Lecture11

Thursday, October 18, 2018 9:26 AM

Sub-Gaussian RVs

LAST TIME:

- Markov:

$$\mathbb{P}[\,|m| \geq t] \leq \frac{\mathbb{E}[|m|]}{t} \quad \forall t \geq 0$$

- Chebyshev: $\mathbb{E}[m] = 0$

$$\mathbb{P}[\,|m| \geq t] \leq \frac{\sigma^2}{t^2}, \; \forall t > 0 \; ; \; \sigma^2 = \mathbb{V}[m]$$

- Chernoff:

$$\mathbb{P}[\,|m| \geq t] \leq \min_{\lambda > 0} \frac{\mathbb{E}[e^{\lambda m}]}{e^{\lambda t}}$$

- $m \geq 0$

$$\mathbb{E}[m] = \int_0^\infty \mathbb{P}[m > t]\, dt$$

- $S_N = \frac{1}{N} \sum m_i \,, \quad m_i \sim \mathcal{N}(\bar{m}, \sigma^2)$

$$\mathbb{P}[\,|S_N - \bar{m}| \geq t] \leq 2e^{-N\frac{t^2}{2\sigma^2}}$$

$$\Updownarrow$$

$$\mathbb{P}\left[|S_N - \bar{m}| < t\right] \geq 1 - 2e^{-N\frac{t^2}{2\sigma^2}}$$

$$\Updownarrow$$

$$|S_N - \bar{m}| < t$$

with probability greater than $1 - 2e^{-N\frac{t^2}{2\sigma^2}}$

TODAY: Sub-Gaussian.

Def: (Sub-Gaussian RVs).

- m is sub-gaussian if

$$\underbrace{\mathbb{E}\left[e^{\lambda m}\right]}_{\text{MGF of } m} \leq \underbrace{e^{\frac{\lambda^2\sigma^2}{2}}}_{\text{MGF of } N(0,\sigma^2)}$$

- m is sometimes called a σ-sub-gaussian
- m " " a sub-gaussian with proxy σ^2.

Prop 13.1: If m is a σ-sub-gaussian then

Prop 13.1: If m is a σ-sub-gaussian then

1) $\mathbb{E}[m] = 0$

2) $\mathbb{V}[m] \leq \sigma^2$

Proof:

1) $$\mathbb{E}[e^{\lambda m}] \leq e^{\frac{\lambda^2\sigma^2}{2}}$$

Taylor of $e^{\lambda m}$ // Dominated Convergence Thm DCT $\qquad$ || Taylor expansion

$$\sum_{n=0}^{\infty} \frac{\lambda^n}{n!}\mathbb{E}[m^n] \leq \sum_{n=0}^{\infty} \frac{\lambda^{2n}\sigma^{2n}}{2^n n!}$$

$$\Downarrow$$

$$\sum_{n=1}^{\infty} \frac{\lambda^n}{n!}\mathbb{E}[m^n] \leq \sum_{n=1}^{\infty} \frac{\lambda^{2n}\sigma^{2n}}{2^n n!}$$

- take $\lambda > 0$, divide λ both sides

$$\frac{1}{1!}\mathbb{E}[m] + \dots \leq \frac{\lambda}{2 \times 1!}\sigma^2 + \dots$$

$$\Downarrow \text{ take } \lambda \to 0$$

$$\mathbb{E}[m] \leq 0$$

- Similarly take $\lambda < 0$

$$\Downarrow$$

$\rightarrow \mathbb{E}[m] = 0$

$$\Big) \to \mathbb{E}[m] = 0$$

$$\Downarrow$$

$$\mathbb{E}[m] \geq 0$$

2) divide both sides $\lambda^2 \to \mathbb{V}[m] \leq \sigma^2$

Thm 13.1 (Sub-gaussian properties)

- Let m be a RV, then the following are equivalent.

1) There exists c_1:

$$\mathbb{P}[|m| \geq t] \leq 2e^{-t^2/2c_1^2} \quad \forall t \geq 0$$

2) There exists c_2:

$$\|m\|_p := \left(\mathbb{E}[|m|^p]\right)^{1/p} \leq c_2 \sqrt{p} \quad \forall p \geq 1$$

3) There exists c_3:

$$\underset{\text{MGF of } m^2}{\underbrace{\mathbb{E}\left[e^{\lambda^2 m^2}\right]}} \leq e^{c_3^2 \lambda^2} \quad \forall |\lambda| \leq \frac{1}{c_3}$$

Moreover if $\mathbb{E}[m] = 0$, then all the above are also equivalent to

4) There exists σ :

$$\mathbb{E}[e^{\lambda m}] \le e^{\sigma^2 \lambda^2 / 2} \quad \forall \lambda \in \mathbb{R}$$

Proof : 1 => 2, 2 => 3, 3 => 1, 4 => 1

1 => 2 :

$$\mathbb{E}[|m|^p] = \int_0^\infty \mathbb{P}[|m|^p > t]\, dt$$

|| change of variables

$$\int_0^\infty \mathbb{P}[|m| > u]\, p u^{p-1}\, du$$

$\le$ using 1)

$$2 \int_0^\infty p u^{p-1} e^{-\frac{u^2}{2 c_1^2}}\, du$$

|| change of variable

$$(\sqrt{2} c_1)^p\, 2 \int_0^\infty p t^{p-1} e^{-t^2}\, dt$$

|| def Gamma function

$$(\sqrt{2}\, c_1)^p \; p \; \Gamma(p/2)$$

$\leq$ Stirling's approximation ?

$$(\sqrt{2}\, c_1)^p \; p \; (p/2)^{p/2}$$

$\Rightarrow$ taking the pth-root both sides

$$\|m\|_p \leq \sqrt{2}\, c_1 \; p^{1/p} \sqrt{\frac{p}{2}}$$

$\leq$ $p^{1/p} < 2$

$$2\, c_1 \sqrt{p}$$

2) $\Rightarrow$ 3),

$$\mathbb{E}\left[e^{\lambda^2 m^2}\right] \underset{\text{MCT}}{\overset{\text{Taylor}}{=}} 1 + \sum_{k=1}^{\infty} \frac{\lambda^{2k}}{k!} \mathbb{E}\left[m^{2k}\right]$$

$\leq$ use 2)

$$1 + \sum_{k=1}^{\infty} \frac{\lambda^{2k}}{k!} (8 c_1^2)^k k^k$$

$\leq$ $\boxed{\text{Stirling} \quad k! \geq \left(\frac{k}{e}\right)^k}$

$$1 + \sum_{k=1}^{\infty} \left(8 e c_1^2 \lambda^2\right)^k$$

$$\| \quad 8 e c_1^2 \lambda^2$$

$$\frac{1}{1 - 8 e c_1^2 \lambda^2}$$

$$\frac{1}{1-x} \leq e^{2x} \qquad \wedge \qquad 8 e c_1^2 \lambda^2 \leq \frac{1}{2}$$

$$x \leq \frac{1}{2}$$

$$e^{16 e c_1^2 \lambda^2}$$

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3) ⇒ 1) :

$$\mathbb{P}[|m| \geq t] = \mathbb{P}[e^{m^2} \geq e^{t^2}]$$

$$\wedge \quad \text{Markov}$$

$$\frac{\mathbb{E}[e^{m^2}]}{e^{t^2}}$$

$$\wedge \quad \text{using } 3)$$

$$e^{c_3^2}$$

$$\overline{e^{t^2}}$$

easy to
to find c_1

$$\Rightarrow \; \| \; - \frac{t^2}{2c_1^2}$$
$$2e$$