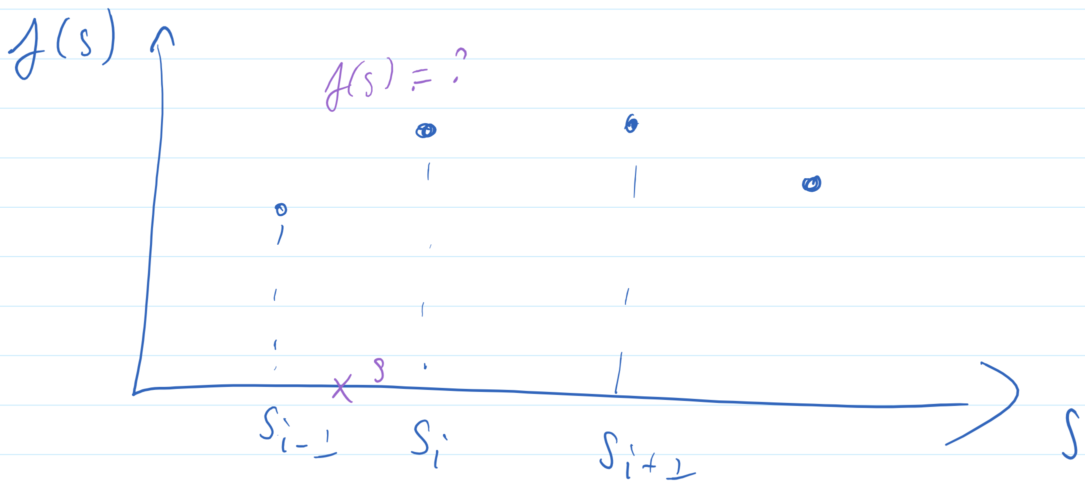
PRIOR ELICITATION (VERY SUBJECTIVE)

I. Smooth priors:



given $[f(s_0), \dots, f(s_n)]$ + "smooth" $f(s)$

How do we estimate $f(s)$ at any points??

- if $f(s)$ is smooth then $f(s)$ should be close to $f(s_{i-1})$ and $f(s_i)$. A simple way to convey this is, if set $m(s) = f(s)$,

$$(*) \quad \boxed{m_i = \frac{m_{i-1} + m_{i+1}}{2} + w_i}$$

$i = 1, \dots, n-1$

w_i ↓ innovative term due to the fact that we are not sure about relation between

assume:

$w_i \sim N(0, \gamma^2)$ ↓ known.

identically independently

identically independently distributed (i.i.d.) known.

relation between m_i, m_{i-1}, m_{i+1}

then

$\vec{m} = [m_0, \dots, m_n]^T$ is a random vector

$\boxed{\text{what is the distribution of } \vec{m}\ ??} = \pi_{prior}(\vec{m})$

- From (*) we have:

$$\underbrace{\frac{1}{2}\begin{bmatrix} -1 & 2 & -1 & 0 & \cdots & & \\ 0 & -1 & 2 & -1 & 0 & \cdots & \\ & & & \ddots & \ddots & & \\ & & \cdots & 0 & -1 & 2 & -1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} m_0 \\ \vdots \\ m_n \end{bmatrix}}_{\vec{m}} = \underbrace{\begin{bmatrix} w_1 \\ \vdots \\ w_{n-1} \end{bmatrix}}_{\vec{w}} \sim \begin{matrix} N(0,\gamma^2) \times \\ N(0,\gamma^2) \times \cdots \\ \vdots \\ N(0,\gamma^2) \end{matrix}$$

$$\parallel ??$$

$$= N(\vec{0}, \gamma^2 I)$$

$$\parallel ??$$

$$\propto \exp\left(-\frac{1}{2\gamma^2} \vec{w}^T \vec{w}\right)$$

$$\Updownarrow$$

$$\underset{\in \mathbb{R}^{(n+1)\times(n-1)}}{A} \vec{m} = \vec{w} \sim N(\vec{0}, \gamma^2 I)$$

$$\Big\Downarrow \quad \text{assume "} A^{-1} \text{ exists"} \quad (\text{fixed later})$$

$$\vec{m} = A^{-1} \vec{w}$$

??

$$m = A \omega$$

$$\Rightarrow \quad \pi_{prior}(\vec{m}) \overset{??}{\sim} \pi_{\vec{\omega}}(A\vec{m})$$

$$\propto \exp\left(-\frac{1}{2\gamma^2} \vec{\omega}^T \vec{\omega}\right)$$

$$\propto \exp\left(-\frac{1}{2\gamma^2} \vec{m}^T A^T A \vec{m}\right)$$

"Rigorously":

$$\int_B \pi_{prior}(\vec{m})\, dm := \int_B \pi(\vec{m})\, d\vec{m}$$

$\Big\|$ def of probability

$$\int_{A^{-1}(B)} \pi_{\vec{\omega}}(\vec{\omega})\, d\vec{\omega}$$

$\Big\|$ $d\vec{\omega} =$ "$\det(A)$" $d\vec{m}$

$$\int_B \pi_{prior}(\vec{m})\, d\vec{m} = \int_B \pi_{\vec{\omega}}(A\vec{m}) \text{ "}\det(A)\text{" } d\vec{m}$$

$\Big\Downarrow$ $\forall B$, "$\det(A)$" $=$ const

$$\pi_{prior}(\vec{m}) \propto \pi(A\vec{m})$$

$$\pi_{prior}(\vec{m}) \propto \pi_{\vec{w}}(A\vec{m})$$

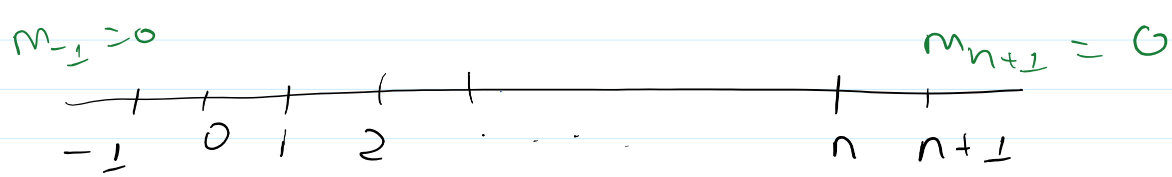
$$\propto \exp\left(-\frac{1}{2\gamma^2}\,\vec{m}^T A^T A\,\vec{m}\right)$$

- <u>Remark</u>: A is not square and this is a direct consequence of the fact that we haven't specified the smoothness at the boundary!

DIRICHLET BOUNDARY CONDITIONS

- Let's say that we believe that the function f is "close" to zero at the boundary. Then one way to convey this belief is to extend by zero:

$$m_{-1} = 0\ , \quad m_{n+1} = 0$$



$$m_i := f(x_i)$$

Thus we can use the same expression:

$$m_0 = \frac{m_1 + \overset{=0}{m_{-1}}}{2} + w_0 \quad \Big| \quad w_0 \sim N(0,\gamma^2)$$

$$m_n = \frac{m_{n-1} + \overset{=0}{m_{n+1}}}{2} + w_n \quad \Big| \; w_n \sim N(0, \gamma^2)$$

Hence:

$$\underbrace{\frac{1}{2}\begin{bmatrix} 2 & -1 & 0 & \cdots & & \\ -1 & 2 & -1 & 0 & \cdots & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & \ddots & \ddots \end{bmatrix}}_{A_D} \underbrace{\begin{bmatrix} m_0 \\ \vdots \\ m_n \end{bmatrix}}_{\vec{m}} = \underbrace{\begin{bmatrix} w_0 \\ \vdots \\ w_n \end{bmatrix}}_{\vec{w}} \sim N(\vec{0}, \gamma^2 I)$$

$$A_D \times \vec{m} = \vec{w}$$

$\Downarrow$ same arguments

side note:

$$\vec{x} = \text{randn}(3) \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad x_i \sim N(0, 1)$$

?

$$? \qquad \begin{pmatrix} x_2 \\ x_3 \end{pmatrix}$$

$$N(\vec{0}, I^{3\times 3})$$

$$x \sim \exp\left(-\frac{1}{2\sigma^2} x^2\right) \text{``}=\text{''} N(0, \sigma^2)$$

$$\vec{x} \sim \exp\left(-\frac{1}{2\sigma^2} \vec{x}^T \vec{x}\right) \text{``}=\text{''} N(\vec{0}, \sigma^2 I)$$

$$\vec{x} \sim \exp\left(-\frac{1}{2\sigma^2} \vec{x}^T L \vec{x}\right) = N(\vec{0}, \sigma L^{-1})$$

For our case

$$\vec{m} \sim \exp\left(-\frac{1}{2\gamma^2} \vec{m}^T A_D^T A_D \vec{m}\right)$$

$$\text{``}=\text{''}$$

$$N\left(\vec{0}, \gamma^2 (A_D^T A_D)^{-1}\right)$$

whittening · process

$$\text{let} \quad \vec{y} = \gamma A_D^{-1} \vec{x} \quad \text{where } \vec{x} \sim N(0, I)$$

linearity of $\mathbb{E}$

$$* \quad \mathbb{E}[\vec{y}] \stackrel{\downarrow}{=} A_D^{-1} \mathbb{E}[\vec{x}] = 0$$

$$* \quad \mathbb{V}[\vec{y}] = \mathbb{E}\left[(\vec{y} - \mathbb{E}[\vec{y}])(\vec{y} - \mathbb{E}[\vec{y}])^T\right]$$

$$= \mathbb{E}[\vec{y}\vec{y}^T] = \gamma^2 \mathbb{E}\left[A_D^{-1} \vec{x} \vec{x}^T A_D^{-T}\right]$$

$$= \mathbb{E}[\vec{y}\,\vec{y}'] = \gamma^2 \mathbb{E}[A_D^{-1}\,\vec{x}\,\vec{x}'\,A_D^{-1'}]$$

|| linearity of $\mathbb{E}$ =

$$\gamma^2 A_D^{-1}\,\mathbb{E}[\vec{x}\,\vec{x}^T]\,A_D^{-T}$$

|| $\mathbb{E}[\vec{x}\,\vec{x}^T] = \mathbb{V}[\vec{x}] = I$

$$\gamma^2 A_D^{-1} A_D^{-T} = \gamma^2 (A_D^T A_D)^{-1}$$

Homework:

1)

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \sim N\left(\begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix}, \begin{bmatrix} a_1 & a_2 \\ a_2 & a_3 \end{bmatrix}\right)$$

a) what is the marginal distribution of x_1

b) // // // x_2

2)

$$\vec{x} = \begin{bmatrix} \vec{x}_1 \\ \vec{x}_2 \end{bmatrix} \sim N\left(\begin{bmatrix} \bar{\vec{x}}_1 \\ \bar{\vec{x}}_2 \end{bmatrix}, \begin{bmatrix} A_1 & A_2 \\ A_2^T & A_3 \end{bmatrix}\right)$$

a) // // // $\vec{x}_1$?

$\underline{x_1}$?

b) " " "

$\vec{x}_2$?

Note. $x \sim \pi(x) \Rightarrow \int_X \pi(x)\,dx = \underline{1}$

NON-ZERO DIRICHLET

- In this case, one way to express nonzero Dirichlet boundary condition is to say

$$m_0 \sim N\left(0, \frac{\gamma^2}{\delta_0^2}\right)$$

$$m_1 = \frac{m_0 + m_2}{2} + N(0, \gamma^2)$$

$$\vdots$$

$$m_{n-1} = \frac{m_{n-2} + m_n}{2} + N(0, \gamma^2)$$

$$m_n \sim N\left(0, \frac{\gamma^2}{\delta_n^2}\right)$$

$$\Downarrow \quad \delta_0 m_0 = w_0 \sim N(0, \gamma^2)$$
$$\delta_n m_n = w_n \sim N(0, \gamma^2)$$

$$\begin{bmatrix} 2\delta_0 & 0 & \cdots \\ -1 & 2 & -1 \;\; 0 \cdots \end{bmatrix} \begin{bmatrix} \\ \\ \end{bmatrix} \begin{bmatrix} w_0 \\ \vdots \end{bmatrix}$$

$$\frac{1}{\gamma}\begin{bmatrix} -1 & 2 & -1 & 0 & \cdots \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \ddots \\ & & \cdots & 0 & 2\delta_n \end{bmatrix}\begin{bmatrix} \vdots \\ \vdots \\ m_n \end{bmatrix} = \begin{bmatrix} \vdots \\ \vdots \\ w_n \end{bmatrix} \sim N(0, \gamma^2 I)$$

$$\underbrace{A_L}\ \times\ \underbrace{\vec{m}} = \vec{w}$$

$$\Big\Downarrow \text{ Same process}$$

$$\boxed{\pi(\vec{m}) \propto \exp\left(-\frac{1}{2\gamma^2}\vec{m}^T A_L^T A_L \vec{m}\right)}$$

Remark. given γ, it is not sufficient to determine $\pi(\vec{m})$ because $\delta_0, \delta_n = ??$ In order to have a more uniform uncertainty we can make δ_0, δ_n = uncertainty of the midpoint

So if we define $L = (A_L^T A_L)^{-1}$, then this means

$$\boxed{\delta_0^2 = \delta_n^2 = L\left(\lfloor \tfrac{n}{2} \rfloor, \lfloor \tfrac{n}{2} \rfloor\right)}$$

This is a nonlinear eqn of $\alpha = \delta_0^2 = \delta_n^2$

$$\Big\Downarrow \text{ relax, simplify}$$

$$\Big\Downarrow \text{ relax, simplify}$$

$$\delta_0^2 = \delta_n^2 = L_D\left(\lfloor \tfrac{n}{2} \rfloor, \lfloor \tfrac{n}{2} \rfloor\right)$$

where $L_D = \left(A_D^{T} A_D\right)^{-1}$

NON-SMOOTH PRIORS

- One way to convey the fact that the unknown function is non-smooth is

$$m_i = m_{i-1} + w_i \;\left(\sim N(0, \gamma^2)\right)$$

$$\Big\Downarrow \quad \forall i = 1, \ldots, n$$

$$m_0 = 0$$

$$\underbrace{\begin{bmatrix} 1 & 0 & \cdots & & \\ -1 & 1 & 0 & \cdots & \\ & \ddots & \ddots & & \\ & & & & \\ \cdots & & 0 & -1 & 1 \end{bmatrix}}_{A_N} \times \underbrace{\begin{bmatrix} m_1 \\ \vdots \\ m_n \end{bmatrix}}_{\vec{m}} = \underbrace{\begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}}_{\vec{w}} \sim N(\vec{0}, \gamma^2 I)$$

$$A_N \times \vec{m} = \vec{w}$$

$$\Big\Downarrow \text{ same process}$$

$$\pi_{prior}(\vec{m}) \propto \exp\left(-\frac{1}{\gamma^2}\vec{m}^T A_N^T A_N \vec{m}\right)$$

$$\pi_{prior}(\vec{m}) \propto \exp\left(-\frac{1}{2\gamma^2}\vec{m}^T A_N^T A_N \vec{m}\right)$$

- Moreover if we "know" that there is a bigger jump at $i = J$, $1 \leq J \leq n$

$$m_i = m_J = m_{J-1} + \omega_J \left(\sim N\left(0, \frac{\gamma^2}{\theta^2}\right)\right)$$

$$\Downarrow$$

A_N becomes

$$\tilde{A}_N = \begin{bmatrix} 1 & & & & 0 \\ & \ddots & & & \\ & & \theta & & \\ & & & \ddots & \\ 0 & & & & 1 \end{bmatrix} \times A_N$$

at the J row

J column.