The Standard Model and Beyond

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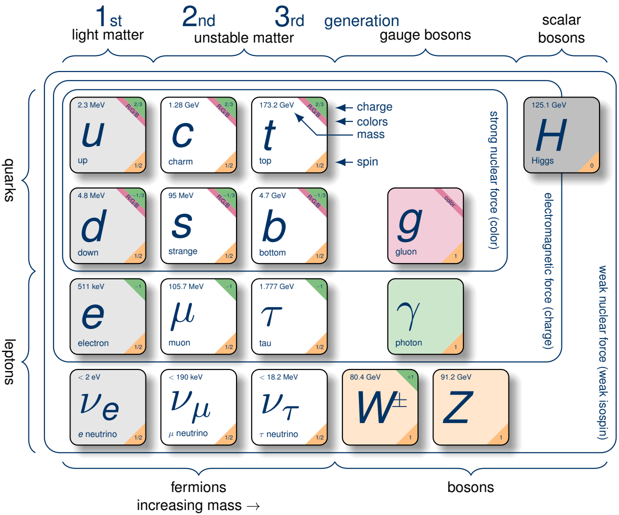
October 30-November 3, 2023

Contents

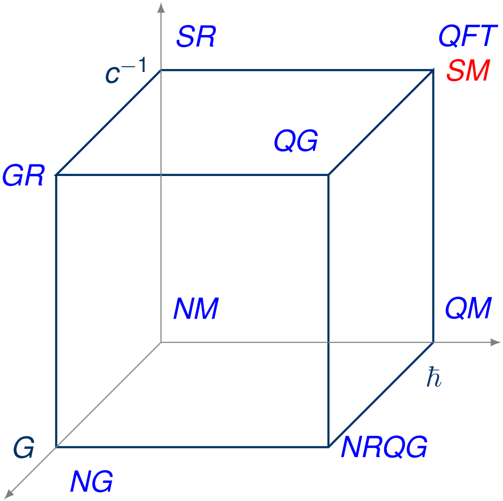
Standard Model (SM)

The Standard Model of particle physics is the theory describing the electromagnetic, weak and strong interactions of the known elementary particles.

Elementary particles and their interactions



SM is the landscape of theoretical physics



Some features of the SM

- Interactions are described by the exchange of spin 1 vector fields.
- Matter is described by spin 1/2 fields, known as fermions. There are two different kinds: Quarks, that interact strongly, and Leptons, that do not.
- Quarks and leptons are grouped into three families.
- Each species of quarks (flavor) is available in 3 colors.
- Quarks and gluons never show up as free particles, they form bounded states known as hadrons.

Principles of the SM

- Correspondence
- Minimalism
- Unitarity
- Renormalizability
- Gauge Principle

Symmetry

The guiding principle in the formulation of the SM is symmetry.
Symmetry $\equiv$ Invariance of a system S under a transformation T

$$S \xrightarrow{T} S' = S \tag{1}$$

The SM is built with different kinds of symmetries:

- Space-Time: Poincaré, Lorentz, CPT
- Gauge: $SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$
- Accidental: B, L, custodial

Conventions

Natural Units

In high energy physics it is convenient to adopt a system of units in which

$$c = \hbar = 1 \tag{2}$$

such that E, p, m, x^{-1} y t^{-1} can be described with one unit (usually eV). In this system, the Action

$$S = \int d^4x\mathcal{L} \tag{3}$$

is adimensional. Since

$$[d^4x] = [m]^{-4},$$

the lagrangian density has units

$$[\mathcal{L}] = [m]^4.$$

Notation Special Relativity

- Invariant interval

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = dt^2 - dx^2 - dy^2 - dz^2. \tag{4}$$

- Metric tensor

$$\eta_{\mu\nu} = \eta^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1). \tag{5}$$

$$\delta^\mu_\nu \equiv \eta^{\mu\sigma}\eta_{\nu\sigma} = \begin{cases} 1, & \mu = \nu \\ 0, & \mu \neq \nu \end{cases} \tag{6}$$

- Four-Vectors

$$A^\mu = (A^0, \mathbf{A}), \qquad A_\mu = \eta_{\mu\nu} A^\nu = (A^0, -\mathbf{A}), \tag{7}$$

- Lorentz invariant product

$$A^\mu B_\mu = A^0 B^0 - \mathbf{A} \cdot \mathbf{B}, \tag{8}$$

- Space-time derivatives

$$\partial_\mu = \left(\tfrac{\partial}{\partial t}, \nabla\right), \qquad \partial^\mu = \left(\tfrac{\partial}{\partial t}, -\nabla\right), \tag{9}$$

$$\partial_\mu \partial^\mu = \frac{\partial^2}{\partial t^2} - \nabla^2 \tag{10}$$

$$\partial_\mu A^\mu = \frac{\partial A^0}{\partial t} + \nabla \cdot \mathbf{A} \tag{11}$$

- Levi-Civita symbol

$$\epsilon^{\mu\nu\rho\sigma}, \quad \text{with} \quad \epsilon^{0123} = 1 = -\epsilon_{0123}, \tag{12}$$

$$\epsilon^{\mu\nu\rho\sigma}\epsilon_{\mu\nu\rho\sigma} = -24, \qquad \epsilon^{\mu\nu\rho\sigma}\epsilon_{\mu\nu\rho\tau} = -6\delta^\sigma_\tau, \tag{13}$$

$$\epsilon^{\mu\nu\rho\sigma}\epsilon_{\mu\nu\tau\omega} = -2\left(\delta^\rho_\tau\delta^\sigma_\omega - \delta^\rho_\omega\delta^\sigma_\tau\right). \tag{14}$$

The SM is a Quantum Field Theory. A field is a set of quantities defined at every point of space and time $(\mathbf{x}, t)$:

$$\phi_a(\mathbf{x}, t) \equiv \phi_a(x), \tag{15}$$

where a is a set of discrete indices.

The most familiar examples of fields from classical physics are the electric and magnetic fields, $\mathbf{E}(\mathbf{x}, t)$ and $\mathbf{B}(\mathbf{x}, t)$. Both of these fields are spatial 3-vectors.

In the SM each of the elementary particles is described by a field that transforms as an irreducible representation (irrep) of the group of space-time symmetries in special relativity.

Space-time symmetries

The Poincaré Group is the group of transformations $x^\mu \to x'^\mu$ leaving invariant the interval $(ds)^2 = \eta_{\mu\nu} dx^\mu dx^\nu$, and can be split in two distinct pieces:

- Spacetime Translations $x'^\mu = x^\mu + a^\mu$, with constant parameters a^μ.
- The group of transformations $x'^\mu = L^\mu{}_\nu x^\nu$ defined by the condition

$$L^\mu{}_\rho \eta_{\mu\nu} L^\nu{}_\sigma = \eta_{\rho\sigma}. \tag{16}$$

This group is known as $O(1,3)$ or Lorentz Group

Lorentz Group

In turn, Lorentz Group $O(1,3)$ can be split into four disconnected pieces, namely:

- The continuous subgroup $SO(1,3)^+$ or Restricted Lorentz Group (RLG), with elements $\Lambda^{\mu}{}_{\nu}$. This subgroup contains proper orthochronous transformations (those continuously connected with the identity $\det\Lambda = 1$ and preserving the direction of time $\Lambda^0{}_0 > 0$).

- Improper orthochronous transformations, described by $[P\Lambda]^{\mu}{}_{\nu}$, where the transformation

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{17}$$

can be identified as space inversion, or Parity, since $x'^{\mu} = P^{\mu}{}_{\nu} x^{\nu}$ implies $t' = t$ and $\mathbf{x}' = -\mathbf{x}$. This sector is characterized by $\det P\Lambda = -1$ and $[P\Lambda]^{0}{}_{0} > 0$.

- Improper heterochronous transformations, described by $[T\Lambda]^{\mu}{}_{\nu}$, with

$$T = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{18}$$

  which implements Time Reversal, in the sense $x'^{\mu} = T^{\mu}{}_{\nu} x^{\nu}$ implies $t' = -t$ and $\mathbf{x}' = \mathbf{x}$. In this case $\det T\Lambda = -1$ and $[T\Lambda]^{0}{}_{0} < 0$.
- Proper heterochronous transformations, described by $[PT\Lambda]^{\mu}{}_{\nu}$, with $\det PT\Lambda = 1$ and $[PT\Lambda]^{0}{}_{0} < 0$.

Thus the isometries (for vectors with fixed origin) in Minkowski space-time are described by the continuous RLG $SO(1,3)^+$ together with the discrete transformations P and T that belong to the quotient group

$$O(1,3)/SO(1,3)^+ \simeq Z_2 \otimes Z_2 = \{\mathbb{I}, P, T, PT\}\,. \tag{19}$$

Lorentz Algebra

The continuous $SO(1,3)^+$ group contains 6 parameters, describing

- 3 rotations, parameterized in cartesian coordinates by

$$\Lambda(\theta_1,0) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta_1 & -\sin\theta_1 \\ 0 & 0 & \sin\theta_1 & \cos\theta_1 \end{pmatrix}, \quad (20)$$

$$\Lambda(\theta_2,0) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_2 & 0 & \sin\theta_2 \\ 0 & 0 & 1 & 0 \\ 0 & -\sin\theta_2 & 0 & \cos\theta_2 \end{pmatrix}, \quad (21)$$

$$\Lambda(\theta_3,0) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_3 & -\sin\theta_3 & 0 \\ 0 & \sin\theta_3 & \cos\theta_3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (22)$$

- 3 boosts, that can be parameterized in terms of 3 rapidities φ_i, defined by $v_i = \tanh\varphi_i$, as

$$\Lambda(0,\varphi_1) = \begin{pmatrix} \cosh\varphi_1 & \sinh\varphi_1 & 0 & 0 \\ \sinh\varphi_1 & \cosh\varphi_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (23)$$

$$\Lambda(0,\varphi_2) = \begin{pmatrix} \cosh\varphi_2 & 0 & \sinh\varphi_2 & 0 \\ 0 & 1 & 0 & 0 \\ \sinh\varphi_2 & 0 & \cosh\varphi_2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (24)$$

$$\Lambda(0,\varphi_3) = \begin{pmatrix} \cosh\varphi_3 & 0 & 0 & \sinh\varphi_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\varphi_3 & 0 & 0 & \cosh\varphi_3 \end{pmatrix}. \quad (25)$$

Thus any element of $SO(1,3)^+$ in its defining representation can be written as

$$\Lambda(\boldsymbol{\theta},\boldsymbol{\varphi}) = e^{-i(\boldsymbol{\theta}\cdot\mathbf{J}+\boldsymbol{\varphi}\cdot\mathbf{K})}, \tag{26}$$

where the generators are given by

$$J_i = i\,\frac{\partial\Lambda(\boldsymbol{\theta},\boldsymbol{\varphi})}{\partial\theta_i}\bigg|_{\boldsymbol{\theta},\boldsymbol{\varphi}\to\mathbf{0}}, \qquad K_i = i\,\frac{\partial\Lambda(\boldsymbol{\theta},\boldsymbol{\varphi})}{\partial\varphi_i}\bigg|_{\boldsymbol{\theta},\boldsymbol{\varphi}\to\mathbf{0}}. \tag{27}$$

Explicitly,

$$J_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}, \qquad K_1 = \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$
$$J_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \qquad K_2 = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$
$$J_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad K_3 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}. \tag{28}$$

The algebra satisfied by these generators is

$$[J_i, J_j] = i\epsilon_{ijk} J_k \ , \quad [J_i, K_j] = i\epsilon_{ijk} K_k \ , \quad [K_i, K_j] = -i\epsilon_{ijk} J_k \ . \tag{29}$$

In terms of an antisymmetric tensor $\mathcal{J}^{\mu\nu} = -\mathcal{J}^{\nu\mu}$ defined as

$$J_i \equiv \frac{1}{2}\epsilon_{ijk}\mathcal{J}^{jk}, \qquad K_i \equiv \mathcal{J}^{0i}, \tag{30}$$

the Lorentz algebra takes the form

$$[\mathcal{J}^{\mu\nu}, \mathcal{J}^{\rho\sigma}] = i(\eta^{\mu\sigma}\mathcal{J}^{\nu\rho} + \eta^{\nu\rho}\mathcal{J}^{\mu\sigma} - \eta^{\mu\rho}\mathcal{J}^{\nu\sigma} - \eta^{\nu\sigma}\mathcal{J}^{\mu\rho}), \tag{31}$$

and the defining elements of the Lorentz group can be written as

$$\Lambda = e^{-\frac{i}{2}\Omega_{\mu\nu}\mathcal{J}^{\mu\nu}} = e^{-i(\boldsymbol{\theta}\cdot\mathbf{J}+\boldsymbol{\varphi}\cdot\mathbf{K})}, \tag{32}$$

with $\Omega_{\mu\nu} = -\Omega_{\nu\mu}$, and $\theta_i \equiv \frac{1}{2}\epsilon_{ijk}\Omega_{jk}$, $\varphi_i \equiv \Omega_{0i}$.

Another useful representation of the Lorentz generators can be found by defining the linear combinations

$$A_i = \frac{1}{2}(J_i + iK_i), \quad B_i = \frac{1}{2}(J_i - iK_i). \tag{33}$$

In terms of **A** and **B**, the Lorentz algebra becomes

$$[A_i, A_j] = i\epsilon_{ijk}A_k, \quad [B_i, B_j] = i\epsilon_{ijk}B_k, \quad [A_i, B_j] = 0, \tag{34}$$

which is locally isomorphic to two copies of $\mathfrak{su}(2)$. We often quote that result as

$$SO(1,3)^+ \simeq SU(2)_A \otimes SU(2)_B. \tag{35}$$

We can further classify the irreps of the RLG in terms of $SU(2)_A$ and $SU(2)_B$, by noticing that

- Irreps of each $SU(2)_A \otimes SU(2)_B$ can described by the states $\{|a, m_a\rangle\,|b, m_b\rangle\}$, with

$$\mathbf{A}^2\,|a, m_a\rangle = a(a+1)\,|a, m_a\rangle\,,\; A_3\,|a, m_a\rangle = m_a\,|a, m_a\rangle\,, \tag{36}$$

$$\mathbf{B}^2\,|b, m_b\rangle = b(b+1)\,|b, m_b\rangle\,,\; B_3\,|b, m_b\rangle = m_b\,|b, m_b\rangle\,. \tag{37}$$

- RLG irreps have dimension $(2a+1)(2b+1)$ and can be labeled by the two half integers (a, b).

RLG irreps form the following tower of states

$$
\begin{array}{c}
(0,0) \\
(\tfrac{1}{2},0) \quad (0,\tfrac{1}{2}) \\
(1,0) \quad (\tfrac{1}{2},\tfrac{1}{2}) \quad (0,1) \\
(\tfrac{3}{2},0) \quad (1,\tfrac{1}{2}) \quad (\tfrac{1}{2},1) \quad (0,\tfrac{1}{2}) \\
(2,0) \quad (\tfrac{3}{2},\tfrac{1}{2}) \quad (1,1) \quad (\tfrac{1}{2},\tfrac{3}{2}) \quad (0,2)
\end{array}
$$

Remarkably, in the Standard Model only four of them are included: Scalars, Spin 1/2 fermions and Gauge bosons. We can explicitly identify the form of the Lorentz group representation $S(\Lambda)$ and the Lorentz generators $\mathcal{J}^{\mu\nu}$ for the fields transforming under irreps of the RLG.

If we denote a general field transforming under some irreducible representation of the HGL as ϕ_a, we can write a finite Lorentz transformation as

$$\phi'_a(x') = [S(\Lambda)]_a{}^b \phi_b(x) = [e^{-\frac{i}{2}\Omega_{\mu\nu}\mathcal{J}^{\mu\nu}}]_a{}^b \phi_b(x), \tag{38}$$

with $x'^\mu = \Lambda^\mu{}_\nu x^\nu$.

Scalar Field $(0, 0)$

Defined by the transformation rule $\phi'(x') = \phi(x)$, we can easily conclude that $S_0(\Lambda) = 1$, and $\mathcal{J}_0^{\mu\nu} = 0$.

Left-handed Weyl Spinor $(\frac{1}{2}, 0)$

This field transforms according to

$$\mathbf{B} = \frac{1}{2}(\mathbf{J} - i\mathbf{K}) = 0 \Rightarrow \mathbf{J} = i\mathbf{K} \qquad \Rightarrow \mathbf{A} = \frac{1}{2}(\mathbf{J} + i\mathbf{K}) = \mathbf{J} = \frac{\boldsymbol{\sigma}}{2}$$

$$S_L(\Lambda) = e^{-i\frac{\boldsymbol{\sigma}}{2}\cdot\boldsymbol{\theta} - \frac{\boldsymbol{\sigma}}{2}\cdot\boldsymbol{\varphi}},$$

We can identify its generators as $\mathcal{J}_L^{ij} = \epsilon_{ijk}\frac{\sigma^k}{2}, \quad \mathcal{J}_L^{0i} = -i\frac{\sigma^i}{2}$. Defining $\sigma^\mu = (1, \boldsymbol{\sigma})$ and $\overline{\sigma}^\mu = (1, -\boldsymbol{\sigma})$, we have

$$\mathcal{J}_L^{\mu\nu} = \frac{i}{4}(\sigma^\mu\overline{\sigma}^\nu - \sigma^\nu\overline{\sigma}^\mu), \tag{39}$$

and the transformation rule for the left-handed fermion becomes

$$\phi'_a(x') = [S_L(\Lambda)]_a{}^b\phi_b(x) = [e^{-\frac{i}{2}\Omega_{\mu\nu}\mathcal{J}_L^{\mu\nu}}]_a{}^b\phi_b(x). \tag{40}$$

Right-handed Weyl Spinor $(0, \frac{1}{2})$

$$\mathbf{A} = \frac{1}{2}(\mathbf{J} + i\mathbf{K}) = 0 \Rightarrow \mathbf{J} = -i\mathbf{K} \qquad \Rightarrow \mathbf{B} = \frac{1}{2}(\mathbf{J} - i\mathbf{K}) = \mathbf{J} = \frac{\boldsymbol{\sigma}}{2}$$
$$S_R(\Lambda) = e^{-i\frac{\boldsymbol{\sigma}}{2}\cdot\boldsymbol{\theta} + \frac{\boldsymbol{\sigma}}{2}\cdot\boldsymbol{\varphi}},$$

and therefore $\mathcal{J}_R^{ij} = \epsilon_{ijk}\frac{\sigma^k}{2}$, $\mathcal{J}_R^{0i} = i\frac{\sigma^i}{2}$, or equivalently

$$\mathcal{J}_R^{\mu\nu} = \frac{i}{4}(\overline{\sigma}^\mu\sigma^\nu - \overline{\sigma}^\nu\sigma^\mu). \tag{41}$$

The Lorentz transformation of a right-handed spinor is written as

$$\psi'^{\dot{a}}(x') = [S_R(\Lambda)]^{\dot{a}}{}_{\dot{b}}\psi^{\dot{b}}(x) = [e^{-\frac{i}{2}\Omega_{\mu\nu}\mathcal{J}_R^{\mu\nu}}]^{\dot{a}}{}_{\dot{b}}\psi^{\dot{b}}(x). \tag{42}$$

Notice that

$$S_R^{-1}(\Lambda) = S_L^\dagger(\Lambda), \quad S_L^{-1}(\Lambda) = S_R^\dagger(\Lambda). \tag{43}$$

Dirac spinor

The Dirac spinor transforms in the $(\frac{1}{2},0)\oplus(0,\frac{1}{2})$ representation of the RLG. Thus in the chiral basis, its RLG rep is

$$S(\Lambda)=\begin{pmatrix} S_L(\Lambda) & 0 \\ 0 & S_R(\Lambda)\end{pmatrix}=\begin{pmatrix} e^{-i\frac{\boldsymbol{\sigma}}{2}\cdot\boldsymbol{\theta}-\frac{\boldsymbol{\sigma}}{2}\cdot\boldsymbol{\varphi}} & 0 \\ 0 & e^{-i\frac{\boldsymbol{\sigma}}{2}\cdot\boldsymbol{\theta}+\frac{\boldsymbol{\sigma}}{2}\cdot\boldsymbol{\varphi}}\end{pmatrix}. \tag{44}$$

In this basis, the Dirac matrices are given by

$$\gamma^\mu=\begin{pmatrix} 0 & \sigma^\mu \\ \overline{\sigma}^\mu & 0\end{pmatrix}, \tag{45}$$

and the Lorentz generators can be written as

$$\mathcal{J}^{\mu\nu}_{1/2}=\frac{i}{4}[\gamma^\mu,\gamma^\nu]. \tag{46}$$

Vector Field

Defined by the transformation rule $A'^{\mu}(x') = \Lambda^{\mu}{}_{\nu} A^{\nu}(x)$, we have in this case $[S_1(\Lambda)]^{\mu}{}_{\nu} = \Lambda^{\mu}{}_{\nu}$, with Lorentz generators

$$[\mathcal{J}_1^{\rho\sigma}]^{\mu}{}_{\nu} = i[\eta^{\rho\mu}\delta^{\sigma}_{\nu} - \eta^{\sigma\mu}\delta^{\rho}_{\nu}]. \tag{47}$$

Ingredients of the SM

Classified by their Space-time representation

- Scalars

$$\mathcal{L}^{\text{free}}_{\text{scalar}} = \partial^\mu \phi^\dagger \partial_\mu \phi - m^2 \phi^\dagger \phi. \tag{48}$$

- Spin $1/2$ fermions

$$\begin{aligned} \mathcal{L}^{\text{free}}_{\text{fermion}} &= i\overline{\psi}\gamma^\mu \partial_\mu \psi - m\overline{\psi}\psi \\ &= i\overline{\psi}_L \gamma^\mu \partial_\mu \psi_L + i\overline{\psi}_R \gamma^\mu \partial_\mu \psi_R - m\overline{\psi}_R \psi_L - m\overline{\psi}_L \psi_R, \end{aligned} \tag{49}$$

where $\bar{\psi} = \psi^\dagger \gamma^0$, and

$$\psi_L = P_L \psi \equiv \frac{1}{2}\left(1 - \gamma^5\right)\psi, \qquad \psi_R = P_R \psi \equiv \frac{1}{2}\left(1 + \gamma^5\right)\psi,$$

with $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = -\frac{i}{4!}\epsilon^{\mu\nu\rho\sigma}\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma$.

- Gauge Boson (Abelian)

$$\mathcal{L}_{\text{fgauge}}^{\text{free}} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu}, \qquad F^{\mu\nu} = \partial^{\mu} A^{\nu} - \partial^{\nu} A^{\mu}, \tag{50}$$

invariant under the local transformation

$$A^{\mu} \to A'^{\mu} = A^{\mu} + \partial^{\mu} \Omega(x). \tag{51}$$

Interactions: The Gauge Principle

In the SM, interactions are introduced by the requirement that the Gauge invariance of the vector fields is satisfied by the whole theory, including scalars and fermions. The procedure to promote a Global symmetry to a local one can be illustrated by the paradigmatic theory of Quantum Electrodynamics (QED).
The Dirac Lagrangian

$$\mathcal{L}_{\text{fermion}}^{\text{free}} = \overline{\psi}\,(i\gamma^{\mu}\partial_{\mu} - m)\,\psi, \tag{52}$$

has an internal global $U(1)$ symmetry $\psi \to e^{-i\alpha}\psi$ and $\overline{\psi} \to e^{+i\alpha}\overline{\psi}$ with real constant α.

The gauge prescription couples the Dirac and Electromagnetic fields by replacing the ordinary derivative in Dirac theory by the Covariant Derivative

$$D_\mu \equiv \partial_\mu + ieQ_f A_\mu, \tag{53}$$

that uses the Gauge field A_μ as a connection in order to make the theory invariant under the gauge transformations

$$A_\mu(x) \to A_\mu(x) + \partial_\mu \Omega(x), \tag{54}$$

$$\psi(x) \to e^{-ieQ_f\Omega(x)}\psi(x), \qquad \overline{\psi}(x) \to e^{+ieQ_f\Omega(x)}\overline{\psi}(x). \tag{55}$$

Explicitly

$$\begin{aligned} D'_\mu\psi' &= \partial_\mu\left(e^{-ieQ_f\Omega}\psi\right) + ieQ_f(A_\mu + \partial_\mu\Omega)e^{-ieQ_f\Omega}\psi \\ &= e^{-ieQ_f\Omega}\partial_\mu\psi - ieQ_f e^{-ieQ_f\Omega}\psi\partial_\mu\Omega + ieQ_f e^{-ieQ_f\Omega}(A_\mu + \partial_\mu\Omega)\psi \\ &= e^{-ieQ_f\Omega}\left[\partial_\mu + ieQ_f A_\mu\right]\psi = e^{-ieQ_f\Omega}D_\mu\psi. \end{aligned} \tag{56}$$

The resulting interacting theory is

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \overline{\psi}\left(i\gamma^{\mu}D_{\mu} - m\right)\psi, \\ &= -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \overline{\psi}\left(i\gamma^{\mu}\partial_{\mu} - m\right)\psi - eQ_f\overline{\psi}\gamma^{\mu}\psi A_{\mu}.\end{aligned} \tag{57}$$

Similarly, the Quantum Electrodynamics of scalars is given by

$$\begin{aligned}\mathcal{L} =& -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + (D^{\mu}\phi)^{\dagger}(D_{\mu}\phi) - m^2\phi^{\dagger}\phi \\ =& -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \partial^{\mu}\phi^{\dagger}\partial_{\mu}\phi - m^2\phi^{\dagger}\phi \\ & - ieQ_s\left[\phi^{\dagger}(\partial^{\mu}\phi) - (\partial^{\mu}\phi^{\dagger})\phi\right]A_{\mu} + e^2Q_s^2\phi^{\dagger}\phi A^{\mu}A_{\mu}.\end{aligned} \tag{58}$$

Standard Model Gauge Symmetries

The gauge symmetry of the SM is the direct product

$$SU(3)_C \times SU(2)_W \times U(1)_Y. \tag{59}$$

The $SU(3)_C$ symmetry describes the color interaction, and the $SU(2)_W \times U(1)_Y$ symmetry defines a unified model of the weak and electroweak interactions that undergoes a spontaneous symmetry breakdown to $U(1)_{EM}$ at low energies.

As in the case of the space-time symmetries, the fields describing elementary particles in the SM transform as irreps of the Gauge Fields.

Standard Model Field Content

Vector Fields

- Gluons G^{μ}_{a}, $a = 1, \ldots, 8$ associated to $SU(3)_C$,
- Non abelian Gauge bosons W^{μ}_{i}, $i = 1, 2, 3$ related to $SU(2)_W$,
- One abelian gauge boson B^{μ} corresponding to $U(1)_Y$.

Left-handed Weyl spinors transforming as isodoublets

$$
\begin{aligned}
L_{iL} &= \left\{ \begin{pmatrix} \nu_e \\ e \end{pmatrix}, \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}, \begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix} \right\}_L, \\
Q_{iL} &= \left\{ \begin{pmatrix} u \\ d \end{pmatrix}, \begin{pmatrix} c \\ s \end{pmatrix}, \begin{pmatrix} t \\ b \end{pmatrix} \right\}_L,
\end{aligned} \tag{60}
$$

Right-handed Weyl spinors transforming as isosinglets

$$e_{iR} = \{e, \mu, \tau\}_R\,, \quad u_{iR} = \{u, c, t\}_R\,, \quad d_{iR} = \{d, s, b\}_R\,. \tag{61}$$

Scalar Fields

One isodoublet transforming as singlet under color

$$\Phi = \begin{pmatrix} w^+ \\ \phi^0 \end{pmatrix}. \tag{62}$$

Field	$SU(3)_C$	$SU(2)_W$	$U(1)_Y$
Q_{iL}	**3**	**2**	$1/6$
u_{iR}	**3**	**1**	$2/3$
d_{iR}	**3**	**1**	$-1/3$
L_{iL}	**1**	**2**	$-1/2$
e_{iR}	**1**	**1**	-1
Φ	**1**	**2**	$1/2$

Table: Standard Model particle content ($i = 1, 2, 3$ represent generation indices).

Denoting T_i and Y to the generators of $SU(2)_W$ and $U(1)_Y$, the algebra they satisfy is

$$[T_i, T_j] = i\epsilon_{ijk} T_k, \quad [T_i, Y] = 0, \tag{63}$$

where $T_i = \tau_i/2$, with τ_i are the Pauli matrices for the fundamental representation of $SU(2)_W$. The electric charge generator is embedded into the Electroweak symmetry according to the Gell-Mann–Nishijima formula

$$Q = T_3 + Y. \tag{64}$$

The local invariance of the SM lagrangian $\mathcal{L}$ is implemented by the covariant derivative

$$D^\mu = \partial^\mu + igT_i W_i^\mu + ig' Y B^\mu. \tag{65}$$

Thus, the kinetic lagrangian density for fermion fields is (ommiting generation indices)

$$\begin{aligned}\mathcal{L}_{\text{fermion}} =& \overline{L}_L i\gamma^\mu D_\mu L_L + \overline{e}_R i\gamma^\mu D_\mu e_R \\ &+ \overline{Q}_L i\gamma^\mu D_\mu Q_R + \overline{u}_R i\gamma^\mu D_\mu u_R + \overline{d}_R i\gamma^\mu D_\mu d_R.\end{aligned} \tag{66}$$

Explicitly, for the leptonic sector we have

$$\begin{aligned}\mathcal{L}_{\text{lepton}} =& \overline{L}_L i\gamma^\mu \left(\partial_\mu + \frac{i}{2} g \tau_i W_{\mu\, i} - \frac{i}{2} g' B_\mu\right) L_L \\ &+ \overline{e}_R i\gamma^\mu \left(\partial_\mu - ig' B_\mu\right) e_R,\end{aligned} \tag{67}$$

and for quarks

$$\begin{aligned}\mathcal{L}_{\text{quark}} &= \overline{Q}_L i\gamma^\mu \left(\partial_\mu + \frac{i}{2} g \tau_i W_{\mu\, i} + \frac{i}{6} g' B_\mu\right) Q_L \\ &+ \overline{u}_R i\gamma^\mu \left(\partial_\mu + \frac{2i}{3} g' B_\mu\right) u_R \\ &+ \overline{d}_R i\gamma^\mu \left(\partial_\mu - \frac{i}{3} g' B_\mu\right) d_R. \end{aligned} \tag{68}$$

The kinetic lagrangian density for gauge fields is simply

$$\mathcal{L}_{\text{gauge}} = -\frac{1}{4} W_i^{\mu\nu} W_{\mu\nu\, i} - \frac{1}{4} B^{\mu\nu} B_{\mu\nu} \tag{69}$$

where

$$W_k^{\mu\nu} = \partial^\mu W_k^\nu - \partial^\nu W_k^\mu - g\epsilon_{ijk} W_i^\mu W_j^\nu \,, \tag{70}$$

and

$$\begin{aligned} B^{\mu\nu} &= D^\mu B^\nu - D^\nu B^\mu \\ &= \partial^\mu B^\nu - \partial^\nu B^\mu . \end{aligned}$$

The charged current interaction involves the non diagonal generators of $SU(2)_W$

$$\mathcal{L}_{\text{charged}} = i^2 g \left\{ \overline{L}_L \gamma^\mu \left(\frac{\tau_1}{2} W_{\mu\,1} + \frac{\tau_2}{2} W_{\mu\,2} \right) L_L + \overline{Q}_L \gamma^\mu \left(\frac{\tau_1}{2} W_{\mu\,1} + \frac{\tau_2}{2} W_{\mu\,2} \right) Q_L \right\} . \tag{71}$$

Upon defining the charge eigenstates

$$W_\mu^\pm = \frac{1}{\sqrt{2}} \left(W_{\mu\,1} \mp i W_{\mu\,2} \right) , \tag{72}$$

we can write

$$\mathcal{L}_{\text{charged}} = -\frac{g}{\sqrt{2}} \left[\overline{\nu}_L \gamma^\mu W_\mu^+ e_L + \overline{e}_L \gamma^\mu W_\mu^- \nu_L + \overline{u}_L \gamma^\mu W_\mu^+ d_L + \overline{d}_L \gamma^\mu W_\mu^- u_L \right] \tag{73}$$

Or, equivalently

$$\mathcal{L}_{\text{charged}} = -\frac{g}{2\sqrt{2}}\left[J_W^{\mu} W_{\mu}^{-} + J_W^{\mu\,\dagger} W_{\mu}^{+}\right] \tag{74}$$

with

$$\begin{aligned} J_W^{\mu} =& 2\left[\overline{e}_L \gamma^{\mu} \nu_L + \overline{d}_L \gamma^{\mu} u_L\right], \\ J_W^{\mu\,\dagger} =& 2\left[\overline{\nu}_L \gamma^{\mu} e_L + \overline{u}_L \gamma^{\mu} d_L\right]. \end{aligned} \tag{75}$$

Charged Current Features

- Contains only left-handed fermion fields.
- It is not diagonal in flavor space.

Similarly, the neutral current interaction is given by

$$
\begin{aligned}
\mathcal{L}_{\text{neutral}} &= -g\left[\overline{L}_L\gamma^\mu \frac{\tau_3}{2} W_{\mu\,3} L_L + \overline{Q}_L\gamma^\mu \frac{\tau_3}{2} W_{\mu\,3} Q_L\right] \\
&\quad + \frac{g'}{2}\left[\overline{L}_L\gamma^\mu B_\mu L_L + 2\overline{e}_R\gamma^\mu B_\mu e_R\right] \\
&\quad - \frac{g'}{2}\left[\frac{1}{3}\overline{Q}_L\gamma^\mu B_\mu Q_L + \frac{4}{3}\overline{u}_R\gamma^\mu B_\mu u_R - \frac{2}{3}\overline{d}_R\gamma^\mu B_\mu d_R\right]. \qquad (76)
\end{aligned}
$$

In component fields, the leptonic sector can be rewritten as

$$
\begin{aligned}
\mathcal{L}^{\text{neutral}}_{\text{leptons}} &= \overline{\nu}_L\gamma^\mu\nu_L\left(-\frac{g}{2}W_{\mu\,3} + \frac{g'}{2}B_\mu\right) \\
&\quad + \overline{e}_L\gamma^\mu e_L\left(\frac{g}{2}W_{\mu\,3} + \frac{g'}{2}B_\mu\right) + g'\overline{e}_R\gamma^\mu e_R B_\mu, \qquad (77)
\end{aligned}
$$

while for quarks we have

$$\begin{aligned}\mathcal{L}^{\text{neutral}}_{\text{quarks}} &= \overline{u}_L\gamma^\mu u_L\left(-\frac{g}{2}W_{\mu\,3}-\frac{g'}{6}B_\mu\right)+\overline{d}_L\gamma^\mu d_L\left(\frac{g}{2}W_{\mu\,3}-\frac{g'}{6}B_\mu\right)\\ &\quad -g'\left[\frac{2}{3}\overline{u}_R\gamma^\mu u_R B_\mu-\frac{1}{3}\overline{d}_R\gamma^\mu d_R B_\mu\right]. \qquad (78)\end{aligned}$$

Features of the Neutral Current

- Involves fermion fields of both chiralities.
- Is diagonal is flavor space.

Spontaneous symmetry breaking

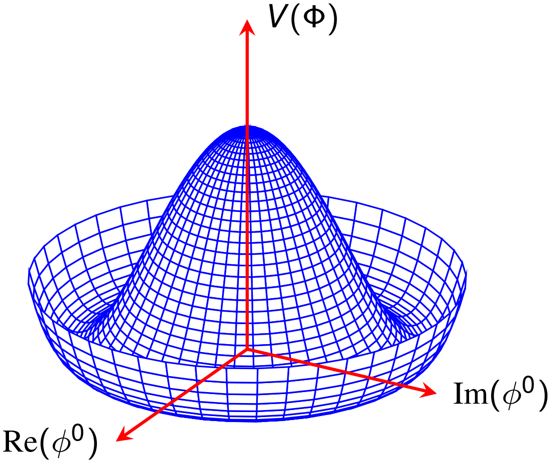
The scalar sector is described by the lagrangian density

$$\mathcal{L}_\Phi = (D^\mu \Phi)^\dagger (D_\mu \Phi) - V(\Phi). \tag{79}$$

The spontaneous symmetry breaking of the electroweak symmetry occurs if the potential is chosen as

$$V(\Phi) = -\mu^2 \Phi^\dagger \Phi + \lambda \left(\Phi^\dagger \Phi\right)^2 \tag{80}$$

with $\mu^2 > 0$ and $\lambda > 0$.


$V(\Phi)$
$\mathrm{Im}(\phi^0)$
$\mathrm{Re}(\phi^0)$

In this configuration there are infinitely many degenerate vacua characterized by the vacuum expectation value (VEV)

$$\langle \Phi \rangle = \frac{v e^{i\alpha}}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{81}$$

We can take $\alpha = 0$, since its value can change arbitrarily by a $U(1)_Y$ transformation, yielding

$$\langle \Phi \rangle = \frac{v}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{82}$$

The value of v is fixed by the minimum condition

$$0 = \left. \frac{\partial V}{\partial(\sqrt{2}\mathrm{Re}\phi^0)} \right|_{\Phi = \langle \Phi \rangle} = \frac{\partial}{\partial v} \left(-\frac{1}{2}\mu^2 v^2 + \frac{1}{4}\lambda v^4 \right) = -\mu^2 v + \lambda v^3. \tag{83}$$

In the broken phase the only solution is $v^2 = \mu^2/\lambda$.

In order for the VEV to be compatible with the preservation of $U(1)_{em}$ after electroweak symmetry breaking (EWSB), it must be invariant under an infinitesimal $U(1)_{em}$ transformation

$$e^{iQ\epsilon}\langle\Phi\rangle = (1+iQ\epsilon)\langle\Phi\rangle = \langle\Phi\rangle \ ,$$

or, equivalently, Q must annihilate the vacuum $Q\langle\Phi\rangle = 0$. Explicitly,

$$\begin{aligned} Q\langle\Phi\rangle &= (T_3+Y)\langle\Phi\rangle \\ &= \frac{1}{2}\left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right]\begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix} = 0. \end{aligned}$$

Unitary Gauge

A convenient parameterization for the scalar doublet is

$$\Phi = \begin{pmatrix} \omega^+ \\ \phi^0 \end{pmatrix} = e^{-iT_i' \xi_i(x)/v} \begin{pmatrix} 0 \\ \frac{\rho}{\sqrt{2}} \end{pmatrix}, \tag{84}$$

Where T_i' are the three broken generators T_1, T_2 and $T_3 - Y$, and the fields ξ_i, ρ are real. We can remove the ξ_i fields by performing a transformation with $U_{SU(2)} = e^{i\tau_i\theta_i(x)/2}$, and $U_{U(1)} = e^{-\theta/2}$, taking $\theta_i(x) = \xi_i(x)/v$ and $\theta = \xi_3(x)/v$,

$$\Phi \to U_{SU(2)} U_{U(1)} \Phi = \begin{pmatrix} 0 \\ \frac{v+H}{\sqrt{2}} \end{pmatrix}. \tag{85}$$

This choice of gauge, known as Unitary Gauge is useful in identifying the physical spectrum of the theory.

In the unitary gauge the kinetic lagrangian for Φ takes the simple form

$$\begin{aligned}\mathcal{L}_{\Phi}^{\text{kin}} =& (D^\mu\Phi)^\dagger(D_\mu\Phi)\\ =& \frac{1}{2}\partial^\mu H\partial_\mu H + \frac{g^2}{4}(v+H)^2 W^{\mu\,+}W_\mu^- \\ &+ \frac{(g^2+g'^2)}{8}(v+H)^2\left(\frac{gW_3^\mu - g'B^\mu}{\sqrt{g^2+g'^2}}\right)^2 .\end{aligned} \tag{86}$$

We can further reduce the last term by noticing that

$$\left(\frac{gW_3^\mu - g'B^\mu}{\sqrt{g^2+g'^2}}\right)^2 = \frac{1}{g^2+g'^2}\begin{pmatrix} W_3^\mu & B^\mu \end{pmatrix}\begin{pmatrix} g^2 & -gg' \\ -gg' & g'^2 \end{pmatrix}\begin{pmatrix} W_{\mu\,3} \\ B_\mu \end{pmatrix} \quad (87)$$

can be diagonalized by an orthogonal transformation of the form

$$\begin{pmatrix} W_{\mu\,3} \\ B_\mu \end{pmatrix} = \begin{pmatrix} c_W & s_W \\ -s_W & c_W \end{pmatrix}\begin{pmatrix} Z_\mu \\ A_\mu \end{pmatrix}, \quad (88)$$

with $c_W \equiv \cos\theta_W$, $s_W \equiv \sin\theta_W$, and θ_W as the weak (or Weinberg) angle.

The solution is

$$s_W = \frac{g'}{\sqrt{g^2 + g'^2}}, \quad c_W = \frac{g}{\sqrt{g^2 + g'^2}}. \tag{89}$$

The physical gauge bosons are

$$\begin{aligned}\begin{pmatrix} Z_\mu \\ A_\mu \end{pmatrix} &= \begin{pmatrix} c_W & -s_W \\ s_W & c_W \end{pmatrix} \begin{pmatrix} W_{\mu\,3} \\ B_\mu \end{pmatrix} \\ &= \frac{1}{\sqrt{g^2 + g'^2}} \begin{pmatrix} g & -g' \\ g' & g \end{pmatrix} \begin{pmatrix} W_{\mu\,3} \\ B_\mu \end{pmatrix}.\end{aligned} \tag{90}$$

and the kinetic scalar term becomes

$$\begin{aligned}\mathcal{L}_\Phi^{\text{kin}} =& \frac{1}{2}\partial^\mu H \partial_\mu H + \frac{g^2}{4}(v + H)^2 W^{\mu\,+} W_\mu^- \\ &+ \frac{(g^2 + g'^2)}{8}(v + H)^2 Z^\mu Z_\mu,\end{aligned} \tag{91}$$

We can read directly the mass terms for the mediators of the weak interaction

$$\frac{g^2 v^2}{4} W^{\mu\,+} W^{-}_{\mu} + \frac{1}{2}\frac{(g^2 + g'^2) v^2}{4} Z^{\mu} Z_{\mu} \\ = M_W^2 W^{\mu\,+} W^{-}_{\mu} + \frac{1}{2} M_Z^2 Z^{\mu} Z_{\mu}, \tag{92}$$

with squared masses

$$M_W^2 = \frac{g^2 v^2}{4}, \qquad M_Z^2 = \frac{(g^2 + g'^2) v^2}{4}, \tag{93}$$

related by

$$M_W = c_W M_Z. \tag{94}$$

Writing the neutral current interactions for leptons in terms of Z^μ and A^μ we have

$$\begin{aligned}\mathcal{L}^{\text{neutral}}_{\text{leptons}} &= -\frac{\sqrt{g^2+g'^2}}{2}\overline{\nu}_L\gamma^\mu\nu_L Z_\mu \\ &+\overline{e}_L\gamma^\mu e_L\left(\frac{g}{2}c_W Z_\mu - \frac{g'}{2}s_W Z_\mu\right) - g' s_W \overline{e}_R\gamma^\mu e_R Z_\mu \\ &+\frac{gg'}{\sqrt{g^2+g'^2}}\overline{e}\gamma^\mu e A_\mu. \end{aligned} \tag{95}$$

The last line can be identified with the electromagnetic current for charged leptons $-eQ_e\overline{e}\gamma^\mu e A_\mu$, yielding

$$e = \frac{gg'}{\sqrt{g^2+g'^2}} = g s_W = g' c_W. \tag{96}$$

Similarly, the interaction between the charged leptons and Z^μ simplifies to

$$\begin{aligned}&\left[\overline{e}_L\gamma^\mu e_L\left(\frac{g}{2}c_W-\frac{g'}{2}s_W\right)-g's_W\overline{e}_R\gamma^\mu e_R\right]Z_\mu\\&=-\frac{\sqrt{g^2+g'^2}}{2}\left[-\overline{e}_L\gamma^\mu e_L+2s_W^2\overline{e}\gamma^\mu e\right]Z_\mu.\end{aligned} \tag{97}$$

Including quarks we can write the QED and weak neutral current interactions as

$$\mathcal{L}_{\text{neutral}} = -eJ_Q^{\mu}A_{\mu} - \frac{g}{2c_W}J_Z^{\mu}Z_{\mu}, \tag{98}$$

with

$$J_Q^{\mu} = \frac{2}{3}\overline{u}\gamma^{\mu}u - \frac{1}{3}\overline{d}\gamma^{\mu}d - \overline{e}\gamma^{\mu}e = \sum_{\psi} Q_{\psi}\bar{\psi}\gamma_{\mu}\psi, \tag{99}$$

and

$$\begin{aligned} J_Z^{\mu} &= \overline{u}_L\gamma^{\mu}u_L - \overline{d}_L\gamma^{\mu}d_L + \overline{\nu}_L\gamma^{\mu}\nu_L - \overline{e}_L\gamma^{\mu}e_L - 2s_W^2 J_Q^{\mu} \\ &= \sum_{\psi}\bar{\psi}\gamma_{\mu}\left[(1-\gamma^5)T_{\psi}^3 - 2Q_{\psi}s_W^2\right]\psi. \end{aligned} \tag{100}$$

We can write the Higgs potential in terms of the VEV as

$$V(\Phi) = \lambda\left(\Phi^\dagger\Phi - \frac{v^2}{2}\right)^2 - \frac{\lambda v^4}{4}. \tag{101}$$

In unitary gauge, it reads

$$V(\Phi) = \lambda\left[v^2H^2 + vH^3 + \frac{H^4}{4}\right] - \frac{\lambda v^4}{4}. \tag{102}$$

From the quadratic term in H we can read directly the squared mass of the physical scalar.

$$\frac{1}{2}M_H^2H^2 = \lambda v^2H^2 \quad \Rightarrow \quad M_H^2 = 2\lambda v^2. \tag{103}$$

Fermion masses

The Yukawa interactions describe all the possible contractions between scalars and fermions that are invariant under all the defining symmetries of the model, and are given by

$$\mathcal{L}_{\text{Yuk}} = -\,\overline{L}_L \Phi y_e e_R - \overline{Q}_L \Phi y_d d_R - \overline{Q}_L \tilde{\Phi} y_u u_R + \text{h.c.}, \tag{104}$$

where y_e, y_u and y_d are 3×3 constant complex matrices, and $\tilde{\Phi}$ is the charge conjugate of Φ defined as

$$\tilde{\Phi} = i\tau_2 \Phi^* = \begin{pmatrix} \phi^{0*} \\ -w^- \end{pmatrix}, \qquad w^- = w^{+*}, \tag{105}$$

with $Y(\tilde{\Phi}) = -1/2$.

After EWSB, mass terms are induced for all charged fermions in the model. As usual, the mass spectrum is easily found in unitary gauge, where

$$\mathcal{L}_{\text{Yuk}} = -\,\overline{e}_L \frac{y_e(v+H)}{\sqrt{2}} e_R - \overline{d}_L \frac{y_d(v+H)}{\sqrt{2}} d_R - \overline{u}_L \frac{y_u(v+H)}{\sqrt{2}} u_R + \text{h.c.} \tag{106}$$

In general, the mass matrices

$$M_e = y_e \frac{v}{\sqrt{2}}, \quad M_d = y_d \frac{v}{\sqrt{2}}, \quad M_u = y_u \frac{v}{\sqrt{2}}, \tag{107}$$

are non-diagonal 3×3 complex matrices, meaning that the fermion states with well defined mass do not in general coincide with the interaction states. For massless neutrinos, M_e can be defined to be diagonal.

In the following discussion we will denote the SM fields in the interaction basis with a 0 superscript. The quark Yukawa interactions in the unitary gauge read

$$\mathcal{L}_{\text{Yuk}}^{\text{quark}} = -\,\overline{u}_L^0 M_u \left(1 + \frac{H}{v}\right) u_R^0 - \overline{d}_L^0 M_d \left(1 + \frac{H}{v}\right) d_R^0 + \text{h.c.} \tag{108}$$

We need to diagonalize the complex matrices M^u and M^d in order to identify the physical fields. This can be done through a bi-unitary transformation:

$$M^u = V_L^u m^u V_R^{u\dagger}, \qquad M^d = V_L^d m^d V_R^{d\dagger}, \tag{109}$$

where $V_{L,R}^u$ and $V_{L,R}^d$ are 3×3 unitary matrices and m^u, m^d are diagonal with non-negative entries.

This bi-unitary transformation defines a basis for mass eigenstate fields given by

$$u_{L,\,R} \equiv {V^{u}_{L,\,R}}^{\dagger} u^{0}_{L,\,R}, \qquad d_{L,\,R} \equiv {V^{d}_{L,\,R}}^{\dagger} d^{0}_{L,\,R}, \tag{110}$$

such that

$$\mathcal{L}^{\text{quark}}_{\text{Yuk}} = -\,\overline{u}_L m_u \left(1 + \frac{H}{v}\right) u_R - \overline{d}_L m_d \left(1 + \frac{H}{v}\right) d_R + \mathrm{h.c.} \tag{111}$$

We can now rewrite the quark gauge interactions in terms of the mass eigenstates. For the neutral currents we have

$$\mathcal{L}^{\text{neutral}}_{\text{quarks}} = \left[-eJ^\mu_Q A_\mu - \frac{g}{2c_W} J^\mu_Z Z_\mu\right]_{\text{quarks}}, \tag{112}$$

with

$$J^\mu_Q|_{\text{quarks}} = \frac{2}{3}\overline{u}^0\gamma^\mu u^0 - \frac{1}{3}\overline{d}^0\gamma^\mu d^0 = \frac{2}{3}\overline{u}\gamma^\mu u - \frac{1}{3}\overline{d}\gamma^\mu d, \tag{113}$$

$$\begin{aligned} J^\mu_Z|_{\text{quarks}} &= \overline{u}^0_L\gamma^\mu u^0_L - \overline{d}^0_L\gamma^\mu d^0_L - 2s^2_W J^\mu_Q|_{\text{quarks}} \\ &= \overline{u}_L\gamma^\mu u_L - \overline{d}_L\gamma^\mu d_L - 2s^2_W J^\mu_Q|_{\text{quarks}}. \end{aligned} \tag{114}$$

Thus, both the electromagnetic current and the weak neutral one take the same form in the mass and flavor basis. This means that both currents are flavor diagonal and family universal. This feature, is known as the GIM (Glashow-Iliopoulos-Maiani) mechanism.

In contrast, for the charged currents, we have

$$\begin{aligned}
\mathcal{L}^{\text{charged}}_{\text{quarks}} &= -\frac{g}{\sqrt{2}}\left[\overline{d}^0_L\gamma^\mu u^0_L W^-_\mu + \overline{u}^0_L\gamma^\mu d^0_L W^+_\mu\right] \\
&= -\frac{g}{\sqrt{2}}\left[\overline{d}_L\gamma^\mu V^{d\dagger}_L V^u_L u_L W^-_\mu + \overline{u}_L\gamma^\mu V^{u\dagger}_L V^d_L d_L W^+_\mu\right] \\
&\equiv -\frac{g}{\sqrt{2}}\left[\overline{d}_L\gamma^\mu V^\dagger_{\text{CKM}} u_L W^-_\mu + \overline{u}_L\gamma^\mu V_{\text{CKM}} d_L W^+_\mu\right],
\end{aligned} \tag{115}$$

where we have defined the unitary Cabibbo-Kobayashi-Maskawa (CKM) matrix

$$V_{\text{CKM}} = V^{u\dagger}_L V^d_L = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}, \tag{116}$$

which describes the mismatch between the unitary transformations relating the weak and mass eigenstates for the up and down quarks.

Conventionally, we absorb the action of V_{CKM} on d-type quarks, such that we can write

$$\mathcal{L}_{\text{quarks}}^{\text{charged}} = -\frac{g}{\sqrt{2}}\left[(\bar{u}_L\gamma^\mu d_L^{\text{mix}} + \bar{c}_L\gamma^\mu s_L^{\text{mix}} + \bar{t}_L\gamma^\mu b_L^{\text{mix}})W_\mu^+ + \text{h.c.}\right], \quad (117)$$

by defining the mixed states

$$\begin{pmatrix} d \\ s \\ b \end{pmatrix}_L^{\text{mix}} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \begin{pmatrix} d \\ s \\ b \end{pmatrix}_L . \quad (118)$$

For three families of quarks, V_{CKM} can be parameterized using three mixing angles and one CP violating complex phase. The most popular parameterization is the PDG one

$$\begin{aligned} V_{\text{CKM}} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13} e^{-i\delta_{13}} \\ 0 & 1 & 0 \\ -s_{13} e^{+i\delta_{13}} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} c_{12} c_{13} & s_{12} c_{13} & s_{13}\ e^{-i\delta_{13}} \\ -s_{12} c_{23} - c_{12} s_{23} s_{13}\ e^{i\delta_{13}} & c_{12} c_{23} - s_{12} s_{23} s_{13}\ e^{i\delta_{13}} & s_{23} c_{13} \\ s_{12} s_{23} - c_{12} c_{23} s_{13}\ e^{i\delta_{13}} & -c_{12} s_{23} - s_{12} c_{23} s_{13}\ e^{i\delta_{13}} & c_{23} c_{13} \end{pmatrix}, \end{aligned} \tag{119}$$

with $s_{ij} \equiv \sin\theta_{ij}$ and $c_{ij} \equiv \cos\theta_{ij}$.

Phenomenologically, the magnitudes of the CKM matrix entries are of order

$$V_{\mathrm{CKM}} \sim \begin{pmatrix} 1 & \lambda & \lambda^3 \\ \lambda & 1 & \lambda^2 \\ \lambda^3 & \lambda^2 & 1 \end{pmatrix}, \tag{120}$$

and the best global fit for its parameters is

$$\theta_{12} = 13.04 \pm 0.05^\circ, \qquad \theta_{13} = 0.201 \pm 0.011^\circ,$$
$$\theta_{23} = 2.38 \pm 0.06^\circ, \qquad \delta_{13} = 1.20 \pm 0.08\,\mathrm{rad}.$$

t'Hooft-Feynman Gauge

Unfortunately, unitary gauge is only appropriate to determine the mass spectrum of the theory. If one is to perform a serious calculation, a more sophisticated choice of gauge fixing is required. Let us consider again the scalar sector, with the Higgs doublet written as

$$\Phi = \begin{pmatrix} w^+ \\ \frac{v+H+iz}{\sqrt{2}} \end{pmatrix} .$$

In components, the scalar potential is

$$
\begin{aligned}
V(\Phi) =& \lambda\left(\Phi^\dagger\Phi - \frac{v^2}{2}\right)^2 - \frac{\lambda v^4}{4} \\
=& H^2\lambda v^2 + H^3\lambda v + H^2 + \frac{H^4\lambda}{4} + \lambda w^- w^+ \\
&+ \frac{1}{2}H^2\lambda z^2 + 2H\lambda v w^- w^+ + H\lambda v z^2 + \lambda(w - w^+)^2 \\
&+ \lambda w^- w^+ z^2 + \frac{\lambda z^4}{4} - \frac{\lambda v^4}{4}.
\end{aligned} \tag{121}
$$

We can write again the kinetic scalar lagrangian in components too, defining $C_\mu \equiv \sin 2\theta A_\mu + \cos 2\theta Z_\mu$ and adopting the notation ($A\overleftrightarrow{\partial}_\mu B = A\partial_\mu B - \partial_\mu A B$) we obtain

$$
\begin{aligned}
\mathcal{L}_\Phi^{\text{kin}} = (D^\mu\Phi)^\dagger D_\mu\Phi = & \frac{1}{2}\partial_\mu H\partial^\mu H + \frac{1}{2}\partial_\mu z\partial^\mu z + \partial^\mu w^+\partial_\mu w^- + \frac{g^2}{4c_W^2}w^- w^+ C^\mu C_\mu + \frac{ig}{2c_W}w^+ C^\mu\overleftrightarrow{\partial}_\mu w^- \\
& + \frac{g^2}{4c_W}C^\mu(W_\mu^- w^+ + W_\mu^+ w^-)(v+H) + \frac{ig^2}{4c_W}C^\mu(W_\mu^+ zw^- - W_\mu^- zw^+) + \frac{g^2}{8c_W^2}H^2 Z^\mu Z_\mu \\
& + \frac{g^2 v}{4c_W^2}HZ^\mu Z_\mu + \frac{M_Z^2}{2}Z^\mu Z_\mu + \frac{g^2}{8c_W^2}z^2 Z^\mu Z_\mu + \frac{g}{2c_W}z\overleftrightarrow{\partial}_\mu HZ^\mu - \frac{g}{2c_W}HZ^\mu\partial_\mu z - M_Z Z_\mu\partial^\mu z \\
& - \frac{g^2}{4c_W}Z^\mu(W_\mu^- w^+ + W_\mu^+ w^-)(v+H) + \frac{ig^2}{4c_W}Z^\mu(W_\mu^- zw^+ - W_\mu^+ zw^-) + \frac{ig}{2}\partial^\mu H(W_\mu^- w^+ W_\mu^+ w^-) \\
& + \frac{g}{2}\partial^\mu z(W_\mu^- w^+ + W_\mu^+ w^-) - \frac{ig}{2}(\partial^\mu w^+ W_\mu^- - \partial^\mu w^- W_\mu^+)H + -iM_W(\partial^\mu w^+ W_\mu^- - \partial^\mu w^- W_\mu^+) \\
& - \frac{g}{2}z(\partial^\mu w^- W_\mu^+ + \partial^\mu w^+ W_\mu^-) + \frac{g^2}{4}H^2 W^- W^+ + \frac{g^2 v}{2}HW^- W^+ \\
& + M_W W^- W^+ + \frac{g^2}{4}W^- W^+ z^2 + \frac{g^2}{2}W^- W^+\partial w^- w^+ .
\end{aligned}
\tag{122}
$$

The presence of the terms

$$\partial_\mu w^+ \partial^\mu w^- + \frac{1}{2}\partial_\mu z \partial^\mu z - iM_W(\partial^\mu w^+ W_\mu^- - \partial^\mu w^- W_\mu^+) - M_Z Z_\mu \partial^\mu z \quad (123)$$

indicates that the unphysical fields w^+, w^- and z do propagate and mix with the weak gauge bosons. In order to decouple them, we can introduce gauge-fixing terms of the form

$$\mathcal{L}_{\text{fix}} = -\frac{1}{2}\left(F_A^2 + F_Z^2 + 2F_+F_-\right), \quad (124)$$

$$F_\pm = \frac{1}{\sqrt{\xi^W}}\left(\partial^\mu W_\mu^\pm \pm iM_W \xi^W w^\pm\right),$$

$$F_Z = \frac{1}{\sqrt{\xi^Z}}\left(\partial^\mu Z_\mu + M_Z \xi^Z z\right), \qquad F_A = \frac{1}{\sqrt{\xi^A}}\partial^\mu A_\mu, \quad (125)$$

with arbitrary parameters $\xi^{W,Z,A}$.

The gauge fixing lagrangian contributes with the following terms:

$$\mathcal{L}_{\text{fix}} = -\frac{1}{\xi^W}\left|\partial^\mu W^+_\mu\right|^2 - M_W\xi^W w^+ w^- - iM_W(w^+\partial^\mu W^-_\mu - w^-\partial^\mu W^+_\mu) \\ -\frac{1}{2\xi^Z}\left(\partial^\mu Z_\mu\right)^2 - \frac{M_Z^2\xi^Z}{2}z^2 - M_Z z\partial^\mu Z_\mu - \frac{1}{2\xi^A}\left(\partial^\mu A_\mu\right)^2 , \tag{126}$$

and the undesired terms combine into

$$\partial_\mu w^+\partial^\mu w^- - M_W\xi^W w^+ w^- + \frac{1}{2}\partial_\mu z\partial^\mu z - \frac{M_Z^2\xi^Z}{2}z^2 \\ -iM_W\partial^\mu(w^+W^-_\mu - w^-W^+_\mu) - M_Z\partial^\mu(Z_\mu z).$$

making explicit the fact that the would-be Goldstone bosons are unphysical, while decoupling their mixing with the gauge bosons since it becomes a total derivative.

The remnant pieces of the gauge fixing lagrangian make possible the obtention of the gauge boson propagators, that now read (with $\xi^{W,Z,A} = \xi$)

$$
\begin{aligned}
W^{\pm} &: \quad \frac{i\left[-g^{\mu\nu} + (1-\xi)k^{\mu}k^{\nu}/(k^2 - \xi M_W^2)\right]}{k^2 - M_W^2 + i\epsilon} \\
Z &: \quad \frac{i\left[-g^{\mu\nu} + (1-\xi)k^{\mu}k^{\nu}/(k^2 - \xi M_Z^2)\right]}{k^2 - M_Z^2 + i\epsilon} \\
A &: \quad \frac{i\left[-g^{\mu\nu} + (1-\xi)k^{\mu}k^{\nu}/k^2\right]}{k^2 + i\epsilon} .
\end{aligned}
\tag{127}
$$

Similarly, for the scalars

$$\left|\partial_\mu w^+\right|^2 - \xi M_W^2 \left|w^+\right|^2 + \frac{1}{2}\partial_\mu z \partial^\mu z - \frac{1}{2}\xi M_Z z^2 + \frac{1}{2}\partial_\mu H \partial^\mu H - \frac{1}{2} m_H^2 H^2 \tag{128}$$

the corresponding propagators are

$$\begin{aligned} w^\pm &: \quad i\left[k^2 - (\xi M_W^2) + i\epsilon\right]^{-1} \\ z &: \quad i\left[k^2 - (\xi M_Z^2) + i\epsilon\right]^{-1} \\ H &: \quad i(k^2 - m_H^2 + i\epsilon)^{-1}. \end{aligned} \tag{129}$$

Faddeev-Popov Ghosts

The last piece of the standard model is a set of fictitious fields known as Faddeev-Popov Ghosts, that cancel the contributions of the additional degrees of freedom introduced by the gauge fixing procedure.
The relevant lagrangian reads

$$\mathcal{L}_{\mathrm{gh}} = -g \sum_{\alpha,\beta} \sqrt{\xi^{\alpha}}\, \bar{u}_{\alpha}(x) \frac{\delta F_{\alpha}}{\delta \theta^{\beta}(x)} u_{\beta}(x), \qquad (130)$$

with ghost fields u_A, u_Z, $u_{\pm}$, and $\delta F_{\alpha}/\delta\theta_{\beta}$ as the variation of the gauge-fixing functions under infinitesimal gauge transformations.

The total lagrangian for the ghost and anti-ghost fields is

$$\mathcal{L}_{\rm gh} = \mathcal{L}_{u_A} + \mathcal{L}_{u_Z} + \mathcal{L}_{u_+} + \mathcal{L}_{u_-} . \tag{131}$$

$$\begin{aligned}\mathcal{L}_{u_A} &= -g\sum_\beta \sqrt{\xi^A}\bar{u}_A \frac{\delta F_A}{\delta\theta^\beta} u_\beta \\ &= \partial^\mu \bar{u}_A \partial_\mu u_A + igs_W \partial^\mu \bar{u}_A (u_- W^+_\mu - u_+ W^-_\mu),\end{aligned} \tag{132}$$

$$\begin{aligned}\mathcal{L}_{u_Z} &= -g\sum_\beta \sqrt{\xi^Z}\bar{u}_Z \frac{\delta F_Z}{\delta\theta^\beta} u_\beta \\ &= \partial^\mu \bar{u}_Z \partial_\mu u_Z - M_Z^2 \xi^Z \bar{u}_Z u_Z + igc_W \partial^\mu \bar{u}_Z \left(u_- W^+_\mu - u_+ W^-_\mu\right) \\ &\quad - \frac{gM_Z\xi^Z}{2} H\bar{u}_Z u_Z + \frac{gM_Z\xi^Z}{2}\bar{u}_Z (u_- w^+ + u_+ w^-),\end{aligned} \tag{133}$$

$$\begin{aligned}
\mathcal{L}_{u_\pm} &= -g \sum_\beta \sqrt{\xi^W} \bar{u}_\pm \frac{\delta F_\pm}{\delta \theta^\beta} u_\beta \qquad (134)\\
&= \partial^\mu \bar{u}_\pm \partial_\mu u_\pm - M_W^2 \xi^W \bar{u}_\pm u_\pm \\
&\quad \pm ig\partial^\mu \bar{u}_\pm \left[c_W u_\pm Z_\mu + s_W u_\pm A_\mu - (c_W u_Z + s_W u_A) W_\mu^\pm \right] \\
&\quad - \frac{g M_W \xi^W}{2} \bar{u}_\pm \left[\left(2 s_W u_A + \frac{c_W^2 - s_W^2}{c_W} u_Z \right) \phi^\pm + u_\pm H \pm i u_\pm z \right].
\end{aligned}$$

Notice that the masses for the ghosts coincide with the masses of the would-be Goldstone Bosons.

QCD

The $SU(3)_c$ gauge bosons are described by the following lagrangian density

$$\mathcal{L}_{\text{gluons}} = -\frac{1}{4} G_a^{\mu\nu} G_{\mu\nu\, a} + \frac{\theta_{QCD}}{32\pi^2} g_s^2 G_a^{\mu\nu} \tilde{G}_{\mu\nu\, a} - \frac{1}{2} (F_a^g)^2 + g_s \sqrt{\xi^g} \bar{\omega}_a \frac{\delta F_a^g}{\delta \vartheta_a} \omega_b, \tag{135}$$

with the non-abelian gluon field strength tensor

$$G_a^{\mu\nu} = \partial^\mu G_a^\nu - \partial^\nu G_a^\mu - g_s f_{abc} G_a^\mu G_b^\nu, \quad \tilde{G}_a^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} G_{\rho\sigma\, a}, \tag{136}$$

and a dimensionless constant parameter θ_{QCD}.

The gauge fixing function

$$F_a^g = \frac{1}{\sqrt{\xi^g}} \partial_\mu G_a^\mu, \tag{137}$$

and ω_a are the the ghosts associated with the $SU(3)_c$ transformations $\Omega_{SU(3)} = e^{i\vartheta_a t_a}$, with $[t_a, t_b] = if_{abc} t_c$ and f_{abc} as the structure constants of $SU(3)_c$. Color interactions take place through the covariant derivative

$$\begin{aligned} D^\mu &= \partial^\mu + ig\mathcal{W}^\mu + ig' Y B^\mu + ig_s \mathcal{G}^\mu \\ &\equiv \partial^\mu + ig T_i W_i^\mu + ig' Y B^\mu + +ig_s t_a G_a^\mu, \end{aligned} \tag{138}$$

where the $SU(3)_c$ generators for quarks are $t_a = \lambda_a/2$, with λ_a as the Gell-Mann matrices.

Thanks