Construction of the likelihood

Remark:

if we define

$$d\mu(m) := \pi(m|y)\, d\lambda(m) = \pi(m|y)\, dm$$

$$d\nu(m) := \pi(m)\, d\lambda(m) = \pi(m)\, dm$$

$$= \pi_{prior}(m)\, dm$$

Then we can rewrite the Bayes' formula as

$$\pi(m|y)\, dm = \frac{\pi_{like}(m)}{\pi(y)} \pi_{prior}(m)\, dm$$

$$\Updownarrow \text{ def}$$

$$d\mu(m) = \frac{\pi_{like}(m)}{\pi(y)}\, d\nu(m)$$

$$\Updownarrow \text{ Radon - Nikodym}$$

$$\frac{d\mu}{d\nu}(m) = \frac{\pi_{like}(m)}{\pi(y)} = \frac{\pi(y|m)}{\pi(y)}$$

$\propto \quad \pi_{like}(m) = \pi(y|m)$

proportional

In conclusion :

$$\frac{d\mu}{d\nu}(m) \propto \pi(y|m)$$

valid for finite and ∞ dimensional settings

finite dimensions

$$\pi_{post}(m) \propto \pi(y|m)\,\pi_{prior}(m)$$

ADDITIVE NOISE CASES

Assume that we have a mathematical model that relates the unknown m and the observation:

$$y = h(m)$$

the parameter-to-observable map.

but what we observe/collect is

$$y^{obs} = y + e$$

e: noise/error

additive noise assumption

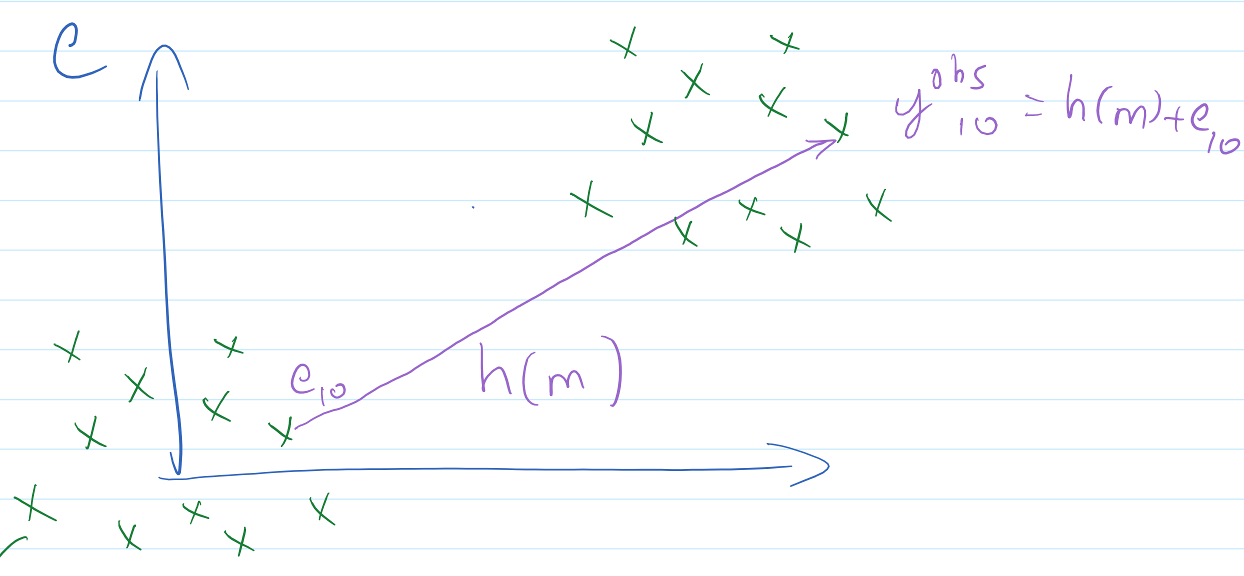
$y = h(m)$ ||

$y = h(m)$ $\Downarrow$ assumption

$$\boxed{y^{obs} = h(m) + e}$$

- Typically we assume that we know the distribution of the noise/error e, i.e.

$$\boxed{e \underbrace{\sim}_{\text{distributed}} \underbrace{\pi_{noise}}_{\text{distribution of noise } e}(e)} = \pi_e(e)$$



random realizations of error

Thus $\boxed{y^{obs} \sim \pi_{like}(y^{obs}) \sim \pi_e(y^{obs} - h(m))}$

Thus $\boxed{y^{obs} \sim \pi_{like}(y^{obs}) \sim \pi_e(y^{obs} - h(m))}$

by observation.

Rigorously we have:

$$\int_A \pi_{like}(y^{obs})\, dy^{obs} =: \mu_{y^{obs}}(A) = \mu_e(A - h(m))$$

$$\int_A \pi_e(e)\, de = \int_{A-h(m)} \pi_e(e)\, de$$

$$de = dy^{obs}$$

$$\int_A \pi_e(y^{obs} - h(m))\, dy^{obs}$$

Thus $$\int_A \pi_{like}(y^{obs})\, dy^{obs} = \int_A \pi_e(y^{obs} - h(m))\, dy^{obs}$$

$\forall A$

$\Downarrow$

$$\boxed{\pi_{like}(y^{obs}) = \pi_e\left(y^{obs} - h(m)\right)}$$

$$\pi_{post}(m) \propto \pi_{like}(y^{obs}) \times \pi_{prior}(m)$$

$$\boxed{\pi_{post}(m) \propto \pi_e\left(y^{obs} - h(m)\right) \times \pi_{prior}(m)}$$

Ex: Consider

$$y^{obs} = A\vec{m} + e$$

where $e \sim N(\bar{e}, \sigma^2) = \pi_e(e)$

$$= \frac{1}{\sqrt{2\pi\sigma}} \exp\left(-\frac{1}{2\sigma^2}(e-\bar{e})^2\right)$$

$$\pi_{like}(m) = \pi_{like}(y^{obs}) = ?$$

we have, from the above result,

$$\pi_{like} = \pi_e\left(y^{obs} - h(\vec{m})\right) = \pi_e\left(y^{obs} - A\vec{m}\right)$$

$$= \frac{1}{\sqrt{2\pi\sigma}} \exp\left(-\frac{1}{2\sigma^2}\left(y^{obs} - \bar{e} - A\vec{m}\right)^2\right)$$

$\in \mathbb{R}^m$ $\in \mathbb{R}^n$ $\in \mathbb{R}^m$

Extension : $\vec{y}^{obs} = A\vec{m} + \vec{e}$

Homework where a) $\vec{e} \sim N(\vec{\bar{e}}, \sigma^2 \underset{\text{identity}}{I})$

b) $\vec{e} \sim N(\vec{\bar{e}}, \underset{\text{positive definite matrix}}{\Sigma})$

$\pi_{like} = ??$