Akshay Kumar

1.1 a) $X = L^2(0,1) \quad Y = L^2(0,1)$

$\bar{m} = m + \alpha \sin(kx)$

$\bar{g} = A\bar{m} = g - \frac{\alpha}{k} + \frac{\alpha}{k}\cos kx$

$\|\bar{g} - g\|_{L^2} = \sqrt{\int_0^1 \left|\frac{\alpha}{k} - \frac{\alpha}{k}\cos kx\right|^2 dx} = \frac{\alpha}{k}\sqrt{\int_0^1 [1 + \cos^2 kx - 2\cos kx]\,dx}$

$= \frac{\cancel{\alpha}}{k}\sqrt{\cancel{\frac{3x}{2} + \sin 2kx}}$

$= \frac{\alpha}{k}\sqrt{\int_0^1 1\,dx + \int_0^1 \cos^2 kx\,dx - \int_0^1 2\cos kx\,dx}$

$= \frac{\alpha}{k}\sqrt{1 + \frac{1}{2} + \frac{\sin 2k}{4k} - \frac{2\sin k}{k}}$

as $k \to \infty$ $\quad \|\bar{g} - g\|_{L^2} = 0$

$\|\bar{m} - m\|_{L^2} = \sqrt{\int_0^1 (\alpha \sin kx)^2 dx}$

$= \alpha\sqrt{\int_0^1 \sin^2 kx\,dx} = \alpha\sqrt{\int_0^1 \frac{1 - \cos 2kx}{2}\,dx}$

$= \alpha\sqrt{\frac{1}{2} - \int_0^1 \frac{\cos 2kx}{2}\,dx} = \alpha\sqrt{\frac{1}{2} - \frac{\sin 2kx}{4k}\Big|_0^1}$

$= \alpha\sqrt{\frac{1}{2} - \frac{\sin 2k}{4k}}$

as $K \to \infty$ $\quad \|\tilde{m} - \bar{m}\|_{L^2} = \alpha \sqrt{\frac{1}{2} - 0} = \frac{\alpha}{\sqrt{2}}$

$\therefore$ Not stable, no-unique solution, ill-posed as small change m does not lead to any change in g.

b) $X = L^2[0,1] \quad Y = H_0^1(0,1)$

$$\|\tilde{g} - g\|_{H_0^1} = \sqrt{\frac{\alpha^2}{K^2}\left(\int_0^1 (1-\cos Kx)^2 dx + \int_0^1 \left(\frac{\alpha}{K} K \sin Kx\right)^2 dx\right.}$$

$$= \sqrt{\frac{\alpha^2}{K^2}\left(\frac{3}{2} + \frac{\sin 2K}{4K} - \frac{2\sin K}{K}\right) + \alpha^2 \int_0^1 \left(\frac{1 - \cos 2Kx}{2}\right) dx}$$

$$= \sqrt{\frac{\alpha^2}{K^2}\left(\frac{3}{2} + \frac{\sin 2K}{4K} - \frac{2\sin K}{K}\right) + \alpha^2 \left(\int_0^1 \frac{1}{2} - \int_0^1 \frac{\cos 2Kx}{2} dx\right)}$$

$$= \sqrt{\frac{\alpha^2}{K^2}\left(\frac{3}{2} + \frac{\sin 2K}{4K} - \frac{2\sin K}{K}\right) + \alpha^2\left(\frac{1}{2} - \frac{\sin 2K}{4K}\right)}$$

as $K \to \infty$ $\quad \|\tilde{g} - g\|_{H_0^1} = \sqrt{0 + \frac{\alpha^2}{2}} = \frac{\alpha}{\sqrt{2}}$

$$\|\bar{m} - m\|_{L^2} = \frac{\alpha}{\sqrt{2}} \quad \text{[From Previous prob]}$$

$\therefore$ Stable, unique, well posed as small change in m leads to small change in g.

1.2. To prove $\{\phi_i\}$ is orthonormal.

A. We have $\{\varphi_i\}$ orthonormal and

$$A^*A\varphi_i = \mu_i^2 \varphi_i \quad \text{and} \quad \mu_i \phi_i = A\varphi_i$$

$$\Rightarrow \phi_i = \frac{A\varphi_i}{\mu_i}$$

$$(\phi_i, \phi_j) = \left(\frac{A\varphi_i}{\mu_i}, \frac{A\varphi_j}{\mu_j}\right)$$

$$= \frac{1}{\mu_i \mu_j}(A\varphi_i, A\varphi_j)$$

$$= \frac{1}{\mu_i \mu_j}(\varphi_i, A^*A\varphi_j)$$

$$= \frac{1}{\mu_i \mu_j}(\varphi_i, \mu_j^2 \varphi_j) = \frac{\mu_j}{\mu_i}\cancelto{0}{(\varphi_i, \varphi_j)}$$

as they are orthonormal

$\therefore \{\phi_i\}$ is orthonormal set.

1.3. To show $N(A) = N(A^*A)$

A To show if $x \in N(A)$ then $x \in N(A^*A)$ and otherwise.

Let $x \in N(A)$ then

$$Ax = 0$$

$$A^*(Ax) = A^*0$$

$$\Rightarrow A^*Ax = 0 \Rightarrow x \in N(A^*A)$$

Let $y \in N(A^*A)$ then

$$A^*Ay = 0$$

$$\Rightarrow A^*(\underbrace{Ay}_{z}) = 0 \Rightarrow A^*z = 0$$

$$z \in N(A^*) = R(A^\perp)$$

Since $z \in R(A^\perp)$ and $z \in R(A)$ as $Ay = z$

z has to be 0. $\Rightarrow$

$\therefore y \in N(A)$ //

1.5 To show (A^*A+I) is Injective.

A. For a function to be injective.

$$(A^*A+I)x = (A^*A+I)y \implies x=y$$

so if $(A^*A+I)x = (A^*A+I)y$

$$(A^*A+I)(x-y)=0$$

Case I $\quad x=y \quad$ trivial

II $\quad x\neq y$ then $\quad A^*A+I=0$

$$A^*A=-I$$

$$\Rightarrow A^*A=-A^{-1}A$$

$$\Rightarrow A^*=-A^{-1}$$ which is not true always.

so $x=y$ is the only solution.
Hence it is injective.

1.4 We find that lower β gives better results even with Noise. The reason is lower β means ~~less loss of information as range is not being compressed much, ie less smoothing~~. The reason is because it offers better stability.

1.4 When β is small both with noise and without noise gave same results. But in other case, noise gave bad results. The reason small β gave similar results without noise is because lower the beta higher the stability.

The shape of the estimated function also looks good when β is small (10^{-10}). But the scale is not good in all the case (Maybe there is a bug in my code)

1.6

$$\min_m \frac{1}{2} \| Am - g \|_\lambda^2 + \frac{K}{2} \| \nabla m \|^2_{L^2(0,1)}$$

$$\min \| Am - g \|^2 + \| \sqrt{K} \nabla m \|^2$$

$$\Rightarrow \quad m = (A^T A + \Gamma^T \Gamma)^{-1} A^T b$$

$$\Gamma = \sqrt{K} \begin{pmatrix} 1 & -1 & 0 & 0 & \cdots \\ 0 & 1 & -1 & 0 & \\ 0 & 0 & 1 & -1 & \\ 0 & & & & \ddots \end{pmatrix}$$

Noise always gives bad result. But here $\beta = 0.2$ gave better result compared to that in 1.4

K is plotted with misfit where
misfit = $\|Am - g\|_2^2$

We find that as K increase misfit ~~decreases~~ increases as we give more importance to $\|\nabla m\|^2$ term.

c) We find Tikhonov gives better result in terms of smoothness compared to Conjugate gradient method. But Both of them were not close to actual results. (May be there is a bug in my code)
The plots don't look good.