STAD94

Poisson PMF: $P(X=k \mid \lambda) = \dfrac{\lambda^{k} e^{-\lambda}}{k!}$

Gamma dist: $f(\lambda \mid \alpha, \beta) = \dfrac{\beta^{\alpha} \lambda^{\alpha-1} e^{-\beta\lambda}}{\Gamma(\alpha)}$, $\lambda > 0$

$\Gamma(\alpha)$ gamma func and α = shape and β rate

We define Poisson-Gamma Mixture

- Suppose $X \mid \lambda \sim \text{Poisson}(\lambda)$
- $\lambda \sim G(\alpha, \beta)$

Marginal dist of X

$$P(X=k) = \int_0^\infty P(X=k \mid \lambda) f(\lambda)\, d\lambda$$

$$\Leftrightarrow P(X=k) = \int_0^\infty \frac{\lambda^{k} e^{-\lambda}}{k!} \cdot \frac{\beta^{\alpha} \lambda^{\alpha-1} e^{-\beta\lambda}}{\Gamma(\alpha)}\, d\lambda$$

$$\Leftrightarrow P(X=k) = \frac{\beta^{\alpha}}{k!\,\Gamma(\alpha)} \int_0^\infty \lambda^{k+\alpha-1} e^{-(\lambda(\beta+1))}\, d\lambda$$

$\int_0^\infty \lambda^{(k+\alpha)-1} e^{-\lambda(\beta+1)}\, d\lambda$ = pdf or kernel of Gamma$(\alpha+k, \beta+1)$

$$= \frac{\Gamma(k+\alpha)}{(\beta+1)^{k+\alpha}}$$

using that

$$P(X=k) = \frac{\Gamma(k+\alpha)}{k!\,\Gamma(\alpha)} \left(\frac{\beta}{\beta+1}\right)^{\alpha} \left(\frac{1}{\beta+1}\right)^{k}$$

Let $p = \frac{1}{\beta+1}$ and $1-p = \frac{\beta}{\beta+1}$

$$P(x=k) = \frac{\Gamma(k+a)}{k!\,\Gamma(a)} (1-p)^{a} p^{k}$$

This becomes the pmf of neg bin with $r = \alpha$ (successes) and $p = \frac{1}{\beta+1}$ (success prob)

$$P(X=k) = \frac{\Gamma(k+r)}{k!\,\Gamma(r)} (1-p)^{r} p^{k}$$