BasicProbabilityTheory

Thursday, August 30, 2018 7:47 AM

HOMEWORK:

Chapter 1: mathematical theory of Inverse problem

TODAY

1) probability theory: random variables, probability space

2) state space + density + push-forward probability density / measure

3) conditional probability + Bayes' theorem/formula

4) mean, variances, etc.

- Chapter 1 looks at deterministic inverse problems
- The "main" disadvantage of a deterministic approach is that it is NOT trivial to take into account the uncertainties in the measurements mathematical models. In other words, it is not clear how to incorporate our lack of of knowledge
- Bayesian inverse problem is a method that can take all uncertainties (lack of knowledge) ignorance into account.

- We are going to use random variables to express our uncertainties (ignorance).

Def: An event is called <u>deterministic</u> if its outcome is completely determined / known.

Ex: Tomorrow is Wed.

Def: A random event is the "complement" of the deterministic event, that is, the outcome of a random event is not completely determined or known (not fully predictable).

Ex: roll dice, flip coin.

Remark. $\boxed{\text{randomess} = \text{ignorance}}$

- There are many to express ignorance

1) fuzzy set method

⋮

* $\boxed{\text{Probability theory}}$ focus of this class

$\Downarrow$

<u>Subjective</u> probality: (different person may have different probability)

Def: We say an event $A \subset \Omega$ happens with probabily $\mathbb{P}[A]$ if:

$$0 \leq \mathbb{P}[A] \leq 1$$

$$0 \leq \mathbb{P}\{A\} \leq 1$$

for sure A does not happen

for sure A happens

Def: $(\Omega, \mathcal{F}, \mathbb{P})$ is defined as a probability space where:

+ Ω: sample space: all the possibilities of the outcomes of elementary event ω

Ex: flipping - coin: elementary event

$$\Omega = \{head, tail\}$$

+ $\mathcal{F}$: σ - algebra: contains all the measurable events $A \subset \Omega$

$\Updownarrow$

an event that we can assign a probability.

Ex: flipping-coin: (fair, unbiased)

$$\mathcal{F} = \{\phi, \{head\}, \{tail\}, \Omega\}$$

$$\mathbb{P}[\{\phi\}] = 0$$

$$\mathbb{P}[\{head\}] = 1/2$$

$$\mathbb{P}[\{tail\}] = 1/2$$

$$\mathbb{P}[\{\Omega\}] = 1$$

Def: Independant events: $A, B \in \mathcal{F}$ are called independent if

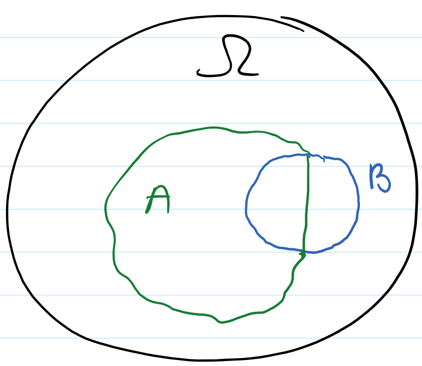
$$\mathbb{P}[A \cap B] = \mathbb{P}[A] \times \mathbb{P}[B]$$

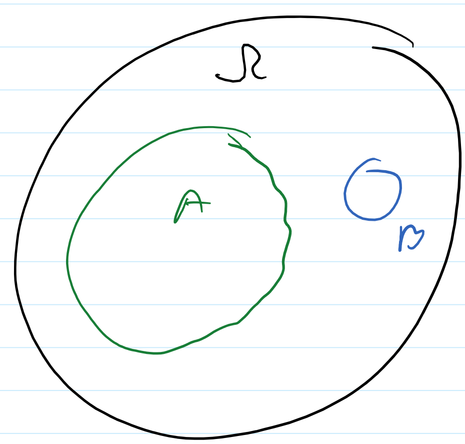
Def: Conditional probability.

The conditional probability of A given the fact that B has already happened is

$$\boxed{\mathbb{P}[A \mid B] = \frac{\mathbb{P}[A \cap B]}{\mathbb{P}[B]}}$$

Kolmogorov.





$\mathbb{P}[A \mid B] = 1/2$

$\mathbb{P}[A \mid B] = 0$

Ex: Roll a dice

B: event of getting face bigger than 4

A: event of getting face 6

$\mathbb{P}[A \mid B] = ?$ assuming $\mathbb{P}[\{A = i\}] = 1/6$

$i = 1, \ldots, 6$

* Kolmogorov:

..., 6

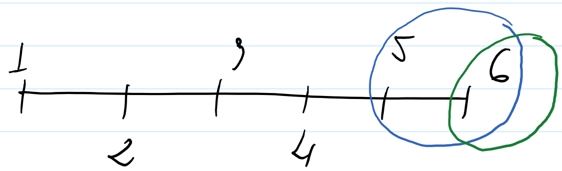
* Kolmogorov:

$$\mathbb{P}[A|B] = \frac{\mathbb{P}[A \cap B]}{\mathbb{P}[B]} \stackrel{\downarrow}{=} \frac{\mathbb{P}[A]}{\mathbb{P}[B]} = \frac{1/6}{2/6} = \frac{1}{2}$$

$A \subset B$

* $B = \{5, 6\}$

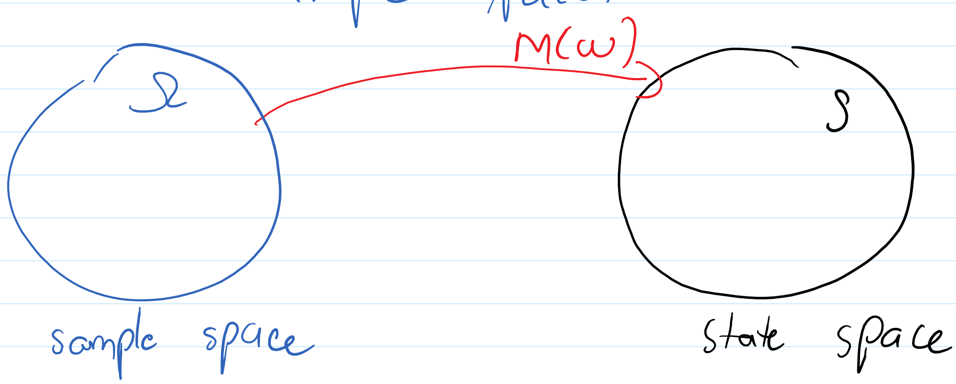
$A = \{6\}$



$$\mathbb{P}[A|B] = 1/2.$$

Random variables

- In practice we are working in spaces that are not the sample space:



Def: State Space S is the set containing all the possible outcomes of a random variable.

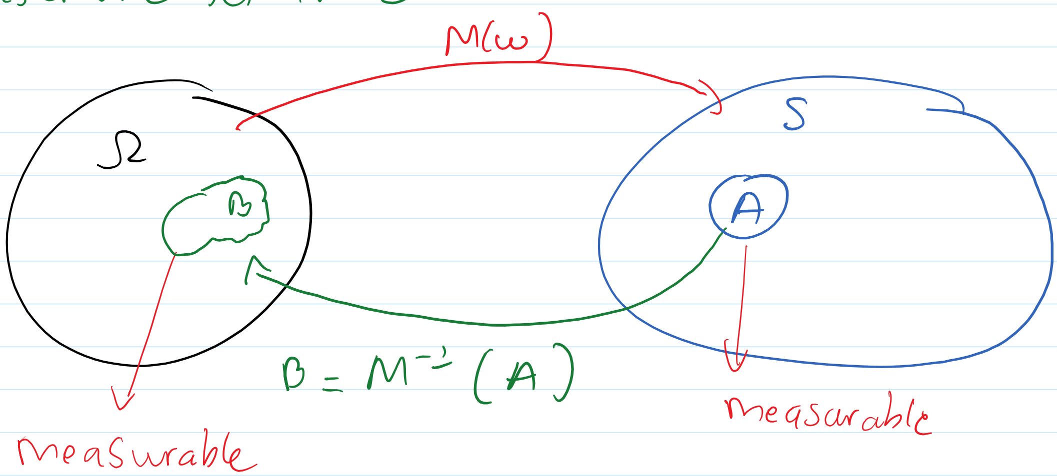
Def: Random variable is a "measurable" map from $\Omega \to S$ (assume M is onto)

$$M : \Omega \to S$$

$$\omega \mapsto M(\omega) \in S$$

$M(\omega)$ is random because its value/outcome is unknown/uncertain:

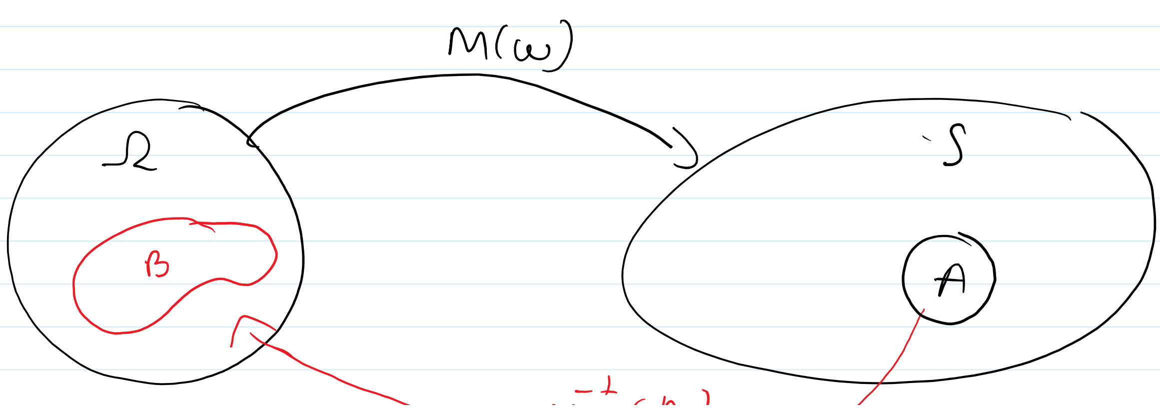
Def. $M : \Omega \to S$ is called a measurable map if the inverse image of any measurable set in S, through M, is also a measurable set in Ω

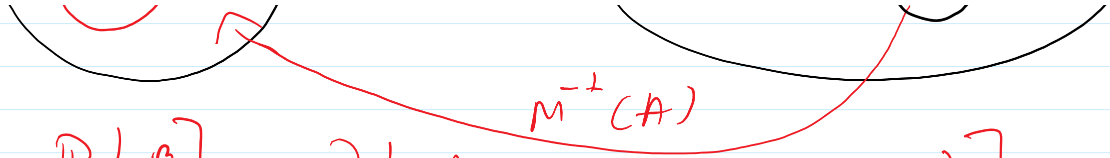


* compared with def continuous map!

Def: The probability distribution (distribution/law) of a random variable $M(\omega)$ is defined as

$$\boxed{\mu_M(A) := \mathbb{P}[M^{-1}(A)] = \mathbb{P}[\{M(\omega) \in A\}]}$$





$$\mathbb{P}[B] = \mathbb{P}[\{\omega : M(\omega) \in A\}]$$

$$= \mathbb{P}[\{M(\omega) \in A\}]$$

Ex.

$$\mu_M(S) = \mathbb{P}[M^{-1}(S)] = \mathbb{P}[\Omega] = 1$$

$$\mu_M(\{\phi\}) = \mathbb{P}[M^{-1}(\phi)] = \mathbb{P}[\phi] = 0$$

$$\Rightarrow \quad 0 \le \mu_M(A) \le 1$$

$$\Downarrow \text{def}$$

$$\boxed{\mu_M \text{ is a probability measure}}$$