Computing the Influence of a Gravitational Wave on an Electromagnetic Field

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Introduction

Maxwell's Equations fully describe electromagnetism in flat spacetime. Their resulting wave equation dictates the behavior of the electromagnetic four-potential. I explore how this is influenced when taking general relativistic effects into consideration. In particular, I compute new solutions for the potential as influenced by a passing gravitational wave. I present a numerical model, whose inputs include arbitrarily adjustable boundary conditions, which computes and animates solutions of a novel generalization to curved spacetime of the electromagnetic wave equation. The model is useful because analytical solutions to this problem only exist for certain boundary conditions and perturbative solutions only approximate successfully with when certain parameters are constrained. I present a model that can find numerical solutions in general.

Flat Spacetime Computational Model

$$\nabla^2 u = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \quad (1) \qquad \mathrm{nD}_x(f_n) = \frac{f_{n+1} - f_{n-1}}{2\Delta x} \quad (2)$$

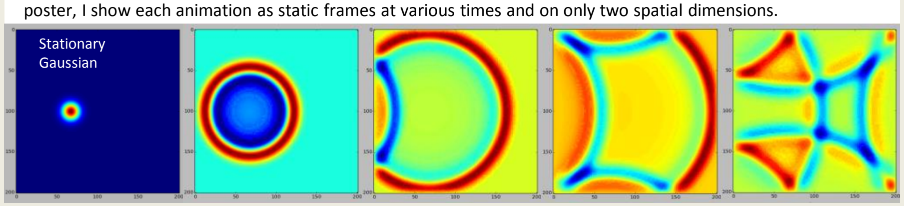
The standard wave equation in Minkowski spacetime is given by Eq.(1). In order to build a numerical model, I must discretize everything. I model continuous spacetime as a (3+1)-dimensional grid, with spatial step sizes $\Delta x = \Delta y = \Delta z = \Delta$ and temporal step size Δt. Defining the function f as the displacement u restricted to this discrete domain and defining the numerical partial derivative as Eq.(2), the wave equation can be numerically approximated to an arbitrarily high degree of accuracy for smooth functions. This yields a recursively-defined function that generates a solution for any initial state with given boundary conditions.

$$f_{m,n,t} = K(f_{m,n+1,t-1} + f_{m,n-1,t-1} + f_{m+1,n,t-1} + f_{m-1,n,t-1} - 4f_{m,n,t-1}) + 2f_{m,n,t-1} - f_{m,n,t-2}$$

For brevity, this is just the (2+1)-dimensional form of the equation. The accumulated parameter K will be discussed below.

$$K \equiv c^2\frac{(\Delta t)^2}{\Delta^2}$$

It represents the evolution (in space and time) of a component of the four-potential and can also be interpreted as a component of the electric field. I formulate this model in Python to solve the wave equation at every point in a finite region of discretized spacetime. When run, the program shows an animation of the solution. On this poster, I show each animation as static frames at various times and on only two spatial dimensions.



Numerical Stability

The parameter K that appeared in the discretized equation depends on the speed of the electromagnetic wave and the step sizes of the numerical model. Essentially, it can be thought of as the square of the ratio between the wave speed and the information propagation speed $\Delta/\Delta t$ of the model. This numerical approximation can only successfully approach the true solution if the information propagation speed is faster than the wave speed. Therefore, in order to maintain stability, $K \leq 1$. This constraint arises from the Courant–Friedrichs–Lewy condition. The lower we set K, the better our time-resolution, but the smaller the solvable time interval in a given runtime.

Switching Gravity On

Now that the computational backbone is in place, it is time to advance the physics. A gravitational wave propagating along, say, the *z* direction sinusoidally stretches and contracts space in the *x* direction while doing the same, but in antiphase, along the *y* direction. As all appreciable sources of gravitational waves (such as, ideally, very closely orbiting compact objects) are extremely far away, I will treat them as plane waves. With this, we have a new form of the wave equation in terms of $\mathbf{g}_S$, the space component of the metric tensor.

$$(\partial_x \quad \partial_y \quad \partial_z)\,\mathbf{g}_S \begin{pmatrix}\partial_x \\ \partial_y \\ \partial_z\end{pmatrix} u = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}, \qquad \mathbf{g}_S = \begin{pmatrix}1+f & 0 & 0 \\ 0 & 1-f & 0 \\ 0 & 0 & 1\end{pmatrix}, \qquad f = \epsilon\cos(kz - kct)$$

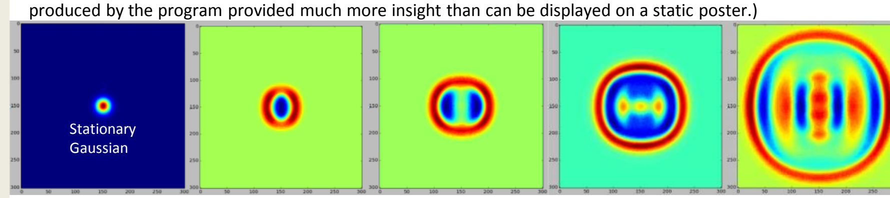
Notice that this assumes that the gravitational wave comes in from the *z* direction. When generalized for any angle, this equation, in the fixed grid point coordinates *<t,X,Y,Z>*, becomes the following equation, where the tensor **T** is the Euler angles rotation conjugate of $\mathbf{g}_S$ and *f(t,z)* is similarly transformed to *<t,X,Y,Z>*.

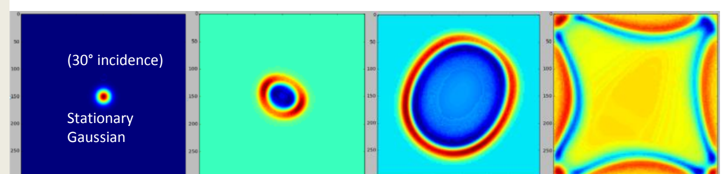
$$\boxed{\nabla^T \mathbf{T} \nabla u = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \quad \text{where } f = f(t, X, Y, Z)}$$

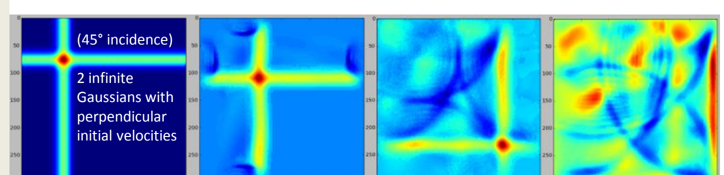
To create the numerical model that answers our original question, all that remains is to discretize this. The resulting final (3+1)-dimensional equation is too long to show here, but it is achieved with the same discretization tools used in the flat spacetime model.

Computational Results

What has so far been achieved was itself a major purpose of this project: to build a program whose every input parameter (gravitational wave amplitude and period, initial conditions, etc.) can be adjusted and outputs an animated solution of the displacement of a component of the electric field in the presence of a gravitational wave. We now come to the nominal primary goal – the actual results of my numerical model. (The animations produced by the program provided much more insight than can be displayed on a static poster.)







Even in this limited medium, we see interesting effects such as self-interference and asymmetrical evolution. The last example is an interferometer setup like LIGO, but crucially is now no longer restricted to simple boundaries.