Bayesian Estimation

* It uses the Bayes theory.
* Recall: Bayes Theorm.

For two events A and B (A and B are not independent events),

$$p(B|A) = \frac{p(A|B)\,p(B)}{p(A)}$$

* Data: $\{x_n\}_{n=1}^{N}$
* The objective: Find an estimate of a parameter value θ using Bayes' Rule for the given data $\{x_n\}_{n=1}^{N}$
* Using Bayes' Rule for data (denoted as $D = \{x_n\}_{n=1}^{N}$)

$$\pi_{\theta|X}(\theta|D) = \frac{f_{X|\theta}(D|\theta)\,\pi(\theta)}{f_X(D)}$$

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$\pi(\theta)$: Prior Distribution

$\pi_{\theta/x}(\theta/D)$: Posterior distribution

$f_{x|\theta}(D|\theta)$: Likelihood

$f_x(D)$: Normalizing factor

* Prior: our belief about possible values of θ (or distribution)
* Posterior: $\longrightarrow$ $\{x_n\}_{n=1}^N$ are observed, the new information on θ has obtained. Then, updating prior in the light of $\{x_n\}_{n=1}^N$ is the posterior distribution $\pi_{\theta|x}(\theta|D)$
* $f_x(D)$: Normalizing costant

$$f_x(x_1 \ldots x_N) = \int P(\theta|X)\, p(\theta)\, d\theta$$

* $\boxed{\pi_{\theta|x}(\theta|D) \propto f_x(D/\theta)\, \pi_\theta(\theta)}$

How can we estimate $\pi_{\theta/x}(\theta/D)$?

How can we estimate $\pi_{\theta|X}(\theta|D)$?

Example:

Data: $x_1 \dots x_n$

Distribution: $N(\mu, \sigma^2)$

μ: unknown and σ^2 is known.

Prior: $\mu \sim N(\mu_0, \sigma_0^2)$ is available.

* $f_X(x_1 \dots x_n / \mu)$

$$= \frac{1}{(2\pi\sigma^2)^{n/2}} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{N}(x_i-\mu)^2}$$

* $$\pi(\mu) = \frac{1}{(2\pi\sigma_0^2)^{1/2}} e^{-\frac{1}{2\sigma_0^2}(\mu-\mu_0)^2}$$

* Joint Probability distribution

$f(x_1 \dots x_n, \mu)$

$$= \frac{1}{(2\pi\sigma^2)^{n/2}} \frac{1}{(2\pi\sigma_0^2)^{1/2}} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{N}(x_i-\mu)^2 - \frac{1}{2\sigma_0^2}(\mu-\mu_0)^2}$$

$$= \frac{1}{(2\pi\sigma^2)^{N/2}\sqrt{2\pi\sigma_0^2}} e^{-\frac{1}{2}\left[\left(\frac{1}{\sigma_0^2}+\frac{N}{\sigma^2}\right)\mu^2 - \left(\frac{\sigma^2}{N}\mu_0 + \sigma_0^2\bar{x}\right)\mu\right]}$$

$h(\cdot)$ is a function of $x_1 \ldots x_n, \sigma^2, \mu_0$

$$= \overbrace{\quad}^{h(\cdot)} e^{-\frac{1}{2}\left(\frac{1}{\sigma_0^2}+\frac{N}{\sigma^2}\right)\left[\mu^2 - 2\left(\frac{\sigma^2\mu_0/N + \sigma_0^2\bar{x}}{\left(\frac{1}{\sigma_0^2}+\frac{N}{\sigma^2}\right)}\right)\mu\right]}$$

$$\underline{\text{Def}} \quad e^{-\frac{1}{2\sigma_N^2}\left[\mu^2 - 2\mu\mu_N\right]} \overbrace{\tilde{h}(\cdot)}^{h(\cdot)}$$

Then, the posterior distribution for μ is also a normal distribution with

$$\frac{1}{\sigma_N^2} = \frac{1}{\sigma_0^2} + \frac{N}{\sigma^2}$$

$$\Longleftrightarrow \sigma_N^2 = \frac{\sigma_0^2\sigma^2}{\sigma^2 + N\sigma_0^2}$$

$$\& \quad \mu_N = \frac{\frac{\sigma^2\mu_0}{N} + \sigma_0^2\bar{x}}{\frac{1}{\sigma_0^2} + \frac{N}{\sigma^2}}$$

Now, Bayes Estimation of θ

$\tilde{\theta} = E[\theta \mid \text{Sample}]$ or

mean of posterior distribution

Note that the mean can be directly determined if $f(\theta|D)$ is a known probability distribution.