Building a Numerical Model of the Influence of a Gravitational Wave on an Electromagnetic Wave

The Physics behind PhotonGravityWavePerpendicularCase.py and PhotonGravityWave(2+1)-D.py

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Let us define a left-handed[1] coordinate system along whose xy plane electromagnetic waves can travel. Consider a gravitational wave propagating along the z-axis. Let us further define the x- and y-axes to be along the polarization axes of the gravitational wave. This has the effect of stretching and contracting space along those 2 axes. So, the wave equation in the presence of a gravitational wave can be written (taking out the the time component because it will not be affected) as

$$\begin{pmatrix}\partial_x & \partial_y & \partial_z\end{pmatrix} \mathbf{M} \begin{pmatrix}\partial_x \\ \partial_y \\ \partial_z\end{pmatrix} u = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}, \tag{1}$$

where $\mathbf{M}$ is the space component of the metric tensor and is represented by

$$\mathbf{M} = \begin{pmatrix}1+f & & \\ & 1-f & \\ & & 1\end{pmatrix}, \quad \text{where } f = \epsilon\cos(kz - kct). \tag{2}$$

Note that when the amplitude ϵ is 0, Eq. 1 reduces to the standard wave equation with no gravitational wave (i.e. flat spacetime),

$$\nabla^2 u = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}. \tag{3}$$

Let us generalize this for any inclination of the gravitational wave. We need not generalize for azimuthal angle because we can always define a planar axis along the azimuthal direction of the wave. First, let us define a left-handed coordinate system $\langle x\prime, y\prime, z\prime\rangle$ attached to the gravitational wave such that $x\prime$ and $y\prime$ are along its polarization axes with $z\prime$ along its propagation direction. Second, we must also define a left-handed coordinate system $\langle X, Y, Z\rangle$ fixed to our frame of interest whose XY plane contains those electromagnetic waves. The latter is the frame our computational algorithms will use; the rows and columns of the numerical method form the discretized representation of the plane. The row indices and column indices correspond to Y-values and X-values respectively with (0,0) as the top-left corner. Note that our initial coordinate system $\langle x, y, z\rangle$ was equivalent to both $\langle X, Y, Z\rangle$ and $\langle x\prime, y\prime, z\prime\rangle$ as that was the case where the gravitational wave is perpendicular to the plane. In our new frame, our tensor $\mathbf{M}$ takes $f = \epsilon\cos(kz\prime - kct)$ with t unchanged.

We know that we can always find a rotation matrix $\mathbf{R}$ such that

$$\begin{pmatrix}X \\ Y \\ Z\end{pmatrix} = \mathbf{R}\begin{pmatrix}x\prime \\ y\prime \\ z\prime\end{pmatrix}. \tag{4}$$

Since the rotation matrix must be orthogonal, $\mathbf{R}^{-1} = \mathbf{R}^T$. Therefore, transposing both sides of Eq. 4 shows that $\begin{pmatrix}X & Y & Z\end{pmatrix} = \begin{pmatrix}x\prime & y\prime & z\prime\end{pmatrix}\mathbf{R}^{-1}$. We can now rewrite the left hand side of Eq. 1 as

$$\begin{aligned} LHS &= \begin{pmatrix}\partial_{x\prime} & \partial_{y\prime} & \partial_{z\prime}\end{pmatrix} \mathbf{R}^{-1}\mathbf{R}\ \mathbf{M}\ \mathbf{R}^{-1}\mathbf{R}\begin{pmatrix}\partial_{x\prime} \\ \partial_{y\prime} \\ \partial_{z\prime}\end{pmatrix} u \\ &= \begin{pmatrix}\partial_X & \partial_Y & \partial_Z\end{pmatrix} \mathbf{R}\ \mathbf{M}\ \mathbf{R}^{-1}\begin{pmatrix}\partial_X \\ \partial_Y \\ \partial_Z\end{pmatrix} u. \end{aligned} \tag{5}$$

[1]This orientation is more convenient for our computational system because its natural 2D origin is at the top-left corner.

Let us define another tensor $\mathbf{T} \equiv \mathbf{R}\mathbf{M}\mathbf{R}^{-1}$, the conjugate of $\mathbf{M}$ by rotation. We can characterize the rotation required to go from the gravitational wave coordinate system to our computational coordinate system as Euler angle rotations about $z\prime, x\prime, z\prime$, but we have enough freedom in our coordinates to be able to always orient our $\langle X, Y, Z\rangle$ such that the last roll rotation about the new $z\prime$ axis does nothing. Thus, for an angle of incidence of θ,[2]

$$\mathbf{R} = \mathbf{R}_{z\prime}\mathbf{R}_{x\prime} = \begin{pmatrix} \cos(\theta) & -\cos(\theta)\sin(\theta) & \sin^2(\theta) \\ \sin(\theta) & \cos^2(\theta) & -\cos(\theta)\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{pmatrix} \tag{6}$$

and hence

$$\mathbf{T} = \begin{pmatrix} \frac{1}{8}(4f\cos(2\theta) + f\cos(4\theta) + 3f + 8) & \frac{1}{8}f(6\sin(2\theta) + \sin(4\theta)) & f\sin^2(\theta)\cos(\theta) \\ \frac{1}{8}f(6\sin(2\theta) + \sin(4\theta)) & \frac{1}{8}(-8f\cos(2\theta) - f\cos(4\theta) + f + 8) & -f\sin(\theta)\cos^2(\theta) \\ f\sin^2(\theta)\cos(\theta) & -f\sin(\theta)\cos^2(\theta) & \frac{1}{2}(f\cos(2\theta) - f + 2) \end{pmatrix} \tag{7}$$

We have introduced the parameter θ to convert between our two coordinate systems as our computations require all terms to be in the $\langle X, Y, Z\rangle$ system, but recall that $f = f(t, z\prime) = \epsilon\cos(kz\prime - kct)$. We therefore need to solve Eq. 4 for the primed column vector with $\mathbf{R}$ as above about $x\prime$,

$$\begin{pmatrix} x\prime \\ y\prime \\ z\prime \end{pmatrix} = \mathbf{R}^T \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} X\cos(\theta) + Y\sin(\theta) \\ \frac{1}{2}(-X\sin(2\theta) + Y\cos(2\theta) + 2Z\sin(\theta) + Y) \\ X\sin^2(\theta) + \cos(\theta)(Z - Y\sin(\theta)) \end{pmatrix}$$

Thus,

$$f = f(t, X, Y, Z) = \epsilon\cos\Big(k\big(X\sin^2(\theta) + \cos(\theta)\big(Z - Y\sin(\theta)\big) - ct\big)\Big). \tag{8}$$

This puts $LHS = \nabla^T\mathbf{T}\nabla u$, the compact form of Eq. 5, entirely in terms of our computational coordinates X, Y, Z. Hence, we have the desired form of the generalized (3+1)-dimensional equation describing the displacement $u(t, X, Y, Z)$ of an electromagnetic field (or, in fact, anything else obeying Eq. 3, the wave equation) interacting with a gravitational wave,

$$\boxed{\nabla^T\mathbf{T}\nabla u = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \qquad \text{with } f = f(t, X, Y, Z).} \tag{9}$$

To numerically compute results, we must discretize Eq. 9. Our process for this will build on that described in WaveEquation.pdf. The electromagnetic waves are contained in the XY plane of interest. We therefore need only concern ourselves with the case $f = f(t, X, Y, 0) = \epsilon\cos\big(k\big(X\sin^2(\theta) - Y\cos(\theta)\sin(\theta) - ct\big)\big)$. We define a numerical displacement function $u_{m,n,o,t}$ where m, n, o, and t are the row, column, depth-step, and time-step indices, respectively. This means that, in our graphically conveniently left-handed system, m, n, o, and t are the discretizations of the Y-, X-, Z-, and t-coordinates, respectively. Let us generalize the numerical first derivative from WaveEquation.pdf (using the notation and names defined there) to numerical partial derivatives. We define nD such that, for a function $g(t, X, Y, Z)$,

$$\begin{aligned} \mathrm{nD}_X(g_{m,n,o,t}) &\equiv \frac{g_{m,n+1,o,t} - g_{m,n-1,o,t}}{2\Delta} \approx \frac{\partial g}{\partial X}, \\ \mathrm{nD}_Y(g_{m,n,o,t}) &\equiv \frac{g_{m+1,n,o,t} - g_{m-1,n,o,t}}{2\Delta} \approx \frac{\partial g}{\partial Y}, \\ &\text{and similarly for } Z \text{ (index } o\text{, step size } \Delta) \text{ and } t \text{ (index } t\text{, step size } \Delta t). \end{aligned} \tag{10}$$

[2]Note that θ is defined clockwise when viewed from above the YZ plane because our coordinate system is left-handed.

Using Eq. 7, we know that the left hand side of our main result, Eq. 9, is given by

$$\begin{aligned}
LHS = & \frac{\partial}{\partial X}\Big[\frac{1}{8}(4f\cos(2\theta)+f\cos(4\theta)+3f+8)\frac{\partial u}{\partial X}+\frac{1}{8}f(6\sin(2\theta)+\sin(4\theta))\frac{\partial u}{\partial Y}+f\sin^2(\theta)\cos(\theta)\frac{\partial u}{\partial Z}\Big] \\
& +\frac{\partial}{\partial Y}\Big[\frac{1}{8}f(6\sin(2\theta)+\sin(4\theta))\frac{\partial u}{\partial X}+\frac{1}{8}(-8f\cos(2\theta)-f\cos(4\theta)+f+8)\frac{\partial u}{\partial Y}-f\sin(\theta)\cos^2(\theta)\frac{\partial u}{\partial Z}\Big] \\
& +\frac{\partial}{\partial Z}\Big[f\sin^2(\theta)\cos(\theta)\frac{\partial u}{\partial X}-f\sin(\theta)\cos^2(\theta)\frac{\partial u}{\partial Y}+\frac{1}{2}(f\cos(2\theta)-f+2). \qquad (11)
\end{aligned}$$

Now, we discretize Eq. 11 in terms of our previously defined numerical derivatives. For later convenience, let us multiply Eqs. 10 by the spatial step size Δ and multiply the discretized form of Eq. 11 by Δ^2 so as to put LHS in terms of two functions, $\Delta\cdot\mathrm{nD}_X$ and $\Delta\cdot\mathrm{nD}_Y$, independent of parameters. This gives us

$$\begin{aligned}
\Delta^2\cdot LHS_{m,n,o,t} = {} & \Delta\cdot\mathrm{nD}_X\Big[\frac{1}{8}(4f\cos(2\theta)+f\cos(4\theta)+3f+8)\Delta\cdot\mathrm{nD}_X(u_{m,n,o,t}) \\
& \quad +\frac{1}{8}f(6\sin(2\theta)+\sin(4\theta))\Delta\cdot\mathrm{nD}_Y(u_{m,n,o,t})+f\sin^2(\theta)\cos(\theta)\Delta\cdot\mathrm{nD}_Z(u_{m,n,o,t})\Big] \\
& +\Delta\cdot\mathrm{nD}_Y\Big[\frac{1}{8}f(6\sin(2\theta)+\sin(4\theta))\Delta\cdot\mathrm{nD}_X(u_{m,n,o,t}) \\
& \quad +\frac{1}{8}(-8f\cos(2\theta)-f\cos(4\theta)+f+8)\Delta\cdot\mathrm{nD}_Y(u_{m,n,o,t})-f\sin(\theta)\cos^2(\theta)\Delta\cdot\mathrm{nD}_Z(u_{m,n,o,t})\Big] \\
& +\Delta\cdot\mathrm{nD}_Z\Big[f\sin^2(\theta)\cos(\theta)\Delta\cdot\mathrm{nD}_X(u_{m,n,o,t})-f\sin(\theta)\cos^2(\theta)\Delta\cdot\mathrm{nD}_Y(u_{m,n,o,t}) \\
& \quad +\frac{1}{2}(f\cos(2\theta)-f+2)\Delta\cdot\mathrm{nD}_Z(u_{m,n,o,t})\Big]. \qquad (12)
\end{aligned}$$

Now, we can explicitly write Eq. 12 by applying Eqs. 10 distributively.[3] The expressions are now exceedingly long, but below is the (2+1)-dimensional XY slice of the final explicit equation (with the coefficients put back as the components of $\mathbf{T}$ that were written out in Eq. 12) taken at time $t-1$ for later programming convenience.

$$\begin{aligned}
\Delta^2\cdot LHS_{m,n,t-1} = \frac{1}{4}\Big(& T_{0,0_{m,n,t-1}}(u_{m,n+2,t-1}-2u_{m,n,t-1}+u_{m,n-2,t-1}) \\
& +T_{0,1_{m,n,t-1}}(u_{m+1,n+1,t-1}-u_{m+1,n-1,t-1}-u_{m-1,n+1,t-1}+u_{m-1,n-1,t-1}) \\
& +T_{1,0_{m,n,t-1}}(u_{m+1,n+1,t-1}-u_{m-1,n+1,t-1}-u_{m+1,n-1,t-1}+u_{m-1,n-1,t-1}) \\
& +T_{1,1_{m,n,t-1}}(u_{m+2,n,t-1}-2u_{m,n,t-1}+u_{m-2,n,t-1})\Big) \qquad (13)
\end{aligned}$$

The right hand side of Eq. 9 is easily discretized in time. Equating that to LHS gives us the fully discretized form of Eq. 9,

$$\frac{1}{\Delta^2}\ \Delta^2\cdot LHS_{m,n,o,t-1} = \frac{1}{c^2}\ \mathrm{nD}_t(\mathrm{nD}_t(u_{m,n,o,t-1})). \qquad (14)$$

Finally, applying Eqs. 10 to Eq. 14 and collecting parameters on one side, we have our desired iterative equation that solves the wave equation in the presence of a gravitational wave.

$$\boxed{u_{m,n,o,t} = \left(c\ \frac{\Delta t}{\Delta}\right)^2 \Delta^2\cdot LHS_{m,n,o,t-1}+2u_{m,n,o,t-1}-u_{m,n,o,t-2}} \qquad (15)$$

with $\Delta^2\cdot LHS$ given by Eq. 12.

The manipulation that allowed us to program the numerical partial derivative functions independently of parameters now gives rise to the accumulated parameter

$$K \equiv \left(c\ \frac{\Delta t}{\Delta}\right)^2. \qquad (16)$$

[3]It is easy to check that this useful trick is allowed because $\mathrm{nD}(kg) = k\mathrm{nD}(g)$ for scalar k as nD is, in principle, a linear transformation.

This is the same $K = c^2/c_i^2$ that we discussed in detail in WaveEquation.pdf. Its constraint conditions apply here as well.