|\Psi_{m+p} - \Psi_m\|_{n,\mathbb{R}}}{1 + \|\Psi_{m+p} - \Psi_m\|_{n,\mathbb{R}}} = \sum_{n=0}^\infty 2^{-n} \frac{\frac{C_n}{m+1}}{1 + \frac{C_n}{m+1}} \\
&\leq \sum_{n=0}^\infty 2^{-n} \frac{C_n}{m+1} = \frac{1}{m+1} \sum_{n=0}^\infty \frac{C_n}{2^n}
\end{aligned}$$

Since $\phi \in \mathscr{D}(\mathbb{R}) = C_c^\infty(\mathbb{R})$, then all derivatives $\partial^\alpha \phi$ are bounded.
Hence, $C := \sum_{n=0}^\infty \frac{C_n}{2^n} < \infty \Rightarrow$ as $m \to \infty$, $d_{\mathscr{D}(\mathbb{R})}(\Psi_m, \Psi_{m+p}) \leq \frac{C}{m+1} \to 0$.
Therefore, $\{\Psi_m\}$ is a Cauchy sequence.

claim : $\Psi = \lim_{m\to\infty} \Psi_m \notin (\mathscr{D}(\mathbb{R}), d_{\mathscr{D}(\Omega)})$.
Define $\Psi(x) = \lim_{m\to\infty} \Psi_m(x) = \sum_{k=1}^{\infty} \frac{1}{k}\phi(x-k)$.
For $x < 1, \Psi(x) = 0$.
For $x \geq 1, \Psi(x) = \frac{1}{k}\phi(x-k)$ on $[k, k+1]$ for each k.
Since $supp(\Psi) = [1, \infty)$, which is not compact, so $\Psi \notin \mathscr{D}(\mathbb{R})$. $\square$

Definition 5.1.6.
Given a set X, a topology $\mathscr{T}$ is a collection of subsets of X satisfies :
1. $\varnothing, X \in \mathscr{T}$
2. $\bigcup_{\alpha\in I} A_\alpha \in \mathscr{T}, \forall A_\alpha \in \mathscr{T}$
3. $\bigcap_{n=1}^{N} A_n \in \mathscr{T}, \forall A_n \in \mathscr{T}, N \in \mathbb{N}.$
If $\mathscr{T}$ is a topology on X, then the pair $(X, \mathscr{T})$ is called a topology space.
Each element in $\mathscr{T}$ is called an open set of X (with respect to $\mathscr{T}$).
Note : we usually equip $\mathscr{D}(\Omega)$ by another (locally convex) topology $\mathscr{T}$ in which "Cauchy sequence" do converge, i.e. "complete".

Definition 5.1.7.
We refer the linear mapping $T : (\mathscr{D}(\Omega), \mathscr{T}) \to (\mathbb{R}, |\cdot|)$ as the linear functional on $(\mathscr{D}(\Omega), \mathscr{T})$, and D be an open set in $\mathbb{R}$.
Define the preimage by $T^{-1}(D) := \{f \in \mathscr{D}(\Omega); T(f) \in D\}$.
We called such linear functional T is continuous (with respect to $\mathscr{T}$) if $T^{-1}(D) \in \mathscr{T}$ for each open set D.
The set of continuous linear functionals on $(\mathscr{D}(\Omega), \mathscr{T})$ is denoted by $\mathscr{D}'(\Omega)$ and its elements are called distributions on Ω.

The following lemma gives equivalent characterization of continuity of linear functionals on $\mathscr{D}(\Omega)$.

Lemma 5.1.3.
Let T be a linear functional on $(\mathscr{D}(\Omega), \mathscr{T})$, then the following are equivalent :
1. T is continuous with respect to $\mathscr{T}$
2. $\lim_{j\to\infty} T(\varphi_j) = 0$ *whenever* $\varphi_j \to 0$ *in* $(\mathscr{D}(\Omega), \mathscr{T})$
3. $T|_{\mathscr{D}_K}$ *is continuous for each compact set $K \subset \Omega$ with respect to the metric*
$$d_K = d_{\mathscr{D}(\Omega)} := \sum_{n=0}^{\infty} 2^{-n} \frac{\|\varphi-\psi\|_{n,\Omega}}{1+\|\varphi-\psi\|_{n,\Omega}},\ \|\varphi\|_{n,\Omega} := \sum_{|\alpha|\le n} \|\partial^\alpha \varphi\|_{L^\infty(\Omega)}.$$

For simplicity, we usually denote $\mathscr{D}(\Omega)$, or even $C_c^\infty(\Omega)$, to represent the topological space $(\mathscr{D}(\Omega), \mathscr{T})$, as we will not focus on its topological aspect in this lecture note.

Example 5.1.4.
Each element $f \in L^1_{loc}(\Omega)$ can be identify with $T_f \in \mathscr{D}'(\Omega)$ defined by
$T_f(\varphi) := \displaystyle\int_{\mathbb{R}^n} f(x)\varphi(x)\ dx, \forall \varphi \in C_c^\infty(\Omega)$ *with the estimate*
$|T_f(\varphi)| \le \displaystyle\int_K |f(x)\varphi(x)|\ dx \le ||\varphi||_{L^\infty(K)} \int_K |f(x)|\ dx, \forall \varphi \in \mathscr{D}_K$ *(by Lma 5.1.3 (3)),*
$\forall K \subset \Omega$ *is compact subset, i.e., $K \Subset \Omega$.*
Hence, $T_f \in \mathscr{D}'(\Omega)$.
Therefore, one can simply write $L^1_{loc}(\Omega) \subset \mathscr{D}'(\Omega)$.

Definition 5.1.8.
We say that $T \in L^1_{loc}(\Omega)$ *if a distribution* $T \in \mathscr{D}'(\Omega)$ *can be written in the form of* $T = T_f$, *for some* $f \in L^1_{loc}(\Omega)$.
Note : we have proved the relation between locally-L^1 *and distribution.*

Example 5.1.5. (*Dirac measure*)
Fix a point $x_0 \in \Omega$.
One can verify that the linear functional $T(\varphi) := \varphi(x_0)$ *is indeed continuous, i.e.,* $T \in \mathscr{D}'(\Omega)$.
We usually denote such distribution T *by* δ_{x_0}.
However, $\delta_{x_0} \notin L^1_{loc}(\Omega)$ *if we consider the identification given in Example 5.1.4.*
Suppose the contrary, $\exists f \in L^1_{loc}(\Omega) \ni \varphi(x_0) = \delta_{x_0}(\varphi) = \int_\Omega f(x)\varphi(x)\ dx, \forall \varphi \in C_c^\infty(\Omega)$.
If we choose $\varphi \in C_c^\infty(\Omega \setminus \{x_0\})$, *i.e.,* $\varphi(x_0) = 0, \forall \varphi \in C_c^\infty(\Omega)$,
then by Lemma 5.1.1 that $f = 0$ *a.e. in* Ω. $\rightarrow\leftarrow$

Definition 5.1.9. *Define the distribution derivatives*
For any $T \in \mathscr{D}'(\Omega)$, *the distribution derivative* $\partial^\alpha T \in \mathscr{D}'(\Omega)$ *of* T *is defined by* $(\partial^\alpha T)(\varphi) := (-1)^{|\alpha|} T(\partial^\alpha \varphi), \forall \varphi \in \mathscr{D}(\Omega)$.
Note : unlike weak derivatives, distribution derivative always exist.

Consider the special case of distribution function $T \in L^1_{loc}(\Omega)$,
i.e., $\exists! f \in L^1_{loc}(\Omega) \ni T(\varphi) = \int_\Omega f(x)\varphi(x)dx, \forall \varphi \in C_c^\infty(\Omega)$.
Then $(-1)^{|\alpha|} T(\partial^\alpha \varphi) = (-1)^{|\alpha|} \int_\Omega f \partial^\alpha \varphi = \int_\Omega \partial^\alpha f \varphi$.
If f

Example 5.1.6.
Let H *be the Heaviside function.*
Since $H \in L^1_{loc}(\mathbb{R})$, *then we can identify it with the distribution*
$T_H(\varphi) = \int_\mathbb{R} H(x)\varphi(x)\ dx = \int_0^\infty \varphi(x)\ dx, \forall \varphi \in C_c^\infty(\mathbb{R})$.
By definition, one sees that its distributional derivative of order 1, here we denoted by T'_H, *is given by* $T'_H(\varphi) = -T_H(\varphi') = -\int_0^\infty \varphi'(x)\ dx = \varphi(0) = \delta_0(\varphi), \forall \varphi \in C_c^\infty(\mathbb{R})$.
Hence, $T'_H = \delta_0 \in \mathscr{D}'(\mathbb{R})$, *i.e.,* δ_0 *is its distributional derivative.*
Therefore, the weak derivative of H *does not exist.*

Algorithm 3 Standard steps to check weak derivative exists or not

1. compute the distribution derivative.
2. if the function is $L^1_{loc}(\Omega)$, check its derivative is in $L^1_{loc}(\Omega)$ or not.

From now on, we will denote ∂^α the distribution derivative without explicitly mentioning.
In addition, we will use the term "derivative", "differentiation" without mention "distribution sense" explicitly.

Exercise 5.5.
Prove that $\forall c \in \mathbb{R}, (e^{-c|x|})' = -ce^{-cx}H(x) + ce^{cx}H(-x)$ *in* $\mathscr{D}'(\mathbb{R})$.

Proof.
Let $f(x) = e^{-c|x|} = \begin{cases} e^{-cx}, & x \geq 0 \\ e^{cx}, & x < 0 \end{cases}$.
Define the distribution T_f by $T_f(\varphi) = \displaystyle\int_{\mathbb{R}} e^{-c|x|}\varphi(x)\ dx, \forall \varphi \in C_c^\infty(\mathbb{R})$.
Then the distribution derivative $(e^{-c|x|})' = T_f'$ satisfies
$\displaystyle\int_{\mathbb{R}} \big(e^{-c|x|}\big)'\varphi(x)\ dx = -\int_{\mathbb{R}} e^{-c|x|}\varphi'(x)\ dx.$
By integration by parts, let $u = e^{-cx}, dv = \varphi'(x)dx$, and $u = e^{cx}, dv = \varphi'(x)dx$,

$$\begin{aligned}
\int_{\mathbb{R}} e^{-c|x|}\varphi'(x)\ dx = & \int_0^\infty e^{-cx}\varphi'(x)\ dx + \int_{-\infty}^0 e^{cx}\varphi'(x)\ dx \\
= & \left(e^{-cx}\varphi(x)\Big|_0^\infty - \int_0^\infty (-ce^{-cx})\varphi(x)dx\right) \\
& + \left(e^{cx}\varphi(x)\Big|_{-\infty}^0 - \int_{-\infty}^0 (ce^{cx})\varphi(x)dx\right) \\
= & \left(-\varphi(0) + c\int_0^\infty e^{-cx}\varphi(x)dx\right) + \left(\varphi(0) - c\int_{-\infty}^0 e^{cx}\varphi(x)dx\right)
\end{aligned}$$

$\displaystyle\Rightarrow \int_{\mathbb{R}} \big(e^{-c|x|}\big)'\varphi(x)\ dx = -\int_{\mathbb{R}} e^{-c|x|}\varphi'(x)\ dx = -c\int_0^\infty e^{-cx}\varphi(x)dx + c\int_{-\infty}^0 e^{cx}\varphi(x)dx$
claim : compute the distribution of RHS.
Since $H(x) = 1$ for $x > 0$, and $H(-x) = 1$ for $x < 0$.

$$\begin{aligned}
&\int_{\mathbb{R}} \big(-ce^{-cx}H(x) + ce^{cx}H(-x)\big)\varphi(x)dx \\
=&\int_{-\infty}^0 \big(-ce^{-cx}H(x) + ce^{cx}H(-x)\big)\varphi(x)dx + \int_0^\infty \big(-ce^{-cx}H(x) + ce^{cx}H(-x)\big)\varphi(x)dx \\
=&\int_{-\infty}^0 -ce^{-cx}H(x)\varphi(x)dx + \int_{-\infty}^0 ce^{cx}H(-x)\varphi(x)dx \\
&+ \int_0^\infty -ce^{-cx}H(x)\varphi(x)dx + \int_0^\infty ce^{cx}H(-x)\varphi(x)dx \\
=&0 + c\int_{-\infty}^0 e^{cx}H(-x)\varphi(x)dx - c\int_0^\infty e^{-cx}H(x)\varphi(x)dx + 0 \\
=&-c\int_0^\infty e^{-cx}H(x)\varphi(x)dx + c\int_{-\infty}^0 e^{cx}H(-x)\varphi(x)dx
\end{aligned}$$

Therefore, $\forall c \in \mathbb{R}, (e^{-c|x|})' = -ce^{-cx}H(x) + ce^{cx}H(-x)$ in $\mathscr{D}'(\mathbb{R})$. $\square$

Exercise 5.6.

Let $f:\mathbb{R}\to\mathbb{R}$ *be defined by* $f(x)=\begin{cases} x\ln|x|-x, & \text{for } x\neq 0,\\ 0, & \text{for } x=0\end{cases}$

Prove that f *is a continuous function and compute its distributional derivative* f'.

Proof.

claim : $f\in C(\mathbb{R})$.

$$\begin{aligned}
\lim_{x\to 0^-} f(x) &= \lim_{x\to 0^-}\big(x\ln(-x)-x\big)=\lim_{x\to 0^-}\big(x\ln(-x)\big)-\lim_{x\to 0^-}x=\lim_{x\to 0^-}\frac{\ln(-x)}{1/x}-0\\
&= \lim_{x\to 0^-}\frac{1/(-x)}{1/(-x^2)} \text{ (by L'Hopital's rule)}\\
&= \lim_{x\to 0^-} x = 0 = f(0)
\end{aligned}$$

$$\begin{aligned}
\lim_{x\to 0^+} f(x) &= \lim_{x\to 0^+}\big(x\ln x-x\big)=\lim_{x\to 0^+}\big(x\ln x\big)-\lim_{x\to 0^+}x=\lim_{x\to 0^+}\frac{\ln x}{1/x}-0\\
&= \lim_{x\to 0^+}\frac{1/x}{1/(-x^2)} \text{ (by L'Hopital's rule)}\\
&= \lim_{x\to 0^+} -x = 0 = f(0)
\end{aligned}$$

Hence, $f\in C(\mathbb{R})$.

Define the distribution T_f by $T_f(\varphi)=\int_{\mathbb{R}} f(x)\varphi(x)\ dx, \forall\varphi\in C_c^\infty(\mathbb{R})$.

Then the distribution derivative $f'=T_f'$ satisfies $\int_{\mathbb{R}} f'\varphi=-\int_{\mathbb{R}} f\varphi', \forall\varphi\in C_c^\infty(\mathbb{R})$.

By integration by parts,

$$\begin{aligned}
\int_{\mathbb{R}} f(x)\varphi'(x) = &\int_0^\infty \big(x\ln x-x\big)\varphi'(x)dx+\int_{-\infty}^0\big(x\ln(-x)-x\big)\varphi'(x)dx\\
= &\left(\big(x\ln x-x\big)\varphi(x)\Big|_0^\infty-\int_0^\infty\big(\ln x+1-1\big)\varphi(x)dx\right)\\
&+\left(\big(x\ln(-x)-x\big)\varphi(x)\Big|_{-\infty}^0-\int_{-\infty}^0\big(\ln(-x)+1-1\big)\varphi(x)dx\right)
\end{aligned}$$

Since $\varphi(0)=0$ and $\lim_{x\to 0}(x\ln x-x)=0$, then

$$\int_{\mathbb{R}} f(x)\varphi'(x)=-\int_0^\infty \ln x\varphi(x)dx-\int_{-\infty}^0\ln(-x)\varphi(x)dx$$

$$\Rightarrow \int_{\mathbb{R}} f'\varphi=-\int_{\mathbb{R}} f\varphi'=\int_{\mathbb{R}}\ln|x|\varphi'$$

Therefore, $f'=\ln|x|$. ☐

Exercise 5.7.
Define $f \in L^1_{loc}(\mathbb{R})$ *as* $f(x) = \begin{cases} x+a, & \text{if } x > 0 \\ -x, & \text{if } x < 0 \end{cases}$.
Determine whether the weak derivative of f *exists or not for each* $a \in \mathbb{R}$.

Proof.
If $x > 0$, then $f'(x) = 1$. If $x < 0$, then $f'(x) = -1$.
However, f is not defined at $x = 0$.
Hence, f has a jump discontinuous point at $x = 0$.
$f(0^+) - f(0^-) = \lim\limits_{x \to 0^+} f(x) - \lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^+}(x+a) - \lim\limits_{x \to 0^-}(-x) = a - 0.$
For $\varphi \in C_c^\infty(\mathbb{R})$, by integration by parts,

$$\begin{aligned}
\int_{\mathbb{R}} f(x)\varphi'(x)\ dx = & \int_0^\infty f(x)\varphi'(x)\ dx + \int_{-\infty}^0 f(x)\varphi'(x)\ dx \\
= & \int_0^\infty (x+a)\varphi'(x)\ dx + \int_{-\infty}^0 (-x)\varphi'(x)\ dx \\
= & \left((x+a)\varphi(x)\Big|_0^\infty - \int_0^\infty \varphi(x)\ dx \right) \\
& + \left((-x)\varphi(x)\Big|_{-\infty}^0 - \int_{-\infty}^0 -\varphi(x)\ dx \right) \\
= & -a\varphi(0) - \int_0^\infty \varphi(x)\ dx + \int_{-\infty}^0 \varphi(x)\ dx
\end{aligned}$$

Suppose the weak derivative $g(x) = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases}$.
Then $-\int_{\mathbb{R}} g(x)\varphi(x)dx = -\int_0^\infty \varphi(x)dx - \int_{-\infty}^0 -\varphi(x)dx$.
Hence, $-a\varphi(0) - \int_0^\infty \varphi(x)dx + \int_{-\infty}^0 \varphi(x)dx = \int_{-\infty}^0 \varphi(x)dx - \int_0^\infty \varphi(x)dx$.
$\Rightarrow -a\varphi(0) = 0$ if $a = 0$ (since $\varphi(0)$ is arbitrary).
If $a = 0$, the weak derivative exists and the weak derivative of f is sign function.
If $a \neq 0$, it does not exist as a function. $\square$

Exercise 5.8.
Let $f : \mathbb{R} \to \mathbb{R}$ *be defined by* $f(x) := sgn(x)\sqrt{|x|}, \forall x \in \mathbb{R}$, *where*

$$sgn(x) := \begin{cases} 1, & x > 0, \\ 0, & x = 0, \\ -1, & x < 0. \end{cases}$$

Show that the (order one) weak derivative f' *exists, and compute it.*

Proof.
claim : $f \in L^1_{loc}(\mathbb{R})$.
For $x > 0$,
$f(x) = 1 \cdot \sqrt{x}$ and $\displaystyle\int_K \sqrt{x}dx = \frac{2}{3}|K|^{3/2} < \infty$, for any compact set $K \subset \mathbb{R}$.
For $x < 0$,
$f(x) = -1 \cdot \sqrt{-x}$ and $\displaystyle\int_K -\sqrt{-x}dx = \frac{2}{3}|K|^{3/2}$, for any compact set $K \subset \mathbb{R}$.
claim : compute the classical derivative
For $x > 0$, $f(x) = \sqrt{x} \Rightarrow f'(x) = \dfrac{1}{2}x^{-1/2}$.
For $x < 0$, $f(x) = -\sqrt{-x} \Rightarrow f'(x) = \dfrac{1}{2}(-x)^{-1/2}$.
At $x = 0$,

$$\lim_{x\to 0^-} \frac{f(x) - f(0)}{x} = \lim_{x\to 0^-} \frac{-\sqrt{-x} - 0}{x} = \lim_{x\to 0^-} \frac{-1}{\sqrt{-x}} = -\infty$$

$$\lim_{x\to 0^+} \frac{f(x) - f(0)}{x} = \lim_{x\to 0^+} \frac{\sqrt{x} - 0}{x} = \lim_{x\to 0^+} \frac{1}{\sqrt{x}} = \infty$$

Define the weak derivative $g(x) = \frac{1}{2\sqrt{|x|}}, x \neq 0$.
claim : verify $\displaystyle\int_{\mathbb{R}} g(x)\varphi(x)dx = -\int_{\mathbb{R}} f(x)\varphi'(x)dx, \forall \varphi \in C_c^\infty(\mathbb{R})$.
By integration by parts,

$$\begin{aligned}
\int_{\mathbb{R}} f(x)\varphi'(x)dx = & \int_{-\infty}^{0} -\sqrt{-x}\varphi'(x)dx + \int_{0}^{\infty} \sqrt{x}\varphi'(x)dx \\
= & \left(-\sqrt{-x}\varphi(x)\Big|_{-\infty}^{0} - \int_{-\infty}^{0} \frac{1}{2}(-x)^{\frac{-1}{2}}\varphi(x)dx \right) \\
& + \left(\sqrt{x}\varphi(x)\Big|_{0}^{\infty} - \int_{0}^{\infty} \frac{1}{2}x^{\frac{-1}{2}}\varphi(x)dx \right) \\
= & -\left(\int_{-\infty}^{0} \frac{1}{2}(-x)^{\frac{-1}{2}}\varphi(x)dx + \int_{0}^{\infty} \frac{1}{2}x^{\frac{-1}{2}}\varphi(x)dx \right)
\end{aligned}$$

$$\Rightarrow \int_{\mathbb{R}} g(x)\varphi(x)dx = -\int_{\mathbb{R}} f(x)\varphi'(x)dx = \int_{\mathbb{R}} \frac{1}{2\sqrt{|x|}}\varphi(x)dx, \forall \varphi \in C_c^\infty(\mathbb{R})$$

Therefore, the weak derivative exists and $f'(x) = \frac{1}{2\sqrt{|x|}}$. □

Exercise 5.9.
Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x,y) := H(x) + H(y), \forall (x,y) \in \mathbb{R}^2$, where H is the Heaviside function given by $H(t) := \begin{cases} 1, & t > 0 \\ 0, & t \leq 0. \end{cases}$
We denote the multiindices $\alpha = (1,1)$ and $\beta = (1,0)$.
1. Prove that the weak derivatives $\partial^\alpha f$ and $\partial^{\alpha+\beta} f$ exist, and compute them.
2. Prove that the weak derivatives $\partial^\beta f$ does not exist. Compute its distributional derivative $\partial^\beta f$.

Proof.
claim : $\partial^\alpha f$ exist.
i.e., find a function $g \ni \int_{\mathbb{R}^2} f(x,y)\partial_x\partial_y\varphi(x,y)dxdy = (-1)^{|\alpha|} \int_{\mathbb{R}^2} g(x,y)\varphi(x,y)dxdy.$
Since $|\alpha| = 1 + 1 = 2$, then $\partial^{\alpha+\beta} f$ exist.
i.e., find a function $g \ni \int_{\mathbb{R}^2} f(x,y)\partial_x\partial_y\varphi(x,y)dxdy = \int_{\mathbb{R}^2} g(x,y)\varphi(x,y)dxdy.$
By Fubini Thm and for each $\varphi \in C_c^\infty(\mathbb{R}^2)$,

$$\begin{aligned}
\int_{\mathbb{R}^2} f(x,y)\partial_x\partial_y\varphi(x,y)dxdy = & \int_{\mathbb{R}^2} \big(H(x) + H(y)\big)\partial_x\partial_y\varphi(x,y)dxdy \\
= & \int_{\mathbb{R}^2} H(x)\partial_x\partial_y\varphi(x,y)dxdy + \int_{\mathbb{R}^2} H(y)\partial_x\partial_y\varphi(x,y)dxdy \\
= & \int_{-\infty}^{\infty}\left(\int_0^\infty \partial_x\partial_y\varphi(x,y)dx\right)dy \\
& + \int_{-\infty}^{\infty}\left(\int_0^\infty \partial_y\partial_x\varphi(x,y)dy\right)dx \\
= & \int_{-\infty}^{\infty}\left(\partial_y\varphi(x,y)\Big|_{x=0}^{\infty}\right)dy + \int_{-\infty}^{\infty}\left(\partial_x\varphi(x,y)\Big|_{y=0}^{\infty}\right)dx \\
= & \int_{-\infty}^{\infty}\Big(0 - \partial_y\varphi(0,y)\Big)dy + \int_{-\infty}^{\infty}\Big(0 - \partial_x\varphi(x,0)\Big)dx
\end{aligned}$$

Suppose $\partial_x\partial_y f = g = 0$. Then $\int_{\mathbb{R}^2} 0\varphi(x,y)dxdy = 0$.

$$\begin{aligned}
-\int_{\mathbb{R}} \partial_y\varphi(0,y)dxdy - \int_{\mathbb{R}} \partial_x\varphi(x,0)dx &= -\varphi(0,y)\big|_{y=-\infty}^{\infty} - \varphi(x,0)\big|_{x=-\infty}^{\infty} \\
&= 0 - 0 = 0 = \int_{\mathbb{R}^2} 0\varphi(x,y)dxdy
\end{aligned}$$

Hence, $\partial^\alpha f = 0$ exists.
claim : $\partial^{\alpha+\beta} f$ exist.
Since $\alpha + \beta = (1,1) + (1,0) = (2,1)$, then $\partial^{\alpha+\beta} f = \partial_x^2\partial_y f$ with $|\alpha + \beta| = 3$.
i.e., find a function $g \ni \int_{\mathbb{R}^2} f(x,y)\partial_x^2\partial_y\varphi(x,y)dxdy = -\int_{\mathbb{R}^2} g(x,y)\varphi(x,y)dxdy.$

By Fubini Thm and for each $\varphi \in C_c^\infty(\mathbb{R}^2)$,

$$
\begin{aligned}
\int_{\mathbb{R}^2} f(x,y)\partial_x^2\partial_y\varphi(x,y)dxdy = &\int_{\mathbb{R}^2} H(x)\partial_x^2\partial_y\varphi(x,y)dxdy + \int_{\mathbb{R}^2} H(y)\partial_x^2\partial_y\varphi(x,y)dxdy \\
= &\int_{-\infty}^{\infty}\Big(\int_0^\infty \partial_x^2\partial_y\varphi(x,y)dx\Big)dy \\
&+\int_{-\infty}^{\infty}\Big(\int_0^\infty \partial_y\partial_x^2\varphi(x,y)dy\Big)dx \\
= &\int_{-\infty}^{\infty}\Big(\partial_x\partial_y\varphi(x,y)\Big|_{x=0}^{\infty}\Big)dy + \int_{-\infty}^{\infty}\Big(\partial_x^2\varphi(x,y)\Big|_{y=0}^{\infty}\Big)dx \\
= &\int_{-\infty}^{\infty}\Big(0-\partial_x\partial_y\varphi(0,y)\Big)dy + \int_{-\infty}^{\infty}\Big(0-\partial_x^2\varphi(x,0)\Big)dx
\end{aligned}
$$

Suppose $g=0$. Then $-\int_{\mathbb{R}^2} 0\cdot\varphi dxdy = 0$.

$$
\begin{aligned}
&\int_{-\infty}^{\infty}\Big(0-\partial_x\partial_y\varphi(0,y)\Big)dy + \int_{-\infty}^{\infty}\Big(0-\partial_x^2\varphi(x,0)\Big)dx \\
=&\partial_x\varphi(0,y)\Big|_{y=-\infty}^{\infty} + \partial_x^2\varphi(x,0)\Big|_{x=-\infty}^{\infty} = 0+0=0
\end{aligned}
$$

Hence, $\partial^{\alpha+\beta}f = 0$ exist.
claim : the weak derivatives $\partial^\beta f$ does not exist.
Suppose $\partial^\beta f$ exists, say $g \in L^1_{loc}(\mathbb{R}^2)$.
By Fubini Thm and for each $\varphi \in C_c^\infty(\mathbb{R}^2)$,

$$
\begin{aligned}
\int_{\mathbb{R}^2} g(x,y)\varphi(x,y) &= (-1)^{|\beta|}\int_{\mathbb{R}^2} f(x,y)\partial_x\varphi(x,y) \\
&= (-1)^{1+0}\int_{\mathbb{R}^2}(H(x)+H(y))\partial_x\varphi(x,y) \\
&= -\Big(\int_{\mathbb{R}^2} H(x)\partial_x\varphi(x,y) + \int_{\mathbb{R}^2} H(y)\partial_x\varphi(x,y)\Big) \\
&= -\int_{-\infty}^{\infty}\int_0^\infty \partial_x\varphi(x,y)dxdy - \int_{-\infty}^{\infty}\int_0^\infty \partial_x\varphi(x,y)dydx \\
&= -\int_{-\infty}^{\infty}\varphi(0,y)dy - \int_{-\infty}^{\infty}\partial_x\varphi(x,0)dx
\end{aligned}
$$

Then $\int_{\mathbb{R}^2} g(x,y)\varphi(x,y) = 0, \forall\varphi \in C_c^\infty(\{(x,y)\in\mathbb{R}^2; x\neq 0 \text{ and } y\neq 0\})$.
By Lma 3.1.3, $g=0$ a.e. in $\{(x,y)\in\mathbb{R}^2; x\neq 0 \text{ and } y\neq 0\}$
$\Rightarrow g = 0$ a.e. in $\mathbb{R}^2$. $\rightarrow\leftarrow$
claim : Compute its distributional derivative $\partial^\beta f$.
Since $f(x,y) = H(x)+H(y)$, then $f \in L^1_{loc}(\mathbb{R}^2)$ for any variable x or y is fixed.

The distribution of f is $T_f(\varphi) = \int_{\mathbb{R}^2} f\varphi, \forall \varphi \in C_c^\infty(\mathbb{R}^2)$.
Then the distribution derivative $\partial^\beta f = \partial_x T_f$.
Hence, by Fubini Thm and for each $\varphi \in C_c^\infty(\mathbb{R}^2)$,

$$\begin{aligned}
\partial_x T_f(\varphi) = (-1)^{1+0} T_f(\partial_x \varphi) &= -\int_{\mathbb{R}^2} f \partial_x \varphi \\
&= -\int_{\mathbb{R}^2} (H(x) + H(y)) \partial_x \varphi(x,y) dxdy \\
&= -\left(\int_{\mathbb{R}^2} H(x) \partial_x \varphi(x,y) dxdy + \int_{\mathbb{R}^2} H(y) \partial_x \varphi(x,y) dxdy \right) \\
&= -\int_{-\infty}^{\infty} \int_0^{\infty} \partial_x \varphi(x,y) dxdy - \int_{-\infty}^{\infty} \int_0^{\infty} \partial_x \varphi(x,y) dydx \\
&= -\int_{-\infty}^{\infty} \varphi(0,y) dy - \int_{-\infty}^{\infty} \partial_x \varphi(x,0) dx
\end{aligned}$$

Hence, the distributional derivative $\partial^\beta f = -\int_{-\infty}^{\infty} \varphi(0,y)dy - \int_{-\infty}^{\infty} \partial_x \varphi(x,0)dx$. $\square$

Exercise 5.10.
Let $f:\mathbb{R}^2\to\mathbb{R}$ *be defined by* $f(x,y):=H(x)-sgn(y),\forall(x,y)\in\mathbb{R}^2$ *with*
$H(t):=\begin{cases}1, & t>0\\ 0, & t\le 0.\end{cases}$ *and* $sgn(x):=\begin{cases}1, & x>0,\\ 0, & x=0,\\ -1, & x<0.\end{cases}$
We denote the multiindex $\alpha=(\alpha_1,\alpha_2)\in{\mathbb{Z}_{\ge 0}}^2$.
Prove that the weak derivative $\partial^\alpha f$ *exists iff* $\alpha_1\ge 1$ *and* $\alpha_2\ge 1$.

Proof.
Suppose the weak derivative $\partial^\alpha f$ exists.
By Fubini Thm and for each $\varphi(x,y)\in C_c^\infty(\mathbb{R}^2)$,

$$\begin{aligned}\int_{\mathbb{R}^2} f\partial^\alpha\varphi &= (-1)^{|\alpha|}\int_{\mathbb{R}^2}\partial^\alpha f\varphi\\
&= (-1)^{\alpha_1+\alpha_2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\partial_x^{\alpha_1}\partial_y^{\alpha_2}f(x,y)\varphi(x,y)dxdy\\
&= (-1)^{\alpha_1+\alpha_2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\partial_x^{\alpha_1}\partial_y^{\alpha_2}(H(x)-sgn(y))\varphi(x,y)dxdy\\
&= (-1)^{\alpha_1+\alpha_2}\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\partial_x^{\alpha_1}\partial_y^{\alpha_2}H(x)\varphi(x,y)dxdy-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\partial_x^{\alpha_1}\partial_y^{\alpha_2}sgn(y)\varphi(x,y)dxdy\right]\\
&= (-1)^{\alpha_1+\alpha_2}\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\partial_x^{\alpha_1}\partial_y^{\alpha_2}H(x)\varphi(x,y)dxdy-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\partial_x^{\alpha_1}\partial_y^{\alpha_2}sgn(y)\varphi(x,y)dydx\right]\\
&= (-1)^{\alpha_1+\alpha_2}\Bigg[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\partial_x^{\alpha_1}\partial_y^{\alpha_2}H(x)\varphi(x,y)dxdy\\
&\quad-\left(\int_{-\infty}^{\infty}\int_{-\infty}^{0}\partial_x^{\alpha_1}\partial_y^{\alpha_2}\varphi(x,y)dydx+\int_{-\infty}^{\infty}\int_{0}^{\infty}\partial_x^{\alpha_1}\partial_y^{\alpha_2}\varphi(x,y)dydx\right)\Bigg]\end{aligned}$$

Hence, $\alpha_1\ge 1$ and $\alpha_2\ge 1$.
Suppose $\alpha_1\ge 1$ and $\alpha_2\ge 1$.
The distribution of f is $T_f(\varphi)=\int_{\mathbb{R}^2}f\varphi,\forall\varphi\in C_c^\infty(\mathbb{R}^2)$.
Then the distribution derivative $\partial^\alpha f=\partial_x^{\alpha_1}\partial_y^{\alpha_2}T_f$.
By Fubini Thm and for each $\varphi(x,y)\in C_c^\infty(\mathbb{R}^2)$,

$$
\begin{aligned}
\partial^\alpha f = {} & \partial_x^{\alpha_1}\partial_y^{\alpha_2}T_f(\varphi) = (-1)^{\alpha_1+\alpha_2}\int_{\mathbb{R}^2} f\partial_x^{\alpha_1}\partial_y^{\alpha_2}\varphi \\
= {} & (-1)^{\alpha_1+\alpha_2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\partial_x^{\alpha_1}\partial_y^{\alpha_2}f(x,y)\varphi(x,y)dxdy \\
= {} & (-1)^{\alpha_1+\alpha_2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\partial_x^{\alpha_1}\partial_y^{\alpha_2}(H(x)-sgn(y))\varphi(x,y)dxdy \\
= {} & (-1)^{\alpha_1+\alpha_2}\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\partial_x^{\alpha_1}\partial_y^{\alpha_2}H(x)\varphi(x,y)dxdy - \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\partial_x^{\alpha_1}\partial_y^{\alpha_2}sgn(y)\varphi(x,y)dxdy\right] \\
= {} & (-1)^{\alpha_1+\alpha_2}\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\partial_x^{\alpha_1}\partial_y^{\alpha_2}H(x)\varphi(x,y)dxdy - \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\partial_x^{\alpha_1}\partial_y^{\alpha_2}sgn(y)\varphi(x,y)dydx\right] \\
= {} & (-1)^{\alpha_1+\alpha_2}\Bigg[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\partial_x^{\alpha_1}\partial_y^{\alpha_2}H(x)\varphi(x,y)dxdy \\
& -\left(\int_{-\infty}^{\infty}\int_{-\infty}^{0}\partial_x^{\alpha_1}\partial_y^{\alpha_2}\varphi(x,y)dydx + \int_{-\infty}^{\infty}\int_{0}^{\infty}\partial_x^{\alpha_1}\partial_y^{\alpha_2}\varphi(x,y)dydx\right)\Bigg] \\
= {} & (-1)^{\alpha_1+\alpha_2}\Bigg[\int_{-\infty}^{\infty}\int_{0}^{\infty}\partial_x^{\alpha_1}\partial_y^{\alpha_2}\varphi(x,y)dxdy \\
& -\left(\int_{-\infty}^{\infty}\int_{-\infty}^{0}\partial_x^{\alpha_1}\partial_y^{\alpha_2}\varphi(x,y)dydx + \int_{-\infty}^{\infty}\int_{0}^{\infty}\partial_x^{\alpha_1}\partial_y^{\alpha_2}\varphi(x,y)dydx\right)\Bigg]
\end{aligned}
$$

Since $\alpha_1 \geq 1$ and $\alpha_2 \geq 1$, then the integral exists.
Hence, the weak derivative $\partial^\alpha f$ exists. $\square$

5.2 Definition and Elementary Properties of the Sobolev Spaces

Definition 5.2.1.
Let $\Omega \subset \mathbb{R}^n$ be an open set, and $p \in \mathbb{R}$ with $1 \le p \le \infty$.
For each $m \in \mathbb{N}$, we define the Sobolev spaces
$W^{m,p}(\Omega) := \{u \in L^p(\Omega); \partial^\alpha u \in L^p(\Omega), \forall \alpha \text{ with } |\alpha| \le m\}$.
In fact, $W^{m,p}(\Omega)$ is a normed space with respect to the norm $\|\cdot\|_{W^{m,p}(\Omega)}$ given by
$$\|u\|_{W^{m,p}(\Omega)} = \left(\sum_{|\alpha|\le m} \left\|\partial^\alpha u\right\|^p_{L^p(\Omega)}\right)^{\frac{1}{p}}, \|u\|_{W^{m,\infty}(\Omega)} = \max_{|\alpha|\le m} \left\|\partial^\alpha u\right\|_{L^\infty(\Omega)}.$$
The elements in $W^{m,p}(\Omega)$ also called the Sobolev functions.

Lemma 5.2.1.
Let $1 \le p \le \infty, m \in \mathbb{N}$ and $\Omega \subset \mathbb{R}^n$ be an open set.
Then the Sobolev space $W^{m,p}(\Omega)$ is Banach.

Similar to L^p-functions, Sobolev functions also can be approximated by smooth functions.

Lemma 5.2.2.
Let $\Omega \subset \mathbb{R}^n$ be an open set, $1 \le p < \infty$, and $m \in \mathbb{N}$.
Given any $f \in W^{m,p}(\Omega)$, $\exists$ a sequence $\{f_k\}_{k\in\mathbb{N}}$ in $C^\infty(\Omega) \cap W^{m,p}(\Omega) \ni$
$f_n \to f$ in $W^{m,p}(\Omega)$, i.e., $\lim\limits_{k\to\infty} \left\|f_k - f\right\|_{W^{m,p}(\Omega)=0}$.
Note : 收斂速度和 m 有關

Example 5.2.1.
$f_k(x) = x^k$ on $(0,1)$.

Question : 找一個 sequence f_k 保證邊界也連續
Note : in this lemma, the approximation functions $\{f_k\}$ are smooth only in the interior of Ω. To have the smoothness up to the boundary of the approximation sequence, we need to make a smoothness assumption on the boundary $\partial\Omega$.

Definition 5.2.2.
For each $k \in \mathbb{N} \cup \{\infty\}$, we define
$C^k(\bar{\Omega}) := \{f|_{\bar{\Omega}}; f \in C^k(U) \text{ for some open set } U \supset \bar{\Omega}\}$.

Lemma 5.2.3.
Let $\Omega \subset \mathbb{R}^n$ be an open set, $1 \le p < \infty$, and $m \in \mathbb{N}$.
Given any $f \in W^{m,p}(\Omega)$,
$\exists$ a sequence $\{f_k\}_{k=1}^\infty$ in $C^\infty(\bar{\Omega}) \ni \lim\limits_{k\to\infty} \|f_k - f\|_{W^{m,p}(\Omega)} = 0$,
i.e., $f_n \to f$ in $W^{m,p}(\Omega)$.

Before introducing the Sobolev embeddings, we introduce the following concept :

Definition 5.2.3.
Let X and Y be two Banach spaces.
The space X is continuous embedded in Y if $\exists\ c \in \mathbb{R} \ni \|v\|_Y \leq c\|v\|_X, \forall v \in X$.
The space X is compactly embedded in Y if $\exists\ c \in \mathbb{R} \ni \|v\|_Y \leq c\|v\|_X, \forall v \in X$ holds and each bounded sequence in X has a convergent subsequence in Y.

Many authors (including myself) simply denote $X \subset Y$ if the Banach space X is continuous embedded in another Banach space Y, despite that X is not necessarily a subset of Y.
We will also denote $X \Subset Y$ or $X \hookrightarrow Y$ if X is compactly embedded in Y.
Here and after (including the next theorem), we will use these notations without mentioning explicitly.
Let $\lfloor x \rfloor$ denotes the integer part of x, and we have the following theorem :

Theorem 5.2.4. (*Sobolev embedding theorems*)
Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$. Then TFAS :
1. if $k < \frac{n}{p}$, then $W^{k,p}(\Omega) \Subset L^q(\Omega)$ for any $q < p^*$ and $W^{k,p}(\Omega) \subset L^q(\Omega)$ when $q \leq p^*$, where $\frac{1}{p^*} = \frac{1}{p} - \frac{k}{n}$.
2. if $k = \frac{n}{p}$, then $W^{k,p}(\Omega) \Subset L^q(\Omega)$ for any $q < \infty$.
3. if $k > \frac{n}{p}$ (i.e., weak derivative 超過臨界值), then

$$\begin{cases} W^{k,p}(\Omega) \Subset C^{k-\lfloor \frac{n}{p} \rfloor -1,\beta}(\Omega), \forall \beta \in \left[0, \left\lfloor \dfrac{n}{p} \right\rfloor + 1 - \dfrac{n}{p}\right) \\ W^{k,p}(\Omega) \subset C^{k-\lfloor \frac{n}{p} \rfloor -1,\beta}(\Omega),\ \text{with } \beta = \begin{cases} \left\lfloor \dfrac{n}{p} \right\rfloor + 1 - \dfrac{n}{p}, & \text{if } \dfrac{n}{p} \notin \mathbb{Z} \\ \text{any positive number } < 1, & \text{if } \dfrac{n}{p} \in \mathbb{Z} \end{cases} \end{cases}$$

Note : $k - \lfloor \frac{n}{p} \rfloor - 1$ 是 Calculus 中的微分，β 是最高次項微分是 Hoelder conti i.e., $|f(x) - f(y)| \leq L|x-y|^\beta$.

Example 5.2.2.
For $p = 2, k = 1$.
$W^{1,2}(\Omega) \subset L^{p^*}(\Omega)$ with $\frac{1}{p^*} = \frac{1}{2} - \frac{1}{n} < \frac{1}{2} \Rightarrow p^* > 2$.
Hence, $W^{1,2}(\Omega) \subset L^2(\Omega)$.

Example 5.2.3.
For $p = 2, k > \frac{n}{2}$. $W^{k,2} \Subset C^{k-\lfloor \frac{n}{2} \rfloor -1}$.
For $n = 2$, $W^{k,2} \Subset C^{k-2}$.

Remark : Thm 5.2.4 is also valid for $W^{k,p}$-spaces with $k \in \mathbb{R}$ for precise definitions. Here we will cover these topics in this lecture note. Part (c) of Thm 5.2.4 in particular gives some sufficient condition in terms of weak derivatives to guarantee the well-definitedness of the strong/classical derivatives.

It is important to mention that the proof of Thm 5.2.4 is based on the existence of the bounded linear extension operator
$E : W^{k,p}(\Omega) \to W^{k,p}(\mathbb{R}^n)$ for any $k \in \mathbb{N} \cup \{0\}$ and for any $1 \le p \le \infty$.
In fact, the operator norm of the extension operator can be explicitly given :

Theorem 5.2.5.
Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$.
Then $\exists$ a constant $C = C(\Omega) > 1 \ni \forall k \in \mathbb{N}, 1 \le p \le \infty$,
$$\left(\frac{k}{C}\right)^k \le \inf_E \|E\|_{W^{k,p}(\Omega) \to W^{k,p}(\mathbb{R}^n)} = \inf_E \left(\sup_{0 \ne f \in C_c^\infty(\Omega)} \frac{\|Ef\|_{W^{k,p}(\Omega)}}{\|f\|_{W^{k,p}(\Omega)}} \right) \le (Ck)^k,$$
where the infimum is taken over all extension operator $E : W^{k,p}(\Omega) \to W^{k,p}(\mathbb{R}^n)$.
Note : C is independent of both k and p.

In fact, the integration by parts also holds true for weak derivatives !
(which generalized Cor 4.1.9)

Theorem 5.2.6. (*Integration by parts*)
Let Ω be a bounded Lipschitz domain in Rn and given $1 \le p < \infty$.
The mapping $Tr : C^\infty(\bar{\Omega}) \to C^\infty(\partial\Omega), Tr(f) = f\big|_{\partial\Omega}$ can be uniquely extended to a bounded surjective linear operator $W^{1,p}(\Omega) \to Tr(W^{1,p}(\Omega)) \subset L^p(\partial\Omega)$.
Furthermore, $\forall \varphi \in (C^1(\mathbb{R}^n))^n$ and $f \in W^{1,p}(\Omega)$, we have
$$\int_\Omega f(x) div(\varphi(x))dx = -\int_\Omega \nabla f(x) \cdot \varphi(x)dx + \int_{\partial\Omega} (\nu \cdot \varphi)Tr(f)d\mathscr{H}^{n-1},$$
where ν is the unit outer normal to $\partial\Omega$.

Remark :
Refer the advance monograph for the precise meaning of ν, which is well-defined for $\mathscr{H}^{n-1}$-a.e. on $\partial\Omega$.
The function $Tr(f)$ given in Thm 5.2.6 is called the trace of f on $\partial\Omega$.
We usually still denote $d\mathscr{H}^{n-1}$ by dS_x.
If there is no ambiguity, we sometime omit the notation the trace operator and simply write it as $\displaystyle\int_\Omega f(x) div(\varphi(x))dx = -\int_\Omega \nabla f(x) \cdot \varphi(x)dx + \int_{\partial\Omega} (\nu \cdot \varphi)dS_x$.

Giving some remarks on convolution.
The following lemma exhibit the (strong) differentiability of convolution :

Lemma 5.2.7.
Let $g \in L^1_{loc}(\mathbb{R}^n)$ and $f \in C_c^m(\mathbb{R}^n)$ for some integer $m \in \mathbb{Z}_{\geq 0}$.
Then $f * g \in C^m(\mathbb{R}^n)$ and $\partial^\alpha(f * g) = (\partial^\alpha f) * g, \forall$ multi-indices α with $|\alpha| \leq m$.

Lemma 5.2.8. *Exhibits the (weak) differentiability of convolution*
Let $\rho \in L^1(\mathbb{R}^n)$ and $v \in W^{m,p}(\mathbb{R}^n)$ with $1 \leq p < \infty$ and $m \in \mathbb{N}$.
Then $\rho * v \in W^{m,p}(\mathbb{R}^n)$ and $\partial^\alpha(\rho * v) = \rho * \partial^\alpha v, \forall \alpha$ with $|\alpha| \leq m$.

Definition 5.2.4.
Let A be a subset of a Banach function space equipped with the norm $\|\cdot\|$.
The set A is called a algebra (with respect to a.e. pointwise multiplication)
if $\exists c \in \mathbb{R}^+ \ni \|uv\| \leq c\|u\|\|v\|, \forall u, v \in A$.
In other words, A is closed with respect to multiplication operator.

Unfortunately, when $p > 1$, the space $W^{m,p}(\mathbb{R}^n)$ is not an algebra for $mp \leq n$, and when $p = 1$, the space $W^{m,1}(\mathbb{R}^n)$ is not an algebra for $m < n$.
However, we have the following theorem, and the proof is based on the Sobolev embedding (Thm 5.2.4).

Theorem 5.2.9.
Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ or $\Omega = \mathbb{R}^n$.
If $p > 1$ and $mp > n$, then $W^{m,p}(\Omega)$ is an algebra.
If $p = 1$ and $m \geq n$, then $W^{m,p}(\Omega)$ is an algebra.

Exercise 5.11.
Suppose $(W_1, \|\cdot\|_{W_1})$ and $(W_2, \|\cdot\|_{W_2})$ are Banach spaces.
Show that $\|\cdot\|_{W_1 \cap W_2} := \|\cdot\|_{W_1} + \|\cdot\|_{W_2}$ and $\|\cdot\|'_{W_1 \cap W_2} := \max\{\|\cdot\|_{W_1}, \|\cdot\|_{W_2}\}$ are norms, and they are equivalent.
In addition, show that $(W_1 \cap W_2, \|\cdot\|_{W_1} + \|\cdot\|_{W_2})$ is a Banach space, equivalently $(W_1 \cap W_2, \|\cdot\|'_{W_1 \cap W_2}$ is a Banach space.

Proof.
claim : $\|\cdot\|_{W_1 \cap W_2} := \|\cdot\|_{W_1} + \|\cdot\|_{W_2}$ is a norm.
Given $x, y \in W_1 \cap W_2$ and any scalar α.
Since $\|\cdot\|_{W_1}$ and $\|\cdot\|_{W_2}$ are norms, then $\|x\|_{W_1} \geq 0$ and $\|x\|_{W_2} \geq 0$.
Hence, $\|x\|_{W_1 \cap W_2} = \|x\|_{W_1} + \|x\|_{W_2} \geq 0 + 0 \geq 0$.
Suppose $\|x\|_{W_1 \cap W_2} = 0$.
Since $\|x\|_{W_1} \geq 0$ and $\|x\|_{W_2} \geq 0$.
Then $\|x\|_{W_1} = 0 = \|x\|_{W_2} \Rightarrow x = 0$.
Suppose $x = 0$.
Then $\|x\|_{W_1} = 0 = \|x\|_{W_2} \Rightarrow 0 = 0 + 0 = \|\cdot\|_{W_1} + \|\cdot\|_{W_2} = \|\cdot\|_{W_1 \cap W_2}$.

$$\begin{aligned}\|\alpha x\|_{W_1 \cap W_2} = \|\alpha x\|_{W_1} + \|\alpha x\|_{W_2} &= |\alpha|\|x\|_{W_1} + |\alpha|\|x\|_{W_2} \\ &= |\alpha|(\|x\|_{W_1} + \|x\|_{W_2}) = |\alpha|\|x\|_{W_1 \cap W_2}\end{aligned}$$

$\|x+y\|_{W_1\cap W_2} = \|x+y\|_{W_1} + \|x+y\|_{W_2}$
$\qquad\qquad \le \|x\|_{W_1} + \|y\|_{W_1} + \|x\|_{W_2} + \|y\|_{W_2} = \|x\|_{W_1\cap W_2} + \|y\|_{W_1\cap W_2}$
claim : $\|\cdot\|'_{W_1\cap W_2} := \max\{\|\cdot\|_{W_1}, \|\cdot\|_{W_2}\}$ is a norm.
Given $x, y \in W_1 \cap W_2$ and any scalar α.
Since $\|\cdot\|_{W_1}$ and $\|\cdot\|_{W_2}$ are norms, then $\|x\|_{W_1} \ge 0$ and $\|x\|_{W_2} \ge 0$.
Hence, $\|x\|'_{W_1\cap W_2} = \max\{\|x\|_{W_1}, \|x\|_{W_2}\} \ge 0$.
Suppose $\|x\|'_{W_1\cap W_2} = \max\{\|x\|_{W_1}, \|x\|_{W_2}\} = 0$.
Since $\|x\|_{W_1} \ge 0$ and $\|x\|_{W_2} \ge 0$.
Then $\|x\|_{W_1} = 0 = \|x\|_{W_2} \Rightarrow x = 0$.
Suppose $x = 0$.
Then $\|x\|_{W_1} = 0 = \|x\|_{W_2} \Rightarrow 0 = \max\{0, 0\} = \max\{\|x\|_{W_1}, \|x\|_{W_2}\} = \|x\|'_{W_1\cap W_2}$.
$\|\alpha x\|'_{W_1\cap W_2} = \max\{\|\alpha x\|_{W_1}, \|\alpha x\|_{W_2}\}$
$\qquad = \max\{|\alpha|\|x\|_{W_1}, |\alpha|\|x\|_{W_2}\} = |\alpha| \max\{\|x\|_{W_1}, \|x\|_{W_2}\} = |\alpha|\|x\|'_{W_1\cap W_2}$

$$\begin{aligned}
\|x+y\|'_{W_1\cap W_2} &= \max\{\|x+y\|_{W_1}, \|x+y\|_{W_2}\} \\
&\le \max\{\|x\|_{W_1} + \|y\|_{W_1}, \|x\|_{W_2} + \|y\|_{W_2}\} \\
&\le \max\{\|x\|_{W_1}, \|x\|_{W_2}\} + \max\{\|y\|_{W_1}, \|y\|_{W_2}\} \\
&\quad = \|x\|'_{W_1\cap W_2} + \|y\|'_{W_1\cap W_2}
\end{aligned}$$

claim : $\|\cdot\|_{W_1\cap W_2}$ and $\|\cdot\|'_{W_1\cap W_2}$ are equivalent.
Given $x \in W_1 \cap W_2$.
$\|x\|'_{W_1\cap W_2} = \max\{\|x\|_{W_1}, \|x\|_{W_2}\}$
$\qquad \le \|x\|_{W_1} + \|x\|_{W_2} = \|x\|_{W_1\cap W_2}$
$\|x\|_{W_1\cap W_2} = \|x\|_{W_1} + \|x\|_{W_2}$
$\qquad \le 2\max\{\|x\|_{W_1}, \|x\|_{W_2}\} = 2\|x\|'_{W_1\cap W_2}$
Hence, $\|x\|'_{W_1\cap W_2} \le \|x\|_{W_1\cap W_2} \le 2\|x\|'_{W_1\cap W_2}$.
claim : $(W_1 \cap W_2, \|\cdot\|_{W_1} + \|\cdot\|_{W_2})$ is a Banach space.
Given a Cauchy sequence $\{x_k\}_{k=1}^{\infty}$ in $(W_1 \cap W_2, \|\cdot\|_{W_1} + \|\cdot\|_{W_2})$.
Since $(W_1, \|\cdot\|_{W_1})$ and $(W_2, \|\cdot\|_{W_2})$ are Banach spaces, then
given $\varepsilon > 0, \exists N_1, N_2 \in \mathbb{N} \ni \forall m > n \ge N = \max\{N_1, N_2\}$,
$\|x_m - x_n\|_{W_1} < \frac{\varepsilon}{2}$ and $\|x_m - x_n\|_{W_2} < \frac{\varepsilon}{2}$.
Hence, $\forall m > n \ge \mathbb{N}, \|x_m - x_n\|_{W_1\cap W_2} = \|x_m - x_n\|_{W_1} + \|x_m - x_n\|_{W_2} < \varepsilon$,
i.e., the Cauchy sequence in $(W_1 \cap W_2, \|\cdot\|_{W_1} + \|\cdot\|_{W_2})$ is converge.
Therefore, $(W_1 \cap W_2, \|\cdot\|_{W_1} + \|\cdot\|_{W_2})$ is a Banach space.
claim : $(W_1 \cap W_2, \|\cdot\|'_{W_1\cap W_2})$ is a Banach space.
Given a Cauchy sequence $\{x_k\}_{k=1}^{\infty}$ in $(W_1 \cap W_2, \|\cdot\|'_{W_1\cap W_2})$.
Since $(W_1, \|\cdot\|_{W_1})$ and $(W_2, \|\cdot\|_{W_2})$ are Banach spaces and $\|x\|'_{W_1\cap W_2} \le \|x\|_{W_1\cap W_2}$,
then given $\varepsilon > 0, \exists N_1, N_2 \in \mathbb{N} \ni \forall m > n \ge N = \max\{N_1, N_2\}$,
$\|x_m - x_n\|_{W_1} < \frac{\varepsilon}{2}$ and $\|x_m - x_n\|_{W_2} < \frac{\varepsilon}{2}$.
Hence, $\forall m > n \ge \mathbb{N}$,

$$\begin{aligned}\|x_m - x_n\|'_{W_1\cap W_2} &= \max\{\|x_m - x_n\|_{W_1}, \|x_m - x_n\|_{W_2}\} \\ &\le \|x_m - x_n\|_{W_1} + \|x_m - x_n\|_{W_2} = \|x_m - x_n\|_{W_1\cap W_2} < \varepsilon,\end{aligned}$$

i.e., the Cauchy sequence in $(W_1 \cap W_2, \|\cdot\|'_{W_1\cap W_2})$ is converge.
Therefore, $(W_1 \cap W_2, \|\cdot\|'_{W_1\cap W_2})$ is a Banach space. $\square$

It is important to mention that the proof following theorem does not involve the Sobolev embedding (Thm 5.2.4).

Theorem 5.2.10.
Let $1 \le p \le +\infty$ and $1 \le m \in \mathbb{N}$.
The Banach space $W^{m,p}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ equipped with the norm $\|\cdot\|_{W^{m,p}(\mathbb{R}^n)} \cap \|\cdot\|_{L^\infty(\mathbb{R}^n)}$ forms an algebra.
In addition, $W^{m,p}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ is the maximal algebra in $W^{m,p}(\mathbb{R}^n)$,
i.e., A is an algebra in $W^{m,p}(\mathbb{R}^n) \Rightarrow A \subset W^{m,p}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$.

Remark :
Since the Stein's extension operator $W^{m,p}(\Omega) \cap L^\infty(\Omega)$ is continuous for bounded Lipschitz domain Ω, then Thm 5.2.10 also hold true by replacing $\mathbb{R}^n$ with any bounded Lipschitz domain Ω.
However, Thm 5.2.10 cannot be generalized for arbitrary bounded domain.

We finally close this section by mentioning that the result in Thm 5.2.10 can be extended for exponents m and p in real numbers.
Such results are called the Kato-Ponce inequality

5.3 Hilbert Space

Definition 5.3.1.
Let H be a vector space.
We say that a mapping $(\cdot,\cdot) : H \times H \to \mathbb{R}$ is a bilinear form
if $(\cdot, u)$ is linear for each fixed $u \in H$ and $(v, \cdot)$ is linear for each fixed $v \in H$.
A scalar product or inner product is a bilinear form $(\cdot,\cdot) : H \times H \to \mathbb{R} \ni$
(1) Positive definiteness : $(u,u) \geq 0, \forall u \in H$ and $(u,u) = 0$ iff $u = 0$
(2) Symmetry : $(u,v) = (v,u), \forall u, v \in H$.
In this case, we call the pair $(H, (\cdot,\cdot))$ an inner product space.

Exercise 5.12.
Show the Cauchy-Schwartz inequality : $|(u,v)| \leq (u,u)^{\frac{1}{2}}(v,v)^{\frac{1}{2}}, \forall u, v \in H$.
In addition, show that the function $\|\cdot\|$ defined by $\|u\| := (u,u)^{\frac{1}{2}}, \forall u \in H$ is a norm, which satisfies the parallelogram law :
$\left\|\dfrac{u+v}{2}\right\|^2 + \left\|\dfrac{u-v}{2}\right\|^2 = \dfrac{1}{2}\Big(\|u\|^2 + \|v\|^2\Big), \forall u, v \in H.$

Proof.
claim : $|(u,v)| \leq (u,u)^{\frac{1}{2}}(v,v)^{\frac{1}{2}}, \forall u, v \in H$.
Given $u, v \in H$ and $\lambda \in \mathbb{R}$.
$$\begin{aligned} 0 \leq (u - \lambda v, u - \lambda v) &= (u, u - \lambda v) - \lambda(v, u - \lambda v) \\ &= (u,u) - \lambda(u,v) - \lambda(v,u) + \lambda^2(v,v) \end{aligned}$$
If $(v,v) = 0$, then $v = 0 \Rightarrow |(u,0)| = 0 \leq (u,u)^{\frac{1}{2}} \cdot 0 = 0$.
Set $\lambda = \frac{(u,v)}{(v,v)}$ with $(v,v) \neq 0$.
$$0 \leq (u,u) - \frac{(u,v)^2}{(v,v)} - \frac{(u,v)^2}{(v,v)} + \frac{(u,v)^2}{(v,v)} \Rightarrow (u,v)^2 \leq (u,u)(v,v)$$
Hence, $|(u,v)| \leq (u,u)^{\frac{1}{2}}(v,v)^{\frac{1}{2}}, \forall u, v \in H$.
claim : the function $\|\cdot\|$ defined by $\|u\| := (u,u)^{\frac{1}{2}}, \forall u \in H$ is a norm.
Given $u, v \in H$ and any scalar λ.
$\|u\| = (u,u)^{\frac{1}{2}} \geq 0$ by H is a inner product space.
If $u = 0$, then $\|u\| = \|0\| = (0,0)^{\frac{1}{2}} = 0$.
If $\|u\| = 0$, then $0 = \|u\| = (u,u)^{\frac{1}{2}} \Rightarrow 0 = (u,u) \Rightarrow u = 0$.
$\|\lambda u\| = (\lambda u, \lambda u)^{\frac{1}{2}} = \big(|\lambda|^2(u,u)\big)^{\frac{1}{2}} = |\lambda|(u,u)^{\frac{1}{2}} = |\lambda|\|u\|$
$$\begin{aligned} \|u+v\|^2 &= (u+v, u+v) = (u,u) + (u,v) + (v,u) + (v,v) \\ &\leq \|u\|^2 + 2\|u\|\|v\| + \|v\|^2 \text{ (by Cauchy-Schwarz)} = (\|u\| + \|v\|)^2 \end{aligned}$$
Hence, $\|\cdot\|$ is a norm.
claim : $\left\|\dfrac{u+v}{2}\right\|^2 + \left\|\dfrac{u-v}{2}\right\|^2 = \dfrac{1}{2}\Big(\|u\|^2 + \|v\|^2\Big), \forall u, v \in H.$
Given $u, v \in H$.

$$
\begin{aligned}
\left\|\frac{u+v}{2}\right\|^2+\left\|\frac{u-v}{2}\right\|^2 &= \left(\frac{u+v}{2},\frac{u+v}{2}\right)+\left(\frac{u-v}{2},\frac{u-v}{2}\right)\\
&= \frac{1}{4}\Big((u,u)+(v,u)+(u,v)+(v,v)\Big)\\
&\quad+\frac{1}{4}\Big((u,u)-(v,u)-(u,v)+(v,v)\Big)\\
&= \frac{1}{2}\Big((u,u)+(v,v)\Big)=\frac{1}{2}\Big(\|u\|^2+\|v\|^2\Big)
\end{aligned}
$$

Hence, $\left\|\frac{u+v}{2}\right\|^2+\left\|\frac{u-v}{2}\right\|^2=\frac{1}{2}\Big(\|u\|^2+\|v\|^2\Big), \forall u,v\in H.$ □

Definition 5.3.2.
Given an inner product space $(H,(\cdot,\cdot))$, and induce the function $\|\cdot\|$ defined by $\|u\|:=(u,u)^{\frac{1}{2}}, \forall u\in H$ is a norm.
If $(H,\|\cdot\|)$ is complete, then we called H is a Hilbert space.

Example 5.3.1.
Let Ω be any open set in $\mathbb{R}^n$.
Define $L^2(\Omega):=\left\{f:\Omega\to\mathbb{R} \text{ with } \int_\Omega \big|f(x)\big|^2dx<\infty\right\}$.
The mapping $(\cdot,\cdot)_{L^2(\Omega)}:L^2(\Omega)\times L^2(\Omega)\to\mathbb{R}$ and $(u,v)_{L^2(\Omega)}:=\int_\Omega u(x)v(x)dx$, $\forall u,v\in L^2(\Omega)$ is a scalar product, and we denote $\|u\|_{L^2(\Omega)}=\sqrt{(u,u)_{L^2(\Omega)}}, \forall u\in L^2(\Omega)$ is the corresponding norm.
In fact, $L^2(\Omega)$ is complete with respect to the norm $\|\cdot\|_{L^2(\Omega)}$.

Exercise 5.13.
Let $(H,\|\cdot\|)$ be a normed space.
Suppose the norm $\|\cdot\|$ satisfies $\left\|\frac{u+v}{2}\right\|^2+\left\|\frac{u-v}{2}\right\|^2=\frac{1}{2}\Big(\|u\|^2+\|v\|^2\Big), \forall u,v\in H$.
Define $(u,v):=\frac{1}{2}\Big(\|u+v\|^2-\|u\|^2-\|v\|^2\Big), \forall u,v\in H$. Prove that
1. $(u,u)=\|u\|^2, (u,v)=(v,u), (-u,v)=-(u,v)$ and $(u,2v)=2(u,v), \forall u,v\in H$.
2. $(u+v,w)=(u,w)+(v,w), \forall u,v,w\in H$.
[Hint: use the parallelogram law successively with $(i)u=\tilde u, v=\tilde v$,
$(ii)u=\tilde u+\tilde w, v=\tilde v+\tilde w, (iii)u=\tilde u+\tilde v+\tilde w, v=\tilde w$]
3. $(\lambda u,v)=\lambda(u,v), \forall\lambda\in\mathbb{R}$ and $u,v\in H$.
[Hint: Consider first the case $\lambda\in\mathbb{N}$, then $\lambda\in\mathbb{Q}$, and finally $\lambda\in\mathbb{R}$]
4. $(\cdot,\cdot)$ is a scalar product on H.

Proof.
claim : $(u,u)=\|u\|^2, \forall u\in H$.

Given $u \in H$.
$(u,u) = \frac{1}{2}\bigg(\|u+u\|^2 - \|u\|^2 - \|u\|^2\bigg) = \frac{1}{2}\|2u\|^2 = \frac{1}{2}\cdot 2\|u\|^2 = \|u\|^2.$
claim : $(u,v) = (v,u), \forall u,v \in H.$
Given $u,v \in H$.
$(u,v) = \frac{1}{2}\bigg(\|u+v\|^2 - \|u\|^2 - \|v\|^2\bigg) = \frac{1}{2}\bigg(\|v+u\|^2 - \|v\|^2 - \|u\|^2\bigg) = (v,u).$
claim : $(-u,v) = -(u,v), \forall u,v \in H.$
Given $u,v \in H$.
Since $\left\|\frac{u+v}{2}\right\|^2 + \left\|\frac{u-v}{2}\right\|^2 = \frac{1}{2}\bigg(\|u\|^2 + \|v\|^2\bigg)$ multiply 4 to both side, then we get $\|v+u\|^2 + \|v-u\|^2 = 2(\|u\|^2 + \|v\|^2) \Rightarrow \|v-u\|^2 = 2(\|u\|^2 + \|v\|^2) - \|v+u\|^2.$

$$
\begin{aligned}
(-u,v) &= \frac{1}{2}\bigg(\|-u+v\|^2 - \|-u\|^2 - \|v\|^2\bigg) = \frac{1}{2}\bigg(\|v-u\|^2 - \|u\|^2 - \|v\|^2\bigg) \\
&= \frac{1}{2}\bigg(2(\|u\|^2 + \|v\|^2) - \|v+u\| - \|u\|^2 - \|v\|^2\bigg) \\
&= \frac{1}{2}\bigg(-\|v+u\| + \|u\| + \|v\|\bigg) \\
&= \frac{-1}{2}\bigg(\|u+v\| - \|u\| - \|v\|\bigg) = -(u,v).
\end{aligned}
$$

claim : $(u,2v) = 2(u,v), \forall u,v \in H.$
Given $u,v \in H$.
Let $u = \tilde{u} + \tilde{v}, v = \tilde{v}$.
We use the parallelogram law can get $\|\tilde{u} + 2\tilde{v}\|^2 + \|\tilde{u}\|^2 = 2(\|u+v\|^2 + \|v\|^2).$

$$
\begin{aligned}
(u,2v) &= \frac{1}{2}\bigg(\|u+2v\|^2 - \|u\|^2 - \|2v\|^2\bigg) = \frac{1}{2}\bigg(\|u+2v\|^2 - \|u\|^2 - 4\|v\|^2\bigg) \\
&= \frac{1}{2}\bigg(2\|u+v\|^2 + 2\|v\|^2 - \|u\|^2 - \|u\|^2 - 4\|v\|^2\bigg) \\
&= 2\cdot\frac{1}{2}\bigg(\|u+v\|^2 - \|u\|^2 - \|v\|^2\bigg) = 2(u,v)
\end{aligned}
$$

claim : $(u+v,w) = (u,w) + (v,w), \forall u,v,w \in H.$
Case 1 : let $u = \tilde{u}, v = \tilde{v}$.

$$\left\|\frac{\tilde{u}+\tilde{v}}{2}\right\|^2 + \left\|\frac{\tilde{u}-\tilde{v}}{2}\right\|^2 = \frac{1}{2}\bigg(\|\tilde{u}\|^2 + \|\tilde{v}\|^2\bigg)$$

$\Rightarrow \|\tilde{u}+\tilde{v}\|^2 + \|\tilde{u}-\tilde{v}\|^2 = 2(\|\tilde{u}\|^2 + \|\tilde{v}\|^2)$
Case 2 : let $u = \tilde{u} + \tilde{w}, v = \tilde{v} + \tilde{w}$.

$$\left\|\frac{(\tilde{u}+\tilde{w}) + (\tilde{v}+\tilde{w})}{2}\right\|^2 + \left\|\frac{(\tilde{u}+\tilde{w}) - (\tilde{v}+\tilde{w})}{2}\right\|^2 = \frac{1}{2}\bigg(\|\tilde{u}+\tilde{w}\|^2 + \|\tilde{v}+\tilde{w}\|^2\bigg)$$

$\Rightarrow \|\tilde{u}+\tilde{w}+\tilde{v}+\tilde{w}\|^2 + \|\tilde{u}-\tilde{v}\|^2 = 2(\|\tilde{u}+\tilde{w}\|^2 + \|\tilde{v}+\tilde{w}\|^2)$

Case 3 : let $u = \tilde{u} + \tilde{v} + \tilde{w}, v = \tilde{w}$.

$$\left\| \frac{(\tilde{u} + \tilde{v} + \tilde{w}) + \tilde{w})}{2} \right\|^2 + \left\| \frac{(\tilde{u} + \tilde{v} + \tilde{w}) - \tilde{w}}{2} \right\|^2 = \frac{1}{2}\left(\|\tilde{u} + \tilde{v} + \tilde{w}\|^2 + \|\tilde{w}\|^2 \right)$$

$\Rightarrow \|\tilde{u} + \tilde{v} + \tilde{w} + \tilde{w}\|^2 + \|\tilde{u} + \tilde{v}\|^2 = 2(\|\tilde{u} + \tilde{v} + \tilde{w}\|^2 + \|\tilde{w}\|^2)$

Given $u, v \in H$.

From Case 2 and combine Case 1, we get

$\|u + w + v + w\|^2 + 2(\|u\|^2 + \|v\|^2) - \|u + v\|^2 = 2(\|u + w\|^2 + \|v + w\|^2)$

$\Rightarrow \|u + w + v + w\|^2 - \|u + v\|^2 = 2(\|u + w\|^2 + \|v + w\|^2) - 2(\|u\|^2 + \|v\|^2)$

The formula from Case 3 minus the formula we get above,

$\|u + v\|^2 = \|u + v + w\|^2 + \|w\|^2 + \|u\|^2 + \|v\|^2 - \|u + w\|^2 - \|v + w\|^2$.

Hence, calculate can get

$$\begin{aligned}
(u + v, w) &= \frac{1}{2}\left(\|u + v + w\|^2 - \|u + v\|^2 - \|w\|^2 \right) \\
&= \frac{1}{2}\left(\|u + w\|^2 + \|v + w\|^2 - \|u\|^2 - \|v\|^2 - 2\|w\|^2 \right) \\
&= \frac{1}{2}\left(\|u + w\|^2 - \|u\|^2 - \|w\|^2 \right) + \frac{1}{2}\left(\|v + w\|^2 - \|v\|^2 - \|w\|^2 \right) \\
&= (u, w) + (v, w)
\end{aligned}$$

claim : $(\lambda u, v) = \lambda(u, v), \forall \lambda \in \mathbb{R}$ and $u, v \in H$.

Given $u, v \in H$.

Case 1 : For $\lambda \in \mathbb{N}$, we use math induction.

For $\lambda = 1, (1 \cdot u, v) = (u, v)$.

Assume $\lambda \in \mathbb{N}$ holds.

For $\lambda + 1$,

$((\lambda + 1)u, v) = (\lambda u + u, v) = (\lambda u, v) + (u, v) = \lambda(u, v) + (u, v) = (\lambda + 1)(u, v)$.

Case 2 : For $\lambda \in \mathbb{Q}$.

Write $\lambda = \frac{m}{n}$ with $m, n \in \mathbb{N}$.

$n(\dfrac{m}{n}u, v) = (mu, v) = m(u, v) \Rightarrow (\dfrac{m}{n}u, v) = \dfrac{m}{n}(u, v)$

Case 3 : For $\lambda \in \mathbb{R}$.

For any $\lambda \in \mathbb{R}$, choose a sequence of rational $\lambda_n \in \mathbb{Q}, n \in \mathbb{N}$ such that $\lambda_n \to \lambda$.

$(\lambda u, v) = (\lim\limits_{n \to \infty} \lambda_n u, v) = \lim\limits_{n \to \infty} \lambda_n (u, v) = \lambda(u, v)$

Hence, $(\lambda u, v) = \lambda(u, v), \forall \lambda \in \mathbb{R}$ and $u, v \in H$.

claim : $(\cdot, \cdot)$ is a scalar product on H.

Since $(H, \| \cdot \|)$ is a norm space, then $(u, u) = 0$ iff $u = 0$.

We have proved $(u, v) = (v, u), \forall u, v \in H$ above.

Hence, $(\cdot, \cdot)$ is a scalar product on H. $\square$

Exercise 5.14. (L^p *is not a Hilbert space for* $p \neq 2$)
Show that $\|f\|_{L^p(\Omega)}$ *satisfies the parallelogram law* :

$$\left\|\frac{u+v}{2}\right\|^p_{L^p(\Omega)} + \left\|\frac{u-v}{2}\right\|^p_{L^p(\Omega)} = \frac{1}{2}\left(\|u\|^p_{L^p(\Omega)} + \|v\|^p_{L^p(\Omega)}\right), \forall u, v \in H \text{ iff } p = 2.$$

[*Hint: Use functions with disjoint supports*]

Proof.
Let A be a measurable set $\ni 0 < |A| < |\Omega|$.
Choose a measurable set $B \ni A \cap B = \varnothing$ with $0 < |B| < |\Omega|$.
Let $u = \chi_A$ and $v = \chi_B$.
Then $u + v = \chi_A + \chi_B = \chi_{A\cup B}$, which is 1 on $A \cup B$ and 0 elsewhere.

$$\begin{aligned}
\left\|\frac{u+v}{2}\right\|^p_{L^p(\Omega)} + \left\|\frac{u-v}{2}\right\|^p_{L^p(\Omega)} &= \int_\Omega \left|\frac{\chi_{A\cup B}}{2}\right|^p + \int_\Omega \left|\frac{\chi_{A\setminus B}}{2}\right|^p \\
&= \frac{1}{2^p}\left(\int_\Omega |\chi_{A\cup B}|^p + \int_\Omega |\chi_{A\setminus B}|^p\right) \\
&= \frac{1}{2^p}\left(\int_{A\cup B} 1^p + \int_{A\setminus B} 1^p\right) \\
&= \frac{1}{2^p}\left((|A| + |B|) + \big||A| + |B|\big|\right) \\
&= \frac{1}{2^p}\left(2(|A| + |B|)\right) = \frac{1}{2^{p-1}}\left(|A| + |B|\right)
\end{aligned}$$

$$\begin{aligned}
\frac{1}{2}\left(\|u\|^p_{L^p(\Omega)} + \|v\|^p_{L^p(\Omega)}\right) &= \frac{1}{2}\left(\int_\Omega |u|^p + \int_\Omega |v|^p\right) \\
&= \frac{1}{2}\left(\int_\Omega |\chi_A|^p + \int_\Omega |\chi_B|^p\right) \\
&= \frac{1}{2}\left(\int_A 1^p + \int_B 1^p\right) \\
&= \frac{1}{2}\left(|A| + |B|\right)
\end{aligned}$$

If $|A| = |B|$, then $\frac{1}{2^{p-1}}(2|A|) = \frac{1}{2^{p-2}}|A| = \frac{1}{2}(2|A|) = |A| \Rightarrow p = 2$.
If $|A| \neq |B|$, then $\frac{1}{2^{p-1}} = \frac{1}{2} \Rightarrow p = 2$.
Hence, the parallelogram law holds only at $p = 2$.
Suppose $p = 2$.

$$
\begin{aligned}
\left\|\frac{u+v}{2}\right\|^2_{L^2(\Omega)} + \left\|\frac{u-v}{2}\right\|^2_{L^2(\Omega)} &= \left(\frac{u+v}{2}, \frac{u+v}{2}\right)_{L^2(\Omega)} + \left(\frac{u-v}{2}, \frac{u-v}{2}\right)_{L^2(\Omega)} \\
&= \int_\Omega \left(\frac{u+v}{2}\right)\left(\frac{u+v}{2}\right) + \int_\Omega \left(\frac{u-v}{2}\right)\left(\frac{u-v}{2}\right) \\
&= \frac{1}{4}\int_\Omega \Big((u^2+2uv+v^2)+(u^2-2uv+v^2)\Big) \\
&= \frac{1}{2}\left(\int_\Omega u^2 + \int_\Omega v^2\right) = \frac{1}{2}\Big((u,u)_{L^2(\Omega)} + (v,v)_{L^2(\Omega)}\Big) \\
&= \frac{1}{2}\Big(\|u\|^2_{L^2(\Omega)} + \|v\|^2_{L^2(\Omega)}\Big)
\end{aligned}
$$

Hence, the parallelogram law holds at $p = 2$. □

Definition 5.3.3.
Let Ω be any open set in $\mathbb{R}^n$, then we denote $H^m(\Omega) := W^{m,2}(\Omega), \forall m \in \mathbb{N}$.
In this case, the norm reads $\|u\|_{H^m(\Omega)} = \left(\sum_{|\alpha|\le m} \|\partial^\alpha u\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}}$.

Exercise 5.15.
Define $(u,v)_{H^m(\Omega)} = \sum_{|\alpha|\le m} (\partial^\alpha u, \partial^\alpha v)_{L^2(\Omega)}$.
Show that $(\cdot,\cdot)_{H^m(\Omega)}$ is the scalar product.

Proof.
claim : $(u,u)_{H^m(\Omega)} = \|u\|^2_{H^m(\Omega)}, \forall u \in H^m(\Omega)$.
Given $u \in H^m(\Omega)$.

$$
\begin{aligned}
(u,u)_{H^m(\Omega)} &= \sum_{|\alpha|\le m} (\partial^\alpha u, \partial^\alpha u)_{L^2(\Omega)} \\
&= \sum_{|\alpha|\le m} \int_\Omega (\partial^\alpha u)^2 \\
&= \sum_{|\alpha|\le m} \left(\left(\int_\Omega (\partial^\alpha u)^2\right)^{\frac{1}{2}}\right)^2 \\
&= \sum_{|\alpha|\le m} \|\partial^\alpha u\|^2_{L^2(\Omega)} = \|u\|^2_{H^m(\Omega)}.
\end{aligned}
$$

claim : $(u,v)_{H^m(\Omega)} = (v,u)_{H^m(\Omega)}, \forall u,v \in H^m(\Omega)$.
Given $u, v \in H^m(\Omega)$.

$$\begin{aligned}(u,v)_{H^m(\Omega)} &= \sum_{|\alpha|\leq m}(\partial^\alpha u,\partial^\alpha v)_{L^2(\Omega)}\\ &= \sum_{|\alpha|\leq m}\int_\Omega(\partial^\alpha u)(\partial^\alpha v)\\ &= \sum_{|\alpha|\leq m}\int_\Omega(\partial^\alpha v)(\partial^\alpha u)\\ &= \sum_{|\alpha|\leq m}(\partial^\alpha v,\partial^\alpha u)_{L^2(\Omega)} = (v,u)_{H^m(\Omega)}\end{aligned}$$

claim : $(-u,v)_{H^m(\Omega)} = -(u,v)_{H^m(\Omega)}, \forall u,v\in H^m(\Omega)$.
Given $u,v\in H^m(\Omega)$.
By differentiation is linear, then we have

$$\begin{aligned}(-u,v)_{H^m(\Omega)} &= \sum_{|\alpha|\leq m}(\partial^\alpha(-u),\partial^\alpha v)_{L^2(\Omega)}\\ &= \sum_{|\alpha|\leq m}\int_\Omega(\partial^\alpha(-u))(\partial^\alpha v)\\ &= -\sum_{|\alpha|\leq m}\int_\Omega(\partial^\alpha u)(\partial^\alpha v)\\ &= -\sum_{|\alpha|\leq m}(\partial^\alpha u,\partial^\alpha v)_{L^2(\Omega)} = -(u,v)_{H^m(\Omega)}\end{aligned}$$

claim : $(u,2v)_{H^m(\Omega)} = 2(u,v)_{H^m(\Omega)}, \forall u,v\in H^m(\Omega)$.
By differentiation is linear, then we have

$$\begin{aligned}(u,2v)_{H^m(\Omega)} &= \sum_{|\alpha|\leq m}(\partial^\alpha u,\partial^\alpha(2v))_{L^2(\Omega)}\\ &= \sum_{|\alpha|\leq m}\int_\Omega(\partial^\alpha u)(\partial^\alpha(2v))\\ &= 2\sum_{|\alpha|\leq m}\int_\Omega(\partial^\alpha u)(\partial^\alpha v)\\ &= 2(u,v)_{H^m(\Omega)}\end{aligned}$$

claim : $(u+v,w)_{H^m(\Omega)} = (u,w)_{H^m(\Omega)} + (v,w)_{H^m(\Omega)}, \forall u,v,w\in H^m(\Omega)$.
Given $u,v,w\in H^m(\Omega)$.
By differentiation is linear, then we have

$$
\begin{aligned}
(u+v,w)_{H^m(\Omega)} &= \sum_{|\alpha|\le m} (\partial^\alpha(u+v),\partial^\alpha w)_{L^2(\Omega)}\\
&= \sum_{|\alpha|\le m} (\partial^\alpha u+\partial^\alpha v,\partial^\alpha w)_{L^2(\Omega)}\\
&= \sum_{|\alpha|\le m} \Big((\partial^\alpha u,\partial^\alpha w)_{L^2(\Omega)}+(\partial^\alpha v,\partial^\alpha w)_{L^2(\Omega)}\Big)\\
&= \sum_{|\alpha|\le m} (\partial^\alpha u,\partial^\alpha w)_{L^2(\Omega)}+\sum_{|\alpha|\le m}(\partial^\alpha v,\partial^\alpha w)_{L^2(\Omega)}\\
&= (u,w)_{H^m(\Omega)}+(v,w)_{H^m(\Omega)}
\end{aligned}
$$

claim : $(\lambda u,v)_{H^m(\Omega)}=\lambda(u,v)_{H^m(\Omega)},\forall u,v\in H^m(\Omega)$ and $\lambda\in\mathbb{R}$.
Given $u,v\in H^m(\Omega)$.
By differentiation is linear, then we have

$$
\begin{aligned}
(\lambda u,v)_{H^m(\Omega)} &= \sum_{|\alpha|\le m} (\partial^\alpha(\lambda u),\partial^\alpha v)_{L^2(\Omega)}\\
&= \sum_{|\alpha|\le m} (\lambda\partial^\alpha u,\partial^\alpha v)_{L^2(\Omega)}\\
&= \lambda\sum_{|\alpha|\le m} (\partial^\alpha u,\partial^\alpha v)_{L^2(\Omega)}\\
&= \lambda(u,v)_{H^m(\Omega)}.
\end{aligned}
$$

For $\lambda\in\mathbb{N}$, we can use induction.
For $\lambda=1,(u,v)_{H^m(\Omega)}=(v,u)_{H^m(\Omega)}$ from above.
Suppose $\lambda\in\mathbb{N}$, the equation holds.
For $\lambda+1$,

$$
\begin{aligned}
((\lambda+1)u,v)_{H^m(\Omega)} &= \sum_{|\alpha|\le m} (\partial^\alpha((\lambda+1)u),\partial^\alpha v)_{L^2(\Omega)}\\
&= \sum_{|\alpha|\le m} (\partial^\alpha(\lambda u+u),\partial^\alpha v)_{L^2(\Omega)}\\
&= \sum_{|\alpha|\le m} (\partial^\alpha(\lambda u),\partial^\alpha v)_{L^2(\Omega)}+\sum_{|\alpha|\le m}(\partial^\alpha u,\partial^\alpha v)_{L^2(\Omega)}\\
&= \lambda(u,v)_{H^m(\Omega)}+(u,v)_{H^m(\Omega)}=(\lambda+1)(u,v)_{H^m(\Omega)}.
\end{aligned}
$$

For $\lambda\in\mathbb{Q}$.
Write $\lambda=\frac{m}{n}$ with $m,n\in\mathbb{N}$.

$$
\begin{aligned}
n\left(\frac{m}{n}u, v\right)_{H^m(\Omega)} &= n\sum_{|\alpha|\le m}\left(\partial^\alpha(\frac{m}{n}u), \partial^\alpha v\right)_{L^2(\Omega)}\\
&= n\sum_{|\alpha|\le m}\left((\frac{m}{n})\partial^\alpha u, \partial^\alpha v\right)_{L^2(\Omega)}\\
&= n(\frac{1}{n})\sum_{|\alpha|\le m}\left(m\partial^\alpha u, \partial^\alpha v\right)_{L^2(\Omega)}\\
&= (mu, v)_{H^m(\Omega)} = m(u, v)_{H^m(\Omega)}
\end{aligned}
$$

$\Rightarrow \left(\frac{m}{n}u, v\right)_{H^m(\Omega)} = \frac{m}{n}(u, v))_{H^m(\Omega)}$

For any $\lambda \in \mathbb{R}$, choose a sequence of rational $\lambda_n \in \mathbb{Q}, n \in \mathbb{N}$ such that $\lambda_n \to \lambda$.

$(\lambda u, v)_{H^m(\Omega)} = (\lim_{n\to\infty} \lambda_n u, v)_{H^m(\Omega)} = \lim_{n\to\infty} \lambda_n (u, v)_{H^m(\Omega)} = \lambda(u, v)_{H^m(\Omega)}$

Hence, $(\lambda u, v)_{H^m(\Omega)} = \lambda(u, v)_{H^m(\Omega)}, \forall \lambda \in \mathbb{R}$ and $u, v \in H$.

claim : $(\cdot,\cdot)$ is a scalar product on H.

Suppose $(u, u)_{H^m(\Omega)} = 0$.

Then $0 = (u, u)_{H^m(\Omega)} = \sum_{|\alpha|\le m}(\partial^\alpha u, \partial^\alpha u)_{L^2(\Omega)} = \sum_{|\alpha|\le m}\int_\Omega (\partial^\alpha u)(\partial^\alpha u) \ge 0$.

Hence, $u = 0$.

Suppose $u = 0$.

$(u, u)_{H^m(\Omega)} = \sum_{|\alpha|\le m}(\partial^\alpha u, \partial^\alpha u)_{L^2(\Omega)} = \sum_{|\alpha|\le m}\int_\Omega (\partial^\alpha u)(\partial^\alpha u) = 0$.

Hence, $(u, u)_{H^m(\Omega)} = 0$ iff $u = 0$.

Since $(H, \|\cdot\|)$ is a norm space, then $(u, u)_{H^m(\Omega)} = 0$ iff $u = 0$.

We have proved $(u, v)_{H^m(\Omega)} = (v, u)_{H^m(\Omega)}, \forall u, v \in H$ above.

Hence, $(\cdot,\cdot)$ is a scalar product on H. $\square$

By introducing the gradient $\nabla u(x) = (\partial_1 u(x), \ldots, \partial_n u(x))$, we see that

$$
\begin{aligned}
\int_\Omega \nabla u(x)\cdot\nabla v(x)dx &= \int_\Omega \sum_{i=1}^n \partial_i u(x)\partial_i v(x)dx\\
&= \sum_{i=1}^n\int_\Omega \partial_i u(x)\partial_i v(x)dx = \sum_{i=1}^n(\partial_i u, \partial_i v)_{L^2(\Omega)}.
\end{aligned}
$$

Hence, it is convenient to define

$(\nabla u, \nabla v)_{L^2(\Omega)} := \int_\Omega \nabla u(x)\cdot\nabla v(x)dx$ and $\|\nabla u\|_{L^2(\Omega)} := \sqrt{(\nabla u, \nabla u)_{L^2(\Omega)}}$.

Therefore, the scalar product and norm on $H^1(\Omega)$ can be expressed as

$(u, v)_{H^1(\Omega)} = (u, v)_{L^2(\Omega)} + (\nabla u, \nabla v)_{L^2(\Omega)}$ and $\|u\|_{H^1(\Omega)} = \left(\|u\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}}$.

We introduce the Hessian matrix $\nabla^{\otimes 2}u(x) \equiv \nabla \otimes \nabla u(x)$ with entries $(\nabla^{\otimes 2}u(x))_{ij} = \partial_i \partial_j u(x)$.

Exercise 5.16.
For each column vectors $a, b \in \mathbb{R}^{n\times 1}$, we define the juxtaposition $a \otimes b \in \mathbb{R}^{n\times n}$ (i.e., an $n \times n$ matrix with entries in $\mathbb{R}$) by $a \otimes b = ab^T$.
Compute each entry $(a \otimes b)_{ij}$ of the $n \times n$ matrix $a \otimes b$.

Proof.
$(a \otimes b)_{ij} = \displaystyle\sum_{k=1}^{1} a_{ik}(b^T)_{kj} = \sum_{k=1}^{1} a_{ik}b_{jk} = a_i b_j.$ □

Exercise 5.17.
Let $e_j \in \mathbb{R}^n$ be the j^{th} column of the identity matrix I_n.
Show that $\displaystyle\sum_{k=1}^{n} e_k \otimes e_k = I_n$.

Proof.
Let $1 \le i, j \le n$.

$$\begin{aligned}\left(\sum_{k=1}^{n} e_k \otimes e_k\right)_{ij} &= \sum_{k=1}^{n}(e_k \otimes e_k)_{ij} = \sum_{k=1}^{n}(e_k({e_k}^T))_{ij} = \sum_{k=1}^{n}\sum_{m=1}^{1}(e_k)_{im}(e_k)_{jm} \\ &= \sum_{k=1}^{n}(e_k)_i(e_k)_j = I_n\end{aligned}$$

Hence, $\displaystyle\sum_{k=1}^{n} e_k \otimes e_k = I_n$. □

Exercise 5.18.
Let $u, v \in \mathbb{R}^n$ and consider the matrix $A := I_n + u \otimes v$, which is called the rank-one perturbation of identity.
Determine the relation between u and v to guarantee A^{-1} exists, and compute A^{-1}.

Proof.
claim : A is singular iff $v^T u = -1$
Observe that A is singular iff $\det(A) = 0$ iff 0 is eigenvalue of $A = I + u \otimes v = uv^T$.
Suppose λ is an eigenvalue of uv^T.
$(uv^T)x = \lambda x \Rightarrow v^T(uv^T)x = v^T\lambda x \Rightarrow (v^T u)v^T x = \lambda(v^T x) \Rightarrow v^T u = \lambda$
Then $v^T u$ and uv^T has the same eigenvalue.
Note that $A = I + u \otimes v = uv^T = p(uv^T)$, where $p(t) = 1 + t$ is a polynomial.
Then the eigenvalue of $A = 1 + \lambda = 0 \Rightarrow$ the eigenvalue of $uv^T = v^T u = -1$.
Hence, A^{-1} exists iff $v^T u \neq -1$.

claim : $A^{-1} = I - \dfrac{1}{1+v^Tu}uv^T$, if $v^Tu \neq -1$.

$$
\begin{aligned}
AA^{-1} &= (I+uv^T)(I - \frac{uv^T}{1+v^Tu}) \\
&= I - \frac{uv^T}{1+v^Tu} + uv^T - \frac{(v^Tu)uv^T}{1+v^Tu} \\
&= I + uv^T - \frac{(1+v^Tu)uv^T}{1+v^Tu} \\
&= I + uv^T - uv^T = I
\end{aligned}
$$

Hence, $A^{-1} = I - \dfrac{1}{1+v^Tu}uv^T$, if $v^Tu \neq -1$. □

We see that

$$
\begin{aligned}
\int_\Omega \nabla^{\otimes 2}u(x) : \nabla^{\otimes 2}v(x)dx &= \int_\Omega \sum_{i,j=1}^{n} \partial_i\partial_j u(x)\partial_i\partial_j v(x)dx \\
&= \sum_{i,j=1}^{n} \int_\Omega \partial_i\partial_j u(x)\partial_i\partial_j v(x)dx \\
&= \sum_{i,j=1}^{n} (\partial_i\partial_j u, \partial_i\partial_j v)_{L^2(\Omega)}.
\end{aligned}
$$

Hence, it is convenient to define

$(\nabla^{\otimes 2}u, \nabla^{\otimes 2}v)_{L^2(\Omega)} := \displaystyle\int_\Omega \nabla^{\otimes 2}u(x) : \nabla^{\otimes 2}v(x)dx$,

$\|\nabla^{\otimes 2}u\|_{L^2(\Omega)} := \sqrt{(\nabla^{\otimes 2}u, \nabla^{\otimes 2}u)_{L^2(\Omega)}}$,

$(u,v)_{H^2(\Omega)} = (u,v)_{L^2(\Omega)} + (\nabla u, \nabla v)_{L^2(\Omega)} + (\nabla^{\otimes 2}u, \nabla^{\otimes 2}v)_{L^2(\Omega)}$,

and $\|u\|_{H^2(\Omega)} = \sqrt{\|u\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)} + \|\nabla^{\otimes 2}u\|^2_{L^2(\Omega)}}$.

The $\|\cdot\|_{H^2(\Omega)}$-norm given by $\|u\|_{H^2(\Omega)} = \sqrt{\|u\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)} + \|\nabla^{\otimes 2}u\|^2_{L^2(\Omega)}}$ is actually equivalent to the $\|\cdot\|_{H^2(\Omega)}$-norm given by $\|u\|_{H^m(\Omega)} = \left(\displaystyle\sum_{|\alpha|\leq m} \|\partial^\alpha u\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}}$ in the following sense :

Definition 5.3.4.
Let $\|\cdot\|_1$ and $\|\cdot\|_2$ are norms on the vector space X.
We say that $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent if
$\exists$ constant $c > 0 \ni \dfrac{1}{c}\|u\|_1 \leq \|u\|_2 \leq c\|u\|_1, \forall u \in X$.

Similarly, by introducing the k-tensor $\nabla^{\otimes k}u(x)$ with entries $(\nabla^{\otimes k}u(x))_{i_1 i_2 \cdots i_k} =$

$\partial_{i_1}\partial_{i_2}\cdots\partial_{i_k}u(x)$, we see that

$$\begin{aligned}\int_\Omega \nabla^{\otimes k}u(x) \overset{(k)}{\cdot} \nabla^{\otimes k}v(x)dx &= \int_\Omega \sum_{i_1,i_2,\ldots,i_k=1}^{n} \partial_{i_1}\partial_{i_2}\cdots\partial_{i_k}u(x)\partial_{i_1}\partial_{i_2}\cdots\partial_{i_k}v(x)dx \\ &= \sum_{i_1,i_2,\ldots,i_k=1}^{n} \int_\Omega \partial_{i_1}\partial_{i_2}\cdots\partial_{i_k}u(x)\partial_{i_1}\partial_{i_2}\cdots\partial_{i_k}v(x)dx \\ &= \sum_{i_1,i_2,\ldots,i_k=1}^{n} (\partial_{i_1}\partial_{i_2}\cdots\partial_{i_k}u, \partial_{i_1}\partial_{i_2}\cdots\partial_{i_k}v)_{L^2(\Omega)}.\end{aligned}$$

Then it is convenient to define $\nabla^{\otimes 0}u := u$, and for each $k \in \mathbb{N}$ that

$(\nabla^{\otimes k}u, \nabla^{\otimes k}v)_{L^2(\Omega)} := \displaystyle\int_\Omega \nabla^{\otimes k}u(x) \overset{(k)}{\cdot} \nabla^{\otimes k}v(x)dx,$

and $\|\nabla^{\otimes k}u\|_{L^2(\Omega)} := \sqrt{(\nabla^{\otimes k}u, \nabla^{\otimes k}u)_{L^2(\Omega)}}$,

where we define the scalar products and norm by

$(u, v)_{H^m(\Omega)} = \displaystyle\sum_{k=0}^{m}(\nabla^{\otimes k}u, \nabla^{\otimes k}v)_{L^2(\Omega)}$ and $\|u\|_{H^m(\Omega)} = \left(\displaystyle\sum_{k=0}^{m} \|\nabla^{\otimes k}u\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}}$.

Note :

the $\|\cdot\|_{H^m(\Omega)}$-norm is actually equivalent to the $\|\cdot\|_{H^2(\Omega)}$-norm by def'n 5.3.3.

Theorem 5.3.1. (*Trace Thm*) 說明邊界 *is well-defined*

Let Ω be a bounded domain in $\mathbb{R}^n$ with $C^{0,1}$ boundary $\partial\Omega$,

i.e. Ω is a bounded Lipschitz domain in $\mathbb{R}^n$, i.e., Ω is locally Lipsscitz.

The mapping $Tr : C^\infty(\overline{\Omega}) \to C^\infty(\partial\Omega), Tr(u) = u\big|_{\partial\Omega}$ extends to a unique bounded linear surjective mapping $H^1(\Omega) \to H^{\frac{1}{2}}(\partial\Omega)$,

where $H^{\frac{1}{2}}(\partial\Omega) := Tr(H^1(\Omega)) \subset L^2(\partial\Omega)$), which is a Hilbert space equipped with the quotient norm $\|g\|_{H^{\frac{1}{2}}(\partial\Omega)} = \inf\limits_{u\in H^1(\Omega), Tr(u)=g} \|u\|_{H^1(\Omega)}$.

Exercise 5.19.

Verify that $\|\cdot\|_{H^{\frac{1}{2}}(\Omega)}$ is a norm.

Example 5.3.2. 如何 *restrict on the boundary*？

Let $H^{\frac{1}{2}}(\Gamma) := \{\chi_\Gamma f; f \in H^{\frac{1}{2}}(\partial\Omega)\}$, and $\|f\|_{H^{\frac{1}{2}}(\Gamma)} = \|\chi_\Gamma f\|_{H^{\frac{1}{2}}(\partial\Omega)}$.

Then $H^{\frac{1}{2}}(\Gamma) = \{f\big|_\Gamma; \in H^{\frac{1}{2}}(\partial\Omega)\}$, and $\|f\|_{H^{\frac{1}{2}}(\Gamma)} = \inf\limits_{F\in H^{\frac{1}{2}}(\partial\Omega), F|_\Gamma = f} \|F\|_{H^{\frac{1}{2}}(\partial\Omega)}$.

Note : $H^{\frac{1}{2}}(\partial\Omega) \supset H^{\frac{1}{2}}(\Gamma)$ and $\|f\|_{H^{\frac{1}{2}}(\partial\Omega)} \le \|f\|_{H^{\frac{1}{2}}(\Gamma)}$.

It is also possible to define the "traces" and "normal derivatives" on $\partial\Omega$ for H^m-functions (see Thm 3.2.10), and similar results for higher order derivatives :

Theorem 5.3.2. (*Trace theorem in general*)
Let $m \in \mathbb{Z}_{\geq 2}$ and Ω be a bounded $C^{m-1,1}$ domain in $\mathbb{R}^n$.
The mapping $u \in C^\infty(\overline{\Omega}) \mapsto (u\big|_{\partial\Omega}, \partial_\nu u\big|_{\partial\Omega} \in C^\infty(\partial\Omega) \times C^\infty(\partial\Omega)$,
where $\partial_\nu u := \nu \cdot \nabla u$, extends to a unique bounded linear surjective mapping $H^m(\Omega) \to H^{m-\frac{1}{2}}(\partial\Omega) \times H^{m-\frac{3}{2}}(\partial\Omega)$, where for each $1 \leq k \leq m$, the space $H^{k-\frac{1}{2}}(\partial\Omega) := Tr(H^k(\Omega)) \subset L^2(\partial\Omega)$, which is a Hilbert space equipped with the quotient norm $\|g\|_{H^{k-\frac{1}{2}}(\partial\Omega)} = \inf\limits_{u \in H^k(\Omega), Tr(u)=g} \|u\|_{H^k(\Omega)}$.

Corollary 5.3.3. (*Density*) 將Ω內部的性質推到∂Ω仍然保持住
Let Ω be the bounded Lipschitz domain in $\mathbb{R}^n$ and let $m \in \mathbb{N}$.
Given any $f \in H^m(\Omega)$, $\exists$ a sequence $\{f_k\}_{k\in\mathbb{N}} \subset C^\infty(\overline{\Omega}) \ni \lim\limits_{k\to\infty} \|f_k - f\|_{H^m(\Omega)} = 0$, i.e., $f_n \to f$ in $H^m(\Omega)$.

It is natural to consider the following subspace of $H^m(\Omega)$:

Definition 5.3.5. 定義如何保住邊界
For each $m \in \mathbb{N}$, we define $H_0^m(\Omega)$ be the closure of $C_c^\infty(\Omega)$ with respect to the norm of $\|\cdot\|_{H^m(\Omega)}$, i.e., $H_0^m(\Omega) := \overline{C_c^\infty(\Omega)}^{H^m(\Omega)} \Rightarrow$ zero-extension.

Lemma 5.3.4. *The relation between $H_0^m(\Omega)$ and $H^m(\Omega)$*
Let Ω is a bounded Lipschitz domain in $\mathbb{R}^n$, then $H_0^1(\Omega) = \{u \in H^1(\Omega); u\big|_{\partial\Omega} = 0\}$.
If Ω is a bounded $C^{1,1}$ domain in $\mathbb{R}^n$, then
$H_0^1(\Omega) = \{u \in H^1(\Omega); u\big|_{\partial\Omega} = \partial_\nu u\big|_{\partial\Omega} = 0\}$.
邊界性質夠好，則可將內部性質推到邊界
Note : $H_0^1(\Omega) \cap H_0^2(\Omega) = \{u \in H^2(\Omega); u\big|_{\partial\Omega} = 0\}$.

Remark :
The subscript 0 in $H_0^m(\Omega)$ means the zero boundary value.
Hence, here we denote $C_c^\infty(\Omega)$ rather than $C_0^\infty(\Omega)$ to avoid confusion.
Since $H_0^1(\Omega) \cap H^2(\Omega) = \{u \in H^2(\Omega); u\big|_{\partial\Omega} = 0\}$, for any bounded $C^{1,1}$ domain $\Omega \subset \mathbb{R}^n$, then $H_0^1(\Omega) \cap H^2(\Omega) \neq H_0^2(\Omega)$.

Example 5.3.3.
For each parameter $\omega \geq 0$, we consider the function $v(t,x) = e^{i\omega t}u(x)$.
It is not difficult to see that $(\partial_t^2 - c^2\Delta)v(t,x) = -c^2 e^{i\omega t}(\Delta + k^2)u(x)$ with $k = \dfrac{\omega}{c}$.
Since $c \neq 0$ and $e^{i\omega t} \neq 0, \forall t \in \mathbb{R}$, then it is natural to consider the following second order elliptic PDE on a bounded Lipschitz domain Ω : $\begin{cases} (\Delta + k^2)u = f \text{ in } \Omega, \\ u\big|_{\partial\Omega} = 0. \end{cases}$

Remark :
When $k = 0$, we call $(\Delta + k^2)u = f$ in Ω, $u\big|_{\partial\Omega} = 0$ the Poisson equation.
When $k > 0$, we call $(\Delta + k^2)u = f$ in Ω, $u\big|_{\partial\Omega} = 0$ the Helmholtz equation, which described the acoustic wave with fixed wave number $k > 0$.
The term $e^{i\omega t}$ is called the time-harmonic, and thus we also called the Helmholtz equation the time-harmonic wave equation.

In view of the integration by parts (Thm 5.2.6), one first formally compute that
$\displaystyle\int_\Omega f\phi = \int_\Omega \Delta u\phi + k^2 \int_\Omega u\phi = -\int_\Omega \nabla u \cdot \nabla\phi + k^2 \int_\Omega u\phi, \forall\phi \in C_c^\infty(\Omega).$

Definition 5.3.6. *In view of the definition of $H_0^1(\Omega)$ and Lma 5.3.4*
We say that u is a weak solution of $(\Delta + k^2)u = f$ in Ω with $u\big|_{\partial\Omega} = 0$
if $u \in H_0^1(\Omega)$ and $(f, \phi)_{L^2(\Omega)} = -(\nabla u, \nabla\phi)_{L^2(\Omega)} + k^2(u, \phi)_{L^2(\Omega)}, \forall\phi \in H_0^1(\Omega)$, for any pre-given $f \in L^2(\Omega)$.

One sees that $(f, \phi)_{L^2(\Omega)}$ is actually well-defined for $f, \phi \in L^2(\Omega)$, and we have $\phi \in H_0^1(\Omega)$, then it is natural to ask :
Given any $\phi \in H_0^1(\Omega)$, whether the term $T_f(\phi) := (f, \phi)_{L^2(\Omega)}$ still make sense for lower regularity f ?

By Exercise 4.6, and the density lemma (Cor 5.3.3), we have
$\|f\|_{L^2(\Omega)} = \sup\limits_{0\not\equiv\phi\in L^2(\Omega)} \dfrac{(f, \phi)_{L^2(\Omega)}}{\|\phi\|_{L^2(\Omega)}} = \sup\limits_{0\not\equiv\phi\in C_c^\infty(\Omega)} \dfrac{(f, \phi)_{L^2(\Omega)}}{\|\phi\|_{L^2(\Omega)}}.$
This is actually a special case of the following general notion :

Definition 5.3.7.
Let X and Y be two Banach spaces.
An unbounded linear operator from X into Y is a linear map $\mathscr{L} : dom(\mathscr{L}) \subset X \to Y$ defined on a linear spaces $dom(L) \subset X$ with values Y.
The linear space $dom(\mathscr{L})$ is called the domain of L.
If $Y = \mathbb{R}$ or $Y = \mathbb{C}$, then we called $\mathscr{L}$ is a linear functional on the domain $dom(\mathscr{L})$.
Note : unbounded linear operator = linear mapping

Example 5.3.4.
Let $\Delta : H^2(\Omega) \to L^2(\Omega)$.
Then $dom(\Delta) : H^2(\Omega) \subset L^2(\Omega)$.
Note : $dom(\Delta)$ can $= H^3(\Omega)$ or others, so $dom(\Delta)$ is not unique.

Definition 5.3.8.
One says that $\mathscr{L}$ is bounded (or continuous)
if dom($\mathscr{L}$) $= X$ and $\exists$ constant $c \geq 0 \ni \|\mathscr{L}u\|_Y \leq c\|u\|_X, \forall u \in X$.
The norm of a bounded operator is defined by
$\|\mathscr{L}\|_{X\to Y} := \inf\{c \geq 0; \|\mathscr{L}u\|_Y \leq c\|u\|_X, \forall u \in X\} \equiv \sup\limits_{u\neq 0} \dfrac{\|\mathscr{L}u\|_Y}{\|u\|_X}$.
If $Y = \mathbb{R}$ or $Y = \mathbb{C}$, one says that such $\mathscr{L}$ is a bounded (or continuous) linear functional on X.

Exercise 5.20.
Let X and Y are normed spaces.
One says that $\mathscr{L} : X \to Y$ is bounded (or continuous) if $\exists$ constant $c \geq 0 \ni \|\mathscr{L}u\|_Y \leq c\|u\|_X, \forall u \in X$.
Verify that $\|\mathscr{L}\|_{X\to Y} := \inf\{c \geq 0; \|\mathscr{L}u\|_Y \leq c\|u\|_X, \forall u \in X\}$ is a norm.

Proof.
claim : $\|\mathscr{L}\|_{X\to Y} \geq 0$.
Since $\|\mathscr{L}\|_{X\to Y} = \inf\{c \geq 0; \|\mathscr{L}u\|_Y \leq c\|u\|_X, \forall u \in X\} \geq 0$, then $\|\mathscr{L}\|_{X\to Y} \geq 0$.
claim : $\|\mathscr{L}\|_{X\to Y} = 0$ iff $\mathscr{L} = 0$.
Suppose $\inf\{c \geq 0; \|\mathscr{L}u\|_Y \leq c\|u\|_X, \forall u \in X\} = \|\mathscr{L}\|_{X\to Y} = 0$.
Then given $\varepsilon > 0$, $\exists\ c \geq 0 \ni \|\mathscr{L}u\|_Y \leq c\|u\|_X < \varepsilon\|u\|_X, \forall u \in X$.
For $\varepsilon \to 0^+, \|\mathscr{L}u\|_Y = 0, \forall u \in X$.
Hence, $\mathscr{L} = 0$.
Suppose $\mathscr{L} = 0$.
Then $\mathscr{L}u = 0, \forall u \in X \Rightarrow \|\mathscr{L}u\|_Y = 0$.
Hence, $\|\mathscr{L}u\|_Y \leq c\|u\|_X$ holds for any $c \geq 0 \Rightarrow \|\mathscr{L}\|_{X\to Y} = 0$.
claim : $\|\alpha\mathscr{L}\|_{X\to Y} = |\alpha|\|\mathscr{L}\|_{X\to Y}, \forall f \in \mathscr{L}$ and scalar α.
Given a scalar α and $f \in \mathscr{L}$.
$\|\alpha f\|_{X\to Y} = \inf\{c \geq 0; \|(af)u\|_Y \leq c\|u\|_X, \forall u \in X\}$.
For any $u \in X$, $\|(\alpha f)u\|_Y = \|\alpha(fu)\|_Y = |\alpha|\|fu\|_Y$.
Let $\|fu\|_Y \leq k\|u\|_X$, for some $k \geq 0$.
Then $\|(\alpha f)u\|_Y = |\alpha|\|fu\|_Y \leq |\alpha|k\|u\|_X$.
Hence, if k satisfies $\|fu\|_Y \leq k\|u\|_X$, then $c = |\alpha|k$ satisfies $\|(\alpha f)u\|_Y \leq c\|u\|_X$.
The norm of αf is $\|\alpha f\|_{X\to Y} = \inf\{c \geq 0; \|(\alpha f)u\|_Y \leq c\|u\|_X\}$.
The set of c for αf is $\{|\alpha|k; k \geq 0, \|fu\|_Y \leq k\|u\|_X\}$.
$$\begin{aligned}\inf\{|\alpha|k; k \geq 0, \|fu\|_Y \leq k\|u\|_X\} &= |\alpha| \inf\{k \geq 0; \|fu\|_Y \leq k\|u\|_X\} \\ &= |\alpha|\|f\|_{X\to Y}\end{aligned}$$
Hence, $\|\alpha f\|_{X\to Y} = |\alpha|\|f\|_{X\to Y}$.
claim : $\|f + g\|_{X\to Y} \leq \|f\|_{X\to Y} + \|g\|_{X\to Y}, \forall f, g \in \mathscr{L}$
Given $f, g \in \mathscr{L}$.
By the triangle inequality in the normed space Y, for any $u \in X$,
$\|(f + g)u\|_Y = \|fu + gu\|_Y \leq \|fu\|_Y + \|gu\|_Y$.

If $\|fu\|_Y \le c_1\|u\|_X$ and $\|gu\|_Y \le c_2\|u\|_X$, for some c_1, c_2, then
$\|(f+g)u\|_Y = \|fu+gu\|_Y \le \|fu\|_Y + \|gu\|_Y \le c_1\|u\|_X + c_2\|u\|_X = (c_1+c_2)\|u\|_X$.
Since $\|f\|_{X\to Y} = \inf\{c \ge 0; \|fu\|_Y \le c\|u\|_X\}$ and similar with $\|g\|_{X\to Y}$, then for any $\varepsilon > 0, \exists\ c_1 \ge \|f\|_{X\to Y}$ and $c_2 \ge \|g\|_{X\to Y} \ni \|f+g\|_{X\to Y} \le c_1 + c_2$.
Taking the infimum over all such c_1, c_2, we get $\|f+g\|_{X\to Y} \le \|f\|_{X\to Y} + \|g\|_{X\to Y}$.
Therefore, $\|\mathscr{L}\|_{X\to Y} := \inf\{c \ge 0; \|\mathscr{L}u\|_Y \le c\|u\|_X, \forall u \in X\}$ is a norm. $\square$

Definition 5.3.9.
Let H be a Hilbert space.
The dual space H^* of H is a Hilbert space consists of all bounded linear functional on H, with norm $\|\cdot\|_{H^*} = \|\cdot\|_{H\to\mathbb{R}}$.

Example 5.3.5.
The equality $\|f\|_{L^2(\Omega)} = \sup\limits_{0\not\equiv\phi\in L^2(\Omega)} \dfrac{(f,\phi)_{L^2(\Omega)}}{\|\phi\|_{L^2(\Omega)}} = \sup\limits_{0\not\equiv\phi\in C_c^\infty(\Omega)} \dfrac{(f,\phi)_{L^2(\Omega)}}{\|\phi\|_{L^2(\Omega)}}$ means that f can be identify with the bounded linear functional $T_f(\phi) := (f,\phi)_{L^2(\Omega)}, \forall\phi \in L^2(\Omega)$, i.e., $L^2(\Omega) = (L^2(\Omega))^*$.

Answer the question :
Since we have $\phi \in H_0^1(\Omega)$, then by the density lemma (Cor 5.3.3),
the equation $\|f\|_{L^2(\Omega)} = \sup\limits_{0\not\equiv\phi\in L^2(\Omega)} \dfrac{(f,\phi)_{L^2(\Omega)}}{\|\phi\|_{L^2(\Omega)}} = \sup\limits_{0\not\equiv\phi\in C_c^\infty(\Omega)} \dfrac{(f,\phi)_{L^2(\Omega)}}{\|\phi\|_{L^2(\Omega)}}$ suggests us to define the following quantity :
$\|f\|_{H^{-1}(\Omega)} := \sup\limits_{\phi\not\equiv 0} \dfrac{\int_\Omega f\phi}{\|\phi\|_{H_0^1(\Omega)}} \equiv \sup\limits_{\|\phi\|_{H_0^1(\Omega)}=1} \displaystyle\int_\Omega f\phi$.
Here we write $\displaystyle\int_\Omega f\phi$ rather than $(f,\phi)_{L^2(\Omega)}$ because here f may not in $L^2(\Omega)$.
We see that $\|f\|_{H^{-1}(\Omega)} = \|T_f\|_{(H_0^1(\Omega))^*}$.
Hence, we immediately can define $H^{-1}(\Omega) = (H_0^1(\Omega))^*$.

Definition 5.3.10.
For each bounded Lipschitz domain Ω in $\mathbb{R}^n$, we define $H^{-1}(\Omega) := \{$

From the above discussions, we obtain that $H_0^1(\Omega) \subset L^2(\Omega) \cong (L^2(\Omega))^* \subset H^{-1}(\Omega)$.
From definition, one can easily note that $\left|\displaystyle\int_\Omega fg\right| \le \|f\|_{H^{-1}(\Omega)}\|g\|_{H_0^1(\Omega)}$.
Warning note :
However, one should aware that in general,
$\displaystyle\int_\Omega |fg|$ cannot be bounded above by $\|f\|_{H^{-1}(\Omega)}\|g\|_{H_0^1(\Omega)}$.
It is interesting to compare this with $\|f\|_{L^p(\Omega)} = \sup\limits_{\|g\|_{L^q(\Omega)}=1} \displaystyle\int_\Omega fg$.

Remark :
Similarly, for each bounded smooth (for simplicity) domain Ω in $\mathbb{R}^n$, one can define $H^{-m}(\Omega) := (H_0^m(\Omega))^*$, for each $m \in \mathbb{N}$.
More precisely, $H^{-m} := \{f \in \mathscr{D}'(\Omega); T_f \in (H_0^m(\Omega))^*\}$.
Similarly, we obtain the triplet $H_0^m(\Omega) \subset L^2(\Omega) \cong (L^2(\Omega))^* \subset H^{-m}(\Omega), \forall m \in \mathbb{N}$.

This ideas also can be extended for real numbers $m \geq 0$ and bounded Lipschitz domains.
Here we also remark that $(H^m(\Omega))^*$, for $0 \leq m \in \mathbb{R}$ can be characterized in terms of quotient norm.

Note : $(H^m(\Omega))^* \subsetneq H^{-m}(\Omega)$ and $(H^{-m}(\Omega))^* = H_0^m(\Omega)$.

Now exhibit the following remarkable fact for all bounded linear functionals on Hilbert spaces :

Theorem 5.3.5. (*Riesz-Frechet representation Thm*)
Given any $\varphi \in H^*$.
Then $\exists$ *a unique* $v \in H$ *with* $\|v\|_H = \|\varphi\|_{H^*} \ni \varphi(u) = (v, u)_H, \forall u \in H$.

Example 5.3.6.
Let $H = H_0^1(\Omega)$, *支點為* $L^2(\Omega)$. *Then* $(H_0^1(\Omega))^* = H^{-1}(\Omega)$.
Given any $\varphi \in H^{-1}(\Omega)$.
Then $\exists$! $v \in H_0^1(\Omega)$ *with* $\|v\|_{H^{-1}(\Omega)} = \|\varphi\|_{H_0^1(\Omega)} \ni$
$\langle \varphi, u \rangle = (v, u)_{H_0^1(\Omega)} = (\nabla v, \nabla u)_{L^2(\Omega)}, \forall u \in H_0^1(\Omega)$.

Remark :
Due to the above Riesz-Frechet theorem, we also denote $\varphi(u) = \langle \varphi, u \rangle$, or more precisely, $\varphi(u) = \langle \varphi, u \rangle_{H^* \otimes H}$, and we call $\langle \cdot, \cdot \rangle$ the duality pair.

The Riesz-Frechet representation theorem (Thm 5.3.5) defines the isometry ι : $H^* \to H$ by $\iota(\varphi) := v$.
Similar to Sobolev embeddings (Thm 5.2.4), this suggests us to identify H and H^*, but this sometimes cause some troubles.
For example, if we identify both $L^2(\Omega) \cong (L^2(\Omega))^*$ and $H_0^1(\Omega) \cong (H_0^1(\Omega))^* = H^{-1}(\Omega)$, then the triplet $H_0^1(\Omega) \subset L^2(\Omega) \cong (L^2(\Omega))^* \subset H^{-1}(\Omega)$ implies $L^2 \cong H_0^1(\Omega) \cong H^{-1}(\Omega)$, which obviously make no sense.
In typical situation, we usually identify $L^2(\Omega) = (L^2())^*$ and not identify $H_0^1(\Omega)$ with its dual $H^{-1}(\Omega)$ despite we have the Riesz-Frechet representation theorem (Thm 5.3.5).

Exercise 5.21.
Let H *be a complex vector space.*
We say that a mapping $(\cdot, \cdot) : H \times H \to \mathbb{C}$ *is a complex inner product if*
1. Linearity :
$(a_1 u_1 + a_2 u_2, v)_H = a_1 (u_1, v)_H + a_2 (u_2, v)_H, \forall a_1, a_2 \in \mathbb{C}$ *and* $\forall u_1, u_2, v \in H$.

2. Sesquilinearity :
$(u, a_1v_1 + a_2v_2)_H = \overline{a_1}(u, v_1)_H + \overline{a_2}(u, v_2)_H, \forall a_1, a_2 \in \mathbb{C}$ *and* $\forall u, v_1, v_2 \in H$.
3. Positive definiteness : $(u, u) \geq 0, \forall u \in H$ *and* $(u, u) = 0$ *iff* $u = 0$.
4. Skew symmetry : $(u, v) = \overline{(v, u)}, \forall u, v \in H$.
Show that $\|u\|_H := \sqrt{(u, u)_H}$ *defines a norm.*
Determine the parallelogram law for $\| \cdot \|_H$.

Proof.
claim : $\|u\|_H \geq 0, \forall u \in H$.
Given $u \in H$. Since $(u, u)_H \geq 0$, then $\|u\|_H = \sqrt{(u, u)_H} \geq 0$.
claim : $\|u\|_H = 0$ iff $u = 0$.
Given $u \in H$.
Since $\|u\|_H = 0 \Rightarrow (u, u)_H = 0$, then $u = 0$.
Since $u = 0$, then $\|u\|_H = (u, u)_H{}^{\frac{1}{2}} = 0$.
claim : $\|\lambda u\|_H = |\lambda| \|u\|_H$.
Given $\lambda \in \mathbb{C}$ and $u \in H$.
$\|\lambda u\|_H = \sqrt{(\lambda u, \lambda u)_H} = \sqrt{\lambda\overline{\lambda}(u, u)_H} = \sqrt{\lambda\overline{\lambda}}\|u\|_H = |\lambda| \|u\|_H$.
claim : $\|u + v\|_H \leq \|u\|_H + \|v\|_H$.
$$\begin{aligned} \|u + v\|_H^2 &= (u + v, u + v)_H = (u, u)_H + (u, v)_H + (v, u)_H + (v, v)_H \\ &= (\|u\|_H + \|v\|_H)^2. \end{aligned}$$
Hence, $\|u\|_H := \sqrt{(u, u)_H}$ defines a norm.
claim : the parallelogram law for $\| \cdot \|_H$.
Given $u, v \in H$.
$$\begin{aligned} \|u + v\|_H^2 + \|u - v\|_H^2 &= (u + v, u + v)_H + (u - v, u - v)_H \\ &= [(u, u)_H + (u, v)_H + (v, u)_H + (v, v)_H] \\ &\quad + [(u, u)_H - (u, v)_H - (v, u)_H + (v, v)_H] \\ &= 2(u, u)_H^2 + 2(v, v)_H^2 = 2\|u\|_H^2 + 2\|v\|_H^2 \\ &= 2(\|u\|_H^2 + \|v\|_H^2) \end{aligned}$$
Hence, the parallelogram law for $\| \cdot \|_H$ holds. □

5.4 Solving Elliptic PDE for Small Wave Number

Now turn back to the Helmholtz equation $(\Delta + k^2)u = f$ in Ω and $u\big|_{\partial\Omega} = 0$, and we now can ask the following question (in a proper way).
Question (See also Question in 5.6 for a slightly general case) :
Suppose Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, and let $f \in H^{-1}(\Omega)$ and $k \geq 0$.
Can we find a weak solution u of the Helmholtz equation $\begin{cases} (\Delta + k^2)u = f \text{ in } \Omega \\ u\big|_{\partial\Omega} = 0 \end{cases}$?
More precisely, can we find $u \in H_0^1(\Omega)$ satisfies
$(\nabla u, \nabla v)_{L^2(\Omega)} - k^2(u, v)_{L^2(\Omega)} = -\langle f, v\rangle_{H^{-1}\otimes H_0^1(\Omega)}, \forall v \in H_0^1(\Omega)$ or not ?
In addition, is the solution unique ?

In view of the above formulation, it is natural to introduce the following notions :

Definition 5.4.1.
A bilinear form $a(\cdot,\cdot) : H \times H \to \mathbb{R}$ is said to be
(a) continuous if $\exists\ c > 0 \ni |a(u,v)| \leq c\|u\|_H\|v\|_H, \forall u, v \in H$.
(b) coercive if $\exists\ \alpha > 0 \ni a(v,v) \geq \alpha\|v\|_H^2, \forall v \in H$.
(c) symmetry if $a(u,v) = a(v,u), \forall u, v \in H$.
In this case, the coercive bilinear form a is also said to be positive definite.

Remark for finite dimensional case in Linear Algebra :
We say that a symmetric matrix $A \in \mathbb{R}^{n\times n}$ is positive definite
if $Au \cdot u = u^T Au > 0, \forall 0 \neq u \in \mathbb{R}^n$.
Since A is symmetric, then it is unitary diagonalizable (iff A is normal, i.e., $AA^T = A^T A$), i.e., $\exists$ an invertible matrix Q with $Q^{-1} = Q^T \ni A = QDQ^T$, where D is a diagonal matrix.
Hence, we can write
$u^T Au = u^T QDQ^T u = D(Q^T u) \cdot (Q^T u) = D(Q^{-1}u) \cdot (Q^{-1}u) > 0, \forall 0 \neq u \in \mathbb{R}^n$
$\Rightarrow Dv \cdot v > 0, \forall 0 \neq v \in \mathbb{R}^n$.
Therefore, all entries in D, they called the eigenvalue of A, must be positive.
This explains why the condition $Au \cdot u = u^T Au > 0, \forall 0 \neq u \in \mathbb{R}^n$ called positive definite, and so is the above definition for infinite dimensional case.

We now exhibit the following remarkable result, which is a very simple and efficient tool for solving linear elliptic PDE :

Theorem 5.4.1. (*Lax-Milgram*)
Assume that $a(\cdot,\cdot)$ is continuous coercive bilinear form on H.
Then, given any $\varphi \in H^*$, $\exists$ a unique element $u \in H \ni a(u,v) = \langle \varphi, v\rangle, \forall v \in H$.
Moreover, if a is symmetric, then u is characterized by the property $u \in H$,
$\frac{1}{2}a(u,u) - \langle \varphi, u\rangle = \min_{v\in H}\left\{\frac{1}{2}a(v,v) - \langle \varphi, v\rangle\right\}$.

Remark :
In the language of the calculus of variations, one says that $a(u,v) = \langle \varphi, v\rangle, \forall v \in H$ is the Euler equation associated with the minimization problem :
$\frac{1}{2}a(u,u) - \langle \varphi, u\rangle = \min_{v\in H}\left\{\frac{1}{2}a(v,v) - \langle \varphi, v\rangle\right\}, \forall u \in H$.

In order to answer Question in the beginning of this subsection, it is now natural to consider $a(u,v) := (\nabla u, \nabla v)_{L^2(\Omega)} - k^2(u,v)_{L^2(\Omega)}, \varphi = -f \in H^{-1}(\Omega)$.
It is easy to verify that $a : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ is a continuous and symmetric bilinear form.
Goal : To verify its coercivity, we need the following lemma :

Lemma 5.4.2. (*Poincare's Inequality*)
Let Ω be a bounded open set.
Then $\exists$ a constant C, depending on $\Omega \ni \|u\|_{L^2(\Omega)} \leq C\|\nabla u\|_{L^2(\Omega)}, \forall u \in H_0^1(\Omega)$.

Exercise 5.22.
Let I be a bounded interval in $\mathbb{R}$.
Show that there exists a constant $C \ni \|u\|_{L^2(I)} \leq C\|u'\|_{L^2(I)}, \forall u \in H_0^1(I)$, where C depends on the length of the interval $|I| < \infty$.
(*Hint : This result is not optimal, you will see the optimal inequality in the proof.*)

Definition 5.4.2. *By Poincare inequality and the density result* (*Cor 5.3.3*)
One can define a positive number, called the fundamental tone of Ω, by
$\lambda_1 := \inf_{0\not\equiv u \in C_c^\infty(\Omega)} \frac{\|\nabla u\|^2_{L^2(\Omega)}}{\|u\|^2_{L^2(\Omega)}} = \inf_{0\not\equiv u \in H_0^1(\Omega)} \frac{\|\nabla u\|^2_{L^2(\Omega)}}{\|u\|^2_{L^2(\Omega)}} > 0.$

The quotient $\frac{\|\nabla u\|^2_{L^2(\Omega)}}{\|u\|^2_{L^2(\Omega)}}$ sometimes also referred as the Rayleigh quotient.

Hence, $a(u,u) = \|\nabla u\|^2_{L^2(\Omega)} - k^2\|u\|^2_{L^2(\Omega)} \geq \left(1 - \frac{k^2}{\lambda_1}\right)\|\nabla u\|^2_{L^2(\Omega)}$, which means that a is coercive when $k^2 < \lambda_1$.

Conclusion :
By using the Lax-Milgram theorem (Thm 3.4.4), we can give some partial answers to the question at the begining of the subsection 5.4.

Theorem 5.4.3.
Let Ω *be a bounded Lipschitz domain in* $\mathbb{R}^n$ *and let* $f \in H^{-1}(\Omega)$.
If $k^2 < \lambda_1$, *then exists a unique* $u \in H_0^1(\Omega)$ *satisfies*
$(\nabla u, \nabla v)_{L^2(\Omega)} - k^2(u,v)_{L^2(\Omega)} = -\langle f, v\rangle_{H^{-1}\otimes H_0^1(\Omega)}, \forall v \in H_0^1(\Omega)$ *satisfying*
$F(u) = \min\limits_{v\in H_0^1(\Omega)} F(v), F(v) = \dfrac{1}{2}\Bigg(\|\nabla v\|^2_{L^2(\Omega)} - k^2\|v\|^2_{L^2(\Omega)}\Bigg) + \langle f, v\rangle_{H^{-1}(\Omega)\otimes H_0^1(\Omega)}.$