Matrix Solutions

Related code in file `matrixSolution.py`.

Given an equation $U = V$ with U and V having a differing number of constants, say $abx = xbx$, we can convert it into a vector, $(a, b, x)^T = (x, b, x)^T$. This equation has the solution $x = a$, and a matrix can be used that represents this equation and its solution. We need a matrix A such that $Au = Av$, where u and v are the vectors representing the variables in the equation.

The above example has matrix

$$A = \begin{bmatrix} a & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & a \end{bmatrix}$$

defined under the multiplication operation:

- $c * c = c$
- $c * x = c$
- $c * 1 = c$

where $c \in \{a, b\}$. We must have that the matrix is diagonal, and cannot contain both a and b.

One equation can also have many matrices. For example, the above equation could have matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & a \end{bmatrix}$$

To mitigate this, we require that the $(i, i)^{\text{th}}$ entry in the matrix is 1 if and only if the i^{th} letters in each side of the equation are both constants.

Note that one matrix can correspond to multiple equations. For example, the above matrix can also be used to solve $xba = aba$.

Since one matrix can correspond to multiple equations, we can construct an equivalence relation between equations. Define $(U, V) \sim (W, X)$ if and only if there exists a matrix A such that $Au = Av$ and $Aw = Ax$. This partitions the set of all equations of the same length n, S_n, into equivalence classes.

The question is now the following. Given a matrix A, can we recover *all* (or at least some) of the equations that correspond to it? For example, for the above matrix

$$A = \begin{bmatrix} a & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & a \end{bmatrix}$$

defined under the multiplication operation, we have that the first element must be either a or x (since $a * b$ is not defined), but not both a.

- $a ** = x **$
- $x ** = x **$

The second element must be either a or b (since it must be a constant).

- $* a * = * a *$
- $* b * = * b *$

The third element must be either a or x by above.

- $** a = ** x$
- $** x = ** a$
- $** x = ** x$

This leads to 2 choices for first letter, 2 for second, and 3 for third, so 12 total equations.

- $aaa = xax$
- $aax = xaa$**
- $aax = xax$
- $aba = xbx$
- $abx = xba$**
- $abx = xbx$
- $xaa = xax$
- $xax = xaa$*
- $xax = xax$*
- $xba = xbx$
- $xbx = xba$*
- $xbx = xbx$*

After redundancies removed (*), we have 8 equations. After removing equations with both sides having the same number of constants (**), we have 6 equations.

Hence, the matrix A corresponds to these 6 equations.

- $aaa = xax$
- $aax = xax$
- $aba = xbx$
- $abx = xbx$
- $xaa = xax$

- $xba = xbx$

Note that we can represent a matrix as a tuple with the diagonal elements, to make things easier to express. e.g. the above matrix A would be represented as the tuple $(a, 1, a)$.

Future thoughts

Notice that the more "rules" we add on the matrix / tuple, the more information we encode, but the more the tuple just represents the equation itself. For example, adding another symbol for the i^{th} elements both being x will further disambiguate equations and give more information in the matrix.

We want to strike a careful balance between encoding too little and encoding too much, since we don't want to just end up essentially having a unique tuple for each equation.

α, β symbols

Similar to the a and 1 symbol, we introduce a new symbol that can be in the matrix - α and β. An α is in the i^{th} position in the matrix if and only if the i^{th} letters are both x, and the solution is $x = a$. This is symmetric with β, except if the solution is $x = b$.

Counting

We have the following formula to count all possible equations for a given value of n. Fix a matrix with some k number of a elements. This gives

$$\sum_{i=0}^{\lceil \frac{k}{2} \rceil - 1} \binom{k}{i}$$

possible equations that are contributed by the a elements. The remaining values are either 1 or α. α contributes one equation, 1 contributes two. To count the number of equations contributed, we have

$$\sum_{j=0}^{n-k} 2^j \binom{n-k}{j}.$$

This multiplied by two to account for a or b, and then multiplied by $\binom{n}{k}$ gives all possible equations for a fixed k. Thus, the number of all possible equations for a fixed n is

$$\sum_{k=0}^{n} \left(2\binom{n}{k} \times \sum_{i=0}^{\lceil \frac{k}{2} \rceil - 1} \binom{k}{i} \times \sum_{j=0}^{n-k} 2^j \binom{n-k}{j} \right).$$

Note that this only generates equations which lead to equations with differing number of constants on either side.