Why all this? We can establish an isomorphism between the following "fields"

(Space of operators composed as "polynomials" of backshift operators , composition of operators , Addition of operators)

(Space of rational fcns of polynomials , Multiplication of rational fcns of polynomials , Addition of rational fcns of polynomials)

In other words, we can "treat" backshift operators and coefficients as if we're dealing with polynomials!

(c) Second order GF coefficients

Let $X_t = \sum_{l=0}^{\infty} G_l a_{t-l}$ where $G_l = g_1 \lambda_1^l + g_2 \lambda_2^l$

and $|\lambda_1| < 1$, $|\lambda_2| < 1$, with $g_1 + g_2 = 1$ (to satisfy the Wold's decomposition formalism

$$X_t = \left[\sum_{l=0}^{\infty} (g_1 \lambda_1^l + g_2 \lambda_2^l) B^l \right] a_t = \quad \text{- - - - -} \quad (*)$$

$$= \left[g_1 \sum_{l=0}^{\infty} \lambda_1^l B^l \right] a_t + \left[g_2 \sum_{l=0}^{\infty} \lambda_2^l B^l \right] a_t =$$

$$= \left[g_1 (1-\lambda_1 B)^{-1} + g_2 (1-\lambda_2 B)^{-1} \right] a_t =$$

$$= \left[\frac{g_1}{1-\lambda_1 B} + \frac{g_2}{1-\lambda_2 B} \right] a_t = \frac{(g_1+g_2) - (g_1\lambda_2 + \lambda_1 g_2) B}{1 - (\lambda_1+\lambda_2)B + \lambda_1\lambda_2 B^2}$$

$$\Rightarrow \left(1 - \overbrace{(\lambda_1+\lambda_2)}^{\phi_1} B + \overset{-\phi_2}{\lambda_1\lambda_2} B^2\right) X_t = \left[(g_1+g_2) - (g_1\lambda_2 + g_2\lambda_1)\right.$$

$$\underbrace{(1 - \phi_1 B - \phi_2 B^2) X_t = (1 - \theta_1 B)\, a_t}_{\text{It's an ARMA(2,1) model!}}$$

We'd get the same result by observing

$$X_t = \left[\sum_{l=0}^{\infty} (g_1 \lambda_1^l + g_2 \lambda_2^l) B^l \right] a_t$$

and algebraically showing that for $\phi_1 = \lambda_1 + \lambda_2$ and $\phi_2 = -\lambda_1\lambda_2$, we'd get that

$$X_t - \phi_1 X_{t-1} - \phi_2 X_{t-2} = a_t - \theta_1 a_{t-1} \qquad (**)$$

where $\theta_1 = g_1 \lambda_2 + g_2 \lambda_1$

Actually, algebraic manipulations of (*) would give (**) even if $|\lambda_1|$ or $|\lambda_2|$ is bigger than but the resulting random process would not be WSS (it doesn't even have a finite variance).

(d) Green's fcn coefficients and Wold's decomposition for an ARMA(2,1) model

Let X_t be a random process such that

$$X_t - \phi_1 X_{t-1} - \phi_2 X_{t-2} = a_t - \theta_1 a_{t-1} \quad \cdots \quad \star$$

and let ϕ_1 and ϕ_2 be such that roots of the poly

$$s^2 - \phi_1 s - \phi_2 = 0 \qquad (\star\star)$$

are all inside the unit circle (roots of $(\star\star)$ λ_1 and λ_2 are such that $|\lambda_i| < 1$, $i=1,2$). Then, we can decompose X_t in the following way.

Since $(1-\phi_1 B - \phi_2 B^2) = (1-\lambda_1 B)(1-\lambda_2 B)$

then

$$(1-\lambda_1 B)(1-\lambda_2 B) X_t = (1-\theta_1 B) a_t$$

/ Apply $(1-\lambda_1 B)^{-1}(1-\lambda_2 B)$ on both sides

$$X_t = \frac{(1-\theta_1 B)}{(1-\lambda_1 B)(1-\lambda_2 B)} a_t$$

which can be decomposed as (Partial fraction exp

$$X_t = \left[\frac{g_1}{1-\lambda_1 B} + \frac{g_2}{1-\lambda_2 B} \right] a_t \qquad \star\star$$

where $\quad g_1 = \dfrac{\lambda_1 - \theta_1}{\lambda_1 - \lambda_2} \quad$ and $\quad g_2 = \dfrac{\lambda_2 - \theta_1}{\lambda_2 - \lambda_1}$

Representation $\star\star$ actually decomposes X_t in the following way

$$X_t = \left[g_1 (1 + \lambda_1 B + \lambda_1^2 B^2 + \dots + \lambda_1^\ell B^\ell + \dots) + \right.$$

$$\left. + g_2 (1 + \lambda_2 B + \lambda_2^2 B^2 + \dots + \lambda_2^\ell B^\ell + \dots) \right] a_t \qquad \left(\begin{smallmatrix}* * \\ *\end{smallmatrix}\right)$$

which is a Wold's decomposition with GF coefficients being

$$G_\ell = g_1 \lambda_1^\ell + g_2 \lambda_2^\ell \qquad \begin{smallmatrix}\star\star \\ \star\end{smallmatrix}$$

Note that if we have an ARMA(2,1) model of form $\star$, we can still represent X_t in the form $\left(\begin{smallmatrix}* * \\ *\end{smallmatrix}\right)$, with GF coefficients given by $\begin{smallmatrix}\star\star \\ \star\end{smallmatrix}$, but the resulting process is NOT stationary if any of the characteristic roots λ_1 or λ_2 is outside the unit circle (there's no variance and one cannot strictly speak about any limits in an infinite summation that is Wold's decomposition

---

Example: (part of an exam question)

For the model found in part (a), please find the corresponding impulse response of the system which when driven by white noise yielded the time-series modeled in part (a)

Note: Model that should have been found in part (a) was

$$\underbrace{X_t - 0.9 X_{t-1} + 0.2 X_{t-2}} = a_t - 0.3 a_{t-1} \qquad a_t \sim NIID(0, \sigma_a^2)$$

$$s^2 - 0.9 s + 0.2 = 0 \Rightarrow \lambda_{1/2} = \frac{1}{2}\left(0.9 \pm \sqrt{0.9^2 - 4 \cdot 0.2}\right)$$

$$= \begin{cases} 0.5 = \lambda_1 \\ 0.4 = \lambda_2 \end{cases}$$

$$G_l = g_1 \lambda_1^l + g_2 \lambda_2^l = g_1 \cdot 0.5^l + g_2 \cdot 0.4^l$$

where $g_1 = \dfrac{\lambda_1 - \theta_1}{\lambda_1 - \lambda_2} = 2$; $g_2 = \dfrac{\lambda_2 - \theta_1}{\lambda_2 - \lambda_1} = -1$

Another way of doing that problem (using B operator)

$$(1-0.9B+0.2B^2)X_t = (1-0.3B)a_t$$

$$\Rightarrow X_t = \frac{1-0.3B}{1-0.9B+0.2B^2}a_t \overset{\text{P.F.E.}}{=} \left(\frac{g_1}{1-\lambda_1 B}+\frac{g_2}{1-\lambda_2 B}\right)a_t$$

where $\lambda_{1/2}$ are roots of $s^2-0.9s+0.2$, i.e. $\lambda_1=0.5$, $\lambda_2=0.4$

and $g_1 = \frac{\lambda_1-\theta_1}{\lambda_1-\lambda_2} = 2$, $g_2 = \frac{\lambda_2-\theta_1}{\lambda_2-\lambda_1} = -1$

$$\Rightarrow X_t = \left(\frac{2}{1-0.5B}-\frac{1}{1-0.4B}\right)a_t = \left[\sum_{\ell=0}^{\infty}(2\cdot 0.5^\ell B^\ell - 0.4^\ell B^\ell)\right]a_t$$

$$= \left[\sum_{\ell=0}^{\infty} G_\ell B^\ell\right]a_t \quad \text{where}$$

$$G_\ell = 2\cdot 0.5^\ell - 0.4^\ell$$

(e) General n^{th} order GF coefficients

Let $X_t = \sum_{l=0}^{\infty} G_l a_{t-l}$ where $a_t \sim$ WSS white noise e.g. $a_t \sim$ NIID $N(0, \sigma_a^2)$

and $G_l = g_1 \lambda_1^l + g_2 \lambda_2^l + \dots + g_n \lambda_n^l$, with $|\lambda_i| < 1$, $i = 1, 2, \dots, n$

and $g_1 + g_2 + \dots + g_n = 1$ (to satisfy Wold's decomposition formalism). Then

$$X_t = \sum_{l=0}^{\infty} (g_1 \lambda_1^l B^l + g_2 \lambda_2^l B^l + \dots + g_n \lambda_n^l B^l) a_t =$$

$$= \left[\frac{g_1}{1 - \lambda_1 B} + \frac{g_2}{1 - \lambda_2 B} + \dots + \frac{g_n}{1 - \lambda_n B} \right] a_t =$$

MA part

$$= \frac{g_1 (1-\lambda_2 B) \dots (1-\lambda_n B) + g_2 (1-\lambda_1 B)(1-\lambda_3 B) \dots (1-\lambda_n B) + \dots + g_n (1-\lambda_1 B) \dots (1-\lambda_{n-1} B)}{(1-\lambda_1 B)(1-\lambda_2 B) \dots (1-\lambda_n B)}$$ ← AR part

This leads to an ARMA$(n, n-1)$ model of the form

$$(1 - \phi_1 B - \dots - \phi_n B^n) X_t = (1 - \theta_1 B - \theta_2 B^2 - \dots - \theta_{n-1} B^{n-1}) a_t$$

(f) Wold's decomposition (Green's fcn coefficients) for an ARMA(n, n-1) model

Let $X_t - \phi_1 X_{t-1} - \dots - \phi_n X_{t-n} = a_t - \theta_1 a_{t-1} - \dots - \theta_{n-1} a_{t-n+1}$

where $a_t \sim$ wss white noise with 0 mean and variance σ_a^2 (e.g. $a_t \sim NIID\ \mathcal{N}(0, \sigma_a^2)$), and all roots of the polynomial $s^n - \phi_1 s^{n-1} - \phi_2 s^{n-2} - \dots - \phi_n = 0$ are inside the unit circle. Then

$$(1 - \phi_1 B - \phi_2 B^2 - \dots - \phi_n B^n) X_t = (1 - \theta_1 B - \dots - \theta_{n-1} B^{n-1}) a_t$$

$$\Rightarrow X_t = \frac{1 - \theta_1 B - \theta_2 B^2 - \dots - \theta_{n-1} B^{n-1}}{1 - \phi_1 B - \phi_2 B^2 - \dots - \phi_n B^n} a_t =$$

$$= \frac{(1 - \theta_1 B - \theta_2 B^2 \dots - \theta_{n-1} B^{n-1})}{(1 - \lambda_1 B)(1 - \lambda_2 B) \dots (1 - \lambda_n B)} a_t \quad \text{------} \quad (1)$$

where λ_i, $i = 1, 2, \dots, n$ are roots of the AR characteristic polynomial $s^n - \phi_1 s^{n-1} - \dots - \phi_n = 0$. Operation (1) can now be represented as:

$$X_t = \left(\frac{g_1}{1-\lambda_1 B} + \frac{g_2}{1-\lambda_2 B} + \dots + \frac{g_n}{1-\lambda_n B} \right) a_t$$

where coefficients g_i, $i=1,2,\dots,n$ are obtained using partial fraction expansion (PFE) and are given by Eqn. (3.1.26) in our textbook.

Thus, since $|\lambda_i|<1$ for $i=1,2,\dots,n$, the equation above gives

$$X_t = \left(\sum_{l=0}^{\infty} G_l B^l \right) a_t$$

where

$$G_l = g_1 \lambda_1^l + g_2 \lambda_2^l + \dots + g_n \lambda_n^l$$

(i.e. the corresponding Wold's decomposition has n^{th} order G.F. coefficients).