Lecture5

Thursday, September 20, 2018 9:24 AM

- At this we assume that we have constructed (been given) the likelihood $\pi_{like}(\vec{m})$ and the prior $\pi_{prior}(\vec{m})$. From the Bayes' theorem we have

$$\pi_{post}(\vec{m}) \propto \pi_{like}(\vec{m}) \times \pi_{prior}(\vec{m})$$

1) $\vec{y}^{obs} = A\vec{m} + \vec{e}$

$\vec{e} \sim N(\vec{0}, \Sigma)$

(Σ could be $\sigma^2 I$)

2) $\pi_{prior}(\vec{m}) \propto \exp\left(-\frac{1}{2\gamma^2}\|L\vec{m}\|^2\right)$

$\|L\vec{m}\|^2 = (\vec{m}^T L^T L \vec{m})$

$$\pi_{post}(\vec{m}) \propto \exp\left(-\frac{1}{2}\|\vec{y}^{obs} - A\vec{m}\|^2_{\Sigma}\right) \times \exp\left(-\frac{1}{2\gamma^2}\|L\vec{m}\|^2\right)$$

$\|\vec{y}^{obs} - A\vec{m}\|^2_{\Sigma} = (\vec{y}^{obs} - A\vec{m})^T \Sigma^{-1} (\vec{y}^{obs} - A\vec{m})$

$$\propto \exp\left[-\underbrace{\left(\frac{1}{2}\|y^{obs} - A\vec{m}\|^2_{\Sigma} + \frac{1}{2\gamma^2}\|L\vec{m}\|^2\right)}\right]$$

$T(\vec{m})$: potential function

$\Downarrow$

$$\boxed{\pi_{post}(\vec{m}) \propto \exp(-T(\vec{m}))}$$

$\Downarrow$

$$\boxed{-\log \pi_{post}(\vec{m}) \propto T(\vec{m})}$$

Completing square

$$ax^2 + bx + c$$

1) "Baby":
$$a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)$$
$$\Updownarrow$$
$$a\left(x + \frac{b}{2a}\right)^2 + C$$

2) "PhD"
$$x\,a\,x + b\,x + c$$
$$\left(x + \frac{a^{-1}b}{2}\right) a \left(x + a^{-1}\frac{b}{2}\right) + C$$

MAXIMUM A POSTERIORI (MAP) POINT

MAP point is the point at which π_{post} is largest (most likely point = "point with highest probability")

$$\vec{m}_{MAP} := \underset{\vec{m}}{\operatorname{argmax}} \ \pi_{post}(\vec{m})$$

$$:= \underset{\vec{m}}{\operatorname{argmin}} \ T(\vec{m})$$

$$\boxed{\vec{m}_{MAP} = \underset{\vec{m}}{\operatorname{argmin}} \ \frac{1}{2} \|\vec{y}^{obs} - A\vec{m}\|_{\Sigma}^{2} + \frac{1}{2\gamma^2} \|L\vec{m}\|^{2}}$$

Remarks

1) $\vec{m}_{MAP}$ is the global minimizer of the potential function

2) For smooth priors: $L\vec{m} \sim \Delta m$, then

$$T(\vec{m}) \propto \frac{1}{2} \|\vec{y}^{obs} - A\vec{m}\|_{\Sigma}^{2} + \frac{1}{2\gamma^2} \|\Delta m\|^{2}$$

then $\vec{m}_{map}$ must have bounded derivaties upto second order (two-time differenticable)

3) Non-smooth prior:

$$T(\vec{m}) \propto \frac{1}{2} \|\vec{y}^{obs} - A\vec{m}\|_{\Sigma}^{2} + \frac{1}{2\gamma^2} \|\nabla \cdot m\|^{2}$$

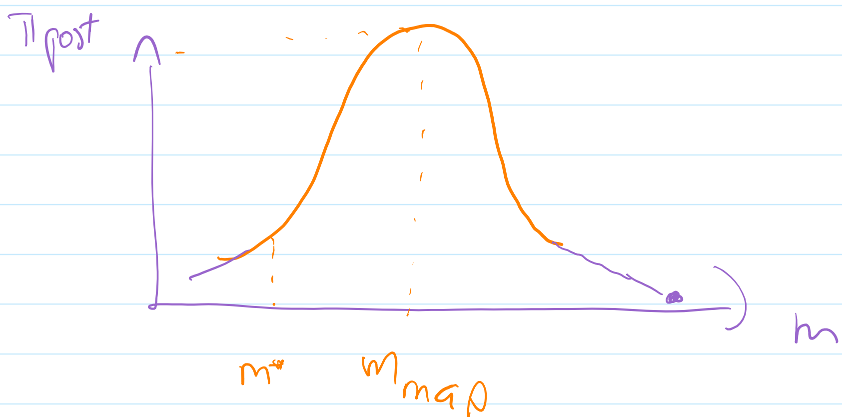
then $\vec{m}_{map}$ just needs to be differentiable.

$\Rightarrow$ less smooth.

4) The map point can be understood as a solution to a regularized deterministic inverse problem with $\| L \vec{m} \|^2$ as a regularization.

$$\Downarrow$$

Bayesian can be understood as a "generalization" of deterministic inverse problems. It provides a number of solutions each with certain probability



m^* is a possible solution but with less certainty (it is less likely that m^* is an outcome).

It can be shown that

$$\boxed{\vec{m}_{map} = \left(A^T \Sigma^{-1} A + \frac{1}{\sigma^2} L^T L \right)^{-1} A^T \Sigma^{-1} \vec{y}^{obs}}$$