Group Theory for Particle Physics

Carlos A. Vaquera-Araujo

CONACYT - DCI Universidad de Guanajuato - DCPIHEP

October 16-20, 2023

Contents

Symmetry and Group Theory

Symmetry $\equiv$ Invariance of a system (S) under a transformation (T)

$$S \xrightarrow{T} S' = S \tag{1}$$

Types of transformations

- Discrete vs Continuous
- Global vs Local
- Finite vs Infinite
- Compact vs Non-compact
- Space-Time vs Internal

Symmetries in Classical Mechanics

Noether's Theorem

Continuous Symmetries $\Rightarrow$ Conserved Quantities.

Simple Version

$$S = \int dt L(q_i, \dot{q}_i) \tag{2}$$

$$\begin{aligned}
\Rightarrow \delta S &= \int dt \left\{ \frac{\partial L}{\partial q_i} \delta q_i + \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i \right\} \\
\Rightarrow \delta S &= \int dt \left\{ \frac{d}{dt} \left[\frac{\partial L}{\partial \dot{q}_i} \right] \delta q_i + \frac{\partial L}{\partial \dot{q}_i} \frac{d}{dt} \left[\delta q_i \right] \right\} \qquad (3) \\
\delta S &= \int dt \frac{d}{dt} \left[\frac{\partial L}{\partial \dot{q}_i} \delta q_i \right]
\end{aligned}$$

Imposing the invariance of the system under arbitrary variations

$$\delta S = 0 \Rightarrow \frac{\partial L}{\partial \dot{q}_i}\delta q_i = \text{constant} \tag{4}$$

In general, for an infinitesimal variation $\delta q_i = \xi_{ik}(q_j, \dot{q}_j)\varepsilon_k$, we define the Conserved Charge

$$Q_k \equiv \frac{\partial L}{\partial \dot{q}_i}\xi_{ik}(q_j, \dot{q}_j). \tag{5}$$

Symmetries in Quantum Mechanics

Complete Set of Commuting Observables(CSCO)

The operators $A, B, C, \ldots$ constitute a Complete Set of Commuting Observables if they commute among themselves. Then they have a common eigenbasis.

Dynamics

The Hamiltonian dictates the time evolution of the system

- Schrödinger: $i\hbar\frac{\partial}{\partial t}|\psi\rangle = H|\psi\rangle$,
- Heisenberg: $\frac{dO}{dt} = i[H, O]$.

Thus, for conservative systems H is part of the CSCO $= \{H, A, B, C, \ldots\}$. And the rest of operators in CSCO commute with H, and are conserved

$$[H, A] = [H, B] = [H, C] = \cdots = 0. \tag{6}$$

Transformations and symmetries

In QM, the transformations are implemented by operators acting on states:

$$|\psi\rangle \xrightarrow{T} |\psi'\rangle = U_T |\psi\rangle . \tag{7}$$

Invariance under a transformation must leave probabilities unchanged

$$|\langle\phi'|\psi'\rangle|^2 = |\langle U\phi|U\psi\rangle|^2 = |\langle\phi| U^\dagger U |\psi\rangle|^2 = |\langle\phi|\psi\rangle|^2 \tag{8}$$

Thus symmetries are implemented by Unitary or anti-unitary operators (Wigner 1932).

$$U^\dagger U = \mathbb{I}. \tag{9}$$

Under a unitary transformation, operators change under the corresponding similarity transformations

$$O\left|\psi\right\rangle \to O'\left|\psi'\right\rangle = U(O\left|\psi\right\rangle) = UOU^\dagger U\left|\psi\right\rangle, \qquad O' = UOU^\dagger. \tag{10}$$

A transformation represents a Symmetry for an operator if it preserves the Hamiltonian

$$H' = UHU^\dagger = H \Rightarrow [H, U] = 0. \tag{11}$$

Besides, if the Unitary operator implementing the symmetry can be written as the exponential of some hermitian operators X_i:

$$U = e^{-i\alpha_i X_i}, \quad [H, U] = 0 \Rightarrow [H, X_i] = 0. \tag{12}$$

Then those operators are natural candidates to be part of the CSCO.

Group Theory: Natural Language of Symmetry

A Group $(G, \star)$, is a set $\{g_i\}$, and a binary operation $\star$ subject to the following axioms

1. Closure: $\forall\ g_i,\ g_j \in G$, $g_j \star g_j \in G$.
2. Associativity: $\forall\ g_i, g_j, g_k \in G$, then $(g_i \star g_j) \star g_k = g_i \star (g_j \star g_k)$.
3. Identity: $\exists\ g = e \in G$, such that $\forall g_i \in G,\ e \star g_i = g_i \star e = g_i$.
4. Inverse: $\forall g \in G, \exists\ h \in G$, such that $h \star g = g \star h = e$, (o $h = g^{-1}$).

Abelian vs Non-Abelian

The group axioms do not imply $g_i \star g_j = g_j \star g_i$. If $g_i \star g_j \neq g_j \star g_i$ for some $g_i,\ g_j \in G$, then the group G is non-abelian.

Example Z_2

$(Z_2, \star)$	e	a
e	e	a
a	a	e

Representations

In particle physics we are often interested in what is called a representation of the group. A representation is a map between the group elements g_i, and linear operators $D(g_i)$, that preserves the group multiplication

$$D(g_i) \cdot D(g_j) = D(g_i \star g_j), \tag{13}$$

and the identity

$$D(e) = \mathbb{I}. \tag{14}$$

Example Z_2

$(Z_2, \star)$	e	a
e	e	a
a	a	e

Representations

- $D(e) = 1$, $D(a) = -1$ (faithful, a bijection)
- $D(e) = 1$, $D(a) = 1$ (trivial, all elements map to one)
- $D(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $D(a) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ (reducible, direct sum)
- ...

The same group can have a lot of different representations. Some of them are more fundamental.

If $D(g_i)^{(1)}$ and $D(g_i)^{(2)}$ are two representations of g_i, then

$$D(g_i) = \begin{pmatrix} D(g_i)^{(1)} & 0 \\ 0 & D(g_i)^{(2)} \end{pmatrix} \tag{15}$$

is also a representation. A representation is Irreducible if its elements cannot be written simultaneously as block diagonal matrices under a similarity transformation $D'(g_i) = S^{-1}D'(g_i)S$.

A few definitions in group theory are:

1. Order: The order of the group is the number of elements that belong to G.
2. Suppose that $g_i, g_j \in G$, and that $f(g_i), f(g_j) \in H$ where H is some other group. If the composition rule satisfies

$$f(g_i)f(g_j) = f(g_i \star g_j) \tag{16}$$

   we say that G is Homomorphic to H. This is a fancy way of saying that the two groups have a similar structure. If, besides, the mapping is one-to-one, the groups are Isomorphic.

Direct Product

As we will often work in matrix representations, it is convenient to introduce the Direct product of matrices. If A is $m \times m$ and B is $n \times n$, then $A \otimes B$ is an $mn \times mn$ defined as

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots a_{1m}B & \\ a_{21}B & a_{22}B & \cdots a_{2m}B & \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \cdots a_{mm}B & \end{pmatrix} . \tag{17}$$

Lie Groups

A Lie Group is a group that is also a differentiable manifold. A manifold is a space that locally resembles Euclidean space. In particle physics one of the most common symmetries is described by the Lie group $U(1)$. It emerges for example in the Klein-Gordon Lagrangian for a complex scalar.

$$\mathcal{L} = \frac{1}{2}\partial^{\mu}\phi^{\dagger}\partial_{\mu}\phi - \frac{m^2}{2}\phi^{\dagger}\phi. \tag{18}$$

This Lagrangian is invariant under the global transformations

$$\phi \rightarrow \phi' = e^{-i\alpha}\phi, \qquad \phi^{\dagger} \rightarrow \phi'^{\dagger} = \phi^{\dagger}e^{i\alpha}. \tag{19}$$

In this case the elements of the group $U(1)$ are parameterized by the phases $e^{-i\alpha}$. The group is continuous and compact, since $\alpha \in [0, 2\pi)$ and the group axioms are satisfied:

1. Closure: $e^{-i\alpha_1} e^{-i\alpha_2} = e^{-i(\alpha_1+\alpha_2)}$.
2. Associativity: The product of complex numbers is associative.
3. Identity: $e^{-i0} = 1$.
4. Inverse: $e^{-i\alpha} e^{+i\alpha} = 1$.

Rotations in 2 dimensions

Proper rotations in $\mathbb{R}^2$ about the origin are given by elements of $SO(2)$. An active counterclockwise rotation by an angle θ is given by the matrix

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}. \tag{20}$$

For Lie groups beyond $U(1)$, one can consider the tangent space at the identity, and that defines the corresponding Lie algebra of the group. For $SO(2)$, an infinitesimal transformation near the identity is given by

$$R(\epsilon) = \begin{pmatrix} 1 & -\epsilon \\ \epsilon & 1 \end{pmatrix} \equiv \mathbf{1} - i\epsilon X, \qquad X = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = i\frac{dR(\theta)}{d\theta}\bigg|_{\theta=0}. \tag{21}$$

The important observation is the fact that a finite transformation can be reconstructed from many infinitesimal ones. Formally we can write a finite transformation as

$$R(\theta) = \lim_{N\to\infty} \left[\mathbf{1} - i\frac{\theta}{N} X \right]^N \equiv e^{-i\theta X} = \sum_{k=0}^{\infty} \frac{(-i\theta X)^k}{k!}, \tag{22}$$

where the exponential is defined by its series expansion. We can explicitly check this fact as follows:

$$\begin{aligned} e^{-i\theta X} &= e^{-i^2\theta \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}} = e^{\theta \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \theta \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} - \frac{1}{2}\theta^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{1}{3!}\theta^3 \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + \cdots \\ &= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}. \end{aligned} \tag{23}$$

Lie algebras

The previous results can be generalized for a Lie group G represented by matrix elements D_n and parameterized by $\alpha^i \in \mathbb{R}$, $i = 1, \ldots, M$, such that $D_n(\boldsymbol{\alpha})\big|_{\alpha^i=0} = \mathbf{1}$. Thus, for infinitesimal transformations, we have

$$D_n(\delta\boldsymbol{\alpha}) = \mathbf{1} + \delta\alpha^i \frac{\partial D_n(\boldsymbol{\alpha})}{\partial \alpha^i}\Big|_{\boldsymbol{\alpha}=\mathbf{0}} \equiv \mathbf{1} - i\delta\alpha^i X^i. \tag{24}$$

the constant matrices

$$X^i \equiv i\frac{\partial D_n}{\partial \alpha^i}\Big|_{\boldsymbol{\alpha}=\mathbf{0}}. \tag{25}$$

are called Generators and in this course we include a (highly convention-dependent) factor of i in their definition.

We can thus build finite transformations from the composition of many infinitesimal ones

$$\lim_{N\to\infty}(1 - i\delta\alpha^i X^i)^N = \lim_{N\to\infty}\left(1 - i\frac{\alpha^i}{N}X^i\right)^N, \tag{26}$$

or formally, by exponentiation of the generators

$$D_n(\boldsymbol{\alpha}) \equiv \lim_{N\to\infty}\left(1 - i\frac{\alpha^i}{N}X^i\right)^N = e^{-i\alpha^i X^i} = e^{-i\boldsymbol{\alpha}\cdot\mathbf{X}}. \tag{27}$$

The Lie algebra $\mathfrak{g}$ determined by the generators of G can be obtained by examining the closure of the group elements, and is given by

$$[X^i, X^j] = if_{ijk} X^k, \tag{28}$$

with real antisymmetric coefficients f_{ijk}, known as structure constants, that constitute by themselves a representation of the algebra called the *adjoint representation*, since the M $M \times M$ matrices

$$[T^a]_{bc} \equiv -if_{abc} \tag{29}$$

satisfy

$$[T^a, T^b] = if_{abc} T^c, \tag{30}$$

due to the Jacobi Identity.

Rotations in $\mathbb{R}^3$

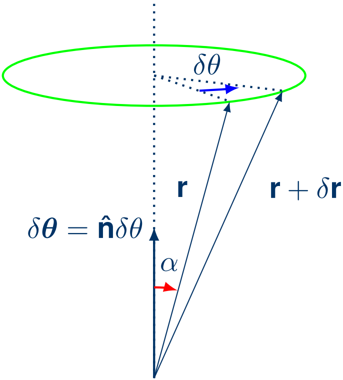


Figure: Rotation In $\mathbb{R}^3$

$$\delta\mathbf{r} = \delta\boldsymbol{\theta} \times \mathbf{r} = \delta\theta\hat{\mathbf{n}} \times \mathbf{r}. \tag{31}$$

We can identify the Generators of $SO(3)$ by comparing

$$\delta x_j = x'_j - x_j = \delta\theta \epsilon_{jik} n_i x_k = -i\theta n_i(-i\epsilon_{ijk}) x_k, \tag{32}$$

with

$$x'_j = (e^{-i\delta\boldsymbol{\theta}\cdot\mathbf{J}})_{jk} x_k, \tag{33}$$

leading to

$$[J^i]_{jk} = -i\epsilon_{jk}. \tag{34}$$

Explicitly, the generators are given by

$$J^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad J^2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \tag{35}$$

$$J^3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

which satisfy the commutation relations

$$[J^i, J^j] = i\epsilon_{ijk} J^k. \tag{36}$$

Notice that the generators are already in the adjoint representation.

Finite Rotations Around Cartesian Axes

Finite rotations can be obtained by the exponentiation of the Generators $R(\theta, \hat{\mathbf{n}}) = e^{-i\theta\hat{\mathbf{n}}\cdot\mathbf{J}}$. In particular, rotations around the cartesian axes are given by

$$R(\theta, \mathbf{e}_1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}, \quad R(\theta, \mathbf{e}_2) = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix},$$
$$R(\theta, \mathbf{e}_3) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{37}$$

$SU(2)$

Another Lie group closely related to $SO(3)$ is $SU(2)$. We can identify its Generators from its defining properties:

1. The group elements are Unitary: $U = e^{-i\boldsymbol{\theta}\cdot\mathbf{j}}$, $U^\dagger U = 1$. This implies that the generators j^i are Hermitian $j^{i\dagger} = j^i$.
2. The group elements have unit determinant $\det U = 1$. Thus, the generators are traceless $\mathrm{tr}\left(j^i\right) = 0$.

A particular basis for the most general Hermitian traceless 2×2 matrices that generate the $SU(2)$ group is

$$j^1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{\sigma^1}{2}, \quad j^2 = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \frac{\sigma^2}{2}, \quad j^3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{\sigma^3}{2}, \tag{38}$$

where we have introduced the Pauli matrices σ^i.

$SO(3)$ and $SU(2)$

The Pauli matrices satisfy the following relations:

$$[\sigma^i, \sigma^j] = 2i\epsilon_{ijk}\sigma^k, \qquad \{\sigma^i, \sigma^j\} = 2\delta_{ij}\mathbf{1}, \qquad \sigma^i\sigma^j = \delta_{ij}\mathbf{1} + i\epsilon_{ijk}\sigma^k. \quad (39)$$

We have chosen the normalization of $SU(2)$ generators in such a way that their algebra coincides with that of $SO(3)$:

$$[j^i, j^j] = i\epsilon_{ijk}j^k. \quad (40)$$

This means that the group $SO(3)$ is locally isomorphic to $SU(2)$.

Although $SO(3)$ and $SU(2)$ are locally equivalent, there is a global difference. Take for example a 2π rotation around the z axis:

- For vectors under $SO(3)$ is implemented by $e^{(-2\pi i J^3)} = \mathbf{1}_{3\times 3}$
- For spinors under $SU(2)$, however $e^{(-2\pi i j^3)} = -\mathbf{1}_{2\times 2}$.

Thus, two complete rotations are needed to return a spinor to its original state $e^{(-4\pi i j^3)} = \mathbf{1}_{2\times 2}$. $SU(2)$ is topologically the simply connected double cover of $SO(3)$. This relation justifies the fact that the generators of $SO(3)$ can be written as

$$[J^i]_{jk} = -i\epsilon_{ijk}, \tag{41}$$

corresponding to the adjoint representation of $SU(2)$.

General $SU(2)$ Transformation

A finite rotation about a fixed axis $\hat{\mathbf{n}}$ is given in the spinor representation as

$$U(\theta, \hat{\mathbf{n}}) = e^{-i\theta\hat{\mathbf{n}}\cdot\mathbf{j}} = e^{-\frac{i}{2}\theta\hat{\mathbf{n}}\cdot\boldsymbol{\sigma}} = \sum_{k=0}^{\infty} \frac{(-i\theta\hat{\mathbf{n}}\cdot\boldsymbol{\sigma}/2)^k}{k!}$$
$$= \sum_{m=0}^{\infty} \frac{(-1)^m(\theta/2)^{2m}(\hat{\mathbf{n}}\cdot\boldsymbol{\sigma})^{2m}}{(2m)!} - i\sum_{m=0}^{\infty} \frac{(-1)^m(\theta/2)^{2m+1}(\hat{\mathbf{n}}\cdot\boldsymbol{\sigma})^{2m+1}}{(2m+1)!}. \tag{42}$$

This expression can be further reduced using

$$(\hat{\mathbf{n}}\cdot\boldsymbol{\sigma})^2 = n^i n^j \sigma^i \sigma^j = n^i n^j (\delta_{ij}\mathbf{1} + i\epsilon_{ijk}\sigma^k) = \sum_{i=1}^{3} (n^i)^2 = 1. \tag{43}$$

Therefore, the most general finite $SU(2)$ transformation becomes

$$\begin{aligned} U(\theta, \hat{\mathbf{n}}) &= \sum_{m=0}^{\infty} \frac{(-1)^m (\theta/2)^{2m}}{(2m)!} - i\hat{\mathbf{n}} \cdot \boldsymbol{\sigma} \sum_{m=0}^{\infty} \frac{(-1)^m (\theta/2)^{2m+1}}{(2m+1)!} \\ &= \cos\frac{\theta}{2} - i \sin\frac{\theta}{2} \hat{\mathbf{n}} \cdot \boldsymbol{\sigma}, \end{aligned} \tag{44}$$

with

$$U^{\dagger}(\theta, \hat{\mathbf{n}}) = U^{-1}(\theta, \hat{\mathbf{n}}) = \cos\frac{\theta}{2} + i \sin\frac{\theta}{2} \hat{\mathbf{n}} \cdot \boldsymbol{\sigma}. \tag{45}$$

Map between $SO(3)$ and $SU(2)$

The explicit form of the mapping between $SO(3)$ and $SU(2)$ can be found by the observation that the components of a real vector $\mathbf{r}$ in $\mathbb{R}^3$ can be packed in a traceless 2×2 matrix with the aid of the sigma matrices

$$\mathbf{r} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \rightarrow \mathbf{x} \equiv (\boldsymbol{\sigma} \cdot \mathbf{r}) = \begin{pmatrix} x_3 & x_1 - ix_2 \\ x_1 + ix_2 & -x_3 \end{pmatrix}, \tag{46}$$

with inverse given by

$$x_i = \frac{1}{2} \mathrm{Tr}\{\sigma^i \mathbf{x}\}. \tag{47}$$

One can explicitly check that for the transformations

$$U = e^{-i\boldsymbol{\theta}\cdot\mathbf{j}}, \quad R = e^{-i\boldsymbol{\theta}\cdot\mathbf{J}}, \quad \text{same parameters } \boldsymbol{\theta}, \tag{48}$$

the following equalities hold:

- $U(\boldsymbol{\sigma}\cdot\mathbf{r})U^\dagger = \boldsymbol{\sigma}\cdot(R\mathbf{r})$,
- $\det\left[U(\boldsymbol{\sigma}\cdot\mathbf{r})U^\dagger\right] = \det[(\boldsymbol{\sigma}\cdot\mathbf{r})] = -|R\mathbf{r}|^2 = -|\mathbf{r}|^2$.

Thus, the $SU(2) \to SO(3)$ mapping is

$$R_{ij}(U) = \frac{1}{2}\operatorname{tr}\{\sigma^i U \sigma^j U^\dagger\} = 2\operatorname{tr}\{j^i U j^j U^\dagger\}. \tag{49}$$

Irreducible representations of $SU(2)$

The construction of representations of $SU(2)$, labeled by their spin j or by their dimension $\mathbf{2j+1}$, is done as in Quantum Mechanics.
The first step in the recipe is to identify the Cartan Subalgebra, the maximum number of independent operators that can be simultaneously diagonalized. For our problem at hand it is easy to prove the existence of a Cuadratic Casimir

$$\mathbf{J}^2 = J^i J^i. \tag{50}$$

A Casimir invariant is a function $f(\mathbf{X})$ of the group generators X^i that commutes with them, $[f(\mathbf{X}), X^i] = 0$. Thus, since $[\mathbf{J}^2, J^i] = 0$, the elements of the Cartan Subalgebra in our case are $\{\mathbf{J}^2, J^3\}$.

The second step is to build a basis of states and to set up the eigenvalue problem. We will label the states of our basis as usual in Quantum Mechanics $|j, m\rangle$, where m is the eigenvalue of J^3 according to

$$J^3|j, m\rangle = m|j, m\rangle, \tag{51}$$

and j is a common label for all the states that belong to the same multiplet, and coincides with the value of the largest eigenvalue of J^3 (and is also related to the eigenvalue of $\mathbf{J}^2$, as we will show below), such that

$$J^3|j, j\rangle = j|j, j\rangle. \tag{52}$$

The third step is to switch from the cartesian basis to a spherical polarization one one by defining the ladder operators

$$J^{\pm} \equiv J^1 \pm iJ^2, \qquad (J^-)^\dagger = J^+. \tag{53}$$

In terms of these operators, the $\mathfrak{su}(2)$ algebra becomes

$$[J^3, J^{\pm}] = \pm J^{\pm}, \qquad [J^+, J^-] = 2J^3. \tag{54}$$

Using the $\mathfrak{su}(2)$ algebra, the action of $J^3 J^{\pm}$ on the states $|j,m\rangle$ can be computed as

$$J^3 J^{\pm}|j,m\rangle = (J^{\pm}J^3 \pm J^{\pm})|j,m\rangle = (m \pm 1)J^{\pm}|j,m\rangle, \tag{55}$$

meaning that the state $J^{\pm}|j,m\rangle$ is an eigenvector of J^3, with eigenvalue $m \pm 1$, *i.e.* $J^{\pm}|j,m\rangle$ must be proportional to the state $|j, m \pm 1\rangle$.

The fourth step consists on identifying the states of a multiplet by studying how are they connected by the repeated action of the ladder operators. Clearly the state $|j, j\rangle$ is the last step of the ladder, in the sense that it is the state with the maximum m value (there is no state $|j, j+1\rangle$ by the definition of j) and therefore

$$J^{+}|j, j\rangle = 0. \tag{56}$$

Moreover, using the identity

$$\mathbf{J}^2 - (J^3)^2 = (J^1)^2 + (J^2)^2 = \frac{1}{2}(J^{+}J^{-} + J^{-}J^{+}) = \frac{1}{2}[J^{+}(J^{+})^{\dagger} + (J^{+})^{\dagger}J^{+}], \tag{57}$$

and denoting by $C_2(j)$ the eigenvalue of $\mathbf{J}^2\,|j, m\rangle = C_2(j)\,|j, m\rangle$, we have

$$\begin{aligned}\langle j, m|\,[\mathbf{J}^2 - (J^3)^2]\,|j, m\rangle =& \frac{1}{2}\,\langle j, m|\,[J^{+}(J^{+})^{\dagger} + (J^{+})^{\dagger}J^{+}]\,|j, m\rangle \\ =& \frac{1}{2}|(J^{+})^{\dagger}\,|j, m\rangle\,|^2 + \frac{1}{2}|J^{+}\,|j, m\rangle\,|^2 \geq 0\end{aligned} \tag{58}$$

This relation implies that there is also a lower bound for the lowest value of m , namely $m_{min} = j' \geq -\sqrt{C_2(j)}$, and therefore, there is a state annihilated by J^-,

$$J^- |j, j'\rangle = 0. \tag{59}$$

We can further use the algebra to massage the following identity

$$\begin{aligned} \mathbf{J}^2 - (J^3)^2 &= \frac{1}{2}(J^+ J^- + J^- J^+) = \frac{1}{2}([J^+, J^-] + 2J^- J^+), \\ &= J^3 + J^- J^+, \end{aligned} \tag{60}$$

into

$$J^- J^+ = \mathbf{J}^2 - (J^3)^2 - J^3. \tag{61}$$

An analogous calculation yields

$$J^+ J^- = \mathbf{J}^2 - (J^3)^2 + J^3. \tag{62}$$

The above relations are useful to determine the eigenvalue $C_2(j)$ of the $\mathbf{J}^2$ operator by evaluating

$$\begin{aligned} J^- J^+ \left|j,j\right\rangle =& [\mathbf{J}^2 - (J^3)^2 - J^3] \left|j,j\right\rangle = (C_2(j) - j^2 - j) \left|j,j\right\rangle = 0 \\ J^+ J^- \left|j,j'\right\rangle =& [\mathbf{J}^2 - (J^3)^2 + J^3] \left|j,j'\right\rangle = (C_2(j) - j'^2 + j') \left|j,j\right\rangle = 0, \end{aligned} \tag{63}$$

$$\Rightarrow \; C_2(j) = j(j+1), \quad j' = -j. \tag{64}$$

Finally, both ends of the ladder must be connected by a finite number of steps. Thus, for some integer n, we have

$$(J_+)^n \left|j,-j\right\rangle \sim \left|j,j\right\rangle, \Rightarrow \; -j+n = j, \Rightarrow \; j = \frac{n}{2}. \tag{65}$$

This means that the label j takes semi-integer values.

The last step in our recipe is to normalize the states. By setting C^+_{jm} as a complex constant and imposing $\langle j', m'|j, m\rangle = \delta_{j'j}\delta_{m'm}$, we have

$$\begin{aligned}
&J^+ |j, m\rangle = C^+_{jm} |j, m+1\rangle \,, \\
&\langle j, m| (J^+)^\dagger J^+ |j, m\rangle = |C^+_{jm}|^2 = \langle j, m| [\mathbf{J}^2 - (J^3)^2 - J^3] |j, m\rangle \\
&|C^+_{jm}|^2 = j(j+1) - m(m+1) = (j-m)(j+m+1).
\end{aligned} \tag{66}$$

The overall phase of C^+_{jm} is matter of convention. In the following we will adopt the Condon-Shortley convention (real positive C^+_{jm} coefficients), yielding

$$J^+ |j, m\rangle = \sqrt{(j-m)(j+m+1)}\, |j, m+1\rangle \,, \tag{67}$$

$$J^- |j, m\rangle = \sqrt{(j+m)(j-m+1)}\, |j, m-1\rangle \,. \tag{68}$$

Summarizing, the irreps of $SU(2)$ are determined by the following relations

$$\mathbf{J}^2 |j, m\rangle = j(j+1) |j, m\rangle , \quad J^3 |j, m\rangle = m |j, m\rangle , \tag{69}$$

$$J^{\pm} |j, m\rangle = \sqrt{(j \mp m)(j \pm m + 1)} |j, m \pm 1\rangle , \tag{70}$$

$$j = \frac{n}{2}, \; n \in \mathbb{Z}, \quad m = -j, -j+1, \ldots, j-1, j. \tag{71}$$

Each subspace with fixed j, described by the set of states

$$\{|j, -j\rangle , |j, -j+1\rangle , \ldots , |j, j-1\rangle , |j, j\rangle\}$$

is the minimal subspace transforming under the $SU(2)$ irrep of dimension $\mathbf{2j+1}$.

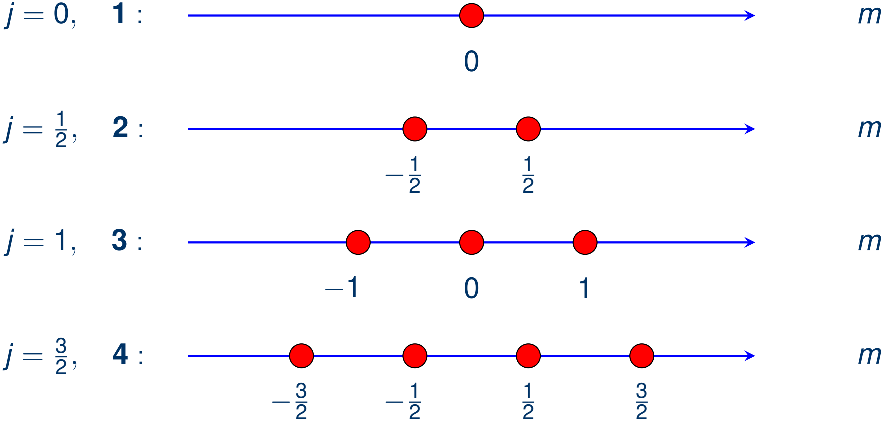


Figure: Lowest dimensional multiplets of $SU(2)$.

The explicit form of an irrep of $SU(2)$ is then determined by either the value of j or the dimensionality of the representation. The rotation matrix

$$D_{(j)}(\theta, \hat{\mathbf{n}}) = e^{-i\mathbf{J}_{(j)}\cdot\hat{\mathbf{n}}\,\theta} \tag{72}$$

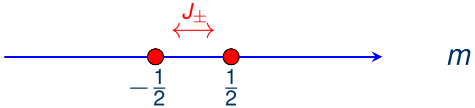
is a $(2j+1)\times(2j+1)$ matrix for the irrep of spin j, and the corresponding generators are also $(2j+1)\times(2j+1)$ matrices with elements

$$\begin{aligned}
\langle j, m'|J^3_{(j)}|j, m\rangle &= m\,\delta_{m',m} && (73)\\
\langle j, m'|J^+_{(j)}|j, m\rangle &= \sqrt{(j+m+1)(j-m)}\,\delta_{m',m+1} && (74)\\
\langle j, m'|J^-_{(j)}|j, m\rangle &= \sqrt{(j+m)(j-m+1)}\,\delta_{m',m-1}. && (75)
\end{aligned}$$

Examples

$SU(2)$ rep **2**, $j = 1/2$, Fundamental

$j = \frac{1}{2}$:



$$\left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \left|\tfrac{1}{2},-\tfrac{1}{2}\right\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{76}$$

$$J_3^{(1/2)} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad J_+^{(1/2)} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad J_-^{(1/2)} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \tag{77}$$

$$\begin{aligned} J_+^{(1/2)}\left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle &= 0, & J_-^{(1/2)}\left|\tfrac{1}{2},-\tfrac{1}{2}\right\rangle &= 0, \\ J_-^{(1/2)}\left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle &= \left|\tfrac{1}{2},-\tfrac{1}{2}\right\rangle, & J_+^{(1/2)}\left|\tfrac{1}{2},-\tfrac{1}{2}\right\rangle &= \left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle. \end{aligned} \tag{78}$$

In terms of the original components,

$$J_1 = \frac{1}{2}(J_- + J_+) \qquad \text{y} \qquad J_2 = \frac{i}{2}(J_- - J_+), \tag{79}$$

$$J_1^{(1/2)} = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; J_2^{(1/2)} = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \; J_3^{(1/2)} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{80}$$

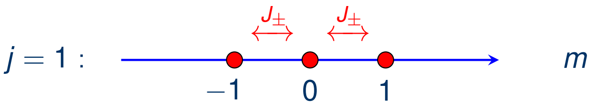
recovering

$$\mathbf{J}^{(1/2)} = \frac{\boldsymbol{\sigma}}{2}. \tag{81}$$

Thus, the **2** rep of $SU(2)$ is

$$D^{(1/2)}(\hat{\mathbf{n}}, \theta) = \exp\left[-i\tfrac{\boldsymbol{\sigma}}{2} \cdot \hat{\mathbf{n}}\,\theta\right] = \cos\tfrac{\theta}{2}\mathbb{I} - i\sin\tfrac{\theta}{2}\boldsymbol{\sigma} \cdot \hat{\mathbf{n}}. \tag{82}$$

$SU(2)$ rep **3**, $j = 1$, Adjoint



$$|1,1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad |1,0\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad |1,-1\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{83}$$

$$J_1^{(1)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, J_2^{(1)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, J_3^{(1)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{84}$$

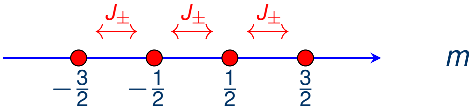
Notice that previously we had identified the adjoint generators as $(\tilde{J}_i)_{jk} = -i\epsilon_{ijk}$:

$$\tilde{J}_1^{(1)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \tilde{J}_2^{(1)} = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \tilde{J}_3^{(1)} = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{85}$$

There is a unitary modal matrix S that relates both sets of generators through a similarity transformation $\tilde{J}_i^{(1)} = S^\dagger J_i^{(1)} S$.

$SU(2)$ rep **4**, $j = 3/2$

$j = \frac{3}{2}$:



$$
\begin{aligned}
\left|\tfrac{3}{2},\tfrac{3}{2}\right\rangle &= \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \qquad & \left|\tfrac{3}{2},\tfrac{1}{2}\right\rangle &= \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \\
\left|\tfrac{3}{2},-\tfrac{1}{2}\right\rangle &= \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \qquad & \left|\tfrac{3}{2},-\tfrac{3}{2}\right\rangle &= \begin{pmatrix}0\\0\\0\\1\end{pmatrix},
\end{aligned}
\tag{86}
$$

$$J_1^{(3/2)} = \begin{pmatrix} 0 & \frac{\sqrt{3}}{2} & 0 & 0 \\ \frac{\sqrt{3}}{2} & 0 & 1 & 0 \\ 0 & 1 & 0 & \frac{\sqrt{3}}{2} \\ 0 & 0 & \frac{\sqrt{3}}{2} & 0 \end{pmatrix}, \; J_2^{(3/2)} = \begin{pmatrix} 0 & -\frac{\sqrt{3}}{2}i & 0 & 0 \\ \frac{\sqrt{3}}{2}i & 0 & -i & 0 \\ 0 & i & 0 & -\frac{\sqrt{3}}{2}i \\ 0 & 0 & \frac{\sqrt{3}}{2}i & 0 \end{pmatrix},$$

$$J_3^{(3/2)} = \begin{pmatrix} \frac{3}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \\ 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & -\frac{3}{2} \end{pmatrix}. \tag{87}$$

Angular Momentum Composition

Direct Product of Representations

What irreducible states $|j, m\rangle$ can be obtained by the direct product of $|j_1, m_1\rangle$ and $|j_2, m_2\rangle$?

Clues

- The original states live in different Hilbert spaces

$$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2,$$

- A new representation is obtained from the direct product of the original irreps

$$D(\hat{\mathbf{n}}, \theta) = D^{(j_1)}(\hat{\mathbf{n}}, \theta) \otimes D^{(j_2)}(\hat{\mathbf{n}}, \theta)$$

- The resulting representation is in general reducible

$$D(\hat{\mathbf{n}},\theta) = D^{(j_1)}(\hat{\mathbf{n}},\theta) \otimes D^{(j_2)}(\hat{\mathbf{n}},\theta) = \bigoplus_{j=|j_2-j_1|}^{j_2+j_1} D^{(j)}(\hat{\mathbf{n}},\theta)$$

- The total angular momentum is $\mathbf{J} = \mathbf{J}_1 \otimes 1_2 + 1_1 \otimes \mathbf{J}_2$
- Direct product basis of $\mathcal{H}$:

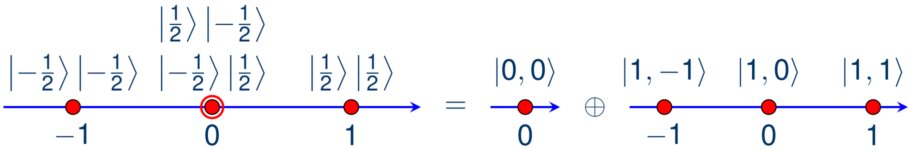
$$|j_1, m_1\rangle_1 \otimes |j_2, m_2\rangle_2 \equiv |j_1, m_1\rangle\, |j_2, m_2\rangle$$

- In the direct product basis, J_3 eigenvalues are

$$\begin{aligned} J_3|j_1, m_1\rangle|j_2, m_2\rangle &\equiv (J_3|j_1, m_1\rangle)\,|j_2, m_2\rangle + |j_1, m_1\rangle\,(J_3|j_2, m_2\rangle) \\ &= (m_1|j_1, m_1\rangle)\,|j_2, m_2\rangle + |j_1, m_1\rangle\,(m_2|j_2, m_2\rangle) \\ &= (m_1 + m_2)\,|j_1, m_1\rangle|j_2, m_2\rangle \end{aligned} \tag{88}$$

- Ladder operators $J_\pm$ connect all states belonging to the same multiplet (states transforming under the same irrep).

Example: $\mathbf{2} \otimes \mathbf{2}$



Omitting the j_1 and j_2 labels, the state with the maximum J_3 eigenvalue is

$$\left|\tfrac{1}{2}\right\rangle \left|\tfrac{1}{2}\right\rangle \tag{89}$$

with

$$\begin{aligned} J_3 \left|\tfrac{1}{2}\right\rangle \left|\tfrac{1}{2}\right\rangle &\equiv \left(J_3 \left|\tfrac{1}{2}\right\rangle\right) \left|\tfrac{1}{2}\right\rangle + \left|\tfrac{1}{2}\right\rangle \left(J_3 \left|\tfrac{1}{2}\right\rangle\right) \\ &= \left(\tfrac{1}{2} + \tfrac{1}{2}\right) \left|\tfrac{1}{2}\right\rangle \left|\tfrac{1}{2}\right\rangle = \left|\tfrac{1}{2}\right\rangle \left|\tfrac{1}{2}\right\rangle \equiv |1,1\rangle. \end{aligned} \tag{90}$$

Operating with J^-:

$$J^- \left|\tfrac{1}{2}\right\rangle \left|\tfrac{1}{2}\right\rangle = \left|\tfrac{1}{2}\right\rangle \left|-\tfrac{1}{2}\right\rangle + \left|-\tfrac{1}{2}\right\rangle \left|\tfrac{1}{2}\right\rangle \tag{91}$$

$$J_3 \frac{1}{\sqrt{2}} \left[\left|\tfrac{1}{2}\right\rangle \left|-\tfrac{1}{2}\right\rangle + \left|-\tfrac{1}{2}\right\rangle \left|\tfrac{1}{2}\right\rangle\right] = 0 \tag{92}$$

$$\Rightarrow |1,0\rangle = \frac{1}{\sqrt{2}} \left[\left|\tfrac{1}{2}\right\rangle \left|-\tfrac{1}{2}\right\rangle + \left|-\tfrac{1}{2}\right\rangle \left|\tfrac{1}{2}\right\rangle\right]. \tag{93}$$

...and finally

$$J^- \frac{1}{\sqrt{2}} \left[\left|\tfrac{1}{2}\right\rangle \left|-\tfrac{1}{2}\right\rangle + \left|-\tfrac{1}{2}\right\rangle \left|\tfrac{1}{2}\right\rangle\right] = \frac{2}{\sqrt{2}} \left|-\tfrac{1}{2}\right\rangle \left|-\tfrac{1}{2}\right\rangle \tag{94}$$

$$J_3 \left|-\tfrac{1}{2}\right\rangle \left|-\tfrac{1}{2}\right\rangle = - \left|-\tfrac{1}{2}\right\rangle \left|-\tfrac{1}{2}\right\rangle \tag{95}$$

$$\Rightarrow |1,-1\rangle = \left|-\tfrac{1}{2}\right\rangle \left|-\tfrac{1}{2}\right\rangle \tag{96}$$

Completes a triplet with $j = 1$ (rep **3**): $\{|1,1\rangle, |1,0\rangle, |1,-1\rangle\}$.

The orthogonal combination

$$\frac{1}{\sqrt{2}}\left[\left|\tfrac{1}{2}\right\rangle\left|-\tfrac{1}{2}\right\rangle-\left|-\tfrac{1}{2}\right\rangle\left|\tfrac{1}{2}\right\rangle\right] \tag{97}$$

satisfies

$$J_3\frac{1}{\sqrt{2}}\left[\left|\tfrac{1}{2}\right\rangle\left|-\tfrac{1}{2}\right\rangle-\left|-\tfrac{1}{2}\right\rangle\left|\tfrac{1}{2}\right\rangle\right]=0, \tag{98}$$

$$J_\pm\frac{1}{\sqrt{2}}\left[\left|\tfrac{1}{2}\right\rangle\left|-\tfrac{1}{2}\right\rangle-\left|-\tfrac{1}{2}\right\rangle\left|\tfrac{1}{2}\right\rangle\right]=0, \tag{99}$$

and therefore, forms a singlet

$$|0,0\rangle=\frac{1}{\sqrt{2}}\left[\left|\tfrac{1}{2}\right\rangle\left|-\tfrac{1}{2}\right\rangle-\left|-\tfrac{1}{2}\right\rangle\left|\tfrac{1}{2}\right\rangle\right]. \tag{100}$$

Summarizing,

$$\mathbf{2} \otimes \mathbf{2} = \mathbf{1} \oplus \mathbf{3} \tag{101}$$

In general, for $SU(2)$

$$\begin{aligned}[\mathbf{2j_1 + 1}] \otimes [\mathbf{2j_2 + 1}] =& [\mathbf{2(j_1 + j_2) + 1}] \oplus [\mathbf{2(j_1 + j_2 - 1) + 1}] \\ & \oplus \cdots \oplus [\mathbf{2|j_1 - j_2| + 1}]\end{aligned} \tag{102}$$

Clebsch-Gordan decomposition

$$|j, m\rangle = \sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} |j_1, j_2, m_1, m_2\rangle\langle j_1, j_2, m_1, m_2|j, m\rangle \tag{103}$$

with $|j_1, j_2, m_1, m_2\rangle \equiv |j_1, m_1\rangle|j_2, m_2\rangle$.

The products $\langle j_1, j_2, m_1, m_2|j, m\rangle$ are known as Clebsch-Gordan coefficients.

Example: $\mathbf{3} \otimes \mathbf{3}$

For $j_1 = j_2 = 1$ (omitting again these labels),we can build the following states

- $\mathbf{5}$ ($j = 2$)

$$\begin{aligned}
|2,2\rangle &= |1\rangle\,|1\rangle \\
|2,1\rangle &= \frac{1}{\sqrt{2}}\left(|1\rangle\,|0\rangle + |0\rangle\,|1\rangle\right) \\
|2,0\rangle &= \frac{1}{\sqrt{6}}\left(|1\rangle\,|-1\rangle + 2\,|0\rangle\,|0\rangle + |-1\rangle\,|1\rangle\right) \\
|2,-1\rangle &= \frac{1}{\sqrt{2}}\left(|-1\rangle\,|0\rangle + |0\rangle\,|-1\rangle\right) \\
|2,-2\rangle &= |-1\rangle\,|-1\rangle
\end{aligned} \tag{104}$$

- **3** ($j = 1$)

$$
\begin{aligned}
|1,1\rangle &= \frac{1}{\sqrt{2}}\left(|1\rangle\,|0\rangle - |0\rangle\,|1\rangle\right) \\
|1,0\rangle &= \frac{1}{\sqrt{2}}\left(|1\rangle\,|-1\rangle - |-1\rangle\,|1\rangle\right) \\
|1,-1\rangle &= \frac{1}{\sqrt{2}}\left(|0\rangle\,|-1\rangle - |-1\rangle\,|0\rangle\right)
\end{aligned}
\tag{105}
$$

- **1** ($j = 0$)

$$
|0,0\rangle = \frac{1}{\sqrt{3}}\left(|1\rangle\,|-1\rangle - |0\rangle\,|0\rangle + |-1\rangle\,|1\rangle\right) \tag{106}
$$

$$
\mathbf{3} \otimes \mathbf{3} = \mathbf{1} \oplus \mathbf{3} \oplus \mathbf{5} \tag{107}
$$

Thanks