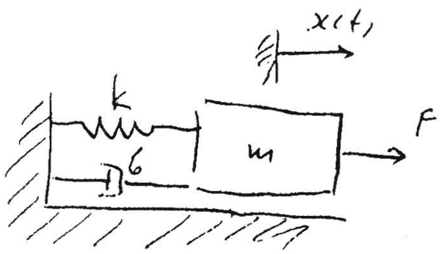
Equidistant Sampling of Canonical 2nd Order Continuous Time Systems Driven by White Noise

$$X''(t) + 2\xi\omega_n X'(t) + \omega_n^2 X(t) = U(t) \qquad t \in R \quad (**)$$

$X(t)$ - response of the system $\qquad$ U - input into the system

ξ - damping ratio $\qquad$ ω_n - natural frequency



Example of a mass-spring-damper system

$$m\ddot{x}(t) + b\dot{x}(t) + kx(t) = F(t)$$

$$\ddot{x}(t) + \underbrace{\frac{b}{m}}_{2\xi\omega_n}\dot{x}(t) + \underbrace{\frac{k}{m}}_{\omega_n^2}x(t) = \underbrace{\frac{1}{m}F(t)}_{U(t)}$$

$$\omega_n^2 = \frac{k}{m} \qquad \xi = \frac{b}{2\sqrt{km}}$$

I.e. 2^{nd} order cont. time differential eqn of form $(**)$ describes a 1 deg. of freedom vibration system!

Solution to $(**)$ can be found as

$$X(t) = \int_0^\infty G(u)\,U(t-u)\,du$$

$G(u)$ – impulse response of $(**)$ (i.e. when one replaces $U(t)$ with $\delta(t)$)

$$G(t) = C_1 e^{\mu_1 t} + C_2 e^{\mu_2 t}$$

μ_1 & μ_2 are roots of

$$s^2 + 2\xi\omega_n s + \omega_n^2 = 0$$

$$\mu_{1,2} = -\xi\omega_n \pm \sqrt{\omega_n^2(\xi^2-1)} =$$

$$= -\xi\omega_n \pm \omega_n\sqrt{\xi^2-1}$$

C_1 & C_2 can be found from initial conditions $G(0) = 0$ $G'(0) = 1$

(remember that theorem from last time)

$G(t)$ for a continuous differential eqn of the form

$$X^{(n)}(t) + \alpha_{n-1} X^{(n-1)}(t) + \dots + \alpha_0 X(t) = U(t)$$

can be found by solving

$$G^{(n)}(t) + \alpha_{n-1} G^{(n-1)}(t) + \dots + \alpha_0 G(t) = 0$$

with initial conditions $G(0) = 0$, $G'(0) = 0, \dots, G^{(n-2)}(0) = 0$,

$$G^{(n-1)}(0) = 1$$

$$G(t) = \frac{e^{\mu_1 t} - e^{\mu_2 t}}{\mu_1 - \mu_2}$$

For $\xi > 1$, μ_1 & μ_2 are real numbers

$$G(t) = \frac{e^{-\xi\omega_n t} \sinh[\omega_n\sqrt{\xi^2-1}\, t]}{\omega_n\sqrt{\xi^2-1}}$$

For $\xi < 1$, $\mu_1 = \mu_2^* \in \mathbb{C}$

$$G(t) = \frac{e^{-\xi \omega_n t} \sin[\omega_n \sqrt{1-\xi^2} \cdot t]}{\omega_n \sqrt{1-\xi^2}}$$

Special case → when $\xi = 1$, then $G(t) = C_1 e^{-\omega_n t} + C_2 t e^{-\omega_n t}$

$$G(t) = t \cdot e^{-\omega_n t}$$

We will not explicitly analyze this situation.

Stability conditions

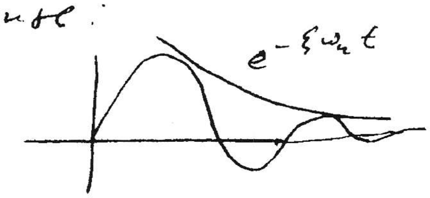
i) $\xi \omega_n >$ (i.e. $Re\,\mu_1 = Re\,\mu_2 < 0$) => stable system

ii) $\xi \omega_n = 0$ => marginally stable system

iii) $\xi \omega_n < 0$ (i.e. $Re\,\mu_1 = Re\,\mu_2 > 0$) => unstable system

Character of the impulse response:

(a) $0 < \xi < 1$ oscillatory



(b) $\xi > 1$ → sum of 2 exponentials

If the vibration system is driven by white noise, i.e. if the system model is

$$X''(t) + 2\xi\omega_n X'(t) + \omega_n^2 X(t) = Z(t)$$

(***)

where $E[Z(t)] = 0$, $E[Z(t) Z(t+s)] = \sigma_Z^2 \delta(s)$, then the resulting model is a 2^{nd} order continuous-time auto-regressive model $A(2)$

$$\gamma(s) = \int_0^\infty G(t) G(t+s)\, dt = \frac{\sigma_Z^2}{2\mu_1\mu_2(\mu_1^2 - \mu_2^2)} (\mu_2 e^{\mu_1 s} - \mu_1 e^{\mu_2 s})$$

(s is assumed to be positive)

$$\gamma(0) = -\frac{\sigma_Z^2}{2\mu_1\mu_2(\mu_1+\mu_2)} = \frac{\sigma_Z^2}{4\zeta\omega_n^3}$$

<u>Uniform sampling of $A(2)$ models:</u>

I Impulse response equivalent sampling

II Covariance function equivalent sampling

I $\quad G(\Delta l) = G_l$

$$G(\Delta l) = \frac{e^{\mu_1 \Delta l}}{\mu_1 - \mu_2} - \frac{e^{\mu_2 \Delta l}}{\mu_1 - \mu_2} =$$

$$= \frac{1}{\mu_1 - \mu_2} (e^{\mu_1 \Delta})^l + \frac{1}{\mu_2 - \mu_1} (e^{\mu_2 \Delta})^l =$$

$$= C_1 \lambda_1^l + C_2 \lambda_2^l \rightarrow \text{Green's fcn of an ARMA(2,1) model}$$

$$\lambda_1 = e^{\mu_1 \Delta} \qquad \lambda_2 = e^{\mu_2 \Delta}$$

$$\phi_1 = \lambda_1 + \lambda_2 = e^{-\xi \omega_n \Delta} \left(e^{\omega_n \Delta \sqrt{\xi^2 - 1}} + e^{-\omega_n \Delta \sqrt{\xi^2 - 1}} \right)$$

$$\phi_2 = -\lambda_1 \lambda_2 = -e^{-(\mu_1 + \mu_2)\Delta} = -e^{-2\xi \omega_n \Delta}$$

Parameter θ_1 cannot be solved to yield the form we've seen so far (ARMA(2,1)). Note that $G(0) = 0$ and hence, we must have $G_0 = 0$, not 1!

Let's see what we're getting.

$$G_\ell = G(\ell \Delta) = \frac{1}{\mu_1 - \mu_2} \lambda_1^\ell - \frac{1}{\mu_1 - \mu_2} \lambda_2^\ell$$

$$\lambda_1 = e^{\mu_1 \Delta}, \quad \lambda_2 = e^{\mu_2 \Delta}$$

$$X_t = \left(\sum_\ell G_\ell B^\ell \right) a_t =$$

$$= \frac{1}{\mu_1-\mu_2}\left(\sum_l \lambda_1^l B^l\right)a_t - \frac{1}{\mu_1-\mu_2}\left(\sum \lambda_2^l B^l\right)a_t =$$

$$= \frac{1}{\mu_1-\mu_2}\left[\frac{1}{1-\lambda_1 B} - \frac{1}{1-\lambda_2 B}\right]a_t =$$

$$= \frac{1}{\mu_1-\mu_2}\left(\frac{1-\lambda_2 B - 1 + \lambda_1 B}{1-(\lambda_1+\lambda_2)B + \lambda_1\lambda_2 B^2}\right)a_t =$$

$$= \frac{1}{\mu_1-\mu_2}\left(\frac{\overbrace{(\lambda_1-\lambda_2)}^{\theta_1} B}{1-\underbrace{(\lambda_1+\lambda_2)}_{\phi_1}B + \underbrace{\lambda_1\lambda_2}_{\phi_2} B^2}\right)a_t$$

$$\phi_1 = \lambda_1+\lambda_2 = e^{-\xi\omega_n\Delta}\left(e^{\Delta\omega_n\sqrt{\xi^2-1}} + e^{-\Delta\omega_n\sqrt{\xi^2-1}}\right)$$

$$\phi_2 = -\lambda_1\lambda_2 = -e^{-2\xi\omega_n\Delta}$$

$$\theta_1 = \frac{\lambda_1-\lambda_2}{\mu_1-\mu_2} = \frac{e^{-\xi\omega_n\Delta}\left(e^{\omega_n\Delta\sqrt{\xi^2-1}} - e^{-\omega_n\Delta\sqrt{\xi^2-1}}\right)}{2\omega_n\sqrt{\xi^2-1}}$$

The resulting model is $\boxed{X_t - \phi_1 X_{t-1} - \phi_2 X_{t-2} = \theta_1 a_{t-1}}$

II Covariance function equivalent sampling

$$\gamma(k\Delta) = \gamma_k \Rightarrow \gamma_k = \frac{\sigma_z^2}{2\mu_1\mu_2(\mu_1^2-\mu_2^2)} \cdot$$

$$\cdot \left[\mu_2 e^{\mu_1 \Delta k} - \mu_1 e^{\mu_2 \Delta k}\right] =$$

$$d_1 \lambda_1^k + d_2 \lambda_2^k \quad \text{where}$$

$$\lambda_1 = e^{\mu_1 \Delta}, \quad \lambda_2 = e^{\mu_2 \Delta}$$

$\Rightarrow$ Once again, we have a second order AR portion of our model and parameters ϕ_1 and ϕ_2 will be the same as they were when we pursued the impulse responce equivalence

$$\phi_1 = \lambda_1 + \lambda_2 = e^{-\xi\omega_n\Delta}\left(e^{-\omega_n\Delta\sqrt{\xi^2-1}} + e^{\omega_n\Delta\sqrt{\xi^2-1}}\right); \quad \phi_2 = -\lambda_1\lambda_2 =$$

$$= -e^{-2\xi\omega_n\Delta}$$

$$d_1 = \frac{\sigma_z^2}{2\mu_2(\mu_1^2-\mu_2^2)} \qquad d_2 = \frac{\sigma_z^2}{2\mu_2(\mu_2^2-\mu_1^2)}$$

Remember, variance decomposition coeffs of ARMA(2,1) model $X_t - \phi_1 X_{t-1} - \phi_2 X_{t-2} = a_t - \theta_1 a_{t-1}$

are

$$d_1 = \frac{\sigma_a^2(\lambda_1-\theta_1)}{(\lambda_1-\lambda_2)^2}\left[\frac{\lambda_1-\theta_1}{1-\lambda_1^2} - \frac{\lambda_2-\theta_1}{1-\lambda_1\lambda_2}\right] \qquad (1)$$

$$d_2 = \frac{\sigma_a^2(\lambda_2-\theta_1)}{(\lambda_1-\lambda_2)^2}\left[\frac{\lambda_2-\theta_1}{1-\lambda_2^2} - \frac{\lambda_1-\theta_1}{1-\lambda_1\lambda_2}\right] \qquad (2)$$

Solving (1) & (2) for σ_a^2 & θ_1 yields (Sec. 7.3.2.)

$$\theta_1^2 + 2\rho\theta_1 + 1 = 0 \qquad \left(2\rho = -\left(\theta_1 + \frac{1}{\theta_1}\right) \to (3)\right)$$

where

$$\rho = \frac{1}{2}\,\frac{-\mu_1(1+\lambda_1^2)(1-\lambda_2^2) + \mu_2(1+\lambda_2^2)(1-\lambda_1^2)}{\mu_1\lambda_1(1-\lambda_2^2) - \mu_2\lambda_2(1-\lambda_1^2)}$$

Look at 3 → If θ_1 is a solution, so is $\frac{1}{\theta_1}$ (there are 2 solutions). Always pick $|\theta_1|<1$ to have an invertible model (unless there is a reason ~~to~~ think otherwise).

$$\sigma_a^2 = \frac{\mu_2(1+\lambda_2^2)(1-\lambda_1^2) - \mu_1(1+\lambda_1^2)(1-\lambda_2^2)}{2\mu_1\mu_2(\mu_1^2-\mu_2^2)(1+\theta_1^2)} \sigma_z^2$$

See sec. 7.3.2. for more detailed & case-specific (but friendly equations).

Let us now do the problem of transforming ARMA models into A models (discrete into continuous).

<u>Case 1</u>

λ_1 & λ_2 are real $\Rightarrow$ μ_1 and μ_2 must be real too

$$\text{Let } \mu_{1/2} = -a \pm b = -\xi\omega_n \pm \omega_n\sqrt{\xi^2-1}$$

$$\phi_1 = \lambda_1 + \lambda_2 = e^{-a\Delta}(e^{b\Delta} + e^{-b\Delta})$$

$$\phi_2 = -\lambda_1\lambda_2 = -e^{-2a\Delta} \Rightarrow a = -\frac{1}{2\Delta}\ln(-\phi_2)$$

$$\frac{\phi_1}{\sqrt{-\phi_2}} = e^{b\Delta} + e^{-b\Delta} \Rightarrow (e^{b\Delta})^2 - \frac{\phi_1}{\sqrt{-\phi_2}} e^{b\Delta} + 1 = 0$$

$$e^{b\Delta} = \frac{1}{2}\left[\frac{\phi_1}{\sqrt{-\phi_2}} \pm \sqrt{\frac{\phi_1^2}{-\phi_2} - 4}\right] \Rightarrow b = \frac{1}{\Delta}\ln\frac{1}{2}\left[\frac{\phi_1}{\sqrt{-\phi_2}} \pm \sqrt{\frac{\phi_1^2}{-\phi_2} - 4}\right]$$

In the case $\lambda_1 = \lambda_2^* \notin \mathbb{R}$ (i.e. λ-s are complex conjugates

$$\phi_1 = \lambda_1 + \lambda_2 = e^{-\xi \omega_n \Delta} 2 \cos(\omega_n \sqrt{1-\xi^2}\,\Delta)$$

$$\phi_2 = -\lambda_1 \lambda_2 = -e^{-2\xi\omega_n \Delta}$$

$$\left.\begin{aligned} \ln(-\phi_2) &= -2\xi\omega_n\Delta \\ \arccos\left(\frac{\phi_1}{2\sqrt{-\phi_2}}\right) &= (\omega_n\sqrt{1-\xi^2}\,)\cdot\Delta \end{aligned}\right\} \quad \frac{1}{4}(\ln(-\phi_2))^2 + \left(\arccos\left(\frac{\phi_1}{2\sqrt{-\phi_2}}\right)\right)^2 = \cancel{\xi^2\omega_n^2\Delta^2} + \omega_n^2\Delta^2 - \cancel{\omega_n^2\xi^2\Delta^2}$$

$$\Rightarrow \omega_n = \frac{1}{\Delta}\sqrt{\frac{[\ln(-\phi_2)]^2}{4} + \left[\cos^{-1}\left(\frac{\phi_1}{2\sqrt{-\phi_2}}\right)\right]^2}$$

Similarly

$$\xi = \sqrt{\frac{[\ln(-\phi_2)]^2}{\left\{[\ln(-\phi_2)]^2 + 4\left[\cos^{-1}\left(\frac{\phi_1}{2\sqrt{-\phi_2}}\right)\right]^2\right\}}}$$

This is great, but arc-cosine is NOT a unique function $\Rightarrow$ ξ and ω_n will not be unique!

$$\boxed{\cos\alpha = x \qquad \arccos x = \alpha + 2k\pi}$$

This is even more obvious if we try to express roots of the A(2) characteristic poly

$$s^2 + 2\xi\omega_n s + \omega_n^2 = 0$$

in the form

$$\mu_{1,2} = a \pm bj \qquad a = \operatorname{Re}\mu_1 \qquad b = \operatorname{Im}\mu_1$$

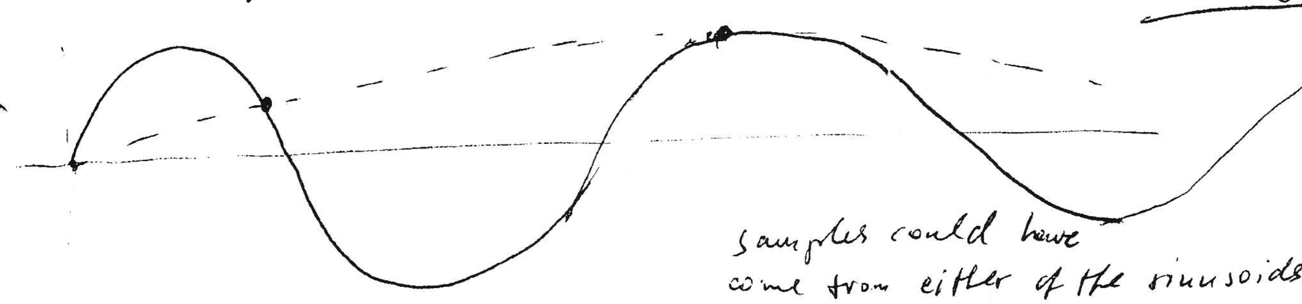
In this case

$$a = -\frac{\ln(-\phi_2)}{2\Delta} = -\xi\omega_n$$

$$b = \frac{\left| \pm\cos^{-1}\left(\frac{\phi_1}{2\sqrt{-\phi_2}}\right) + 2n\pi \right|}{\Delta} \quad n \in \mathbb{Z} \qquad (*)$$

Coefficient b is not unique & the discrete time series described by ARMA(2,1) could have been generated by several continuous-time systems.

aliasing



samples could have come from either of the sinusoids!