Duality and optimality in multiobjective optimization

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Report

The aim of this work is to make some investigations concerning duality for multiobjective optimization problems. In order to do this we study first the duality for scalar optimization problems by using the conjugacy approach. This allows us to attach three different dual problems to a primal one. We examine the relations between the optimal objective values of the duals and verify, under some appropriate assumptions, the existence of strong duality. Closely related to the strong duality we derive the optimality conditions for each of these three duals.

By means of these considerations, we study the duality for two vector optimization problems, namely, a convex multiobjective problem with cone inequality constraints and a special fractional programming problem with linear inequality constraints. To each of these vector problems we associate a scalar primal and study the duality for it. The structure of both scalar duals give us an idea about how to construct a multiobjective dual. The existence of weak and strong duality is also shown.

We conclude our investigations by making an analysis over different duality concepts in multiobjective optimization. To a general multiobjective problem with cone inequality constraints we introduce other six different duals for which we prove weak as well as strong duality assertions. Afterwards, we derive some inclusion results for the image sets and, respectively, for the maximal elements sets of the image sets of these problems. Moreover, we show under which conditions they become identical.

A general scheme containing the relations between the six multiobjective duals and some other duals mentioned in the literature is derived.

Keywords

perturbation functions; conjugate duality; optimality conditions; location problems with demand sets; duality in multiobjective convex optimization; duality in multiobjective fractional programming; Pareto-efficient solutions and properly efficient solutions; weak, strong and converse duality; sets of maximal elements

Contents

Chapter 1

Introduction

Over the last fifty years the theory of duality in multiobjective optimization has experienced a very distinct development. Depending on the type of the objective functions and, especially, on the type of efficiency used, different duality concepts have been studied.

In this work we propose a new duality approach for general convex multiobjective problems with cone inequality constraints in finite dimensional spaces. The main and most fruitful idea for constructing the multiobjective dual is to establish first a dual problem to the scalarized primal. The suitable scalar dual problem is obtained by means of the conjugacy approach (cf. [19]), by using a special perturbation of the primal problem.

This opens us the possibility to take in the first part a deeper look on the usage of the conjugacy approach in the theory of duality for scalar optimization problems.

We conclude the thesis by generalizing the new approach and relating it to some other theories of duality encountered in previous works.

1.1 An overview on the literature dealing with duality in multiobjective optimization

The first results concerning duality in vector optimization were obtained by Gale, Kuhn and Tucker [24] in 1951. They established some theorems of duality in multiple objective linear programming, namely, for linear programming problems with a matrix-valued linear objective function. Further theories of duality in the linear case have been developed by Kornbluth [45], Rödder [63] and Isermann [36], [37].

A first description of the relations between the duality concepts of Gale, Kuhn and Tucker, Isermann and Kornbluth was given by Isermann in [38]. In [39] Ivanov and Nehse did also investigate the relations between some duality concepts in the linear multiobjective optimization, in fact, those of Gale, Kuhn and Tucker [24], Schönefeld [68], Isermann [36], [37], Rödder [63] and Fiala [22].

Concerning the duality for linear vector optimization problems, let us also mention an alternative approach recently introduced by Galperin and Jimenez Guerra [27]. It bases on the very controversial balance set method described by Galperin in [25] (see also the papers of Ehrgott, Hamacher, Klamroth, Nickel, Schöbel and Wiecek [18] and Galperin [26]) and proves that to an unique optimal primal vector the non-scalarized dual problem presents a cluster of corresponding optimal dual vectors.

For non-linear vector optimization problems we notice that the development of the theories of duality followed on many different directions. We will enumerate here some of them alongside their most representative papers.

Let us start by emphasizing the paper of Tanino and Sawaragi [74], where the authors examine the duality for vector optimization problems using the concept of conjugate maps. They extended the theory which was fully developed in scalar optimization by Rockafellar [62] to the case of multiobjective optimization, by introducing the new concepts of conjugate map and subgradient for a vector-valued map and, otherwise, for a set-valued map. This encouraged many authors to introduce different conjugate duality theories for set-valued optimization problems, which can be seen as generalizations of the vector optimization problems. Among the contributions on this field we remind the papers of Brumelle [12], Kawasaki [44], for problems in finite dimensional spaces, Postolică [60], [61], Tanino [73], Corley [14] and Song [71] for problems in general partially ordered topological vector spaces.

Another very important approach in the theory of duality for convex vector optimization problems in general partially ordered topological vector spaces has been introduced in the beginning of the eighties by Jahn [40] (see also Jahn [41] and Nehse and Göpfert [30]). It generalizes the concept of Schönefeld [68] by using the duality theory described by Van Slyke and Wets in [76]. For linear vector optimization problems, Isermann's duality [37] can be obtained as a particular case of this approach. We want also to mention here another extension of the duality theory of Van Slyke and Wets [76] for vector optimization problems, which has been considered by Nieuwenhuis in [58].

In finite dimensional spaces, among the most important contributions to the theory of duality we praise two approaches introduced by Nakayama in [54] (see also Sawaragi, Nakayama and Tanino [65] and Nakayama [55]). The first one bases on the theory presented by Tanino and Sawaragi in [75] which uses the so-called vector-valued Lagrangian functions. Besides convexity assumptions for the sets and the functions involved in the formulation of the primal problem, the authors impose the fulfilment of some compactness and, respectively, continuity assumptions. Because of the fact that just convexity assumptions are imposed, the second approach described by Nakayama in [54] is more general. Moreover, it turns out to be another generalization of the duality concept of Isermann from the linear case. For both approaches in [54], some geometric considerations have been given. Two other important contributions in this direction, which also use the vector-valued Lagrangian functions, are the papers of Luc [48], [49].

The last two duality concepts we recall here concern the multiobjective optimization problems in finite dimensional spaces, the inequality constraints being defined by the use of the non-negative orthant as ordering cone. They extend the results of Wolfe [94] and Mond and Weir [52] for scalar convex programs to vector programs. Weir had first introduced in [90] these duals in the differentiable case and, then, Weir and Mond (in [92], [93] and together with Egudo in [17]) have weakened the initial assumptions by formulating and proving the duality also in the non-differentiable case, under generalized convexity assumptions and without requiring any constraint qualification.

1.2 A description of the contents

In this section we will give a description of how is this work organized.

Chapter 2 is devoted to the study of the theory of conjugate duality in scalar optimization. We begin by giving a short description of this technique and then we adapt it to an optimization problem with cone inequality constraints given in a finite dimensional space. To this problem we associate three conjugate dual problems, two of them proving to be the well-known Lagrange and Fenchel dual problems. This approach has the property that the so-called "weak duality" always holds, namely, the optimal objective value of the primal problem is greater than or equal to the optimal objective value of the dual problem. Concerning these three duals we establish in the general case ordering relations between their optimal objective values.

Moreover, we verify under appropriate assumptions some equality relations between the optimal objective values and prove that these assumptions guarantee the so-called "strong duality". As usual, by strong duality we suppose that the optimal objective values of the primal and the dual problems coincide and that the dual problem has an optimal solution. In order to achieve strong duality, we require some convexity assumptions of the sets and functions involved, and some regularity conditions called "constraint qualifications". On the other hand, we also show how it is possible to weaken these assumptions in a way that the equality between the optimal objective values of the three dual problems and, otherwise, the above mentioned strong duality results still hold.

This part can also be seen as a contribution to a subject proposed by Magnanti in [50], regarding the connections between the Lagrange and Fenchel duality concepts. In order to complete our investigations we establish necessary and sufficient optimality conditions for the primal and dual problems, closely connected to the strong duality.

In the second part of the chapter we deal in a general normed space with an optimization problem, the objective function being a composite of a convex and componentwise increasing function with a vector convex function. Using again the conjugacy approach, we construct a dual problem to it, prove the existence of strong duality and derive the optimality conditions. Using the general result we introduce then a dual problem and the optimality conditions for a single facility location problem in which the existing facilities are represented by sets of points. This part of the thesis was motivated by the paper of Nickel, Puerto and Rodriguez - Chia [57], where the authors give a geometrical characterization of the set of optimal solutions. The classical Weber problem and minmax problem with demand sets are studied as particular cases of the general one.

In chapter 3 we draw our attention to the duality for vector optimization problems in finite dimensional spaces. The chapter contains two different parts, the first one devoted to the duality for a general convex multiobjective problem with cone inequality constraints and the second one devoted to a particular multiobjective fractional problem with linear inequality constraints. In both cases the ordering cone in the objective space is the non-negative orthant.

The general convex multiobjective problem with cone inequality constraints has the following formulation

$$(P) \quad \underset{x \in \mathcal{A}}{\text{v-min}}\, f(x),$$

$$\mathcal{A} = \left\{ x \in \mathbb{R}^n : g(x) = (g_1(x), \dots, g_k(x))^T \underset{K}{\leqq} 0 \right\},$$

where $f(x) = (f_1(x), \dots, f_m(x))^T$, $f_i : \mathbb{R}^n \to \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$, $i = 1, ..., m$, are proper functions, $g_j : \mathbb{R}^n \to \mathbb{R}$, $j = 1, ..., k$, and $K \subseteq \mathbb{R}^k$ is assumed to be a convex closed cone with $int(K) \neq \emptyset$, defining a partial ordering according to $x_2 \underset{K}{\leqq} x_1$ if and only if $x_1 - x_2 \in K$.

Our aim is to present a new duality approach for (P), the vector objective functions of the dual problem being represented in closed form by conjugate functions of the primal objective functions and of the functions describing the constraints. To (P) we associate a scalar problem (P^λ) for which we construct, using the conjugacy approach described in chapter 2, a dual problem, (D^λ). We show the existence of strong duality and derive the optimality conditions, which are used later to obtain duality assertions regarding the original and dual multiobjective problem. The structure of the scalar dual (D^λ) is formulated in terms of conjugate functions and gives us an idea about how to construct a multiobjective dual (D) to (P). The existence of weak and, under certain conditions, of strong duality between (P) and (D) is shown. We notice that these concepts represent an extension of the concepts of weak and strong duality from scalar optimization to the multiobjective case.

Afterwards, we show that this duality approach generalizes our former investigations referring duality for vector optimization problems with convex objective functions and linear inequality constraints (cf. WANKA AND BOŢ [83], [84]). The duality for multiobjective problems with convex objective functions and positive semidefinite constraints is also derived, as a particular case of the general theory developed in this first part.

The multiobjective problem considered in the second part of the chapter has linear inequality constraints and the objective functions are ratios

$$(P_r) \quad \operatorname*{v-min}_{x\in\mathcal{A}_r} \left(\frac{f_1^2(x)}{g_1(x)}, \dots, \frac{f_m^2(x)}{g_m(x)}\right)^T,$$

$$\mathcal{A}_r = \left\{ x \in \mathbb{R}^n : Cx \underset{\mathbb{R}^l_+}{\leqq} b \right\}.$$

C is a $l \times n$ matrix with real entries, the functions f_i and $g_i, i = 1, \dots, m$, mapping from $\mathbb{R}^n$ into $\mathbb{R}$, are assumed to be convex and concave, respectively, such that for all $x \in \mathcal{A}_r$ and $i = 1, ..., m$, $f_i(x) \geq 0$ and $g_i(x) > 0$ are fulfilled.

In order to formulate a dual for (P_r) we study first the duality for a scalar problem (P_r^λ) obtained from the multiobjective primal via linear scalarization. Duality considerations for such kind of problems had also been published by SCOTT AND JEFFERSON [69], by using the duality in geometric programming. Unlike [69], we use again the conjugacy approach. This allows us to construct a scalar dual problem (D_r^λ), which turns out to have a form adapted for generating in a natural way a multiobjective dual (D_r). Moreover, by use of the optimality conditions, we can prove the existence of weak and strong duality. We conclude this second part by particularizing the problem to the case of quadratic-linear fractional programming problems.

The aim of the fourth chapter is to investigate the relations between different dual problems in the theory of vector optimization. As a primal problem we consider the multiobjective problem (P) introduced in the first part of chapter 3, to which we associate again a scalar problem (P^λ). We introduce to (P^λ) by the same scheme as used in the second chapter three scalar conjugate duals. These are then the starting point for formulating six different multiobjective duals to (P), for which we prove the existence of weak and strong duality. Between the six duals one can recognize a generalization of the dual introduced in chapter 3 and, on the other hand, the dual presented by JAHN in [40] and [41], here in the finite dimensional case.

For the multiobjective duals we derive some inclusion results between the image sets of the objective functions on the admissible sets and between their maximal elements sets, respectively. By giving some counter-examples we show that these sets are not always equal. Otherwise, we show under which conditions they become identical.

A complete analysis of the duals introduced here, which also includes a comparison with the duals of NAKAYAMA (cf. [54], [55]), WOLFE (cf. [90], [93]) and WEIR AND MOND (cf. [90], [92]) is available in the last part of the chapter.

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Chapter 2

Conjugate duality in scalar optimization

One of the most fruitful theories of duality in convex optimization bases on the concept of conjugate functions. This concept is due to FENCHEL [21] and ROCKAFELLAR [62] in the finite dimensional case and was further developed by MOREAU [53] to cover the case of infinite dimensions. In their book, EKELAND AND TEMAM [19] have presented a very detailed description of this theory. Given an optimization problem, they embed it in a family of perturbed problems and, using conjugate functions, they associate a dual problem to it.

In the first part of the chapter we adapt this very flexible theory to an optimization problem with cone inequality constraints in a finite dimensional space. For it we consider three different conjugate dual problems: the well-known Lagrange and Fenchel dual problems (denoted by (D_L^s) and (D_F^s), respectively,) and a "combination" of the above two, which we call the Fenchel-Lagrange dual problem (denoted by (D_{FL}^s)). It is relatively easy to show that in each case the so-called "weak duality" holds, namely, the optimal objective value $inf(P^s)$ of the primal problem (P^s) is always greater than or equal to each of the optimal objective values of the considered dual problems. Moreover, among the optimal objective values of these three dual problems, $sup(D_{FL}^s)$ is the smallest. By some counter-examples we show that, in general, an ordering between $sup(D_L^s)$ and $sup(D_F^s)$ cannot be established.

For the three dual problems we also verify, under some appropriate assumptions, the existence of equality relations between their optimal objective values. We prove that these assumptions guarantee the so-called "strong duality", in fact, that the optimal objective values of the primal and the dual problems coincide and that the dual problems have optimal solutions. By means of strong duality some necessary and sufficient optimality conditions for each of these problems are established.

In the last section of the chapter, in order to show that the conjugate duality theory can be adapted to a variety of situations, we consider in a general normed space the optimization problem with the objective function being a composite of a convex and componentwise increasing function with a vector convex function. Perturbing the primal problem in an appropriate way we obtain, by means of the conjugate approach, a dual problem to it. The existence of strong duality is shown and the optimality conditions are derived. Using the general result we introduce then the dual problem and the optimality conditions for the single facility location problem in a general normed space in which the existing facilities are represented by sets of points. This approach was motivated by the paper of NICKEL, PUERTO AND RODRIGUEZ-CHIA [57]. The classical Weber problem and minmax problem with demand sets are studied as particular instances.

2.1 The constrained optimization problem and its conjugate duals

2.1.1 Problem formulation

Let $X \subseteq \mathbb{R}^n$ be a nonempty set and $K \subseteq \mathbb{R}^k$ a nonempty closed convex cone with $int(K) \neq \emptyset$. The set $K^* := \{k^* \in \mathbb{R}^k : k^{*T}k \geq 0, \forall k \in K\}$ is the dual cone of K. Consider the partial ordering "$\underset{K}{\leqq}$" induced by K in $\mathbb{R}^k$, namely, for $y, z \in \mathbb{R}^k$ we have that $y \underset{K}{\leqq} z$, iff $z - y \in K$. Let be $f : \mathbb{R}^n \to \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ and $g = (g_1, \ldots, g_k)^T : \mathbb{R}^n \to \mathbb{R}^k$. The optimization problem we investigate in this paper is

$$(P^s) \quad \inf_{x \in G} f(x),$$

where

$$G = \left\{ x \in X : g(x) \underset{K}{\leqq} 0 \right\}.$$

In the following we suppose that the feasible set G is nonempty. Assume further that $dom(f) = X$, where $dom(f) := \{x \in \mathbb{R}^n : f(x) < +\infty\}$.

The problem (P^s) is said to be the primal problem and its optimal objective value is denoted by $inf(P^s)$.

Definition 2.1 *An element $\bar{x} \in G$ is said to be an optimal solution for (P^s) if $f(\bar{x}) = inf(P^s)$.*

The aim of this section is to construct different dual problems to (P^s). To do so, we use an approach described by Ekeland and Temam in [19], which is based on the theory of conjugate functions. Therefore, let us first consider the general optimization problem without constraints

$$(PG^s) \quad \inf_{x \in \mathbb{R}^n} F(x),$$

with F a mapping from $\mathbb{R}^n$ into $\overline{\mathbb{R}}$.

Definition 2.2 *The function $F^* : \mathbb{R}^n \to \overline{\mathbb{R}}$, defined by*

$$F^*(p^*) = \sup_{x \in \mathbb{R}^n} \left\{ p^{*T}x - F(x) \right\},$$

is called the conjugate function of F.

Remark 2.1 By the assumptions we made for f, we have

$$f^*(p^*) = \sup_{x \in \mathbb{R}^n} \left\{ p^{*T}x - f(x) \right\} = \sup_{x \in X} \left\{ p^{*T}x - f(x) \right\}.$$

The approach in [19] is based on the construction of a so-called perturbation function $\Phi : \mathbb{R}^n \times \mathbb{R}^m \to \overline{\mathbb{R}}$ with the property that $\Phi(x, 0) = F(x)$ for each $x \in \mathbb{R}^n$. Here, $\mathbb{R}^m$ is the space of the perturbation variables. For each $p \in \mathbb{R}^m$ we obtain then a new optimization problem

$$(PG^s_p) \quad \inf_{x \in \mathbb{R}^n} \Phi(x, p).$$

For $p \in \mathbb{R}^m$, the problem (PG^s_p) is called the perturbed problem of (PG^s).

By Definition 2.2, the conjugate of Φ is the function $\Phi^* : \mathbb{R}^n \times \mathbb{R}^m \to \overline{\mathbb{R}}$,

$$\begin{aligned}\Phi^*(x^*, p^*) &= \sup_{\substack{x\in\mathbb{R}^n,\\ p\in\mathbb{R}^m}} \left\{(x^*, p^*)^T(x,p) - \Phi(x,p)\right\} \\ &= \sup_{\substack{x\in\mathbb{R}^n,\\ p\in\mathbb{R}^m}} \left\{x^{*T}x + p^{*T}p - \Phi(x,p)\right\}. \end{aligned} \tag{2. 1}$$

Now we can define the following optimization problem

$$(DG^s) \quad \sup_{p^*\in\mathbb{R}^m} \{-\Phi^*(0, p^*)\}.$$

The problem (DG^s) is called the dual problem to (PG^s) and its optimal objective value is denoted by $sup(DG^s)$.

This approach has an important property: between the primal and the dual problem weak duality always holds. The following theorem proves this fact.

Theorem 2.1 *([19]) The relation*

$$-\infty \leq sup(DG^s) \leq inf(PG^s) \leq +\infty \tag{2. 2}$$

always holds.

Proof. Let $p^* \in \mathbb{R}^m$. From (2. 1), we obtain

$$\begin{aligned}\Phi^*(0, p^*) &= \sup_{\substack{x\in\mathbb{R}^n,\\ p\in\mathbb{R}^m}} \{0^T x + p^{*T}p - \Phi(x,p)\} \\ &= \sup_{\substack{x\in\mathbb{R}^n,\\ p\in\mathbb{R}^m}} \{p^{*T}p - \Phi(x,p)\} \\ &\geq \sup_{x\in\mathbb{R}^n} \{p^{*T}0 - \Phi(x,0)\} = \sup_{x\in\mathbb{R}^n} \{-\Phi(x,0)\}.\end{aligned}$$

This means that, for each $p^* \in \mathbb{R}^m$ and $x \in \mathbb{R}^n$, it holds

$$-\Phi^*(0, p^*) \leq \Phi(x, 0) = F(x),$$

which implies that $sup(DG^s) \leq inf(PG^s)$. □

Our next aim is to show how can we apply this approach to the constrained optimization problem (P^s). Therefore, let $F : \mathbb{R}^n \to \overline{\mathbb{R}}$ be the function given by

$$F(x) = \begin{cases} f(x), & \text{if} \quad x \in G, \\ +\infty, & \text{otherwise.} \end{cases}$$

The primal problem (P^s) is then equivalent to

$$(PG^s) \quad \inf_{x\in\mathbb{R}^n} F(x),$$

and, since the perturbation function $\Phi : \mathbb{R}^n \times \mathbb{R}^m \to \overline{\mathbb{R}}$ satisfies $\Phi(x, 0) = F(x)$ for each $x \in \mathbb{R}^n$, we obtain that

$$\Phi(x, 0) = f(x), \quad \forall x \in G \tag{2. 3}$$

and

$$\Phi(x, 0) = +\infty, \quad \forall x \in \mathbb{R}^n \setminus G. \tag{2. 4}$$

In the following we study, for special choices of the perturbation function, some dual problems to (P^s).

2.1.2 The Lagrange dual problem

For the beginning, let the function $\Phi_L : \mathbb{R}^n \times \mathbb{R}^k \to \overline{\mathbb{R}}$ be defined by

$$\Phi_L(x,q) = \begin{cases} f(x), & \text{if} \quad x \in X, \quad g(x) \underset{K}{\leqq} q, \\ +\infty, & \text{otherwise,} \end{cases}$$

with the perturbation variable $q \in \mathbb{R}^k$. It is obvious that the relations (2. 3) and (2. 4) are fulfilled. For the conjugate of Φ_L we have

$$\begin{aligned} \Phi_L^*(x^*,q^*) &= \sup_{\substack{x\in\mathbb{R}^n,\\ q\in\mathbb{R}^k}} \left\{ x^{*T}x + q^{*T}q - \Phi_L(x,q) \right\} \\ &= \sup_{\substack{x\in X, q\in\mathbb{R}^k,\\ g(x) \underset{K}{\leqq} q}} \left\{ x^{*T}x + q^{*T}q - f(x) \right\}. \end{aligned}$$

In order to calculate this expression we introduce the variable s instead of q, by $s = q - g(x) \in K$. This implies

$$\begin{aligned} \Phi_L^*(x^*,q^*) &= \sup_{x\in X, s\in K} \left\{ x^{*T}x + q^{*T}[s + g(x)] - f(x) \right\} \\ &= \sup_{x\in X} \left\{ x^{*T}x + q^{*T}g(x) - f(x) \right\} + \sup_{s\in K} q^{*T}s \\ &= \begin{cases} \sup\limits_{x\in X} \left\{ x^{*T}x + q^{*T}g(x) - f(x) \right\}, & \text{if} \quad q^* \in -K^*, \\ +\infty, & \text{otherwise.} \end{cases} \end{aligned}$$

As we have seen, the dual of (P^s) obtained by the perturbation function Φ_L is

$$(D_L^s) \quad \sup_{q^*\in\mathbb{R}^k} \{-\Phi_L^*(0,q^*)\},$$

and, since

$$\sup_{q^*\in -K^*} \left\{ -\sup_{x\in X} [q^{*T}g(x) - f(x)] \right\} = \sup_{q^*\in -K^*} \left\{ \inf_{x\in X} [-q^{*T}g(x) + f(x)] \right\},$$

the dual has the following form

$$(D_L^s) \quad \sup_{q^* \underset{K^*}{\geqq} 0} \inf_{x\in X} \left[f(x) + q^{*T}g(x) \right]. \tag{2. 5}$$

The problem (D_L^s) is actually the well-known Lagrange dual problem. Its optimal objective value is denoted by $sup(D_L^s)$ and Theorem 2.1 implies

$$sup(D_L^s) \leq inf(P^s). \tag{2. 6}$$

We are now interested to obtain dual problems for (P^s), different from the classical Lagrange problem.

2.1.3 The Fenchel dual problem

Let us consider the perturbation function $\Phi_F : \mathbb{R}^n \times \mathbb{R}^n \to \overline{\mathbb{R}}$ given by

$$\Phi_F(x,p) = \begin{cases} f(x+p), & \text{if} \quad x \in G, \\ +\infty, & \text{otherwise,} \end{cases}$$

with the perturbation variable $p \in \mathbb{R}^n$. The relations (2. 3) and (2. 4) are also fulfilled and it holds

$$\begin{aligned}\Phi_F^*(x^*, p^*) &= \sup_{\substack{x \in \mathbb{R}^n, \\ p \in \mathbb{R}^n}} \left\{ x^{*T}x + p^{*T}p - \Phi_F(x,p) \right\} \\ &= \sup_{\substack{x \in X, p \in \mathbb{R}^n, \\ g(x) \underset{K}{\leqq} 0}} \left\{ x^{*T}x + p^{*T}p - f(x+p) \right\}.\end{aligned}$$

Introducing a new variable $r = x + p \in \mathbb{R}^n$, we have

$$\begin{aligned}\Phi_F^*(x^*, p^*) &= \sup_{\substack{x \in X, r \in \mathbb{R}^n, \\ g(x) \underset{K}{\leqq} 0}} \left\{ x^{*T}x + p^{*T}(r - x) - f(r) \right\} \\ &= \sup_{r \in \mathbb{R}^n} \{ p^{*T}r - f(r) \} + \sup_{\substack{x \in X, \\ g(x) \underset{K}{\leqq} 0}} \{ (x^* - p^*)^T x \} \\ &= f^*(p^*) - \inf_{\substack{x \in X, \\ g(x) \underset{K}{\leqq} 0}} \left\{ (p^* - x^*)^T x \right\} = f^*(p^*) - \inf_{x \in G} \left\{ (p^* - x^*)^T x \right\}.\end{aligned}$$

Now the dual of (P^s)

$$(D_F^s) \quad \sup_{p^* \in \mathbb{R}^n} \{ -\Phi_F^*(0, p^*) \}$$

can be written in the form

$$(D_F^s) \quad \sup_{p^* \in \mathbb{R}^n} \left\{ -f^*(p^*) + \inf_{\substack{x \in X, \\ g(x) \underset{K}{\leqq} 0}} p^{*T}x \right\}.$$

Denoting by

$$\chi_G(x) = \begin{cases} 0, & \text{if} \quad x \in G, \\ +\infty, & \text{otherwise,} \end{cases}$$

the indicator function of the set G, we have that $\chi_G^*(-p^*) = -\inf_{x \in G} p^{*T}x$. The dual (D_F^s) becomes then

$$(D_F^s) \quad \sup_{p^* \in \mathbb{R}^n} \{ -f^*(p^*) - \chi_G^*(-p^*) \}. \tag{2. 7}$$

Let us call (D_F^s) the Fenchel dual problem and denote its optimal objective value by $sup(D_F^s)$. The weak duality

$$sup(D_F^s) \leq inf(P^s) \tag{2. 8}$$

is also fulfilled by Theorem 2.1.

2.1.4 The Fenchel-Lagrange dual problem

Another dual problem, different from (D_L^s) and (D_F^s), can be obtained considering the perturbation function as a combination of the functions Φ_L and Φ_F. Let this be defined by $\Phi_{FL} : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^k \to \overline{\mathbb{R}}$,

$$\Phi_{FL}(x, p, q) = \begin{cases} f(x+p), & \text{if} \quad x \in X, \quad g(x) \underset{K}{\leqq} q, \\ +\infty, & \text{otherwise,} \end{cases}$$

with the perturbation variables $p \in \mathbb{R}^n$ and $q \in \mathbb{R}^k$. Φ_{FL} satisfies the relations (2. 3) and (2. 4) and its conjugate is

$$\begin{aligned}\Phi^*_{FL}(x^*, p^*, q^*) &= \sup_{\substack{x\in\mathbb{R}^n,\\ p\in\mathbb{R}^n, q\in\mathbb{R}^k}} \left\{x^{*T}x + p^{*T}p + q^{*T}q - \Phi_{FL}(x,p,q)\right\} \\ &= \sup_{\substack{x\in X, g(x) \underset{K}{\leqq} q,\\ p\in\mathbb{R}^n, q\in\mathbb{R}^k}} \left\{x^{*T}x + p^{*T}p + q^{*T}q - f(x+p)\right\}.\end{aligned}$$

Like in the previous subsections, we introduce the new variables $r = x + p \in \mathbb{R}^n$ and $s = q - g(x) \in K$. Then we have

$$\begin{aligned}\Phi^*_{FL}(x^*, p^*, q^*) &= \sup_{\substack{r\in\mathbb{R}^n, s\in K,\\ x\in X}} \left\{x^{*T}x + p^{*T}(r-x) + q^{*T}[s+g(x)] - f(r)\right\} \\ &= \sup_{r\in\mathbb{R}^n} \left\{p^{*T}r - f(r)\right\} + \sup_{x\in X}\left\{(x^*-p^*)^T x + q^{*T}g(x)\right\} \\ &+ \sup_{s\in K} q^{*T}s.\end{aligned}$$

Computing the first supremum we get

$$\sup_{r\in\mathbb{R}^n} \left\{p^{*T}r - f(r)\right\} = f^*(p^*),$$

while for the last it holds

$$\sup_{s\in K} q^{*T}s = \begin{cases} 0, & \text{if } q^* \in -K^*, \\ +\infty, & \text{otherwise.}\end{cases}$$

In this case, the dual problem

$$(D^s_{FL}) \quad \sup_{\substack{p^*\in\mathbb{R}^n,\\ q^*\in\mathbb{R}^k}} \{-\Phi^*_{FL}(0, p^*, q^*)\}$$

becomes

$$(D^s_{FL}) \quad \sup_{\substack{p^*\in\mathbb{R}^n,\\ q^*\in -K^*}} \left\{-f^*(p^*) - \sup_{x\in X}[-p^{*T}x + q^{*T}g(x)]\right\}$$

or, equivalently,

$$(D^s_{FL}) \quad \sup_{\substack{p^*\in\mathbb{R}^n,\\ q^* \underset{K^*}{\geqq} 0}} \left\{-f^*(p^*) + \inf_{x\in X}[p^{*T}x + q^{*T}g(x)]\right\}. \tag{2. 9}$$

We will call (D^s_{FL}) the Fenchel-Lagrange dual problem and denote its optimal objective value by $sup(D^s_{FL})$. By Theorem 2.1, the weak duality, i.e.

$$sup(D^s_{FL}) \le inf(P^s) \tag{2. 10}$$

also holds.

2.2 The relations between the optimal objective values of the duals

2.2.1 The general case

As we have seen in the previous section, the optimal objective values of the dual problems (D^s_L), (D^s_F) and (D^s_{FL}) are less than or equal to the optimal objective value of the primal problem (P^s). This fact is true for the general case, without any special assumptions concerning the functions f and g or the set X. In the following we are going to prove some relations between the optimal objective values of the dual problems introduced so far, under the same general assumptions. The first one refers to the problems (D^s_L) and (D^s_{FL}).

Proposition 2.1 *The inequality $sup(D^s_L) \geq sup(D^s_{FL})$ holds.*

Proof. Let $q^* \underset{K^*}{\geqq} 0$ and $p^* \in \mathbb{R}^n$ be fixed. By the definition of the conjugate function, we have for each $x \in X$, the so-called inequality of Young (cf. [19])

$$f^*(p^*) \geq p^{*T}x - f(x),$$

or, equivalently,

$$f(x) \geq p^{*T}x - f^*(p^*).$$

Adding to both sides the term $q^{*T}g(x)$, we obtain for each $x \in X$,

$$f(x) + q^{*T}g(x) \geq -f^*(p^*) + p^{*T}x + q^{*T}g(x).$$

This means that for all $q^* \underset{K^*}{\geqq} 0$ and $p^* \in \mathbb{R}^n$, it holds

$$\inf_{x\in X}[f(x) + q^{*T}g(x)] \geq -f^*(p^*) + \inf_{x\in X}[p^{*T}x + q^{*T}g(x)]. \tag{2. 11}$$

We can calculate now the supremum over $p^* \in \mathbb{R}^n$ and $q^* \underset{K^*}{\geqq} 0$ and this implies

$$\sup_{q^* \underset{K^*}{\geqq} 0} \inf_{x\in X} \left[f(x) + q^{*T}g(x)\right] \geq \sup_{\substack{p^*\in\mathbb{R}^n, \\ q^* \underset{K^*}{\geqq} 0}} \left\{-f^*(p^*) + \inf_{x\in X}[p^{*T}x + q^{*T}g(x)]\right\}.$$

The last inequality is in fact $sup(D^s_L) \geq sup(D^s_{FL})$ and the proof is complete. □

Let us give now two examples which show that the inequality in Proposition 2.1 may be strict.

Example 2.1 Let be $K = \mathbb{R}_+$, $X = [0, +\infty) \subseteq \mathbb{R}$, $f : \mathbb{R} \to \overline{\mathbb{R}}$, $g : \mathbb{R} \to \mathbb{R}$, defined by

$$f(x) = \begin{cases} -x^2, & \text{if} \quad x \in X, \\ +\infty, & \text{otherwise,} \end{cases}$$

and

$$g(x) = x^2 - 1.$$

The optimal objective value of the Lagrange dual is

$$\begin{aligned} sup(D^s_L) &= \sup_{q^*\geq 0} \inf_{x\geq 0}[-x^2 + q^*(x^2 - 1)] \\ &= \sup_{q^*\geq 0} \inf_{x\geq 0}[(q^* - 1)x^2 - q^*] \\ &= \sup_{q^*\geq 1}(-q^*) = -1. \end{aligned}$$

For (D^s_{FL}) we have

$$\begin{aligned} sup(D^s_{FL}) &= \sup_{\substack{p^*\in\mathbb{R}, \\ q^*\geq 0}} \left\{-\sup_{x\geq 0}[p^*x + x^2] + \inf_{x\geq 0}[p^*x + q^*(x^2 - 1)]\right\} \\ &= \sup_{\substack{p^*\in\mathbb{R}, \\ q^*\geq 0}} \left\{-\infty + \inf_{x\geq 0}[p^*x + q^*(x^2 - 1)]\right\} \\ &= -\infty. \end{aligned}$$

It is obvious that between the optimal objective values of the Lagrange and Fenchel-Lagrange duals the strict inequality $sup(D_L^s) = -1 > -\infty = sup(D_{FL}^s)$ holds.

Example 2.2 Let be now $K = \mathbb{R}_+$, $X = [0, +\infty) \subseteq \mathbb{R}, f : \mathbb{R} \to \overline{\mathbb{R}}$, $g : \mathbb{R} \to \mathbb{R}$, defined by

$$f(x) = \begin{cases} x^2, & \text{if } \quad x \in X, \\ +\infty, & \text{otherwise,} \end{cases}$$

and

$$g(x) = 1 - x^2.$$

Then we get

$$\begin{aligned} sup(D_L^s) &= \sup_{q^* \geq 0} \inf_{x \geq 0} [x^2 + q^*(1 - x^2)] \\ &= \sup_{q^* \geq 0} \inf_{x \geq 0} [(1 - q^*)x^2 + q^*] \\ &= \sup_{0 \leq q^* \leq 1} (q^*) = 1 \end{aligned}$$

and

$$\begin{aligned} sup(D_{FL}^s) &= \sup_{\substack{p^* \in \mathbb{R}, \\ q^* \geq 0}} \left\{ - \sup_{x \geq 0} [p^* x - x^2] + \inf_{x \geq 0} [p^* x + q^*(1 - x^2)] \right\} \\ &= \sup_{\substack{p^* \geq 0, \\ q^* = 0}} \left\{ - \frac{(p^*)^2}{4} + \inf_{x \geq 0} [p^* x] \right\} \\ &= \sup_{p^* \geq 0} \left[- \frac{(p^*)^2}{4} \right] = 0. \end{aligned}$$

The strict inequality $sup(D_L^s) = 1 > 0 = sup(D_{FL}^s)$ is again fulfilled.

The next result states an inequality between the optimal objective values of the problems (D_F^s) and (D_{FL}^s).

Proposition 2.2 *The inequality* $sup(D_F^s) \geq sup(D_{FL}^s)$ *holds.*

Proof. Let $p^* \in \mathbb{R}^n$ be fixed. For each $q^* \underset{K^*}{\geqq} 0$ we have

$$\inf_{x \in X} \left[p^{*T} x + q^{*T} g(x) \right] \leq \inf_{\substack{x \in X, \\ g(x) \underset{K}{\leqq} 0}} \left[p^{*T} x + q^{*T} g(x) \right] \leq \inf_{\substack{x \in X, \\ g(x) \underset{K}{\leqq} 0}} p^{*T} x.$$

Then, for every $p^* \in \mathbb{R}^n$,

$$\sup_{q^* \underset{K^*}{\geqq} 0} \inf_{x \in X} \left[p^{*T} x + q^{*T} g(x) \right] \leq \inf_{\substack{x \in X \\ g(x) \underset{K}{\leqq} 0}} p^{*T} x = -\chi_G^*(-p^*). \tag{2. 12}$$

By adding $-f^*(p^*)$ to both sides one obtains

$$-f^*(p^*) + \sup_{q^* \underset{K^*}{\geqq} 0} \inf_{x \in X} \left[p^{*T} x + q^{*T} g(x) \right] \leq -f^*(p^*) - \chi_G^*(-p^*), \ \forall p^* \in \mathbb{R}^n.$$

This last inequality implies

$$\sup_{\substack{p^* \in \mathbb{R}^n, \\ q^* \underset{K^*}{\geqq} 0}} \left\{ -f^*(p^*) + \inf_{x \in X} [p^{*T} x + q^{*T} g(x)] \right\} \leq \sup_{p^* \in \mathbb{R}^n} \left\{ -f^*(p^*) - \chi_G^*(-p^*) \right\},$$

or, equivalently, $sup(D_F^s) \geq sup(D_{FL}^s)$. $\square$

As for Proposition 2.1, we consider two examples which show that the inequality $sup(D_F^s) \geq sup(D_{FL}^s)$ may be strict.

Example 2.3 For $K = \mathbb{R}_+$, $X = [0, +\infty) \subseteq \mathbb{R}$, let be $f : \mathbb{R} \to \overline{\mathbb{R}}$, $g : \mathbb{R} \to \mathbb{R}$, defined by

$$f(x) = \begin{cases} x, & \text{if} \quad x \in X, \\ +\infty, & \text{otherwise,} \end{cases}$$

and

$$g(x) = 1 - x^2.$$

For the Fenchel dual problem we have

$$\begin{aligned} sup(D_F^s) &= \sup_{p^* \in \mathbb{R}} \left\{ -\sup_{x \geq 0}[p^*x - x] + \inf_{\substack{x \geq 0 \\ 1-x^2 \leq 0}} p^*x \right\} \\ &= \sup_{p^* \in \mathbb{R}} \left[\inf_{x \geq 0} (1 - p^*)x + \inf_{x \geq 1} p^*x \right] \\ &= \sup_{0 \leq p^* \leq 1} (p^*) = 1. \end{aligned}$$

But, the optimal objective value of the Fenchel-Lagrange dual is

$$\begin{aligned} sup(D_{FL}^s) &= \sup_{\substack{p^* \in \mathbb{R}, \\ q^* \geq 0}} \left\{ -\sup_{x \geq 0}[p^*x - x] + \inf_{x \geq 0}[p^*x + q^*(1 - x^2)] \right\} \\ &= \sup_{\substack{p^* \geq 0, \\ q^* = 0}} \left[\inf_{x \geq 0} (1 - p^*)x \right] = \sup_{0 \leq p^* \leq 1} 0 = 0, \end{aligned}$$

and, from here, it follows that $sup(D_F^s) = 1 > 0 = sup(D_{FL}^s)$.

Example 2.4 The following example has been presented in [20], but only regarding the Lagrange dual. Let be $K = \mathbb{R}_+$,

$$X = \left\{ x = (x_1, x_2)^T \in \mathbb{R}^2 : 0 \leq x_1 \leq 2, \begin{array}{ll} 3 \leq x_2 \leq 4 & \text{for} \quad x_1 = 0 \\ 1 < x_2 \leq 4 & \text{for} \quad x_1 > 0 \end{array} \right\}$$

a subset of $\mathbb{R}^2$ and the functions $f : \mathbb{R}^2 \to \overline{\mathbb{R}}$, $g : \mathbb{R}^2 \to \mathbb{R}$, defined by

$$f(x_1, x_2) = \begin{cases} x_2, & \text{if} \quad x = (x_1, x_2)^T \in X, \\ +\infty, & \text{otherwise,} \end{cases}$$

and

$$g(x_1, x_2) = x_1.$$

A straightforward calculation shows that the optimal objective value of the Fenchel dual is

$$\begin{aligned} sup(D_F^s) &= \sup_{(p_1^*, p_2^*) \in \mathbb{R} \times \mathbb{R}} \left\{ -f^*(p_1^*, p_2^*) + \inf_{\substack{(x_1, x_2)^T \in X, \\ x_1 \leq 0}} (p_1^*x_1 + p_2^*x_2) \right\} \\ &= \sup_{(p_1^*, p_2^*) \in \mathbb{R} \times \mathbb{R}} \left\{ - \sup_{(x_1, x_2)^T \in X} [p_1^*x_1 + p_2^*x_2 - x_2] + \inf_{3 \leq x_2 \leq 4} p_2^*x_2 \right\} \\ &= 3. \end{aligned}$$

On the other hand, for the optimal objective value of the Fenchel-Lagrange dual we have

$$\begin{aligned} sup(D^s_{FL}) &= \sup_{\substack{p_1^*\in\mathbb{R}, p_2^*\in\mathbb{R},\\ q^*\geq 0}} \left\{ -f^*(p_1^*, p_2^*) + \inf_{x\in X} [(p_1^* + q^*)x + p_2^* x_2] \right\} \\ &= 1. \end{aligned}$$

So, the strict inequality $sup(D^s_F) = 3 > 1 = sup(D^s_{FL})$ is verified.

Remark 2.2 Let us notice that, in general, an ordering between the optimal objective values of the problems (D^s_L) and (D^s_F) cannot be established. In Example 2.1 one can obtain that $sup(D^s_F) = -\infty$, which means that $sup(D^s_L) = -1 > -\infty = sup(D^s_F)$. Otherwise, in Example 2.4 we have $sup(D^s_L) = 1$ (see also [20]), and in this situation the inverse inequality $sup(D^s_F) = 3 > 1 = sup(D^s_L)$ holds.

2.2.2 The equivalence of the dual problems (D^s_L) and (D^s_{FL})

In this subsection we prove that in the case of a convex programming problem the optimal objective values of the Lagrange dual problem (D^s_L) and the Fenchel-Lagrange dual problem (D^s_{FL}) are equal. To do this, we define first the following notion.

Definition 2.3 *Let be $X \subseteq \mathbb{R}^n$ a nonempty and convex set. The function $g : \mathbb{R}^n \to \mathbb{R}^k$ is said to be convex on X relative to the cone K if $\forall x, y \in X$, $\forall \lambda \in [0, 1]$,*

$$\lambda g(x) + (1 - \lambda)g(y) - g(\lambda x + (1 - \lambda)y) \in K.$$

If the function $g : \mathbb{R}^n \to \mathbb{R}^k$ is convex on $\mathbb{R}^n$ relative to the cone K, then we say that g is convex relative to the cone K.

In the following theorem we prove that under convexity assumptions for the functions f and g the gap between the optimal objective values of the Lagrange dual and Fenchel-Lagrange dual vanishes.

Theorem 2.2 *Assume that X is a convex set, f is a convex function on X and $g = (g_1, \ldots, g_k)^T$ is convex on X relative to the cone K. It holds then*

$$sup(D^s_L) = sup(D^s_{FL}).$$

Proof. We prove actually a much "stronger" result, in fact, that under these assumptions for every $q^* \in \mathbb{R}^k, q^* \underset{K^*}{\geqq} 0$, the following equality is true

$$\inf_{x\in X} \left[f(x) + q^{*T} g(x)\right] = \sup_{p^*\in\mathbb{R}^n} \left\{ -f^*(p^*) + \inf_{x\in X} [p^{*T} x + q^{*T} g(x)] \right\}. \qquad (2.\ 13)$$

Therefore, let be $q^* \in \mathbb{R}^k, q^* \underset{K^*}{\geqq} 0$ fixed. We denote by $\alpha := \inf_{x\in X} [f(x) + q^{*T} g(x)]$. Obviously, $\alpha \in [-\infty, +\infty)$.

From (2. 11) we have for every $p^* \in \mathbb{R}^n$ the following relation

$$\inf_{x\in X} \left[f(x) + q^{*T} g(x)\right] \geq -f^*(p^*) + \inf_{x\in X} \left[p^{*T} x + q^{*T} g(x)\right],$$

which implies that

$$\alpha \geq \sup_{p^*\in\mathbb{R}^n} \left\{ -f^*(p^*) + \inf_{x\in X} [p^{*T} x + q^{*T} g(x)] \right\}. \qquad (2.\ 14)$$

If $\alpha = -\infty$, then the term in the right side of (2. 14) must also be $-\infty$ and, in this case, (2. 13) is fulfilled.

Let us assume now that $\alpha > -\infty$. So, the sets

$$A = \{(x, \mu) : x \in X, \mu \in \mathbb{R}, f(x) \leq \mu\} \subseteq \mathbb{R}^{n+1}$$

and

$$B = \{(x, \mu) : x \in X, \mu \in \mathbb{R}, \mu + q^{*T} g(x) \leq \alpha\} \subseteq \mathbb{R}^{n+1}.$$

are nonempty and convex. According to Lemma 7.3 in [62], the relative interior of A is nonempty and can be written as

$$ri(A) = \{(x, \mu) : x \in ri(X), f(x) < \mu < +\infty\}.$$

Let us now prove that

$$ri(A) \cap B = \emptyset. \tag{2. 15}$$

Therefore, assume that there exists $x' \in ri(A) \cap B$. This means that x' belongs to $ri(X)$, with the properties that $f(x') < \mu$ and $\mu + q^{*T} g(x') \leq \alpha$. The last inequalities lead us to $f(x') + q^{*T} g(x') < \alpha$, which contradicts the definition of α. Consequently, the intersection $ri(A) \cap B$ must be empty.

Because $ri(B) \subseteq B$, (2. 15) implies that $ri(A) \cap ri(B) = \emptyset$. By a well-known separation theorem in finite dimensional spaces (see for instance Theorem 11.3 in [62]), the sets A and B can be properly separated, that is, there exists a vector $(p^*, \mu^*) \in \mathbb{R}^n \times \mathbb{R} \setminus \{(0, 0)\}$ and $\alpha^* \in \mathbb{R}$ such that

$$p^{*T} x + \mu^* \mu \leq \alpha^* \leq p^{*T} y + \mu^* r, \quad \forall (x, \mu) \in A, \ (y, r) \in B, \tag{2. 16}$$

and

$$\inf\{p^{*T} x + \mu^* \mu : (x, \mu) \in A\} < \sup\{p^{*T} y + \mu^* r : (y, r) \in B\}. \tag{2. 17}$$

It is easy to see that $\mu^* \leq 0$. Let us show that $\mu^* \neq 0$. Suppose by contradiction that $\mu^* = 0$. This means that $p^* \neq 0$ and, by (2. 16), it follows that for every $x \in X$, $p^{*T} x = \alpha^*$. But this relation contradicts (2. 17) and, so, μ^* must be nonzero.

Dividing relation (2. 16) by $-\mu^*$ one obtains

$$p_0^{*T} x - \mu \leq \alpha_0^* \leq p_0^{*T} y - r, \quad \forall (x, \mu) \in A, \ (y, r) \in B, \tag{2. 18}$$

where $p_0^* := -\frac{1}{\mu^*} p^*$ and $\alpha_0^* := -\frac{1}{\mu^*} \alpha^*$. Since for every $x \in X$ the pair $(x, f(x)) \in A$, by (2. 18) we obtain that

$$p_0^{*T} x - f(x) \leq \alpha_0^*, \quad \forall x \in X,$$

and taking the supremum of the left hand side over all $x \in X$ we get

$$f^*(p_0^*) \leq \alpha_0^*. \tag{2. 19}$$

Similarly, since for every $x \in X$ the pair $(x, \alpha - q^{*T} g(x))$ is in B, by (2. 18) we also obtain

$$\alpha_0^* \leq p_0^{*T} x - \alpha + q^{*T} g(x), \quad \forall x \in X,$$

therefore,

$$\alpha_0^* + \alpha \leq \inf_{x \in X} \left[p_0^{*T} x + q^{*T} g(x)\right]. \tag{2. 20}$$

Combining the relations (2. 19) and (2. 20) it follows

$$\alpha \leq -f^*(p_0^*) + \inf_{x \in X} \left[p_0^{*T} x + q^{*T} g(x)\right],$$

which leads us together with (2. 14) to

$$\alpha = \sup_{p^* \in \mathbb{R}^n} \left\{ -f^*(p^*) + \inf_{x \in X} [p^{*T} x + q^{*T} g(x)] \right\}.$$

In conclusion, (2. 13) holds for each $q^* \in \mathbb{R}^k, q^* \underset{K^*}{\geqq} 0$, and, from here, we obtain that $sup(D_L^s) = sup(D_{FL}^s)$. □

Remark 2.3 In the Examples 2.1 and 2.2, we have for $K = \mathbb{R}_+$ that g is convex but f is not convex and that f is convex but g is not convex, respectively. Let us notice that in both situations the inequality $sup(D_L^s) \geq sup(D_{FL}^s)$ holds strictly. It means that the convexity of one of the functions without the convexity of the other one is not sufficient in order to have $sup(D_L^s) = sup(D_{FL}^s)$.

Remark 2.4 Let us consider again the Example 2.4. Obviously, X is a convex set and f and g are convex functions ($K = \mathbb{R}_+$). The optimal objective value of (P^s) is

$$inf(P^s) = \inf\{x_2 : (x_1, x_2)^T \in X, x_1 \leq 0\} = \inf\{x_2 : x_1 = 0, 3 \leq x_2 \leq 4\} = 3,$$

and it holds

$$inf(P^s) = sup(D_F^s) = 3 > 1 = sup(D_L^s) = sup(D_{FL}^s).$$

We conclude that the fulfilment of the convexity assumptions for f and g is not enough, neither to have equality between the optimal objective values of the three duals, nor to obtain strong duality.

2.2.3 The equivalence of the dual problems (D_F^s) and (D_{FL}^s)

The goal of this section is to investigate some necessary conditions in order to ensure equality between the optimal objective values of the duals (D_F^s) and (D_{FL}^s).

Therefore, we consider the following constraint qualification

$$(CQ^s) \;\Big|\; \text{there exists an element } x' \in X \text{ such that } g(x') \in -int(K).$$

In the next theorem we show that the so-called generalized Slater condition (CQ^s) together with the convexity of g on X relative to the cone K imply the existence of equality between $sup(D_F^s)$ and $sup(D_{FL}^s)$.

Theorem 2.3 *Assume that X is a convex set, $g = (g_1, ..., g_k)^T$ is convex on X relative to the cone K and the constraint qualification (CQ^s) is fulfilled. Then it holds*

$$sup(D_F^s) = sup(D_{FL}^s).$$

Proof. For $p^* \in \mathbb{R}^n$ fixed, we prove first that

$$\sup_{q^* \underset{K^*}{\geqq} 0} \inf_{x \in X} \left[p^{*T} x + q^{*T} g(x)\right] = \inf_{x \in G} p^{*T} x. \tag{2. 21}$$

Let be $\beta := \inf_{x \in G} p^{*T} x$. Because of $G \neq \emptyset$, $\beta \in [-\infty, +\infty)$.

If $\beta = -\infty$, then by (2. 12) it follows that

$$\sup_{q^* \underset{K^*}{\geqq} 0} \inf_{x \in X} \left[p^{*T} x + q^{*T} g(x)\right] = -\infty = \inf_{x \in G} p^{*T} x.$$

Suppose now that $-\infty < \beta < +\infty$. It is easy to check that the system

$$\begin{cases} p^{*T}x - \beta < 0, \\ g(x) \in -K, \\ x \in X, \end{cases}$$

has no solution. Therefore, the system

$$\begin{cases} p^{*T}x - \beta < 0, \\ g(x) \in -int(K), \\ x \in X, \end{cases}$$

has no solution too.

Define the vector-valued function $G : \mathbb{R}^n \to \mathbb{R} \times \mathbb{R}^k$, given by $G(x) = (p^{*T}x - \beta, g(x))$, and let S be the closed convex cone $S := [0, +\infty) \times K$. Let us notice that G is a convex function on X relative to the cone S and that there is no $x \in X$ such that $G(x) \in -int(S)$. Using now an alternative theorem of Farkas-Gordan type (see for instance Theorem 3.4.2 in [15]), it follows that there exists $(u^*, q^*) \in [0, +\infty) \times K^* \setminus \{(0, 0)\}$ such that

$$u^*(p^{*T}x - \beta) + q^{*T}g(x) \geq 0, \quad \forall x \in X. \tag{2. 22}$$

We show now that $u^* \neq 0$. For this, suppose by contradiction that $u^* = 0$. We have then $q^* \neq 0$ and, by (2. 22),

$$q^{*T}g(x) \geq 0, \quad \forall x \in X. \tag{2. 23}$$

The constraint qualification (CQ^s) being fulfilled, it follows that there exists an $x' \in X$ such that $g(x') \in -int(K)$, or, equivalently, $-g(x') \in int(K)$. From the so-called positive lemma (see Lemma 3.4.1 in [15]) it holds then $q^{*T}(-g(x')) > 0$ and, from here, $q^{*T}g(x') < 0$, contradiction to (2. 23).

This means that $u^* \neq 0$ and dividing relation (2. 22) by u^* we obtain

$$p^{*T}x - \beta + q_0^{*T}g(x) \geq 0, \quad \forall x \in X, \tag{2. 24}$$

with $q_0^* := \frac{1}{u^*}q^*$. The last relation implies

$$\sup_{q^* \underset{K^*}{\geqq} 0} \inf_{x \in X} \left[p^{*T}x + q^{*T}g(x)\right] \geq \beta,$$

which together with (2. 12) leads to (2. 21).

To finish the proof we have to add, for $p^* \in \mathbb{R}^n$, in the both sides of (2. 21) the term $-f^*(p^*)$ and this becomes

$$\begin{aligned} -f^*(p^*) + \sup_{q^* \underset{K^*}{\geqq} 0} \inf_{x \in X} \left[p^{*T}x + q^{*T}g(x)\right] &= -f^*(p^*) + \inf_{\substack{x \in X, \\ g(x) \underset{K}{\leqq} 0}} p^{*T}x \\ &= -f^*(p^*) - \chi_G^*(-p^*), \ \forall p^* \in \mathbb{R}^n. \end{aligned}$$

Taking now the supremum in both sides over $p^* \in \mathbb{R}^n$, we obtain the equality $sup(D_F^s) = sup(D_{FL}^s)$. This completes the proof. □

Remark 2.5 In the Examples 2.3 and 2.4, we have for $K = \mathbb{R}_+$ that (CQ^s) is fulfilled, but g is not convex and that (CQ^s) is not fulfilled, but g is convex, respectively. We notice that in both situations the inequality $sup(D_F^s) \geq sup(D_{FL}^s)$ holds strictly. It means that the fulfilment of (CQ^s) without the convexity of g or the convexity of g without the fulfilment of (CQ^s) are not sufficient in order to have

$sup(D_F^s) = sup(D_{FL}^s)$.

Remark 2.6 In the Example 2.1, we have $K = \mathbb{R}_+$, $X = [0 + \infty)$ a convex set, $g : \mathbb{R} \to \mathbb{R}$, $g(x) = x^2 - 1$ a convex function and that the constraint qualification (CQ^s) is fulfilled (take for instance $x' = \frac{1}{2}$). The optimal objective value of (P^s) is

$$inf(P^s) = \inf_{x \in G} f(x) = \inf_{x \in [0,1]} (-x^2) = -1$$

and it verifies

$$inf(P^s) = sup(D_L^s) = -1 > -\infty = sup(D_F^s) = sup(D_{FL}^s).$$

This means that, even if the hypotheses of Theorem 2.3 are fulfilled, neither we have equality between the optimal objective values of the three duals, nor the strong duality is attained.

The question of the existence of some necessary conditions for which, both, the optimal objective values of (D_L^s), (D_F^s) and (D_{FL}^s) are equal and the strong duality is attained, will be answered in the next section. Until then we show in the following subsection that the equality $sup(D_F^s) = sup(D_{FL}^s)$ holds even under "weaker" assumptions than those imposed in Theorem 2.3.

2.2.4 Some weaker assumptions for the equivalence of the dual problems (D_F^s) and (D_{FL}^s)

In a recent work, Boţ, Kassay and Wanka [9] established some relations between the optimal objective values of (D_L^s), (D_F^s) and (D_{FL}^s) for a class of generalized convex programming problems. In the same context we succeed to weaken the convexity and the regularity assumptions considered in Theorem 2.3 in a way that $sup(D_F^s)$ and $sup(D_{FL}^s)$ still remain equal. In order to present these results we recall the concepts of nearly convex sets and nearly convex functions introduced by Green and Gustin [31] and Aleman [1], respectively.

Definition 2.4 *A subset $X \subseteq \mathbb{R}^m$ is called nearly convex if there exists a constant $0 < \alpha < 1$ such that for each $x, y \in X$ it follows that $\alpha x + (1 - \alpha)y \in X$.*

Obviously, each convex set is nearly convex, but the contrary is not true, since for instance the set $\mathbb{Q} \subset \mathbb{R}$ of all rational numbers is nearly convex (with $\alpha = 1/2$) but not a convex set.

Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ and $g : \mathbb{R}^n \to \mathbb{R}^k$ be given functions, and let D, E be nonempty subsets of $\mathbb{R}^n$ such that $D \subseteq dom(f)$. We denote the epigraph of f on D by $epi(f; D)$, i.e. the set $\{(x, r) \in D \times \mathbb{R} : f(x) \leq r\}$. Furthermore, if $C \subseteq \mathbb{R}^k$ is a nonempty convex cone, the epigraph of g on E relative to the cone C is the set

$$epi_C(g; E) := \{(x, v) \in E \times \mathbb{R}^k : g(x) \underset{C}{\leqq} v\},$$

where "$\underset{C}{\leqq}$ denotes the partial ordering relation induced by C.

Now we can define the following concepts.

Definition 2.5 *The function f is said to be nearly convex on D, if $epi(f; D)$ is a nearly convex set. Moreover, the vector-valued function g is said to be nearly convex on E relative to the cone C, if $epi_C(g; E)$ is a nearly convex set.*

It is obvious that in case D or/and E are convex sets and f or/and g are convex functions in the usual sense on D and E, respectively, then they are also nearly convex. An interesting fact is that it is possible to give an example for nearly convex, but not convex function defined on a convex set. This will be related to the functional equation of Cauchy.

Example 2.5 Let $F : \mathbb{R} \to \mathbb{R}$ be any discontinuous solution of the Cauchy functional equation, i.e. F satisfies

$$F(x+y) = F(x) + F(y), \quad \forall x, y \in \mathbb{R}.$$

(Such a solution exists, see [32].)

It is easy to deduce that F is nearly convex on $\mathbb{R}$ with constant $1/2$. However, F is not convex (even more: there is no interval in $\mathbb{R}$ on which F is convex) due to the lack of continuity.

Now we are ready to present a more general theorem which gives the equality between the optimal objective values of the problems (D_F^s) and (D_{FL}^s) (see Theorem 3.1 in Boţ, Kassay and Wanka [9]). Therefore, let us consider instead of K the closed convex cone C, without asking that $int(C)$ must be non-empty.

Theorem 2.4 *Suppose that $g : \mathbb{R}^n \to \mathbb{R}^k$ is a nearly convex function on the set $X \subseteq \mathbb{R}^n$ relative to the closed convex cone $C \subseteq \mathbb{R}^k$. Furthermore, suppose that there exists an element $y_0 \in aff(g(X))$ such that*

$$g(X) \subseteq y_0 + aff(C) \tag{2. 25}$$

and the (Slater type) regularity condition

$$0 \in g(X) + ri(C) \tag{2. 26}$$

holds. Then $sup(D_F^s) = sup(D_{FL}^s)$.

In Theorem 2.4, $aff(C)$ represents the affine hull of the set C.

Remark 2.7 Assuming that the hypotheses of Theorem 2.3 are fulfilled, it is obvious that g is nearly convex on X relative to K. Moreover, because $int(K) \neq \emptyset$, it follows that $aff(K) = \mathbb{R}^k$ and $ri(K) = int(K)$. Therefore, the constrained qualifications (2. 25) and (2. 26) hold and, so, the hypotheses of Theorem 2.4 are also verified.

Another modality to weaken the constraint qualification (CQ^s) in Theorem 2.3 has been presented by Wanka and Boţ in [86], but in the case $K = \mathbb{R}_+^k$. In order to recall it, let us introduce for $g = (g_1, ..., g_k)^T : \mathbb{R}^n \to \mathbb{R}^k$ the following sets: L the set of those $i \in \{1, ..., k\}$ for which g_i is an affine function and N the set of those $i \in \{1, ..., k\}$ for which g_i is not an affine function. By using these notations we can formulate the following constraint qualification for (P^s) in the case $K = \mathbb{R}_+^k$ (see also condition (R_5^X) in section 4.3 in [20])

(CQ_{ln}^s) | there exists an element $x' \in ri(X)$ such that $g(x') < 0$ for $i \in N$ and $g(x') \leq 0$ for $i \in L$.

Theorem 2.5 states the equality between $sup(D_F^s)$ and $sup(D_{FL}^s)$ in the case $K = \mathbb{R}_+^k$, if we substitute (CQ^s) by (CQ_{ln}^s) (for the proof see Wanka and Boţ [86]).

Theorem 2.5 *Assume that X is a convex set, $K = \mathbb{R}_+^k$, the functions g_i, $i = 1, ..., k$ are convex on X and the constraint qualification (CQ_{ln}^s) is fulfilled. Then it holds*

$$sup(D_F^s) = sup(D_{FL}^s).$$

2.3 Strong duality and optimality conditions

2.3.1 Strong duality for (D_L^s), (D_F^s) and (D_{FL}^s)

In the Theorems 2.2 and 2.3 we have presented some necessary conditions in order to have equality between the optimal objective values of the Fenchel-Lagrange dual and the Lagrange dual and of the Fenchel-Lagrange dual and the Fenchel dual, respectively. Combining the hypotheses of both theorems it follows, obviously, the equality of the optimal objective values of the three duals. Meanwhile, under the same conditions, it can be shown that these optimal objective values are also equal with $inf(P^s)$, conducting us, in the case when $inf(P^s)$ is finite, to strong duality. Let us remind here that strong duality means that the optimal objective values of the primal and dual problems coincide and that the dual problem has an optimal solution. This fact is proved by the following theorem.

Theorem 2.6 *Assume that $X \subseteq \mathbb{R}^n$ is a convex set, f is convex on X and g is convex on X relative to the cone K. If the constraint qualification (CQ^s) is fulfilled, then it holds*

$$inf(P^s) = sup(D_L^s) = sup(D_F^s) = sup(D_{FL}^s). \qquad (2.\ 27)$$

Moreover, if $inf(P^s) > -\infty$, then all the duals have an optimal solution. We represent this by replacing in (2. 27) "sup" by "max", namely,

$$inf(P^s) = max(D_L^s) = max(D_F^s) = max(D_{FL}^s). \qquad (2.\ 28)$$

Proof. By Theorem 2.2 and Theorem 2.3 we obtain

$$sup(D_L^s) = sup(D_F^s) = sup(D_{FL}^s). \qquad (2.\ 29)$$

Because $G = \{x \in X : g(x) \underset{K}{\leqq} 0\} \neq \emptyset$, it holds $inf(P^s) \in [-\infty, +\infty)$. If $inf(P^s) = -\infty$, then by (2. 6), (2. 8) and (2. 10) we have

$$sup(D_L^s) = sup(D_F^s) = sup(D_{FL}^s) = -\infty = inf(P^s).$$

Suppose now $-\infty < infP < +\infty$. The system

$$\begin{cases} f(x) - inf(P^s) < 0, \\ g(x) \underset{K}{\leqq} 0, \\ x \in X, \end{cases}$$

has then no solution. In a similar way as in the proof of Theorem 2.3 we obtain an element $q_0^* \in \mathbb{R}^k$, $q_0^* \underset{K^*}{\geqq} 0$, such that

$$f(x) - inf(P^s) + q_0^{*T} g(x) \geq 0, \quad \forall x \in X,$$

or, equivalently,

$$\inf_{x \in X} \left[f(x) + q_0^{*T} g(x) \right] \geq inf(P^s). \qquad (2.\ 30)$$

The latter relation and (2. 6) imply

$$\begin{aligned} inf(P^s) &\geq sup(D_L^s) = \sup_{q^* \underset{K^*}{\geqq} 0} \inf_{x \in X} \left[f(x) + q^{*T} g(x) \right] \\ &\geq \inf_{x \in X} \left[f(x) + q_0^{*T} g(x) \right] \geq inf(P^s), \end{aligned} \qquad (2.\ 31)$$

which leads us, together with (2. 29), to

$$inf(P^s) = sup(D_L^s) = sup(D_F^s) = sup(D_{FL}^s).$$

Moreover, $q_0^* \in \mathbb{R}^k$ is an optimal solution to the Lagrange dual (D_L^s).

As in the proof of Theorem 2.2 we can obtain now for the vector $q_0^* \in \mathbb{R}^k$, $q_0^* \underset{K^*}{\geqq} 0$, an element $p_0^* \in \mathbb{R}^n$ such that

$$\begin{aligned}\inf_{x\in X}\left[f(x) + q_0^{*T} g(x)\right] &= \sup_{p^*\in\mathbb{R}^n}\left\{-f^*(p^*) + \inf_{x\in X}[p^{*T}x + q_0^{*T}g(x)]\right\}\\ &= -f^*(p_0^*) + \inf_{x\in X}\left[p_0^{*T}x + q_0^{*T}g(x)\right].\end{aligned}$$

By (2. 31) we have

$$inf(P^s) = sup(D_{FL}^s) = \inf_{x\in X}\left[f(x) + q_0^{*T}g(x)\right] = -f^*(p_0^*) + \inf_{x\in X}\left[p_0^{*T}x + q_0^{*T}g(x)\right] \tag{2. 32}$$

and, therefore, (p_0^*, q_0^*) is an optimal solution to (D_{FL}^s).

It remains to show that p_0^* is actually an optimal solution to the Fenchel dual (D_F^s). By (2. 12), (2. 27) and (2. 32), it results that

$$-f^*(p_0^*) - \chi_G^*(-p_0^*) \geq -f^*(p_0^*) + \inf_{x\in X}\left[p_0^{*T}x + q_0^{*T}g(x)\right] = inf(P^s) = sup(D_F^s).$$

On the other hand, from (2. 8) we have

$$inf(P^s) \geq sup(D_F^s) = \sup_{p^*\in\mathbb{R}^n}\{-f^*(p^*) - \chi_G^*(-p^*)\} \geq -f^*(p_0^*) - \chi_G^*(-p_0^*).$$

Combining the last two inequalities we conclude that

$$inf(P^s) = sup(D_F^s) = -f^*(p_0^*) - \chi_G^*(-p_0^*)$$

and, so, p_0^* is an optimal solution to (D_F^s). This completes the proof. □

Example 2.6 For $X = \mathbb{R}$ and $K = \mathbb{R}_+$, let the functions $f : \mathbb{R} \to \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$, be given by $f(x) = e^{-x}$ and $g(x) = -x$. The functions f and g are convex, the constraint qualification (CQ^s) is fulfilled and the optimal objective value of the primal problem (P^s), $inf(P^s) = \inf_{x\geq 0} e^{-x} = 0$ is finite. Then $sup(D_L^s) = sup(D_F^s) = sup(D_{FL}^s) = 0$, $q_0^* = 0$ is an optimal solution to (D_L^s), $(p_0^*, q_0^*) = (0, 0)$ is an optimal solution to (D_{FL}^s), $p_0^* = 0$ is an optimal solution to (D_F^s), but the primal problem (P^s) has no solution. We observe that it is possible to have strong duality, without the need for the primal problem to have an optimal solution.

In the last part of this subsection we will weaken the hypotheses of Theorem 2.6 in a way that the strong duality results still hold. The first result relies on the concept of nearly convexity introduced in subsection 2.2.4. In order to present it let us consider instead of K the convex closed cone C, without imposing that $int(C) \neq \emptyset$. As usual, by the epigraph of the objective function $f : \mathbb{R} \to \overline{\mathbb{R}}$ we denote the set

$$epi(f) = \{(x, r) \in \mathbb{R}^n \times \mathbb{R} : f(x) \leq r\} = \{(x, r) \in X \times \mathbb{R} : f(x) \leq r\}.$$

Theorem 2.7 *(see Theorem 3.3 in [9]) Suppose that f is nearly convex on the set X, g is nearly convex on the set X relative to the closed convex cone $C \subseteq \mathbb{R}^k$ and that the constraint qualifications* (2. 25) *and* (2. 26) *hold. Assume further $ri(epi(f)) \neq \emptyset$ and $ri(G) \neq \emptyset$. Then*

$$inf(P^s) = sup(D_L^s) = sup(D_F^s) = sup(D_{FL}^s).$$

Moreover, if $inf(P^s) > -\infty$, then all dual problems (D_L^s), (D_F^s) and (D_{FL}^s) have optimal solutions.

Remark 2.8 If X is a convex set, f is a convex function on X and g is convex on X relative to the cone C, then the convexity of the sets $epi(f)$ and G follows immediately. In this case both sets have non-empty relative interiors.

Remark 2.9 We want to notice here that in the past many results concerning duality for generalized convex programming problems have been published (see for instance [15], [23], [29], [42]). In these works the authors deal only with the Lagrange dual problem. But, Theorem 2.7 gives some necessary conditions, "weaker" than those considered in Theorem 2.6, such that strong duality holds not just for (D_L^s), but also for (D_F^s) and (D_{FL}^s).

Remark 2.10 One can see that in Theorem 2.7 (like in Theorem 2.4) the closed convex cone C does not need to have a non-empty interior. This means that, taking $C = \mathbb{R}_+^r \times \{0_{\mathbb{R}^{k-r}}\}$ with $r < k$, the problem $\inf\{f(x) : x \in X, g(x) \in -C\}$ becomes an optimization problem with inequality and equality constraints. So, Theorem 2.7 "covers" also this type of primal optimization problems. Because of the assumption $int(K) \neq \emptyset$, Theorem 2.6 is "too strong" in order to be applied to such a class of problems with, both, inequality and equality constraints. Nevertheless, in the next chapters we will work in the initial frame, by assuming the convexity of the sets and functions involved and the fulfilment of a constraint qualification of the same type like (CQ^s). The scalar duality will be used then as a good tool for the study of the duality in vector optimization.

The last theorem in this subsection also "weakens" Theorem 2.6, in the case $K = \mathbb{R}_+^k$, by using instead of (CQ^s) the weaker constraint qualification (CO_{ln}^s).

Theorem 2.8 *(see Theorem 3 in [86]) Assume that X is a convex set and f, g_i, $i = 1, ..., m$, are convex on X. If the constraint qualification (CO_{ln}^s) is fulfilled, then it holds*

$$inf(P^s) = sup(D_L^s) = sup(D_F^s) = sup(D_{FL}^s).$$

Moreover, if $inf(P^s) > -\infty$, then all dual problems (D_L^s), (D_F^s) and (D_{FL}^s) have optimal solutions.

2.3.2 Optimality conditions

In this subsection we complete our investigations by presenting the necessary and sufficient optimality conditions for the primal and dual problems, closely connected with the strong duality. Let us start with the optimality conditions for the Lagrange dual.

Theorem 2.9 *(a) Let the assumptions of Theorem 2.6 be fulfilled and let $\bar{x}$ be a solution to (P^s). Then there exists an element $\bar{q}^* \in \mathbb{R}^k, \bar{q}^* \underset{K^*}{\geqq} 0$, solution to (D_L^s), such that the following optimality conditions are satisfied*

$$\begin{array}{llcl} (i) & f(\bar{x}) & = & \inf\limits_{x \in X} [f(x) + \bar{q}^{*T} g(x)], \\ (ii) & \bar{q}^{*T} g(\bar{x}) & = & 0. \end{array}$$

(b) Let $\bar{x}$ be admissible to (P^s) and $\bar{q}^*$ be admissible to (D_L^s), satisfying (i) and (ii). Then $\bar{x}$ is a solution to (P^s), $\bar{q}^*$ is a solution to (D_L^s) and strong duality holds.

Proof.

(a) By Theorem 2.6, there exists an element $\bar{q}^* \underset{K^*}{\geqq} 0$, solution to (D_L^s), such that

$$f(\bar{x}) = inf(P^s) = sup(D_L^s) = \inf_{x \in X} \left[f(x) + \bar{q}^{*T} g(x) \right].$$

Because

$$\inf_{x \in X} \left[f(x) + \bar{q}^{*T} g(x) \right] \leq f(\bar{x}) + \bar{q}^{*T} g(\bar{x})] \leq f(\bar{x}),$$

it follows that

$$f(\bar{x}) + \bar{q}^{*T} g(\bar{x}) = f(\bar{x}),$$

which implies $\bar{q}^{*T} g(\bar{x}) = 0$. So, the relations (i) and (ii) are proved.

(b) By (i) and (ii), we obtain that

$$sup(D_L^s) \geq \inf_{x \in X} \left[f(x) + \bar{q}^{*T} g(x) \right] = f(\bar{x}) = inf(P^s),$$

which leads us together with (2. 6) to the expected conclusion. □

The next theorem gives us the optimality conditions for the Fenchel dual problem.

Theorem 2.10 *(a) Let the assumptions of Theorem 2.6 be fulfilled and let $\bar{x}$ be a solution to (P^s). Then there exists an element $\bar{p}^* \in \mathbb{R}^n$, solution to (D_F^s), such that the following optimality conditions are satisfied*

$$\begin{array}{lrcl} (i) & f(\bar{x}) + f^*(\bar{p}^*) & = & \bar{p}^{*T} \bar{x}, \\ (ii) & \bar{p}^{*T} \bar{x} & = & -\chi_G^*(-\bar{p}^*). \end{array}$$

(b) Let $\bar{x}$ be admissible to (P^s) and $\bar{p}^*$ be admissible to (D_F^s), satisfying (i) and (ii). Then $\bar{x}$ is a solution to (P^s), $\bar{p}^*$ is a solution to (D_F^s) and strong duality holds.

Proof.

(a) Again, by Theorem 2.6, there exists an element $\bar{p}^* \in \mathbb{R}^n$, solution to (D_F^s), such that

$$f(\bar{x}) = inf(P^s) = sup(D_F^s) = -f^*(\bar{p}^*) - \chi_G^*(-\bar{p}^*).$$

This last equality yields after some transformations

$$f(\bar{x}) + f^*(\bar{p}^*) - \bar{p}^{*T} \bar{x} + \bar{p}^{*T} \bar{x} + \chi_G^*(-\bar{p}^*) = 0. \tag{2. 33}$$

Because of the inequality of Young, $f(\bar{x}) + f^*(\bar{p}^*) \geq \bar{p}^{*T} \bar{x}$, and $\bar{p}^{*T} \bar{x} + \chi_G^*(-\bar{p}^*) \geq 0$, it results that (i) and (ii) must be true.

(b) We complete the proof by observing that

$$sup(D_F^s) \geq -f^*(\bar{p}^*) - \chi_G^*(-\bar{p}^*) = f(\bar{x}) \geq inf(P^s),$$

which leads us together with (2. 8) to the conclusion. □

Finally, in Theorem 2.11 we formulate the optimality conditions for the Fenchel-Lagrange dual problem.

Theorem 2.11 *(a) Let the assumptions of Theorem 2.6 be fulfilled and let $\bar{x}$ be a solution to (P^s). Then there exists an element $(\bar{p}^*, \bar{q}^*)$, $\bar{p}^* \in \mathbb{R}^n, \bar{q}^* \underset{K^*}{\geqq} 0$, solution to (D^s_{FL}), such that the following optimality conditions are satisfied*

$$\begin{array}{lrcl} (i) & f(\bar{x}) + f^*(\bar{p}^*) & = & \bar{p}^{*T}\bar{x}, \\ (ii) & \bar{q}^{*T}g(\bar{x}) & = & 0, \\ (iii) & \bar{p}^{*T}\bar{x} & = & \inf\limits_{x\in X}[\bar{p}^{*T}x + \bar{q}^{*T}g(x)]. \end{array}$$

(b) Let $\bar{x}$ be admissible to (P^s) and $(\bar{p}^*, \bar{q}^*)$ be admissible to (D^s_{FL}), satisfying (i), (ii) and (iii). Then $\bar{x}$ is a solution to (P^s), $(\bar{p}^*, \bar{q}^*)$ is a solution to (D^s_{FL}) and strong duality holds.

Proof.

(a) Let be $\bar{x}$ an optimal solution to (P^s). Theorem 2.6 assures the existence of an optimal solution $(\bar{p}^*, \bar{q}^*) \in \mathbb{R}^n \times \mathbb{R}^k, \bar{q}^* \underset{K^*}{\geqq} 0$, to (D^s_{FL}) such that

$$inf(P^s) = f(\bar{x}) = -f^*(\bar{p}^*) + \inf_{x\in X}\left[\bar{p}^{*T}x + \bar{q}^{*T}g(x)\right]$$

or, equivalently,

$$f(\bar{x})+f^*(\bar{p}^*)-\bar{p}^{*T}\bar{x}+\bar{p}^{*T}\bar{x}+\bar{q}^{*T}g(\bar{x})-\inf_{x\in X}\left[\bar{p}^{*T}x + \bar{q}^{*T}g(x)\right]-\bar{q}^{*T}g(\bar{x}) = 0. \quad (2.\ 34)$$

On the other hand, the following inequalities hold

$$\begin{array}{rcl} f(\bar{x}) + f^*(\bar{p}^*) - \bar{p}^{*T}\bar{x} & \geq & 0, \\ \bar{p}^{*T}\bar{x} + \bar{q}^{*T}g(\bar{x}) - \inf\limits_{x\in X}\left[\bar{p}^{*T}x + \bar{q}^{*T}g(x)\right] & \geq & 0, \\ -\bar{q}^{*T}g(\bar{x}) & \geq & 0. \end{array}$$

By (2. 34) it follows that all these inequalities have to be in fact fulfilled as equalities. This conducts us to the optimality conditions (i), (ii) and (iii).

(b) All calculations done within part (a) may be carried out in the inverse direction starting from (i), (ii) and (iii). Then, $\bar{x}$ solves (P^s), $(\bar{p}^*, \bar{q}^*)$ solves (D^s_{FL}) and the strong duality holds. □

2.4 Duality for composed convex functions with applications in location theory

2.4.1 Motivation

In this section we show the usefulness of the conjugacy approach in the study of the duality for optimization problems not just in finite dimensional spaces, but also in general normed spaces. This part of the work has been motivated by a paper of Nickel, Puerto and Rodriguez-Chia [57]. The authors have studied there a single facility location problem in a general normed space in which the existing facilities are represented by sets of points. For this problem, a geometrical characterization of the set of optimal solutions have been given.

Our intention is to construct a dual problem to the optimization problem treated in [57] and for its particular instances, the Weber problem and the minmax problem with demand sets. Afterwards, we derive the optimality conditions for all these problems, via strong duality.

In order to do this, we consider a more general optimization problem, in fact, a problem with the objective function being a composite of a convex and componentwise increasing function with a convex vector function. Applying the conjugacy approach and using some appropriate perturbation we construct a dual problem to it. The dual is formulated in terms of conjugate functions and the existence of strong duality is shown. Afterwards, we particularize the results for the location problems in [57]. An extension of these considerations concerning duality in the vector case can be found in [89].

In the past, optimization problems with the objective function being a composed convex function have been treated by different authors. We recall here the works [34] and [35], where the form of the subdifferential of a composed convex function has been described, and also [13] and [47], where some results with regard to duality have been given. Concerning duality, VOLLE studied in a recent paper [78] the same problem as a particular case of a d.c. programming problem. But, the dual introduced in [78], as well as the dual problems presented in [13] and [47] are different from the dual we present in the following.

2.4.2 The optimization problem with a composed convex function as objective function

Let now $(X, \|\cdot\|)$ be a normed space, $g_i : X \to \mathbb{R}, i = 1, \ldots, m$, convex and continuous functions and $f : \mathbb{R}^m \to \mathbb{R}$ a convex and componentwise increasing function, i.e. for $y = (y_1, \ldots, y_m)^T, z = (z_1, \ldots, z_m)^T \in \mathbb{R}^m$,

$$y_i \geq z_i, i = 1, \ldots, m \Rightarrow f(y) \geq f(z).$$

The optimization problem which we consider here is the following one

$$(P^c) \quad \inf_{x \in X} f(g(x)),$$

where $g : X \to \mathbb{R}^m$, $g(x) = (g_1(x), \ldots, g_m(x))^T$.

In order to construct a dual problem to (P^c) we consider the following perturbation function $\Psi : \underbrace{X \times \ldots \times X}_{m+1} \times \mathbb{R}^m \to \mathbb{R}$,

$$\Psi(x, q, d) = f((g_1(x + q_1), \ldots, g_m(x + q_m))^T + d),$$

where $q = (q_1, \ldots, q_m) \in X \times \ldots \times X$ and $d \in \mathbb{R}^m$ are the perturbation variables.

Then the dual problem to (P^c), obtained by using the perturbation function Ψ, is

$$(D^c) \quad \sup_{\substack{p_i \in X^*, i=1,\ldots,m, \\ \lambda \in \mathbb{R}^m}} \{-\Psi^*(0, p, \lambda)\},$$

where $\Psi^* : \underbrace{X^* \times \ldots \times X^*}_{m+1} \times \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\}$ is the conjugate function of Ψ and $p_i, i = 1, ..., m$, $\lambda \in \mathbb{R}^m$, are the dual variables.

If Y is a Hausdorff locally convex vector space, then the conjugate function of $h : Y \to \mathbb{R}$ is the function $h^* : Y^* \to \mathbb{R} \cup \{+\infty\}$, defined by $h^*(y^*) = \sup_{y \in Y} \{\langle y^*, y \rangle - h(y)\}$, where Y^* is the topological dual to Y and $\langle \cdot, \cdot \rangle$ is the bilinear pairing between Y^* and Y.

The conjugate function of Ψ can be then calculated by the following formula

$$\Psi^*(x^*, p, \lambda) = \sup_{\substack{q_i \in X, i=1,\dots,m,\\ x\in X, d\in\mathbb{R}^m}} \Big\{ \langle x^*, x\rangle + \sum_{i=1}^m \langle p_i, q_i\rangle + \langle \lambda, d\rangle - f((g_1(x+q_1),\dots,g_m(x+q_m))^T + d)\Big\}.$$

To find these expression we introduce, first, the new variable t instead of d and, then, the new variables r_i instead of q_i, by

$$t := d + (g_1(x+q_1),\dots,g_m(x+q_m))^T \in \mathbb{R}^m$$

and

$$r_i := x + q_i \in X, i = 1, ..., m.$$

This implies

$$\begin{aligned}
\Psi^*(x^*, p, \lambda) &= \sup_{\substack{q_i \in X, i=1,\dots,m,\\ x\in X, t\in\mathbb{R}^m}} \Bigg\{ \langle x^*, x\rangle + \sum_{i=1}^m \langle p_i, q_i\rangle \\
&\quad + \big\langle \lambda, t - (g_1(x+q_1),\dots,g_m(x+q_m))^T \big\rangle - f(t)\Bigg\} \\
&= \sup_{\substack{r_i \in X, i=1,\dots,m,\\ x\in X}} \Bigg\{ \langle x^*, x\rangle + \sum_{i=1}^m \langle p_i, r_i - x\rangle \\
&\quad - \big\langle \lambda, (g_1(r_1),\dots,g_m(r_m))^T \big\rangle \Bigg\} + \sup_{t\in\mathbb{R}^m}\{\langle \lambda, t\rangle - f(t)\} \\
&= \sum_{i=1}^m \sup_{r_i\in X}\{\langle p_i, r_i\rangle - \lambda_i g_i(r_i)\} + \sup_{x\in X}\left\langle x^* - \sum_{i=1}^m p_i, x\right\rangle \\
&\quad + f^*(\lambda) \\
&= f^*(\lambda) + \sum_{i=1}^m (\lambda_i g_i)^*(p_i) + \sup_{x\in X}\left\langle x^* - \sum_{i=1}^m p_i, x\right\rangle.
\end{aligned}$$

Now we have to consider $x^* = 0$ and, so, the dual problem of (P^c) has the following form

$$(D^c) \quad \sup_{\substack{\lambda\in\mathbb{R}^m, p_i\in X^*,\\ i=1,\dots,m}} \left\{ -f^*(\lambda) - \sum_{i=1}^m (\lambda_i g_i)^*(p_i) + \inf_{x\in X}\left\langle \sum_{i=1}^m p_i, x\right\rangle \right\}.$$

It is obvious that if $\sum\limits_{i=1}^m p_i \neq 0_{X^*}$, then $\inf\limits_{x\in X}\left\langle \sum\limits_{i=1}^m p_i, x\right\rangle = -\infty$ and, so, in order to have supremum in (D^c), we must require that $\sum\limits_{i=1}^m p_i = 0$.

By this, the dual problem of (P^c) becomes

$$(D^c) \quad \sup_{\substack{\lambda\in\mathbb{R}^m, p_i\in X^*,\\ i=1,\dots,m, \sum_{i=1}^m p_i = 0}} \left\{ -f^*(\lambda) - \sum_{i=1}^m (\lambda_i g_i)^*(p_i) \right\}. \tag{2. 35}$$

Let us point out that between (P^c) and (D^c) weak duality holds, i.e. $inf(P^c) \geq sup(D^c)$. Here, $inf(P^c)$ and $sup(D^c)$ represent the optimal objective values of the problems (P^c) and (D^c), respectively. The existence of weak duality can be shown in the same way like in Theorem 2.1 in the finite dimensional case.

In order to prove the existence of strong duality ($inf(P^c) = max(D^c)$), namely, that the optimal objective values are equal and the dual has an optimal solution, we have to verify the stability of the primal problem (P^c) (cf. [19]). Therefore, we prove that the stability criterion described in Proposition III.2.3 in [19] is fulfilled. We start by enunciating the following proposition.

Proposition 2.3 *The function* $\Psi : \underbrace{X \times \ldots \times X}_{m+1} \times \mathbb{R}^m \to \mathbb{R}$,

$$\Psi(x, q, d) = f((g_1(x + q_1), \ldots, g_m(x + q_m))^T + d)$$

is convex.

The convexity of Ψ follows from the convexity of the functions f and g and the fact that f is a componentwise increasing function.

Theorem 2.12 (strong duality for (D^c)) *If $inf(P^c) > -\infty$, then the dual problem (D^c) has an optimal solution and strong duality holds, i.e.*

$$inf(P^c) = max(D^c).$$

Proof. By Proposition 2.3 we have that the perturbation function Ψ is convex. Moreover, $inf(P^c)$ is a finite number and the function

$$(q_1, \ldots, q_m, d) \longrightarrow \Psi(0, q_1, \ldots, q_m, d)$$

is finite and continuous at $(\underbrace{0, ..., 0}_{m}, 0_{\mathbb{R}^m}) \in \underbrace{X \times \ldots \times X}_{m} \times \mathbb{R}^m$. This means that the stability criterion in Proposition III.2.3 in [19] is fulfilled, which implies that the problem (P^c) is stable. Finally, the Propositions IV.2.1 and IV.2.2 in [19] lead us to the desired conclusions. $\square$

The last part of this section is devoted to the presentation of the optimality conditions for the problem (P^c). They are derived by using the equality between the optimal objective values of the primal and dual problem.

Theorem 2.13 (optimality conditions for (P^c))
(a) Let $\bar{x} \in X$ be a solution to (P^c). Then there exist $\bar{p}_i \in X^*, i = 1, ..., m$, and $\bar{\lambda} \in \mathbb{R}^m$ such that $(\bar{\lambda}, \bar{p}_1, \ldots, \bar{p}_m)$ is an optimal solution to (D^c) and the following optimality conditions are satisfied

$$(i) \quad f(g(\bar{x})) + f^*(\bar{\lambda}) = \sum_{i=1}^m \bar{\lambda}_i g_i(\bar{x}),$$

$$(ii) \quad \bar{\lambda}_i g_i(\bar{x}) + (\bar{\lambda}_i g_i)^*(\bar{p}_i) = \langle \bar{p}_i, \bar{x} \rangle, i = 1, \ldots, m,$$

$$(iii) \quad \sum_{i=1}^m \bar{p}_i = 0.$$

(b) If $\bar{x} \in X$, $(\bar{\lambda}, \bar{p}_1, \ldots, \bar{p}_m)$ is admissible to (D^c) and (i)-(iii) are satisfied, then $\bar{x}$ is an optimal solution to (P^c), $(\bar{\lambda}, \bar{p}_1, \ldots, \bar{p}_m)$ is an optimal solution to (D^c) and strong duality holds

$$f(g(\bar{x})) = -f^*(\bar{\lambda}) - \sum_{i=1}^m (\bar{\lambda}_i g_i)^*(\bar{p}_i).$$

Proof.

(a) By Theorem 2.12, it follows that there exist $\bar{p}_i \in X^*, i = 1, ..., m$, and $\bar{\lambda} \in \mathbb{R}^m$ such that $(\bar{\lambda}, \bar{p}_1, \dots, \bar{p}_m)$ is an optimal solution to (D^c) and the optimal objective values of (P^c) and (D^c) are equal. This means that $\sum\limits_{i=1}^m \bar{p}_i = 0$ and

$$f(g(\bar{x})) = -f^*(\bar{\lambda}) - \sum_{i=1}^m (\bar{\lambda}_i g_i)^*(\bar{p}_i). \tag{2. 36}$$

The last equality is equivalent to

$$0 = f(g(\bar{x})) + f^*(\bar{\lambda}) - \sum_{i=1}^m \bar{\lambda}_i g_i(\bar{x}) + \sum_{i=1}^m \left[\bar{\lambda}_i g_i(\bar{x}) + (\bar{\lambda}_i g_i)^*(\bar{p}_i) - \langle \bar{p}_i, \bar{x}\rangle\right]. \tag{2. 37}$$

From the definition of the conjugate functions we can derive, also in this case, the following inequalities (cf. [19])

$$f(g(\bar{x})) + f^*(\bar{\lambda}) \geq \bar{\lambda}^T g(\bar{x}) = \sum_{i=1}^m \bar{\lambda}_i g_i(\bar{x}) \tag{2. 38}$$

and

$$\bar{\lambda}_i g_i(\bar{x}) + (\bar{\lambda}_i g_i)^*(\bar{p}_i) \geq \langle \bar{p}_i, \bar{x}\rangle, i = 1, \dots, m. \tag{2. 39}$$

By (2. 37) it follows that the inequalities in (2. 38) and (2. 39) must become equalities, leading us to the conclusion.

(b) All the calculations and transformations done within part (a) may be carried out in the inverse direction starting from the conditions (i), (ii) and (iii). Thus the equality (2. 36) results and therefore $\bar{x}$ solves (P^c) and $(\bar{\lambda}, \bar{p}_1, \dots, \bar{p}_m)$ solves (D^c). □

2.4.3 The case of monotonic norms

In this section we consider a first particularization of the problem (P^c). Let be $\Phi : \mathbb{R}^m \to \mathbb{R}$ a monotonic norm on $\mathbb{R}^m$. Recall that a norm Φ is said to be monotonic (cf. [2]), if

$$\forall\ u, v \in \mathbb{R}^m, |u_i| \leq |v_i|, i = 1, \dots, m \Rightarrow \Phi(u) \leq \Phi(v).$$

Let be now the following optimization problem

$$(P_\Phi^c) \quad \inf_{x \in X} \Phi^+(g(x)),$$

where $\Phi^+ : \mathbb{R}^m \to \mathbb{R}, \Phi^+(t) := \Phi(t^+)$, with $t^+ = (t_1^+, \dots, t_m^+)^T$ and $t_i^+ = \max\{0, t_i\}$, $i = 1, \dots, m$.

Proposition 2.4 *([10]) The function $\Phi^+ : \mathbb{R}^m \to \mathbb{R}$ is convex and componentwise increasing.*

By the approach described in subsection 2.4.2 we obtain then as a dual problem to (P_Φ^c) the following optimization problem

$$(D_\Phi^c) \quad \sup_{\substack{\lambda \in \mathbb{R}^m, p_i \in X^*, \\ i=1,\dots,m, \sum\limits_{i=1}^m p_i = 0}} \left\{ -(\Phi^+)^*(\lambda) - \sum_{i=1}^m (\lambda_i g_i)^*(p_i) \right\}.$$

Proposition 2.5 *The conjugate function $(\Phi^+)^* : \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\}$ of Φ^+ verifies*

$$(\Phi^+)^*(\lambda) = \begin{cases} 0, & \text{if } \lambda \underset{\mathbb{R}^m_+}{\geqq} 0 \text{ and } \Phi^0(\lambda) \leq 1, \\ +\infty, & \text{otherwise,} \end{cases}$$

where Φ^0 is the dual norm of Φ in $\mathbb{R}^m$ and "$\underset{\mathbb{R}^m_+}{\geqq}$" is the partial ordering induced by the non-negative orthant $\mathbb{R}^m_+$.

Proof. Let be $\lambda \in \mathbb{R}^m$. For $t \in \mathbb{R}^m$, we have $|t_i| \geq |t_i^+|, i = 1, \dots, m$, which implies that $\Phi(t) \geq \Phi(t^+)$ and

$$\Phi^*(\lambda) = \sup_{t \in \mathbb{R}^m} \{\lambda^T t - \Phi(t)\} \leq \sup_{t \in \mathbb{R}^m} \{\lambda^T t - \Phi^+(t)\} = (\Phi^+)^*(\lambda). \tag{2. 40}$$

On the other hand, the conjugate of the norm Φ verifies the following formula (cf. [62])

$$\Phi^*(\lambda) = \sup_{t \in \mathbb{R}^m} \{\lambda^T t - \Phi(t)\} = \begin{cases} 0, & \text{if } \Phi^0(\lambda) \leq 1, \\ +\infty, & \text{otherwise.} \end{cases} \tag{2. 41}$$

If $\Phi^0(\lambda) > 1$, by (2. 40) and (2. 41), we have $+\infty = \Phi^*(\lambda) \leq (\Phi^+)^*(\lambda)$. From here, $(\Phi^+)^*(\lambda) = +\infty$.

Suppose now that $\Phi^0(\lambda) \leq 1$. If there exists an $i_0 \in \{1, \dots, m\}$ such that $\lambda_{i_0} < 0$, then we have

$$\begin{aligned} (\Phi^+)^*(\lambda) &= \sup_{t \in \mathbb{R}^m} \{\lambda^T t - \Phi^+(t)\} = \sup_{t \in \mathbb{R}^m} \{\lambda^T t - \Phi(t^+)\} \\ &\geq \sup_{t_{i_0} < 0} \{\lambda^T (0, \dots, t_{i_0}, \dots, 0)^T - \Phi((0, \dots, t_{i_0}, \dots, 0)^+)\} \\ &= \sup_{t_{i_0} < 0} \lambda_{i_0} t_{i_0} = +\infty. \end{aligned}$$

As in the previous case, $(\Phi^+)^*(\lambda) = +\infty$.

Finally, let be $\Phi^0(\lambda) \leq 1$ and $\lambda \underset{\mathbb{R}^m_+}{\geqq} 0$. For every $t \in \mathbb{R}^m$, it holds $\lambda^T t \leq \lambda^T t^+$ and $\lambda^T t^+ \leq \Phi(t^+)$. By the last two relations, the conjugate function of Φ^+ verifies the inequality

$$(\Phi^+)^*(\lambda) = \sup_{t \in \mathbb{R}^m} \{\lambda^T t - \Phi(t^+)\} \leq \sup_{t \in \mathbb{R}^m} \{\lambda^T t^+ - \Phi(t^+)\} \leq 0.$$

Again, by (2. 40) and (2. 41), we have $(\Phi^+)^*(\lambda) \geq \Phi^*(\lambda) = 0$. Therefore, we must have $(\Phi^+)^*(\lambda) = 0$. This concludes the proof. □

By Proposition 2.5 we obtain for (D^c_Φ) the following formulation

$$(D^c_\Phi) \quad \sup_{\substack{\lambda \in \mathbb{R}^m_+, p_i \in X^*, i=1,\dots,m, \\ \sum_{i=1}^m p_i = 0, \Phi^0(\lambda) \leq 1}} \left\{ -\sum_{i=1}^m (\lambda_i g_i)^*(p_i) \right\}.$$

In the objective function of the dual (D^c_Φ) we separate now the terms for which $\lambda_i > 0$ from the terms for which $\lambda_i = 0$. Then the dual can be written as

$$(D^c_\Phi) \quad \sup_{\substack{p_i \in X^*, i=1,\dots,m, \sum_{i=1}^m p_i = 0, \\ \Phi^0(\lambda) \leq 1, I \subseteq \{1,\dots,m\}, \\ \lambda_i > 0 (i \in I), \lambda_i = 0 (i \notin I)}} \left\{ -\sum_{i \in I} (\lambda_i g_i)^*(p_i) - \sum_{i \notin I} (0)^*(p_i) \right\}. \tag{2. 42}$$

For $i \notin I$, it holds

$$0^*(p_i) = \sup_{x\in X}\{\langle p_i, x\rangle - 0\} = \sup_{x\in X}\langle p_i, x\rangle = \begin{cases} 0, & \text{if } p_i = 0, \\ +\infty, & \text{otherwise,} \end{cases}$$

and this means that, in order to have supremum in (D^c_Φ), we must take $p_i = 0,\ \forall i \notin I$. Then the dual problem becomes

$$(D^c_\Phi) \quad \sup_{\substack{\Phi^0(\lambda)\leq 1, I\subseteq\{1,\ldots,m\},\\ \lambda_i>0(i\in I),\lambda_i=0(i\notin I),\\ p_i\in X^*, i\in I, \sum\limits_{i\in I} p_i=0}} \left\{-\sum_{i\in I}(\lambda_i g_i)^*(p_i)\right\}.$$

For $\lambda_i > 0, i \in I$, let us apply the following property of the conjugate functions $(\lambda_i g_i)^* = \lambda_i g_i^*\left(\frac{1}{\lambda_i}p_i\right),\ \forall i \in I$ (cf. [19]). Denoting $p_i := \frac{1}{\lambda_i}p_i$, we obtain finally the following formulation for the dual of (P^c_Φ)

$$(D^c_\Phi) \quad \sup_{(I,\lambda,p)\in Y_\Phi}\left\{-\sum_{i\in I}\lambda_i g_i^*(p_i)\right\},$$

with

$$Y_\Phi = \Bigg\{(I,\lambda,p) \quad : \quad I \subseteq \{1,\ldots,m\}, \lambda = (\lambda_1,\ldots,\lambda_m)^T, p = (p_1,\ldots,p_m),$$
$$\Phi^0(\lambda) \leq 1, \lambda_i > 0(i\in I), \lambda_i = 0(i \notin I), \sum_{i\in I}\lambda_i p_i = 0\Bigg\}.$$

In Proposition 2.4 we have shown that Φ^+ is a convex and componentwise increasing function. Moreover, one can observe that the optimal objective value of (P^c_Φ), $inf(P^c_\Phi)$, is finite, being greater than or equal to zero. This fact, together with Theorem 2.12, permits us to formulate the following strong duality theorem for the problems (P^c_Φ) and (D^c_Φ).

Theorem 2.14 (strong duality for (D^c_Φ)) *The dual problem (D^c_Φ) has an optimal solution and strong duality holds, i.e.*

$$inf(P^c_\Phi) = max(D^c_\Phi).$$

As for the general problem (P^c), we can derive the optimality conditions for (P^c_Φ). The proof of the next theorem can be found in [10].

Theorem 2.15 (optimality conditions for (P^c_Φ))
(a) Let $\bar{x} \in X$ be a solution to (P^c_Φ). Then there exists $(\bar{I}, \bar{\lambda}, \bar{p}) \in Y_\Phi$, solution to (D^c_Φ), such that the following optimality conditions are satisfied

$$(i) \quad \bar{I} \subseteq \{1,\ldots,m\}, \bar{\lambda}_i > 0(i\in\bar{I}), \bar{\lambda}_i = 0(i\notin\bar{I}),$$

$$(ii) \quad \Phi^0(\bar{\lambda}) \leq 1, \sum_{i\in\bar{I}}\bar{\lambda}_i\bar{p}_i = 0,$$

$$(iii) \quad \Phi^+(g(\bar{x})) = \sum_{i\in\bar{I}}\bar{\lambda}_i g_i(\bar{x}),$$

$$(iv) \quad g_i(\bar{x}) + g_i^*(\bar{p}_i) = \langle\bar{p}_i, \bar{x}\rangle, i \in \bar{I}.$$

(b) If $\bar{x} \in X$, $(\bar{I}, \bar{\lambda}, \bar{p}) \in Y_\Phi$ and (i)-(iv) are satisfied, then $\bar{x}$ is an optimal solution to (P_Φ), $(\bar{I}, \bar{\lambda}, \bar{p}) \in Y_\Phi$ is an optimal solution to (D_Φ) and strong duality holds

$$\Phi^+(g(\bar{x})) = -\sum_{i \in \bar{I}} \bar{\lambda}_i g_i^*(\bar{p}_i).$$

Remark 2.11 In Theorem 2.15 we do not exclude the possibility that the set $\bar{I}$ could be empty. This would mean that $\bar{\lambda} = 0$ and, from (iii), $\Phi^+(g(\bar{x})) = 0$. But, this can happen only if the following equivalent relations are true

$$\Phi(g(\bar{x})^+) = 0 \Leftrightarrow g^+(\bar{x}) = 0 \Leftrightarrow g_i^+(\bar{x}) = 0, i = 1, \ldots, m \Leftrightarrow g_i(\bar{x}) \leq 0, i = 1, \ldots, m.$$

2.4.4 The location model involving sets as existing facilities

After we studied in the previous subsections the duality for two quite general optimization problems, we consider now the problem treated by Nickel, Puerto and Rodriguez-Chia in [57]. This problem is a single facility location problem in a general normed space in which the existing facilities are represented by sets.

Let $\mathcal{A} = \{A_1, \ldots, A_m\}$ be a family of convex sets in X such that $\bigcap_{i=1}^{m} cl(A_i) = \emptyset$. For $i = 1, ..., m$, we consider $g_i : X \to \mathbb{R}, g_i(x) = d_i(x, A_i)$, where

$$d_i(x, A_i) = \inf\{\gamma_i(x - a_i) : a_i \in A_i\}.$$

Here, γ_i is a continuous norm on X, for $i = 1, \ldots, m$. This means that the functions $g_i, i = 1, ..., m$, are convex and continuous on X.

Let $d : X \to \mathbb{R}^m$ be the vector function defined by

$$d(x) = (d_1(x, A_1), \ldots, d_m(x, A_m))^T.$$

The location problem with sets as existing facilities studied in [57] is

$$(P_\Phi^c(\mathcal{A})) \quad \inf_{x \in X} \Phi(d(x)).$$

Because

$$\Phi^+(d(x)) = \Phi(d^+(x)) = \Phi(d(x)),\ \forall x \in X,$$

we can write $(P_\Phi^c(\mathcal{A}))$ in the equivalent form

$$(P_\Phi^c(\mathcal{A})) \quad \inf_{x \in X} \Phi^+(d(x)).$$

Therefore, the problem $(P_\Phi^c(\mathcal{A}))$ turns out to be a particular case of (P_Φ^c) and its dual has then the following form

$$(D_\Phi^c(\mathcal{A})) \quad \sup_{(I,\lambda,p) \in Y_\Phi(\mathcal{A})} \left\{ -\sum_{i \in I} \lambda_i d_i^*(p_i) \right\},$$

with

$$Y_\Phi(\mathcal{A}) = \Bigg\{ (I, \lambda, p) \quad : \quad I \subseteq \{1, \ldots, m\}, \lambda = (\lambda_1, \ldots, \lambda_m)^T, p = (p_1, \ldots, p_m),$$
$$\Phi^0(\lambda) \leq 1, \lambda_i > 0 (i \in I), \lambda_i = 0 (i \notin I), \sum_{i \in I} \lambda_i p_i = 0 \Bigg\}.$$

By the use of the Theorems 2.14 and 2.15 we can give for $(P_\Phi^c(\mathcal{A}))$ and $(D_\Phi^c(\mathcal{A}))$ the strong duality theorem and the optimality conditions.

Theorem 2.16 (strong duality for $(D^c_\Phi(\mathcal{A}))$) *The dual problem $(D^c_\Phi(\mathcal{A}))$ has an optimal solution and strong duality holds, i.e.*

$$inf(P^c_\Phi(\mathcal{A})) = max(D^c_\Phi(\mathcal{A})).$$

Theorem 2.17 (optimality conditions for $(P^c_\Phi(\mathcal{A}))$)
(a) Let $\bar{x} \in X$ be a solution to $(P^c_\Phi(\mathcal{A}))$. Then there exists $(\bar{I}, \bar{\lambda}, \bar{p}) \in Y_\Phi(\mathcal{A})$, solution to $(D^c_\Phi(\mathcal{A}))$, such that the following optimality conditions are satisfied

$$\begin{aligned}
(i) &\quad \bar{I} \subseteq \{1, \ldots, m\}, \bar{I} \neq \emptyset, \bar{\lambda}_i > 0 (i \in \bar{I}), \bar{\lambda}_i = 0 (i \notin \bar{I}),\\
(ii) &\quad \Phi^0(\bar{\lambda}) = 1, \sum_{i \in \bar{I}} \bar{\lambda}_i \bar{p}_i = 0,\\
(iii) &\quad \Phi(d(\bar{x})) = \sum_{i \in \bar{I}} \bar{\lambda}_i d_i(\bar{x}, A_i),\\
(iv) &\quad \bar{x} \in \partial d_i^*(\bar{p}_i), i \in \bar{I}.
\end{aligned}$$

(b) If $\bar{x} \in X$, $(\bar{I}, \bar{\lambda}, \bar{p}) \in Y_\Phi(\mathcal{A})$ and (i) - (iv) are satisfied, then $\bar{x}$ is an optimal solution to $(P^c_\Phi(\mathcal{A}))$, $(\bar{I}, \bar{\lambda}, \bar{p}) \in Y_\Phi(\mathcal{A})$ is an optimal solution to $(D^c_\Phi(\mathcal{A}))$ and strong duality holds

$$\Phi(d(\bar{x})) = \sum_{i \in \bar{I}} \bar{\lambda}_i d_i(\bar{x}, A_i) = -\sum_{i \in \bar{I}} \bar{\lambda}_i d_i^*(\bar{p}_i).$$

Proof.

(a) By Theorem 2.15 follows that there exists $(\bar{I}, \bar{\lambda}, \bar{p}) \in Y_\Phi(\mathcal{A})$, solution to $(D^c_\Phi(\mathcal{A}))$, such that

$$\begin{aligned}
(i') &\quad \bar{I} \subseteq \{1, \ldots, m\}, \bar{\lambda}_i > 0 (i \in \bar{I}), \bar{\lambda}_i = 0 (i \notin \bar{I}),\\
(ii') &\quad \Phi^0(\bar{\lambda}) \leq 1, \sum_{i \in \bar{I}} \bar{\lambda}_i \bar{p}_i = 0,\\
(iii') &\quad \Phi^+(d(\bar{x})) = \sum_{i \in \bar{I}} \bar{\lambda}_i d_i(\bar{x}, A_i),\\
(iv') &\quad d_i(\bar{x}, A_i) + d_i^*(\bar{p}_i) = \langle \bar{p}_i, \bar{x} \rangle, i \in \bar{I}.
\end{aligned}$$

We prove that $(\bar{I}, \bar{\lambda}, \bar{p})$ satisfies the relations (i)-(iv). If $\bar{I}$ were empty, then by Remark 2.11, it would follow that

$$g_i(\bar{x}) = d_i(\bar{x}, A_i) = 0, i = 1, \ldots, m.$$

But, this would imply that $\bar{x}$ belongs to $\bigcap_{i=1}^{m} cl(A_i)$, which contradicts the hypothesis $\bigcap_{i=1}^{m} cl(A_i) = \emptyset$. By this, relation (i) is proved.

From (iii') we have that

$$\Phi^+(d(\bar{x})) = \Phi(d(\bar{x})) = \sum_{i \in \bar{I}} \bar{\lambda}_i d_i(\bar{x}, A_i), \tag{2. 43}$$

and (iii) is also proved.

From (iv') we have that $\bar{p}_i \in \partial d_i(\bar{x}, A_i)$, for $i \in \bar{I}$ (cf. [19]). On the other hand, the distance function d_i being convex and continuous verifies the following (cf. [19] and [95])

$$\bar{p}_i \in \partial d_i(\bar{x}, A_i) \Leftrightarrow \bar{x} \in \partial d_i^*(\bar{p}_i), \ \forall i \in \bar{I},$$

that proves (iv).

In order to finish the proof we have to show that $\Phi^0(\bar{\lambda}) = 1$. By the definition of the dual norm it holds

$$\Phi^0(\bar{\lambda}) = \sup_{\substack{\Phi(v)\leq 1,\\ v\in\mathbb{R}^m}} |\left\langle \bar{\lambda}, v \right\rangle|.$$

Because $\bigcap\limits_{i=1}^{m} cl(A_i) = \emptyset$ it holds $\Phi(d(\bar{x})) > 0$. Let be $\bar{v} := \frac{1}{\Phi(d(\bar{x}))} d(\bar{x}) \in \mathbb{R}^m$. We have $\Phi(\bar{v}) = 1$ and, by (iii) and (2. 43),

$$\Phi^0(\bar{\lambda}) \geq \left\langle \bar{\lambda}, \bar{v} \right\rangle = \frac{\sum\limits_{i\in\bar{I}} \bar{\lambda}_i d_i(\bar{x}, A_i)}{\Phi(d(\bar{x}))} = 1.$$

This last inequality, together with (ii'), gives $\Phi^0(\bar{\lambda}) = 1$.
(b) All the calculations and transformations done within part (a) may be carried out in the inverse direction. □

Remark 2.12

(a) Lemma 3.3 in [57] which characterizes the solutions of $(P^c_\Phi(\mathcal{A}))$ can be automatically obtained by means of the optimality conditions given in Theorem 2.17.

(b) In [57] the authors made the assumption that the sets $A_i, i = 1, ..., m$, have to be compact. As one can see, in order to formulate the strong duality theorem and the optimality conditions for $(P^c_\Phi(\mathcal{A}))$, the compactness of the sets $A_i, i = 1, ..., m$, is not necessary.

In the last two subsections we consider the Weber problem and the minmax problem with infimal distances and sets as existing facilities. For these problems we formulate the duals and present the optimality conditions. Therefore we write both problems, equivalently, in a form which appears to be a particularization of the problem $(P^c_\Phi(\mathcal{A}))$.

2.4.5 The Weber problem with infimal distances

The Weber problem with infimal distances for the data $\mathcal{A}$ is

$$(P^c_W(\mathcal{A})) \quad \inf_{x\in X} \sum_{i=1}^{m} w_i d_i(x, A_i),$$

where $d_i(x, A_i) = \inf\limits_{a_i\in A_i} \gamma_i(x - a_i)$, $i = 1, ..., m$, and $w_i > 0$, $i = 1, ..., m$, are positive weights.

We introduce now, for $i = 1, ..., m$, the continuous norms $\gamma'_i : X \to \mathbb{R}$, $\gamma'_i = w_i\gamma_i$ and the corresponding distance functions $d'_i(\cdot, A_i) : X \to \mathbb{R}$, $d'_i(x, A_i) = \inf\limits_{a_i\in A_i} \gamma'_i(x - a_i)$. This means that

$$d'_i(x, A_i) = \inf_{a_i\in A_i} \gamma'_i(x - a_i) = w_i d_i(x, A_i), i = 1, \ldots, m. \tag{2. 44}$$

By (2. 44) the primal problem $(P^c_W(\mathcal{A}))$ becomes

$$(P^c_W(\mathcal{A})) \quad \inf_{x\in X} \sum_{i=1}^{m} d'_i(x, A_i) = \inf_{x\in X} l_1(d'(x)),$$

where $d' : X \to \mathbb{R}^m$, $d'(x) = (d'_1(x, A_1), \ldots, d'_m(x, A_m))^T$ and $l_1 : \mathbb{R}^m \to \mathbb{R}, l_1(\lambda) = \sum\limits_{i=1}^{m} |\lambda_i|$. One may easy observe that the l_1-norm is a monotonic norm.

Then the dual problem of $(P_W^c(\mathcal{A}))$ is

$$(D_W^c(\mathcal{A})) \quad \sup_{(I,\lambda,p)\in Y_W(\mathcal{A})} \left\{ -\sum_{i\in I} \lambda_i (d'_i)^*(p_i) \right\},$$

with

$$Y_W(\mathcal{A}) = \Bigg\{ (I, \lambda, p) \quad : \quad I \subseteq \{1, \ldots, m\}, \lambda = (\lambda_1, \ldots, \lambda_m)^T, p = (p_1, \ldots, p_m),$$
$$l_1^0(\lambda) \leq 1, \lambda_i > 0 (i \in I), \lambda_i = 0 (i \notin I), \sum_{i\in I} \lambda_i p_i = 0 \Bigg\}.$$

For $i = 1, \ldots, m$, we have that (cf. [19]) $(d'_i)^*(p_i) = (w_i d_i)^*(p_i) = w_i d_i^* \left(\frac{1}{w_i} p_i \right)$. Otherwise, the dual norm of the l_1-norm is $l_1^0(\lambda) = \max\limits_{i=1,\ldots,m} |\lambda_i|$. Denoting $p_i := \frac{1}{w_i} p_i, i = 1, \ldots, m$, we obtain the following formulation

$$(D_W^c(\mathcal{A})) \quad \sup_{(I,\lambda,p)\in Y_W(\mathcal{A})} \left\{ -\sum_{i\in I} \lambda_i w_i d_i^*(p_i) \right\},$$

with

$$Y_W(\mathcal{A}) = \Bigg\{ (I, \lambda, p) \quad : \quad I \subseteq \{1, \ldots, m\}, \lambda = (\lambda_1, \ldots, \lambda_m)^T, p = (p_1, \ldots, p_m),$$
$$\max_{i\in I} \lambda_i \leq 1, \lambda_i > 0 (i \in I), \lambda_i = 0 (i \notin I), \sum_{i\in I} \lambda_i w_i p_i = 0 \Bigg\}.$$

Let us give now the strong duality theorem and the optimality conditions for $(P_W^c(\mathcal{A}))$ and its dual $(D_W^c(\mathcal{A}))$ (for the proofs see [10]).

Theorem 2.18 (strong duality for $(D_W^c(\mathcal{A}))$) *The dual problem $(D_W^c(\mathcal{A}))$ has an optimal solution and strong duality holds, i.e.*

$$inf(P_W^c(\mathcal{A})) = max(D_W^c(\mathcal{A})).$$

Theorem 2.19 (optimality conditions for $(P_W^c(\mathcal{A}))$)
(a) Let $\bar{x} \in X$ be a solution to $(P_W^c(\mathcal{A}))$. Then there exists $(\bar{I}, \bar{\lambda}, \bar{p}) \in Y_W(\mathcal{A})$, optimal solution to $(D_W^c(\mathcal{A}))$, such that the following optimality conditions are satisfied

$$(i) \quad \bar{I} \subseteq \{1, \ldots, m\}, \bar{I} \neq \emptyset, \bar{\lambda}_i = 1 (i \in \bar{I}), \bar{\lambda}_i = 0 (i \notin \bar{I}),$$

$$(ii) \quad \sum_{i\in \bar{I}} w_i \bar{p}_i = 0,$$

$$(iii) \quad \sum_{i=1}^{m} w_i d_i(\bar{x}, A_i) = \sum_{i\in \bar{I}} w_i d_i(\bar{x}, A_i),$$

$$(iv) \quad \bar{x} \in \partial d_i^*(\bar{p}_i), i \in \bar{I}.$$

(b) If $\bar{x} \in X$, $(\bar{I}, \bar{\lambda}, \bar{p}) \in Y_W(\mathcal{A})$ and (i)-(iv) are satisfied, then $\bar{x}$ is an optimal solution to $(P_W^c(\mathcal{A}))$, $(\bar{I}, \bar{\lambda}, \bar{p}) \in Y_W(\mathcal{A})$ is an optimal solution to $(D_W^c(\mathcal{A}))$ and strong duality holds

$$\sum_{i=1}^{m} w_i d_i(\bar{x}, A_i) = \sum_{i\in \bar{I}} w_i d_i(\bar{x}, A_i) = -\sum_{i\in \bar{I}} \bar{\lambda}_i w_i d_i^*(\bar{p}_i).$$

We finish this subsection considering a particular instance of the Weber problem $(P_W^c(\mathcal{A}))$. Therefore, we assume that A_i is a singleton, in fact, that $A_i = \{x_i\}$, where $x_i \in X$, $i = 1, ..., m$. Moreover, we assume that the norms γ_i are all equal with $\|\cdot\|$, the norm which equips the space X. The problem $(P_W^c(\mathcal{A}))$ becomes then

$$\inf_{x \in X} \sum_{i=1}^{m} w_i \|x - x_i\|,$$

which is the standard so-called Weber location problem in a normed space.

For the conjugate of the function d_i, $i = 1, ..., m$, we have

$$\begin{aligned} d_i^*(p_i) &= \sup_{x \in X} \{\langle p_i, x\rangle - \|x - x_i\|\} = \sup_{x \in X} \{\langle p_i, x - x_i\rangle - \|x - x_i\|\} + \langle p_i, x_i\rangle \\ &= \begin{cases} \langle p_i, x_i\rangle, & \text{if } \|p_i\|^0 \leq 1, \\ +\infty, & \text{otherwise,} \end{cases} \end{aligned}$$

where $\|\cdot\|^0$ represents the dual norm of $\|\cdot\|$. The dual problem $(D_W^c(\mathcal{A}))$ can be now written as

$$\sup_{\substack{(I,\lambda,p) \in Y_W(\mathcal{A}), \\ \|p_i\|^0 \leq 1, i=1,\dots,m}} \left\{ -\sum_{i \in I} \lambda_i w_i \langle p_i, x_i\rangle \right\},$$

with

$$\begin{aligned} Y_W(\mathcal{A}) = \Bigg\{ (I, \lambda, p) \quad : \quad & I \subseteq \{1, \dots, m\}, \lambda = (\lambda_1, \dots, \lambda_m)^T, p = (p_1, \dots, p_m), \\ & \max_{i \in I} \lambda_i \leq 1, \lambda_i > 0 (i \in I), \lambda_i = 0 (i \notin I), \sum_{i \in I} \lambda_i w_i p_i = 0 \Bigg\}. \end{aligned}$$

Denoting $q_i := -\lambda_i p_i$, for $i = 1, ..., m$, the dual of the standard location problem in a normed space becomes

$$\sup_{\substack{q_i \in X^*, \|q_i\|^0 \leq 1, \\ i=1,\dots,m, \\ \sum_{i=1}^m w_i q_i = 0}} \left\{ \sum_{i=1}^{m} w_i \langle q_i, x_i\rangle \right\}.$$

The first works which deal with duality for location problems and where this result also appears are those of Kuhn [46] in finite dimensional spaces and Rubinstein [64] in general Banach spaces. For further results concerning duality for the scalar location problem see also the paper of Wanka [79].

2.4.6 The minmax problem with infimal distances

The last optimization problem that we consider in this section is the minmax problem with infimal distances for the data $\mathcal{A}$,

$$(P_H^c(\mathcal{A})) \quad \inf_{x \in X} \max_{i=1,\dots,m} w_i d_i(x, A_i),$$

where $d_i(x, A_i) = \inf_{a_i \in A_i} \gamma_i(x - a_i)$, $i = 1, ..., m$, and $w_i > 0$, $i = 1, ..., m$, are positive weights.

As for the Weber problem studied above let be, for $i = 1, ..., m$, the continuous norms $\gamma_i' : X \to \mathbb{R}$, $\gamma_i' = w_i \gamma_i$ and the corresponding distance functions $d_i'(\cdot, A_i) : X \to \mathbb{R}$, $d_i'(x, A_i) = \inf_{a_i \in A_i} \gamma_i'(x - a_i)$.

This means that the equality in (2. 44) remains true and, so, the primal problem $(P_H^c(\mathcal{A}))$ becomes

$$(P_H^c(\mathcal{A})) \quad \inf_{x\in X} \max_{i=1,\ldots,m} d_i'(x, A_i) = \inf_{x\in X} l_\infty(d'(x)),$$

where $d' : X \to \mathbb{R}^m$, $d'(x) = (d_1'(x, A_1), \ldots, d_m'(x, A_m))^T$ and $l_\infty : \mathbb{R}^m \to \mathbb{R}, l_\infty(\lambda) = \max\limits_{i=1,\ldots,m} |\lambda_i|$. The l_∞-norm is also a monotonic norm and its dual norm is $l_\infty^0(\lambda) = \sum\limits_{i=1}^{m} |\lambda_i|$.

Then the dual problem of $(P_H^c(\mathcal{A}))$ is

$$(D_H^c(\mathcal{A})) \quad \sup_{(I,\lambda,p)\in Y_H(\mathcal{A})} \left\{ -\sum_{i\in I} \lambda_i w_i d_i^*(p_i) \right\},$$

with

$$Y_H(\mathcal{A}) = \Bigg\{ (I, \lambda, p) \quad : \quad I \subseteq \{1, \ldots, m\}, \lambda = (\lambda_1, \ldots, \lambda_m)^T, p = (p_1, \ldots, p_m),$$
$$\sum_{i\in I} \lambda_i \leq 1, \lambda_i > 0 (i \in I), \lambda_i = 0 (i \notin I), \sum_{i\in I} \lambda_i w_i p_i = 0 \Bigg\}.$$

Like for the Weber problem we can give the strong duality theorem and formulate the optimality conditions (for the proofs see [10]).

Theorem 2.20 (strong duality for $(D_H^c(\mathcal{A}))$ *The dual problem* $(D_H^c(\mathcal{A}))$ *has an optimal solution and strong duality holds, i.e.*

$$inf(P_H^c(\mathcal{A})) = max(D_H^c(\mathcal{A})).$$

Theorem 2.21 (optimality conditions for $(P_H^c(\mathcal{A})))$
(a) Let $\bar{x} \in X$ *be a solution to* $(P_H^c(\mathcal{A}))$*. Then there exists* $(\bar{I}, \bar{\lambda}, \bar{p}) \in Y_H(\mathcal{A})$*, optimal solution to* $(D_H^c(\mathcal{A}))$*, such that the following optimality conditions are satisfied*

$$(i) \quad \bar{I} \subseteq \{1, \ldots, m\}, \bar{I} \neq \emptyset, \bar{\lambda}_i > 0 (i \in \bar{I}), \bar{\lambda}_i = 0 (i \notin \bar{I}),$$

$$(ii) \quad \sum_{i\in\bar{I}} \bar{\lambda}_i = 1, \sum_{i\in\bar{I}} w_i \bar{\lambda}_i \bar{p}_i = 0,$$

$$(iii) \quad \max_{i=1,\ldots,m} w_i d_i(\bar{x}, A_i) = w_i d_i(\bar{x}, A_i), \ \forall i \in \bar{I},$$

$$(iv) \quad \bar{x} \in \partial d_i^*(\bar{p}_i), i \in \bar{I}.$$

(b) If $\bar{x} \in X$*,* $(\bar{I}, \bar{\lambda}, \bar{p}) \in Y_H(\mathcal{A})$ *and (i)-(iv) are satisfied, then* $\bar{x}$ *is an optimal solution to* $(P_H^c(\mathcal{A}))$*,* $(\bar{I}, \bar{\lambda}, \bar{p}) \in Y_H(\mathcal{A})$ *is an optimal solution to* $(D_H^c(\mathcal{A}))$ *and strong duality holds*

$$\max_{i=1,\ldots,m} w_i d_i(\bar{x}, A_i) = \sum_{i\in\bar{I}} \bar{\lambda}_i w_i d_i(\bar{x}, A_i) = -\sum_{i\in\bar{I}} \bar{\lambda}_i w_i d_i^*(\bar{p}_i).$$

Chapter 3

Duality for multiobjective convex optimization problems

The third chapter of this work deals with duality in multiobjective optimization. It contains two different parts referring to two different types of vector optimization problems, namely, a general convex multiobjective problem with cone inequality constraints (cf. Wanka and Boţ [85]) and a particular multiobjective fractional programming problem with linear inequality constraints (cf. Wanka and Boţ [87]). In both cases, the basic idea is to establish a dual problem to an scalarized problem associated to the multiobjective primal. The scalar dual is formulated in terms of conjugate functions and its structure gives an idea about how to construct a multiobjective dual in a natural way. The existence of weak and, under certain conditions, of strong duality between the primal and the dual problem is shown.

3.1 A new duality approach

3.1.1 Motivation

The duality approach for general convex multiobjective optimization problems, which we present here, may be seen as a rigorous application of conjugate duality to such problems. The objective function of the dual is represented in a closed form, wherein the conjugate of the objective functions of the primal problem as well as the conjugates of the functions describing the set of constraints appear in a clear and natural way. The dual constraint adopts a simple form of only two conditions, a bilinear inequality and a scalar product to be zero.

In this representation, this dual problem differs from other known formulations of multiobjective duals found in the literature. Otherwise, it extends our former investigations concerning duality for vector optimization problems with convex objective functions and linear inequality constraints (cf. Wanka and Boţ [83], [84]). We also notice that the duality results presented in [83] and [84] generalize some previous results established in the past by different authors for more special problems, in particular, multiobjective location and control-approximation problems (cf. Tammer and Tammer [72], Wanka [81], [80], [82]).

Among the theories dealing with different duality approaches for similar multiobjective optimization problems we mention as a representative selection those developed by Jahn [40], [41], Nakayama [54], [55] and Weir and Mond [90],

[92], [93]. An comprehensive analysis of these duality concepts will be done in the next chapter.

In the approach we present here the new idea is to use a dual problem of the scalarized primal obtained by means of the conjugacy duality theory (cf. chapter 2). The scalar dual problem turns out to have a form adapted for generating in a natural way a conjugate multiobjective dual problem to the original one that allows to prove weak and strong duality. Moreover, a converse duality assertion will be also verified. In the last part of the section some special cases of vector optimization problems with linear constraints, which can be obtained from the general result are summarized. On the other hand, a dual for the multiobjective convex semidefinite programming problem is presented.

3.1.2 Problem formulation

The primal multiobjective optimization problem with cone inequality constraints which we consider here is the following one

$$(P) \quad \operatorname*{v\text{-}min}_{x \in \mathcal{A}} f(x),$$

$$\mathcal{A} = \left\{ x \in \mathbb{R}^n : g(x) \underset{K}{\leqq} 0 \right\},$$

$$f(x) = (f_1(x), \dots, f_m(x))^T,$$

$$g(x) = (g_1(x), \dots, g_k(x))^T.$$

For $i = 1, ..., m$, $f_i : \mathbb{R}^n \to \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ are proper and convex functions with the property that $\bigcap\limits_{i=1}^{m} ri(dom(f_i)) \neq \emptyset$, where $ri(dom(f_i))$ represents the relative interior of the set $dom(f_i) = \{x \in \mathbb{R}^n : f_i(x) < +\infty\}$. The function $g : \mathbb{R}^n \to \mathbb{R}^k$ is convex relative to the cone $K \subseteq \mathbb{R}^k$. K is a convex closed cone with $int(K) \neq \emptyset$ which defines a partial ordering on $\mathbb{R}^k$ according to $x_1 \underset{K}{\leqq} x_2$ if and only if $x_2 - x_1 \in K$.

The "v-min" term means that we ask for Pareto-efficient solutions of the problem (P). This kind of solutions is obtained by using the dominance structure given by the non-negative orthant $\mathbb{R}^m_+ = \{x = (x_1, ..., x_m)^T \in \mathbb{R}^m : x_i \geq 0, i = 1, \dots, m\}$ on $\mathbb{R}^m$.

Definition 3.1 *An element $\bar{x} \in \mathcal{A}$ is said to be efficient (or Pareto-efficient) with respect to (P) if from $f(\bar{x}) \underset{R^m_+}{\geqq} f(x)$, for $x \in \mathcal{A}$, follows $f(\bar{x}) = f(x)$.*

Another kind of solutions which we use in this chapter are the properly efficient solutions. This a strengthened solution concept and, in order to introduce it, we use the definition given by Geoffrion [28].

Definition 3.2 *An element $\bar{x} \in \mathcal{A}$ is said to be properly efficient with respect to (P) if it is efficient and if there exists a number $M > 0$ such that for each $i \in \{1, ..., m\}$ and $x \in \mathcal{A}$ satisfying $f_i(x) < f_i(\bar{x})$, there exists at least one $j \in \{1, ..., m\}$ such that $f_j(\bar{x}) < f_j(x)$ and*

$$\frac{f_i(\bar{x}) - f_i(x)}{f_j(x) - f_j(\bar{x})} \leq M.$$

Other well-known definitions for the concept of proper efficiency have been given by Borwein [7], Benson [6] and Henig [33] for vector optimization problems in general partially ordered vector spaces and/or with the ordering cone a general closed convex cone. But, in our case all these four concepts are equivalent (see for instance the results presented in section 3.1.2 in the book of Sawaragi, Nakayama, Tanino [65]) and, more than that, they can be characterized via scalarization, as we do in the following definition (see Theorem 3.4.1 and Theorem 3.4.2 in [65]).

Definition 3.3 *An element $\bar{x} \in \mathcal{A}$ is said to be properly efficient with respect to (P) if there exists $\lambda = (\lambda_1, \ldots, \lambda_m)^T \in int(R^m_+)$ (i.e. $\lambda_i > 0, i = 1, \ldots, m$) such that $\sum\limits_{i=1}^m \lambda_i f_i(\bar{x}) \leq \sum\limits_{i=1}^m \lambda_i f_i(x)$, $\forall x \in \mathcal{A}$.*

3.1.3 Duality for the scalarized problem

In order to study the duality for the multiobjective problem (P) we study first the duality for the scalarized problem

$$(P^\lambda) \quad \inf_{x \in \mathcal{A}} \sum_{i=1}^m \lambda_i f_i(x),$$

where $\lambda = (\lambda_1, \ldots, \lambda_m)^T$ is a fixed vector in $int(\mathbb{R}^m_+)$.

For $\tilde{f} : \mathbb{R}^n \to \overline{\mathbb{R}}$, $\tilde{f}(x) = \sum\limits_{i=1}^m \lambda_i f_i(x)$, the problem (P^λ) can be written as

$$(P^\lambda) \quad \inf_{x \in \mathcal{A}} \tilde{f}(x),$$

where $\mathcal{A} = \left\{ x \in \mathbb{R}^n : g(x) \underset{K}{\leqq} 0 \right\}$. Then the Fenchel-Lagrange dual problem (cf. subsection 2.1.4) of the problem (P^λ) is

$$(D^\lambda) \quad \sup_{\substack{\tilde{p} \in \mathbb{R}^n, \\ q \underset{K^*}{\geqq} 0}} \left\{ -\tilde{f}^*(\tilde{p}) + \inf_{x \in \mathbb{R}^n} \left[\tilde{p}^T x + q^T g(x) \right] \right\}.$$

Replacing $\tilde{f}$ by its formula, we get

$$(D^\lambda) \quad \sup_{\substack{\tilde{p} \in \mathbb{R}^n, \\ q \underset{K^*}{\geqq} 0}} \left\{ - \left(\sum_{i=1}^m \lambda_i f_i \right)^* (\tilde{p}) + \inf_{x \in \mathbb{R}^n} \left[\tilde{p}^T x + q^T g(x) \right] \right\}.$$

Because of $\bigcap\limits_{i=1}^m ri(dom(f_i)) \neq \emptyset$ we have (cf. Theorem 16.4 in [62])

$$\left(\sum_{i=1}^m \lambda_i f_i \right)^* (\tilde{p}) = \inf \left\{ \sum_{i=1}^m (\lambda_i f_i)^*(\tilde{p}_i) : \sum_{i=1}^m \tilde{p}_i = \tilde{p} \right\}$$

and the dual (D^λ) becomes

$$(D^\lambda) \quad \sup_{\substack{\tilde{p} \in \mathbb{R}^n, q \underset{K^*}{\geqq} 0, \\ \tilde{p}_i \in \mathbb{R}^n, \sum\limits_{i=1}^m \tilde{p}_i = \tilde{p}}} \left\{ - \sum_{i=1}^m (\lambda_i f_i)^*(\tilde{p}_i) + \inf_{x \in \mathbb{R}^n} \left[\tilde{p}^T x + q^T g(x) \right] \right\}.$$

But $(\lambda_i f_i)^*(\tilde{p}_i) = \lambda_i f_i^*(\frac{\tilde{p}_i}{\lambda_i})$, for $i = 1, \ldots, m$, and, therefore, we can make the substitutions $p_i := \frac{\tilde{p}_i}{\lambda_i}, i = 1, \ldots, m$. So, $\tilde{p} = \sum\limits_{i=1}^m \lambda_i p_i$ and omitting $\tilde{p}$ we obtain for the dual of (P^{λ})

$$(D^{\lambda}) \quad \sup_{\substack{p_i\in\mathbb{R}^n, i=1,\dots,m,\\ q \underset{K^*}{\geqq} 0}} \left\{ -\sum_{i=1}^{m} \lambda_i f_i^*(p_i) + \inf_{x\in\mathbb{R}^n} \left[\left(\sum_{i=1}^{m} \lambda_i p_i \right)^T x + q^T g(x) \right] \right\},$$

or, equivalently,

$$(D^{\lambda}) \quad \sup_{\substack{p_i\in\mathbb{R}^n, i=1,\dots,m,\\ q \underset{K^*}{\geqq} 0}} \left\{ -\sum_{i=1}^{m} \lambda_i f_i^*(p_i) - (q^T g)^* \left(-\sum_{i=1}^{m} \lambda_i p_i \right) \right\}.$$

The reason why we consider the dual in this form is because, as one can see in the next subsection, (D^{λ}) will suggest us the form of the dual for the vector problem (P).

Let us notice that we use here the Fenchel-Lagrange duality concept introduced in chapter 2 even if one of the assumptions imposed there, $dom(\tilde{f}) = X$, is not fulfilled. For the problem (P^{λ}), we have $dom(\tilde{f}) = \bigcap_{i=1}^{m} dom(f_i) \subseteq \mathbb{R}^n = X$. But one can verify that in this situation the strong duality results presented in chapter 2 still remain valid. In conclusion, by means of the strong duality results proved there, we are able to present a strong duality theorem for (P^{λ}) and (D^{λ}). Therefore, we need the following constraint qualification

(CQ) | there exists an element $x' \in \bigcap_{i=1}^{m} dom(f_i)$ such that $g(x') = (g_1(x'), \dots, g_m(x'))^T \in -int(K)$.

According to Theorem 2.6 and Theorem 16.4 in [62], we can formulate the following strong duality theorem.

Theorem 3.1 *Let the optimal objective value of (P^{λ}) be finite and assume that there exists an element $x' \in \bigcap_{i=1}^{m} dom(f_i)$ such that $g(x') \in -int(K)$ (i.e. the constraint qualification (CQ) is fulfilled). Then the dual problem (D^{λ}) has an optimal solution and strong duality holds*

$$inf(P^{\lambda}) = max(D^{\lambda}).$$

For later investigations we need the optimality conditions regarding the scalar problem (P^{λ}) and its dual (D^{λ}). They can be derived in the same way as we did in the proof of Theorem 2.11. The following theorem gives us these conditions (for the proof see [85]).

Theorem 3.2 *(a) Let the constraint qualification (CQ) be fulfilled and let $\bar{x}$ be a solution to (P^{λ}). Then there exists $(\bar{p}, \bar{q}), \bar{p} = (\bar{p}_1, \dots, \bar{p}_m) \in \mathbb{R}^n \times \dots \times \mathbb{R}^n, \bar{q} \underset{K^*}{\geqq} 0$, optimal solution to (D^{λ}), such that the following optimality conditions are satisfied*

$$(i) \quad f_i^*(\bar{p}_i) + f_i(\bar{x}) = \bar{p}_i^T \bar{x}, \quad i = 1, \dots, m,$$

$$(ii) \quad \bar{q}^T g(\bar{x}) = 0,$$

$$(iii) \quad \left(\sum_{i=1}^{m} \lambda_i \bar{p}_i \right)^T \bar{x} = \inf_{x\in\mathbb{R}^n} \left[\left(\sum_{i=1}^{m} \lambda_i \bar{p}_i \right)^T x + \bar{q}^T g(x) \right].$$

(b) Let $\bar{x}$ be admissible to (P^λ) and $(\bar{p}, \bar{q})$ be admissible to (D^λ), satisfying (i), (ii) and (iii). Then $\bar{x}$ is an optimal solution to (P^λ), $(\bar{p}, \bar{q})$ is an optimal solution to (D^λ) and strong duality holds

$$\sum_{i=1}^m \lambda_i f_i(\bar{x}) = -\sum_{i=1}^m \lambda_i f_i^*(\bar{p}_i) + \inf_{x \in \mathbb{R}^n} \left[\left(\sum_{i=1}^m \lambda_i \bar{p}_i \right)^T x + \bar{q}^T g(x) \right].$$

Remark 3.1 Using the definition of the conjugate functions, relation (iii) in Theorem 3.2 (a) can be written equivalently in the following form

$$(\bar{q}^T g)^* \left(-\sum_{i=1}^m \lambda_i \bar{p}_i \right) = -\left(\sum_{i=1}^m \lambda_i \bar{p}_i \right)^T \bar{x}. \tag{3. 1}$$

3.1.4 The multiobjective dual problem

Now we are able to formulate a multiobjective dual to (P). The dual (D) will be a vector maximum problem and for it Pareto-efficient solutions in the sense of maximum are considered. After we introduce the multiobjective dual (D) we prove the weak and strong duality theorems.

The dual multiobjective optimization problem (D) is

$$(D) \quad \underset{(p,q,\lambda,t)\in\mathcal{B}}{\text{v-max}} \, h(p, q, \lambda, t),$$

with

$$h(p, q, \lambda, t) = \begin{pmatrix} h_1(p, q, \lambda, t) \\ \vdots \\ h_m(p, q, \lambda, t) \end{pmatrix},$$

$$h_j(p, q, \lambda, t) = -f_j^*(p_j) - (q_j^T g)^* \left(-\frac{1}{m\lambda_j} \sum_{i=1}^m \lambda_i p_i \right) + t_j, j = 1, \ldots, m,$$

the dual variables

$$p = (p_1, \ldots, p_m) \in \mathbb{R}^n \times ... \times \mathbb{R}^n, q = (q_1, \ldots, q_m) \in \mathbb{R}^k \times ... \times \mathbb{R}^k,$$

$$\lambda = (\lambda_1, \ldots, \lambda_m)^T \in \mathbb{R}^m, t = (t_1, \ldots, t_m)^T \in \mathbb{R}^m,$$

and the set of constraints

$$\mathcal{B} = \left\{ (p, q, \lambda, t) : \lambda \in int(\mathbb{R}^m_+), \quad \sum_{i=1}^m \lambda_i q_i \underset{K^*}{\geqq} 0, \quad \sum_{i=1}^m \lambda_i t_i = 0 \right\}. \tag{3. 2}$$

Definition 3.4 *An element $(\bar{p}, \bar{q}, \bar{\lambda}, \bar{t}) \in \mathcal{B}$ is said to be efficient (or Pareto-efficient) with respect to (D) if from $h(p, q, \lambda, t) \underset{R^m_+}{\geqq} h(\bar{p}, \bar{q}, \bar{\lambda}, \bar{t})$, for $(p, q, \lambda, t) \in \mathcal{B}$, follows $h(p, q, \lambda, t) = h(\bar{p}, \bar{q}, \bar{\lambda}, \bar{t})$.*

The following theorem states the weak duality assertion for the vector problems (P) and (D).

Theorem 3.3 *There is no $x \in \mathcal{A}$ and no $(p, q, \lambda, t) \in \mathcal{B}$ fulfilling $h(p, q, \lambda, t) \underset{R^m_+}{\geqq} f(x)$ and $h(p, q, \lambda, t) \neq f(x)$.*

Proof. We assume that there exist $x \in \mathcal{A}$ and $(p, q, \lambda, t) \in \mathcal{B}$ such that $f_i(x) \leq h_i(p, q, \lambda, t), \forall i \in \{1, \ldots, m\}$ and $f_j(x) < h_j(p, q, \lambda, t)$ for at least one $j \in \{1, \ldots, m\}$. This implies

$$\sum_{i=1}^{m} \lambda_i f_i(x) < \sum_{i=1}^{m} \lambda_i h_i(p, q, \lambda, t). \tag{3. 3}$$

On the other hand, we have

$$\sum_{i=1}^{m} \lambda_i h_i(p, q, \lambda, t) = -\sum_{i=1}^{m} \lambda_i f_i^*(p_i) - \sum_{i=1}^{m} \lambda_i (q_i^T g)^* \left(-\frac{1}{m\lambda_i} \sum_{i=1}^{m} \lambda_i p_i \right) + \sum_{i=1}^{m} \lambda_i t_i.$$

For f_i and $q_i^T g, i = 1, \ldots, m$, we can apply the inequality of Young

$$\begin{aligned} -f_i^*(p_i) &\leq f_i(x) - p_i^T x, \\ -(q_i^T g)^* \left(-\frac{1}{m\lambda_i} \sum_{i=1}^{m} \lambda_i p_i \right) &\leq q_i^T g(x) + \left(\frac{1}{m\lambda_i} \sum_{i=1}^{m} \lambda_i p_i \right)^T x \end{aligned}$$

and, so, we obtain

$$\begin{aligned} \sum_{i=1}^{m} \lambda_i h_i(p, q, \lambda, t) &\leq \sum_{i=1}^{m} \lambda_i f_i(x) - \left(\sum_{i=1}^{m} \lambda_i p_i \right)^T x \\ &+ \sum_{i=1}^{m} \lambda_i \left[q_i^T g(x) + \left(\frac{1}{m\lambda_i} \sum_{i=1}^{m} \lambda_i p_i \right)^T x \right] \\ &= \sum_{i=1}^{m} \lambda_i f_i(x) + \left(\sum_{i=1}^{m} \lambda_i q_i \right)^T g(x) \\ &\leq \sum_{i=1}^{m} \lambda_i f_i(x). \end{aligned}$$

The resulting inequality $\sum\limits_{i=1}^{m} \lambda_i h_i(p, q, \lambda, t) \leq \sum\limits_{i=1}^{m} \lambda_i f_i(x)$ contradicts relation (3. 3).
□

The following theorem expresses the so-called strong duality between the two multiobjective problems (P) and (D).

Theorem 3.4 *Assume the existence of an element $x' \in \bigcap\limits_{i=1}^{m} dom(f_i)$ fulfilling $g(x') \in -int(K)$. Let $\bar{x}$ be a properly efficient element to (P). Then there exists an efficient solution $(\bar{p}, \bar{q}, \bar{\lambda}, \bar{t}) \in \mathcal{B}$ to the dual (D) and the strong duality $f(\bar{x}) = h(\bar{p}, \bar{q}, \bar{\lambda}, \bar{t})$ holds.*

Proof. Assume $\bar{x}$ to be properly efficient to (P). From Definition 3.3 there follows the existence of a corresponding vector $\bar{\lambda} = (\bar{\lambda}_1, \ldots, \bar{\lambda}_m)^T \in int(\mathbb{R}^m_+)$ such that $\bar{x}$ solves the scalar problem

$$(P^{\bar{\lambda}}) \quad \inf_{x \in \mathcal{A}} \sum_{i=1}^{m} \bar{\lambda}_i f_i(x).$$

The constraint qualification (CQ) being fulfilled, by Theorem 3.2, there exists $(\tilde{p}, \tilde{q})$ an optimal solution to the dual $(D^{\bar{\lambda}})$ such that the optimality conditions (i), (ii) and (iii) are satisfied.

By means of $\bar{x}$ and $(\tilde{p}, \tilde{q})$ we construct now an efficient solution $(\bar{p}, \bar{q}, \bar{\lambda}, \bar{t})$ to (D). Therefore, let $\bar{\lambda} = (\bar{\lambda}_1, \ldots, \bar{\lambda}_m)^T$ be the vector given by the proper efficiency of $\bar{x}$ and $\bar{p} = (\bar{p}_1, \ldots \bar{p}_m) := (\tilde{p}_1, \ldots \tilde{p}_m) = \tilde{p}$. It remains us to define $\bar{q} = (\bar{q}_1, \ldots \bar{q}_m)$ and $\bar{t} = (\bar{t}_1, \ldots \bar{t}_m)^T$.

Let, for $i = 1, \ldots, m$, be

$$\begin{aligned} \bar{q}_i &:= \frac{1}{m\bar{\lambda}_i}\tilde{q} \in \mathbb{R}^k, \\ \bar{t}_i &:= \bar{p}_i^T \bar{x} + (\bar{q}_i^T g)^* \left(-\frac{1}{m\bar{\lambda}_i} \sum_{i=1}^m \bar{\lambda}_i \bar{p}_i \right) \in \mathbb{R}. \end{aligned} \tag{3. 4}$$

For $(\bar{p}, \bar{q}, \bar{\lambda}, \bar{t})$ it holds $\bar{\lambda} \in int(\mathbb{R}^m_+)$, $\sum\limits_{i=1}^m \bar{\lambda}_i \bar{q}_i = \tilde{q} \underset{K^*}{\geqq} 0$ and

$$\begin{aligned} \sum_{i=1}^m \bar{\lambda}_i \bar{t}_i &= \left(\sum_{i=1}^m \bar{\lambda}_i \bar{p}_i \right)^T \bar{x} + \sum_{i=1}^m \bar{\lambda}_i \left(\frac{1}{m\bar{\lambda}_i} \tilde{q}^T g \right)^* \left(-\frac{1}{m\bar{\lambda}_i} \sum_{i=1}^m \bar{\lambda}_i \bar{p}_i \right) \\ &= \left(\sum_{i=1}^m \bar{\lambda}_i \bar{p}_i \right)^T \bar{x} + \sum_{i=1}^m \bar{\lambda}_i \frac{1}{m\bar{\lambda}_i} (\tilde{q}^T g)^* \left(-\sum_{i=1}^m \bar{\lambda}_i \bar{p}_i \right) \\ &= \left(\sum_{i=1}^m \bar{\lambda}_i \bar{p}_i \right)^T \bar{x} + (\tilde{q}^T g)^* \left(-\sum_{i=1}^m \bar{\lambda}_i \bar{p}_i \right) \\ &= 0 \quad (\text{by } (3.\ 1)). \end{aligned}$$

In conclusion, the element $(\bar{p}, \bar{q}, \bar{\lambda}, \bar{t})$ is feasible to (D).

It remains to show that the values of the objective functions are equal, namely, that $f(\bar{x}) = h(\bar{p}, \bar{q}, \bar{\lambda}, \bar{t})$. Therefore, we prove that $f_i(\bar{x}) = h_i(\bar{p}, \bar{q}, \bar{\lambda}, \bar{t})$ holds, for each $i = 1, \ldots, m$. For this we use the relation (i) in Theorem 3.2 and the equations (3. 4). Then it holds

$$\begin{aligned} h_i(\bar{p}, \bar{q}, \bar{\lambda}, \bar{t}) &= -f_i^*(\bar{p}_i) - (\bar{q}_i^T g)^* \left(-\frac{1}{m\bar{\lambda}_i} \sum_{i=1}^m \bar{\lambda}_i \bar{p}_i \right) + \bar{t}_i \\ &= -f_i^*(\bar{p}_i) - (\bar{q}_i^T g)^* \left(-\frac{1}{m\bar{\lambda}_i} \sum_{i=1}^m \bar{\lambda}_i \bar{p}_i \right) + \bar{p}_i^T \bar{x} \\ &+ (\bar{q}_i^T g)^* \left(-\frac{1}{m\bar{\lambda}_i} \sum_{i=1}^m \bar{\lambda}_i \bar{p}_i \right) = -f_i^*(\bar{p}_i) + \bar{p}_i^T \bar{x} = f_i(\bar{x}). \end{aligned}$$

The maximality of $(\bar{p}, \bar{q}, \bar{\lambda}, \bar{t})$ is given by Theorem 3.3. □

Remark 3.2 In [11] Boţ and Wanka have introduced a duality approach for the vector optimization problem with a convex objective function and d.c. constraints

$$(P_{dc}) \quad \underset{x \in \mathcal{A}_{dc}}{\text{v-min}}\, f(x),$$

$$\mathcal{A}_{dc} = \{x \in X : g_i(x) - h_i(x) \leq 0, i \in 1, ..., k\},$$
$$f(x) = (f_1(x), \ldots, f_m(x))^T.$$

In the formulation of (P_{dc}), X is a real Hausdorff locally convex vector space, $f_i : X \to \overline{\mathbb{R}}$, $i = 1, ..., m$, are proper and convex functions and $g_i, h_i : X \to \overline{\mathbb{R}}$, $i \in 1, ..., k$, are extended real-valued convex functions. By using a decomposition formula for the feasible set $\mathcal{A}_{dc}$, which was first mentioned by Martinez-Legaz and Volle in [51], we gave, under some continuity and subdifferentiability assumptions for the involved functions, weak and strong duality statements for (P_{dc}). We want just to notice here that in the convex case, in fact, if $h_i = 0, \forall i \in \{1, ..., k\}$, we rediscovered in [11] the duality results presented above, of course, in the case $K = \mathbb{R}^k_+$.

3.1.5 The converse duality

In this subsection we complete our investigations concerning duality by formulating the converse duality theorem for (P) and (D). Therefore, we introduce some new notations. For each $\lambda \in int(\mathbb{R}^m_+)$, let be

$$\mathcal{B}_\lambda = \left\{ (p, q, t) : \sum_{i=1}^m \lambda_i q_i \underset{K^*}{\geqq} 0, \sum_{i=1}^m \lambda_i t_i = 0 \right\},$$

$$p = (p_1, \dots, p_m), q = (q_1, \dots, q_m), t = (t_1, \dots, t_m)^T,$$

$$p_i \in \mathbb{R}^m, \quad q_i \in \mathbb{R}^k, \quad t_i \in \mathbb{R}, \quad i = 1, \dots, m.$$

Further, let be

$$M = \Big\{ a \in \mathbb{R}^m : \quad \exists \lambda \in int(\mathbb{R}^m_+), \quad \exists (p, q, t) \in \mathcal{B}_\lambda \\ \text{such that } \sum_{i=1}^m \lambda_i a_i = \sum_{i=1}^m \lambda_i h_i(p, q, \lambda, t) \Big\}.$$

For the proof of the converse duality theorem we need the following propositions.

Proposition 3.1 *It holds $h(\mathcal{B}) \cap \mathbb{R}^m = M$.*

Proof. Obviously, $h(\mathcal{B}) \cap \mathbb{R}^m \subseteq M$. We need to prove just the inverse inclusion.

Therefore, let be $a \in M$. Then there exist $\lambda \in int(\mathbb{R}^m_+)$ and $(p, q, t) \in \mathcal{B}_\lambda$ such that $\sum_{i=1}^m \lambda_i a_i = \sum_{i=1}^m \lambda_i h_i(p, q, \lambda, t)$ or, equivalently,

$$\sum_{i=1}^m \lambda_i a_i = -\sum_{i=1}^m \lambda_i f_i^*(p_i) - \sum_{i=1}^m \lambda_i (q_i^T g)^* \left(-\frac{1}{m\lambda_i} \sum_{i=1}^m \lambda_i p_i \right) + \sum_{i=1}^m \lambda_i t_i.$$

Let us define for $i = 1, \dots, m$,

$$\bar{t}_i := a_i + f_i^*(p_i) + (q_i^T g)^* \left(-\frac{1}{m\lambda_i} \sum_{i=1}^m \lambda_i p_i \right) \in \mathbb{R}.$$

It is easy to observe that $\sum_{i=1}^m \lambda_i \bar{t}_i = \sum_{i=1}^m \lambda_i t_i = 0$ and, so, $(p, q, \lambda, \bar{t}) \in \mathcal{B}$.

On the other hand, we have for $i = 1, \dots, m$,

$$a_i = -f_i^*(p_i) - (q_i^T g)^* \left(-\frac{1}{m\lambda_i} \sum_{i=1}^m \lambda_i p_i \right) + \bar{t}_i,$$

which means that $a = h(p, q, \lambda, \bar{t}) \in h(\mathcal{B})$. In conclusion, $M \subseteq h(\mathcal{B}) \cap \mathbb{R}^m$ and the proof is complete. $\square$

Proposition 3.2 *An element $\bar{a} \in \mathbb{R}^m$ is maximal in M if and only if for every $a \in M$ with corresponding $\lambda^a \in int(\mathbb{R}^m_+)$ and $(p^a, q^a, t^a) \in \mathcal{B}_{\lambda^a}$, it holds*

$$\sum_{i=1}^{m} \lambda_i^a \bar{a}_i \geq \sum_{i=1}^{m} \lambda_i^a a_i. \tag{3. 5}$$

Proof. First we show the sufficiency. Assume the existence of some $a \in M$ such that $a \in \bar{a} + \mathbb{R}^m_+ \setminus \{0\}$. For the corresponding $\lambda^a \in int(\mathbb{R}^m_+)$ it holds $\sum\limits_{i=1}^{m} \lambda_i^a \bar{a}_i < \sum\limits_{i=1}^{m} \lambda_i^a a_i$, which contradicts relation (3. 5).

To prove the necessity, let us assume that there exists $b \in \mathbb{R}^m$, $b \in \bar{a} + \mathbb{R}^m_+ \setminus \{0\}$, and $a \in M$ with corresponding $\lambda^a \in int(\mathbb{R}^m_+)$ and $(p^a, q^a, t^a) \in \mathcal{B}_{\lambda^a}$ such that

$$\sum_{i=1}^{m} \lambda_i^a a_i \geq \sum_{i=1}^{m} \lambda_i^a b_i. \tag{3. 6}$$

We will show that this assumption is false.

If in (3. 6) equality holds, $\sum\limits_{i=1}^{m} \lambda_i^a a_i = \sum\limits_{i=1}^{m} \lambda_i^a b_i$, then $b \in M$ and this contradicts the maximality of $\bar{a}$ in M.

If $\sum\limits_{i=1}^{m} \lambda_i^a a_i > \sum\limits_{i=1}^{m} \lambda_i^a b_i$, then we can choose a $c = (c_1, ..., c_m)^T \in \mathbb{R}^m$ such that $c_i > \max\{a_i, b_i\}$, for $i = 1, ..., m$.

Because it holds

$$\sum_{i=1}^{m} \lambda_i^a c_i > \sum_{i=1}^{m} \lambda_i^a a_i > \sum_{i=1}^{m} \lambda_i^a b_i,$$

there exists an $r \in (0, 1)$ such that $\sum\limits_{i=1}^{m} \lambda_i^a a_i = \sum\limits_{i=1}^{m} \lambda_i^a [(1-r)b_i + rc_i]$. This means that $(1-r)b + rc \in M$. On the other hand,

$$(1-r)b + rc = r(c-b) + b \in \mathbb{R}^m_+ \setminus \{0\} + \bar{a} + \mathbb{R}^m_+ \setminus \{0\} \subseteq \bar{a} + \mathbb{R}^m_+ \setminus \{0\}.$$

Our assumption proves to be false because the last inclusion also contradicts the maximality of $\bar{a}$ in M.

Then, for each $b \in \bar{a} + \mathbb{R}^m_+ \setminus \{0\}$ and $a \in M$ with corresponding $\lambda^a \in int(\mathbb{R}^m_+)$ and $(p^a, q^a, t^a) \in \mathcal{B}_{\lambda^a}$, we must have

$$\sum_{i=1}^{m} \lambda_i^a b_i > \sum_{i=1}^{m} \lambda_i^a a_i. \tag{3. 7}$$

From this last relation implies that for each $a \in M$ with corresponding $\lambda^a \in int(\mathbb{R}^m_+)$ and $(p^a, q^a, t^a) \in \mathcal{B}_{\lambda^a}$ it holds

$$\sum_{i=1}^{m} \lambda_i^a \bar{a}_i = \inf \left\{ \sum_{i=1}^{m} \lambda_i^a b_i : b \in \bar{a} + \mathbb{R}^m_+ \setminus \{0\} \right\} \geq \sum_{i=1}^{m} \lambda_i^a a_i,$$

which finishes the proof. □

We are now ready to formulate the converse duality theorem.

Theorem 3.5 *Assume the constraint qualification (CQ) is fulfilled. Suppose that for each $\lambda \in int(\mathbb{R}^m_+)$ the following property holds*

(C) *If* $\inf\limits_{x\in\mathcal{A}} \sum\limits_{i=1}^m \lambda_i f_i(x) > -\infty$, *then there exists an element* $x_\lambda \in \mathcal{A}$ *such that* $\inf\limits_{x\in\mathcal{A}} \sum\limits_{i=1}^m \lambda_i f_i(x) = \sum\limits_{i=1}^m \lambda_i f_i(x_\lambda)$.

(a) Then for any efficient solution $(\bar{p},\bar{q},\bar{\lambda},\bar{t})$ *to* (D) *it holds* $h(\bar{p},\bar{q},\bar{\lambda},\bar{t}) \in cl(f(\mathcal{A}) + \mathbb{R}^m_+)$ *and there exists a properly efficient solution* $\bar{x}_{\bar{\lambda}}$ *to* (P) *such that*

$$\sum_{i=1}^m \bar{\lambda}_i [f_i(\bar{x}_{\bar{\lambda}}) - h_i(\bar{p},\bar{q},\bar{\lambda},\bar{t})] = 0.$$

(b) If, additionally, $f(\mathcal{A})$ *is* $\mathbb{R}^m_+$*-closed (i.e.* $f(\mathcal{A}) + \mathbb{R}^m_+$ *is closed), then there exists* $\bar{x} \in \mathcal{A}$ *a properly efficient solution to* (P) *such that*

$$\sum_{i=1}^m \bar{\lambda}_i f_i(\bar{x}_{\bar{\lambda}}) = \sum_{i=1}^m \bar{\lambda}_i f_i(\bar{x}) \text{ and } f(\bar{x}) = h(\bar{p},\bar{q},\bar{\lambda},\bar{t}).$$

Proof.

(a) Let us denote by $\bar{a} := h(\bar{p},\bar{q},\bar{\lambda},\bar{t})$. From the maximality of $\bar{a}$ in $h(\mathcal{B})$ we have that $\bar{a} \in h(\mathcal{B}) \cap \mathbb{R}^m$. By Proposition 3.1, $\bar{a}$ is maximal in M.

For the beginning, we will prove that $\bar{a} \in cl(f(\mathcal{A}) + \mathbb{R}^m_+)$.

Assume the contrary. Because $cl(f(\mathcal{A}) + \mathbb{R}^m_+)$ is closed and convex, by a well-known separation theorem (see, for instance, Corollary 11.4.1 in [62]), there exist $\lambda^1 \in \mathbb{R}^m \setminus \{0\}$ and $\alpha \in \mathbb{R}$ such that

$$\sum_{i=1}^m \lambda_i^1 \bar{a}_i < \alpha \leq \sum_{i=1}^m \lambda_i^1 d_i, \quad \forall d \in cl(f(\mathcal{A}) + \mathbb{R}^m_+). \tag{3. 8}$$

From (3. 8) it is easy to observe that $\lambda^1 \in \mathbb{R}^m_+ \setminus \{0\}$.

But the fact that $\bar{a} \in M$ assures the existence of an corresponding $\lambda^{\bar{a}} \in int(\mathbb{R}^m_+)$ and $(p^{\bar{a}}, q^{\bar{a}}, t^{\bar{a}}) \in \mathcal{B}_{\lambda^{\bar{a}}}$ such that $\sum\limits_{i=1}^m \lambda_i^{\bar{a}} \bar{a}_i = \sum\limits_{i=1}^m \lambda_i^{\bar{a}} h_i(p^{\bar{a}}, q^{\bar{a}}, t^{\bar{a}})$. Like in the proof of Theorem 3.3, it holds

$$\sum_{i=1}^m \lambda_i^{\bar{a}} \bar{a}_i = \sum_{i=1}^m \lambda_i^{\bar{a}} h_i(p^{\bar{a}}, q^{\bar{a}}, t^{\bar{a}}) \leq \sum_{i=1}^m \lambda_i^{\bar{a}} d_i, \quad \forall d \in cl(f(\mathcal{A}) + \mathbb{R}^m_+). \tag{3. 9}$$

Let be now $s \in (0,1)$ fixed. Considering $\lambda^* = s\lambda^1 + (1-s)\lambda^{\bar{a}} \in int(\mathbb{R}^m_+)$, from (3. 8) and (3. 9) follows

$$\sum_{i=1}^m \lambda_i^* \bar{a}_i < \sum_{i=1}^m \lambda_i^* d_i, \quad \forall d \in cl(f(\mathcal{A}) + \mathbb{R}^m_+),$$

which implies that for each $x \in \mathcal{A}$ there holds

$$\sum_{i=1}^m \lambda_i^* \bar{a}_i < \sum_{i=1}^m \lambda_i^* f_i(x). \tag{3. 10}$$

Relation (3. 10) guarantees that the assumption of condition (C) is fulfilled and this assures further the existence of a solution $x_{\lambda^*} \in \mathcal{A}$ to the problem

$$(P^{\lambda^*}) \ \inf_{x\in\mathcal{A}} \sum_{i=1}^m \bar{\lambda}_i^* f_i(x).$$

The constraint qualification (CQ) being fulfilled, we can construct, like in the proof of Theorem 3.4, an efficient element $(p_{\lambda^*}, q_{\lambda^*}, \lambda^*, t_{\lambda^*})$ to (D) such that

$$f(x_{\lambda^*}) = h(p_{\lambda^*}, q_{\lambda^*}, \lambda^*, t_{\lambda^*}) \in h(\mathcal{B}) \cap \mathbb{R}^m = M.$$

Using the maximality of $\bar{a} \in M$, by Proposition 3.2, we have that

$$\sum_{i=1}^m \lambda_i^* \bar{a}_i \geq \sum_{i=1}^m \lambda_i^* f_i(x_{\lambda^*}),$$

which contradicts the strict inequality in (3. 10). This means that $\bar{a} = h(\bar{p}, \bar{q}, \bar{\lambda}, \bar{t}) \in cl(f(\mathcal{A}) + \mathbb{R}^m_+)$.

Then there exist the sequences $(x^n)_{n \in \mathbb{N}} \subseteq \mathcal{A}$ and $(k^n)_{n \in \mathbb{N}} \subseteq \mathbb{R}^m_+$ with the property that $f(x^n) + k^n$ converges to $\bar{a}$. On the other hand, by the proof of Theorem 3.3, for each $x \in \mathcal{A}$ it holds

$$\sum_{i=1}^m \bar{\lambda}_i f_i(x) \geq \sum_{i=1}^m \bar{\lambda}_i \bar{a}_i.$$

Considering again condition (C), there exists a properly efficient solution $\bar{x}_{\bar{\lambda}} \in \mathcal{A}$ to (P) such that $\inf\limits_{x \in \mathcal{A}} \sum\limits_{i=1}^m \bar{\lambda}_i f_i(x) = \sum\limits_{i=1}^m \bar{\lambda}_i f_i(\bar{x}_{\bar{\lambda}})$.

This means that the following inequalities must hold

$$\sum_{i=1}^m \bar{\lambda}_i \bar{a}_i \leq \sum_{i=1}^m \bar{\lambda}_i f_i(\bar{x}_{\bar{\lambda}}) = \inf_{x \in \mathcal{A}} \sum_{i=1}^m \bar{\lambda}_i f_i(x) \leq \sum_{i=1}^m \bar{\lambda}_i (f_i(x^n) + k_i^n), \ \forall n \in \mathbb{N}.$$

Finally, letting $n \to +\infty$, we have that

$$\sum_{i=1}^m \bar{\lambda}_i f_i(\bar{x}_{\bar{\lambda}}) = \sum_{i=1}^m \bar{\lambda}_i \bar{a}_i = \sum_{i=1}^m \bar{\lambda}_i h_i(\bar{p}, \bar{q}, \bar{\lambda}, \bar{t}).$$

This concludes the proof of the first part.

(b) If $f(\mathcal{A}) + \mathbb{R}^m_+$ is closed, then $\bar{a} \in cl(f(\mathcal{A}) + \mathbb{R}^m_+) = f(\mathcal{A}) + \mathbb{R}^m_+$. According to the weak duality theorem, Theorem 3.3, we have $\bar{a} \in f(\mathcal{A})$ and this implies the existence of an element $\bar{x} \in \mathcal{A}$ such that $f(\bar{x}) = \bar{a} = h(\bar{p}, \bar{q}, \bar{\lambda}, \bar{t})$. It is obvious that $\bar{x}$ is properly efficient and that it holds

$$\sum_{i=1}^m \bar{\lambda}_i f_i(\bar{x}_{\bar{\lambda}}) = \sum_{i=1}^m \bar{\lambda}_i f_i(\bar{x}).$$

□

3.1.6 The convex multiobjective optimization problem with linear inequality constraints

In the past, Wanka and Boţ have studied in [83] and [84] the duality for a multiobjective optimization problem with convex objective functions and linear inequality constraints. The purpose of this subsection is to show that the dual obtained in those papers is actually a special case of the general dual problem (D).

Special case I

Let be $g : \mathbb{R}^n \to \mathbb{R}^k$ defined by $g(x) = Ax + b$, with A being a $k \times n$ matrix with real entries and $b \in \mathbb{R}^k, b \neq 0$.

We consider first the following primal problem

$$(P_I) \quad \underset{x \in \mathcal{A}_I}{\text{v-min}}\, f(x),$$

with

$$\mathcal{A}_I = \left\{ x \in \mathbb{R}^n : Ax + b \underset{K}{\leqq} 0 \right\},$$

and

$$f(x) = (f_1(x), \ldots, f_m(x))^T.$$

Using the linearity of g, we can calculate the conjugate of $q_i^T g, i = 1, \ldots, m$,

$$(q_i^T g)^* \left(-\frac{1}{m\lambda_i} \sum_{i=1}^m \lambda_i p_i \right) = \begin{cases} -q_i^T b, & \text{if} \quad A^T q_i + \frac{1}{m\lambda_i} \sum\limits_{i=1}^m \lambda_i p_i = 0, \\ +\infty, & \text{otherwise.} \end{cases}$$

Thus, the dual of (P_I) is

$$(D_I) \quad \underset{(p,q,\lambda,t) \in \mathcal{B}_I}{\text{v-max}} h(p, q, \lambda, t) = \begin{pmatrix} -f_1^*(p_1) + q_1^T b + t_1 \\ \vdots \\ -f_m^*(p_m) + q_m^T b + t_m \end{pmatrix},$$

with the set of constraints

$$\mathcal{B}_I = \left\{ (p, q, \lambda, t) \quad : \quad \lambda \in int(\mathbb{R}^m_+),\ \sum_{i=1}^m \lambda_i q_i \underset{K^*}{\geqq} 0,\ \sum_{i=1}^m \lambda_i t_i = 0, \right.$$
$$\left. A^T q_i + \frac{1}{m\lambda_i} \sum_{i=1}^m \lambda_i p_i = 0,\ i = 1, \ldots, m \right\}.$$

Next we prove that the images of h on the sets $\mathcal{B}_I$ and

$$\mathcal{B}'_I = \left\{ (p, q', \lambda, t') \quad : \quad \lambda \in int(\mathbb{R}^m_+),\ \sum_{i=1}^m \lambda_i q'_i \underset{K^*}{\geqq} 0, \right.$$
$$\left. \sum_{i=1}^m \lambda_i t'_i = 0,\ \sum_{i=1}^m \lambda_i (A^T q'_i + p_i) = 0 \right\}$$

coincide (i.e. $h(\mathcal{B}_I) = h(\mathcal{B}'_I)$).

It is only to show that $h(\mathcal{B}'_I) \subseteq h(\mathcal{B}_I)$ because the inverse inclusion is obvious. Therefore, let be $(p, q', \lambda, t') \in \mathcal{B}'_I$. Considering for $i = 1, \ldots, m$, $q_i := \frac{1}{m\lambda_i} \sum\limits_{i=1}^m \lambda_i q'_i$ and $t_i := t'_i + q_i^{'T} b - \left(\frac{1}{m\lambda_i} \sum\limits_{i=1}^m \lambda_i q'_i \right)^T b$, we obtain an element $(p, q, \lambda, t) \in \mathcal{B}_I$ such that $h(p, q', \lambda, t') = h(p, q, \lambda, t) \in h(\mathcal{B}_I)$.

The dual of (P_I) is then equivalent to the following problem

$$(D_I) \quad \underset{(p,q',\lambda,t') \in \mathcal{B}'_I}{\text{v-max}} h(p, q', \lambda, t') = \begin{pmatrix} -f_1^*(p_1) + q_1^{'T} b + t'_1 \\ \vdots \\ -f_m^*(p_m) + q_m^{'T} b + t'_m \end{pmatrix}.$$

The last step is to show that (D_I) can be actually written in the following form

$$(D_I) \quad \underset{(p,\delta,\lambda)\in\tilde{\mathcal{B}}_I}{\text{v-max}} \tilde{h}_I(p,\delta,\lambda) = \begin{pmatrix} -f_1^*(p_1) + \delta_1^T b \\ \vdots \\ -f_m^*(p_m) + \delta_m^T b \end{pmatrix},$$

with

$$\tilde{\mathcal{B}}_I = \left\{ (p,\delta,\lambda) : \lambda \in int(\mathbb{R}_+^m), \quad \sum_{i=1}^m \lambda_i \delta_i \underset{K^*}{\geqq} 0, \quad \sum_{i=1}^m \lambda_i (A^T \delta_i + p_i) = 0 \right\}.$$

For $q_i' := \delta_i$ and $t_i' := 0, i = 1, \ldots, m$, it is easy to observe that $\tilde{h}_I(\tilde{\mathcal{B}}_I) \subseteq h(\mathcal{B}_I')$. In order to show the inverse inclusion, let be $(p, q', \lambda, t') \in \mathcal{B}_I'$. Because $b \neq 0$, we can consider a vector $\gamma \in \mathbb{R}^k$ such that $\gamma^T b = 1$. Defining, for $i = 1, \ldots, m$, $\delta_i := q_i' + t_i'\gamma$, it holds $\delta_i^T b = q_i^{'T} b + t_i'$ and $\sum\limits_{i=1}^m \lambda_i \delta_i = \sum\limits_{i=1}^m \lambda_i q_i'$. This means that (p, δ, λ) belongs to $\tilde{\mathcal{B}}_I$ and that $h(p, q', \lambda, t') = \tilde{h}_I(p, \delta, \lambda) \in \tilde{h}_I(\tilde{\mathcal{B}}_I)$.

Special case II

Let us consider the same problem as before but for the case $b = 0$. The first steps are the same and this means that the multiobjective dual for

$$(P_{II}) \quad \underset{x\in\mathcal{A}_{II}}{\text{v-min}} f(x),$$

with

$$\mathcal{A}_{II} = \left\{ x \in \mathbb{R}^n : Ax \underset{K}{\leqq} 0 \right\},$$

and

$$f(x) = (f_1(x), \ldots, f_m(x))^T,$$

is

$$(D_{II}) \quad \underset{(p,q',\lambda,t')\in\mathcal{B}_{II}'}{\text{v-max}} h(p,q',\lambda,t') = \begin{pmatrix} -f_1^*(p_1) + t_1' \\ \vdots \\ -f_m^*(p_m) + t_m' \end{pmatrix},$$

with

$$\mathcal{B}_{II}' = \Bigg\{ (p,q',\lambda,t') \quad : \quad \lambda \in int(\mathbb{R}_+^m), \ \sum_{i=1}^m \lambda_i q_i' \underset{K^*}{\geqq} 0, \\ \sum_{i=1}^m \lambda_i t_i' = 0, \ \sum_{i=1}^m \lambda_i (A^T q_i' + p_i) = 0 \Bigg\}.$$

Substituting $\gamma := \sum\limits_{i=1}^m \lambda_i q_i'$, we obtain the equivalent formulation

$$(D_{II}) \quad \underset{(p,\gamma,\lambda,t)\in\tilde{\mathcal{B}}_{II}}{\text{v-max}} \tilde{h}_{II}(p,\gamma,\lambda,t) = \begin{pmatrix} -f_1^*(p_1) + t_1 \\ \vdots \\ -f_m^*(p_m) + t_m \end{pmatrix},$$

with

$$\tilde{\mathcal{B}}_{II} = \left\{ (p,\gamma,\lambda,t) : \lambda \in int(\mathbb{R}_+^m), \quad \gamma \underset{K^*}{\geqq} 0, \quad \sum_{i=1}^m \lambda_i t_i = 0, \quad -A^T\gamma = \sum_{i=1}^m \lambda_i p_i \right\}.$$

Special case III

The next problem we study here is the vector optimization problem considered in [83] and [84],

$$(P_{III}) \quad \underset{x \in \mathcal{A}_{III}}{\text{v-min}}\, f(x),$$

with

$$\mathcal{A}_{III} = \left\{ x \in \mathbb{R}^n : x \underset{K_0}{\geqq} 0, Ax + b \underset{K_1}{\leqq} 0 \right\},$$

$$f(x) = (f_1(x), \ldots, f_m(x))^T.$$

A is a $k \times n$ matrix with real entries, $b \in \mathbb{R}^k, b \neq 0$, $K_0 \subseteq \mathbb{R}^n$ and $K_1 \subseteq \mathbb{R}^k$ are two convex closed cones. Considering the $(k+n) \times n$ matrix $\bar{A} = \begin{pmatrix} A \\ -I_n \end{pmatrix}$, the vector $\bar{b} = \begin{pmatrix} b \\ 0 \end{pmatrix} \in \mathbb{R}^{k+n}, \bar{b} \neq 0$, and the convex closed cone $K = K_1 \times K_0 \in \mathbb{R}^{k+n}$, we can represent the feasible set of (P_{III}) as $\mathcal{A}_{III} = \left\{ x \in \mathbb{R}^n : \bar{A}x + \bar{b} \underset{K}{\leqq} 0 \right\}$ and, so, we can reduce the problem (P_{III}) to the problem studied as the Special case I.

Then the dual of (P_{III}) becomes

$$(D_{III}) \quad \underset{(p,\bar{\delta},\lambda) \in \bar{\mathcal{B}}_{III}}{\text{v-max}} \bar{h}_{III}(p, \bar{\delta}, \lambda) = \begin{pmatrix} -f_1^*(p_1) + \bar{\delta}_1^T \bar{b} \\ \vdots \\ -f_m^*(p_m) + \bar{\delta}_m^T \bar{b} \end{pmatrix},$$

with

$$\bar{\mathcal{B}}_{III} = \left\{ (p, \bar{\delta}, \lambda) : \lambda \in int(\mathbb{R}^m_+), \quad \sum_{i=1}^m \lambda_i \bar{\delta}_i \underset{K^*}{\geqq} 0, \quad \sum_{i=1}^m \lambda_i (\bar{A}^T \bar{\delta}_i + p_i) = 0 \right\},$$

and the dual variables $p = (p_1, \ldots, p_m) \in \mathbb{R}^n \times \ldots \times \mathbb{R}^n$ and $\bar{\delta} = (\delta^1, \delta^2) \in \mathbb{R}^k \times \mathbb{R}^n$.

Remarking that $\bar{\delta}_i^T \bar{b} = \delta_i^{1T} b$ and $\bar{A}^T \bar{\delta}_i = A^T \delta_i^1 - \delta_i^2, i = 1, \ldots, m$, we obtain for the dual of (P_{III}) the following formulation

$$(D_{III}) \quad \underset{(p,\delta_1,\delta_2,\lambda) \in \bar{\mathcal{B}}_{III}}{\text{v-max}} \bar{h}_{III}(p, \delta_1, \delta_2, \lambda) = \begin{pmatrix} -f_1^*(p_1) + \delta_1^{1T} b \\ \vdots \\ -f_m^*(p_m) + \delta_m^{1T} b \end{pmatrix},$$

with

$$\begin{aligned} \bar{\mathcal{B}}_{III} = \Bigg\{ (p, \delta_1, \delta_2, \lambda) \quad : \quad & \lambda \in int(\mathbb{R}^m_+), \ \sum_{i=1}^m \lambda_i \delta_i^1 \underset{K_1^*}{\geqq} 0, \ \sum_{i=1}^m \lambda_i \delta_i^2 \underset{K_0^*}{\geqq} 0, \\ & \sum_{i=1}^m \lambda_i (A^T \delta_i^1 + p_i) = \sum_{i=1}^m \lambda_i \delta_i^2 \Bigg\}, \end{aligned}$$

or, equivalently,

$$(D_{III}) \quad \underset{(p,\delta,\lambda) \in \tilde{\mathcal{B}}_{III}}{\text{v-max}} \tilde{h}_{III}(p, \delta, \lambda) = \begin{pmatrix} -f_1^*(p_1) + \delta_1^T b \\ \vdots \\ -f_m^*(p_m) + \delta_m^T b \end{pmatrix},$$

with

$$\tilde{\mathcal{B}}_{III} = \left\{ (p, \delta, \lambda) : \lambda \in int(\mathbb{R}^m_+), \quad \sum_{i=1}^m \lambda_i \delta_i \underset{K_1^*}{\geqq} 0, \quad \sum_{i=1}^m \lambda_i (A^T \delta_i + p_i) \underset{K_0^*}{\geqq} 0 \right\}.$$

The dual (D_{III}) is exactly the problem obtained in [83] and [84].

Remark 3.3

(a) The approach presented in [83] and [84] for (P_{III}) assumes, as in the general case, the study of the duality for a scalar problem associated to the multiobjective one, by the use of the conjugacy approach. An extension of this method for set-valued optimization problems have been given in [8].

(b) The constraint qualification (CQ) adapted to the problem (P_{III}) claims the existence of an element $x' \in \mathbb{R}^n$ such that $x' \in int(K_0)$ and $Ax' + b \in -int(K_1)$. But, as we have proved in [83] and [84], in order to have strong duality, it is enough to assume the existence of an element $x' \in \mathbb{R}^n$ such that $x' \in K_0$ and $Ax' + b \in -int(K_1)$.

(c) The problem (P_{III}) has been also considered in [72] for general spaces, but with a special choice of the objective functions, in fact, each $f_i, i = 1, \ldots, m$, being represented as a sum of a norm with a linear function. The converse duality theorem presented there is false. The converse duality theorem proved in subsection 3.1.5 corrects that theorem. Moreover, as shown, the theorem is valid in the very general case, namely, for multiobjective optimization problems with convex objective functions and cone inequality constraints.

3.1.7 The convex semidefinite multiobjective optimization problem

As another particular case of (P), we consider in this subsection the multiobjective semidefinite programming problem with convex objective functions (cf. [88])

$$(P_{SDP}) \quad \underset{x \in \mathcal{A}_{SDP}}{\text{v-min}} \; f(x),$$

with

$$\mathcal{A}_{SDP} = \left\{ x \in \mathbb{R}^n : F(x) = F_0 + \sum_{i=1}^{n} x_i F_i \underset{\mathcal{S}_+^k}{\geqq} 0 \right\},$$

$$f(x) = (f_1(x), f_2(x), ..., f_m(x))^T.$$

For each $i = 1, ..., m$, $f_i : \mathbb{R}^n \to \overline{\mathbb{R}}$ is a proper and convex function, with the property that $\bigcap\limits_{i=1}^{m} ri(dom(f_i)) \neq \emptyset$. We also have that $F_j \in \mathcal{S}^k, \forall j = 0, ..., n$, where $\mathcal{S}^k$ is the linear subspace of the symmetric $k \times k$ matrices with real entries, i.e. $\mathcal{S}^k = \{A \in \mathbb{R}^{k \times k} : A = A^T\}$. On $\mathcal{S}^k$ we consider the scalar product

$$\langle A, B \rangle := \sum_{i,j=1}^{k} A_{j,i} B_{j,i} = Tr\left(A^T \cdot B\right),$$

$$\forall A = (A_{i,j})_{i,j=\overline{1,k}}, B = (B_{i,j})_{i,j=\overline{1,k}} \in \mathbb{R}^{k \times k},$$

where $Tr(A)$ denotes the trace of the matrix with real entries A and "$\cdot$" is the well-known product of matrices.

By S_+^k we denote the cone of the symmetric positive semidefinite $k \times k$ matrices with real entries, i.e.

$$\mathcal{S}_+^k = \left\{ A \in \mathcal{S}^k : y^T \cdot A \cdot y \geq 0, \forall y \in \mathbb{R}^k \right\},$$

which introduces the so-called Löwner partial order on $\mathcal{S}^k$

$$A \underset{\mathcal{S}^k_+}{\geqq} B \Leftrightarrow A - B \in \mathcal{S}^k_+, \text{ for } A, B \in \mathcal{S}^k.$$

So, the constraint $F(x) \underset{\mathcal{S}^k_+}{\geqq} 0$ means actually that $F(x)$ is a symmetric positive semidefinite matrix. Considering $g : \mathbb{R}^n \to \mathcal{S}^k$, $g(x) = -F_0 + \sum\limits_{i=1}^{n} x_i(-F_i)$, we can write the feasible set of (P_{SDP}) as $\mathcal{A}_{SDP} = \left\{ x \in \mathbb{R}^n : g(x) \underset{\mathcal{S}^k_+}{\leqq} 0 \right\}$.

By means of the conjugacy approach, we have obtained in [88] for the scalar optimization problem

$$(P^s_{SDP}) \quad \inf_{x \in \mathcal{A}_{SDP}} \tilde{f}(x),$$

$\tilde{f} : \mathbb{R}^n \to \overline{\mathbb{R}}$ being a proper and convex function, the following dual

$$(D^s_{SDP}) \quad \sup_{Q \underset{\mathcal{S}^k_+}{\geqq} 0} \left\{ -\tilde{f}^* \left(Tr(Q \cdot F_1), Tr(Q \cdot F_2), ..., Tr(Q \cdot F_n) \right) - Tr(Q \cdot F_0) \right\}.$$

One may observe that for $f(x) = c^T x, c \in \mathbb{R}^n$, the problem (D^s_{SDP}) becomes

$$(D^l_{SDP}) \quad \sup_{\substack{Q \underset{\mathcal{S}^k_+}{\geqq} 0, Tr(Q \cdot F_i) = c_i, \\ i=1,...,n}} -Tr(Q \cdot F_0).$$

Here, (D^l_{SDP}) is exactly the dual which appears in the literature dealing with linear semidefinite programming problems (see for instance [56], [70], [77]).

The multiobjective dual problem to (P_{SDP}) formulated by Wanka, Boţ and Grad in [88] is

$$(D_{SDP}) \quad \underset{(p,Q,\lambda,t) \in \mathcal{B}_{SDP}}{\text{v-max}} \begin{pmatrix} -f_1^*(p_1) - \frac{1}{m\lambda_1} Tr\left(Q \cdot F_0\right) + t_1 \\ \vdots \\ -f_m^*(p_m) - \frac{1}{m\lambda_m} Tr\left(Q \cdot F_0\right) + t_m \end{pmatrix},$$

with the dual variables

$$p = (p_1, p_2, ..., p_m) \in \mathbb{R}^n \times ... \times \mathbb{R}^n, Q \in \mathbb{R}^{k \times k},$$

$$\lambda = (\lambda_1, \lambda_2, ..., \lambda_m)^T \in \mathbb{R}^m, t = (t_1, t_2, ..., t_m)^T \in \mathbb{R}^m,$$

and the set of constraints

$$\begin{aligned} \mathcal{B}_{SDP} = \Bigg\{ (p, Q, \lambda, t) \quad : \quad & \lambda \in int(\mathbb{R}^m_+), \quad \sum_{i=1}^{m} \lambda_i t_i = 0, \quad Q \underset{\mathcal{S}^k_+}{\geqq} 0, \\ & \sum_{i=1}^{m} \lambda_i p_i = \left(Tr\left(Q \cdot F_1\right), Tr\left(Q \cdot F_2\right), ..., Tr\left(Q \cdot F_n\right)\right) \Bigg\}. \end{aligned}$$

For (P_{SDP}) and (D_{SDP}) the weak and, under certain conditions, the strong and converse duality theorems are also true (see [88]).

3.2 Multiobjective duality for convex ratios

3.2.1 Motivation

The results we present in this section are motivated by the work of Scott and Jefferson [69]. They have investigated the duality for a particular fractional programming problem having the objective function consisting of a sum of ratios, where the nominators are squared non-negative convex functions and the denominators are positive concave functions. This has to be minimized subject to linear inequality constraints. The method they used in the construction of the dual problem is based on geometric programming duality.

Our aim is to study the duality for a vector programming problem (P_r) with linear inequality constraints, the objective functions being represented by ratios of the form described above. As in the previous section, in order to formulate a multiobjective dual problem (D_r), we study first the duality for a scalar optimization problem obtained from (P_r) via linear scalarization. But, unlike [69], we use in our investigations the conjugacy duality approach, in fact, the Fenchel-Lagrange duality (cf. subsection 2.1.4). Moreover, we verify strong duality under some assumptions weaker than those ones used in [69].

In the theory of multiobjective fractional programming the study of duality is a well developed branch with many theoretical results. In general, these programs deal with ratios of a convex function and a positive concave function. The majority of these works extend different duality approaches for scalar fractional programming problems (see for instance Bector [4] and Schaible [66]) to the vector case. As a fundamental idea, the parametric replacement method of Dinkelbach [16] is in the most cases used. Among the contributions to duality in multiobjective fractional optimization we mention [5], [43] in the differentiable case and [59], [91] in the non-differentiable case.

3.2.2 Problem formulation

We consider the following multiobjective fractional programming problem with linear inequality constraints

$$(P_r) \quad \underset{x\in\mathcal{A}_r}{\text{v-min}} \left(\frac{f_1^2(x)}{g_1(x)}, \ldots, \frac{f_m^2(x)}{g_m(x)}\right)^T,$$

$$\mathcal{A}_r = \left\{ x \in \mathbb{R}^n : Cx \underset{\mathbb{R}^l_+}{\leqq} b \right\}.$$

The functions f_i and g_i, $i = 1, ..., m$, mapping from $\mathbb{R}^n$ into $\mathbb{R}$, are assumed to be convex and concave, respectively. For all $x \in \mathcal{A}_r$ and $i = 1, \ldots, m$, let $f_i(x) \geq 0$ and $g_i(x) > 0$ be fulfilled. By C is denoted a $l \times n$ matrix with real entries, b is a vector in $\mathbb{R}^l$, $b \neq 0_{\mathbb{R}^l}$ and " $\underset{\mathbb{R}^l_+}{\leqq}$ " represents the partial ordering induced by $\mathbb{R}^l_+$ on $\mathbb{R}^l$.

The solution concepts we use for (P_r) are again the Pareto-efficiency and the proper efficiency. Let us recall these notions for this special problem.

Definition 3.5 *An element $\bar{x} \in \mathcal{A}_r$ is said to be efficient (or Pareto-efficient) with respect to (P_r) if from $\frac{f_i^2(\bar{x})}{g_i(\bar{x})} \geq \frac{f_i^2(x)}{g_i(x)}$ for $x \in \mathcal{A}_r$ follows $\frac{f_i^2(\bar{x})}{g_i(\bar{x})} = \frac{f_i^2(x)}{g_i(x)}$, for $i = 1, \ldots, m$.*

Concerning the components of the objective function of (P_r), we want to mention here that, for each $i = 1, \ldots, m$, the function $\frac{f_i^2}{g_i}$ is convex on $\mathcal{A}_r$ (cf. [3]). This permits us to define the notion of proper efficiency again via linear scalarization (see also Definition 3.3).

Definition 3.6 *An element $\bar{x} \in \mathcal{A}_r$ is said to be properly efficient with respect to (P_r) if there exists $\lambda = (\lambda_1, \ldots, \lambda_m)^T \in int(R^m_+)$ such that $\sum\limits_{i=1}^{m} \lambda_i \frac{f_i^2(\bar{x})}{g_i(\bar{x})} \leq \sum\limits_{i=1}^{m} \lambda_i \frac{f_i^2(x)}{g_i(x)}$, $\forall x \in \mathcal{A}_r$.*

3.2.3 The scalar optimization problem

In order to study the duality for (P_r) we consider first the scalarized problem

$$(P_r^\lambda) \quad \inf_{x \in \mathcal{A}_r} \sum_{i=1}^{m} \lambda_i \frac{f_i^2(x)}{g_i(x)},$$

where $\lambda = (\lambda_1, \ldots, \lambda_m)^T$ is a fixed vector in $int(\mathbb{R}^m_+)$.

To (P_r^λ) we associate a new scalar optimization problem $(\tilde{P}_r^\lambda)$, with the property that the optimal objective values of these two problems are equal, i.e. $inf(P_r^\lambda) = inf(\tilde{P}_r^\lambda)$. We formulate a dual to this problem and this will then suggest us how to construct a multiobjective dual problem to (P_r).

Therefore, let us consider for $s = (s_1, \ldots, s_m)^T, t = (t_1, \ldots, t_m)^T \in \mathbb{R}^m$, the following feasible set

$$\tilde{\mathcal{A}}_r = \left\{ (x, s, t) : Cx \underset{\mathbb{R}^l_+}{\leqq} b,\ t_i > 0,\ f_i(x) - s_i \leq 0,\ t_i - g_i(x) \leq 0,\ i = 1, \ldots, m \right\}.$$

For $i = 1, \ldots, m$, let be the functions $\Phi_i : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \to \overline{\mathbb{R}}$

$$\Phi_i(x, s, t) = \begin{cases} \frac{s_i^2}{t_i}, & \text{if } \quad (x, s, t) \in \mathbb{R}^n \times \mathbb{R}^m \times int(\mathbb{R}^m_+), \\ +\infty, & \text{otherwise.} \end{cases}$$

Now, we can introduce the following scalar optimization problem

$$(\tilde{P}_r^\lambda) \quad \inf_{(x,s,t) \in \tilde{\mathcal{A}}_r} \sum_{i=1}^{m} \lambda_i \Phi_i(x, s, t).$$

Lemma 3.1 *It holds $inf(P_r^\lambda) = inf(\tilde{P}_r^\lambda)$.*

Proof. Let be $(x, s, t) \in \tilde{\mathcal{A}}_r$. This means that $x \in \mathcal{A}_r$ and, because of $f_i(x) \geq 0, \forall x \in \mathcal{A}_r$, it holds

$$\sum_{i=1}^{m} \lambda_i \Phi_i(x, s, t) = \sum_{i=1}^{m} \lambda_i \frac{s_i^2}{t_i} \geq \sum_{i=1}^{m} \lambda_i \frac{f_i^2(x)}{g_i(x)} \geq \inf_{x \in \mathcal{A}_r} \sum_{i=1}^{m} \lambda_i \frac{f_i^2(x)}{g_i(x)} = inf(P_\lambda),$$

which implies that $inf(\tilde{P}_r^\lambda) \geq inf(P_r^\lambda)$.

Conversely, let be $x \in \mathcal{A}_r$. Considering $s_i := f_i(x)$ and $t_i := g_i(x)$, for $i = 1, \ldots, m$, one can observe that $(x, s, t) \in \tilde{\mathcal{A}}_r$. Moreover, we have

$$\sum_{i=1}^{m} \lambda_i \frac{f_i^2(x)}{g_i(x)} = \sum_{i=1}^{m} \lambda_i \Phi_i(x, s, t) \geq \inf_{(x,s,t) \in \tilde{\mathcal{A}}_r} \sum_{i=1}^{m} \lambda_i \Phi_i(x, s, t) = inf(\tilde{P}_r^\lambda),$$

and this assures that the opposite inequality, $inf(P_r^\lambda) \geq inf(\tilde{P}_r^\lambda)$, also holds. In conclusion, $inf(P_r^\lambda) = inf(\tilde{P}_r^\lambda)$. □

3.2.4 Fenchel-Lagrange duality for the scalarized problem

In [69] Scott and Jefferson have used an approach based on the theory of geometric programming for finding the dual of a scalar optimization problem with a similar form to $(\tilde{P}_r^\lambda)$. In this subsection we obtain a dual for $(\tilde{P}_r^\lambda)$ using a completely different approach from that in [69]. Otherwise, the regularity condition considered by us is "weaker" than the Slater condition used by Scott and Jefferson.

Let us recall that in subsection 2.1.4 we have associated to the general convex optimization problem

$$(P^s) \quad \inf_{\substack{u \in V, \\ \tilde{g}(u) \leqq_{\mathbb{R}^w_+} 0}} \tilde{f}(u), \tag{3. 11}$$

with $V \subseteq \mathbb{R}^v$ being a nonempty convex set and $\tilde{f} : \mathbb{R}^v \to \overline{\mathbb{R}}$, $\tilde{g} : \mathbb{R}^v \to \mathbb{R}^w$ being convex functions such that $dom(\tilde{f}) = V$, the following so-called Fenchel-Lagrange dual problem

$$(D^s_{FL}) \quad \sup_{\substack{\tilde{p} \in \mathbb{R}^v, \\ \tilde{q} \geqq_{\mathbb{R}^w_+} 0}} \left\{ -\tilde{f}^*(\tilde{p}) + \inf_{u \in V} \left[\langle \tilde{p}, u \rangle + \langle \tilde{q}, \tilde{g}(u) \rangle \right] \right\}. \tag{3. 12}$$

Here, $K = \mathbb{R}^w_+$ is the ordering cone and we denote by $\langle \cdot, \cdot \rangle$ the Euclidean scalar product in the corresponding space, in fact, for $p = (p_1, ..., p_v)^T, u = (u_1, ..., u_v)^T \in \mathbb{R}^v$, $< p, u >= p^T u = \sum_{i=1}^{v} p_i u_i$.

For $\tilde{g}(u) = (\tilde{g}_1(u), \ldots, \tilde{g}_w(u))^T$ consider the sets

$$L = \left\{ i \in \{1, \ldots, w\} : \tilde{g}_i \text{ is an affine function} \right\},$$

$$N = \left\{ i \in \{1, \ldots, w\} : \tilde{g}_i \text{ is not an affine function} \right\},$$

and the following constraint qualification

(CQ^s_{ln}) | there exists an element $u' \in ri(V)$ such that $\tilde{g}_i(u') < 0$ for $i \in N$ and $\tilde{g}_i(u') \leq 0$ for $i \in L$.

By Theorem 2.8 we have that if the optimal objective value of (P^s) is finite and if (CQ^s_{ln}) is fulfilled, then the dual problem (D^s_{FL}) has an optimal solution and strong duality holds, i.e. $inf(P^s) = max(D^s_{FL})$.

We write now the problem $(\tilde{P}_r^\lambda)$ in the form (3. 11). In order to do this, we take $\mathbb{R}^v := \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m$, $\mathbb{R}^w := \mathbb{R}^l \times \mathbb{R}^m \times \mathbb{R}^m$, $V := \mathbb{R}^n \times \mathbb{R}^m \times int(\mathbb{R}^m_+)$ (i.e. $t_i > 0, i = 1, \ldots, m$),

$$\tilde{f}(x, s, t) = \sum_{i=1}^{m} \lambda_i \Phi_i(x, s, t)$$

and

$$\tilde{g}(x, s, t) = (Cx - b, f(x) - s, t - g(x)).$$

It is obvious that V is a nonempty convex set, $\tilde{f}$ is a convex function and $dom(\tilde{f}) = V$. From the convexity of f_i and the concavity of $g_i, i = 1, \ldots, m$, it follows that the function $\tilde{g}$ is convex relative to the cone $\mathbb{R}^w_+$. So, $(\tilde{P}_r^\lambda)$ is a particular case of the general convex optimization problem (P^s).

By (3. 12), (D_{FL}^s) yields the dual of the scalar problem $(\tilde{P}_r^\lambda)$, with the dual variables $\tilde{p} = (p^x, p^s, p^t)$ and $\tilde{q} = (q^x, q^s, q^t)$,

$$(\tilde{D}_r^\lambda) \quad \sup_{\substack{\tilde{p}\in\mathbb{R}^v,\\ \tilde{q}\in\mathbb{R}^w_+}} \left\{ -\sup_{(x,s,t)\in\mathbb{R}^v} \left[\langle \tilde{p}, (x,s,t)\rangle - \sum_{i=1}^m \lambda_i \Phi_i(x,s,t)\right] + \inf_{(x,s,t)\in V} [\langle \tilde{p}, (x,s,t)\rangle + \langle \tilde{q}, (Cx-b, f(x)-s, t-g(x))\rangle] \right\},$$

or, equivalently,

$$(\tilde{D}_r^\lambda) \quad \sup_{\substack{(p^x,p^s,p^t)\in\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^m,\\ (q^x,q^s,q^t)\in\mathbb{R}^l_+\times\mathbb{R}^m_+\times\mathbb{R}^m_+}} \left\{ - \sup_{\substack{(x,s,t)\in\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^m\\ t_i>0, i=1,\dots,m}} \left[\langle p^x, x\rangle + \langle p^s, s\rangle + \langle p^t, t\rangle - \sum_{i=1}^m \lambda_i \frac{s_i^2}{t_i}\right] + \inf_{s\in\mathbb{R}^m} \langle p^s - q^s, s\rangle + \inf_{t\in int(\mathbb{R}^m_+)} \langle p^t + q^t, t\rangle + \inf_{x\in\mathbb{R}^n} [\langle p^x, x\rangle + \langle q^x, Cx-b\rangle + \langle q^s, f(x)\rangle - \langle q^t, g(x)\rangle] \right\}.$$

After some transformations we obtain the following dual problem

$$(\tilde{D}_r^\lambda) \quad \sup_{\substack{p^x\in\mathbb{R}^n, p^s,p^t\in\mathbb{R}^m,\\ q^x\in\mathbb{R}^l_+, q^s,q^t\in\mathbb{R}^m_+}} \left\{ - \sum_{i=1}^m \sup_{\substack{s_i\in\mathbb{R}\\ t_i>0}} \left[\langle p_i^s, s_i\rangle + \langle p_i^t, t_i\rangle - \lambda_i \frac{s_i^2}{t_i}\right] - \sup_{x\in\mathbb{R}^n} \langle p^x, x\rangle + \inf_{x\in\mathbb{R}^n} \left\{ \langle p^x - C^T q^x, x\rangle + \sum_{i=1}^m [q_i^s f_i(x) - q_i^t g_i(x)] \right\} - \langle q^x, b\rangle + \inf_{s\in\mathbb{R}^m} \langle p^s - q^s, s\rangle + \inf_{t\in int(\mathbb{R}^m_+)} \langle p^t + q^t, t\rangle \right\}.$$

Since

$$\sup_{x\in\mathbb{R}^n} \langle p^x, x\rangle = \begin{cases} 0, & \text{if } \ p^x = 0, \\ +\infty, & \text{otherwise,} \end{cases}$$

$$\inf_{s\in\mathbb{R}^m} \langle p^s - q^s, s\rangle = \begin{cases} 0, & \text{if } \ p^s = q^s, \\ -\infty, & \text{otherwise,} \end{cases}$$

and

$$\inf_{t\in int(\mathbb{R}^m_+)} \langle p^t + q^t, t\rangle = \begin{cases} 0, & \text{if } \ p^t + q^t \underset{\mathbb{R}^m_+}{\geqq} 0, \\ -\infty, & \text{otherwise,} \end{cases}$$

in order to obtain supremum in $(\tilde{D}_r^\lambda)$, we have to take $p^x = 0$, $p^s = q^s$ and $p^t + q^t \underset{\mathbb{R}^m_+}{\geqq} 0$.

Moreover, for $i = 1, \dots, m$, we have

$$\sup_{\substack{s_i\in\mathbb{R}\\ t_i>0}} \left[\langle p_i^s, s_i\rangle + \langle p_i^t, t_i\rangle - \lambda_i \frac{s_i^2}{t_i}\right] = \begin{cases} 0, & \text{if } \ \frac{(p_i^s)^2}{4\lambda_i} + p_i^t \leq 0, \\ +\infty, & \text{otherwise.} \end{cases}$$

After all these considerations, the dual problem of $(\tilde{P}_r^\lambda)$ becomes

$$(\tilde{D}_r^\lambda) \quad \sup_{\substack{q^x \in \mathbb{R}_+^l, q^s, q^t \in \mathbb{R}_+^m, \\ p^s = q^s, p^t + q^t \in \mathbb{R}_+^m, \\ \frac{(p_i^s)^2}{4\lambda_i} + p_i^t \leq 0, i=1,\ldots,m}} \left\{ -\langle q^x, b\rangle - \sup_{x\in\mathbb{R}^n} \left[\langle -C^T q^x, x\rangle - \left(\sum_{i=1}^m (q_i^s f_i - q_i^t g_i) \right)(x) \right] \right\},$$

or, equivalently, by using the definition of the conjugate function,

$$\begin{aligned} (\tilde{D}_r^\lambda) \quad & \sup \left\{ -\langle q^x, b\rangle - \left[\sum_{i=1}^m (q_i^s f_i - q_i^t g_i) \right]^* (-C^T q^x) \right\}. \\ & \text{s.t.} \quad (q^x, q^s, q^t) \in \mathbb{R}_+^l \times \mathbb{R}_+^m \times \mathbb{R}_+^m, \\ & \qquad \frac{(q_i^s)^2}{4\lambda_i} \leq q_i^t, i = 1, \ldots, m \end{aligned} \tag{3. 13}$$

Remark 3.4

(a) In (3. 13) the conjugate of the sum can be written in the following form (cf. [62])

$$\left[\sum_{i=1}^m (q_i^s f_i - q_i^t g_i) \right]^* (-C^T q^x) = \inf \left\{ \sum_{i=1}^m (q_i^s f_i)^*(u_i) + \sum_{i=1}^m (-q_i^t g_i)^*(v_i) : \sum_{i=1}^m (u_i + v_i) = -C^T q^x \right\}.$$

(b) For the positive components of the vectors q^s and q^t it holds, for $i = 1, \ldots, m$,

$$(q_i^s f_i)^*(u_i) = q_i^s f_i^* \left(\frac{1}{q_i^s} u_i \right),$$

and

$$(-q_i^t g_i)^*(v_i) = q_i^t (-g_i)^* \left(\frac{1}{q_i^t} v_i \right).$$

Here, it is important to remark that these formulas can be applied in our case even if $q_i^s = 0$ or $q_i^t = 0$. In this situation, in order to obtain supremum in $(\tilde{D}_r^\lambda)$, we must consider $u_i = 0$, $(q_i^s f_i)^*(u_i) = 0$ and $v_i = 0$, $(-q_i^t g_i)^*(v_i) = 0$, respectively. This means that if $q_i^s = 0$ or $q_i^t = 0$, we have to take in the objective function of the dual $(\tilde{D}_r^\lambda)$ instead of $q_i^s f_i^* \left(\frac{1}{q_i^s} u_i \right)$ or, respectively, $q_i^t (-g_i)^* \left(\frac{1}{q_i^t} v_i \right)$, the value 0. Moreover, in the feasible set of the dual problem we have to consider the additional restrictions $u_i = 0$ and $v_i = 0$, respectively.

By Remark 3.4 ((a) and (b)) we obtain the following final form of the scalar dual problem

$$\begin{aligned} (\tilde{D}_r^\lambda) \quad & \sup \left\{ -\langle q^x, b\rangle - \sum_{i=1}^m q_i^s f_i^* \left(\frac{1}{q_i^s} u_i \right) - \sum_{i=1}^m q_i^t (-g_i)^* \left(\frac{1}{q_i^t} v_i \right) \right\}. \\ & \text{s.t.} \quad (q^x, q^s, q^t) \in \mathbb{R}_+^l \times \mathbb{R}_+^m \times \mathbb{R}_+^m, \\ & \qquad \frac{(q_i^s)^2}{4\lambda_i} \leq q_i^t, i = 1, \ldots, m, \\ & \qquad \sum_{i=1}^m (u_i + v_i) + C^T q^x = 0 \end{aligned}$$

We can present the strong duality theorem for the problems $(\tilde{P}_r^\lambda)$ and $(\tilde{D}_r^\lambda)$.

Theorem 3.6 *Let be $\mathcal{A}_r \neq \emptyset$. Then the dual problem $(\tilde{D}_r^\lambda)$ has an optimal solution and strong duality holds*

$$inf(P_r^\lambda) = inf(\tilde{P}_r^\lambda) = max(\tilde{D}_r^\lambda).$$

Proof. The set $\mathcal{A}_r$ being nonempty, by Lemma 3.1 it follows that $inf(P_r^\lambda) = inf(\tilde{P}_r^\lambda) \in \mathbb{R}$. For $x' \in \mathcal{A}_r$ (i.e. $Cx' \underset{\mathbb{R}^l_+}{\leqq} b$), we consider $t_i' := \frac{1}{2}g_i(x') > 0$ and $s_i' := f_i(x') + c_i, (c_i > 0)$, $i = 1, \ldots, m$. The element $u' = (x', s', t')$ belongs to the relative interior of $V = \mathbb{R}^n \times \mathbb{R}^m \times int(\mathbb{R}^m_+)$ and, obviously, it satisfies the constraint qualification (CQ^s_{ln}).

The hypotheses of Theorem 2.8 are verified and this means that $(\tilde{D}_r^\lambda)$ has an optimal solution and the equality $inf(P_r^\lambda) = inf(\tilde{P}_r^\lambda) = max(\tilde{D}_r^\lambda)$ is true. □

In order to investigate the duality for the multiobjective problem (P_r), we need the optimality conditions which result from the equality of the optimal objective values in Theorem 3.6. The following theorem gives us these conditions.

Theorem 3.7 *(a) Let $\hat{x}$ be a solution to (P_r^λ). Then there exists $(\hat{u}, \hat{v}, \hat{q}^x, \hat{q}^s, \hat{q}^t)$, an optimal solution to $(\tilde{D}_r^\lambda)$, such that the following optimality conditions are satisfied*

$$(i) \quad \hat{q}_i^s f_i^*\left(\tfrac{1}{\hat{q}_i^s}\hat{u}_i\right) + \hat{q}_i^s f_i(\hat{x}) = \langle \hat{u}_i, \hat{x}\rangle, \quad i = 1, \ldots, m,$$

$$(ii) \quad \hat{q}_i^t(-g_i)^*\left(\tfrac{1}{\hat{q}_i^t}\hat{v}_i\right) - \hat{q}_i^t g_i(\hat{x}) = \langle \hat{v}_i, \hat{x}\rangle, \quad i = 1, \ldots, m,$$

$$(iii) \quad \langle \hat{q}^x, b - C\hat{x}\rangle = 0,$$

$$(iv) \quad \sum_{i=1}^m (\hat{u}_i + \hat{v}_i) + C^T\hat{q}^x = 0,$$

$$(v) \quad \hat{q}_i^s = 2\lambda_i \tfrac{f_i(\hat{x})}{g_i(\hat{x})}, \quad i = 1, \ldots, m,$$

$$(vi) \quad \hat{q}_i^t = \lambda_i \tfrac{f_i^2(\hat{x})}{g_i^2(\hat{x})}, \quad i = 1, \ldots, m.$$

(b) Let $\hat{x}$ be admissible to (P_r^λ) and $(\hat{u}, \hat{v}, \hat{q}^x, \hat{q}^s, \hat{q}^t)$ be admissible to $(\tilde{D}_r^\lambda)$, satisfying (i)-(vi). Then $\hat{x}$ is an optimal solution to (P_r^λ), $(\hat{u}, \hat{v}, \hat{q}^x, \hat{q}^s, \hat{q}^t)$ is an optimal solution to $(\tilde{D}_r^\lambda)$ and strong duality holds.

Proof.

(a) Assume that $\hat{x}$ is a solution to (P_r^λ). By Theorem 3.6, there exists an optimal solution $(\hat{u}, \hat{v}, \hat{q}^x, \hat{q}^s, \hat{q}^t)$ to $(\tilde{D}_r^\lambda)$ such that $inf(P_r^\lambda) = inf(\tilde{P}_r^\lambda) = max(\tilde{D}_r^\lambda)$ or, equivalently,

$$\begin{aligned}
0 \;=\; & \sum_{i=1}^m \lambda_i \frac{f_i^2(\hat{x})}{g_i(\hat{x})} + \langle \hat{q}^x, b\rangle + \sum_{i=1}^m \hat{q}_i^s f_i^*\left(\frac{1}{\hat{q}_i^s}\hat{u}_i\right) + \sum_{i=1}^m \hat{q}_i^t(-g_i)^*\left(\frac{1}{\hat{q}_i^t}\hat{v}_i\right) \\
=\; & \sum_{i=1}^m \left[\hat{q}_i^s f_i^*\left(\frac{1}{\hat{q}_i^s}\hat{u}_i\right) + \hat{q}_i^s f_i(\hat{x}) - \langle \hat{u}_i, \hat{x}\rangle\right] + \sum_{i=1}^m g_i(\hat{x})\left[\hat{q}_i^t - \frac{(\hat{q}_i^s)^2}{4\lambda_i}\right] \\
+\; & \sum_{i=1}^m \left[\hat{q}_i^t(-g_i)^*\left(\frac{1}{\hat{q}_i^t}\hat{v}_i\right) - \hat{q}_i^t g_i(\hat{x}) - \langle \hat{v}_i, \hat{x}\rangle\right] + \langle \hat{q}^x, b - C\hat{x}\rangle \\
+\; & \sum_{i=1}^m \lambda_i g_i(\hat{x})\left(\frac{f_i(\hat{x})}{g_i(\hat{x})} - \frac{\hat{q}_i^s}{2\lambda_i}\right)^2 + \left\langle \sum_{i=1}^m (\hat{u}_i + \hat{v}_i) + C^T\hat{q}^x, \hat{x}\right\rangle.
\end{aligned} \qquad (3.\ 14)$$

By the definition of the conjugate function and Remark 3.4 (b), the inequality of Young for $i = 1, ..., m$ gives us

$$\hat{q}_i^s f_i^* \left(\frac{1}{\hat{q}_i^s} \hat{u}_i \right) + \hat{q}_i^s f_i(\hat{x}) \geq \langle \hat{u}_i, \hat{x} \rangle , \tag{3. 15}$$

and

$$\hat{q}_i^t (-g_i)^* \left(\frac{1}{\hat{q}_i^t} \hat{v}_i \right) - \hat{q}_i^t g_i(\hat{x}) \geq \langle \hat{v}_i, \hat{x} \rangle . \tag{3. 16}$$

By the inequalities (3. 15), (3. 16), the feasibility of $\hat{x}$ to (P_r^λ) and the feasibility of $(\hat{u}, \hat{v}, \hat{q}^x, \hat{q}^s, \hat{q}^t)$ to $(\tilde{D}_r^\lambda)$, it follows that the terms of the sum in (3. 14) are greater or equal than zero. This means that all of them must be equal to zero and, in conclusion, the optimality conditions (i)-(vi) must be fulfilled.

(b) All the calculations and transformations done before may be carried out in the reverse direction starting from the relations (i)-(vi). □

3.2.5 The multiobjective dual problem

With the above preparation we are able now to formulate a multiobjective dual problem to (P_r). This is introduced by

$$(D_r) \qquad \underset{(u,v,\lambda,\delta,q^s,q^t)\in\mathcal{B}_r}{\text{v-max}} h(u, v, \lambda, \delta, q^s, q^t),$$

with

$$h(u, v, \lambda, \delta, q^s, q^t) = \begin{pmatrix} h_1(u, v, \lambda, \delta, q^s, q^t) \\ \vdots \\ h_m(u, v, \lambda, \delta, q^s, q^t) \end{pmatrix},$$

$$h_j(u, v, \lambda, \delta, q^s, q^t) = -q_j^s f_j^* \left(\frac{1}{q_j^s} u_j \right) - q_j^t (-g_j)^* \left(\frac{1}{q_j^t} v_j \right) - \langle \delta_j, b \rangle ,$$

for $j = 1, \ldots, m$. The dual variables are

$$u = (u_1, \ldots, u_m), v = (v_1, \ldots, v_m), \lambda = (\lambda_1, \ldots, \lambda_m)^T,$$

$$\delta = (\delta_1, \ldots, \delta_m), q^s = (q_1^s, \ldots, q_m^s)^T, q^t = (q_1^t, \ldots, q_m^t)^T,$$

$$u_i \in \mathbb{R}^n, v_i \in \mathbb{R}^n, \lambda_i \in \mathbb{R}, \delta_i \in \mathbb{R}^l, q_i^s \in \mathbb{R}, q_i^t \in \mathbb{R}, i = 1, \ldots, m,$$

and the set of constraints is defined by

$$\begin{aligned} \mathcal{B}_r = \quad & \Big\{ (u, v, \lambda, \delta, q^s, q^t) : \lambda \in int(\mathbb{R}_+^m), \quad q^s, q^t \underset{R_+^m}{\geqq} 0, \quad \sum_{i=1}^m \lambda_i \delta_i \underset{R_+^l}{\geqq} 0, \\ & \sum_{i=1}^m \lambda_i (u_i + v_i + C^T \delta_i) = 0, \quad (q_i^s)^2 \leq 4 q_i^t, \quad i = 1, \ldots, m \Big\}. \end{aligned} \tag{3. 17}$$

Definition 3.7 *An element $(\bar{u}, \bar{v}, \bar{\lambda}, \bar{\delta}, \bar{q}^s, \bar{q}^t) \in \mathcal{B}_r$ is said to be efficient (or Pareto-efficient) with respect to (D_r) if from*

$$h(u, v, \lambda, \delta, q^s, q^t) \underset{R_+^m}{\geqq} h(\bar{u}, \bar{v}, \bar{\lambda}, \bar{\delta}, \bar{q}^s, \bar{q}^t), \quad \textit{for} \quad (u, v, \lambda, \delta, q^s, q^t) \in \mathcal{B}_r,$$

follows $h(u, v, \lambda, \delta, q^s, q^t) = h(\bar{u}, \bar{v}, \bar{\lambda}, \bar{\delta}, \bar{q}^s, \bar{q}^t)$.

The following theorem states the weak duality between the multiobjective problem (P_r) and its dual (D_r).

Theorem 3.8 *There is no $x \in \mathcal{A}_r$ and no $(u, v, \lambda, \delta, q^s, q^t) \in \mathcal{B}_r$ such that $\frac{f_i^2(x)}{g_i(x)} \leq h_i(u, v, \lambda, \delta, q^s, q^t)$, for $i = 1, \ldots, m$, and $\frac{f_j^2(x)}{g_j(x)} < h_j(u, v, \lambda, \delta, q^s, q^t)$ for at least one $j \in \{1, \ldots, m\}$.*

Proof. Let us assume the contrary. This means that there exist $x \in \mathcal{A}_r$ and $(u, v, \lambda, \delta, q^s, q^t) \in \mathcal{B}_r$ such that

$$\sum_{i=1}^{m} \lambda_i \frac{f_i^2(x)}{g_i(x)} < \sum_{i=1}^{m} \lambda_i h_i(u, v, \lambda, \delta, q^s, q^t). \tag{3. 18}$$

On the other hand, applying the inequalities (3. 15) and (3. 16), we have

$$\begin{aligned}
\sum_{i=1}^{m} \lambda_i \frac{f_i^2(x)}{g_i(x)} - \sum_{i=1}^{m} \lambda_i h_i(u, v, \lambda, \delta, q^s, q^t) &= \sum_{i=1}^{m} \lambda_i \frac{f_i^2(x)}{g_i(x)} + \left\langle \sum_{i=1}^{m} \lambda_i \delta_i, b \right\rangle \\
&+ \sum_{i=1}^{m} \lambda_i \left[q_i^s f_i^* \left(\frac{1}{q_i^s} u_i \right) + q_i^t (-g_i)^* \left(\frac{1}{q_i^t} v_i \right) \right] \geq \sum_{i=1}^{m} \lambda_i \frac{f_i^2(x)}{g_i(x)} \\
&+ \left\langle \sum_{i=1}^{m} \lambda_i \delta_i, b \right\rangle + \sum_{i=1}^{m} \lambda_i \left[-q_i^s f_i(x) + q_i^t g_i(x) + \langle u_i + v_i, x \rangle \right] \\
&= \left\langle \sum_{i=1}^{m} \lambda_i \delta_i, b - Cx \right\rangle + \sum_{i=1}^{m} \lambda_i g_i(x) \left[\frac{f_i^2(x)}{g_i^2(x)} - q_i^s \frac{f_i(x)}{g_i(x)} + q_i^t \right] \\
&\geq \left\langle \sum_{i=1}^{m} \lambda_i \delta_i, b - Cx \right\rangle + \sum_{i=1}^{m} \lambda_i g_i(x) \left[\frac{f_i^2(x)}{g_i^2(x)} - q_i^s \frac{f_i(x)}{g_i(x)} + \frac{(q_i^s)^2}{4} \right] \\
&= \left\langle \sum_{i=1}^{m} \lambda_i \delta_i, b - Cx \right\rangle + \sum_{i=1}^{m} \lambda_i g_i(x) \left[\frac{f_i(x)}{g_i(x)} - \frac{q_i^s}{2} \right]^2 \geq 0.
\end{aligned}$$

This contradicts the strict inequality (3. 18). □

The following theorem expresses the strong duality between the problems (P_r) and (D_r).

Theorem 3.9 *If $\bar{x} \in \mathcal{A}_r$ is a properly efficient solution to (P_r), then there exists an efficient solution $(\bar{u}, \bar{v}, \bar{\lambda}, \bar{\delta}, \bar{q}^s, \bar{q}^t) \in \mathcal{B}_r$ to the dual (D_r) such that strong duality $\frac{f_i^2(\bar{x})}{g_i(\bar{x})} = h_i(\bar{u}, \bar{v}, \bar{\lambda}, \bar{\delta}, \bar{q}^s, \bar{q}^t), i = 1, \ldots, m$, holds.*

Proof. From the proper efficiency of $\bar{x}$, by Definition 3.6, we get a vector $\bar{\lambda} = (\bar{\lambda}_1, \ldots, \bar{\lambda}_m)^T \in int(\mathbb{R}_+^m)$ with the property that $\bar{x}$ solves the scalar optimization problem

$$(P_r^{\bar{\lambda}}) \quad \inf_{x \in \mathcal{A}_r} \sum_{i=1}^{m} \bar{\lambda}_i \frac{f_i^2(x)}{g_i(x)}.$$

Theorem 3.7 assures the existence of an optimal solution $(\hat{u}, \hat{v}, \hat{q}^x, \hat{q}^s, \hat{q}^t)$ to the dual $(D_r^{\bar{\lambda}})$ such that (i)-(vi) are satisfied.

Let us construct by means of $\bar{x}$ and $(\hat{u}, \hat{v}, \hat{q}^x, \hat{q}^s, \hat{q}^t)$ a solution to (D_r). Therefore, let be $\hat{q} \in \mathbb{R}^l$ such that $\langle \hat{q}, b \rangle = 1$ (such an $\hat{q}$ exists because $b \neq 0$). For $i = 1, \ldots, m$, let also be $\bar{u}_i := \frac{1}{\bar{\lambda}_i} \hat{u}_i$, $\bar{v}_i := \frac{1}{\bar{\lambda}_i} \hat{v}_i$, $\bar{q}_i^s := \frac{\hat{q}_i^s}{\bar{\lambda}_i}$, $\bar{q}_i^t := \frac{\hat{q}_i^t}{\bar{\lambda}_i}$ and

$$\bar{\delta}_i = \begin{cases} -\frac{1}{\bar{\lambda}_i} \frac{\langle \hat{u}_i + \hat{v}_i, \bar{x} \rangle}{\langle \hat{q}^x, b \rangle} \hat{q}^x, & \text{if } \langle \hat{q}^x, b \rangle \neq 0, \\ \frac{1}{m \bar{\lambda}_i} \hat{q}^x - \frac{\langle \hat{u}_i + \hat{v}_i, \bar{x} \rangle}{\bar{\lambda}_i} \hat{q}, & \text{if } \langle \hat{q}^x, b \rangle = 0. \end{cases}$$

By (iii) and (iv), for $(\bar{u}, \bar{v}, \bar{\lambda}, \bar{\delta}, \bar{q}^s, \bar{q}^t)$, $\bar{\delta} := (\bar{\delta}_1, \ldots, \bar{\delta}_m)$, it holds $\bar{\lambda} \in int(\mathbb{R}^m_+)$, $\bar{q}^s, \bar{q}^t \underset{\mathbb{R}^m_+}{\geqq} 0$, $\sum\limits_{i=1}^m \bar{\lambda}_i \bar{\delta}_i = \hat{q}^x \underset{\mathbb{R}^l_+}{\geqq} 0$ and $\sum\limits_{i=1}^m \bar{\lambda}_i(\bar{u}_i + \bar{v}_i + C^T \bar{\delta}_i) = 0$.

Additionally, by (v) and (vi), we have, for $i = 1, \ldots, m$,

$$(\bar{q}_i^s)^2 = \left(\frac{\hat{q}_i^s}{\bar{\lambda}_i}\right)^2 = 4\frac{f_i^2(\bar{x})}{g_i^2(\bar{x})} = 4\frac{\hat{q}_i^t}{\bar{\lambda}_i} = 4\bar{q}_i^t,$$

and this means that $(\bar{u}, \bar{v}, \bar{\lambda}, \bar{\delta}, \bar{q}^s, \bar{q}^t) \in \mathcal{B}_r$, i.e. it is feasible to (D_r).

On the other hand, by (i)-(ii) and (v)-(vi), for $i = 1, \ldots, m$, it holds

$$h_i(\bar{u}, \bar{v}, \bar{\lambda}, \bar{\delta}, \bar{q}^s, \bar{q}^t) = -\bar{q}_i^s f_i^*\left(\frac{1}{\bar{q}_i^s}\bar{u}_i\right) - \bar{q}_i^t(-g_i)^*\left(\frac{1}{\bar{q}_i^t}\bar{v}_i\right) - \langle \bar{\delta}_i, b\rangle =$$

$$-\frac{\hat{q}_i^s}{\bar{\lambda}_i} f_i^*\left(\frac{1}{\hat{q}_i^s}\hat{u}_i\right) - \frac{\hat{q}_i^t}{\bar{\lambda}_i}(-g_i)^*\left(\frac{1}{\hat{q}_i^t}\hat{v}_i\right) + \frac{1}{\bar{\lambda}_i}\langle \hat{u}_i + \hat{v}_i, \bar{x}\rangle = \frac{\hat{q}_i^s}{\bar{\lambda}_i} f_i(\bar{x})$$

$$-\frac{1}{\bar{\lambda}_i}\langle \hat{u}_i, \bar{x}\rangle - \frac{\hat{q}_i^t}{\bar{\lambda}_i} g_i(\bar{x}) - \frac{1}{\bar{\lambda}_i}\langle \hat{v}_i, \bar{x}\rangle + \frac{1}{\bar{\lambda}_i}\langle \hat{u}_i + \hat{v}_i, \bar{x}\rangle =$$

$$2\frac{f_i^2(\bar{x})}{g_i(\bar{x})} - \frac{f_i^2(\bar{x})}{g_i(\bar{x})} = \frac{f_i^2(\bar{x})}{g_i(\bar{x})}.$$

The maximality of $(\bar{u}, \bar{v}, \bar{\lambda}, \bar{\delta}, \bar{q}^s, \bar{q}^t)$ follows immediately by Theorem 3.8. □

3.2.6 The quadratic-linear fractional programming problem

In the last subsection of this chapter we consider the multiobjective optimization problem for one of the two special cases presented in [69] and we find out how its dual looks like.

As primal multiobjective problem let be

$$(P_{ql}) \quad \underset{x \in \mathcal{A}_{ql}}{\text{v-min}} \left(\frac{x^T Q_1 x}{(d_1)^T x + e_1}, \ldots, \frac{x^T Q_m x}{(d_m)^T x + e_m}\right)^T,$$

$$\mathcal{A}_{ql} = \left\{ x \in \mathbb{R}^n : Cx \underset{\mathbb{R}^l_+}{\leqq} b \right\},$$

where Q_i is a symmetric positive definite $n \times n$ matrix with real entries, $f_i(x) = \sqrt{x^T Q_i x}$ and $g_i(x) = (d_i)^T x + e_i$ are convex functions, for each $i = 1, \ldots, m$.

Let be $d_i \in \mathbb{R}^n, e_i \in \mathbb{R},\ i = 1, \ldots, m$, and the polyhedral set $\mathcal{A}_{ql} = \{x \in R^n : Cx \underset{\mathbb{R}^l_+}{\leqq} b\}$ selected so that $g_i(x) = (d_i)^T x + e_i > 0$, for all $x \in \mathcal{A}_{ql}$.

For the conjugate of f_i and g_i we have, for $i = 1, \ldots, m$,

$$f_i^*\left(\frac{1}{q_i^s}u_i\right) = \begin{cases} 0, & \text{if } \sqrt{u_i^T Q_i^{-1} u_i} \leq q_i^s, \\ +\infty, & \text{otherwise,} \end{cases}$$

and

$$(-g_i)^*\left(\frac{1}{q_i^t}v_i\right) = \begin{cases} e_i, & \text{if } \frac{1}{q_i^t}v_i = -d_i, \\ +\infty, & \text{otherwise.} \end{cases}$$

Owing to the general approach presented within subsection 3.2.5, the dual of (P_{ql}) turns out to be

$$(D_{ql}) \qquad \text{v}-\max \begin{pmatrix} -q_1^t e_1 - \langle \delta_1, b\rangle \\ \vdots \\ -q_m^t e_m - \langle \delta_m, b\rangle \end{pmatrix},$$

$$\text{s.t.} \qquad (u, v, \lambda, \delta, q^t, q^s) \in \mathcal{B}'_{ql},$$

with

$$\mathcal{B}'_{ql} = \Bigg\{ (u, v, \lambda, \delta, q^t, q^s) : \lambda \in int(R^m_+), q^s, q^t \underset{\mathbb{R}^m_+}{\geqq} 0, \sum_{i=1}^m \lambda_i (u_i + v_i + C^T \delta_i) = 0,$$
$$\sum_{i=1}^m \lambda_i \delta_i \underset{\mathbb{R}^l_+}{\geqq} 0,\ (q_i^s)^2 \leq 4 q_i^t,\ \sqrt{u_i^T Q_i^{-1} u_i} \leq q_i^s,\ v_i = -q_i^t d_i,\ i = 1, \dots, m \Bigg\},$$

or, equivalently,

$$(D_{rl}) \qquad \text{v}-\max \begin{pmatrix} -q_1^t e_1 - \langle \delta_1, b\rangle \\ \vdots \\ -q_m^t e_m - \langle \delta_m, b\rangle \end{pmatrix},$$

$$\text{s.t.} \qquad (u, \lambda, \delta, q^t) \in \mathcal{B}_{ql},$$

with

$$\mathcal{B}_{ql} = \Bigg\{ (u, \lambda, \delta, q^t) : \quad \lambda \in int(R^m_+),\ q^t \underset{\mathbb{R}^m_+}{\geqq} 0,\ \sum_{i=1}^m \lambda_i (u_i - d_i q_i^t + C^T \delta_i) = 0,$$
$$\sum_{i=1}^m \lambda_i \delta_i \underset{\mathbb{R}^l_+}{\geqq} 0,\ u_i^T Q_i^{-1} u_i \leq 4 q_i^t,\ i = 1, \dots, m \Bigg\}.$$

Remark 3.5 The problem (P_{ql}) can also be seen as a special case of a general multiobjective fractional convex-concave problem. This means that one can construct a dual to (P_{ql}) by using the approaches given for this kind of multiobjective problems in the literature (see for instance [59] and [91]). But, it turns out that the dual (D_{ql}) is different from the duals obtained by applying the approaches from the papers mentioned above.

Chapter 4

An analysis of some dual problems in multiobjective optimization

In the fourth chapter of this thesis we intend to investigate the relationships between different dual problems that appear in the theory of vector optimization. As primal problem we consider the same multiobjective optimization problem (P) with cone inequality constraints as in the first part of chapter 3. We construct by means of scalarization several multiobjective duals to (P) and relate these new duality concepts to each other and, more than that, to some well-known duality concepts from the literature (cf. [40], [41], [54], [55], [65], [92], [93]).

In the past, ISERMANN also made in [38] an analysis of different duality concepts, but for linear multiobjective optimization problems. He related the duality concept introduced by himself in [36] and [37] to the concepts introduced by GALE, KUHN AND TUCKER in [24] and by KORNBLUTH in [45]. As another important contribution to this field let us remind the paper of IVANOV AND NEHSE [39].

In the beginning we associate to the primal multiobjective optimization problem (P) a scalar one. Then we introduce three scalar dual problems to it, constructed by using the Lagrange, Fenchel and Fenchel-Lagrange duals presented in chapter 2. Starting from them, we formulate six different multiobjective duals and prove the existence of weak and, under certain conditions, of strong duality. Between these six duals one can recognize a generalization of the dual introduced by WANKA AND BOŢ in [85] described in chapter 3 and, also, the dual presented by JAHN in [40] and [41], here in the finite dimensional case.

Afterwards, we derive for the six duals some relations between their image sets and between the maximal elements sets of their image sets, respectively. By giving some counter-examples we also show that these sets are not always equal. On the other hand, we give some conditions for which the sets become identical.

In the last part of the chapter we complete this analysis by including the multiobjective duals of NAKAYAMA [54], [55], WOLFE [90], [93] and WEIR AND MOND [90], [92].

4.1 Preliminaries

The primal optimization problem with cone inequality constraints which we consider in this chapter is again the following one

$$(P) \quad \operatorname*{v-min}_{x \in \mathcal{A}} f(x),$$

$$\mathcal{A} = \left\{ x \in \mathbb{R}^n : g(x) = (g_1(x), \ldots, g_k(x))^T \underset{K}{\leqq} 0 \right\},$$

where $f(x) = (f_1(x), \ldots, f_m(x))^T$, $f_i : \mathbb{R}^n \to \overline{\mathbb{R}}$, $i = 1, ..., m$, are proper functions, $g_j : \mathbb{R}^n \to \mathbb{R}$, $j = 1, ..., k$, and $K \subseteq \mathbb{R}^k$ is assumed to be a convex closed cone with $int(K) \neq \emptyset$, defining a partial ordering according to $x_2 \underset{K}{\leqq} x_1$ if and only if $x_1 - x_2 \in K$. Further on we deal with Pareto-efficient and properly efficient solutions to (P) with respect to the ordering cone $\mathbb{R}^m_+$.

Let us introduce now three quite general assumptions which play an important role in this chapter

(A_f) | the functions f_i, $i = 1, ..., m$, are convex and $\bigcap\limits_{i=1}^{m} ri(dom(f_i)) \neq \emptyset$,

(A_g) | the function g is convex relative to the cone K, i.e. $\forall x_1, x_2 \in \mathbb{R}^n$, $\forall \lambda \in [0, 1], \lambda g(x_1) + (1 - \lambda) g(x_2) - g(\lambda x_1 + (1 - \lambda) x_2) \in K$,

(A_{CQ}) | there exists $x' \in \bigcap\limits_{i=1}^{m} dom(f_i)$ such that $g(x') \in -int(K)$.

We notice that the assumption (A_{CQ}) is nothing else but the constraint qualification (CQ) considered in section 3.1. Within this part of the work we will mention if we are in the general case or if (A_f), (A_g) and/or (A_{CQ}) are assumed to be fulfilled.

Let now $\lambda = (\lambda_1, \ldots, \lambda_m)^T$ be a fixed vector in $int(\mathbb{R}^m_+)$ and the following scalar problem associated to (P)

$$(P^\lambda) \quad \inf_{x \in \mathcal{A}} \sum_{i=1}^{m} \lambda_i f_i(x).$$

By means of the theory developed in chapter 2 we can introduce in the same way like in subsection 3.1.3 the following duals to (P^λ)

$$(D^\lambda_L) \sup_{q \underset{K^*}{\geqq} 0} \inf_{x \in \mathbb{R}^n} \left[\sum_{i=1}^{m} \lambda_i f_i(x) + q^T g(x) \right],$$

$$(D^\lambda_F) \sup_{p_i \in \mathbb{R}^n, i=1,\ldots,m} \left\{ - \sum_{i=1}^{m} \lambda_i f_i^*(p_i) - \chi^*_{\mathcal{A}} \left(- \sum_{i=1}^{m} \lambda_i p_i \right) \right\},$$

and

$$(D^\lambda_{FL}) \sup_{\substack{p_i \in \mathbb{R}^n, i=1,\ldots,m, \\ q \underset{K^*}{\geqq} 0}} \left\{ - \sum_{i=1}^{m} \lambda_i f_i^*(p_i) - (q^T g)^* \left(- \sum_{i=1}^{m} \lambda_i p_i \right) \right\}.$$

According to Theorem 2.6 and Theorem 16.4 in [62] we get the following strong duality theorem.

Theorem 4.1 *Assume that the optimal objective value of (P^λ), $inf(P^\lambda)$, is finite and that the assumptions (A_f), (A_g) and (A_{CQ}) are fulfilled. Then the dual problems (D^λ_L), (D^λ_F) and (D^λ_{FL}) have optimal solutions and strong duality holds*

$$inf(P^\lambda) = max(D^\lambda_L) = max(D^\lambda_F) = max(D^\lambda_{FL}).$$

4.2 The multiobjective dual (D_1) and the family of multiobjective duals (D_α), $\alpha \in \mathcal{F}$

The first multiobjective dual problem to (P) we introduce here is

$$(D_1) \quad \underset{(p,q,\lambda,t)\in\mathcal{B}_1}{\text{v-max}} \; h^1(p,q,\lambda,t),$$

with

$$h^1(p,q,\lambda,t) = \begin{pmatrix} h_1^1(p,q,\lambda,t) \\ \vdots \\ h_m^1(p,q,\lambda,t) \end{pmatrix},$$

$$h_j^1(p,q,\lambda,t) = -f_j^*(p_j) - (q^T g)^* \left(-\frac{1}{\sum\limits_{i=1}^m \lambda_i} \sum_{i=1}^m \lambda_i p_i \right) + t_j, j = 1,...,m,$$

the dual variables

$$p = (p_1,...,p_m) \in \mathbb{R}^n \times ... \times \mathbb{R}^n, q \in \mathbb{R}^k,$$

$$\lambda = (\lambda_1,...,\lambda_m)^T \in \mathbb{R}^m, t = (t_1,...,t_m)^T \in \mathbb{R}^m,$$

and the set of constraints

$$\mathcal{B}_1 = \left\{ (p,q,\lambda,t) : \lambda \in int(\mathbb{R}^m_+), \quad q \underset{K^*}{\geqq} 0, \quad \sum_{i=1}^m \lambda_i t_i = 0 \right\}.$$

Next, we present the weak and strong duality theorems for the multiobjective problems (P) and (D_1).

Theorem 4.2 (weak duality for (D_1)) *There is no $x \in \mathcal{A}$ and no $(p,q,\lambda,t) \in \mathcal{B}_1$ fulfilling $h^1(p,q,\lambda,t) \underset{\mathbb{R}^m_+}{\geqq} f(x)$ and $h^1(p,q,\lambda,t) \neq f(x)$.*

Proof. We assume that there exist $x \in \mathcal{A}$ and $(p,q,\lambda,t) \in \mathcal{B}_1$ such that $f_i(x) \leq h_i^1(p,q,\lambda,t), \forall i \in \{1,\ldots,m\}$ and $f_j(x) < h_j^1(p,q,\lambda,t)$, for at least one $j \in \{1,\ldots,m\}$. This means that we have

$$\sum_{i=1}^m \lambda_i f_i(x) < \sum_{i=1}^m \lambda_i h_i^1(p,q,\lambda,t). \tag{4. 1}$$

On the other hand, by using the inequalities

$$-f_i^*(p_i) \leq f_i(x) - p_i^T x, \; i = 1,...,m,$$

and

$$-(q^T g)^* \left(-\frac{1}{\sum\limits_{j=1}^m \lambda_j} \sum_{j=1}^m \lambda_j p_j \right) \leq q^T g(x) + \frac{1}{\sum\limits_{j=1}^m \lambda_j} \left(\sum_{j=1}^m \lambda_j p_j \right)^T x,$$

we have that

$$\begin{aligned}
\sum_{i=1}^{m} \lambda_i h_i^1(p,q,\lambda,t) &= -\sum_{i=1}^{m} \lambda_i f_i^*(p_i) - \sum_{i=1}^{m} \lambda_i (q^T g)^* \left(-\frac{1}{\sum\limits_{i=1}^{m} \lambda_i} \sum_{j=1}^{m} \lambda_j p_j \right) \\
&+ \sum_{i=1}^{m} \lambda_i t_i \leq \sum_{i=1}^{m} \lambda_i f_i(x) + \left(\sum_{i=1}^{m} \lambda_i q \right)^T g(x) \\
&- \left(\sum_{i=1}^{m} \lambda_i p_i \right)^T x + \left(\sum_{i=1}^{m} \lambda_i p_i \right)^T x \leq \sum_{i=1}^{m} \lambda_i f_i(x).
\end{aligned}$$

The inequality obtained above, $\sum\limits_{i=1}^{m} \lambda_i h_i^1(p,q,\lambda,t) \leq \sum\limits_{i=1}^{m} \lambda_i f_i(x)$, contradicts relation (4. 1). □

Theorem 4.3 (strong duality for (D_1)) *Assume that (A_f), (A_g) and (A_{CQ}) are fulfilled. If $\bar{x}$ is a properly efficient solution to (P), then there exists an efficient solution $(\bar{p}, \bar{q}, \bar{\lambda}, \bar{t}) \in \mathcal{B}_1$ to the dual (D_1) and the strong duality $f(\bar{x}) = h^1(\bar{p}, \bar{q}, \bar{\lambda}, \bar{t})$ holds.*

Proof. If $\bar{x}$ is properly efficient to the problem (P), then there exists a vector $\bar{\lambda} = (\bar{\lambda}_1, \ldots, \bar{\lambda}_m)^T \in int(\mathbb{R}^m_+)$ such that $\bar{x}$ solves the scalar problem

$$(P^{\bar{\lambda}}) \quad \inf_{x \in \mathcal{A}} \sum_{i=1}^{m} \bar{\lambda}_i f_i(x).$$

But, Theorem 3.2 assures the existence of a solution $(\tilde{p}, \tilde{q})$ to the scalar dual $(D^{\bar{\lambda}}_{FL})$ such that the optimality conditions (i), (ii) and (iii) are satisfied.

Considering

$$\bar{p} := \tilde{p}, \bar{q} := \frac{1}{\sum\limits_{i=1}^{m} \bar{\lambda}_i} \tilde{q} \underset{K^*}{\geqq} 0,$$

and

$$\bar{t}_i := \bar{p}_i^T \bar{x} + (\bar{q}^T g)^* \left(-\frac{1}{\sum\limits_{j=1}^{m} \bar{\lambda}_j} \sum_{j=1}^{m} \bar{\lambda}_j \bar{p}_j \right) \in \mathbb{R}, i = 1, ..., m,$$

it holds $\sum\limits_{i=1}^{m} \bar{\lambda}_i \bar{t}_i = 0$ (cf. (3. 1)), so $(\bar{p}, \bar{q}, \bar{\lambda}, \bar{t})$ is feasible to (D_1), i.e. $(\bar{p}, \bar{q}, \bar{\lambda}, \bar{t}) \in \mathcal{B}_1$. Moreover, from Theorem 3.2 (i), it follows that $f_i(\bar{x}) = h_i^1(\bar{p}, \bar{q}, \bar{\lambda}, \bar{t})$, for $i = 1, ..., m$. The maximality of $(\bar{p}, \bar{q}, \bar{\lambda}, \bar{t})$ is given by Theorem 4.2. □

In the second part of the section we introduce a family of dual multiobjective problems to (P). Therefore, let us consider the following family of functions

$$\mathcal{F} = \Big\{ \alpha : int(\mathbb{R}^m_+) \to \mathbb{R}^m_+ \ : \ \alpha(\lambda) = (\alpha_1(\lambda), ..., \alpha_m(\lambda))^T, \text{ such that } \sum_{i=1}^{m} \lambda_i \alpha_i(\lambda) = 1, \ \forall \lambda = (\lambda_1, ..., \lambda_m)^T \in int(\mathbb{R}^m_+) \Big\}$$

For each $\alpha \in \mathcal{F}$ we consider the following dual problem

$$(D_\alpha) \quad \underset{(p,\tilde{q},\lambda,t) \in \mathcal{B}_\alpha}{\text{v-max}} h^\alpha(p, \tilde{q}, \lambda, t),$$

with

$$h^\alpha(p,\tilde{q},\lambda,t) = \begin{pmatrix} h_1^\alpha(p,\tilde{q},\lambda,t) \\ \vdots \\ h_m^\alpha(p,\tilde{q},\lambda,t) \end{pmatrix},$$

$$h_j^\alpha(p,\tilde{q},\lambda,t) = -f_j^*(p_j) - (q_j^T g)^* \left(-\alpha_j(\lambda)\sum_{i=1}^m \lambda_i p_i\right) + t_j, j = 1,\dots,m,$$

the dual variables

$$p = (p_1,\dots,p_m) \in \mathbb{R}^n \times ... \times \mathbb{R}^n, \tilde{q} = (q_1,...,q_m) \in \mathbb{R}^k \times ... \times \mathbb{R}^k,$$

$$\lambda = (\lambda_1,\dots,\lambda_m)^T \in \mathbb{R}^m, t = (t_1,\dots,t_m)^T \in \mathbb{R}^m,$$

and the set of constraints

$$\mathcal{B}_\alpha = \left\{ (p,\tilde{q},\lambda,t) : \lambda \in int(\mathbb{R}^m_+), \quad \sum_{i=1}^m \lambda_i q_i \underset{K^*}{\geqq} 0, \quad \sum_{i=1}^m \lambda_i t_i = 0 \right\}.$$

Let us show now the existence of weak and strong duality between the primal problem and the problems just introduced.

Theorem 4.4 (weak duality for $(D_\alpha), \alpha \in \mathcal{F}$) *For each $\alpha \in \mathcal{F}$ there is no $x \in \mathcal{A}$ and no $(p,\tilde{q},\lambda,t) \in \mathcal{B}_\alpha$ fulfilling $h^\alpha(p,\tilde{q},\lambda,t) \underset{\mathbb{R}^m_+}{\geqq} f(x)$ and $h^\alpha(p,\tilde{q},\lambda,t) \neq f(x)$.*

Proof. Let be $\alpha \in \mathcal{F}$, fixed. We assume that there exist $x \in \mathcal{A}$ and $(p,\tilde{q},\lambda,t) \in \mathcal{B}_\alpha$ such that $f_i(x) \leq h_i^\alpha(p,\tilde{q},\lambda,t), \forall i \in \{1,\dots,m\}$ and $f_j(x) < h_j^\alpha(p,\tilde{q},\lambda,t)$, for at least one $j \in \{1,\dots,m\}$. This means that

$$\sum_{i=1}^m \lambda_i f_i(x) < \sum_{i=1}^m \lambda_i h_i^\alpha(p,\tilde{q},\lambda,t). \tag{4. 2}$$

Applying again the inequality of Young it results

$$\begin{aligned} \sum_{i=1}^m \lambda_i h_i^\alpha(p,\tilde{q},\lambda,t) &= -\sum_{i=1}^m \lambda_i f_i^*(p_i) - \sum_{i=1}^m \lambda_i (q_i^T g)^* \left(-\alpha_i(\lambda)\sum_{j=1}^m \lambda_j p_j\right) \\ &+ \sum_{i=1}^m \lambda_i t_i \leq \sum_{i=1}^m \lambda_i f_i(x) - \left(\sum_{i=1}^m \lambda_i p_i\right)^T x \\ &+ \left(\sum_{i=1}^m \lambda_i q_i\right)^T g(x) + \left(\sum_{i=1}^m \lambda_i \alpha_i(\lambda)\right)\left(\sum_{i=1}^m \lambda_i p_i\right)^T x \\ &\leq \sum_{i=1}^m \lambda_i f_i(x). \end{aligned}$$

But, the inequality $\sum\limits_{i=1}^m \lambda_i h_i^\alpha(p,\tilde{q},\lambda,t) \leq \sum\limits_{i=1}^m \lambda_i f_i(x)$ contradicts relation (4. 2), so the assertion of the theorem holds. □

Theorem 4.5 (strong duality for $(D_\alpha), \alpha \in \mathcal{F}$) *Let be $\alpha \in \mathcal{F}$ and assume that (A_f), (A_g) and (A_{CQ}) are fulfilled. If $\bar{x}$ is a properly efficient solution to (P), then there exists an efficient solution $(\bar{p},\bar{\tilde{q}},\bar{\lambda},\bar{t}) \in \mathcal{B}_\alpha$ to the dual (D_α) and the strong duality $f(\bar{x}) = h^\alpha(\bar{p},\bar{\tilde{q}},\bar{\lambda},\bar{t})$ holds.*

Proof. By the same arguments as in the proof of Theorem 4.3, there exists a vector $\bar{\lambda} = (\bar{\lambda}_1, \ldots, \bar{\lambda}_m)^T \in int(\mathbb{R}^m_+)$ such that $\bar{x}$ solves the scalar problem

$$(P^{\bar{\lambda}}) \quad \inf_{x \in \mathcal{A}} \sum_{i=1}^{m} \bar{\lambda}_i f_i(x),$$

and there exists $(\tilde{p}, \tilde{q})$, solution to $(D^{\bar{\lambda}}_{FL})$, such that the optimality conditions (i), (ii) and (iii) in Theorem 3.2 are satisfied.

Considering

$$\bar{p} := \tilde{p}, \bar{q}_i := \alpha_i(\bar{\lambda})\tilde{q} \in \mathbb{R}^k, i = 1, ..., m,$$

and

$$\bar{t}_i := \bar{p}_i^T \bar{x} + (\bar{q}^T g)^* \left(-\alpha_i(\bar{\lambda}) \sum_{j=1}^{m} \bar{\lambda}_j \bar{p}_j \right) \in \mathbb{R}, i = 1, ..., m,$$

it holds $\sum\limits_{i=1}^{m} \bar{\lambda}_i \bar{q}_i = \tilde{q} \underset{K^*}{\geqq} 0$ and $\sum\limits_{i=1}^{m} \bar{\lambda}_i \bar{t}_i = 0$ (cf. (3. 1)). This means that $(\bar{p}, \bar{\bar{q}}, \bar{\lambda}, \bar{t})$, for $\bar{\bar{q}} = (\bar{q}_1, ..., \bar{q}_m)$, is feasible to (D_α). Moreover, from Theorem 3.2 (i), it follows that $f_i(\bar{x}) = h_i^\alpha(\bar{p}, \bar{\bar{q}}, \bar{\lambda}, \bar{t})$, for $i = 1, ..., m$. The maximality of $(\bar{p}, \bar{\bar{q}}, \bar{\lambda}, \bar{t})$ is given by Theorem 4.4. □

Remark 4.1

(a) The set $\mathcal{B}_\alpha$ does not depend on the function $\alpha \in \mathcal{F}$.

(b) For $\alpha : int(\mathbb{R}^m_+) \to \mathbb{R}^m_+$, $\alpha(\lambda) = \left(\frac{1}{m\lambda_1}, ..., \frac{1}{m\lambda_m} \right)^T$, $\lambda \in int(\mathbb{R}^m_+)$, it holds $\sum\limits_{i=1}^{m} \lambda_i$ $\alpha_i(\lambda) = 1$, which implies that $\alpha \in \mathcal{F}$. The dual problem (D_α) obtained for this choice of the function α is actually the multiobjective dual problem introduced by Wanka and Boţ in [85] and described in section 3.1.

4.3 The multiobjective dual problems (D_{FL}), (D_F), (D_L) and (D_P)

In this section we continue to introduce some other vector dual problems to the primal (P). Therefore, we use the expressions which appear in the formulation of the Lagrange, Fenchel and Fenchel-Lagrange dual problems presented in section 4.1. For all the multiobjective duals we prove the existence of weak and strong duality between them and the primal problem. Let us begin with the following dual problem

$$(D_{FL}) \quad \underset{(p,q,\lambda,y) \in \mathcal{B}_{FL}}{\text{v-max}} h^{FL}(p, q, \lambda, y),$$

with

$$h^{FL}(p, q, \lambda, y) = \begin{pmatrix} h_1^{FL}(p, q, \lambda, y) \\ \vdots \\ h_m^{FL}(p, q, \lambda, y) \end{pmatrix} = \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix},$$

$$h_j^{FL}(p, q, \lambda, y) = y_j, j = 1, ..., m,$$

the dual variables

$$p = (p_1, \ldots, p_m) \in \mathbb{R}^n \times ... \times \mathbb{R}^n, q \in \mathbb{R}^k,$$

$$\lambda = (\lambda_1, \ldots, \lambda_m)^T \in \mathbb{R}^m, y = (y_1, \ldots, y_m)^T \in \mathbb{R}^m,$$

and the set of constraints

$$\mathcal{B}_{FL} = \left\{ (p,q,\lambda,y) : \quad \lambda = (\lambda_1,\dots,\lambda_m)^T \in int(\mathbb{R}^m_+),\ q \underset{K^*}{\geqq} 0, \right.$$
$$\left. \sum_{i=1}^m \lambda_i y_i \leq -\sum_{i=1}^m \lambda_i f_i^*(p_i) - (q^T g)^* \left(-\sum_{i=1}^m \lambda_i p_i \right) \right\}.$$

Theorem 4.6 (weak duality for (D_{FL})**)** *There is no $x \in \mathcal{A}$ and no $(p,q,\lambda,y) \in \mathcal{B}_{FL}$ fulfilling $h^{FL}(p,q,\lambda,y) \underset{\mathbb{R}^m_+}{\geqq} f(x)$ and $h^{FL}(p,q,\lambda,y) \neq f(x)$.*

Proof. We assume that there exist $x \in \mathcal{A}$ and $(p,q,\lambda,y) \in \mathcal{B}_{FL}$ such that $f_i(x) \leq y_i, \forall i \in \{1,\dots,m\}$ and $f_j(x) < y_j$, for at least one $j \in \{1,\dots,m\}$. This means that

$$\sum_{i=1}^m \lambda_i f_i(x) < \sum_{i=1}^m \lambda_i y_i. \tag{4. 3}$$

On the other hand, the inequality of Young for $f_i, i = 1,...,m$, gives us

$$\begin{aligned}
\sum_{i=1}^m \lambda_i y_i &\leq -\sum_{i=1}^m \lambda_i f_i^*(p_i) - (q^T g)^* \left(-\sum_{i=1}^m \lambda_i p_i \right) \\
&\leq \sum_{i=1}^m \lambda_i f_i(x) - \left(\sum_{i=1}^m \lambda_i p_i \right)^T x + q^T g(x) + \left(\sum_{i=1}^m \lambda_i p_i \right)^T x \\
&\leq \sum_{i=1}^m \lambda_i f_i(x).
\end{aligned}$$

But, the inequality $\sum_{i=1}^m \lambda_i y_i \leq \sum_{i=1}^m \lambda_i f_i(x)$ contradicts relation (4. 3). □

Theorem 4.7 (strong duality for (D_{FL})**)** *Assume that (A_f), (A_g) and (A_{CQ}) are fulfilled. If $\bar{x}$ is a properly efficient solution to (P), then there exists an efficient solution $(\bar{p},\bar{q},\bar{\lambda},\bar{y}) \in \mathcal{B}_{FL}$ to the dual (D_{FL}) and the strong duality $f(\bar{x}) = h^{FL}(\bar{p},\bar{q},\bar{\lambda},\bar{y}) = \bar{y}$ holds.*

Proof. If $\bar{x}$ is properly efficient to the problem (P), then there exists a vector $\bar{\lambda} = (\bar{\lambda}_1,\dots,\bar{\lambda}_m)^T \in int(\mathbb{R}^m_+)$ such that $\bar{x}$ solves the scalar problem

$$(P^{\bar{\lambda}}) \quad \inf_{x \in \mathcal{A}} \sum_{i=1}^m \bar{\lambda}_i f_i(x).$$

By the strong duality Theorem 4.1, it results the existence of a solution $(\bar{p},\bar{q})$ to $(D^{\bar{\lambda}}_{FL})$ such that

$$\sum_{i=1}^m \bar{\lambda}_i f_i(\bar{x}) = inf(P^{\bar{\lambda}}) = max(D^{\bar{\lambda}}_{FL}) = -\sum_{i=1}^m \bar{\lambda}_i f_i^*(\bar{p}_i) - (\bar{q}^T g)^* \left(-\sum_{i=1}^m \bar{\lambda}_i \bar{p}_i \right).$$

Taking $\bar{y}_i := f_i(\bar{x})$, for $i = 1,...,m$, we have that $(\bar{p},\bar{q},\bar{\lambda},\bar{y}) \in \mathcal{B}_{FL}$ and $f(\bar{x}) = h^{FL}(\bar{p},\bar{q},\bar{\lambda},\bar{y}) = \bar{y}$. The maximality of $(\bar{p},\bar{q},\bar{\lambda},\bar{y})$ comes from Theorem 4.6. □

Following the same scheme, namely, using the form of the objective functions of the scalar duals (D^λ_F) and (D^λ_L), we can formulate two other dual multiobjective duals to (P),

$$(D_F) \quad \underset{(p,\lambda,y)\in\mathcal{B}_F}{\text{v-max}} h^F(p,\lambda,y),$$

with

$$h^F(p, \lambda, y) = \begin{pmatrix} h_1^F(p, \lambda, y) \\ \vdots \\ h_m^F(p, \lambda, y) \end{pmatrix} = \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix},$$

$$h_j^F(p, \lambda, y) = y_j, j = 1, ..., m,$$

the dual variables

$$p = (p_1, \dots, p_m) \in \mathbb{R}^n \times ... \times \mathbb{R}^n, \lambda = (\lambda_1, \dots, \lambda_m)^T \in \mathbb{R}^m, y = (y_1, \dots, y_m)^T \in \mathbb{R}^m,$$

and the set of constraints

$$\mathcal{B}_F = \Bigg\{ (p, \lambda, y) : \quad \lambda = (\lambda_1, \dots, \lambda_m)^T \in int(\mathbb{R}^m_+), \ p = (p_1, ..., p_m), \\ \sum_{i=1}^m \lambda_i y_i \leq - \sum_{i=1}^m \lambda_i f_i^*(p_i) - \chi^*_{\mathcal{A}} \left(- \sum_{i=1}^m \lambda_i p_i \right) \Bigg\}.$$

and

$$(D_L) \quad \underset{(q,\lambda,y)\in\mathcal{B}_L}{\text{v-max}} \ h^L(q, \lambda, y),$$

with

$$h^L(q, \lambda, y) = \begin{pmatrix} h_1^L(q, \lambda, y) \\ \vdots \\ h_m^L(q, \lambda, y) \end{pmatrix} = \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix},$$

$$h_j^L(q, \lambda, y) = y_j, j = 1, ..., m,$$

the dual variables

$$q \in \mathbb{R}^k, \lambda = (\lambda_1, \dots, \lambda_m)^T \in \mathbb{R}^m, y = (y_1, \dots, y_m)^T \in \mathbb{R}^m,$$

and the set of constraints

$$\mathcal{B}_L = \Bigg\{ (q, \lambda, y) : \quad \lambda = (\lambda_1, \dots, \lambda_m)^T \in int(\mathbb{R}^m_+), \ q \underset{K^*}{\geqq} 0, \\ \sum_{i=1}^m \lambda_i y_i \leq \inf_{x \in \mathbb{R}^n} \left[\sum_{i=1}^m \lambda_i f_i(x) + q^T g(x) \right] \Bigg\}.$$

Next, we show that also for these two dual problems weak and strong duality hold.

Theorem 4.8 (weak duality for (D_F)) *There is no $x \in \mathcal{A}$ and no $(p, \lambda, y) \in \mathcal{B}_F$ fulfilling $h^F(p, \lambda, y) \underset{\mathbb{R}^m_+}{\geqq} f(x)$ and $h^F(p, \lambda, y) \neq f(x)$.*

Proof. We assume that there exist $x \in \mathcal{A}$ and $(p, \lambda, y) \in \mathcal{B}_F$ such that $f_i(x) \leq y_i, \forall i \in \{1, \dots, m\}$ and $f_j(x) < y_j$, for at least one $j \in \{1, \dots, m\}$. Summing them, one gets

$$\sum_{i=1}^m \lambda_i f_i(x) < \sum_{i=1}^m \lambda_i y_i. \qquad (4.\ 4)$$

But

$$\begin{aligned}\sum_{i=1}^{m}\lambda_i y_i &\leq -\sum_{i=1}^{m}\lambda_i f_i^*(p_i)-\chi_{\mathcal{A}}^*\left(-\sum_{i=1}^{m}\lambda_i p_i\right)\\ &\leq \sum_{i=1}^{m}\lambda_i f_i(x)-\left(\sum_{i=1}^{m}\lambda_i p_i\right)^T x+\chi_{\mathcal{A}}(x)+\left(\sum_{i=1}^{m}\lambda_i p_i\right)^T x\\ &= \sum_{i=1}^{m}\lambda_i f_i(x),\end{aligned}$$

and the inequality $\sum\limits_{i=1}^{m}\lambda_i y_i \leq \sum\limits_{i=1}^{m}\lambda_i f_i(x)$ contradicts relation (4. 4). □

Theorem 4.9 (weak duality for (D_L)) *There is no $x\in\mathcal{A}$ and no $(q,\lambda,y)\in\mathcal{B}_L$ fulfilling $h^L(q,\lambda,y) \underset{\mathbb{R}^m_+}{\geqq} f(x)$ and $h^L(q,\lambda,y)\neq f(x)$.*

Proof. We assume that there exist $x\in\mathcal{A}$ and $(q,\lambda,y)\in\mathcal{B}_L$ such that $f_i(x)\leq y_i, \forall i\in\{1,\ldots,m\}$ and $f_j(x)<y_j$, for at least one $j\in\{1,\ldots,m\}$. This gives us that

$$\sum_{i=1}^{m}\lambda_i f_i(x)<\sum_{i=1}^{m}\lambda_i y_i. \tag{4. 5}$$

Again,

$$\begin{aligned}\sum_{i=1}^{m}\lambda_i y_i &\leq \inf_{x\in\mathbb{R}^n}\left[\sum_{i=1}^{m}\lambda_i f_i(x)+q^T g(x)\right]\\ &\leq \sum_{i=1}^{m}\lambda_i f_i(x)+q^T g(x)\\ &\leq \sum_{i=1}^{m}\lambda_i f_i(x),\end{aligned}$$

which contradicts the inequality in (4. 5). □

Theorem 4.10 (strong duality for (D_F)) *Assume that (A_f), (A_g) and (A_{CQ}) are fulfilled. If $\bar{x}$ is a properly efficient solution to (P), then there exists an efficient solution $(\bar{p},\bar{\lambda},\bar{y})\in\mathcal{B}_F$ to the dual (D_F) and the strong duality $f(\bar{x})=h^F(\bar{p},\bar{\lambda},\bar{y})=\bar{y}$ holds.*

Proof. Again, if $\bar{x}$ is properly efficient to problem (P), then there exists a vector $\bar{\lambda}=(\bar{\lambda}_1,\ldots,\bar{\lambda}_m)^T\in int(\mathbb{R}^m_+)$ such that $\bar{x}$ solves the scalar problem

$$(P^{\bar{\lambda}})\quad \inf_{x\in\mathcal{A}}\sum_{i=1}^{m}\bar{\lambda}_i f_i(x),$$

and, moreover, by the strong duality Theorem 4.1, its dual $(D_F^{\bar{\lambda}})$ has a solution $\bar{p}=(\bar{p}_1,...,\bar{p}_m)$. This means that

$$\sum_{i=1}^{m}\bar{\lambda}_i f_i(\bar{x})=inf(P^{\bar{\lambda}})=max(D_F^{\bar{\lambda}})=-\sum_{i=1}^{m}\bar{\lambda}_i f_i^*(\bar{p}_i)-\chi_{\mathcal{A}}^*\left(-\sum_{i=1}^{m}\bar{\lambda}_i\bar{p}_i\right),$$

and, taking $\bar{y}_i:=f_i(\bar{x})$, for $i=1,...,m$, we have that $(\bar{p},\bar{\lambda},\bar{y})\in\mathcal{B}_F$ and $f(\bar{x})=h^F(\bar{p},\bar{\lambda},\bar{y})=\bar{y}$. By Theorem 4.8 it follows that the element $(\bar{p},\bar{\lambda},\bar{y})$ is maximal to (D_F). □

Theorem 4.11 (strong duality for (D_L)) *Assume that (A_f), (A_g) and (A_{CQ}) are fulfilled. If $\bar{x}$ is a properly efficient solution to (P), then there exists an efficient solution $(\bar{q},\bar{\lambda},\bar{y}) \in \mathcal{B}_L$ to the dual (D_L) and the strong duality $f(\bar{x}) = h^L(\bar{q},\bar{\lambda},\bar{y}) = \bar{y}$ holds.*

Proof. As in the proof of Theorem 4.10 there exists a vector $\bar{\lambda} = (\bar{\lambda}_1,\ldots,\bar{\lambda}_m)^T \in int(\mathbb{R}^m_+)$ and $\bar{q} \underset{K^*}{\geqq} 0$, solution to the dual $(D_L^{\bar{\lambda}})$, such that

$$\sum_{i=1}^{m} \bar{\lambda}_i f_i(\bar{x}) = inf(P^{\bar{\lambda}}) = max(D_L^{\bar{\lambda}}) = \inf_{x\in\mathbb{R}^n}\left[\sum_{i=1}^{m}\bar{\lambda}_i f_i(x) + \bar{q}^T g(x)\right].$$

By taking $\bar{y}_i := f_i(\bar{x})$, for $i = 1,...,m$, we have that $(\bar{q},\bar{\lambda},\bar{y}) \in \mathcal{B}_L$ and $f(\bar{x}) = h^L(\bar{q},\bar{\lambda},\bar{y}) = \bar{y}$. By Theorem 4.10 it follows that the element $(\bar{q},\bar{\lambda},\bar{y})$ is maximal to (D_L). □

Remark 4.2 The dual problem (D_L) represents the transcription in finite dimensional spaces of the multiobjective dual introduced by Jahn in [40] and [41].

For the last vector problem which we present here we use exclusively the scalarized problem (P^λ). So, let this dual be

$$(D_P) \quad \underset{(\lambda,y)\in\mathcal{B}_P}{\text{v-max}}\; h^P(\lambda,y),$$

with

$$h^P(\lambda,y) = \begin{pmatrix} h_1^P(\lambda,y) \\ \vdots \\ h_m^P(\lambda,y)\end{pmatrix} = \begin{pmatrix} y_1 \\ \vdots \\ y_m\end{pmatrix},$$

$$h_j^P(\lambda,y) = y_j, j = 1,...,m,$$

the dual variables

$$\lambda = (\lambda_1,\ldots,\lambda_m)^T \in \mathbb{R}^m, y = (y_1,\ldots,y_m)^T \in \mathbb{R}^m,$$

and the set of constraints

$$\mathcal{B}_P = \left\{(\lambda,y) : \lambda = (\lambda_1,\ldots,\lambda_m)^T \in int(\mathbb{R}^m_+), \sum_{i=1}^{m}\lambda_i y_i \leq \inf_{x\in\mathcal{A}}\sum_{i=1}^{m}\lambda_i f_i(x)\right\}.$$

Between the primal problem (P) and the dual (D_P) weak and strong duality hold. We omit the proofs of the following two theorems because of their simplicity.

Theorem 4.12 (weak duality for (D_P)) *There is no $x \in \mathcal{A}$ and no $(\lambda,y) \in \mathcal{B}_P$ fulfilling $h^P(\lambda,y) \underset{\mathbb{R}^m_+}{\geqq} f(x)$ and $h^F(\lambda,y) \neq f(x)$.*

Theorem 4.13 (strong duality for (D_P)) *Assume that (A_f) and (A_g) are fulfilled. If $\bar{x}$ is a properly efficient solution to (P), then there exists an efficient solution $(\bar{\lambda},\bar{y}) \in \mathcal{B}_P$ to the dual (D_P) and the strong duality $f(\bar{x}) = h^P(\bar{\lambda},\bar{y}) = \bar{y}$ holds.*

Remark 4.3 Let us notice that, in order to have strong duality between (P) and (D_P), we do not need the assumption (A_{CQ}) to be fulfilled. Only (A_f) and (A_g) are here necessary, assuring the convexity of the problem (P). This permits us to characterize the properly efficient solutions of (P) via scalarization (cf. Definition 3.3).

4.4 The relations between the duals (D_1), (D_α), $\alpha \in \mathcal{F}$, and (D_{FL})

In this section we examine the relations between the dual problems (D_1), (D_α), $\alpha \in \mathcal{F}$, and (D_{FL}).

For the beginning, let us notice that to find the Pareto-efficient solutions of a multiobjective dual problem means actually to determine the maximal elements of the image set of its objective function on the set of constraints. This is the reason why, in order to compare the duals (D_1), (D_α), $\alpha \in \mathcal{F}$, and (D_{FL}), we analyse the relations between the corresponding image sets. Therefore, let be $D_1 := h^1(\mathcal{B}_1)$, $D_\alpha := h^\alpha(\mathcal{B}_\alpha), \alpha \in \mathcal{F}$, and $D_{FL} := h^{FL}(\mathcal{B}_{FL})$. It is obvious that $D_1 \subseteq \underbrace{\overline{\mathbb{R}} \times ... \times \overline{\mathbb{R}}}_{m}$, $D_\alpha \subseteq \underbrace{\overline{\mathbb{R}} \times ... \times \overline{\mathbb{R}}}_{m}, \alpha \in \mathcal{F}$, and $D_{FL} \subseteq \mathbb{R}^m$.

Proposition 4.1 *For each $\alpha \in \mathcal{F}$ it holds $D_1 \subseteq D_\alpha$.*

Proof. Let be $\alpha \in \mathcal{F}$ fixed and $d = (d_1, ..., d_m)^T$ an element in D_1. Then there exists $(p, q, \lambda, t) \in \mathcal{B}_1$ such that

$$d_j = -f_j^*(p_j) - (q^T g)^* \left(-\frac{1}{\sum\limits_{i=1}^m \lambda_i} \sum_{i=1}^m \lambda_i p_i \right) + t_j, j = 1, ..., m.$$

Let us define now $\bar{p}_j := p_j$, $\bar{\lambda}_j := \lambda_j$, $\bar{q}_j := \alpha_j(\bar{\lambda}) \left(\sum\limits_{i=1}^m \bar{\lambda}_i \right) q$, for $j = 1, ..., m$, and $\bar{\bar{q}} := (\bar{q}_1, ..., \bar{q}_m)$. It holds

$$\sum_{i=1}^m \bar{\lambda}_i \bar{q}_i = \left(\sum_{j=1}^m \bar{\lambda}_j \alpha_j(\bar{\lambda}) \right) \left(\sum_{i=1}^m \bar{\lambda}_i \right) q = \left(\sum_{i=1}^m \bar{\lambda}_i \right) q \underset{K^*}{\geqq} 0,$$

and, for $j = 1, ..., m$,

$$(\bar{q}_j^T g)^* \left(-\alpha_j(\bar{\lambda}) \sum_{i=1}^m \bar{\lambda}_i \bar{p}_i \right) = \left(\alpha_j(\bar{\lambda}) \sum_{i=1}^m \bar{\lambda}_i \right) (q^T g)^* \left(-\frac{1}{\sum\limits_{i=1}^m \bar{\lambda}_i} \sum_{i=1}^m \bar{\lambda}_i \bar{p}_i \right). \quad (4.\ 6)$$

First let us consider the case when $(q^T g)^* \left(-\frac{1}{\sum\limits_{i=1}^m \bar{\lambda}_i} \sum\limits_{i=1}^m \bar{\lambda}_i \bar{p}_i \right) = +\infty$. This means that $d_j = -\infty, \forall j = 1, ..., m$. By taking $\bar{t}_j := t_j$, for $j = 1, ..., m$, we have that $(\bar{p}, \bar{\bar{q}}, \bar{\lambda}, \bar{t}) \in \mathcal{B}_\alpha$ and $d = (-\infty, ..., -\infty)^T = h^\alpha(\bar{p}, \bar{\bar{q}}, \bar{\lambda}, \bar{t}) \in h^\alpha(\mathcal{B}_\alpha) = D_\alpha$.

In the other case, when $(q^T g)^* \left(-\frac{1}{\sum\limits_{i=1}^m \bar{\lambda}_i} \sum\limits_{i=1}^m \bar{\lambda}_i \bar{p}_i \right) \in \mathbb{R}$, let us take, for $j = 1, ..., m$,

$$\bar{t}_j := t_j + (\bar{q}_j^T g)^* \left(-\alpha_j(\bar{\lambda}) \sum_{i=1}^m \bar{\lambda}_i \bar{p}_i \right) - (q^T g)^* \left(-\frac{1}{\sum\limits_{i=1}^m \bar{\lambda}_i} \sum_{i=1}^m \bar{\lambda}_i \bar{p}_i \right) \in \mathbb{R}.$$

From (4. 6) we have $\sum\limits_{i=1}^{m} \bar{\lambda}_i \bar{t}_i = \sum\limits_{i=1}^{m} \lambda_i t_i = 0$, which implies that $(\bar{p}, \bar{\tilde{q}}, \bar{\lambda}, \bar{t}) \in \mathcal{B}_\alpha$. Moreover, for $j = 1, ..., m$,

$$d_j = -f_j^*(\bar{p}_j) - (\bar{q}_j^T g)^* \left(-\alpha_j(\bar{\lambda}) \sum_{i=1}^{m} \bar{\lambda}_i \bar{p}_i \right) + \bar{t}_j.$$

Therefore, $d = h^\alpha(\bar{p}, \bar{\tilde{q}}, \bar{\lambda}, \bar{t}) \in h^\alpha(\mathcal{B}_\alpha) = D_\alpha$ meaning $D_1 \subseteq D_\alpha$. $\square$

Example 4.1 Let be $\alpha \in \mathcal{F}$ fixed, $m = 2, n = 1, k = 1$ and $K = \mathbb{R}_+$. Considering $f_1, f_2 : \mathbb{R} \to \mathbb{R}$, $f_1(x) = f_2(x) = x^2$, $g : \mathbb{R} \to \mathbb{R}$, $g(x) = x^2 - 1$, $\lambda = (2, 1)^T$, $\tilde{q} = (q_1, q_2) = (1, -1)$, $t = (1, -2)^T$, we have that $\lambda_1 q_1 + \lambda_2 q_2 = 1$ and $\lambda_1 t_1 + \lambda_2 t_2 = 0$. For $p = (0, 0)$ it holds $f_1^*(p_1) = f_2^*(p_2) = 0$ and

$$d = (-(q_1 g)^*(0) + t_1, -(q_2 g)^*(0) + t_2)^T = (0, -\infty)^T = h^\alpha(p, \tilde{q}, \lambda, t) \in D_\alpha.$$

But let us notice that $d \notin D_1$. It means that the inclusion $D_1 \subseteq D_\alpha$ may be strict. We denote this by $D_1 \subsetneq D_\alpha$, $\alpha \in \mathcal{F}$.

Example 4.2 Let be again $\alpha \in \mathcal{F}$ fixed, but $m = 2, n = 1, k = 2$ and $K = \mathbb{R}^2$. Considering now $f_1, f_2 : \mathbb{R} \to \mathbb{R}$, $f_1(x) = f_2(x) = 0$, $g_1, g_2 : \mathbb{R} \to \mathbb{R}$,

$$g_1(x) = \begin{cases} 1, & \text{if } x < 0, \\ e^{-x}, & \text{if } x \geq 0, \end{cases}, \qquad g_2(x) = \begin{cases} e^x, & \text{if } x < 0, \\ 1, & \text{if } x \geq 0, \end{cases}$$

$p = (0, 0)$, $q_1 = (1, -1), q_2 = (-1, 1)$, $\tilde{q} = (q_1, q_2)$, $\lambda = (1, 1)^T$ and $t = \left(\frac{1}{2}, -\frac{1}{2}\right)^T$, we have $\lambda_1 q_1 + \lambda_2 q_2 = (0, 0)^T \in K^*$, $\lambda_1 t_1 + \lambda_2 t_2 = 0$ and $f_1^*(0) = f_2^*(0) = 0$.

This means that

$$d = \left(-\frac{1}{2}, -\frac{3}{2}\right)^T = \left(-(q_1^T g)^*(0) + t_1, -(q_2^T g)^*(0) + t_2\right)^T = h^\alpha(p, \tilde{q}, \lambda, t) \in D_\alpha \cap \mathbb{R}^2.$$

Let us show now that $d \notin D_1$. If this were not true, then there would exist a tuple $(\bar{p}, \bar{q}, \bar{\lambda}, \bar{t}) \in \mathcal{B}_1$ such that

$$\begin{pmatrix} -\frac{1}{2} \\ \\ -\frac{3}{2} \end{pmatrix} = \begin{pmatrix} -f_1^*(\bar{p}_1) - (\bar{q}^T g)^* \left(-\frac{\bar{\lambda}_1 \bar{p}_1 + \bar{\lambda}_2 \bar{p}_2}{\bar{\lambda}1 + \bar{\lambda}2} \right) + \bar{t}_1 \\ \\ -f_2^*(\bar{p}_2) - (\bar{q}^T g)^* \left(-\frac{\bar{\lambda}_1 \bar{p}_1 + \bar{\lambda}_2 \bar{p}_2}{\bar{\lambda}1 + \bar{\lambda}2} \right) + \bar{t}_2 \end{pmatrix}.$$

It follows that $f_1^*(\bar{p}_1), f_2^*(\bar{p}_2) \in \mathbb{R}$, but, in order to happen this, we must have $\bar{p}_1 = \bar{p}_2 = 0$, $f_1^*(\bar{p}_1) = f_2^*(\bar{p}_2) = 0$ and, because $\bar{q} \in (\mathbb{R}^2)^* = \{0\}$, $(\bar{q}^T g)^*(0) = 0$. So,

$$\begin{pmatrix} -\frac{1}{2} \\ \\ -\frac{3}{2} \end{pmatrix} = \begin{pmatrix} \bar{t}_1 \\ \\ \bar{t}_2 \end{pmatrix},$$

and, from here,

$$\bar{\lambda}_1 \bar{t}_1 + \bar{\lambda}_2 \bar{t}_2 = -\frac{\bar{\lambda}_1 + 3\bar{\lambda}_2}{2} < 0. \tag{4. 7}$$

Obviously, relation (4. 7) contradicts $\bar{\lambda}_1 \bar{t}_1 + \bar{\lambda}_2 \bar{t}_2 = 0$ and this means that $d = (-\frac{1}{2}, -\frac{3}{2})^T \notin D_1 \cap \mathbb{R}^2$. In conclusion, $D_1 \cap \mathbb{R}^m \subsetneq D_\alpha \cap \mathbb{R}^m$, i.e. the inclusion may be strict.

Proposition 4.2 *For each $\alpha \in \mathcal{F}$, it holds $D_\alpha \cap \mathbb{R}^m \subseteq D_{FL}$.*

Proof. Let be $\alpha \in \mathcal{F}$ fixed and $d = (d_1, ..., d_m)^T$ an element in $D_\alpha \cap \mathbb{R}^m$. Then there exists $(p, \tilde{q}, \lambda, t) \in \mathcal{B}_\alpha$ such that $d = h^\alpha(p, \tilde{q}, \lambda, t)$. From here, by using the inequality of Young for $q_j^T g, j = 1, ..., m$, we have

$$\begin{aligned}
\sum_{j=1}^m \lambda_j d_j &= \sum_{i=j}^m \lambda_j h_j^\alpha(p, \tilde{q}, \lambda, t) = -\sum_{i=j}^m \lambda_j f_j^*(p_j) \\
&- \sum_{j=1}^m \lambda_j (q_j^T g)^* \left(-\alpha_j(\lambda) \sum_{i=1}^m \lambda_i p_i \right) + \sum_{j=1}^m \lambda_j t_j \\
&\leq -\sum_{j=1}^m \lambda_j f_j^*(p_j) + \left(\sum_{j=1}^m \lambda_j q_j \right)^T g(x) \\
&+ \left(\sum_{j=1}^m \lambda_j \alpha_j(\lambda) \right) \left(\sum_{i=1}^m \lambda_i p_i \right)^T x = -\sum_{j=1}^m \lambda_j f_j^*(p_j) \\
&+ \left(\sum_{j=1}^m \lambda_j p_j \right)^T x + \left(\sum_{j=1}^m \lambda_j q_j \right)^T g(x), \ \forall x \in \mathbb{R}^n.
\end{aligned}$$

Then it follows

$$\begin{aligned}
\sum_{j=1}^m \lambda_j d_j &\leq -\sum_{j=1}^m \lambda_j f_j^*(p_j) + \inf_{x \in \mathbb{R}^n} \left[\left(\sum_{i=1}^m \lambda_i p_i \right)^T x + \left(\sum_{j=1}^m \lambda_j q_j \right)^T g(x) \right] \\
&= -\sum_{j=1}^m \lambda_j f_j^*(p_j) - \left(\left(\sum_{j=1}^m \lambda_j q_j \right)^T g \right)^* \left(-\sum_{i=1}^m \lambda_i p_i \right),
\end{aligned}$$

and this means that $(p, \sum\limits_{j=1}^m \lambda_j q_j, \lambda, d) \in \mathcal{B}_{FL}$. In conclusion,

$$d = h^{FL}(p, \sum_{j=1}^m \lambda_j q_j, \lambda, d) \in h^{FL}(\mathcal{B}_{FL}) = D_{FL}.$$

□

Example 4.3 Let be $\alpha \in \mathcal{F}$ fixed, $m = 2, n = 1, k = 1$ and $K = \mathbb{R}$. Considering $f_1, f_2 : \mathbb{R} \to \mathbb{R}$, $f_1(x) = f_2(x) = 0$, $g : \mathbb{R} \to \mathbb{R}$, $g(x) = x^2$, $p = (0, 0)$, $q = 0 \in K^* = \{0\}$, $\lambda = (1, 1)^T$ and $d = (-1, -1)^T$, we have

$$\lambda_1 d_1 + \lambda_2 d_2 = -2 < 0 = -\lambda_1 f_1^*(p_1) - \lambda_2 f_2^*(p_2) - (qg)^*(-\lambda_1 p_1 - \lambda_2 p_2),$$

which implies that $(p, q, \lambda, d) \in \mathcal{B}_{FL}$ and $d = (-1, -1)^T \in D_{FL}$.

Let us show now that $d \notin D_\alpha$. If this were not true, then there would exist $(\bar{p}, \bar{\tilde{q}}, \bar{\lambda}, \bar{t}) \in \mathcal{B}_\alpha$, with $\bar{\tilde{q}} = (\bar{q}_1, \bar{q}_2)$, such that

$$\begin{pmatrix} -1 \\ \\ -1 \end{pmatrix} = \begin{pmatrix} -f_1^*(\bar{p}_1) - (\bar{q}_1 g)^* \left(-\alpha_1(\bar{\lambda})(\bar{\lambda}_1 \bar{p}_1 + \bar{\lambda}_2 \bar{p}_2) \right) + \bar{t}_1 \\ \\ -f_2^*(\bar{p}_2) - (\bar{q}_2 g)^* \left(-\alpha_2(\bar{\lambda})(\bar{\lambda}_1 \bar{p}_1 + \bar{\lambda}_2 \bar{p}_2) \right) + \bar{t}_2 \end{pmatrix}.$$

Again, $f_1^*(\bar{p}_1), f_2^*(\bar{p}_2) \in \mathbb{R}$, but, in order to happen this, we must have $\bar{p}_1 = \bar{p}_2 = 0$ and $f_1^*(\bar{p}_1) = f_2^*(\bar{p}_2) = 0$. In this case, we obtain

$$\begin{pmatrix} -1 \\ \\ -1 \end{pmatrix} = \begin{pmatrix} -(\bar{q}_1 g)^* (0) + \bar{t}_1 \\ \\ -(\bar{q}_2 g)^* (0) + \bar{t}_2 \end{pmatrix}. \tag{4. 8}$$

From (4. 8) it follows $-(\bar{q}_1 g)^*(0) = \inf\limits_{x\in\mathbb{R}}(\bar{q}_1 x^2) \in \mathbb{R}$ and $-(\bar{q}_2 g)^*(0) = \inf\limits_{x\in\mathbb{R}}(\bar{q}_2 x^2) \in \mathbb{R}$, which hold just if $\bar{q}_1 \geq 0$ and $\bar{q}_2 \geq 0$. On the other hand, we have that $\bar{\lambda}_1\bar{q}_1 + \bar{\lambda}_2\bar{q}_2 \in K^* = \{0\}$, whence $\bar{q}_1 = \bar{q}_2 = 0$.

So, $\bar{t}_1 = \bar{t}_2 = -1$ and $\bar{\lambda}_1\bar{t}_1 + \bar{\lambda}_2\bar{t}_2 = -\bar{\lambda}_1 - \bar{\lambda}_2 < 0$, which is a contradiction to $(\bar{p}, \bar{\bar{q}}, \bar{\lambda}, \bar{t}) \in \mathcal{B}_\alpha$. Our assumption that $d \in D_\alpha$ proves to be false. In conclusion, $D_\alpha \cap \mathbb{R}^m \subsetneq D_{FL}$, i.e. the inclusion may be strict.

By the Propositions 4.1, 4.2 and the Examples 4.1-4.3, we have, for each $\alpha \in \mathcal{F}$,

$$D_1 \cap \mathbb{R}^m \subsetneq D_\alpha \cap \mathbb{R}^m \subsetneq D_{FL}. \tag{4. 9}$$

In the last part of the section, let us prove that, even the sets D_1, $D_\alpha, \alpha \in \mathcal{F}$, and D_{FL} may be different (cf. (4. 9)), they have the same maximal elements. In order to do this, we consider their corresponding sets of maximal elements $vmaxD_1$, $vmaxD_\alpha$, $\alpha \in \mathcal{F}$, and $vmaxD_{FL}$, respectively. All these sets are subsets of $\mathbb{R}^m$. By maximal elements we call here the efficient elements of a set, in the sense of maximum, with respect to the non-negative orthant.

We are now able to prove the main results of this section.

Theorem 4.14 *It holds $vmaxD_1 = vmaxD_{FL}$.*

Proof.

"$vmaxD_1 \subseteq vmaxD_{FL}$". Let be $d \in vmaxD_1$. It means that $d \in D_1 \cap \mathbb{R}^m$ and, from (4. 9), we have $d \in D_{FL}$. Then there exists an element $(p, q, \lambda, y) \in \mathcal{B}_{FL}$ such that $y = h^{FL}(p, q, \lambda, y) = d$.

Let us assume now that $d \notin vmaxD_{FL}$. By the definition of the maximal elements, it follows that there exists $(\bar{p}, \bar{q}, \bar{\lambda}, \bar{d}) \in \mathcal{B}_{FL}$ such that $d \in \bar{d} - (\mathbb{R}^m_+ \setminus \{0\})$ ($\bar{d} \gneqq d$), i.e. $d_i \leq \bar{d}_i$, $\forall i = 1, ..., m$, and $d_j < \bar{d}_j$, for at least one $j \in \{1, ..., m\}$. Thus,

$$\sum_{i=1}^m \bar{\lambda}_i d_i < \sum_{i=1}^m \bar{\lambda}_i \bar{d}_i \leq -\sum_{i=1}^m \bar{\lambda}_i f_i^*(\bar{p}_i) - (\bar{q}^T g)^* \left(-\sum_{i=1}^m \bar{\lambda}_i \bar{p}_i\right).$$

Let be now $\bar{\bar{d}} \in \mathbb{R}^m$ such that $\bar{\bar{d}} \in \bar{d} + \mathbb{R}^m_+$ (i.e. $\bar{\bar{d}} \geqq \bar{d}$) and

$$\sum_{i=1}^m \bar{\lambda}_i \bar{\bar{d}}_i = -\sum_{i=1}^m \bar{\lambda}_i f_i^*(\bar{p}_i) - (\bar{q}^T g)^* \left(-\sum_{i=1}^m \bar{\lambda}_i \bar{p}_i\right). \tag{4. 10}$$

Considering $\bar{\bar{q}} := \frac{1}{\sum\limits_{i=1}^m \bar{\lambda}_i} \bar{q} \underset{K^*}{\geqq} 0$ and, for $j = 1, ..., m$,

$$\begin{aligned} \bar{t}_j &:= f_j^*(\bar{p}_j) + (\bar{\bar{q}}^T g)^* \left(-\frac{1}{\sum\limits_{i=1}^m \bar{\lambda}_i} \sum_{i=1}^m \bar{\lambda}_i \bar{p}_i\right) + \bar{\bar{d}}_j \\ &= f_j^*(\bar{p}_j) + \frac{1}{\sum\limits_{i=1}^m \bar{\lambda}_i} (\bar{q}^T g)^* \left(-\sum_{i=1}^m \bar{\lambda}_i \bar{p}_i\right) + \bar{\bar{d}}_j \in \mathbb{R}, \end{aligned}$$

we obtain an element $(\bar{p}, \bar{\bar{q}}, \bar{\lambda}, \bar{t})$ with the properties $\bar{\bar{q}} \underset{K^*}{\geqq} 0$, $\bar{\lambda} \in int(\mathbb{R}^m_+)$ and, by (4. 10),

$$\sum_{i=1}^m \bar{\lambda}_i \bar{t}_i = \sum_{i=1}^m \bar{\lambda}_i \bar{\bar{d}}_i + \sum_{i=1}^m \bar{\lambda}_i f_i^*(\bar{p}_i) + (\bar{q}^T g)^* \left(-\sum_{i=1}^m \bar{\lambda}_i \bar{p}_i\right) = 0.$$

So, $(\bar{p}, \bar{\bar{q}}, \bar{\lambda}, \bar{t}) \in \mathcal{B}_1$, $\bar{\bar{d}} = h^1(\bar{p}, \bar{\bar{q}}, \bar{\lambda}, \bar{t}) \in h^1(\mathcal{B}_1) = D_1$ and

$$\bar{\bar{d}} \in \bar{d} + \mathbb{R}^m_+ \in d + \mathbb{R}^m_+ \setminus \{0\} + \mathbb{R}^m_+ = d + \mathbb{R}^m_+ \setminus \{0\} \ (\bar{\bar{d}} \gneqq d). \tag{4. 11}$$

This contradicts the maximality of d in D_1 and implies that d must be maximal in D_{FL}.

"$vmaxD_{FL} \subseteq vmaxD_1$". Let be now $d \in vmaxD_{FL}$. Then there exist $p_i \in \mathbb{R}^n, i = 1, ..., m$, $q \underset{K^*}{\geqq} 0$ and $\lambda \in int(\mathbb{R}^m_+)$ such that

$$\sum_{i=1}^m \lambda_i d_i \leq -\sum_{i=1}^m \lambda_i f_i^*(p_i) - (q^T g)^* \left(-\sum_{i=1}^m \lambda_i p_i \right).$$

Let be again $\bar{d} \in \mathbb{R}^m$ such that $\bar{d} \in d + \mathbb{R}^m_+$ $(\bar{d} \geqq d)$ and

$$\sum_{i=1}^m \bar{\lambda}_i \bar{d}_i = -\sum_{i=1}^m \lambda_i f_i^*(p_i) - (q^T g)^* \left(-\sum_{i=1}^m \lambda_i p_i \right). \tag{4. 12}$$

For $\bar{p} := p$, $\bar{\lambda} := \lambda$, $\bar{q} := \frac{1}{\sum_{i=1}^m \bar{\lambda}_i} q \underset{K^*}{\geqq} 0$ and

$$\bar{t}_j := f_j^*(\bar{p}_j) + (\bar{q}^T g)^* \left(-\frac{1}{\sum_{i=1}^m \bar{\lambda}_i} \sum_{i=1}^m \bar{\lambda}_i \bar{p}_i \right) + \bar{d}_j \in \mathbb{R}, j = 1, ..., m,$$

we have $\sum_{i=1}^m \bar{\lambda}_i \bar{t}_i = 0$ (cf. (4. 12)), whence $(\bar{p}, \bar{q}, \bar{\lambda}, \bar{t}) \in \mathcal{B}_1$.

Moreover, the value of the objective function on this element is $h^1(\bar{p}, \bar{q}, \bar{\lambda}, \bar{t}) = \bar{d} \in d + \mathbb{R}^m_+$ $(\bar{d} \geqq d)$. On the other hand, (4. 12) assures that $(\bar{p}, \bar{q}, \bar{\lambda}, \bar{d}) \in \mathcal{B}_{FL}$ and the maximality of d in D_{FL} implies that it is impossible to have $d \in \bar{d} - \left(\mathbb{R}^m_+ \setminus \{0\}\right)$ $(\bar{d} \gneqq d)$. Then we must have $h^1(\bar{p}, \bar{q}, \bar{\lambda}, \bar{t}) = \bar{d} = d$ and, so, $d \in h^1(\mathcal{B}_1) = D_1$.

It remains to show that actually $d \in vmaxD_1$. If this does not happen, then there exists an element $(\bar{\bar{p}}, \bar{\bar{q}}, \bar{\bar{\lambda}}, \bar{\bar{t}}) \in \mathcal{B}_1$ such that $\bar{\bar{d}} = h^1(\bar{\bar{p}}, \bar{\bar{q}}, \bar{\bar{\lambda}}, \bar{\bar{t}}) \in d + \left(\mathbb{R}^m_+ \setminus \{0\}\right)$ $(\bar{\bar{d}} \gneqq d)$. But, relation (4. 9) states that $D_1 \cap \mathbb{R}^m \subseteq D_{FL}$ and, from here, $\bar{\bar{d}} \in D_{FL}$. The fact that $\bar{\bar{d}} \in d + \left(\mathbb{R}^m_+ \setminus \{0\}\right)$ $(\bar{\bar{d}} \gneqq d)$ contradicts the maximality of d in D_{FL}. We conclude that $d \in vmaxD_1$. □

Theorem 4.15 *For each $\alpha \in \mathcal{F}$ it holds $vmaxD_\alpha = vmaxD_{FL}$.*

Proof. Let be $\alpha \in \mathcal{F}$ fixed.

"$vmaxD_\alpha \subseteq vmaxD_{FL}$". Let be $d \in vmaxD_\alpha$. Then it holds $d \in D_\alpha \cap \mathbb{R}^m$ and from (4. 9) we have that $d \in D_{FL}$. Let us assume that $d \notin vmaxD_{FL}$. As in the proof of Theorem 4.14, there exists $\bar{\bar{d}} \in D_1$ such that (cf. (4. 11))

$$\bar{\bar{d}} \in d + \mathbb{R}^m_+ \setminus \{0\} \ (\bar{\bar{d}} \gneqq d).$$

But, by (4. 9) we have $\bar{\bar{d}} \in D_\alpha$ and this contradicts the maximality of d in D_α.

"$vmaxD_{FL} \subseteq vmaxD_\alpha$." Let be now $d \in vmaxD_{FL}$. By Theorem 4.14, it follows $d \in vmaxD_1$ and, from here, $d \in D_1$. Using the inclusion in (4. 9) we have $d \in D_\alpha$.

Assuming that $d \notin vmaxD_\alpha$, there must exist an $\bar{\bar{d}} \in D_\alpha$ such that $\bar{\bar{d}} \in d + \mathbb{R}^m_+ \setminus \{0\}$ $(\bar{\bar{d}} \gneqq d)$. On the other hand, because $D_\alpha \cap \mathbb{R}^m \subseteq D_{FL}$, $\bar{\bar{d}} \in D_{FL}$ and this is a contradiction to $d \in vmaxD_{FL}$. So, d must belong to $vmaxD_{\alpha}$. □

From the last two theorems we obtain that, for each $\alpha \in \mathcal{F}$,

$$vmaxD_1 = vmaxD_{\alpha} = vmaxD_{FL}. \tag{4. 13}$$

Remark 4.4 Let us notice that the inclusions in (4. 9) can be strict, even if the assumptions (A_f), (A_g) and (A_{CQ}) are fulfilled (cf. Example 4.2 and Example 4.3). Otherwise, the equality (4. 13) holds without asking the fulfilment of any of these three assumptions. The equality in (4. 13) holds in the most general case.

4.5 The relations between the duals (D_{FL}), (D_F), (D_L) and (D_P)

In order to continue the analysis started in the previous section, let us denote the image sets of the problems (D_F), (D_L) and (D_P) by $D_F := h^F(\mathcal{B}_F)$, $D_L := h^L(\mathcal{B}_L)$ and $D_P := h^P(\mathcal{B}_P)$, respectively. Obviously, from the definition of the multiobjective duals in section 4.4 it follows that D_F, D_L, D_P are subsets of $\mathbb{R}^m$. We also want to notice that during this section we work in the general case.

Proposition 4.3 *It holds*

(a) $D_{FL} \subseteq D_F$.

(b) $D_{FL} \subseteq D_L$.

Proof.

(a) Let be $d = (d_1, ..., d_m)^T \in D_{FL}$. Then there exist $p_i \in \mathbb{R}^n, i = 1, ..., m$, $q \underset{K^*}{\geqq} 0$ and $\lambda \in int(\mathbb{R}^m_+)$ such that

$$\sum_{i=1}^m \lambda_i d_i \leq -\sum_{i=1}^m \lambda_i f_i^*(p_i) - (q^T g)^* \left(-\sum_{i=1}^m \lambda_i p_i \right).$$

By the definition of the conjugate function we have

$$\begin{aligned}
\sum_{i=1}^m \lambda_i d_i &\leq -\sum_{i=1}^m \lambda_i f_i^*(p_i) + \inf_{x \in \mathbb{R}^n} \left[\left(\sum_{i=1}^m \lambda_i p_i \right)^T x + q^T g(x) \right] \\
&\leq -\sum_{i=1}^m \lambda_i f_i^*(p_i) + \inf_{x \in \mathcal{A}} \left[\left(\sum_{i=1}^m \lambda_i p_i \right)^T x + q^T g(x) \right] \\
&\leq -\sum_{i=1}^m \lambda_i f_i^*(p_i) + \inf_{x \in \mathcal{A}} \left(\sum_{i=1}^m \lambda_i p_i \right)^T x \\
&= -\sum_{i=1}^m \lambda_i f_i^*(p_i) - \chi_{\mathcal{A}}^* \left(-\sum_{i=1}^m \lambda_i p_i \right).
\end{aligned}$$

This means that $(p, \lambda, d) \in \mathcal{B}_F$ and $d = h^F(p, \lambda, d) \in h^F(\mathcal{B}_F) = D_F$.

(b) Like in the case (a), let be $d = (d_1, ..., d_m)^T \in D_{FL}$. Again, there exist $p_i \in \mathbb{R}^n, i = 1, ..., m$, $q \underset{K^*}{\geqq} 0$ and $\lambda \in int(\mathbb{R}^m_+)$ such that

$$\sum_{i=1}^m \lambda_i d_i \leq -\sum_{i=1}^m \lambda_i f_i^*(p_i) - (q^T g)^* \left(-\sum_{i=1}^m \lambda_i p_i \right).$$

Applying the inequality of Young (cf. [19]), we get for $f_i, i = 1, ..., m$,

$$-f_i^*(p_i) \leq f_i(x) - p_i^T x, \ \forall x \in \mathbb{R}^n,$$

and for $q^T g$

$$-(q^T g)^* \left(-\sum_{i=1}^m \lambda_i p_i \right) \leq q^T g(x) + \left(\sum_{i=1}^m \lambda_i p_i \right)^T x, \ \forall x \in \mathbb{R}^n.$$

From here,

$$\sum_{i=1}^m \lambda_i d_i \leq \inf_{x \in \mathbb{R}^n} \left[\sum_{i=1}^m \lambda_i f_i(x) + q^T g(x) \right],$$

and this means that $(p, \lambda, d) \in \mathcal{B}_L$ and $d = h^L(p, \lambda, d) \in h^L(\mathcal{B}_L) = D_L$. □

Example 4.4 For $m = 2, n = 1, k = 1$, $K = \mathbb{R}_+$, let us consider the functions $f_1, f_2 : \mathbb{R} \to \overline{\mathbb{R}}$, $g : \mathbb{R} \to \mathbb{R}$, defined by

$$f_1(x) = \begin{cases} x, & \text{if} \quad x \in [0, +\infty), \\ +\infty, & \text{otherwise}, \end{cases}, \qquad f_2(x) = 0,$$

and

$$g(x) = \begin{cases} 1 - x^2, & \text{if} \quad x \in [0, +\infty), \\ 1, & \text{otherwise}. \end{cases}$$

For $p = (p_1, p_2) = (1, 0)$, $\lambda = (1, 1)^T$ and $d = (1, 0)^T$, it holds

$$\lambda_1 d_1 + \lambda_2 d_2 = 1 = -\lambda_1 f_1^*(p_1) - \lambda_2 f_2^*(p_2) + \inf_{g(x) \leq 0} (\lambda_1 p_1 + \lambda_2 p_2) x,$$

and, so, we have that $d = (1, 0)^T \in D_F$.

Let us show now that $d \notin D_{FL}$. If this were not true, then there would exist $\bar{p} = (\bar{p}_1, \bar{p}_2)$, $\bar{q} \geq 0$ and $\bar{\lambda} = (\bar{\lambda}_1, \bar{\lambda}_2)^T \in int(\mathbb{R}^2_+)$ such that

$$\bar{\lambda}_1 \leq -\bar{\lambda}_1 f_1^*(\bar{p}_1) - \bar{\lambda}_2 f_2^*(\bar{p}_2) - (\bar{q} g)^* \left(-\bar{\lambda}_1 \bar{p}_1 - \bar{\lambda}_2 \bar{p}_2 \right). \tag{4. 14}$$

In order to happen this, we must have $\bar{p}_2 = 0$ and $f_2^*(\bar{p}_2) = 0$. Then, from (4. 14),

$$1 \leq -f_1^*(\bar{p}_1) - \left(\frac{\bar{q}}{\bar{\lambda}_1} g \right)^* (-\bar{p}_1). \tag{4. 15}$$

In the case $\bar{q} > 0$, we have that $\left(\frac{\bar{q}}{\bar{\lambda}_1} g \right)^* (-\bar{p}_1) = \sup\limits_{x \in \mathbb{R}} \left[-\bar{p}_1 x - \frac{\bar{q}}{\bar{\lambda}_1} g(x) \right] = +\infty$, which means that $\bar{q}$ must be 0. Then the inequality (4. 15) becomes

$$1 \leq -f_1^*(\bar{p}_1) + \inf_{x \in \mathbb{R}} [\bar{p}_1 x],$$

and, so, it is obvious that $\bar{p}_1$ must be also 0. It remains that

$$1 \leq -f_1^*(0) = \inf_{x \in \mathbb{R}} [f_1(x)] = \inf_{x \geq 0} x = 0,$$

and this is a contradiction. In conclusion, $d = (1,0) \notin D_{FL}$, which means that $D_{FL} \subsetneq D_F$, i.e. the inclusion may be strict.

Example 4.5 Let be now $m = 2, n = 1, k = 1$, $K = \mathbb{R}_+$, and the functions $f_1, f_2 : \mathbb{R} \to \overline{\mathbb{R}}$, $g : \mathbb{R} \to \mathbb{R}$, introduced by

$$f_1(x) = \begin{cases} -x^2, & \text{if} \quad x \in [0, +\infty), \\ +\infty, & \text{otherwise}, \end{cases} \quad , \qquad f_2(x) = 0$$

and

$$g(x) = x^2 - 1.$$

For $q = 1$, $\lambda = (1,1)^T$ and $d = (-1,0)^T$, it holds

$$\lambda_1 d_1 + \lambda_2 d_2 = -1 = \inf_{x \in \mathbb{R}} \left[\lambda_1 f_1(x) + \lambda_2 f_2(x) + q g(x)\right],$$

and this implies that $d = (-1,0)^T \in D_L$.

Like in the previous example, let us show now that $d \notin D_{FL}$. If this were not true, then there would exist $\bar{p} = (\bar{p}_1, \bar{p}_2)$, $\bar{q} \geq 0$ and $\bar{\lambda} = (\bar{\lambda}_1, \bar{\lambda}_2)^T \in int(\mathbb{R}^2_+)$ such that

$$-\bar{\lambda}_1 \leq -\bar{\lambda}_1 f_1^*(\bar{p}_1) - \bar{\lambda}_2 f_2^*(\bar{p}_2) - (\bar{q} g)^* \left(-\bar{\lambda}_1 \bar{p}_1 - \bar{\lambda}_2 \bar{p}_2\right). \tag{4. 16}$$

It holds $\bar{p}_2 = 0$, $f_2^*(\bar{p}_2) = 0$ and, from (4. 16),

$$-1 \leq -f_1^*(\bar{p}_1) - \left(\frac{\bar{q}}{\bar{\lambda}_1} g\right)^* (-\bar{p}_1). \tag{4. 17}$$

But,

$$-f_1^*(\bar{p}_1) = \inf_{x \in \mathbb{R}} \left[f_1(x) - \bar{p}_1 x\right] = \inf_{x \geq 0} \left[-x^2 - \bar{p}_1 x\right] = -\infty,$$

and this contradicts relation (4. 17). So, $d = (-1,0)^T \notin D_{FL}$ and, from here, $D_{FL} \subsetneq D_F$, i.e. the inclusion $D_{FL} \subseteq D_L$ may be strict.

Proposition 4.4 *It holds*

(a) $D_F \subseteq D_P$.

(b) $D_L \subseteq D_P$.

Proof.

(a) Let be $d = (d_1, ..., d_m)^T \in D_F$. Then there exist $p_i \in \mathbb{R}^n, i = 1, ..., m$, and $\lambda \in int(\mathbb{R}^m_+)$ such that

$$\begin{aligned} \sum_{i=1}^m \lambda_i d_i &\leq -\sum_{i=1}^m \lambda_i f_i^*(p_i) - \chi_{\mathcal{A}}^* \left(-\sum_{i=1}^m \lambda_i p_i\right) \\ &= -\sum_{i=1}^m \lambda_i f_i^*(p_i) + \inf_{x \in \mathcal{A}} \left(\sum_{i=1}^m \lambda_i p_i\right)^T x \\ &\leq -\sum_{i=1}^m \lambda_i f_i^*(p_i) + \left(\sum_{i=1}^m \lambda_i p_i\right)^T x, \ \forall x \in \mathcal{A}. \end{aligned}$$

By the inequality of Young for $f_i, i = 1, ..., m$, we obtain

$$\sum_{i=1}^m \lambda_i d_i \leq \sum_{i=1}^m \lambda_i f_i(x), \ \forall x \in \mathcal{A},$$

or, equivalently,

$$\sum_{i=1}^{m} \lambda_i d_i \leq \inf_{x\in\mathcal{A}} \sum_{i=1}^{m} \lambda_i f_i(x).$$

This means that $(\lambda, d) \in \mathcal{B}_P$ and $d = h^P(\lambda, d) \in h^P(\mathcal{B}_P) = D_P$.

(b) Let be again $d = (d_1, ..., d_m)^T \in D_L$, $q \underset{K^*}{\geqq} 0$ and $\lambda \in int(\mathbb{R}^m_+)$ such that

$$\begin{aligned}\sum_{i=1}^{m} \lambda_i d_i &\leq \inf_{x\in\mathbb{R}^n} \left[\sum_{i=1}^{m} \lambda_i f_i(x) + q^T g(x)\right] \\ &\leq \inf_{x\in\mathcal{A}} \left[\sum_{i=1}^{m} \lambda_i f_i(x) + q^T g(x)\right] \\ &\leq \inf_{x\in\mathcal{A}} \sum_{i=1}^{m} \lambda_i f_i(x).\end{aligned}$$

Like before, $(\lambda, d) \in \mathcal{B}_P$ and $d = h^P(\lambda, d) \in h^P(\mathcal{B}_P) = D_P$. □

Remark 4.5 Let us consider again the problem in Example 4.5. We show that $d = (-1, 0)^T \in D_P$, but $d = (-1, 0)^T \notin D_F$. For $\lambda = (1, 1)^T$, it holds

$$\lambda_1 d_1 + \lambda_2 d_2 = -1 = \inf_{x\in\mathcal{A}} \left[\lambda_1 f_1(x) + \lambda_2 f_2(x)\right],$$

and, from here, we have $d = (-1, 0)^T \in D_P$.

Assuming that $d \in D_F$, then there would exist $\bar{p} = (\bar{p}_1, \bar{p}_2)$ and $\bar{\lambda} = (\bar{\lambda}_1, \bar{\lambda}_2)^T \in int(\mathbb{R}^2_+)$ such that

$$-\bar{\lambda}_1 \leq -\bar{\lambda}_1 f_1^*(\bar{p}_1) - \bar{\lambda}_2 f_2^*(\bar{p}_2) + \inf_{g(x)\leq 0} \left(\bar{\lambda}_1 \bar{p}_1 + \bar{\lambda}_2 \bar{p}_2\right) x. \tag{4. 18}$$

But, in order to happen this, we must have $\bar{p}_2 = 0$ and $f_2^*(\bar{p}_2) = 0$ and, so, (4. 18) becomes

$$-1 \leq -f_1^*(\bar{p}_1) + \inf_{x\in[-1,1]} (\bar{p}_1 x).$$

Again, $-f_1^*(\bar{p}_1) = -\infty$ leads us to a contradiction. So, $d = (-1, 0)^T \notin D_F$, and, from here, $D_F \subsetneq D_P$, i.e. the inclusion $D_F \subseteq D_P$ may be strict.

Remark 4.6 We show now that, for the problem presented in Example 4.4, $d = (1, 0)^T \in D_P$, but $d = (1, 0)^T \notin D_L$. For $\lambda = (1, 1)^T$, it holds

$$\lambda_1 d_1 + \lambda_2 d_2 = 1 = \inf_{x\in\mathcal{A}} \left[\lambda_1 f_1(x) + \lambda_2 f_2(x)\right],$$

and, then, we have $d = (1, 0)^T \in D_P$.

Assuming $d \in D_L$, then there would exist $\bar{q} \geq 0$ and $\bar{\lambda} = (\bar{\lambda}_1, \bar{\lambda}_2)^T \in int(\mathbb{R}^2_+)$ such that

$$\bar{\lambda}_1 \leq \inf_{x\in\mathbb{R}} \left[\bar{\lambda}_1 f_1(x) + \bar{\lambda}_2 f_2(x) + \bar{q} g(x)\right] = \inf_{x\geq 0} \left[\bar{\lambda}_1 x + \bar{q}(1 - x^2)\right]. \tag{4. 19}$$

Obviously, (4. 19) is true just if $\bar{q} = 0$ and, in this case, it becomes

$$\bar{\lambda}_1 \leq \inf_{x\geq 0} \left(\bar{\lambda}_1 x\right) = 0,$$

which is a contradiction. From here, $d = (1, 0)^T \notin D_L$, and, so, the inclusion $D_L \subseteq D_P$ may be also strict.

So far we have proved that

$$D_{FL} \subsetneq \begin{array}{c} D_F \\ D_L \end{array} \subsetneq D_P. \qquad (4.\ 20)$$

Remark 4.7 In the Examples 4.4 and 4.5, one may notice that $(1,0)^T \in D_F$, $(1,0)^T \notin D_L$ and $(-1,0)^T \in D_L$, $(-1,0)^T \notin D_F$, respectively. This certifies the fact that, in the general case, between the sets D_F and D_L does not exist any relation of inclusion like those presented in the Propositions 4.3 and 4.4.

From (4. 9) and (4. 20) we can state that in the general case it holds, for every $\alpha \in \mathcal{F}$,

$$D_1 \cap \mathbb{R}^m \subsetneq D_\alpha \cap \mathbb{R}^m \subsetneq D_{FL} \subsetneq \begin{array}{c} D_F \\ D_L \end{array} \subsetneq D_P. \qquad (4.\ 21)$$

In section 4.4 we have proved that even if the inclusions in (4. 9) between D_1, $D_\alpha, \alpha \in \mathcal{F}$, and D_{FL} are strict, their sets of maximal elements are equal (cf. (4. 13)). We show now by some counter-examples that this result does not hold for the maximal elements sets of D_{FL}, D_F, D_L and D_P. Actually, we show that there is no relation of inclusion between $vmaxD_{FL}$, $vmaxD_F$, $vmaxD_L$ and $vmaxD_P$.

Remark 4.8 Let us consider again the problem in Example 4.4. We have shown that $d = (1,0)^T \notin D_{FL}$ and this means that $d = (1,0)^T \notin vmaxD_{FL}$. On the other hand, we have $d = (1,0) \in D_F$ and, moreover, it can be proved that $d = (1,0) \in vmaxD_F$. In conclusion, $vmaxD_F \not\subseteq vmaxD_{FL}$.

For the same example, let be now $\tilde{d} = (0,0)^T$. It can be verified that $\tilde{d} \in vmaxD_{FL}$, which means that $\tilde{d} = (0,0)^T \in D_{FL} \subseteq D_F$. But, because $d = (1,0)^T \in D_F$, it follows that $\tilde{d} \notin vmaxD_F$. So, $vmaxD_{FL} \not\subseteq vmaxD_F$.

Example 4.6 For $m = 2, n = 1, k = 1$, $K = \mathbb{R}_+$, let be $f_1, f_2 : \mathbb{R} \to \overline{\mathbb{R}}$, $g : \mathbb{R} \to \mathbb{R}$, defined by

$$f_1(x) = \begin{cases} x^2, & \text{if} \quad x \in [0,+\infty), \\ +\infty, & \text{otherwise}, \end{cases} \quad , \qquad f_2(x) = 0,$$

and

$$g(x) = \begin{cases} 1 - x^2, & \text{if} \quad x \in [0,+\infty), \\ 1, & \text{otherwise}. \end{cases}$$

For $q = 1$, $\lambda = (1,1)^T$ and $d = (1,0)^T$, we have $(q,\lambda,d) \in \mathcal{B}_L$ and $d \in D_L$. Moreover, $d \in vmaxD_L$. It can be also verified that $d \notin D_{FL}$ and, from here, $d = (1,0)^T \notin vmaxD_{FL}$. This means that $vmaxD_L \not\subseteq vmaxD_{FL}$.

Moreover, it can be shown that $\tilde{d} = (0,0)^T \in vmaxD_{FL}$. But, Proposition 4.3 ($b$) implies that $\tilde{d} = (0,0)^T \in D_{FL} \subseteq D_L$. Obviously, $\tilde{d} \notin vmaxD_L$, otherwise it would contradict the maximality of $d = (1,0)^T$ in D_L. So, $vmaxD_{FL} \not\subseteq vmaxD_L$.

Remark 4.9 For the problem presented in Example 4.5, we have that $d = (-1,0)^T \in D_P$ and, moreover, $d \in vmaxD_P$. Because $d \notin D_F$, we also have that $d \notin vmaxD_F$. In conclusion, $vmaxD_P \not\subseteq vmaxD_F$.

In order to show that $vmaxD_F \not\subseteq vmaxD_P$, let us consider for $m = 2, n = 1, k = 1$, $K = \mathbb{R}_+$, the functions $f_1, f_2 : \mathbb{R} \to \overline{\mathbb{R}}$, $g : \mathbb{R} \to \mathbb{R}$, defined by

$$f_1(x) = \begin{cases} x, & \text{if} \quad x \in (0,+\infty), \\ +\infty, & \text{otherwise}, \end{cases} \quad , \ f_2(x) = 0 \text{ and } g(x) = x.$$

It is easy to verify, for $p = (0,0)$, $\lambda = (1,1)^T$ and $d = (0,0)^T$, that the element (p,λ,d) belongs to $\mathcal{B}_F$, that implies $d = (0,0)^T \in D_F$. Moreover, $d = (0,0)^T \in vmaxD_F$.

By Proposition 4.4 (a), we have $d = (0,0)^T \in D_P$. But, for $\lambda = (1,1)^T$ and $\tilde{d} = (1,0)^T$, $(\lambda, \tilde{d}) \in \mathcal{B}_P$ and, from here, $\tilde{d} = (1,0)^T \in D_P$. So, $d = (0,0)^T \notin vmaxD_P$ and $vmaxD_F \nsubseteq vmaxD_P$.

Remark 4.10 Considering again the problem in Example 4.4, we have $d = (1,0)^T \in D_P$ and $d \notin D_L$. From here, $d \notin vmaxD_L$. Moreover, $d = (1,0)^T \in vmaxD_P$, which shows that $vmaxD_P \nsubseteq vmaxD_L$.

On the other hand, $\tilde{d} = (0,0)^T \in vmaxD_L$ and, by Proposition 4.4 (b), $\tilde{d} \in D_L \subseteq D_P$. Because $d = (1,0)^T \in D_P$, it follows $\tilde{d} = (0,0)^T \notin vmaxD_P$. So, $vmaxD_L \nsubseteq vmaxD_P$.

In conclusion, in the general case, between the sets of maximal elements of D_{FL}, D_F, D_L and D_P a relation of equality or any relation of inclusion does not exist. In this situation, the only valid relation is the relation of inclusion (4. 20).

4.6 Conditions for the equality of the sets D_{FL}, D_F, D_L and D_P

Assuming that (A_f), (A_g) and (A_{CQ}) are satisfied we prove in this section that the relation (4. 20) becomes an equality.

Theorem 4.16 *Let the assumptions (A_g) and (A_{CQ}) be fulfilled. Then it holds $D_{FL} = D_F$.*

Proof. By Proposition 4.3 (a) we have that $D_{FL} \subseteq D_F$.

Let be $d \in D_F$. Then there exist $p = (p_1, ..., p_m) \in \mathbb{R}^n \times ... \times \mathbb{R}^n$ and $\lambda \in int(\mathbb{R}^m_+)$ such that $(p, \lambda, d) \in \mathcal{B}_F$, i.e.

$$\begin{aligned} \sum_{i=1}^m \lambda_i d_i &\leq -\sum_{i=1}^m \lambda_i f_i^*(p_i) - \chi_{\mathcal{A}}^*\left(-\sum_{i=1}^m \lambda_i p_i\right) \\ &= -\sum_{i=1}^m \lambda_i f_i^*(p_i) + \inf_{x \in \mathcal{A}} \left(\sum_{i=1}^m \lambda_i p_i\right)^T x. \end{aligned} \tag{4. 22}$$

For the optimization problem

$$(P^{\lambda p}) \quad \inf_{x \in \mathcal{A}} \left(\sum_{i=1}^m \lambda_i p_i\right)^T x,$$

$$\mathcal{A} = \left\{x \in \mathbb{R}^n : g(x) \underset{K}{\leqq} 0\right\},$$

let us consider its Lagrange dual

$$(D_L^{\lambda p}) \quad \sup_{q \underset{K^*}{\geqq} 0} \inf_{x \in \mathbb{R}^n} \left[\left(\sum_{i=1}^m \lambda_i p_i\right)^T x + q^T g(x)\right].$$

(A_g) and (A_{CQ}) being fulfilled, it follows by Theorem 4.1 that strong duality between $(P^{\lambda p})$ and $(D^{\lambda p})$ holds. This fact ensures the existence of an element $\bar{q} \underset{K^*}{\geqq} 0$, optimal solution to $(D_L^{\lambda p})$, such that

$$\inf_{x \in \mathcal{A}} \left(\sum_{i=1}^m \lambda_i p_i\right)^T x = inf(P^{\lambda p}) = max(D_L^{\lambda p}) = \inf_{x \in \mathbb{R}^n} \left[\left(\sum_{i=1}^m \lambda_i p_i\right)^T x + \bar{q}^T g(x)\right]. \tag{4. 23}$$

From (4. 22) and (4. 23) we have

$$\begin{aligned}\sum_{i=1}^m \lambda_i d_i &\leq -\sum_{i=1}^m \lambda_i f_i^*(p_i) + \inf_{x\in\mathcal{A}} \left(\sum_{i=1}^m \lambda_i p_i\right)^T x \\ &= -\sum_{i=1}^m \lambda_i f_i^*(p_i) + \inf_{x\in\mathbb{R}^n} \left[\left(\sum_{i=1}^m \lambda_i p_i\right)^T x + \bar{q}^T g(x)\right] \\ &= -\sum_{i=1}^m \lambda_i f_i^*(p_i) - (\bar{q}^T g)^* \left(-\sum_{i=1}^m \lambda_i p_i\right).\end{aligned}$$

This means that $(p, \bar{q}, \lambda, d) \in \mathcal{B}_{FL}$ and $d = h^{FL}(p, \bar{q}, \lambda, d) \in h^{FL}(\mathcal{B}_{FL}) = D_{FL}$. □

Remark 4.11 For the problem presented in Example 4.5, one may observe that (A_g) and (A_{CQ}) are fulfilled, $d = (-1, 0)^T \in D_P$, but $d = (-1, 0)^T \notin D_{FL} = D_F$. We conclude that just these two assumptions are not sufficient to have equality between all the sets in (4. 20).

Theorem 4.17 *Let the assumptions (A_f) and (A_g) be fulfilled. Then it holds $D_{FL} = D_L$.*

Proof. By Proposition 4.3 (b) we have that $D_{FL} \subseteq D_L$.

Let be $d \in D_L$. Then there exist $q \underset{K^*}{\geqq} 0$ and $\lambda \in int(\mathbb{R}^m_+)$ such that $(p, \lambda, d) \in \mathcal{B}_L$, i.e.

$$\sum_{i=1}^m \lambda_i d_i \leq \inf_{x\in\mathbb{R}^n} \left[\sum_{i=1}^m \lambda_i f_i(x) + q^T g(x)\right]. \tag{4. 24}$$

Let us consider the function $k : \mathbb{R}^n \to \overline{\mathbb{R}}$, $k(x) = \sum\limits_{i=1}^m \lambda_i f_i(x)$. We have $dom(k) = \bigcap\limits_{i=1}^m dom(f_i)$ and, from (A_f), it follows $ri(dom(k)) = \bigcap\limits_{i=1}^m ri(dom(f_i)) \neq \emptyset$ (cf. [20]). Let us also notice that $dom(q^T g) = \mathbb{R}^n$ and then, by Theorem 31.1 in [62], there exists $\tilde{p} \in \mathbb{R}^n$ such that

$$\inf_{x\in\mathbb{R}^n} \left[\sum_{i=1}^m \lambda_i f_i(x) + q^T g(x)\right] = -\left(\sum_{i=1}^m \lambda_i f_i\right)^* (\tilde{p}) + \inf_{x\in\mathbb{R}^n} \left[\tilde{p}^T x + q^T g(x)\right]. \tag{4. 25}$$

On the other hand, from Theorem 16.4 in [62], there exist $\bar{p}_i \in \mathbb{R}^n, i = 1, ..., m$, such that $\tilde{p} = \sum\limits_{i=1}^m \lambda_i \bar{p}_i$ and

$$\left(\sum_{i=1}^m \lambda_i f_i\right)^* (\tilde{p}) = \sum_{i=1}^m \lambda_i f_i^*(\bar{p}_i).$$

From (4. 24) and (4. 25) we obtain

$$\begin{aligned}\sum_{i=1}^m \lambda_i d_i &\leq \inf_{x\in\mathbb{R}^n} \left[\sum_{i=1}^m \lambda_i f_i(x) + q^T g(x)\right] \\ &= -\left(\sum_{i=1}^m \lambda_i f_i\right)^* (\tilde{p}) + \inf_{x\in\mathbb{R}^n} \left[\tilde{p}^T x + q^T g(x)\right]\end{aligned}$$

$$= -\sum_{i=1}^m \lambda_i f_i^*(\bar{p}_i) + \inf_{x\in\mathbb{R}^n}\left[\left(\sum_{i=1}^m \lambda_i \bar{p}_i\right)^T x + q^T g(x)\right]$$
$$= -\sum_{i=1}^m \lambda_i f_i^*(\bar{p}_i) - (q^T g)^*\left(-\sum_{i=1}^m \lambda_i \bar{p}_i\right).$$

This means that, for $\bar{p} = (\bar{p}_1, ..., \bar{p}_m)$, $(\bar{p}, q, \lambda, d) \in \mathcal{B}_{FL}$ and $d = h^{FL}(\bar{p}, q, \lambda, d) \in h^{FL}(\mathcal{B}_{FL}) = D_{FL}$. □

Example 4.7 For $m = 2, n = 2, k = 1$, $K = \mathbb{R}_+$, we consider the functions $f_1, f_2 : \mathbb{R}^2 \to \overline{\mathbb{R}}$, $g : \mathbb{R}^2 \to \mathbb{R}$, introduced by

$$f_1(x_1,x_2) = \begin{cases} x_2 & \text{if } \ x \in X, \\ +\infty, & \text{otherwise,} \end{cases}$$

$$X = \left\{ x = (x_1,x_2) \in \mathbb{R}^2 : 0 \le x_1 \le 2, \begin{array}{ll} 3 \le x_2 \le 4 & \text{for} \quad x_1 = 0 \\ 1 < x_2 \le 4 & \text{for} \quad x_1 > 0 \end{array} \right\},$$

$$f_2(x_1,x_2) = 0 \text{ and } g(x_1,x_2) = x_1.$$

It can be observed that (A_f) and (A_g) are fulfilled, $d = (3,0)^T \in D_P$, but $d = (3,0)^T \notin D_{FL} = D_L$. Like in Remark 4.11, we can conclude that just the assumptions (A_f) and (A_g) are also not sufficient to have equality between all the sets in (4. 20). The next theorem shows when this fact happens.

Theorem 4.18 *Let the assumptions (A_f), (A_g) and (A_{CQ}) be fulfilled. Then it holds $D_{FL} = D_L = D_F = D_P$.*

Proof. By the Theorems 4.16 and 4.17 we have $D_{FL} = D_L = D_F$. Let us prove now that $D_F = D_P$.

Proposition 4.4 (a) gives us that $D_F \subseteq D_P$. It remains to prove just that the reversed inclusion also holds.

Let be $d \in D_P$. Then there exists $\lambda \in int(\mathbb{R}^m_+)$ such that $(\lambda, d) \in \mathcal{B}_P$, i.e.

$$\sum_{i=1}^m \lambda_i d_i \le \inf_{x\in\mathcal{A}} \sum_{i=1}^m \lambda_i f_i(x). \tag{4. 26}$$

Moreover, by (4. 26) and since (A_f), (A_g) and (A_{CQ}) are true, it follows that the assumptions of the strong duality Theorem 4.1 are fulfilled. Considering for the primal problem

$$(P^\lambda) \quad \inf_{x\in\mathcal{A}} \sum_{i=1}^m \lambda_i f_i(x),$$

its Fenchel dual

$$(D_F^\lambda) \sup_{p_i\in\mathbb{R}^n, i=1,...,m} \left\{ -\sum_{i=1}^m \lambda_i f_i^*(p_i) - \chi_{\mathcal{A}}^*\left(-\sum_{i=1}^m \lambda_i p_i\right)\right\},$$

the last one has a solution and the optimal objective values of both problems are equal. Then there exist $\bar{p}_i \in \mathbb{R}^n, i = 1, ..., m$, such that

$$\inf_{x\in\mathcal{A}} \sum_{i=1}^m \lambda_i f_i(x) = -\sum_{i=1}^m \lambda_i f_i^*(\bar{p}_i) - \chi_{\mathcal{A}}^*\left(-\sum_{i=1}^m \lambda_i \bar{p}_i\right). \tag{4. 27}$$

From (4. 26) and (4. 27) we have

$$\sum_{i=1}^{m}\lambda_i d_i \leq \inf_{x\in\mathcal{A}}\sum_{i=1}^{m}\lambda_i f_i(x) = -\sum_{i=1}^{m}\lambda_i f_i^*(\bar{p}_i) - \chi_{\mathcal{A}}^*\left(-\sum_{i=1}^{m}\lambda_i\bar{p}_i\right),$$

which actually means that, for $\bar{p} = (\bar{p}_1, ..., \bar{p}_m)$, $(\bar{p}, \lambda, d) \in \mathcal{B}_F$ and $d = h^F(\bar{p}, \lambda, d) \in h^F(\mathcal{B}_F) = D_F$. □

As a consequence of Theorem 4.18 we can affirm that, if (A_f), (A_g) and (A_{CQ}) are fulfilled, then (4. 21) becomes, for every $\alpha \in \mathcal{F}$,

$$D_1 \cap \mathbb{R}^m \subsetneq D_\alpha \cap \mathbb{R}^m \subsetneq D_{FL} = D_L = D_F = D_P. \quad (4.\ 28)$$

This last relation, together with (4. 13), gives us for every $\alpha \in \mathcal{F}$,

$$vmaxD_1 = vmaxD_\alpha = vmaxD_{FL} = vmaxD_F = vmaxD_L = vmaxD_P, \quad (4.\ 29)$$

provided that (A_f), (A_g) and (A_{CQ}) hold.

In the next three sections we investigate the relations between the six multiobjective problems considered in this chapter and some well-known dual problems from the literature. We start with the dual introduced by NAKAYAMA.

4.7 Nakayama multiobjective duality

One of the first theories concerning duality for convex multiobjective problems has been developed by NAKAYAMA and can be found in [54], [55] and [65]. If we consider this theory for the primal problem (P), the dual introduced there becomes

$$(D_N) \quad \underset{(U,y)\in\mathcal{B}_N}{\text{v-max}}\ h^N(U, y),$$

with

$$h^N(U, y) = \begin{pmatrix} h_1^N(U, y) \\ \vdots \\ h_m^N(U, y) \end{pmatrix} = \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix},$$

$$h_j^N(U, y) = y_j, j = 1, ..., m,$$

the dual variables

$$U \in \mathcal{U}, y = (y_1, \ldots, y_m)^T \in \mathbb{R}^m,$$

$$\mathcal{U} = \{U : U \text{ is a } m \times k \text{ matrix such that } U \cdot K \subseteq \mathbb{R}_+^m\},$$

and the set of constraints

$$\mathcal{B}_N = \{(U, y) : U \in \mathcal{U} \text{ and there is no } x \in \mathbb{R}^n \text{ such that } y \gneqq f(x) + Ug(x)\}.$$

If $U = \begin{pmatrix} q_1^T \\ \vdots \\ q_m^T \end{pmatrix} \in \mathcal{U}, q_i \in \mathbb{R}^k, i = 1, ..., m$, then for every $k \in K$, it must hold $(q_1^T k, ..., q_m^T k)^T \in \mathbb{R}_+^m$. From here, for $i = 1, ..., m$, $q_i^T k \geq 0, \forall k \in K$, which actually means that $q_i \in K^*$, for $i = 1, ..., m$. By this observation, the dual (D_N) can be written, equivalently, in the following way

$$(D_N) \quad \underset{(q_1,...,q_m,y)\in\mathcal{B}_N}{\text{v-max}}\ h^N(q_1, ..., q_m, y),$$

with

$$h^N(q_1,...,q_m,y) = \begin{pmatrix} h_1^N(q_1,...,q_m,y) \\ \vdots \\ h_m^N(q_1,...,q_m,y) \end{pmatrix} = \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix},$$

$$h_j^N(q_1,...,q_m,y) = y_j, j = 1,...,m,$$

the dual variables

$$q_i \in \mathbb{R}^k, i = 1,...,m, y = (y_1,\ldots,y_m)^T \in \mathbb{R}^m,$$

and the set of constraints

$$\mathcal{B}_N = \Big\{(q_1,...,q_m,y) : \quad q_i \underset{K^*}{\geqq} 0, i = 1,...,m, \text{ and there is no } x \in \mathbb{R}^n \text{ such that } y \gneqq f(x) + (q_1^T g(x),...,q_m^T g(x))^T\Big\}.$$

The proofs of the next two theorems have been given in [54].

Theorem 4.19 (weak duality for (D_N)) *There is no $x \in \mathcal{A}$ and no element $(q_1,...,q_m,y) \in \mathcal{B}_N$ fulfilling $h^N(q_1,...,q_m,y) \underset{\mathbb{R}^m_+}{\geqq} f(x)$ and $h^N(q_1,...,q_m,y) \neq f(x)$.*

Theorem 4.20 (strong duality for (D_N)) *Assume that (A_f), (A_g) and (A_{CQ}) are fulfilled. If $\bar{x}$ is a properly efficient solution to (P), then there exists an efficient solution $(\bar{q}_1,...,\bar{q}_m,\bar{y}) \in \mathcal{B}_N$ to the dual (D_N) and the strong duality $f(\bar{x}) = h^N(\bar{q}_1,...,\bar{q}_m,\bar{y}) = \bar{y}$ holds.*

In order to relate the dual (D_N) to the duals considered in the previous chapters, let us denote by $D_N := h^N(\mathcal{B}_N) \subseteq \mathbb{R}^m$ the image set of the Nakayama multiobjective dual.

Proposition 4.5 *It holds $D_L \subseteq D_N$.*

Proof. Let be $d = (d_1,...,d_m)^T \in D_L$. Then there exist $q \underset{K^*}{\geqq} 0$ and $\lambda \in int(\mathbb{R}^m_+)$ such that $(q,\lambda,d) \in \mathcal{B}_L$, i.e.

$$\sum_{i=1}^m \lambda_i d_i \leq \inf_{x\in\mathbb{R}^n} \left[\sum_{i=1}^m \lambda_i f_i(x) + q^T g(x)\right]. \tag{4. 30}$$

Let be, for $i = 1,...,m$, $\bar{q}_i := \frac{1}{\sum\limits_{i=1}^m \lambda_i} q \underset{K^*}{\geqq} 0$.

We show now that $(\bar{q}_1,...,\bar{q}_m,d) \in \mathcal{B}_N$. If this does not happen, then there exists $x' \in \mathbb{R}^n$ such that $d \gneqq f(x') + (\bar{q}_1^T g(x'),...,\bar{q}_m^T g(x'))^T$. It follows that $\sum\limits_{i=1}^m \lambda_i d_i > \sum\limits_{i=1}^m \lambda_i f_i(x') + q^T g(x')$, but this contradicts the inequality in (4. 30). From here we obtain that $(\bar{q}_1,...,\bar{q}_m,d) \in \mathcal{B}_N$ and $d = h^N(\bar{q}_1,...,\bar{q}_m,d) \in h^N(\mathcal{B}_N) = D_N$. □

Example 4.8 For $m = 2, n = 1, k = 1$, $K = \mathbb{R}_+$, let be $f_1, f_2 : \mathbb{R} \to \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$, defined by $f_1(x) = x$, $f_2(x) = 1$ and $g(x) = -1$.

Considering $q_1 = q_2 = 0$ and $d = (1,0)^T$, it is obvious that there is no $x \in \mathbb{R}^n$ such that

$$d = (1,0)^T \gneqq f(x) + (q_1 g(x), q_2 g(x))^T = (x,1)^T.$$

This means that $d = (1,0)^T \in D_N$.

On the other hand, we have $d \notin D_L$ and, so, $D_L \subsetneq D_N$, i.e. the inclusion $D_L \subseteq D_N$ may be strict.

Example 4.9 For $m = 2, n = 1, k = 1$, $K = \mathbb{R}_+$, let now be $f_1, f_2 : \mathbb{R} \to \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$, defined by

$$f_1(x) = f_2(x) = x,$$

and

$$g(x) = \begin{cases} 1 - x^2, & \text{if} \quad x \in [0, +\infty), \\ 1, & \text{otherwise.} \end{cases}$$

The element $d = (1,1)^T$ belongs to D_F and D_P. We show now that $d \notin D_N$. If this were not true, then there would exist $\bar{q}_1, \bar{q}_2 \geq 0$ such that $(\bar{q}_1, \bar{q}_2, d) \in D_N$, or, equivalently,

$$d = (1,1)^T \gneqq (x + q_1 g(x), x + q_2 g(x))^T \tag{4. 31}$$

would not hold for any $x \in \mathbb{R}$. But, for $i = 1, 2$, $\lim\limits_{x \to -\infty} (x + q_i g(x)) = -\infty$, which means that there exists $x' \in \mathbb{R}$ such that $x + q_1 g(x) < 1$ and $x + q_2 g(x) < 1$. This contradicts (4. 31). The conclusion is that, in general, $D_F \nsubseteq D_N$ and $D_P \nsubseteq D_N$.

Remark 4.12 For the problem introduced in Example 4.8, let us notice that (A_f), (A_g) and (A_{CQ}) are fulfilled. By Theorem 4.18 we have $D_L = D_F = D_P$, and, so, $d = (1,0)^T$ neither belongs to D_F, nor to D_P. But we have shown that $d = (1,0)^T \in D_N$. We conclude that $D_N \nsubseteq D_F$ and $D_N \nsubseteq D_P$.

The last results allow us to extend the relation (4. 21) by introducing the set D_N. We get, for every $\alpha \in \mathcal{F}$,

$$D_1 \cap \mathbb{R}^m \subsetneq D_\alpha \cap \mathbb{R}^m \subsetneq D_{FL} \subsetneq \begin{array}{c} D_F \subsetneq D_P \\ D_L \subsetneq \begin{array}{c} D_P \\ D_N \end{array} \end{array}. \tag{4. 32}$$

If (A_f), (A_g) and (A_{CQ}) are fulfilled, then from (4. 28) and Proposition 4.5 the inclusions in (4. 32) becomes, for every $\alpha \in \mathcal{F}$,

$$D_1 \cap \mathbb{R}^m \subsetneq D_\alpha \cap \mathbb{R}^m \subsetneq D_{FL} = D_L = D_F = D_P \subsetneq D_N. \tag{4. 33}$$

We remind that, if (A_f), (A_g) and (A_{CQ}) are fulfilled, then the maximal elements sets of the first six duals are equal (cf. (4. 29)). The following example shows that, even if the three assumptions are fulfilled, between $vmaxD_N$ and $vmaxD_P$ does not exist any relation of inclusion.

Example 4.10 For $m = 2, n = 2, k = 1$, $K = \mathbb{R}$, let be $f_1, f_2 : \mathbb{R}^2 \to \overline{\mathbb{R}}$, $g : \mathbb{R}^2 \to \mathbb{R}$, defined by

$$f_1(x_1, x_2) = \begin{cases} x_1 & \text{if} \quad x \in X, \\ +\infty, & \text{otherwise,} \end{cases} \quad , \quad f_2(x_1, x_2) = \begin{cases} x_2 & \text{if} \quad x \in X, \\ +\infty, & \text{otherwise,} \end{cases}$$

$$X = \Big\{ x = (x_1, x_2)^T \in \mathbb{R}^2 : x_1, x_2 \geq 0 \text{ such that } x_2 > 0, \text{ if } x_1 \in [0,1) \Big\},$$

and

$$g(x_1, x_2) = 0.$$

We notice that (A_f), (A_g) and (A_{CQ}) are fulfilled.

For $q_1 = q_2 = 0 \in K^* = \{0\}$ and $d = (1,0)^T$ it does not exist $x = (x_1, x_2)^T \in X$ such that $(1,0)^T \gneqq (x_1, x_2)^T$. This means that $(0,0,d) \in \mathcal{B}_N$ and $d \in D_N$.

Let us assume now that there exist $\bar{q}_1, \bar{q}_2 \in K^*$ and $\bar{d} \in \mathbb{R}^2$ such that $(\bar{q}_1, \bar{q}_2, \bar{d}) \in \mathcal{B}_N$ and $\bar{d} \gneqq d = (1,0)$. We have then $\bar{q}_1 = \bar{q}_2 = 0$ and for $\bar{x} = (1,0)^T \in X$ it holds

$$(f_1(\bar{x}) + \bar{q}_1 g(\bar{x}), f_2(\bar{x}) + \bar{q}_2 g(\bar{x}))^T = (\bar{x}_1, \bar{x}_2)^T = (1,0)^T = d \lneqq \bar{d}.$$

It follows that $(\bar{q}_1, \bar{q}_2, \bar{d}) \notin \mathcal{B}_N$, which means that $d = (1,0)^T \in vmaxD_N$.

Let us assume now that $d \in D_P = D_L$. Then there exists $\bar{\lambda} = (\bar{\lambda}_1, \bar{\lambda}_2)^T \in int(R^2_+)$ such that

$$\bar{\lambda}_1 = \bar{\lambda}_1 d_1 + \bar{\lambda}_2 d_2 \leq \inf_{x \in \mathcal{A}} \left[\bar{\lambda}_1 f_1(x) + \bar{\lambda}_2 f_2(x)\right] = \inf_{x \in X} \left(\bar{\lambda}_1 x_1 + \bar{\lambda}_2 x_2\right).$$

Otherwise, for $n \in \mathbb{N}^*$, $(\frac{1}{n}, \frac{1}{n})^T \in X$, it holds

$$\bar{\lambda}_1 \leq \bar{\lambda}_1 \frac{1}{n} + \bar{\lambda}_2 \frac{1}{n}, \ \forall n \in \mathbb{N}^*.$$

If $n \to +\infty$, then we must have $\bar{\lambda}_1 \leq 0$ and this is a contradiction. From here, $d = (1,0)^T \notin D_P$ and, obviously, $d = (1,0)^T \notin vmaxD_P$. In conclusion, $vmaxD_N \nsubseteq vmaxD_P$.

On the other hand, for $\lambda_1 = \lambda_2 = 1$ and $\tilde{d} = (0,0)^T$, we have $\tilde{d} = (0,0)^T \in D_P$ and, moreover, $\tilde{d} = (0,0)^T \in vmaxD_P$.

By Proposition 4.5, $\tilde{d} = (0,0)^T \in D_P \subseteq D_N$ and, because $d = (1,0)^T \in D_N$, it follows $\tilde{d} = (0,0)^T \notin vmaxD_N$. So, $vmaxD_P \nsubseteq vmaxD_N$.

Remark 4.13 In Proposition 5 in [55], Nakayama gave some necessary conditions to have

$$vminP = vmaxD_L = vmaxD_N, \tag{4. 34}$$

where $vminP$ represents the set of the Pareto-efficient solutions of the problem (P).

In order to have (4. 34), this proposition claims that (A_f), (A_g) and (A_{CQ}) must be fulfilled, the problem (P) must have at least one Pareto-efficient solution, all these Pareto-efficient solutions must be properly efficient and the set

$$G = \left\{ (z,y) \in \mathbb{R}^m \times \mathbb{R}^k : \exists x \in X, \text{ s.t. } y \underset{\mathbb{R}^m_+}{\geqq} f(x), z \underset{K}{\geqq} g(x) \right\}$$

must be closed.

4.8 Wolfe multiobjective duality

The next vector dual problem we treat in this chapter is the Wolfe multiobjective dual also well-known in the literature. First it was introduced in the differentiable case by Weir in [90]. Its formulation for the non-differentiable case can be found in [93] and it has been inspired by the Wolfe scalar dual problem for non-differentiable optimization problems (cf. [67], [94]).

The Wolfe multiobjective dual problem has the following formulation

$$(D_W) \quad \underset{(x,q,\lambda) \in \mathcal{B}_W}{\text{v-max}} h^W(x,q,\lambda),$$

with

$$h^W(x,q,\lambda) = \begin{pmatrix} h_1^W(x,q,\lambda) \\ \vdots \\ h_m^W(x,q,\lambda) \end{pmatrix},$$

$$h_j^W(x, q, \lambda) = f_j(x) + q^T g(x), j = 1, ..., m,$$

the dual variables

$$x \in \mathbb{R}^n, q \in \mathbb{R}^k, \lambda = (\lambda_1, \ldots, \lambda_m)^T \in \mathbb{R}^m,$$

and the set of constraints

$$\mathcal{B}_W = \left\{ (x, q, \lambda) : \quad x \in \mathbb{R}^n, \lambda = (\lambda_1, \ldots, \lambda_m)^T \in int(\mathbb{R}^m_+), \sum_{i=1}^m \lambda_i = 1, \right.$$
$$\left. q \underset{K^*}{\geqq} 0, 0 \in \partial\left(\sum_{i=1}^m \lambda_i f_i\right)(x) + \partial(q^T g)(x) \right\}.$$

Here, for a function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$,

$$\partial f(\bar{x}) = \left\{ x^* \in \mathbb{R}^n : f(x) - f(\bar{x}) \geq < x^*, x - \bar{x} >, \ \forall x \in \mathbb{R}^n \right\}$$

represents the subdifferential of f at the point $\bar{x} \in \mathbb{R}^n$.

The following two theorems represent the weak and strong duality theorems. Their proofs can be derived from [90] and [93].

Theorem 4.21 (weak duality for (D_W)) *There is no $x \in \mathcal{A}$ and no element $(y, q, \lambda) \in \mathcal{B}_W$ fulfilling $h^W(y, q, \lambda) \underset{\mathbb{R}^m_+}{\geqq} f(x)$ and $h^W(y, q, \lambda) \neq f(x)$.*

Theorem 4.22 (strong duality for (D_W)) *Assume that (A_f), (A_g) and (A_{CQ}) are fulfilled. If $\bar{x}$ is a properly efficient solution to (P), then there exists $\bar{q} \underset{K^*}{\geqq} 0$ and $\bar{\lambda} \in int(\mathbb{R}^m_+)$ such that $(\bar{x}, \bar{q}, \bar{\lambda}) \in \mathcal{B}_W$ is a properly efficient solution to the dual (D_W) and the strong duality $f(\bar{x}) = h^W(\bar{x}, \bar{q}, \bar{\lambda})$ holds.*

Let us consider now $D_W := h^W(\mathcal{B}_W) \subseteq \mathbb{R}^m$. We study in the general case the relations between D_W and the image sets of the duals introduced so far.

Proposition 4.6 *It holds $D_W \subseteq D_L$.*

Proof. Let be $d = (d_1, ..., d_m)^T \in D_W$. Then there exists $(x, q, \lambda) \in \mathcal{B}_W$ such that $d = h^W(x, q, \lambda) = f(x) + (q^T g(x), ..., q^T g(x))^T$.

From here, it follows

$$\sum_{i=1}^m \lambda_i d_i = \sum_{i=1}^m \lambda_i f_i(x) + \left(\sum_{i=1}^m \lambda_i\right) q^T g(x) = \sum_{i=1}^m \lambda_i f_i(x) + q^T g(x). \tag{4. 35}$$

On the other hand, because $(x, q, \lambda) \in \mathcal{B}_W$, we have

$$0 \in \partial\left(\sum_{i=1}^m \lambda_i f_i\right)(x) + \partial(q^T g)(x),$$

which implies that

$$\sum_{i=1}^m \lambda_i f_i(x) + q^T g(x) \leq \inf_{u \in \mathbb{R}^n} \left[\sum_{i=1}^m \lambda_i f_i(u) + q^T g(u)\right]. \tag{4. 36}$$

From (4. 35) and (4. 36) we obtain

$$\sum_{i=1}^m \lambda_i d_i = \sum_{i=1}^m \lambda_i f_i(x) + q^T g(x) \leq \inf_{u \in \mathbb{R}^n} \left[\sum_{i=1}^m \lambda_i f_i(u) + q^T g(u)\right],$$

which gives us $(q, \lambda, d) \in \mathcal{B}_L$ and $d = h^L(q, \lambda, d) \in h^L(\mathcal{B}_L) = D_L$. $\square$

Example 4.11 For $m = 2, n = 1, k = 1$, $K = \mathbb{R}$, let be $f_1, f_2 : \mathbb{R} \to \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$, defined by $f_1(x) = f_2(x) = x^2$ and $g(x) = 0$.

For $q = 0 \in K^* = \{0\}$, $\lambda = (1, 1)^T$ and $d = (-1, -1)^T$ we have

$$\lambda_1 d_1 + \lambda_2 d_2 = -2 < 0 = \inf_{x \in \mathbb{R}} \left[x^2 + x^2\right] = \inf_{x \in \mathbb{R}} \left[\lambda_1 f_1(x) + \lambda_2 f_2(x) + q^T g(x)\right],$$

which implies that $d = (-1, -1)^T \in D_L$.

We will show that $d = (1, -1)^T \notin D_W$. If this were not true, then there would exists $(\bar{x}, \bar{q}, \bar{\lambda}) \in \mathcal{B}_W$, with $\bar{\lambda} = (\bar{\lambda}_1, \bar{\lambda}_2)^T \in int(\mathbb{R}^2_+)$, $\bar{\lambda}_1 + \bar{\lambda}_2 = 1$, $\bar{q} \in K^* = \{0\}$ such that

$$d = (-1, -1)^T = (f_1(\bar{x}) + \bar{q} g(\bar{x}), f_2(\bar{x}) + \bar{q} g(\bar{x}))^T = (\bar{x}^2, \bar{x}^2)^T.$$

But, this is a contradiction and, so, $D_W \subsetneq D_L$, i.e. the inclusion may be strict. Moreover, by (4. 32), we have $D_P \not\subseteq D_W$ and $D_N \not\subseteq D_W$.

Example 4.12 For $m = 2, n = 1, k = 1$, $K = \mathbb{R}$, let be $f_1, f_2 : \mathbb{R} \to \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$, defined by $f_1(x) = f_2(x) = 0$ and $g(x) = 0$.

For $p = (0, 0)$, $q = 0 \in K^* = \{0\}$, $\lambda = \left(\frac{1}{2}, \frac{1}{2}\right)^T$, $t = (1, -1)^T$, it holds $d = (1, -1)^T \in D_1$. On the other hand, $d = (1, -1)^T \notin D_W$. So, $D_1 \cap \mathbb{R}^m \not\subseteq D_W$, whence, $D_\alpha \cap \mathbb{R}^m \not\subseteq D_W$, $\alpha \in \mathcal{F}$, $D_{FL} \not\subseteq D_W$ and $D_F \not\subseteq D_W$.

Example 4.13 For $m = 2, n = 1, k = 1$, $K = \mathbb{R}_+$, let be $f_1, f_2 : \mathbb{R} \to \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$, defined by $f_1(x) = x^2 - 1$, $f_2(x) = 1 - x^2$ and $g(x) = 0$.

For $x = 0$, $q = 0$ and $\lambda = (\frac{1}{2}, \frac{1}{2})^T$, it holds $(x, q, \lambda) \in \mathcal{B}_W$ and $d = (-1, 1)^T = (f_1(0), f_2(0))^T \in D_W$.

Let us show that $d \notin D_F$. If this were not true, then there would exist $\bar{p} = (\bar{p}_1, \bar{p}_2)$, $\bar{\lambda} = (\bar{\lambda}_1, \bar{\lambda}_2)^T \in int(\mathbb{R}^2_+)$ such that $(\bar{p}, \bar{\lambda}, d) \in \mathcal{B}_F$, i.e.

$$-\bar{\lambda}_1 + \bar{\lambda}_2 \leq -\bar{\lambda}_1 f_1^*(\bar{p}_1) - \bar{\lambda}_2 f_2^*(\bar{p}_2) + \inf_{x \in \mathbb{R}} \left(\bar{\lambda}_1 \bar{p}_1 + \bar{\lambda}_2 \bar{p}_2\right) x. \tag{4. 37}$$

But, $f_2^*(\bar{p}_2) = \sup_{x \in \mathbb{R}} \{\bar{p}_2 x + x^2 - 1\} = +\infty$, and this contradicts the inequality in (4. 37). In conclusion, $D_W \not\subseteq D_F$, and, so, $D_W \not\subseteq D_{FL}$, $D_W \not\subseteq D_\alpha \cap \mathbb{R}^m$, $\alpha \in \mathcal{F}$, and $D_W \not\subseteq D_1 \cap \mathbb{R}^m$ (cf. (4. 32)).

By (4. 32), Proposition 4.6 and Examples 4.11-4.13, we obtain in the general case the following scheme, for every $\alpha \in \mathcal{F}$,

$$\begin{array}{r} D_1 \cap \mathbb{R}^m \subsetneq D_\alpha \cap \mathbb{R}^m \subsetneq D_{FL} \subsetneq \begin{array}{l} D_F \subsetneq D_P \\ D_L \subsetneq \begin{array}{l} D_P \\ D_N \end{array} \end{array} \\ D_W \subsetneq D_L \subsetneq \begin{array}{l} D_P \\ D_N \end{array} \end{array} . \tag{4. 38}$$

For the last part of this section, let us assume that (A_f), (A_g) and (A_{CQ}) are fulfilled.

Proposition 4.7 *If (A_f), (A_g) and (A_{CQ}) are fulfilled, then it holds $D_W \subseteq D_1 \cap \mathbb{R}^m$.*

Proof. Let be $d = (d_1, ..., d_m)^T \in D_W$. Then there exists $(x, q, \lambda) \in \mathcal{B}_W$ such that $d = h^W(x, q, \lambda)$. Because

$$0 \in \partial \left(\sum_{i=1}^{m} \lambda_i f_i\right)(x) + \partial(q^T g)(x) = \sum_{i=1}^{m} \lambda_i \partial f_i(x) + \partial(q^T g)(x),$$

it follows that there exist $p_i \in \mathbb{R}^n, i = 1, ..., m$, such that $p_i \in \partial f_i(x), i = 1, ..., m$, and $-\sum\limits_{i=1}^{m} \lambda_i p_i \in \partial(q^T g)(x)$. As a consequence it follows (cf. [19])

$$f_i^*(p_i) + f_i(x) = p_i^T x, i = 1, ..., m, \tag{4. 39}$$

and

$$(q^T g)^* \left(-\sum_{i=1}^{m} \lambda_i p_i \right) + q^T g(x) = \left(-\sum_{i=1}^{m} \lambda_i p_i \right)^T x. \tag{4. 40}$$

Defining, for $j = 1, ..., m$,

$$t_j := p_j^T x + \left(-\sum_{i=1}^{m} \lambda_i p_i \right)^T x \in \mathbb{R},$$

then $\sum\limits_{i=1}^{m} \lambda_i t_i = 0$ and this means that $(p, q, \lambda, t) \in \mathcal{B}_1$, for $p = (p_1, ..., p_m)$. On the other hand, from (4. 39) and (4. 40) we have, for $j = 1, ..., m$,

$$\begin{aligned}
h_j^1(p, q, \lambda, t) &= -f_j^*(p_j) - (q^T g)^* \left(-\frac{1}{\sum\limits_{i=1}^{m} \lambda_i} \sum_{i=1}^{m} \lambda_i p_i \right) + t_j \\
&= -f_j^*(p_j) - (q^T g)^* \left(-\sum_{i=1}^{m} \lambda_i p_i \right) + t_j \\
&= f_j(x) - p_j^T x + q^T g(x) - \left(-\sum_{i=1}^{m} \lambda_i p_i \right)^T x + t_j \\
&= f_j(x) + q^T g(x) = d_j.
\end{aligned}$$

In conclusion, $d = h^1(p, q, \lambda, t) \in h^1(\mathcal{B}_1) = D_1$. □

Remark 4.14 For the problem described in Example 4.12 the assumptions (A_f), (A_g) and (A_{CQ}) are fulfilled and $d = (1, -1)^T \in D_1 \cap \mathbb{R}^2$, but $d \notin D_W$. This means that even in this case the inclusion $D_W \subseteq D_1 \cap \mathbb{R}^m$ may be strict.

So, if (A_f), (A_g) and (A_{CQ}) are fulfilled, then (4. 38) becomes, for every $\alpha \in \mathcal{F}$,

$$D_W \subsetneqq D_1 \cap \mathbb{R}^m \subsetneqq D_\alpha \cap \mathbb{R}^m \subsetneqq D_{FL} = D_F = D_L = D_P \subsetneqq D_N. \tag{4. 41}$$

Let us recall that in this situation we have, by (4. 29), the following equality, for every $\alpha \in \mathcal{F}$,

$$vmaxD_1 = vmaxD_\alpha = vmaxD_{FL} = vmaxD_F = vmaxD_L = vmaxD_P.$$

The next example shows that, even in this case, the sets $vmaxD_W$ and $vmaxD_P$ are in general not equal.

Example 4.14 For $m = 2, n = 1, k = 1$, $K = \mathbb{R}$, let be $f_1, f_2 : \mathbb{R} \to \overline{\mathbb{R}}$, $g : \mathbb{R} \to \mathbb{R}$, defined by

$$f_1(x) = f_2(x) = \begin{cases} x^2, & \text{if } x \in (0, +\infty), \\ +\infty, & \text{otherwise}, \end{cases} \quad \text{and } g(x) = 0.$$

It is obvious that (A_f), (A_g) and (A_{CQ}) are fulfilled. For $\lambda = (1,1)^T$ and $d = (0,0)^T$, we have $(\lambda, d) \in \mathcal{B}_P$ and $d \in D_P$. Moreover, $d \in vmaxD_P$.

We will show now that $d = (0,0)^T \notin D_W$. If this were not true, then there would exist $(\bar{x}, \bar{q}, \bar{\lambda}) \in \mathcal{B}_W$, with $\bar{\lambda} = (\bar{\lambda}_1, \bar{\lambda}_2)^T \in int(\mathbb{R}^2_+)$, $\bar{\lambda}_1 + \bar{\lambda}_2 = 1$, $\bar{q} \in K^* = \{0\}$ such that

$$d = (0,0)^T = (f_1(\bar{x}) + \bar{q}g(\bar{x}), f_2(\bar{x}) + \bar{q}g(\bar{x}))^T = (f_1(\bar{x}), f_2(\bar{x}))^T.$$

But $f_1(x) = f_2(x) > 0, \forall x \in \mathbb{R}$, and this leads to a contradiction. From here we obtain that $d = (0,0)^T \notin D_W$ and, obviously, $d = (0,0)^T \notin vmaxD_W$.

4.9 Weir-Mond multiobjective duality

The last section of this work is devoted to the study of the so-called Weir-Mond dual optimization problem. It has the following formulation (cf. [90], [92])

$$(D_{WM}) \quad \underset{(x,q,\lambda)\in\mathcal{B}_{WM}}{\text{v-max}} h^{WM}(x,q,\lambda),$$

with

$$h^{WM}(x,q,\lambda) = \begin{pmatrix} h_1^{WM}(x,q,\lambda) \\ \vdots \\ h_m^{WM}(x,q,\lambda) \end{pmatrix},$$

$$h_j^{WM}(x,q,\lambda) = f_j(x), j = 1, ..., m,$$

the dual variables

$$x \in \mathbb{R}^n, q \in \mathbb{R}^k, \lambda = (\lambda_1, \dots, \lambda_m)^T \in \mathbb{R}^m,$$

and the set of constraints

$$\mathcal{B}_{WM} = \left\{ (x,q,\lambda) : \quad x \in \mathbb{R}^n, \lambda = (\lambda_1, \dots, \lambda_m)^T \in int(\mathbb{R}^m_+), \sum_{i=1}^m \lambda_i = 1, q \underset{K^*}{\geqq} 0, \right.$$
$$\left. q^T g(x) \geq 0, 0 \in \partial\left(\sum_{i=1}^m \lambda_i f_i\right)(x) + \partial(q^T g)(x) \right\}.$$

The following theorems state the existence of weak and strong duality (cf. [90], [92]).

Theorem 4.23 (weak duality for (D_{WM})) *There is no $x \in \mathcal{A}$ and no element $(y,q,\lambda) \in \mathcal{B}_{WM}$ fulfilling $h^{WM}(y,q,\lambda) \underset{\mathbb{R}^m_+}{\geqq} f(x)$ and $h^{WM}(y,q,\lambda) \neq f(x)$.*

Theorem 4.24 (strong duality for (D_{WM})) *Assume that (A_f), (A_g) and (A_{CQ}) are fulfilled. If $\bar{x}$ is a properly efficient solution to (P), then there exists $\bar{q} \underset{K^*}{\geqq} 0$ and $\bar{\lambda} \in int(\mathbb{R}^m_+)$ such that $(\bar{x}, \bar{q}, \bar{\lambda}) \in \mathcal{B}_{WM}$ is a properly efficient solution to the dual (D_{WM}) and the strong duality $f(\bar{x}) = h^{WM}(\bar{x}, \bar{q}, \bar{\lambda})$ holds.*

Let be $D_{WM} := h^{WM}(\mathcal{B}_{WM}) \subseteq \mathbb{R}^m$. We are now interested in relating the image set D_{WM} to the image sets which appear in the relation (4. 38).

Proposition 4.8 *It holds $D_{WM} \subseteq D_L$.*

Proof. Let be $d = (d_1, ..., d_m)^T \in D_{WM}$. Then there exists $(x, q, \lambda) \in \mathcal{B}_{WM}$ such that $d = h^{WM}(x, q, \lambda) = f(x)$. Because

$$0 \in \partial \left(\sum_{i=1}^{m} \lambda_i f_i \right)(x) + \partial (q^T g)(x),$$

we have

$$\sum_{i=1}^{m} \lambda_i f_i(x) + q^T g(x) \leq \inf_{u \in \mathbb{R}^n} \left[\sum_{i=1}^{m} \lambda_i f_i(u) + q^T g(u) \right].$$

On the other hand,

$$\sum_{i=1}^{m} \lambda_i d_i = \sum_{i=1}^{m} \lambda_i f_i(x) \leq \sum_{i=1}^{m} \lambda_i f_i(x) + q^T g(x),$$

which implies

$$\sum_{i=1}^{m} \lambda_i d_i \leq \inf_{u \in \mathbb{R}^n} \left[\sum_{i=1}^{m} \lambda_i f_i(u) + q^T g(u) \right].$$

So, $(q, \lambda, d) \in \mathcal{B}_L$ and $d = h^L(q, \lambda, d) \in h^L(\mathcal{B}_L) = D_L$. □

Remark 4.15 For the problem considered in Example 4.11 we have that $d = (-1, -1)^T \in D_L$ and $d \notin D_W$. In a similar way it can be shown that $d = (1, -1)^T \notin D_{WM}$. This means that the inclusion $D_{WM} \subseteq D_L$ may be strict. From here it follows that $D_P \nsubseteq D_{WM}$ and $D_N \nsubseteq D_{WM}$ (cf. (4. 32)).

Remark 4.16 Let us consider now the problem in Example 4.12. It holds $d = (1, -1)^T \in D_1$. But one can verify that $d = (1, -1)^T \notin D_{WM}$, which implies that $D_1 \cap \mathbb{R}^m \nsubseteq D_{WM}$ and, from here, $D_\alpha \cap \mathbb{R}^m \nsubseteq D_{WM}$, $\alpha \in \mathcal{F}$, $D_{FL} \nsubseteq D_{WM}$, $D_F \nsubseteq D_{WM}$ and $D_P \nsubseteq D_{WM}$.

Remark 4.17 For the problem in Example 4.13, we have $d = (-1, 1)^T \notin D_F$ and, obviously, $d = (-1, 1)^T \in D_{WM}$. So, it holds $D_{WM} \nsubseteq D_F$ and, as a consequence, $D_{WM} \nsubseteq D_{FL}$, $D_{WM} \nsubseteq D_\alpha \cap \mathbb{R}^m$, $\alpha \in \mathcal{F}$, and $D_{WM} \nsubseteq D_1 \cap \mathbb{R}^m$.

Next we construct two other examples which show that between D_W and D_{WM} also does not exist any relation of inclusion.

Example 4.15 For $m = 2, n = 1, k = 1$, $K = \mathbb{R}_+$, let be $f_1, f_2 : \mathbb{R} \to \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$, defined by $f_1(x) = f_2(x) = 0$ and $g(x) = x^2 - 1$.

For $x = 0$, $q = 1$ and $\lambda = \left(\frac{1}{2}, \frac{1}{2}\right)^T$, it holds $(x, q, \lambda) \in \mathcal{B}_W$ and

$$d = (-1, -1)^T = (f_1(0) + qg_1(0), f_2(0) + qg_2(0))^T \in D_W.$$

On the other hand, $d \notin D_{WM}$, which means that $D_W \nsubseteq D_{WM}$.

Example 4.16 For $m = 2, n = 1, k = 1$, $K = \mathbb{R}_+$, let be $f_1, f_2 : \mathbb{R} \to \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$, defined by $f_1(x) = x$, $f_2(x) = x$ and $g(x) = -x + 1$.

For $x = \frac{1}{2}$, $q = 1$ and $\lambda = (\frac{1}{2}, \frac{1}{2})^T$, it holds $qg\left(\frac{1}{2}\right) = \frac{1}{2} \geq 0$ and

$$\inf_{x \in \mathbb{R}} [\lambda_1 f_1(x) + \lambda_2 f_2(x) + qg(x)] = 1,$$

which means that $(x, q, \lambda) \in \mathcal{B}_{WM}$ and $d = \left(\frac{1}{2}, \frac{1}{2}\right)^T = (f_1(\frac{1}{2}), f_2(\frac{1}{2}))^T \in D_{WM}$.

Let us prove that $d \notin D_W$. If this were not true, then there would exist $(\bar{x}, \bar{q}, \bar{\lambda}) \in \mathcal{B}_W$ such that

$$d = \left(\frac{1}{2}, \frac{1}{2}\right)^T = (f_1(\bar{x}) + \bar{q}g(\bar{x}), f_2(\bar{x}) + \bar{q}g(\bar{x}))^T = (\bar{x} + \bar{q}(-\bar{x}+1), \bar{x} + \bar{q}(-\bar{x}+1))^T. \tag{4. 42}$$

Because $(\bar{x}, \bar{q}, \bar{\lambda}) \in \mathcal{B}_W$, we have

$$\inf_{x \in \mathbb{R}} [\bar{\lambda}_1 f_1(x) + \bar{\lambda}_2 f_2(x) + \bar{q}g(x)] = \bar{\lambda}_1 f_1(\bar{x}) + \bar{\lambda}_2 f_2(\bar{x}) + \bar{q}g(\bar{x}),$$

or, equivalently,

$$\inf_{x \in \mathbb{R}} [x + \bar{q}(-x+1)] = \bar{x} + \bar{q}(-\bar{x}+1).$$

This is true just if $\bar{q} = 1$. But, in this case, (4. 42) leads us to a contradiction.

In conclusion, $D_{WM} \not\subseteq D_W$.

In the general case we get the following scheme, for every $\alpha \in \mathcal{F}$,

$$\begin{array}{rl}
D_1 \cap \mathbb{R}^m \subsetneq D_\alpha \cap \mathbb{R}^m \subsetneq D_{FL} \subsetneq & \begin{array}{l} D_F \subsetneq D_P \\ D_L \subsetneq \begin{array}{l} D_P \\ D_N \end{array} \end{array} \\
D_W \subsetneq D_L \subsetneq & \begin{array}{l} D_P \\ D_N \end{array} \\
D_{WM} \subsetneq D_L \subsetneq & \begin{array}{l} D_P \\ D_N \end{array}
\end{array} . \tag{4. 43}$$

Let us try now to find out how is this scheme changing under the fulfilment of the assumptions (A_f), (A_g) and (A_{CQ}). From (4. 41) we have, for every $\alpha \in \mathcal{F}$,

$$D_W \subsetneq D_1 \cap \mathbb{R}^m \subsetneq D_\alpha \cap \mathbb{R}^m \subsetneq D_{FL} = D_F = D_L = D_P \subsetneq D_N.$$

Remark 4.18 Let us notice that for the problem formulated in Example 4.15 (A_f), (A_g) and (A_{CQ}) are fulfilled. But, $D_W \not\subseteq D_{WM}$, which implies $D_1 \cap \mathbb{R}^m \not\subseteq D_{WM}$, $D_\alpha \cap \mathbb{R}^m \not\subseteq D_{WM}$, $\alpha \in \mathcal{F}$, and $D_{FL} = D_F = D_L = D_P \not\subseteq D_{WM}$.

Remark 4.19 For the problem presented in Example 4.16 we proved that $d = \left(\frac{1}{2}, \frac{1}{2}\right)^T \in D_{WM}$. By using some calculation techniques concerning conjugate functions, it can be also proved that $d = \left(\frac{1}{2}, \frac{1}{2}\right)^T \notin D_\alpha$, for every $\alpha \in \mathcal{F}$. In conclusion, $D_{WM} \not\subseteq D_\alpha \cap \mathbb{R}^m$, $\alpha \in \mathcal{F}$, and, from here, $D_{WM} \not\subseteq D_1 \cap \mathbb{R}^m$, even if (A_f), (A_g) and (A_{CQ}) are fulfilled.

By the last two remarks, using (4. 41), if (A_f), (A_g) and (A_{CQ}) are fulfilled, we get the following scheme, for every $\alpha \in \mathcal{F}$,

$$D_W \subsetneq D_1 \cap \mathbb{R}^m \subsetneq D_\alpha \cap \mathbb{R}^m \subsetneq D_{FL} = D_F = D_L = D_P \subsetneq D_N,$$

and

$$D_{WM} \subsetneq D_{FL} = D_F = D_L = D_P \subsetneq D_N,$$

and no other relation of inclusion holds between these sets.

Remark 4.20 For the problem in Example 4.14 we have $d = (0,0)^T \in vmaxD_P$, but $d \notin vmaxD_W$ and $d \notin vmaxD_{WM}$. This means that $vmaxD_P \not\subseteq vmaxD_W$ and $vmaxD_P \not\subseteq vmaxD_{WM}$ and we notice that, even if (A_f), (A_g) and (A_{CQ}) are fulfilled, these sets may be different.

Remark 4.21 The question concerning finding some necessary or sufficient conditions for which the sets $vmaxD_P$, $vmaxD_W$ and $vmaxD_{WM}$ coincide is still open.

Theses

1. The central point of this work is represented by the study of the duality for a convex multiobjective optimization problem of the form

$$(P) \quad \underset{x \in \mathcal{A}}{\text{v-min}} \, f(x),$$

$$\mathcal{A} = \left\{ x \in \mathbb{R}^n : g(x) = (g_1(x), \ldots, g_k(x))^T \underset{K}{\leqq} 0 \right\},$$

where $f(x) = (f_1(x), \ldots, f_m(x))^T$, $f_i : \mathbb{R}^n \to \overline{\mathbb{R}}$, $i = 1, ..., m$, are proper functions, $g_j : \mathbb{R}^n \to \mathbb{R}$, $j = 1, ..., k$, and $K \subseteq \mathbb{R}^k$ is assumed to be a convex closed cone which defines a partial ordering on $\mathbb{R}^k$. To (P) is associated the following scalarized optimization problem

$$(P^\lambda) \quad \inf_{x \in \mathcal{A}} \sum_{i=1}^{m} \lambda_i f_i(x),$$

for $\lambda = (\lambda_1, ..., \lambda_m)^T \in int(\mathbb{R}^m_+)$. A scalar dual to it is constructed and the optimality conditions are derived. The structure of the scalar dual suggests the form of the multiobjective dual (D) to (P). Weak, strong and converse duality between (P) and (D) are proved (see also Wanka and Boţ [85]).

2. To study the duality for the scalarized problem (P^λ) the conjugacy approach is used. A deeper look in the usage of this approach in developing duality theories for scalar optimization problems is taken. To the problem

$$(P^s) \quad \inf_{x \in G} f(x),$$

$$G = \left\{ x \in X : g(x) = (g_1(x), \ldots, g_k(x))^T \underset{K}{\leqq} 0 \right\},$$

where $X \subseteq \mathbb{R}^n$ is a non-empty set, $K \subseteq \mathbb{R}^k$ a is non-empty closed convex cone with $int(K) \neq \emptyset$, $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ and $g : \mathbb{R}^n \to \mathbb{R}^k$, three different dual problems are constructed, namely, the well-known Lagrange and Fenchel duals (denoted by (D^s_L) and (D^s_F), respectively) and a "combination" of the above two, called the Fenchel-Lagrange dual (denoted by (D^s_{FL})). The ordering relations between the optimal objective values of the duals are verified and it is proved that, under convexity assumptions on the sets and functions involved and some regularity conditions, they become equal. Moreover, it is shown that these assumptions guarantee the existence of strong duality between (D^s_L), (D^s_F), (D^s_{FL}) and (P^s). By means of strong duality the optimality conditions for each of these problems are established.

3. Concerning the three duals, it is also mentioned how is possible to weaken the convexity and regularity assumptions in a way that the optimal objective

values of (D_L^s), (D_F^s) and (D_{FL}^s) remain equal and the strong duality results still hold. This offers the possibility to include in the above considerations optimization problems (P^s) for which the ordering cone K does not need to have a non-empty interior. On the other hand, instead of the convexity of the sets and functions involved it is enough to consider the weaker concept of nearly convexity (cf. [1], [31]).

4. As another application of the conjugacy approach, the duality for an optimization problem with the objective function being a composite of a convex and componentwise increasing function with a convex vector function

$$(P^c) \quad \inf_{x\in X} f(g(x)) = f(g_1(x), \dots, g_m(x)),$$

where $(X, \|\cdot\|)$ is a normed space, $f : \mathbb{R}^m \to \mathbb{R}$ and $g : X \to \mathbb{R}^m$, $g(x) = (g_1(x), \dots, g_m(x))^T$, is studied. By using some appropriate perturbations a dual problem to (P^c) is constructed. The existence of strong duality is proved and the optimality conditions are derived. For the single facility location problem in which the existing facilities are represented by sets of points (see also [57]) a dual problem and the optimality conditions are introduced. The duality for the classical Weber problem and minmax problem with demand sets is also studied as particular instances of (P^c).

5. The insights concerning duality for the general multiobjective optimization problem (P) give the possibility to deal with the duality for some particular cases of it. Considering the problem with a convex objective vector function and linear inequality constraints, some former duality results are (cf. Wanka and Boţ [83], [84]) rediscovered. On the other hand, a multiobjective dual for the vector problem with a convex objective function and positive semidefinite constraints is proposed.

6. After the same scheme, as in the case of the problem (P), a duality approach is presented for the multiobjective fractional programming problem

$$(P_r) \quad \operatorname*{v-min}_{x\in\mathcal{A}_r} \left(\frac{f_1^2(x)}{g_1(x)}, \dots, \frac{f_m^2(x)}{g_m(x)}\right)^T,$$

$$\mathcal{A}_r = \left\{ x \in \mathbb{R}^n : Cx \underset{\mathbb{R}^l_+}{\leqq} b \right\}.$$

Here, the functions f_i and $g_i, i = 1, ..., m$, mapping from $\mathbb{R}^n$ into $\mathbb{R}$, are assumed to be convex and concave, respectively, such that for all $x \in \mathcal{A}_r$ and $i = 1, \dots, m$, $f_i(x) \geq 0$ and $g_i(x) > 0$ hold. For $\lambda = (\lambda_1, ..., \lambda_m)^T \in int(\mathbb{R}^m_+)$, the scalarized problem

$$(P_r^\lambda) \quad \inf_{x\in\mathcal{A}_r} \sum_{i=1}^m \lambda_i \frac{f_i^2(x)}{g_i(x)}$$

is associated to (P_r) and, by the use of the conjugacy approach, a dual to (P_r^λ) is found. This leads to the formulation of a multiobjective dual (D_r) to (P_r). Weak and strong duality between (P_r) and (D_r) is proved (see also Wanka and Boţ [10]).

7. In addition to (D), for the primal problem (P) with cone inequality constraints, other six multiobjective duals are introduced. Their construction bases on the structure of the Lagrange, Fenchel and Fenchel-Lagrange scalar

duals. Among the six duals one can recognize a generalization of (D) and, on the other hand, the dual introduced by Jahn in [40] and [41], here in the finite dimensional case.

8. In order to relate these duals to each other, some inclusion relations between the image sets of the vector objective functions on their corresponding admissible sets are verified. It is shown by some counter-examples that these sets are not always equal. The same analysis is done for the maximal elements sets of the image sets. Some necessary conditions for which these sets become identical are given.

9. The investigations referring to the six duals of (P) are completed by comparing them to some other duals mentioned in the literature. A general scheme containing the relations between all these duals is derived. This scheme includes the duality concepts of Nakayama (cf. [54], [55]), Wolfe (cf. [90], [93]) and Weir and Mond (cf. [90], [92]).

Index of notation

$\mathbb{N}$	the set of natural numbers
$\mathbb{Q}$	the set of rational numbers
$\mathbb{R}$	the set of real numbers
$\overline{\mathbb{R}}$	the extended set of real numbers
$\mathbb{R}^{k\times k}$	the set of $k \times k$ matrices with real entries
$\mathbb{R}^m_+$	the non-negative orthant of $\mathbb{R}^m$
S^k	the set of symmetric $k \times k$ matrices
S^k_+	the cone of symmetric positive semidefinite $k \times k$ matrices
K^*	the dual cone of the cone K
$int(X)$	the interior of the set X
$ri(X)$	the relative interior of the set X
$cl(X)$	the closure of the set X
$aff(X)$	the affine hull of the set X
$dom(f)$	the domain of the function f
$epi(f)$	the epigraph of the function f
$epi(f; D)$	the epigraph of the function f on the set D
$epi_C(g; E)$	the epigraph of the function g on the set E with respect to the cone C
f^*	the conjugate of the function f
∂f	the subdifferential of the function f
χ_G	the indicator function of the set G
$\underset{K}{\geqq}$	the partial ordering induced by the cone K
$\underset{K^*}{\geqq}$	the partial ordering induced by the dual cone K^*
$\underset{\mathbb{R}^m_+}{\geqq}$	the partial ordering induced by the non-negative orthant $\mathbb{R}^m_+$
$\underset{S^k_+}{\geqq}$	the partial ordering induced by the cone S^k_+
$\langle \cdot, \cdot \rangle$	the bilinear pairing between a topological vector space and its topological dual
$Tr(A)$	the trace of the matrix $A \in \mathbb{R}^{k\times k}$
Φ^0	the dual norm of the norm Φ

$v-min$	the notation for a multiobjective optimization problem in the sense of minimum
$v-max$	the notation for a multiobjective optimization problem in the sense of maximum
$inf(P^s)$	the optimal objective value of the scalar minimum optimization problem (P^s)
$sup(D^s)$	the optimal objective value of the scalar maximum optimization problem (D^s)
$max(D^s)$	the notation for the optimal objective value $sup(D^s)$ when this is attained
$vminA$	the set of the minimal elements of the set $A \subseteq \mathbb{R}^m$ relative to the ordering cone $\mathbb{R}^m_+$
$vmaxA$	the set of the maximal elements of the set $A \subseteq \mathbb{R}^m$ relative to the ordering cone $\mathbb{R}^m_+$
$A \subsetneq B$	the set A is included in the set B but the inclusion may be strict
$A \nsubseteq B$	the set A is not included in the set B
$x \gneqq y$	the notation for $x \underset{\mathbb{R}^m_+}{\geqq} y$, but $x \neq y$, $x, y \in \mathbb{R}^m$

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Lebenslauf

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Erklärung gemäß §6 der Promotionsordnung

Hiermit erkläre ich an Eides Statt, dass ich die von mir eingereichte Arbeit "Duality and optimality in multiobjective optimization" selbstständig und nur unter Benutzung der in der Arbeit angegebenen Hilfsmittel angefertigt habe.

Chemnitz, den 10.01.2003 Radu Ioan Boţ