Lecture17

Thursday, November 15, 2018 9:29 AM

HYPOTHESIS SPACE II

I) Kernel-based integral operators:

- Define, given a kernel $K(x,t)$,

$$(L_K f)(x) = \int_X K(x,t)\, f(t)\, d\pi(t)$$

where π is a probability measure

$$|(L_K f)(x)| = \left| \int_X K(x,t)\, f(t)\, d\pi(t) \right|$$

Cauchy-Schwarz $\leq$

$$\|K_x\|_{L^2(\pi(x))} \, \|f\|_{L^2(\pi(x))}$$

$C_K = \sup_{x,t} \sqrt{|K(x,t)|} \leq \quad \|K_x\|_\infty \leq C_K$

$$C_K \, \|f\|_{L^2(\pi(x))}$$

$\Rightarrow$ (*) $\boxed{|(L_K f)(x)| \leq C_K \|f\|_{L^2(\pi(x))}}$

Thus: ??

$$L_k : L^1(\pi(+)) \to C(X) \hookrightarrow L^\infty(\pi(x))$$

$$\Downarrow$$

$$L_k : L^2(\pi(+)) \hookrightarrow L^2(\pi(x))$$

which is continuous with : operator norm :

$$\|L_k\|_{L^2 \to L^2} \overset{??}{\leq} C_k$$

Prop 19.1: $L_k : L^2(\pi(x)) \to C(X)$ is a compact operator; If, in addition, K is a Mercer kernel, then L_k is self-adjoint positive semi-definite.

Proof: Let B be a bounded set in $L^2(\pi(x))$ such that $\|f\|_{L^2(\pi(x))} \leq M \quad \forall f \in B$.

From (*) we see that $L_k(B)$ is uniformly bounded. What remains is to show that $L_k(B)$ is equi-continuous. To that end we consider:

$$|(L_k f)(x) - (L_k f)(x')|$$

Cauchy-Schwarz $\wedge\!\!/\!\!/$

$$\sup_{t\in X} |k_x(t) - k_{x'}(t)| \, \|f\|_{L^2(\pi(x))}$$

$$\wedge\!\!/\!\!/$$

$$2 C_K \|f\|_{L^2(\pi(x))}$$

$$\wedge\!\!/\!\!/$$

$$2 C_K M$$

$$\Rightarrow \left|(L_K f)(x) - (L_K f)(x')\right| \leq 2 C_K M$$

$$\forall x, x', \forall f \in B$$

$\Rightarrow$ $L_K(B)$ is equi-continuous.

$\Downarrow$ def $\overline{L_K(B)}$ compact by Ascoli-Arzela'

$L_K(B)$ is (pre compact) relatively compact

$\Downarrow$ def

L_K is a compact map!

* Now K is a Mercer kernel. We need to show

1) L_K is self-adjoint

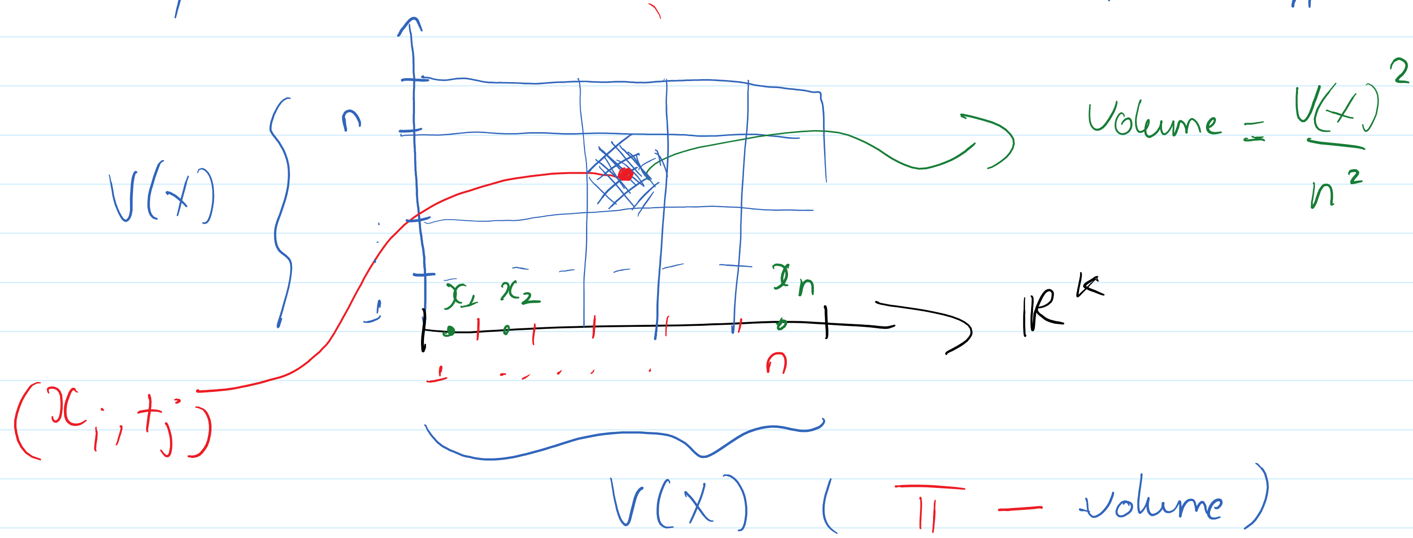
2) L_K is SPD.

1) The self-adjointness is clear by Fubini.

$$(g, L_K f)_{L^2(\pi(x))} \overset{??}{=} (L_K g, f)_{L^2(\pi(x))}$$

2) <u>positive definiteness</u>: (direct consequence of the positive semi-definiteness of K).

Since X is compact subset of $\mathbb{R}^K$, without loss of generality, we can subdivide X into n subsets with equal volumes with centroids $x_1, \dots, x_n$



$$(f, L_K f)_{L^2(\pi(x))} \overset{??}{\geq} 0$$

$\Big\|$ def

$$\int K(x,t)\, f(t)\, f(x)\, d\pi(t)\, d\pi(x)$$

$$\int_{X \times X} K(x,t)\, f(t)\, f(x)\, d\pi(t)\, d\pi(x)$$

$$= \sum_{i,j}^{n} \int_{\square} K(x,t)\, f(x)\, f(t)\, d\pi(x)\, d\pi(t)$$

$=$ rectangular rule + Riemann

$$\lim_{n \to \infty} \sum_{i,j}^{n} \int_{\square} K(x_i, t_j)\, f(x_i)\, f(t_j)\, d\pi(x)\, d\pi(t)$$

$$= \qquad \square = \frac{V(x)^2}{n^2}$$

$$\lim_{n \to \infty} \frac{V(x)^2}{n^2} \sum_{i,j=1}^{n} \underbrace{K(x_i, t_j)\, f(x_i)\, f(t_j)}_{\geq 0}$$

K is Mercer

Thus $(f, L_K f)_{L^2(\pi(x))} \geq 0$

- By Hilbert-Schmidt theorem 1.1, L_K admits a spectral decomposition with eigenpairs $(\lambda_i, \varphi_i)_{i \in \mathbb{N}}$ i.e.,

$$L_k f = \sum a_i \lambda_i \varphi_i$$

for any $f = \sum_{i=1}^{\infty} a_i \varphi_i$