Lecture6

Tuesday, September 25, 2018 9:25 AM

LAST TIME:

- Bayesian framework
- Gaussian noise + Gaussian priors + linear parameter-to-observable map, then

the posterior:

$$\pi_{post} \propto \exp\Big(-\frac{1}{2}\big(\vec{y}^{obs} - A\vec{m}\big)^T \Sigma^{-1}\big(\vec{y}^{obs} - A\vec{m}\big)$$

linear parameter-to observable

noise covariance

$$- \frac{1}{2\gamma^2} \| L\vec{m} \|^2 \Big)$$

$\Downarrow$ exercise

posterior is also a Gaussian

$$\pi_{post} = N\big(\bar{m}_{post}, \Gamma\big)$$

$$\bar{m}_{post} = \mathbb{E}_{\pi_{post}}[m] = \int m \, \pi_{post} \, dm$$

TODAY:

+ non-linear parameter-to-observable map

+ how to compute

$$\bar{m}_{post} = \mathbb{E}_{\pi_{post}}[\vec{m}]$$

$$\text{Covariance} = \mathbb{E}_{\pi_{post}}\left[(\vec{m} - \bar{m}_{post})(\vec{m} - \bar{m}_{post})^T\right]$$

$$\bar{f} = \mathbb{E}_{\pi_{post}}[f(m)]$$

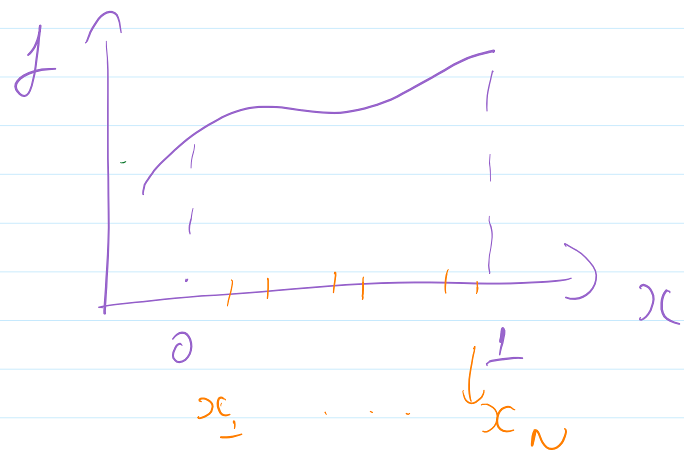
how to compute $\mathbb{E}_{\pi_{post}}[(\cdot)] = \int (\cdot)\,\pi_{post}\,dm$

- For non-linear problems, $\bar{f} = \mathbb{E}_{\pi_{post}}[f(m)]$ is non-trivial problem even when $f(m) = m$

MOTIVATION.

$$\int_0^1 x\,dx$$

$$\|$$



$$\lim \sum_{i}^{N} x_i \,\Delta x_i$$

$N \to \infty$

$\| \quad \Delta x_i = 1/N$

$$\boxed{\lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} x_i = \int_0^1 x\, dx} \quad (*)$$

Question: How do we extend $(*)$ for a general setting:

law of large numbers

$$\boxed{\begin{aligned} \bar{m}_{post} &= \int_X m\, \pi_{post}\, dm \\ &\overset{??}{=} \\ &\lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} m_i \end{aligned}} \quad (**)$$

It turns out that $(**)$ is valid If m_i are independently identically distributed (i.i.d) by $\pi_{post}(m)$.

TODAY   I.I.D random draws

$(*)$ Uniform distribution in $[0,1]$: $U[0,1]$

$m \sim U[0,1]$.

$\mu\,[\,m \subset A\,] = \int 1\, dm$

$m \sim U[0,1]$.

$$\mu_m[m \in A] = \int_A 1 \, dm$$

$\Rightarrow$ 1 is the density of $U[0,1]$

* Matlab: m = rand $\Leftrightarrow$ $m \sim U[0,1]$

(*) standard normal distribution: $N(0,1)$

$m \sim N(0,1)$.

$$\mu_m[m \in A] = \int_A \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}x^2\right) dx$$

$\Rightarrow$ $\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}x^2\right)$ is the density of $N(0,1)$

* Matlab: m = randn $\Leftrightarrow$ $m \sim N(0,1)$

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i.i.d. draws from $\pi(m)$; $m \in \mathbb{R}$

Question: how to draw i.i.d. samples from a general $\pi(m)$, $m \in \mathbb{R}$?

Answer: use CDF sampling

- To understand Cummulative distribution function (CDF) samping we start with definition of CDF.

Def: CDF of $m \sim \pi(m)$ is defined as

$$\phi(w) := \mathbb{P}[m < w] = \int_{-\infty}^{w} \pi(m)\, dm$$

Observations

1) $0 \leq \phi(w) \leq 1$

2) $\phi(w)$ is Non-decreasing

Now define $z = \phi(m)$

Claim: $z \sim U[0,1] \Leftrightarrow \mu_z(z < a) = \int_0^a dz$

Proof:

$$\mu_z(z < a)$$

$\parallel$ def

$$\mathbb{P}[z < a]$$

$\parallel$ def z

$$\mathbb{P}[\{\phi(m) < a\}]$$

ϕ is Non-decreasing

$\parallel$ $\phi(m) < a \Leftrightarrow m < \phi^{-1}(a)$

$$\mathbb{P}[\{m < \phi^{-1}(a)\}]$$

$\parallel$ def

$$\phi_r^{-1}(a)$$

$$\int_{-\infty}^{\phi^{-1}(a)} \pi(m)\, dm \quad \| \ a \in y$$

$m = -\infty, \; z = 0$ || $dz = \phi'(m)\, dm$

$m = \phi^{-1}(a); \; z = a$ || $= \pi(m)\, dm$

$$\int_0^a 1 \, dz$$

- This shows that we can get i.i.d. draws from $\pi(m)$ through $U[0,1]$:

<u>CDF sampling method</u>:

1) draw z_i from $U[0,1]$
2) $m_i = \phi^{-1}(z_i)$

$\pi(m)$ is arbitrary but can only be zero at isolated points in $\mathbb{R}$

* <u>limitations</u>:

1) need to compute ϕ^{-1} (could be nontrivial)
2) It is valid for 1D. Extension to multidimensional cases is only trivial for independent distribution:

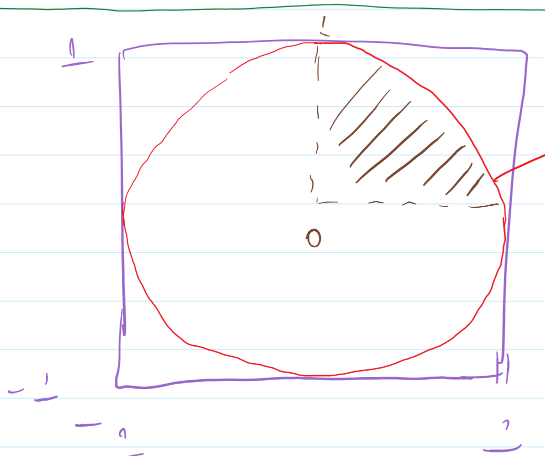
$$\pi(\vec{m}) = \prod_{i=1}^{n} \pi(m_i)$$

ACCEPTANCE - REJECTION SAMPLING

Problem: $m \sim \tilde{\pi}(m) = c\,\pi(m)$

How to draw i.i.d. samples from $\tilde{\pi}(m)$.

Approach: use acceptance - rejection sampling



$x^2 + y^2 \le 1$

* How to compute  ?
* draw a sample $(x_i, y_i) \sim U[0, 1]$
* compute $\alpha = x_i^2 + y_i^2$
* if $\alpha \le 1$ accept otherwise reject.

$$\frac{\text{accepted samples}}{\text{total samples}} \Longleftarrow$$

$\Downarrow$

area of 

- Suppose there exists $q(m)$ and D such that

$$\tilde{\pi} = c\,\pi(m) \le D\,q(m).$$

- it is trivial/easy to draw i.i.d samples from $q(m)$.

A - R sampling algorithm

1) Draw m from $q(m)$

2) Compute

$$\alpha = \frac{\tilde{\pi}(m)}{D\,q(m)} = \frac{c\,\pi(m)}{D\,q(m)}$$

3) Accepte m with probability α