MMSE Approximation for the Sparse Prior

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Computer Science, Technion, Israel

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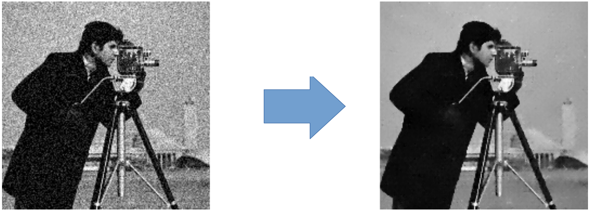
Joint work with Jeremias Sulam, Yaniv Romano, Yue M. Lu and Michael Elad

Noise Removal

Noise Removal



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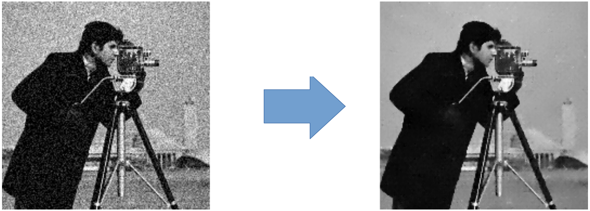

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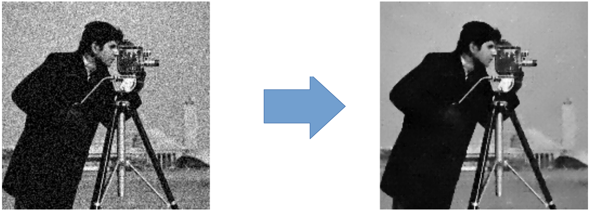
Noise Removal



Why denoising?

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- A simple testing ground for novel concepts in signal processing.
- Can be generalized to other, more complicated applications.

Noise Removal

Noisy Signal

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Clean Estimate

Noise Removal

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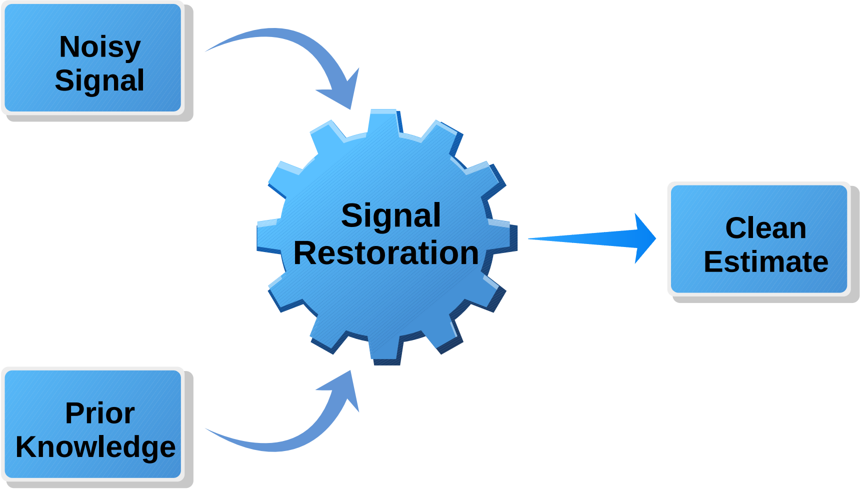

Noise Removal – Bayesian Standpoint

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Denoising

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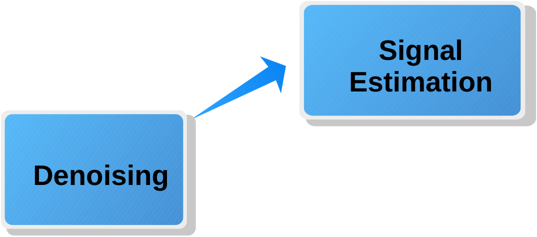

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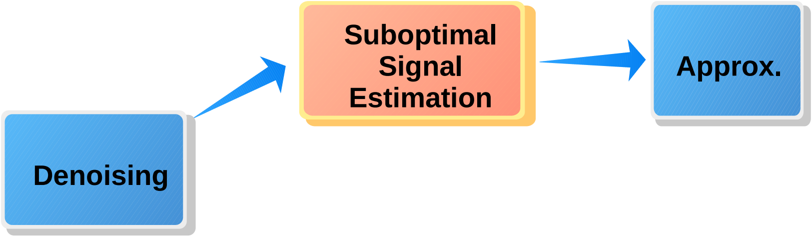

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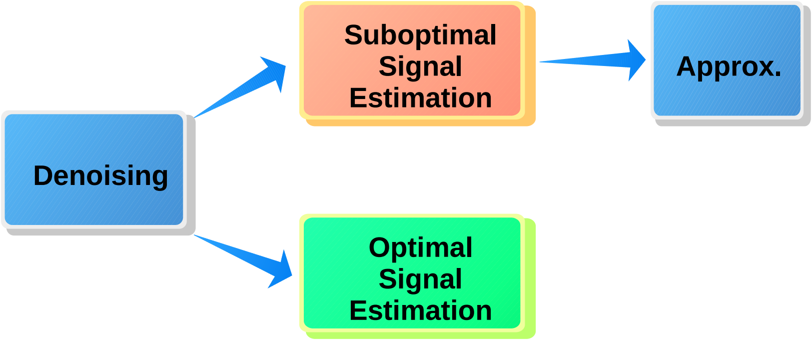

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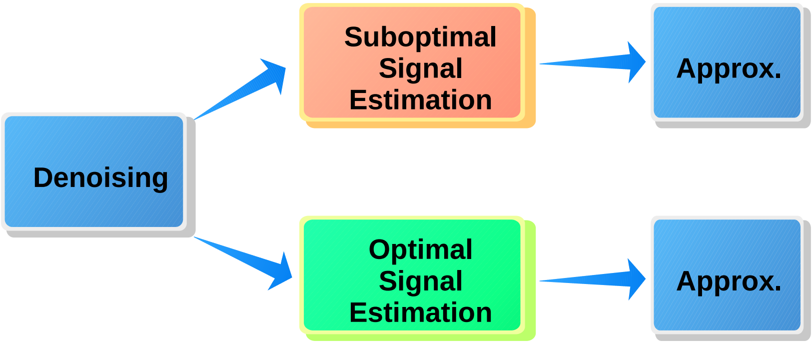

Outline

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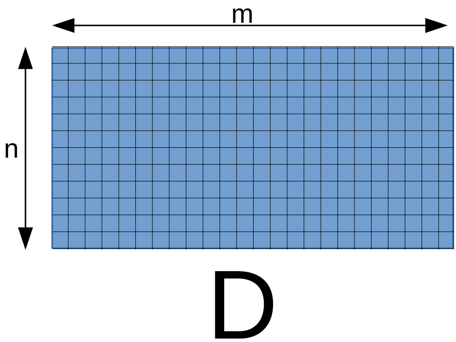
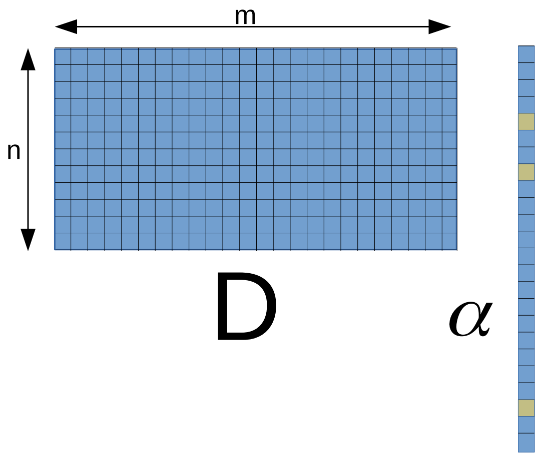
The Generative Model

$\mathbf{D} \in \mathbb{R}^{n \times m}$ is a dictionary with normalized columns.



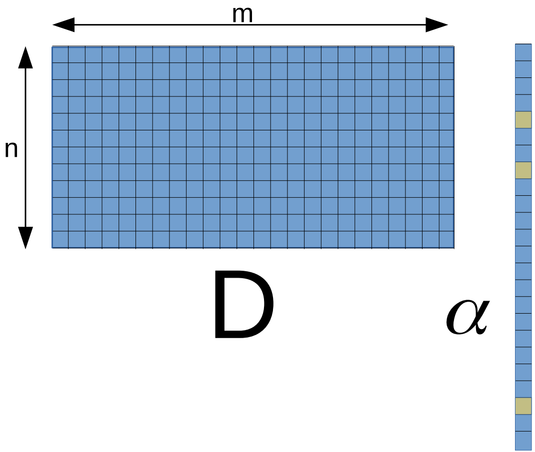
The Generative Model

Each element i in $\boldsymbol{\alpha}$ is non zero with probability $p_i \ll 1$.



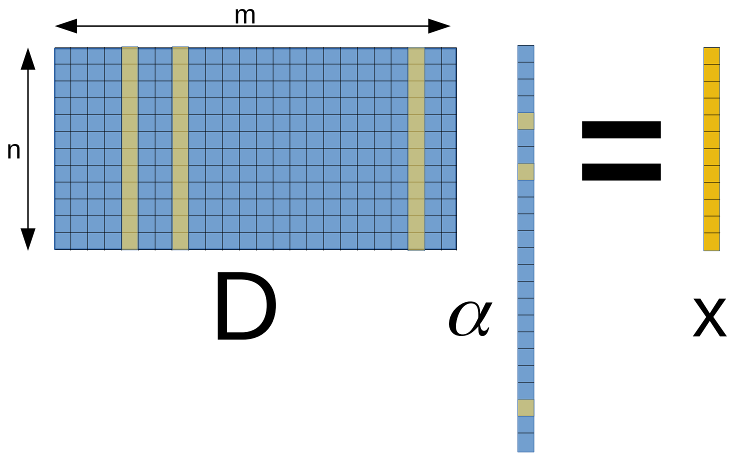
The Generative Model

The non-zero elements of the sparse representation, denoted by $\boldsymbol{\alpha}_s$, are sampled from a Gaussian distribution $\boldsymbol{\alpha}_s | s \sim \mathcal{N}\left(\mathbf{0}, \sigma_{\boldsymbol{\alpha}}^2 \mathbf{I}_{|s|}\right)$.



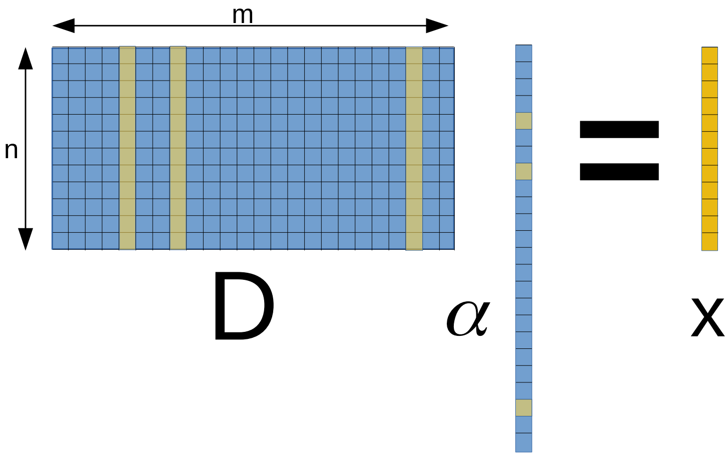
The Generative Model

The product $\mathbf{D}\alpha$ leads to a signal $\mathbf{x}$.



The Generative Model

We are given noisy measurements $\mathbf{y} = \mathbf{D}\boldsymbol{\alpha} + \boldsymbol{\nu}$, where $\boldsymbol{\nu}$ is a white Gaussian noise $\boldsymbol{\nu} \sim \mathcal{N}\left(\mathbf{0}, \sigma_{\boldsymbol{\nu}}^2 \mathbf{I}_n\right)$.



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  - $\boldsymbol{\alpha}_s|\mathbf{y}, s$ is Gaussian: $\boldsymbol{\alpha}_s|\mathbf{y}, s \sim \mathcal{N}\left(\frac{1}{\sigma_\nu^2} \mathbf{Q}_s^{-1} \mathbf{D}_s^T \mathbf{y}, \mathbf{Q}_s^{-1}\right)$.

$$\mathbf{C}_s = \sigma_\alpha^2 \mathbf{D}_s \mathbf{D}_s^T + \sigma_\nu^2 \mathbf{I}_n, \quad \mathbf{Q}_s = \frac{1}{\sigma_\alpha^2} \mathbf{I}_{|s|} + \frac{1}{\sigma_\nu} \mathbf{D}_s^T \mathbf{D}_s$$

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2. The Maximum A-posteriori Probability (MAP) estimator.
3. The Minimum Mean Square Error (MMSE) estimator.

The Bayesian Estimators

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Oracle Estimator

$$\widehat{\boldsymbol{\alpha}}_s^{\text{Oracle}} = \mathbb{E}\left\{\boldsymbol{\alpha}_s | s, \mathbf{y}\right\} = \frac{1}{\sigma_\nu^2}\mathbf{Q}_s^{-1}\mathbf{D}_s^T\mathbf{y}$$

The Bayesian Estimators

MAP Support Estimator

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$$s \in \{0,1\}^m$$

The Bayesian Estimators

MMSE Estimator

$$\widehat{\boldsymbol{\alpha}}_{\text{MMSE}} = \underset{\widehat{\boldsymbol{\alpha}}(\mathbf{y})}{\arg\min}\, \mathbb{E}\left\{ \left\| \widehat{\boldsymbol{\alpha}}\left(\mathbf{y}\right) - \boldsymbol{\alpha} \right\|_2^2 \middle| \mathbf{y} \right\}$$

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$$= \mathbb{E}_{s|\mathbf{y}} \left\{ \mathbb{E}_{\boldsymbol{\alpha}|\mathbf{y},s} \left\{ \boldsymbol{\alpha} | \mathbf{y}, s \right\} \right\}$$

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- Can we do better?

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- Empirically achieves better MSE than OMP even when these conditions are not met.

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$\Rightarrow$ **They are impractical for high dimensional signals**.

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Noise **improves** system performance?

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Dither

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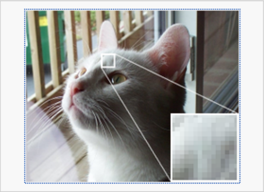

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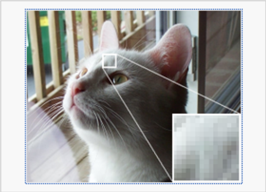

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$$\mathbf{y}$$

The Proposed Algorithm

$\mathbf{n}$

$\mathbf{y}$

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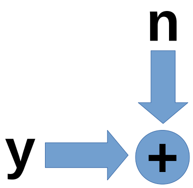

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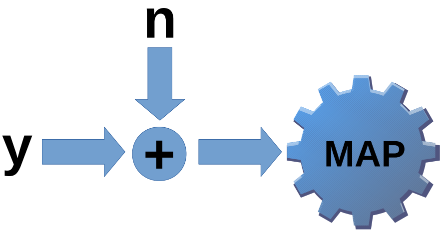

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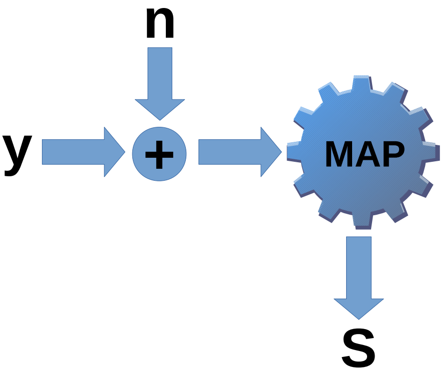

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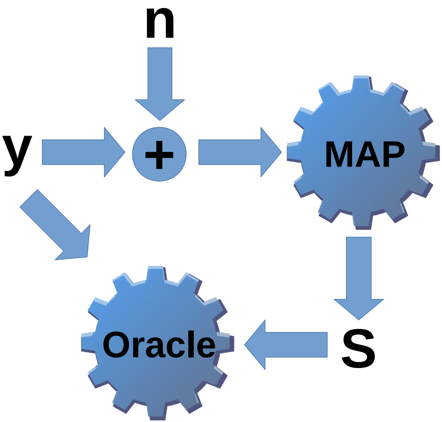

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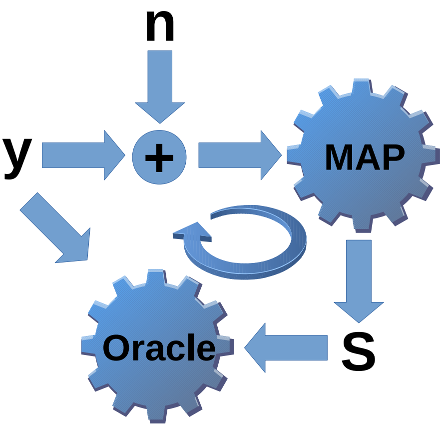

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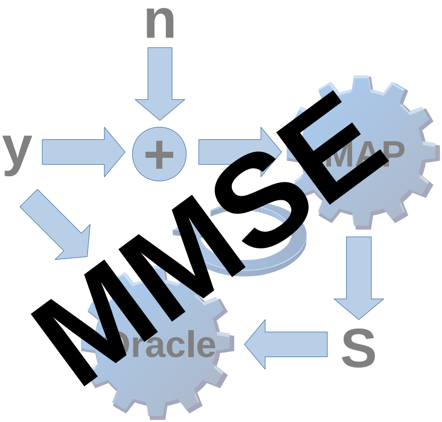

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Algorithm 1: Stochastic Resonance MMSE Approximation

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for $k \in 1 ... K$ **do**

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input : $\mathbf{y}$, $\mathbf{D}$, `PursuitMethod`, σ_n, K

output: $\widehat{\alpha}$

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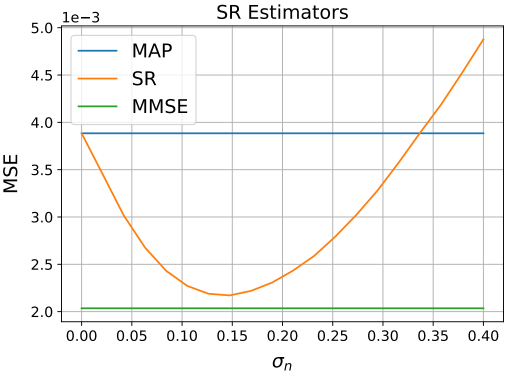

Outline

Unitary Case - Estimators

When $\mathbf{D}$ is a unitary matrix $\left(\mathbf{D}^T\mathbf{D} = \mathbf{I}\right)$, the oracle, MAP and MMSE estimators are element-wise shrinkage operators:

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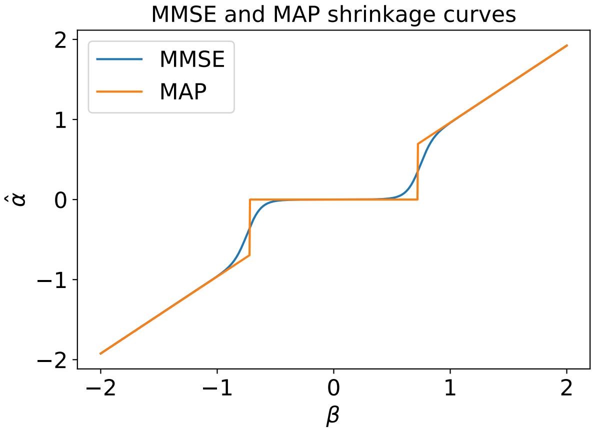

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$Q(x) = \int_x^\infty e^{-\frac{t^2}{2}}\, dt$

Unitary Case – Empirical Performance

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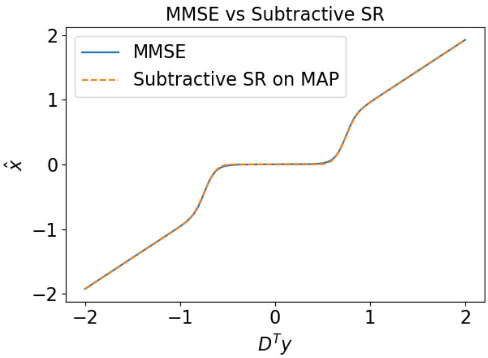

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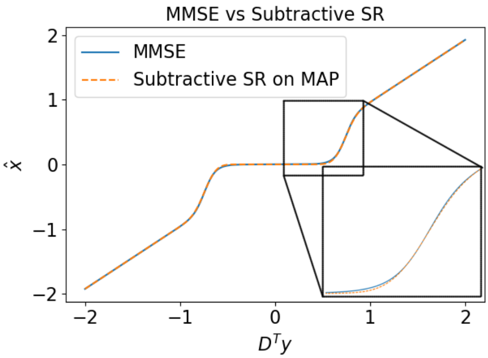

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Are the two the same?

Stochastic Resonance & MMSE

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More information in our paper.

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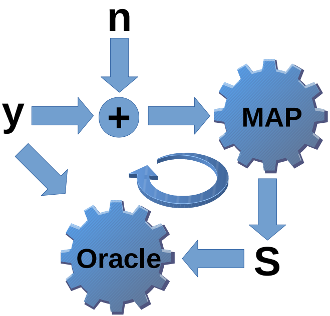

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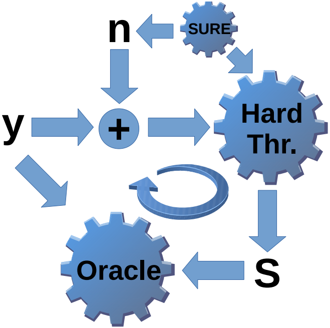

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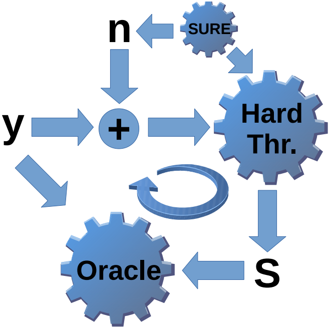

What about non-unitary cases?

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  $\Longrightarrow$ Asymptotically converges to the MMSE!

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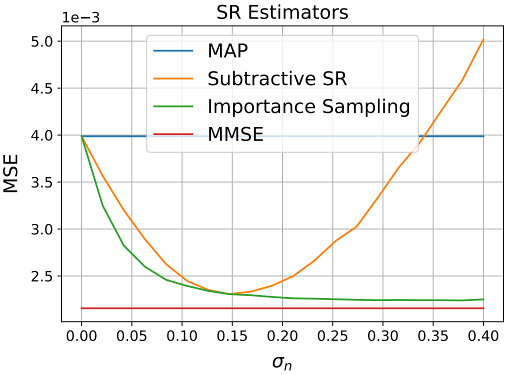

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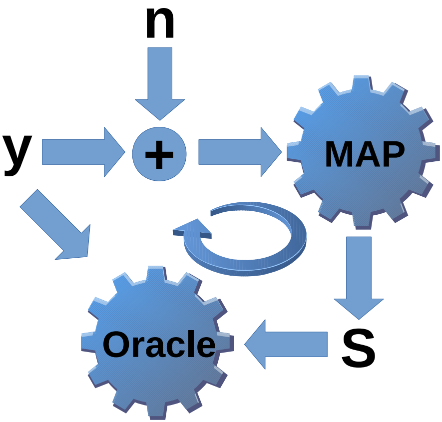

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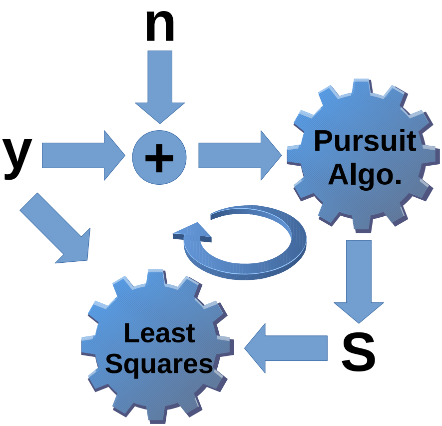

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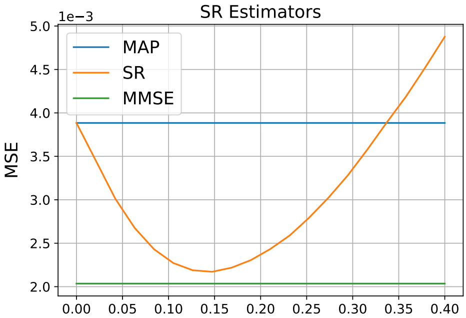

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Use bounded noise formulation for the pursuit:

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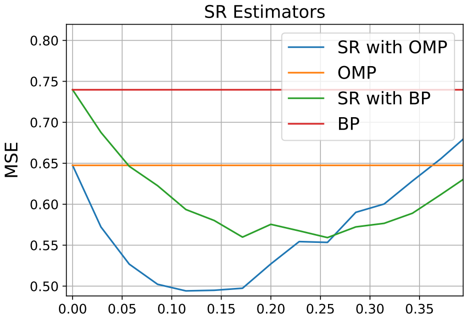

Outline

Image Denoising

Method Used:

[5] Sulam, Jeremias, et al, 2016. "Trainlets: Dictionary learning in high dimensions."
[6] Dai, Wei, and Olgica Milenkovic, 2009. "Subspace pursuit for compressive sensing signal reconstruction."

Image Denoising

Method Used:

- Dataset containing facial images.

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[6] Dai, Wei, and Olgica Milenkovic, 2009. "Subspace pursuit for compressive sensing signal reconstruction."

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- Use SR algorithm using the same SP configuration used in the previous step.

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Image Denoising – Results

Clean Image.

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Noisy image.
PSNR=16.1 dB.

Clean Image.

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Subspace Pursuit.
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Clean Image.

Image Denoising – Results

Noisy image. PSNR=16.1 dB.

Subspace Pursuit. PSNR=26.88 dB.

Stochastic Resonance. PSNR=28.76 dB.

Clean Image.

Image Denoising – Results

Noisy image.
PSNR=16.1 dB.

Subspace Pursuit.
PSNR=26.88 dB.

Stochastic Resonance.
PSNR=28.76 dB.

Clean Image.

$\sim$ 2dB better.

Outline

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- MMSE estimator approximation is attainable, even for large dimensions.

Thank You