6) Given $X = [x_1, x_2 \ldots x_D]^T$, $X \in R^{D \times N}$

a) we have $[X = U \Sigma V^T = U_1 \Sigma_1 V_1^T + U_2 \Sigma_2 V_2^T]$

$U_1 \in R^{D \times M}$ $U_2 \in R^{D \times (D-M)}$

$V_1 \in R^{N \times M}$ $V_2 \in R^{N \times (D-M)}$

Σ_1 = M-dim diagonal matrix $\Sigma_2 \in (D-M)$-dim diagonal matrix.

To prove : $[U_{2,i}^T \cdot X = 0]$, ($U_{2,i}$ = i^{th} column of U_2)

- By proving this we are proving that $U_{2,i}^T$ is null space of X.

- for any column vector x, we can express it using $U \cdot \alpha$, where α is a D-dimensional vector representing the coefficients of x in the basis of U's columns.

- Since x lies in the subspace spanned by U, then the projection of x onto U_2 column is orthogonal to U_1 $\Rightarrow$ $U_1^T \cdot U_{2,i} = 0$

Now,

$$x = U\alpha$$

$$U_{2,i}^T x = (U_{2,i}^T)(U \cdot \alpha)$$

Also, $U_{2,i} = U_2 e_i$ { e_i : standard basis vector }

Now, $U_{2,i}^T \cdot x = e_i^T U_2^T U \alpha$

$\therefore U_1^T U_2 = 0$ So $e_i^T U_2^T U \cdot \alpha = e_i^T U_1^T U \cdot \alpha = 0$

Hence $\boxed{U_{2,i}^T \cdot X = 0}$

b) given $X = U\Sigma V^T$

where $U \in R^{D\times D}$ | $U^TU = I$

$\Sigma \in R^{D\times N}$ | $V^TV = VV^T = I$

$V \in R^{N\times N}$

we have $Z = \Sigma^{-1/2} U^T X$

$$Cov(Z) = \frac{1}{D}[ZZ^T]$$

$$= \frac{1}{D}\left[(\Sigma^{-1/2}U^TX)(\Sigma^{-1/2}U^TX)^T\right]$$

$$= \frac{1}{D}\left[\Sigma^{-1/2}U^TX \cdot X^TU\Sigma^{-1/2}\right]$$

$$= \frac{1}{D}\left[\Sigma^{-1/2}U^T(U\Sigma V^T).(V\Sigma U^T)U\Sigma^{-1/2}\right]$$

$$= \frac{1}{D}\left[\Sigma^{-1/2}(U^TU)(\Sigma^2)(U^TU)\Sigma^{-1/2}\right]$$

$$Cov(Z) = \frac{1}{D}[\Sigma]$$

So, finally we have $Cov(Z) = \frac{1}{D}(\Sigma)$

which is identity matrix after multiplication by $(\frac{1}{D})$