Discretizing and Numerically Solving the Wave Equation: documentation for WaveEquationOptimized.py, PhotonGravityWavePerpendicularCase.py, and PhotonGravityWave4D.py

Varadarajan Srinivasan

For a function $f(x)$ whose discretization is represented as a sequence f_n, we define its numerical derivative with respect to x, $\mathrm{nD}_x(f_n)$, as the average of the forward difference and backward difference.

$$\mathrm{nD}_x(f_n) = \frac{1}{2}\left(\frac{f_n - f_{n-1}}{\Delta x} + \frac{f_{n+1} - f_n}{\Delta x}\right) = \frac{f_{n+1} - f_{n-1}}{2\Delta x} \tag{1}$$

For differentiable functions, this approximates the true derivative of the function to an arbitrarily high accuracy as the step size $\Delta x \to 0$. As explained in DiscretizedLaplaceEquation.pdf, this logically produces the one-dimensional numerical second derivative $\mathrm{nD}^2_x(f_n)$ as well as the numerical Laplacian $\mathrm{nD}^2_{x,y}(f_{m,n})$ for our evenly spaced grid ($\Delta x = \Delta y \equiv \Delta$).

$$\mathrm{nD}^2_x(f_n) = \mathrm{nD}(\mathrm{nD}(f_n)) = \frac{f_{n+1} - 2f_n + f_{n-1}}{\Delta^2} \tag{2}$$

$$\mathrm{nD}^2_{x,y}(f_{m,n}) = \frac{f_{m,n+1} + f_{m,n-1} + f_{m+1,n} + f_{m-1,n} - 4f_{m,n}}{\Delta^2} \tag{3}$$

The Wave Equation for displacement f,

$$\frac{\partial^2 f}{\partial t^2} = c^2 \nabla^2 f, \tag{4}$$

can be numerically approximated using our discretizations as

$$\mathrm{nD}^2_t(f_{m,n,t}) = c^2 \mathrm{nD}^2_{x,y}(f_{m,n,t}). \tag{5}$$

Substituting Eq. 2 and Eq. 3 into Eq. 5 gives us

$$\frac{f_{m,n,t+1} - 2f_{m,n,t} + f_{m,n,t-1}}{(\Delta t)^2} = \frac{c^2}{\Delta^2}(f_{m,n+1,t} + f_{m,n-1,t} + f_{m+1,n,t} + f_{m-1,n,t} - 4f_{m,n,t}) \tag{6}$$

Rearranging and subtracting one time step from every term gives us the desired iterative formula that solves the wave equation for given boundary conditions

$$f_{m,n,t} = K(f_{m,n+1,t-1} + f_{m,n-1,t-1} + f_{m+1,n,t-1} + f_{m-1,n,t-1} - 4f_{m,n,t-1}) + 2f_{m,n,t-1} - f_{m,n,t-2} \tag{7}$$

where

$$K = c^2 \frac{(\Delta t)^2}{\Delta^2} \tag{8}$$