There are 3 problems in total. Please write down your solution on another piece of paper and submit it to your TA on or before Mar. 20, 2018

1. Let A = $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & -2 & 1 \\ 1 & -1 & 1 & 1 \end{bmatrix}$

   (a) Find a singular value decomposition for A

   (b) Find an orthogonal basis for the range of A

   (c) Find an orthogonal basis for the null space of A^T

2. Singular value decomposition of A denote as $A = U\Sigma V^T$. Assume A has l positive singular values, i.e.,

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_l > \sigma_{l+1} = \cdots = \sigma_n = 0$$

   For $1 \leq k \leq l$, obtain Σ_k by setting the singular values $\sigma_{k+1},\ \sigma_{k+2}, ...,\ \sigma_l$ to zeros in the matrix Σ. Let $A_k = U\Sigma_k V^T$. Show $\|A - A_k\| = \sigma_{k+1}$.

3. Our data vector b has been measured with some error. Let b_{true} be the true but unknown data, and let $Ax_{true} = b_{true}$.

   (a) The columns of the matrix $V = [v_1, ..., v_n]$ form an orthonormal basis for n-dimensional space. Let's express the solution x_{true} to the least squares problem as

$$x_{true} = w_1 v_1 + \cdots + w_n v_n$$

   . Determine a formula for w_i $(i = 1, ..., n)$ in terms of $U,\ b_{true}$ and the singular values of A.

   (b) Use the two statements below and the fact that $\|A\| = \sigma_1$

$$A(x - x_{true}) = (b - b_{true} - r) \ means \ \|x - x_{true}\| \leq \frac{1}{\sigma_n}(\|b - b_{true} - r\|)$$

$$b_{true} = Ax_{true} \ means \ \|b_{true}\| = \|Ax_{true}\| \leq \|A\| \, \|x_{true}\|$$

   . Derive an upper bound on $\frac{\|x - x_{true}\|}{\|x_{true}\|}$ in terms of the condition number $\kappa(A) \equiv \frac{\sigma_1}{\sigma_n}$ and $\frac{\|b - b_{true}\|}{\|b_{true}\|}$.