Classical limit theorems

Recall our question:

$$\frac{1}{N}\sum_{i=1}^{N} m_i \xrightarrow[N\to\infty]{??} \int m\,\pi(m)\,dm$$

Central limit theorem (CLT)

- Assume m_i are i.i.d. draws from $\pi(m)$

$$m_i \sim \pi(m)$$

denote $\bar{m} := \mathbb{E}_{\pi(m)}[m] = \int m\,\pi(m)\,dm < \infty$

$$\sigma^2 := \mathbb{E}_{\pi}\left[(m-\bar{m})^2\right] < \infty$$

Then:

$$Z_N(\omega) := \frac{1}{\sigma\sqrt{N}}\left(m_1 + \cdots + m_N\right) - \frac{\bar{m}}{\sigma}\sqrt{N}$$

Converges, in distribution, to a standard normal random variable. In particular:

$$\lim_{N\to\infty} \mathbb{P}\left[Z_N \le a\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{a} \exp\left(-\frac{t^2}{2}\right) dt$$

|| def

$$\mu_{Z_N}[Z_N \le a] = \int_{\{Z_N \le a\}} \pi_{Z_N}(m)\, dm$$

Law of large numbers [LLN]

Assume m_i are i.i.d. draws from $\pi(m)$

$$m_i \sim \pi(m)$$

define

$$\begin{cases} \bar{m} = \mathbb{E}_\pi(m) < \infty \\ \sigma^2 = \mathbb{E}_\pi\left[(m-\bar{m})^2\right] < \infty \end{cases}$$

first two moments of $\pi(m)$

Then:

$$\lim_{N\to\infty} S_N := \frac{1}{N}(m_1 + \cdots + m_N) = \bar{m}$$

ALMOST SURELY = ALMOST EVERYWHERE

$\|$

Convergence with probability 1

HOW ABOUT THE CONVERGENCE RATE OF S_N ??

(This allows us to estimate N for a given error $S_N - \bar{m}$)

* 1st Answer: CLT:

$$Z_N \underset{N\to\infty}{\sim} N(0,1)$$

$$\Downarrow$$

$$\mathbb{V}[Z_N] \approx 1$$

$$\Downarrow \text{ def}$$

$$\int (Z_N - 0)^2 \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{Z_N^2}{2}\right) dZ_N \approx 1$$

def $\|$ back to m_i

$$\int \left[\frac{1}{\sigma\sqrt{N}} \sum m_i - \frac{\bar{m}}{\sigma}\sqrt{N}\right]^2 \pi(m)\, dm$$

LAST TIME:

1) assume $\{m_i\}$ are i.i.d. draws from $\pi(m)$, + $\mathbb{E}_{\pi}[m] = \bar{m} < \infty$

+ $\mathbb{E}_{\pi}[(m-\bar{m})^2] = \mathbb{V}(m) < \infty$

2) CLT:

$$Z_N(\omega) := \frac{1}{\sigma\sqrt{N}} \sum m_i(\omega) - \frac{\bar{m}}{\sigma}\sqrt{N} \xrightarrow[N\to\infty]{\text{in distribution}}$$

a standard normal random variable

2) LLN: :

$$S_N = \frac{1}{N} \sum m_i \xrightarrow[N\to\infty]{a.s.} \bar{m}$$

TODAY: Convergence rate of S_N:

I/ Using CLT:

$$Z_N \overset{N\to\infty}{\sim} N(0,1)$$

$$\Downarrow$$

$$1$$

$$\approx$$

$$V[Z_N]$$

$$\| \text{ def}$$

$$\mathbb{E}\left[(Z_N - \bar{Z}_N)^2 \right]$$

$$\| \quad \bar{Z}_N \approx 0 \text{ since } Z_N \overset{N\to\infty}{\sim} N(0,1)$$

$$\mathbb{E}\left[(Z_N(\omega))^2 \right]$$

$$\| \text{ def } Z_N$$

$$\mathbb{E}\left[\left(\frac{1}{\sigma\sqrt{N}} \sum m_i - \frac{\bar{m}}{\sigma}\sqrt{N} \right)^2 \right]$$

$$\|$$

$$\mathbb{E}\left[\frac{N}{\sigma^2}\left(\frac{\sum m_i}{N} - \bar{m}\right)^2\right]$$

$$\| \text{ def } S_N$$

$$\frac{N}{\sigma^2}\,\mathbb{E}\left[\left(S_N - \bar{m}\right)^2\right]$$

$$\| \quad L^2(\mathbb{P}) := \left\{ f(\omega) : \int_\Omega (f(\omega))^2 \, d\mathbb{P} < \infty \right\}$$

$$\frac{N}{\sigma^2}\,\| S_N - \bar{m} \|^2_{L^2(\mathbb{P})}$$: mean-squared error

Thus we have "shown":

$$\frac{N}{\sigma^2}\,\| S_N - \bar{m} \|^2_{L^2(\mathbb{P})} \approx 1$$

$$\Rightarrow \quad \| S_N - \bar{m} \|_{L^2(\mathbb{P})} \approx \frac{\sigma}{\sqrt{N}}$$

$$\Downarrow$$

$$S_N - \bar{m} \sim \frac{1}{\sqrt{N}}$$

$$\Downarrow$$

$$\boxed{\| S_N - \bar{m} \| = c\,\frac{\sigma}{\sqrt{N}}}$$

$$\Downarrow$$

$$\boxed{\|S_N - \bar{m}\| = c\,\frac{\sigma}{\sqrt{N}}}$$

II/ Direct approach:

$$\|S_N - \bar{m}\|^2_{L^2(\mathbb{P})}$$

$$\| \text{ def } S_N$$

$$\mathbb{E}\left[\left(\frac{1}{N}\sum m_i - \bar{m}\right)\left(\frac{1}{N}\sum m_j - \bar{m}\right)\right]$$

$$\|$$

$$\mathbb{E}\left[\frac{1}{N^2}\left\{\sum (m_i - \bar{m})\right\}\left\{\sum (m_j - \bar{m})\right\}\right]$$

$$\|$$

$$\frac{1}{N^2}\mathbb{E}\left[\sum_i (m_i - \bar{m})(m_i - \bar{m}) + \sum_{i \neq j} (m_i - \bar{m})(m_j - \bar{m})\right]$$

$$\|$$

$$\frac{1}{N^2}\mathbb{E}\left[\sum_i (m_i - \bar{m})(m_i - \bar{m})\right] +$$

$$\frac{1}{N^2}\,\mathbb{E}\Big[\sum_{i\neq j}(m_i-\bar{m})(m_j-\bar{m})\Big]$$

$$\mathbb{E}\Big[\sum_{i\neq j}(m_i-\bar{m})(m_j-\bar{m})\Big]$$

$$=\sum_{i\neq j}\mathbb{E}\big[(m_i-\bar{m})(m_j-\bar{m})\big]$$

$$\overset{\text{i.i.d.}}{=}\sum_{i\neq j}\underbrace{\mathbb{E}[m_i-\bar{m}]}_{=0}\,\underbrace{\mathbb{E}[m_j-\bar{m}]}_{=0}$$

$$\frac{1}{N^2}\sum_i\mathbb{E}\big[(m_i-\bar{m})(m_i-\bar{m})\big]$$

|| def: $\mathbb{E}[(m_i-\bar{m})^2]=\sigma^2$

$$\frac{\sigma^2}{N}$$

Thus:

$$\boxed{\|S_N-\bar{m}\|^2_{L^2(\mathbb{P})}=\frac{\sigma^2}{N}}$$

- So the mean-square error of i.i.d draws (Monte Carlo method) converges to zero with the rate $\frac{1}{\sqrt{N}}$ Independent of the dimension of m (only true if σ DOES NOT depend on the dimension)

- In general:

$$S_N := \frac{1}{N} \sum_i f(m_i) \xrightarrow[N \to \infty]{a.s} \mathbb{E}_{\pi}[f(m)]$$

and the convergence rate is

$$\| S_N - \overline{f(m)} \|_{L^2(\mathbb{P})} = \frac{\sqrt{\mathbb{V}[f(m)]}}{\sqrt{N}}$$