Crash course on Z transforms

$$Z(X_t) = \sum_{i=-\infty}^{\infty} X_i z^{-i}$$

$$Z(X_{t-1}) = \sum_{i=-\infty}^{\infty} X_{i-1} z^{-i} = \sum_{k=-\infty} X_k z^{-(k+1)} = z^{-1} Z(X_t)$$

$z^{-1} \to$ like the back shift operator

<u>Thm 1</u>

$$Z(X_t * Y_t) = Z\left(\sum_{\ell=-\infty}^{\infty} X_\ell Y_{t-\ell}\right) = Z(X_t) \cdot Z(Y_t)$$

Convolution in the time domain means multiplication in the z domain

<u>Note 1</u>: $X_t = G_t * a_t \Rightarrow Z(X_t) = Z(G_t) \cdot Z(a_t)$

Note 2: $X_t - \phi_1 X_{t-1} - \dots - \phi_n X_{t-n} = a_t - \theta_1 a_{t-1} - \dots - \theta_{n-1} a_{t-n+1}$

Applying Z transform on both sides gives

$$(1 - \phi_1 z^{-1} - \phi_2 z^{-2} - \dots - \phi_n z^{-n}) Z(X_t) = (1 - \theta_1 z^{-1} - \dots - \theta_{n-1} z^{-(n-1)}) Z(a_t)$$

$$\Rightarrow \frac{Z(X_t)}{Z(a_t)} = \frac{1-\theta_1 z^{-1} - \dots - \theta_{n-1} z^{-(n-1)}}{1-\phi_1 z^{-1} - \dots - \phi_n z^{-n}}$$

$$Z(X_t) = \frac{1-\theta_1 z^{-1} - \dots - \theta_{n-1} z^{-(n-1)}}{1-\phi_1 z^{-1} - \dots - \phi_n z^{-n}} \, Z(a_t)$$

Similarly, if we drive this system with an impulse instead of a_t, our output, which is G_t satisfies

$$Z(G_t) = \frac{1-\theta_1 z^{-1} - \dots - \theta_{n-1} z^{-(n-1)}}{1-\phi_1 z^{-1} - \dots - \phi_n z^{-n}} \, \cancelto{1}{Z(\delta_t)} =$$

$$= \frac{1-\theta_1 z^{-1} - \dots - \theta_{n-1} z^{-(n-1)}}{1-\phi_1 z^{-1} - \dots - \phi_n z^{-n}}$$

Spectral Estimation

$\boxed{\text{Def}}$ Power spectrum or power spectral density of a random process X_t is the Discrete-Time-Fourier-Transform (DTFT) of the autocorrelation sequence γ_k of that process (X_t is assumed to be W.S.S. → otherwise, spectral density of that process changes over time).

$$P_X(\omega) = \frac{1}{2\pi} \sum_{\ell=-\infty}^{\infty} \gamma_\ell \, e^{-j\omega \ell} \qquad \omega \in [-\pi, \pi]$$

Note $\quad P_X(\omega) = \frac{1}{2\pi} \mathcal{Z}(\gamma_\ell) \Big|_{z = e^{j\omega}}$

(DTFT is essentially z transform of a sequence, evaluated for $z = e^{j\omega}$)

For a white noise process a_t,

$$\gamma_{a_\ell} = \sigma_a^2 \delta_\ell = \begin{cases} \sigma_a^2, & \ell = 0 \\ 0, & \text{otherwise} \end{cases}$$

Hence

$$\boxed{P_a(\omega) = \frac{1}{2\pi} \sigma_a^2} \quad (\text{flat for all } \omega\text{-s})$$

$$\omega \in [-\pi, \pi]$$

For our time-series model

$$X_t = G_t * a_t = \sum_{\ell=0}^{\infty} G_\ell a_{t-\ell}$$

where $a_t \sim NIID\, N(0, \sigma_a^2)$

Then

$$\gamma_\ell = E[X_t X_{t+\ell}] = \sigma_a^2 \sum_{k=-\infty}^{\infty} G_k G_{k+\ell}$$

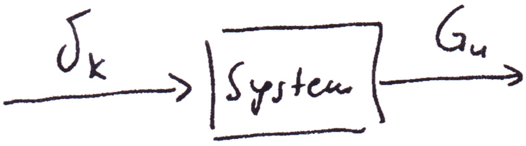
$$\Rightarrow P_x(\omega) = \frac{1}{2\pi} \sum_{\ell=-\infty}^{\infty} \gamma_\ell e^{-j\omega\ell} = \frac{\sigma_a^2}{2\pi} \sum_{\ell=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} G_k G_{k+\ell} e^{-j\omega\ell}$$

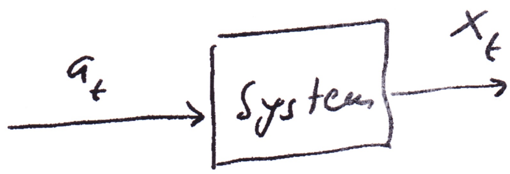
It is easy to show that

$$\sum_{l=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} G_k \, G_{k+l} \, e^{-j\omega l} = \left| \sum_{u=-\infty}^{\infty} G_u \, e^{-j\omega u} \right|^2$$

$$= \left| Z(G_u) \right|^2_{z=e^{j\omega}}$$

$$Z(G_u) = ?$$





$$Z(\delta_k) = 1 \qquad Z(X_{t-1}) = z^{-1} Z(X_t)$$

$$X_t - \phi_1 X_{t-1} - \dots - \phi_n X_{t-n} = a_t - \theta_1 a_{t-1} - \dots - \theta_m a_{t-m} \quad / Z$$

$$(1 - \phi_1 z^{-1} - \phi_2 z^{-2} - \dots - \phi_n z^{-n}) \, Z(X_t) = (1 - \theta_1 z^{-1} - \dots - \theta_m z^{-m}) \, Z(a_t)$$

similarly

$$(1 - \phi_1 z^{-1} - \phi_2 z^{-2} - \dots - \phi_n z^{-n}) \, Z(G_t) = (1 - \theta_1 z^{-1} - \dots - \theta_m z^{-m}) \, \underbrace{Z(\delta_t)}_{1}$$

$$\Rightarrow \mathcal{Z}(G_t) = \frac{1 - \theta_1 z^{-1} - \theta_2 z^{-2} - \dots - \theta_m z^{-m}}{1 - \phi_1 z^{-1} - \dots - \phi_n z^{-n}}$$

This is the so-called discrete-time transfer function of this system (the system that, when driven by white noise, gave us our time series).

$$\Rightarrow P_x(\omega) = \frac{\sigma_a^2}{2\pi} \left| \frac{1 - \theta_1 z^{-1} - \theta_2 z^{-2} - \dots - \theta_m z^{-m}}{1 - \phi_1 z^{-1} - \phi_2 z^{-2} - \dots - \phi_n z^{-n}} \right|^2_{z = e^{-j\omega}}$$

This is the formula from your book!