Akshay Kumar

1.1 a) $X = L^2(0,1)$ $Y = L^2(0,1)$

$\bar{m} = m + \alpha \sin(kx)$

$\bar{g} = A\bar{m} = g - \frac{\alpha}{k} + \frac{\alpha}{k}\cos kx$

$\|\bar{g}-g\|_{L^2} = \sqrt{\int_0^1 \left|\frac{\alpha}{k} - \frac{\alpha}{k}\cos kx\right|^2 dx} = \frac{\alpha}{k}\sqrt{\int_0^1 [1+\cos^2 kx - 2\cos kx]\,dx}$

$= \cancel{\frac{\alpha}{k}\sqrt{\frac{3x}{2} + \sin 2kx}}$

$= \frac{\alpha}{k}\sqrt{\int_0^1 1\,dx + \int_0^1 \cos^2 kx\,dx - \int_0^1 2\cos kx\,dx}$

$= \frac{\alpha}{k}\sqrt{1 + \frac{1}{2} + \frac{\sin 2k}{4k} - \frac{2\sin k}{k}}$
$\to 0$ $\to 0$ $\to 0$

as $k \to \infty$ $\|\bar{g}-g\|_{L^2} = 0$

$\|\bar{m}-m\|_{L^2} = \sqrt{\int_0^1 (\alpha \sin kx)^2 dx}$

$= \alpha\sqrt{\int_0^1 \sin^2 kx\,dx} = \alpha\sqrt{\int_0^1 \frac{1-\cos 2kx}{2}\,dx}$

$= \alpha\sqrt{\frac{1}{2} - \int_0^1 \frac{\cos 2kx}{2}\,dx} = \alpha\sqrt{\frac{1}{2} - \frac{\sin 2kx}{4k}\Big|_0^1}$

$= \alpha\sqrt{\frac{1}{2} - \frac{\sin 2k}{4k}}$

as $K \to \infty$ $\quad \|\bar{m} - \bar{m}\|_{L^2} = \alpha\sqrt{\frac{1}{2} - 0} = \frac{\alpha}{\sqrt{2}}$

$\therefore$ Not stable, no-unique solution, ill-posed as small change m does not lead to any change in g.

b) $X = L^2[0,1] \quad Y = H_0^1(0,1)$

$$\|\bar{g} - g\|_{H_0^1} = \sqrt{\frac{\alpha^2}{K^2}\left(\int_0^1 (1-\cos Kx)^2 dx + \int_0^1 \left(\frac{\alpha}{K} K\sin Kx\right)^2 dx\right.}$$

$$= \sqrt{\frac{\alpha^2}{K^2}\left(\frac{3}{2} + \frac{\sin 2K}{4K} - \frac{2\sin K}{K}\right) + \alpha^2\int_0^1\left(\frac{1-\cos 2Kx}{2}\right)dx}$$

$$= \sqrt{\frac{\alpha^2}{K^2}\left(\frac{3}{2} + \frac{\sin 2K}{4K} - \frac{2\sin K}{K}\right) + \alpha^2\left(\int_0^1 \frac{1}{2} - \int_0^1 \frac{\cos 2Kx}{2}\,dx\right)}$$

$$= \sqrt{\frac{\alpha^2}{K^2}\left(\frac{3}{2} + \frac{\sin 2K}{4K} - \frac{2\sin K}{K}\right) + \alpha^2\left(\frac{1}{2} - \frac{\sin 2K}{4K}\right)}$$

as $K \to \infty$ $\quad \|\bar{g} - g\|_{H_0^1} = \sqrt{0 + \frac{\alpha^2}{2}} = \frac{\alpha}{\sqrt{2}}$

$$\|\bar{m} - m\|_{L^2} = \frac{\alpha}{\sqrt{2}}$$ [From Previous prob]

$\therefore$ Stable, unique, well posed as small change in m leads to small change in g.

1.2. To prove $\{\phi_i\}$ is orthonormal.

A. We have $\{\varphi_i\}$ orthonormal and

$$A^* A \varphi_i = \mu_i^2 \varphi_i \quad \text{and} \quad \mu_i \phi_i = A\varphi_i$$

$$\Rightarrow \phi_i = \frac{A\varphi_i}{\mu_i}$$

$$(\phi_i, \phi_j) = \left(\frac{A\varphi_i}{\mu_i}, \frac{A\varphi_j}{\mu_j}\right)$$

$$= \frac{1}{\mu_i \mu_j} (A\varphi_i, A\varphi_j)$$

$$= \frac{1}{\mu_i \mu_j} (\varphi_i, A^* A \varphi_j)$$

$$= \frac{1}{\mu_i \mu_j} (\varphi_i, \mu_j^2 \varphi_j) = \frac{\mu_j}{\mu_i} \xcancel{(\varphi_i, \varphi_j)}^{0}$$ as they are orthonormal

$\therefore \{\phi_i\}$ is orthonormal set.

1.3. To show $N(A) = N(A^*A)$

A To show if $x \in N(A)$ then $x \in N(A^*A)$ and otherwise.

Let $x \in N(A)$ then

$$Ax = 0$$

$$A^*(Ax) = A^*0$$

$$\Rightarrow A^*Ax = 0 \Rightarrow x \in N(A^*A)$$

Let $y \in N(A^*A)$ then

$$A^*Ay = 0$$

$$\Rightarrow A^*(\underbrace{Ay}_{z}) = 0 \Rightarrow A^*z = 0$$

$$z \in N(A^*) = R(A^\perp)$$

Since $z \in R(A^\perp)$ and $z \in R(A)$ as $Ay = z$

z has to be 0. $\Rightarrow$

$\therefore y \in N(A)$ //

1.5 To show $(A^* A + I)$ is Injective.

A. For a function to be injective.

$$(A^* A + I)x = (A^* A + I)y \Rightarrow x = y$$

so if $(A^* A + I)x = (A^* A + I)y$

$$(A^* A + I)(x - y) = 0$$

Case I $\quad x = y \quad$ trivial

II $\quad x \neq y \quad$ then $\quad A^* A + I = 0$

$$A^* A = -I$$

$$\Rightarrow A^* A = -A^{-1} A$$

$$\Rightarrow A^* = -A^{-1}$$ which is not true always.

so $x = y$ is the only solution.

Hence it is injective.

1.4 We find that lower $\beta$ gives better results even with Noise. The reason is lower $\beta$ means less loss of information as range is not being compressed much, i.e less smoothing. The reason is because it offers better stability.

1.4 When $\beta$ is small both with noise and without noise gave same results. But in other case, noise gave bad results. The reason small $\beta$ gave similar results without noise is because lower the beta higher the stability.

The shape of the estimated function also looks good when $\beta$ is small ($10^{-10}$). But the scale is not good in all the case. (Maybe there is a bug in my code)

1.6

$$\min_{m} \frac{1}{2} \| Am - g \|^2_{\lambda} + \frac{K}{2} \| \nabla m \|^2_{L^2(0,1)}$$

$$\min \| Am - g \|^2 + \| \sqrt{K} \nabla m \|^2$$

$$\Rightarrow \quad m = (A^T A + \Gamma^T \Gamma)^{-1} A^T b$$

$$\Gamma = \sqrt{K} \begin{pmatrix} 1 & -1 & 0 & 0 & \cdots \\ 0 & 1 & -1 & 0 & \\ 0 & 0 & 1 & -1 & \\ 0 & & & & \ddots \end{pmatrix}$$

Noise always gives bad result. But here $\beta = 0.2$ gave better result compared to that in 1.4

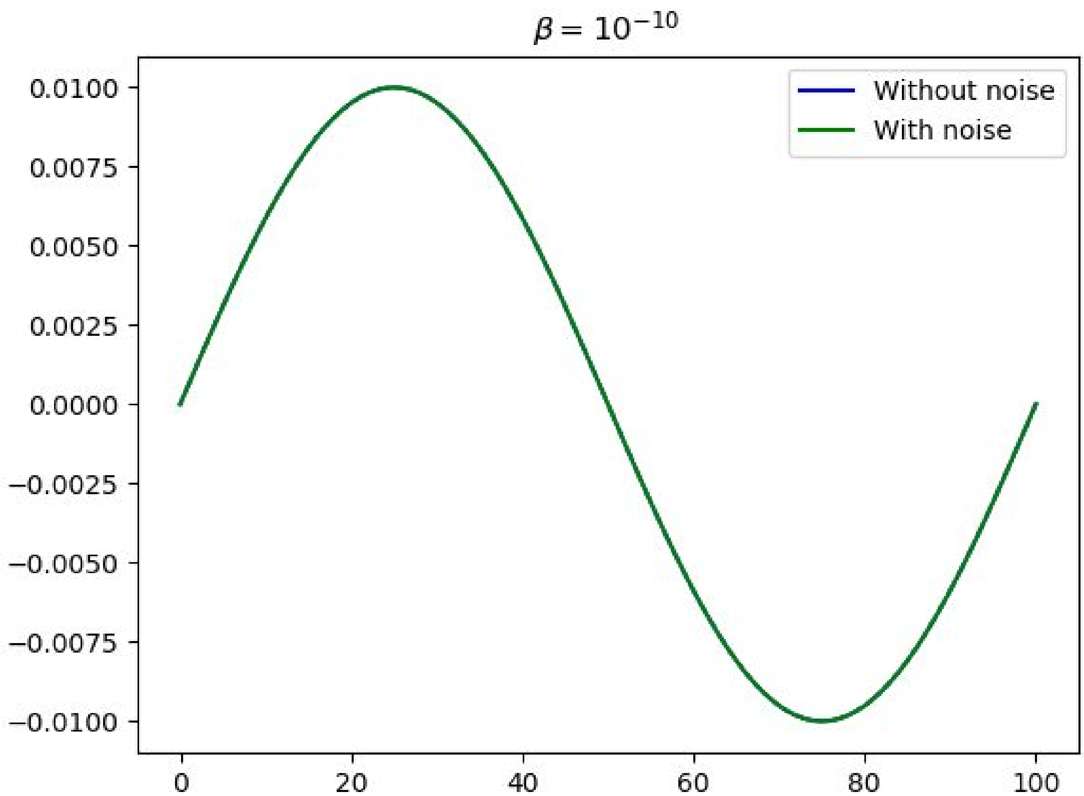
K is plotted with misfit where
misfit = $\|Am - g\|_2^2$

We find that as K increase misfit ~~decreases~~ increases as we give more importance to $\|\nabla m\|^2$ term.

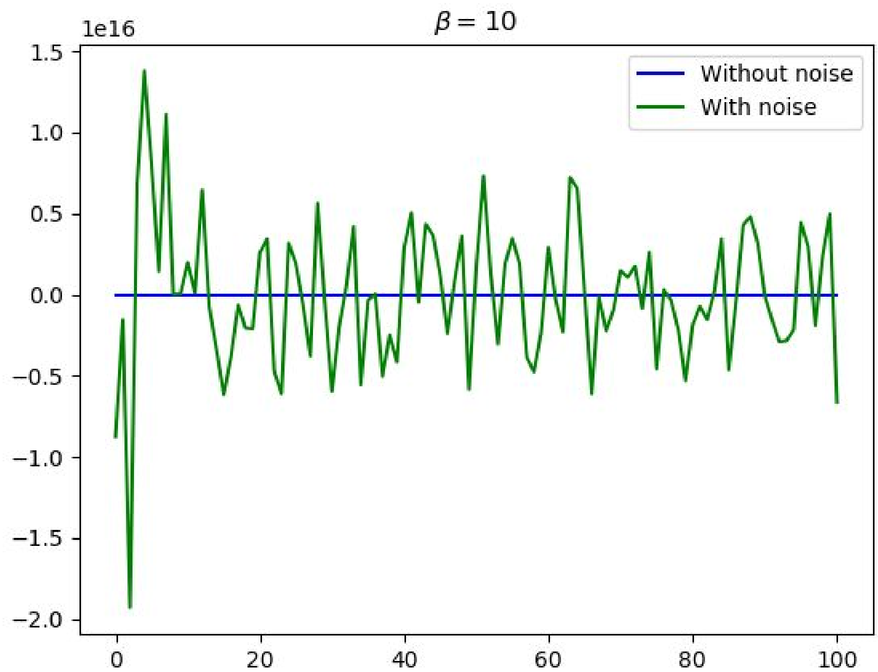
c) We find Tikhonov gives better result interms of

Value closer to actual solution

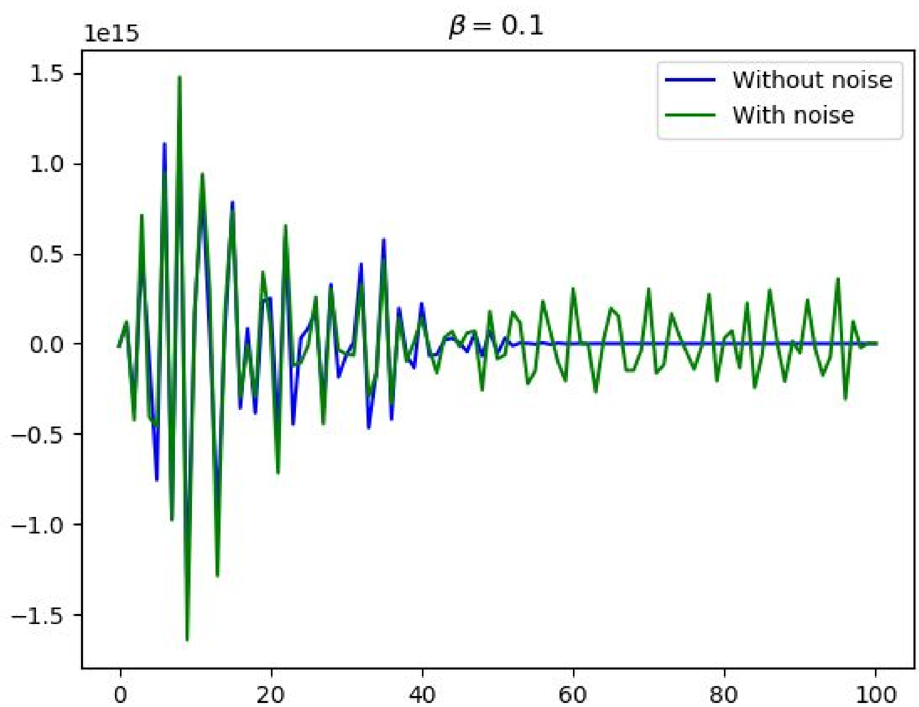
compared to Conjugate gradient method. But Both of them were not close to actual results. (May be there is a bug in my code)
The plots don't look good.



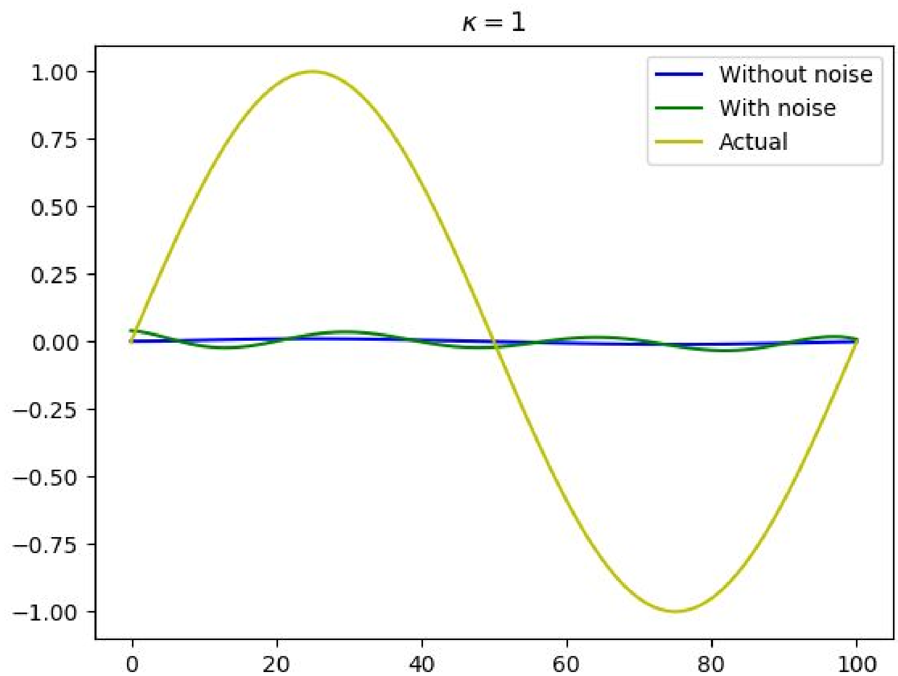

As we can see as the beta value decreases, we get better results.

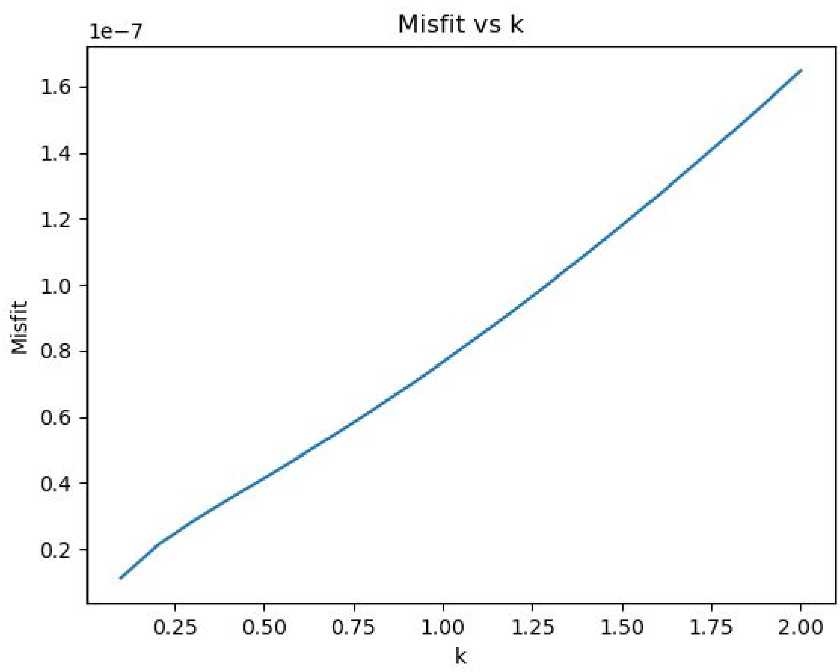


$\beta = 10$
1e16
Without noise
With noise
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
−2.0
0
20
40
60
80
100

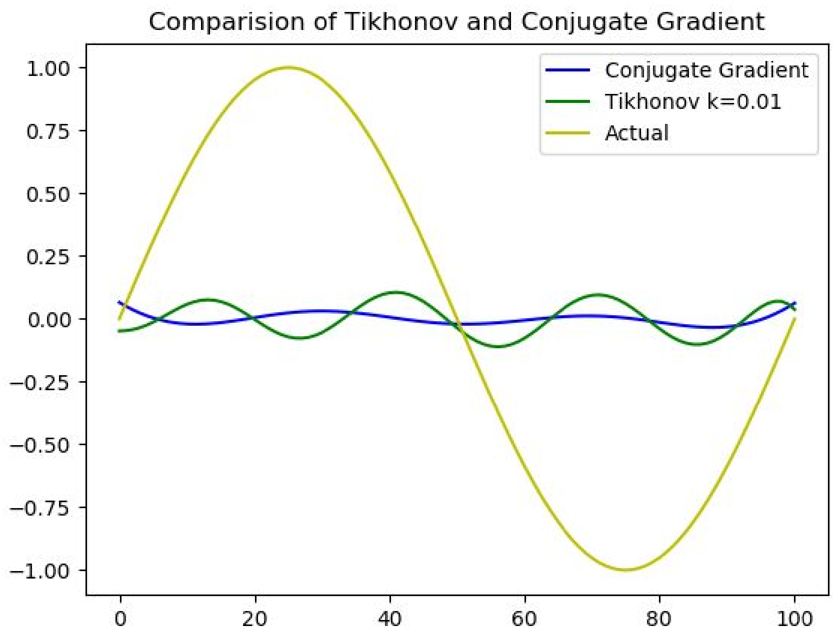



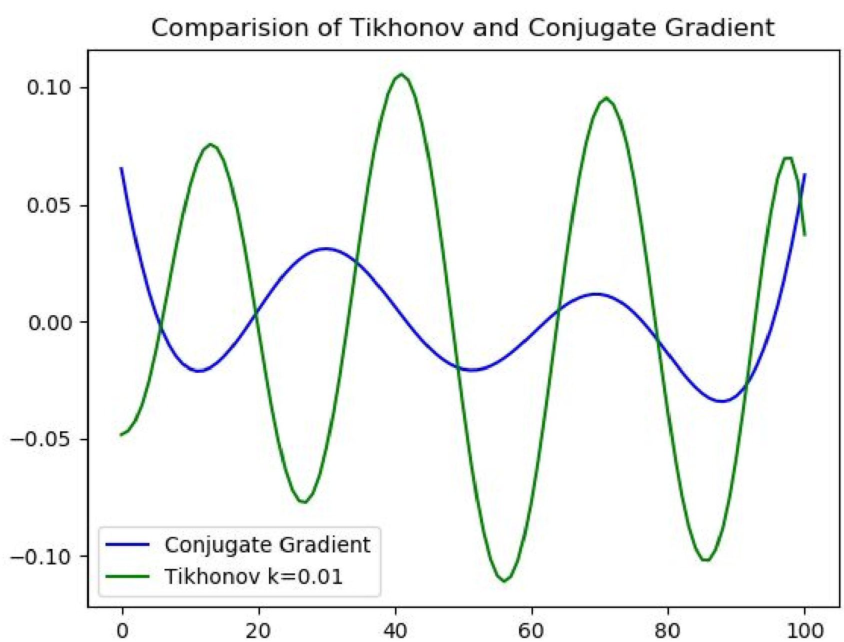
$\beta = 0.1$
1e15
Without noise
With noise
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
0
20
40
60
80
100




As k increase Misfit also increase so k=0.01 was taken as best k value.




As we can see relatively Tikhonov gave better results, though both of them were bad with respect to actual solution.