Midterm Review

Ch. 3. Analysis of time-series modeled through ARMA models

Green's fcn: $X_t = \sum_{k=0}^{\infty} G_k a_{t-k}$ ← orthogonal (Wold's) decomposition

Inverse fcn: $a_t = -\sum_{k=0}^{\infty} I_k X_{t-k}$

Autocovariance fcn $\gamma_k = E[X_t X_{t-k}]$

$\gamma_k = \sigma_a^2 \sum_{j=0}^{\infty} G_j G_{j+k}$ (if you know G_j-s, then you can get γ_k-s).

$G_k = g_1 \lambda_1^k + g_2 \lambda_2^k + \dots + g_n \lambda_n^k \longleftrightarrow$ $X_t - \phi_1 X_{t-1} - \dots - \phi_n X_{t-n} = a_t - \theta_1 a_{t-1} - \dots - \theta_{n-1} a_{t-n+1}$

λ_k - roots of the AR characteristic poly

$$s^n - \phi_1 s^{n-1} - \dots - \phi_n = 0$$

For ARMA(2,1) model (in exam, I will not ask for explicit formulae of models of order higher than 2)

$$X_t - \phi_1 X_{t-1} - \phi_2 X_{t-2} = a_t - \theta_1 a_{t-1}$$

$$G_k = g_1 \lambda_1^k + g_2 \lambda_2^k$$

$$g_1 = \frac{\lambda_1 - \theta_1}{\lambda_1 - \lambda_2} \qquad g_2 = \frac{\lambda_2 - \theta_1}{\lambda_2 - \lambda_1} \qquad \lambda_{1/2} = \frac{1}{2}\left(\phi_1 \pm \sqrt{\phi_1^2 + 4\phi_2}\right)$$

Note $\lambda_1 + \lambda_2 = \phi_1$, $-\lambda_1 \lambda_2 = \phi_2$

If we just need a few G_k coefficients, we can use the implicit method

$$X_t - X_{t-1} + 0.25 X_{t-2} = a_t - 0.4 a_{t-1}$$

$$X_t = \left[\sum_{k=0}^{\infty} G_k B^k\right] a_t$$

$$(1 - B + 0.25 B^2)(G_0 + G_1 B + G_2 B^2 + \dots) a_t = (1 - 0.4B) a_t$$

B^0: $1 = G_0$

B^1: $G_1 - G_0 = -0.4 \Rightarrow G_1 = G_0 - 0.4 = 0.6$

B^2: $G_2 - G_1 + 0.25 G_0 = 0 \Rightarrow G_2 = -0.25 + 0.6 = .35$

etc.

Stability & Invertibility

$$(1-\lambda_1 B)(1-\lambda_2 B)\cdots(1-\lambda_n B)\, X_t = (1-\nu_1 B)(1-\nu_2 B)\cdots(1-\nu_m B)\, a_t$$

i) Stable $\quad \forall i,\ |\lambda_i|<1$

ii) Marginally stable $\quad \forall i,\ |\lambda_i|\le 1$ and if $|\lambda_i|=1$, λ_i is a simple AR characteristic root.

iii) Unstable: $\quad \exists \lambda_i,\ |\lambda_i|>1$ or

$\exists \lambda_i,\ |\lambda_i|=1$ and

λ_i is a root of multiplicity greater than 1

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i) Invertible $\quad \forall i,\ |\nu_i|<1$

ii) Marginally invertible $\quad \forall i,\ |\nu_i|\le 1$ and if $|\nu_i|=1$, ν_i is a simple MA characteristic root.

iii) Non-invertible $\quad \exists \nu_i,\ |\nu_i|>1$

or $\exists \nu_i,\ |\nu_i|=1$ and ν_i is a MA root of multiplicity >1.

<u>Ex.</u> $(1-0.5B)^2 X_t = (1-0.4B)\,a_t \rightarrow$ stable & invertible

- $X_t - 2X_{t-1} + X_{t-2} = a_t - 0.4\,a_{t-1} \rightarrow$ un-stable & invertible

$X_t + X_{t-1} + X_{t-2} = a_t - 2a_{t-1}$ $\quad \lambda_{1,2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} j \quad |\lambda_1| = |\lambda_2| = 1$

NON invertible! $\quad \Rightarrow$ marginally stable!

Chap. 4. - Modeling

$$\underbrace{ARMA(2n+2, 2n+1)}_{A_0} \quad vs \quad \underbrace{ARMA(2n, 2n-1)}_{A_1}$$

unrestricted RSS restricted RSS

$$F = \frac{(A_1 - A_0)/s}{A_0/(N-r)}$$

s - # of restricted parameters
r - # of estimated params $(2n+2, 2n+1)$
N - # of samples

$F \sim F(s, N-r)$
if both models are adequate!

Here, $s = 4$
$r = 2n+2+2n+1+1 = 4n+4$
$\uparrow \mu_x$

$F < F_{0.95}(s, N-r) \Rightarrow$ keep $ARMA(2n, 2n-1)$

$F > F_{0.95}(s, N-r) \rightarrow$ go on to the next test!

<u>Example:</u> $ARMA(4,3)$ vs $ARMA(6,5)$ $N = 500$
$RSS = 1600 = A_1$ $RSS = 1500 = A_0$

$$F = \frac{(A_1 - A_0)/4}{A_0/(500 - 6+5+1)} = 8133$$

$F_{0.95}(4, \infty) = 2.37$

$F > F_{0.95}(4, \infty) \Rightarrow$ reduction is significant $\Rightarrow$ keep modeling

Example: $\underbrace{ARMA(2,1)}_{A_0}$ vs $\underbrace{ARMA(2,0)}_{A_1}$

$$\frac{(A_1 - A_0)/s^{\swarrow 1}}{A_0/(N-\underset{\substack{\uparrow \\ 2+1+1}}{r})} = \frac{A_1 - A_0}{A_0/(N-4)} \sim F(1, N-4)$$

Chap. 5 → Forecasting

Forecasting through orthogonal decomposition & through conditional expectation

$$\hat{X}_t(\ell) = E[X_{t+\ell} \mid X_t, X_{t-1}, \dots, X_0]$$

$$Var[X_t(\ell)] = \sigma_a^2 [1 + G_1^2 + \dots + G_{\ell-1}^2]$$

Obtained through orthogonal decomposition of X_t

Example

$$X_t - 0.7X_{t-1} + 0.12X_{t-2} = a_t - 0.3a_{t-1}$$

$X_{200} = 2$, $X_{199} = -1$ $\quad a_{200} = X_{200} - \hat{X}_{199}(1) = 3$

$\hat{X}_{200}(2) = ?$

$$X_{201} - 0.7\, X_{200} + 0.12\, X_{199} = a_{201} - a_{200} \cdot 0.3 \qquad E[\,\cdot \mid X_{200}, X_{199} \ldots]$$

$$\hat{X}_{200}(1) = 0.7\, X_{200} - 0.12\, X_{199} + 0 - 0.3 \cdot a_{200} = 0.38$$

$$X_{202} - 0.7\, X_{201} + 0.12\, X_{200} = a_{202} - 0.3\, a_{201} \qquad E[\,\cdot \mid t = 200]$$

$$\hat{X}_{200}(2) - 0.7\, \hat{X}_{200}(1) + 0.12\, X_{200} = 0 \Rightarrow$$

$$\hat{X}_{200}(2) = 0.026$$

<u>Remember also</u>: $E[X_{t+\ell} \mid t] = \hat{X}_t(\ell) = G_\ell\, a_t + G_{\ell+1}\, a_{t-1} + \ldots$

Time constant of a 1st order ordinary differential equation modeling how the torque of a DC motor depends on the input voltage is evaluated to be 5 seconds.

(1) Please describe the differential equation governing this system. Please assume the scaling factor with which input comes into the system as being 1 (i.e. assume a canonical 1st order system)

$$\dot{X}(t) + \frac{X(t)}{5} = u(t)$$

(2) If this system is driven by a continuous time white noise with covariance function

$$\gamma(\tau) = 10\delta(\tau)$$

where $\delta(\tau)$ denotes a continuous-time Dirac's delta function, please describe the model of the discrete time-series obtained by equidistantly sampling its response, with sampling interval of 0.2 seconds.

$$X_t - \phi_1 X_{t-1} = a_t$$

$$\phi_1 = e^{-\alpha_0 \Delta} = e^{-\frac{\Delta}{\tau}} = e^{-\frac{0.2}{5}} = 0.96$$

$$\sigma_a^2 = \frac{\sigma_z^2 (1-\phi_1^2)}{2\alpha_0} = \frac{10(1-0.96^2)}{2 \cdot \frac{1}{5}} = 1.96$$