Implicit Method for finding Green's Function Coefficients for ARMA Models

Green's function of the ARMA(2,1) system -- implicit method

$$X_t - \phi_1 X_{t-1} - \phi_2 X_{t-2} = a_t - \theta_1 a_{t-1}$$

or

$$(1- \phi_1 B - \phi_2 B^2)\, X_t = (1- \theta_1 B)\, a_t$$

using

$$X_t = \sum_{j=0}^{\infty} G_j a_{t-j} = \left(\sum_{j=0}^{\infty} G_j B^j\right) a_t$$

$$(1- \phi_1 B - \phi_2 B^2)\left(\sum_{j=0}^{\infty} G_j B^j\right) a_t = (1- \theta_1 B)\, a_t$$

$$(1- \phi_1 B - \phi_2 B^2)(G_0 + G_1 B + G_2 B^2 + \ldots) \equiv (1- \theta_1 B)$$

$$0:\ G_0 = 1$$

$$1:\ G_1 - \phi_1 = -\theta_1 \quad \Rightarrow \quad G_1 = \phi_1 - \theta_1$$

$$(1- \phi_1 B - \phi_2 B^2)\, G_j = 0 \qquad j \geq 2$$

For ARMA(n, n-1) model,

$$(1- \phi_1 B - \phi_2 B^2 - \ldots - \phi_n B^n)(G_0 + G_1 B + G_2 B^2 + \ldots) \equiv (1- \theta_1 B - \theta_2 B^2 - \ldots - \theta_{n-1} B^{n-1})$$

$$0:\ G_0 = 1$$

$$1:\ G_1 - \phi_1 G_0 = -\theta_1$$

$$2:\ G_2 - \phi_1 G_1 - \phi_2 G_0 = -\theta_2$$

$$n\text{-}1:\ G_{n-1} - \phi_1 G_{n-2} - \phi_2 G_{n-3} - \ldots - \phi_{n-1} G_0 = -\theta_{n-1}$$

$$n:\ G_n - \phi_1 G_{n-1} - \phi_2 G_{n-2} - \ldots - \phi_n G_0 = 0$$

$$\text{i.e.,} \quad (1- \phi_1 B - \phi_2 B^2 - \ldots - \phi_n B^n)\, G_j = 0 \qquad j \geq n$$

Discussion on Green's function coefficients for ARMA(2,1) models

$$(1 - \phi_1 B - \phi_2 B^2) = (1 - \lambda_1 B)(1 - \lambda_2 B)$$

$$\lambda_1 + \lambda_2 = \phi_1$$

$$\lambda_1 \lambda_2 = -\phi_2$$

$$\lambda^2 - \phi_1 \lambda - \phi_2 = 0 \qquad \lambda_1,\ \lambda_2 = \frac{1}{2}\left(\phi_1 \pm \sqrt{\phi_1^2 + 4\phi_2}\right)$$

Real distinct roots:

$$X_t = \frac{(1-\theta_1 B)\, a_t}{(1-\phi_1 B - \phi_2 B^2)} = \frac{(1-\theta_1 B)\, a_t}{(1-\lambda_1 B)(1-\lambda_2 B)} = \left[\frac{(\lambda_1 - \theta_1)}{(\lambda_1 - \lambda_2)}\frac{1}{(1-\lambda_1 B)} + \frac{(\lambda_2 - \theta_1)}{(\lambda_2 - \lambda_1)}\frac{1}{(1-\lambda_2 B)}\right] a_t$$

$$= \sum_{j=0}^{\infty}\left[\left(\frac{\lambda_1 - \theta_1}{\lambda_1 - \lambda_2}\right)\lambda_1^j + \left(\frac{\lambda_2 - \theta_1}{\lambda_2 - \lambda_1}\right)\lambda_2^j\right] a_{t-j}$$

$$G_j = \left(\frac{\lambda_1 - \theta_1}{\lambda_1 - \lambda_2}\right)\lambda_1^j + \left(\frac{\lambda_2 - \theta_1}{\lambda_2 - \lambda_1}\right)\lambda_2^j$$

Above explicit form of Green's function can also be derived as the solution of an nth order homogeneous difference equation:

$$(1 - \phi_1 B - \phi_2 B^2)\, G_j = 0$$

with initial conditions of

$$G_0 = 1 \qquad G_1 = \phi_1 - \theta_1$$

The solution of the difference equation is a linear combination of terms, λ^j,

$$G_j = g_1 \lambda_1^j + g_2 \lambda_2^j$$

$$G_0 = g_1 + g_2 = 1$$

$$G_1 = g_1 \lambda_1 + g_2 \lambda_2 = \phi_1 - \theta_1$$

Thus,

$$g_1 = \frac{\lambda_1 - \theta_1}{\lambda_1 - \lambda_2} \qquad g_2 = \frac{\lambda_2 - \theta_1}{\lambda_2 - \lambda_1}$$

Complex Roots:

$$\phi_1^2 + 4\phi_2 < 0$$

$$\lambda_1, \lambda_2 = \frac{1}{2}(\phi_1 \pm \sqrt{\phi_1^2 + 4\phi_2}\ ) = r\,e^{\pm i\,\omega}$$

$$r = |\lambda_1| = |\lambda_2| = \sqrt{-\phi_2}$$

$$\omega = \cos^{-1}\frac{\phi_1}{2\sqrt{-\phi_2}} = \cos^{-1}\frac{(\lambda_1+\lambda_2)}{2\sqrt{\lambda_1\lambda_2}}$$

$$g_1, g_2 = g\,e^{\pm i\,\beta}$$

$$G_j = g_1\lambda_1^j + g_2\lambda_2^j = g\,e^{i\,\beta}(r\,e^{i\,\omega})^j + g\,e^{-i\,\beta}(r\,e^{-i\,\omega})^j$$

* Green's function for special models

AR(2): $\theta_1=0$

$$G_j = \frac{1}{\lambda_1 - \lambda_2}[\,\lambda_1^{j+1} - \lambda_2^{j+1}]$$