Matrix Solutions

Given an equation $U = V$ with U and V having a differing number of constants, say $abx = xbx$, we can convert it into a vector, $(a, b, x)^T = (x, b, x)^T$. This equation has the solution $x = a$, and a matrix can be used that represents this equation and its solution. We need a matrix A such that $Au = Av$, where u and v are the vectors representing the variables in the equation.

The above example has matrix

$$A = \begin{bmatrix} a & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & a \end{bmatrix}$$

defined under the multiplication operation:

- $c * c = c$
- $c * x = c$
- $c * 1 = c$

where $c \in \{a, b\}$. We must have that the matrix is diagonal, and cannot contain both a and b.

One equation can also have many matrices. For example, the above equation could have matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & a \end{bmatrix}$$

To mitigate this, we require that the $(i, i)^{\text{th}}$ entry in the matrix is 1 if and only if the i^{th} letters in each side of the equation are both constants.

Note that one matrix can correspond to multiple equations. For example, the above matrix can also be used to solve $xba = aba$.

Since one matrix can correspond to multiple equations, we can construct an equivalence relation between equations. Define $(U, V) \sim (W, X)$ if and only if there exists a matrix A such that $Au = Av$ and $Aw = Ax$. This partitions the set of all equations of the same length n, S_n, into equivalence classes.

The question is now the following. Given a matrix A, can we recover *all* (or at least some) of the equations that correspond to it? For example, for the above matrix

$$A = \begin{bmatrix} a & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & a \end{bmatrix}$$

defined under the multiplication operation, we have that the first element must be either a or x (since $a * b$ is not defined), but not both a.

- $a * * = x * *$

- $x * * = x * *$

The second element must be either a or b (since it must be a constant).

- $* \, a * = * \, a *$
- $* \, b * = * \, b *$

The third element must be either a or x by above.

- $* * a = * * x$
- $* * x = * * a$
- $* * x = * * x$

This leads to 2 choices for first letter, 2 for second, and 3 for third, so 12 total equations.

- $aaa = xax$
- $aax = xaa$**
- $aax = xax$
- $aba = xbx$
- $abx = xba$**
- $abx = xbx$
- $xaa = xax$
- $xax = xaa$*
- $xax = xax$*
- $xba = xbx$
- $xbx = xba$*
- $xbx = xbx$*

After redundancies removed (*), we have 8 equations. After removing equations with both sides having the same number of constants (**), we have 6 equations.

Hence, the matrix A corresponds to these 6 equations.

- $aaa = xax$
- $aax = xax$
- $aba = xbx$
- $abx = xbx$
- $xaa = xax$
- $xba = xbx$