Lecture 16: Multiresolution Image Analysis

Harvey Rhody
Chester F. Carlson Center for Imaging Science
Rochester Institute of Technology
rhody@cis.rit.edu

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Abstract

Multiresolution analysis provides information on both the spatial and frequency domains. Here we describe multiresolution analysis from a wavelet perspective and provide a simple example.

Multiresolution Analysis

The wavelet transform is the foundation of techniques for analysis, compression and transmission of images.

Mallat (1987) showed that wavelets unify a number of techniques, including subband coding (signal processing), quadrature mirror filtering (speech processing) and pyramidal coding (image processing). The name *multiresolution analysis* has been used for these techniques.

Image Pyramid

Let A be an image of size $N \times N$ where $N = 2^J$.

Let A_{J-1} be formed by smoothing A and then downsampling.

Let $\tilde{A}$ be an approximation of A reconstructed by upsampling and interpolating.

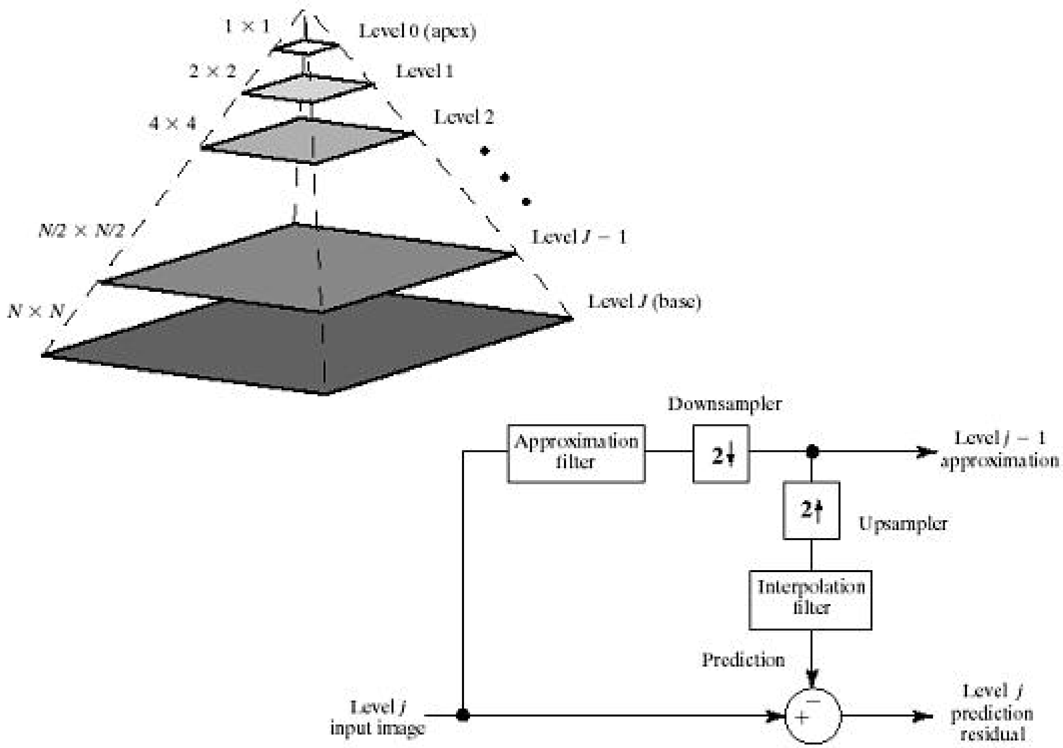
Let $E_J = A - \tilde{A}$. If we record A_{J-1} and E_J we can perfectly reconstruct A.

The process can be repeated, leading to the construction of a pyramid.

The number of pixels in a pyramid with P levels is

$$N^2 \left(1 + \frac{1}{4} + \frac{1}{4^2} + \cdots + \frac{1}{4^P}\right) \leq \frac{4}{3} N^2$$

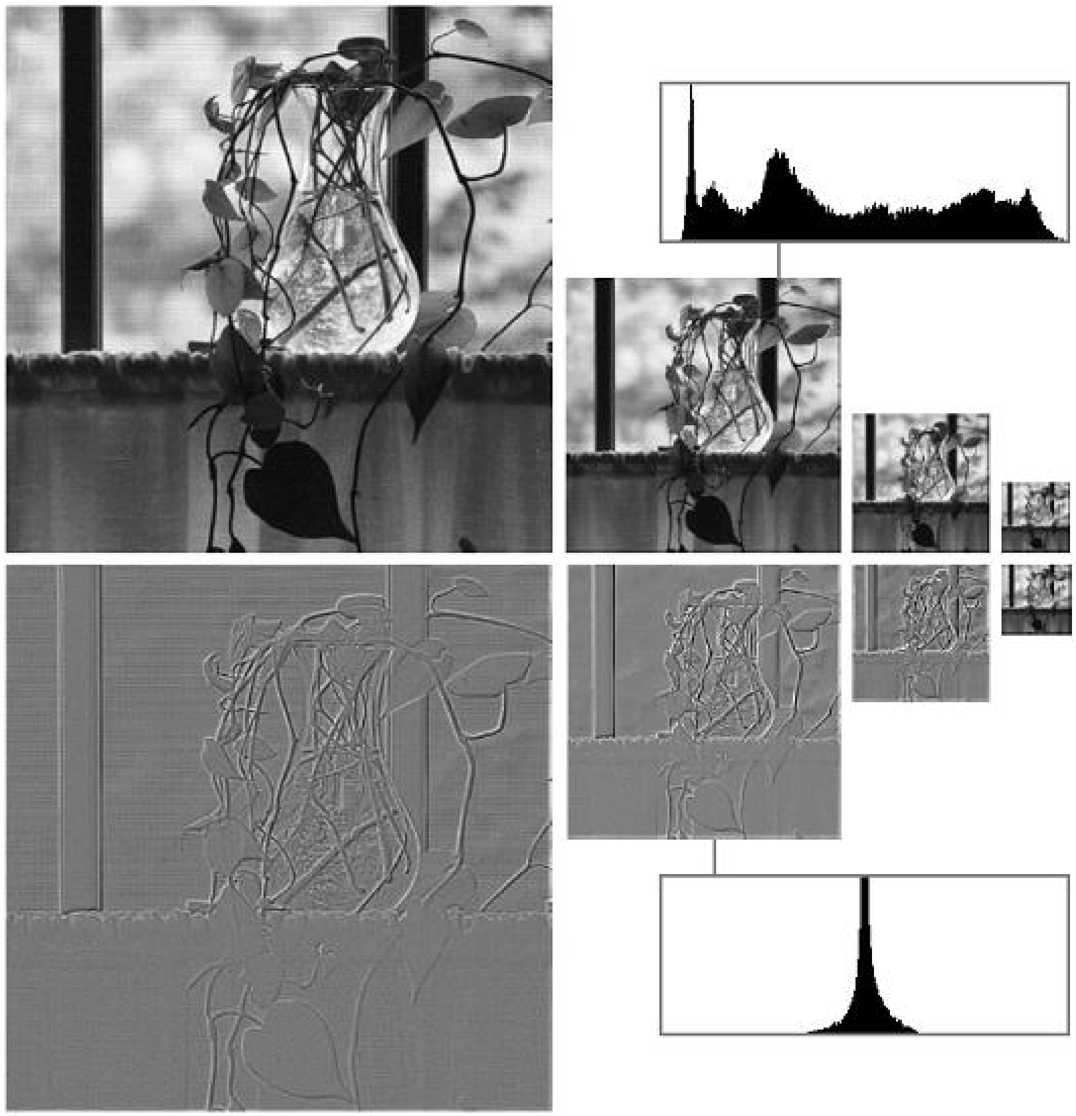
Image Pyramid (cont)



a b

FIGURE 7.2 (a) A pyramidal image structure and (b) system block diagram for creating it.

Image Pyramid (cont)

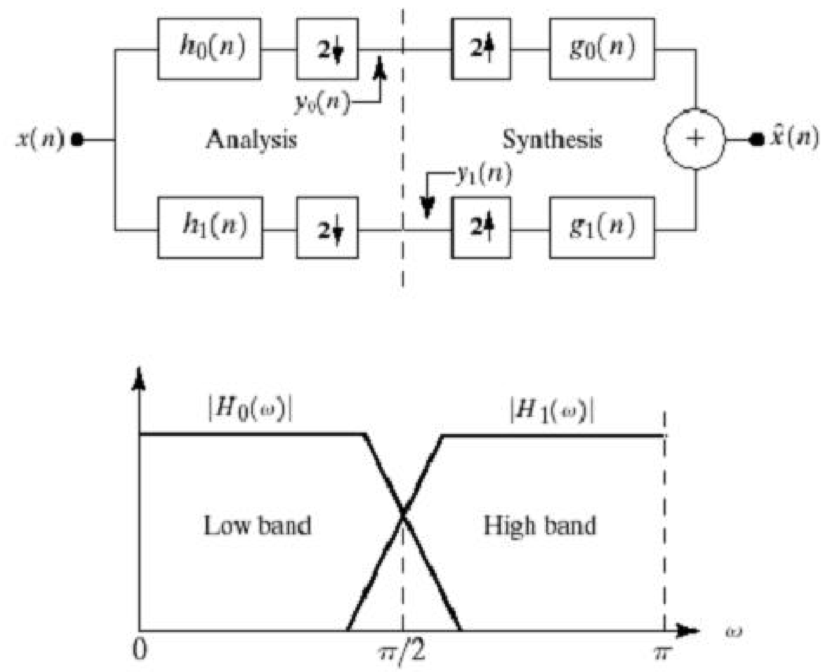


a
b

FIGURE 7.3 Two image pyramids and their statistics: (a) a Gaussian (approximation) pyramid and (b) a Laplacian (prediction residual) pyramid.

One-dimensional Subband Coding

FIGURE 7.4 (a) A two-band filter bank for one-dimensional subband coding and decoding, and (b) its spectrum splitting properties.



Subband Coding (cont)

The filters must have certain symmetry properties to enable perfect reconstruction, $\hat{x}(n) = x(n)$.

Biorthogonal conditions (G&W page 358)

$$\langle h_i(2n-k), g_j(k) \rangle = \delta(i-j)\delta(n),\ i,j = \{0,1\}$$

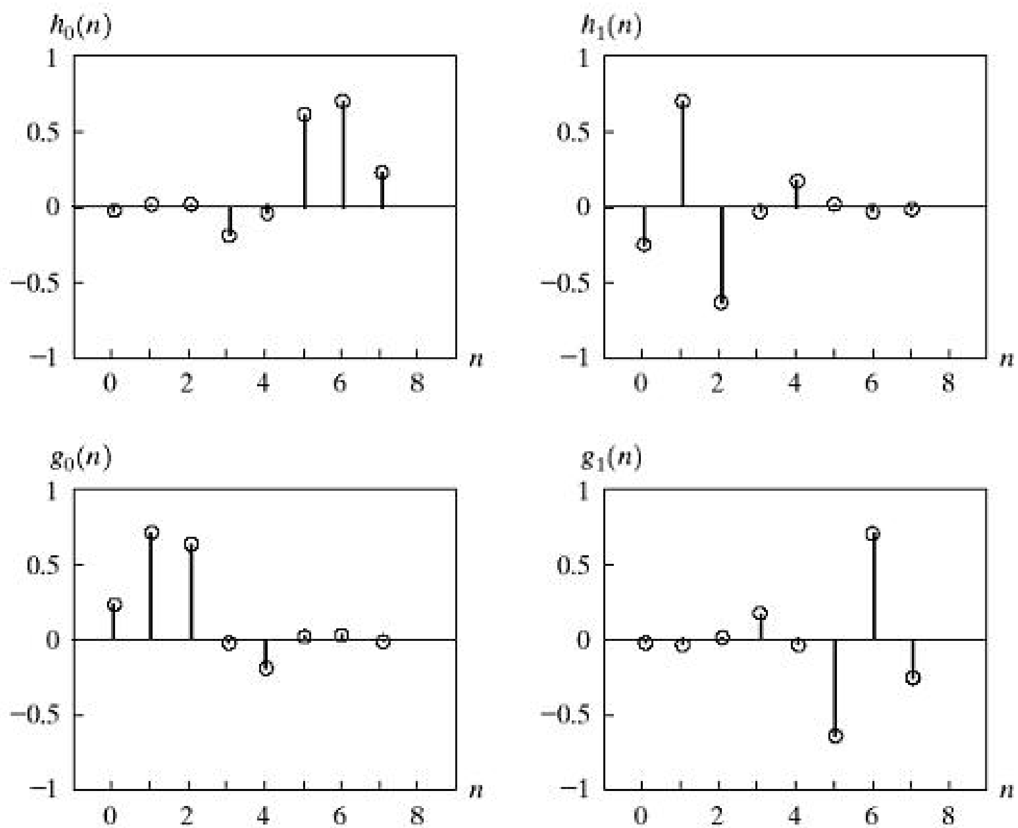
Many filter designs are possible that satisfy these conditions. There is a large and thorough published literature.

An additional condition, used in development of the fast wavelet transform, is that

$$\langle g_i(n), g_j(n+2m) \rangle = \delta(i-j)\delta(m),\ i,j = \{0,1\}$$

One-dimensional Subband Coding

FIGURE 7.6 The impulse responses of four 8-tap Daubechies orthonormal filters.



Two-dimensional Subband Coding

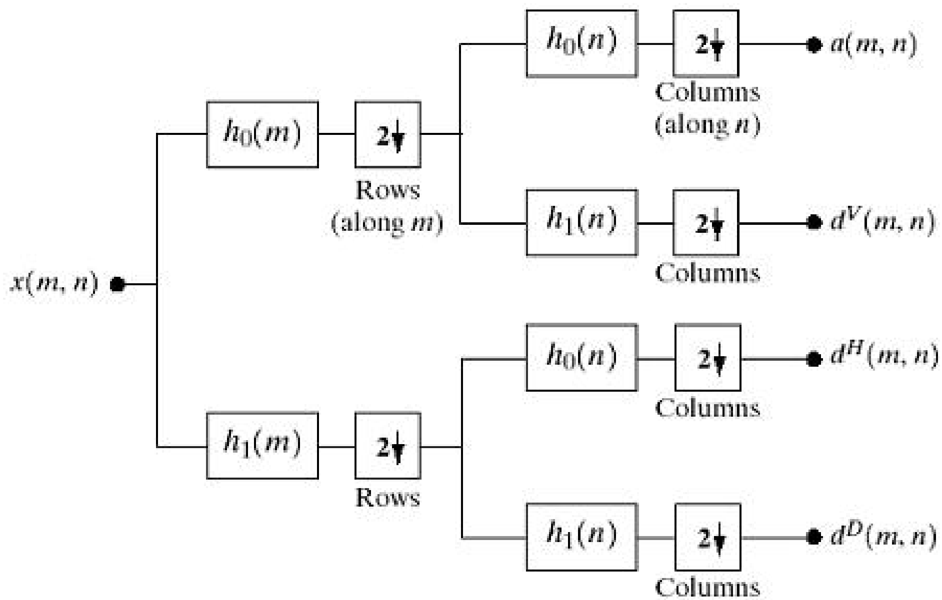


FIGURE 7.5 A two-dimensional, four-band filter bank for subband image coding.

Two-dimensional Subband Coding (cont)

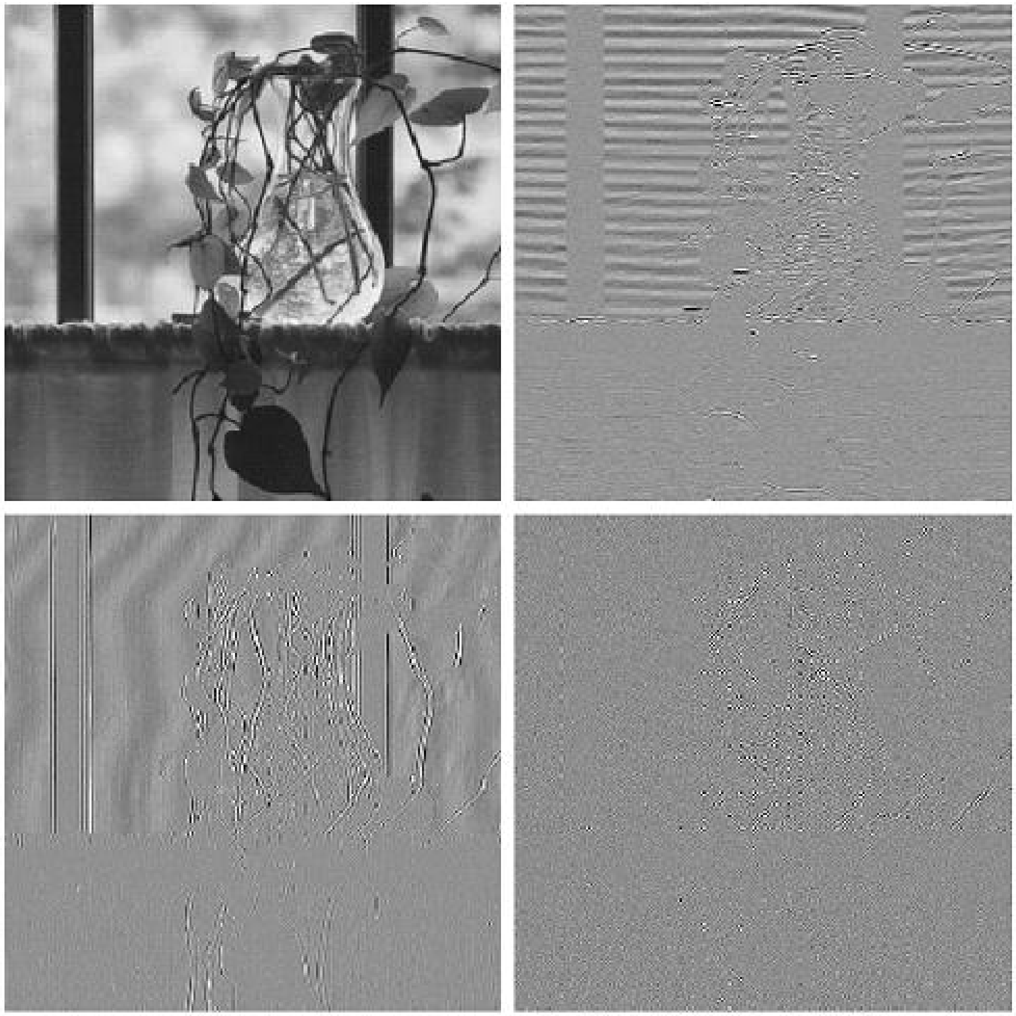


FIGURE 7.7 A four-band split of the vase in Fig. 7.1 using the subband coding system of Fig. 7.5.

Multiresolution Expansions

In MRA *scaling function* are used to construct approximations to a function (or an image).

The approximation has 1/2 the number of samples of the original in each dimension.

Other functions, called *wavelets* are used to encode the difference information between successive approximations.

We will illustrate the theory with 1D functions and then extend them to 2D.

MRA Expansions

A function $f(x)$ can be expanded as

$$f(x) = \sum_k \alpha_k \varphi_k(x)$$

1. The $\varphi_k(x)$ are real-valued expansion functions.
2. The α_k are real-valued expansion coefficients.
3. The set $\{\varphi_k(x)\}$ is the basis for a class of functions.
4. The set $V = \text{Span}_k\{\varphi_k(x)\}$ is the set of all functions that can be expressed this way. V is a vector space.
5. There is a set of *dual* functions $\{\tilde{\varphi}_k(x)\}$ that can be used to compute the coefficients.

$$\alpha_k = \langle \tilde{\varphi}_k(x), f(x) \rangle$$

6. We will show how to construct the set $\{\varphi_k(x)\}$ out of *scaling functions*.

Scaling Functions

Define the basis functions by translating and stretching (or compressing) a function $\varphi(x)$, called the scaling function.

$$\varphi_{j,k}(x) = 2^{j/2}\varphi(2^j x - k)$$

for all integers j, k.

A simple example is provided by the function

$$\varphi(x) = \begin{cases} 1 & 0 \leq x < 1 \\ 0 & \text{elsewhere} \end{cases}$$

This is called the Haar (1910) scaling function.

Haar Function

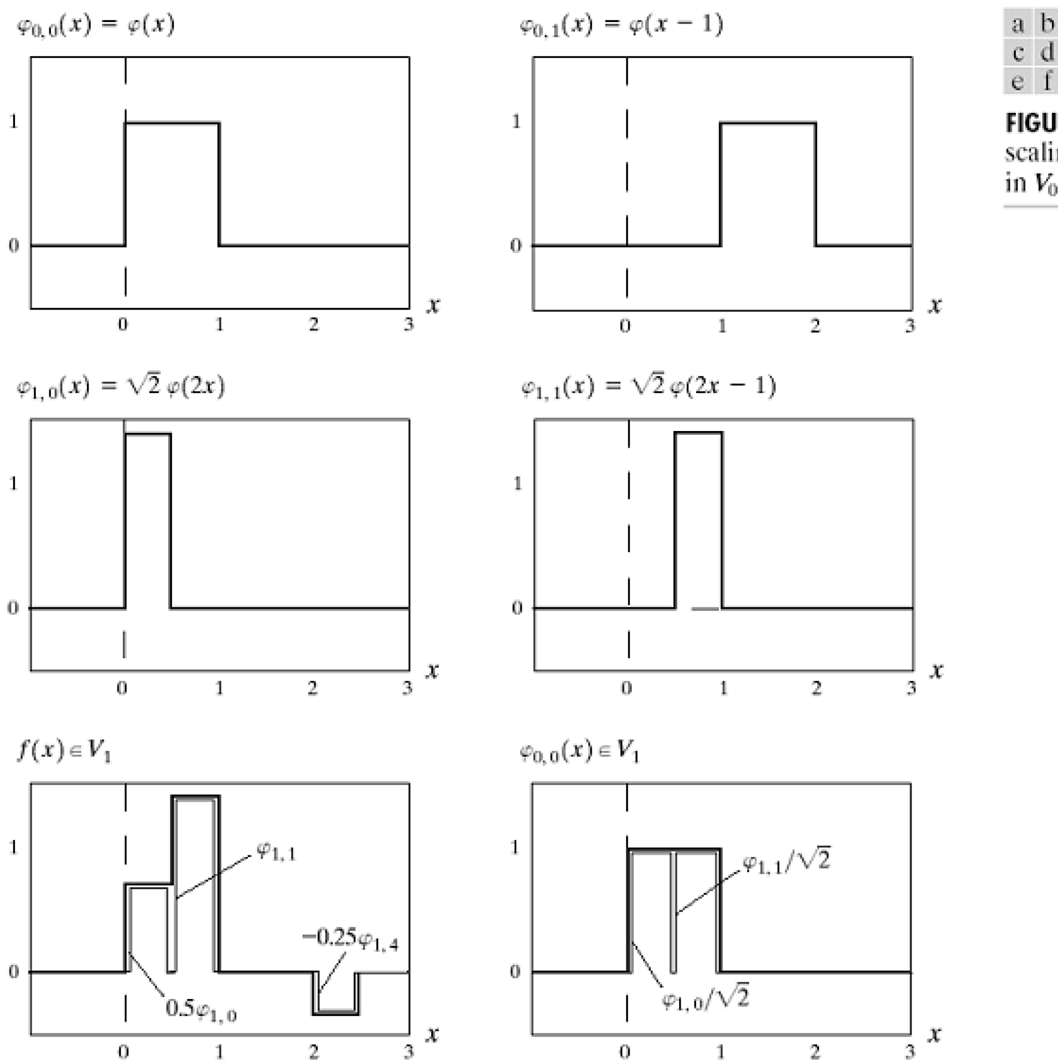


FIGURE 7.9 Haar scaling functions in V_0 in V_1.

Approximation Spaces

The set of scaling functions at any level j can be used to express functions that form a set V_j.

$$f(x) = \sum_k \alpha_k \varphi_{j,k}(x)$$

The function

$$f(x) = 0.5\varphi_{1,0}(x) + \varphi_{1,1}(x) - 0.25\varphi_{1,4}(x)$$

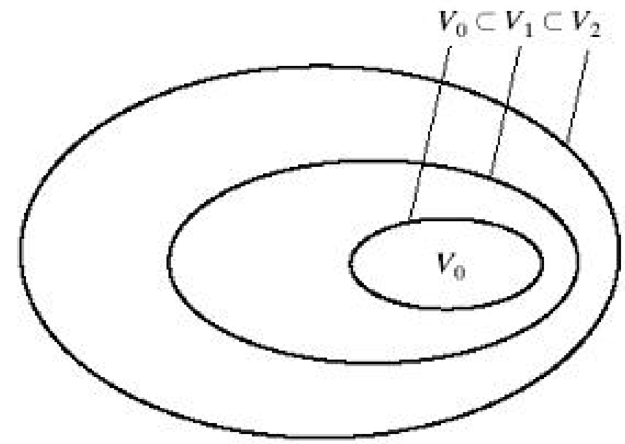
is shown in Figure 7.9(e) above.

The function $\varphi_{0,k}(x)$ can be expressed as

$$\varphi_{0,k}(x) = \frac{1}{\sqrt{2}}\varphi_{1,2k}(x) + \frac{1}{\sqrt{2}}\varphi_{1,2k+1}(x)$$

This is illustrated in Figure 7.9(f) above.

Mallat's Requirements for MRA



1. The scaling function is orthogonal to its integer translates.

2. The subspaces spanned by the scaling function at low scales are nested within those spanned at higher scales.

$$V_{-\infty} \subset \cdots \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_\infty$$

3. The only function that is common to all V_j is $f(x) = 0$.

4. Any function can be represented with arbitrary precision.

MRA Equation

If Mallat's requirements are satisfied then the expansion functions for V_j can be expressed in terms of the expansion functions for V_{j+1}.

$$\varphi_{j,k}(x) = \sum_n \alpha_n \varphi_{j+1,n}(x)$$

Substitute the definition

$$\varphi_{j,n}(x) = 2^{j/2}\varphi(2^j x - n)$$

and replace the coefficients with the notation $h_\varphi(n) = \alpha_n$.

$$\varphi_{j,k}(x) = \sum_n h_\varphi(n) 2^{(j+1)/2} \varphi\left(2^{j+1}x - n\right)$$

Since $\varphi(x) = \varphi_{0,0}(x)$, by setting $(j,k) = (0,0)$ we obtain

$$\varphi(x) = \sum_n h_\varphi(n)\sqrt{2}\varphi(2x - n)$$

This recursive equation is called the MRA equation. It defines the $h(n)$ values.

Haar function

The scaling function coefficients for the Haar function are found by noting that

$$\varphi(x) = \frac{1}{\sqrt{2}}\left[\sqrt{2}\varphi(2x)\right] + \frac{1}{\sqrt{2}}\left[\sqrt{2}\varphi(2x-1)\right]$$

We will find that the coefficients $\left[\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]$ are the foundation of the Haar wavelet transform.

The scaling functions are used to construct approximations to a function $f(x)$ at different resolutions (scales). We need a second set of functions to encode the differences in the approximations. This is the job of the wavelet function, $\psi(x)$.

Approximation Function Spaces

Suppose that a function $f(x) \in V_1$ then it can be expressed in terms of the $\{\varphi_{1,k}(x)\}$ set. However, if we were to try to express it in terms of the $\{\varphi_{0,k}(x)\}$ set we would have a residual error.

$$f(x) = f_0(x) + e_0(x)$$

The error $e_0(x)$ lies within V_1 but is outside V_0. We call the space that contains the residual W_0.

Functions in W_0 can be expanded in another basis set

$$e_0(x) = \sum_k \alpha_k \psi_{0,k}(x-k)$$

The $\psi(x)$ functions are the wavelets.

Approximation Function Spaces

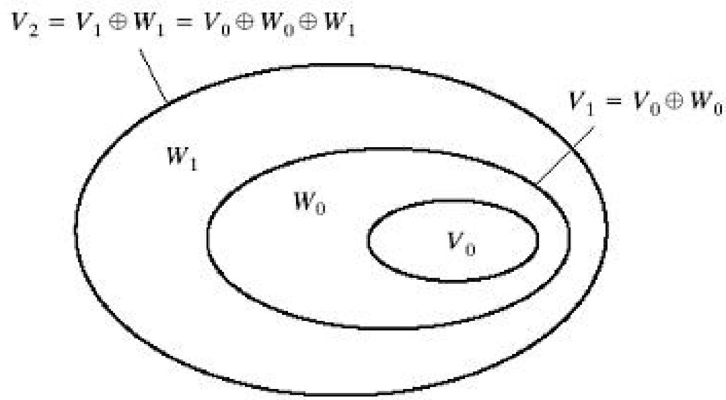


FIGURE 7.11 The relationship between scaling and wavelet function spaces.

In general, $V_{j+1} = V_j \oplus W_j$ The functions in W_j can be expanded in terms of a set of wavelets $\{\psi_{jk}\}$ and the wavelets must be orthogonal to the scaling functions $\{\varphi_{jk}\}$

Any function can be represented by a sequence of approximations which contain more and more detail.

$$L^2(\mathbf{R}) = V_0 \oplus W_0 \oplus W_1 \oplus W_2 \oplus \cdots$$

MRA Wavelet Expansion

We can write an function in $f_{j+1}(x) = f_j(x) + e_j(x)$. But $f_j(x)$ can be written as $f_j(x) = f_{j-1}(x) + e_{j-1}(x)$ so that $f_{j+1}(x) = f_{j-1}(x) + e_{j-1}(x) + e_j(x)$. In this manner, it is possible to expand any function as

$$f_{j+1}(x) = f_0(x) + \sum_{i=0}^{j} e_i(x)$$

Each of the residuals can be expanded using wavelets.

Wavelets satisfy the requirements

$$\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k)$$

$$\psi_{j,k}(x) = \sum_n \alpha_n \varphi_{j+1,n}(x)$$

This leads to a second MRA equation

$$\psi(x) = \sum_n h_\psi(n)\sqrt{2}\varphi(2x - n)$$

Haar Wavelets

It is true in general that $h_\psi(n) = (-1)^n h_\varphi(1-n)$. For the Haar wavelet we found

$$\mathbf{h}_\varphi = [h_\varphi(0), h_\varphi(1)] = \left[\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]$$

Then

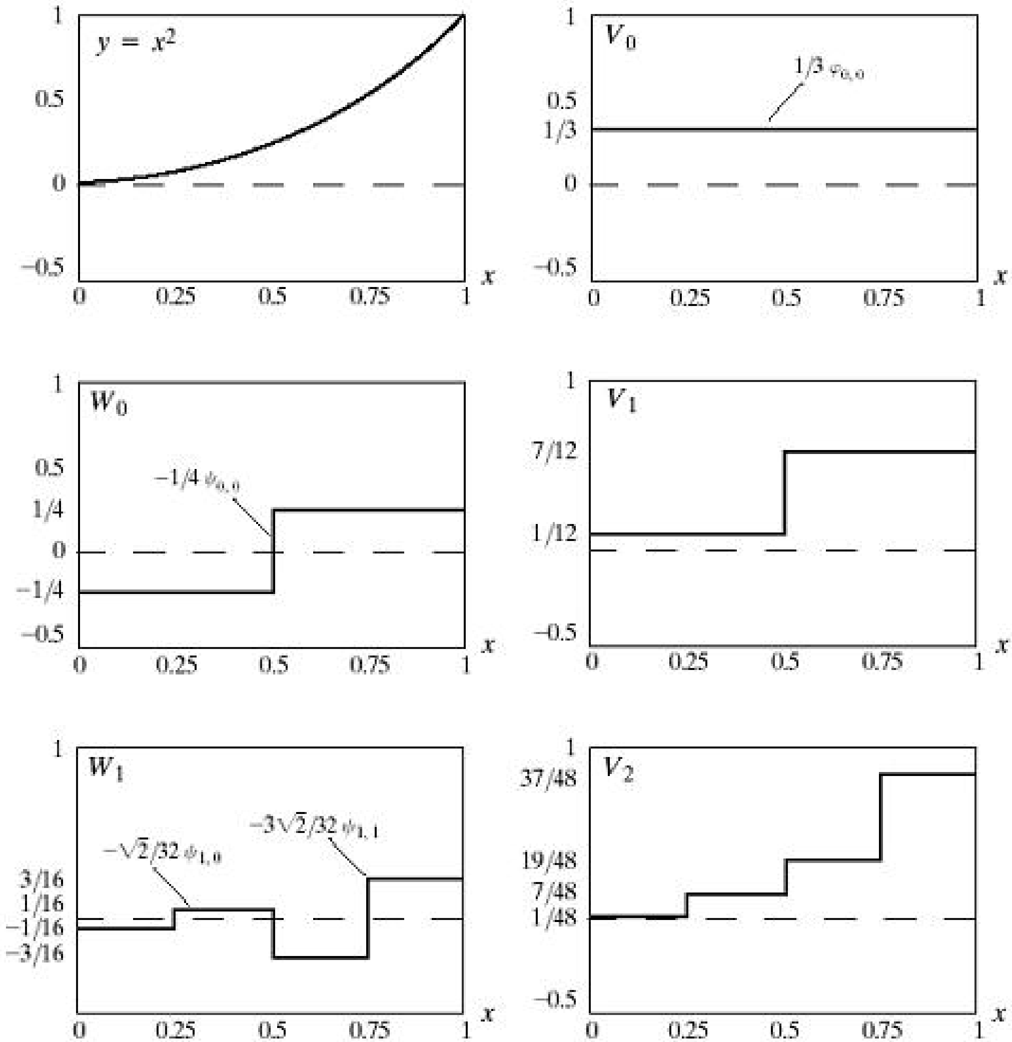
$$h_\psi(0) = (-1)^0 h_\varphi(1-0) = h_\varphi(1) = \frac{1}{\sqrt{2}}$$

$$h_\psi(1) = (-1)^1 h_\varphi(1-1) = -h_\varphi(0) = -\frac{1}{\sqrt{2}}$$

The Haar wavelet is

$$\psi(x) = \varphi(2x) - \varphi(2x-1) = \begin{cases} 1 & 0 \leq x < 0.5 \\ -1 & 0.5 \leq x < 1 \\ 0 & \text{elsewhere} \end{cases}$$

Haar Function Approximation Example

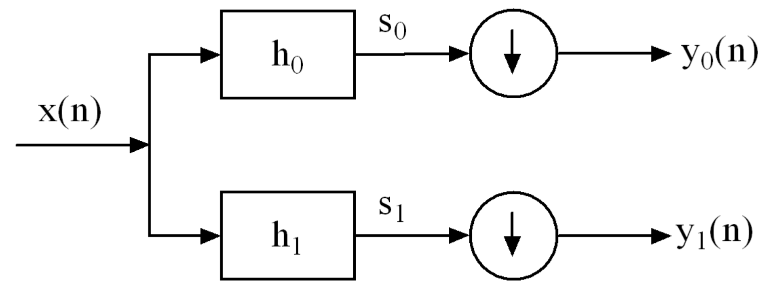


Haar Wavelet Transform

The HWT processes a function $x(n)$ through a pair of filters with impulse response

$$h_0(n) = \frac{1}{\sqrt{2}}[1, 1]$$

$$h_1(n) = \frac{1}{\sqrt{2}}[-1, 1]$$



To illustrate the operation of the system, consider the short input sequence

$$x = [\ 1 \quad -3 \quad 0 \quad 4\ ]$$

The output of the two filters is

$$s_0 = [\ 1 \quad -2 \quad -3 \quad 4 \quad 4\ ]/\sqrt{2}$$

$$s_1 = [\ -1 \quad 4 \quad -3 \quad -4 \quad 4\ ]/\sqrt{2}$$

Haar Example (cont)

After down-sampling

$$y_0 = s_0[1,3] = [-2,4]/\sqrt{2}$$
$$y_1 = s_1[1,3] = [4,-4]/\sqrt{2}$$

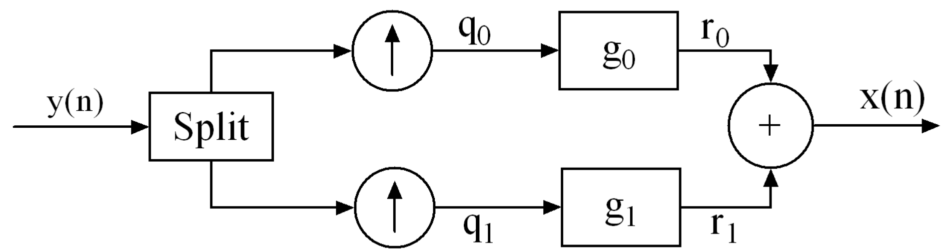
The concatenated values form the output

$$y = [y_0, y_1] = [-2,4,4,-4]/\sqrt{2}$$

Haar Example (cont)

The sequence may be recovered by using the synthesis device to the right with

$$g_0(n) = \frac{1}{\sqrt{2}}[1, 1]$$
$$g_1(n) = \frac{1}{\sqrt{2}}[1, -1]$$



The first step is to split the input sequence into two pieces that correspond to the two analyzer output channels.

$$y = [y_0, y_1] = [-2, 4, 4, -4]/\sqrt{2}$$

$$y_0 = [-2, 4]/\sqrt{2}$$
$$y_1 = [4, -4]/\sqrt{2}$$

Haar Example (cont)

The channels are then up-sampled to form

$$q_0 = [-2, 0, 4, 0]/\sqrt{2}$$

$$q_1 = [4, 0, -4, 0]/\sqrt{2}$$

The channels are then filtered to form

$$r_0 = [-2, -2, 4, 4, 0]/\sqrt{2}$$

$$r_1 = [4, -4, -4, 4, 0]/\sqrt{2}$$

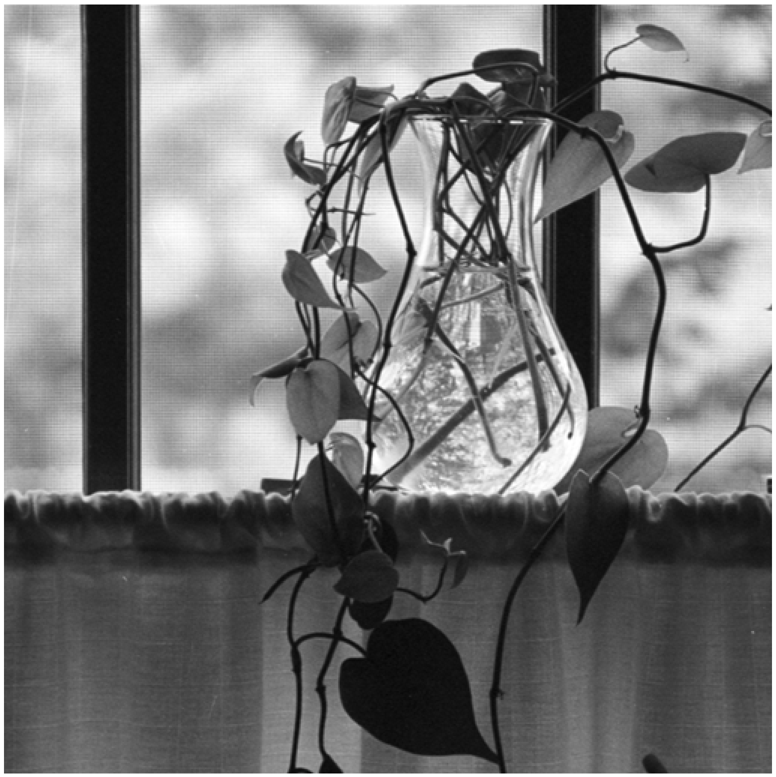
$$r = (r_0 + r_1)/\sqrt{2} = [2, -6, 0, 8, 0]/2$$

The list is trimmed to be of the same length as y to produce the output

$$x = [1, -3, 0, 4]$$

Haar Wavelet Transform – Image Example

The HWT can be applied to each row of an image. The effect is illustrated by the figure on the right.

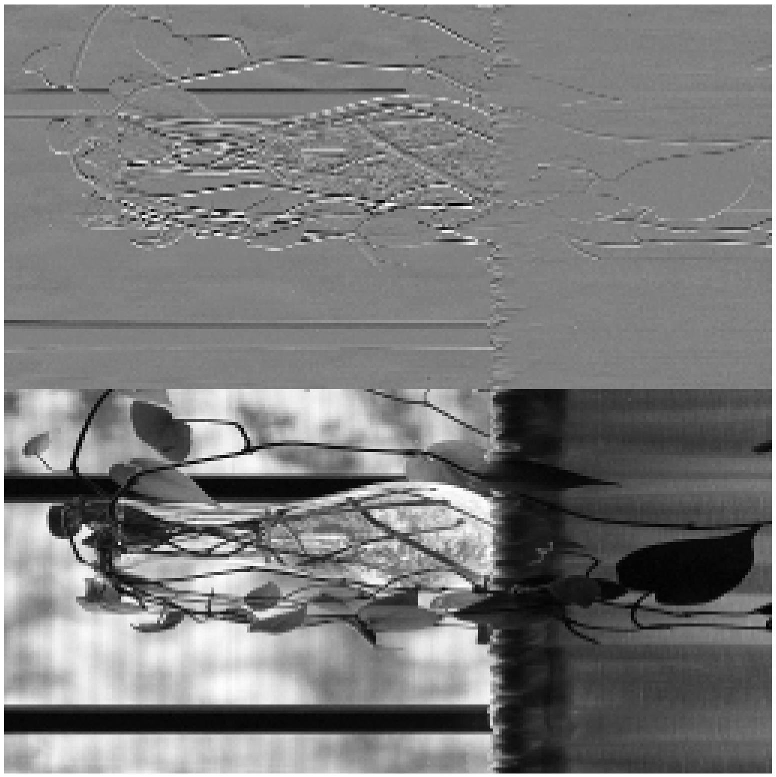


Original Image

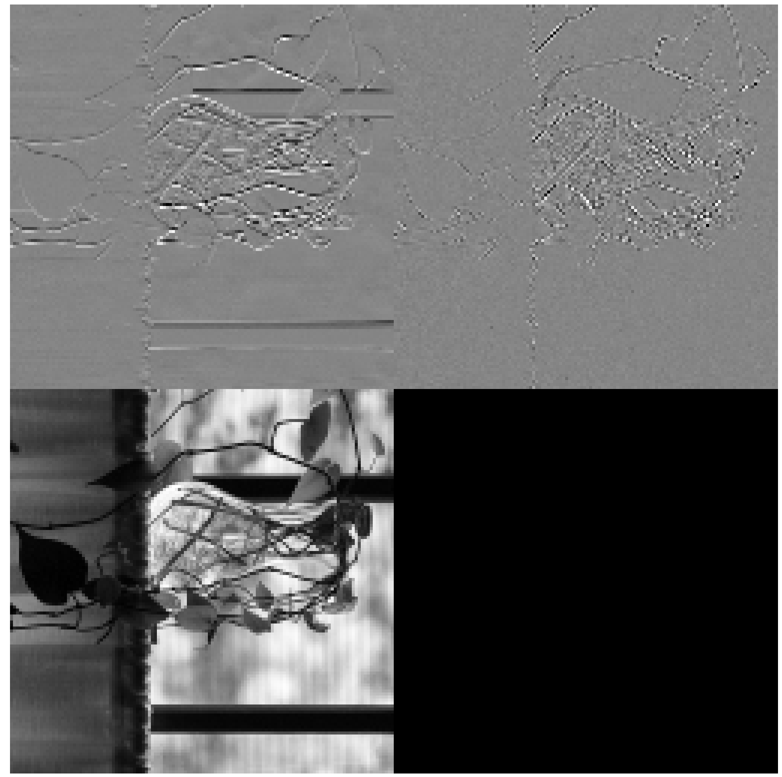
HWT along rows

HWT Example – Pass 2

A second pass can be done on the transposed image. This completes the image coding of slide 8.



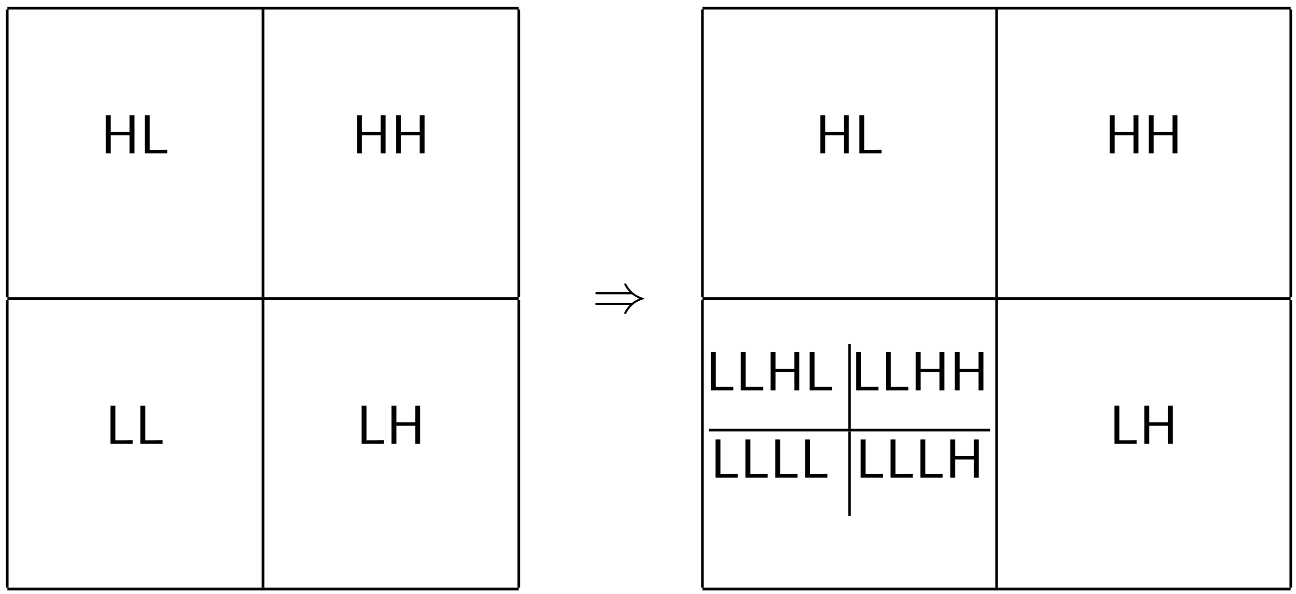
From Pass 1



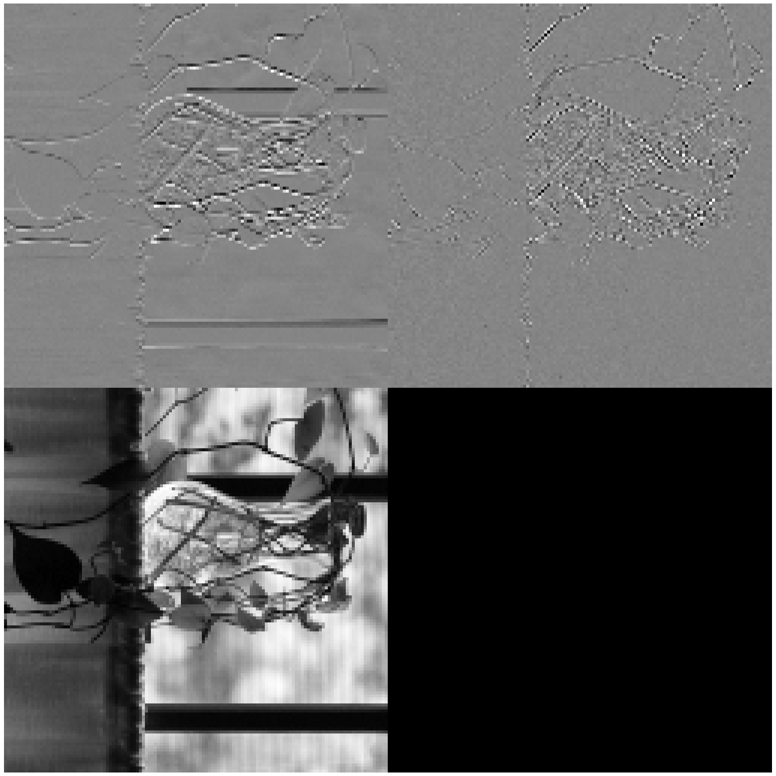
Result of Pass 2

HWT Example – Pass 3

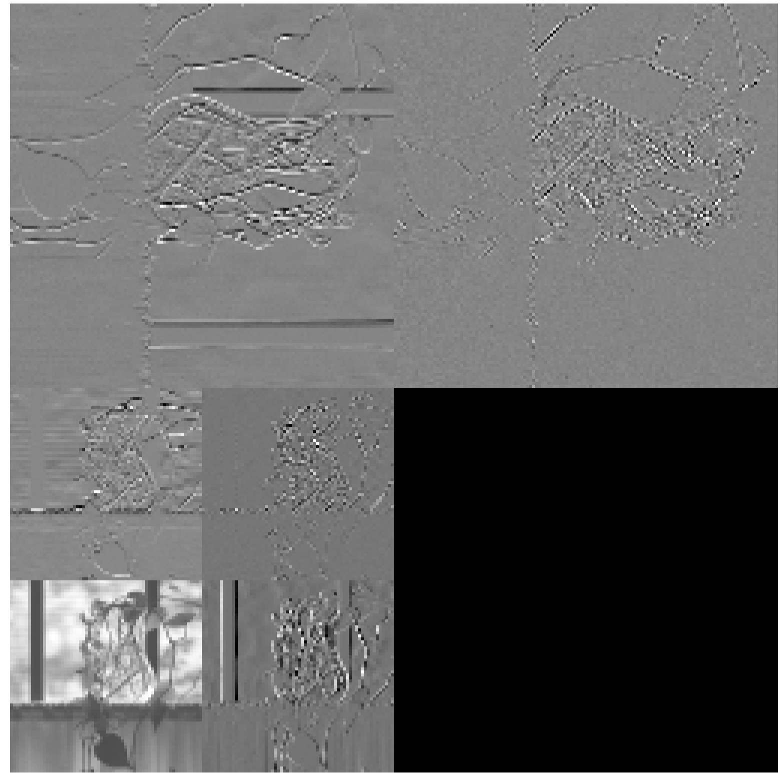
Another pass can be made using the LL image.



HWT Example – Pass 3



From Pass 2



Result of Pass 3 (both rows and columns)