Multiple (Vectorial) Time-Series (Chapter 11)

We will now consider multiple stationary time-series as realizations of a vectorial stationary random process.

* Multiple (vectorial) time-series (TS-s) mostly find application in control theory.
* We can postulate a "vectorial" arma model and use the predictions given by that model to counteract them and achieve "optimal control in the mean squared sense" (minimize variance of the output).

Let's introduce vectorial TS-s using 2 time series X_{1t} and X_{2t}.

If X_{1t} and X_{2t} were modeled independently, I would get something like this

$$X_{1t} = \phi_{11}^* X_{1\,t-1} + \qquad + a_{1t}^*$$

$$X_{2t} = \qquad \phi_{22}^* X_{2\,t-1} + a_{2t}^*$$

If there is interdependency between X_{1t} and X_{2t}, then I may want to postulate a model of the form

$$X_{1t} = \phi_{11}^{*} X_{1\,t-1} + \phi_{12}^{*} X_{2\,t-1} + a_{1\,t}^{*}$$

$$X_{2t} = \phi_{21}^{*} X_{1\,t-1} + \phi_{22}^{*} X_{2\,t-1} + a_{2\,t}^{*}$$

X_{1t} does not only depend on its own previous realization, but also that of X_{2t}

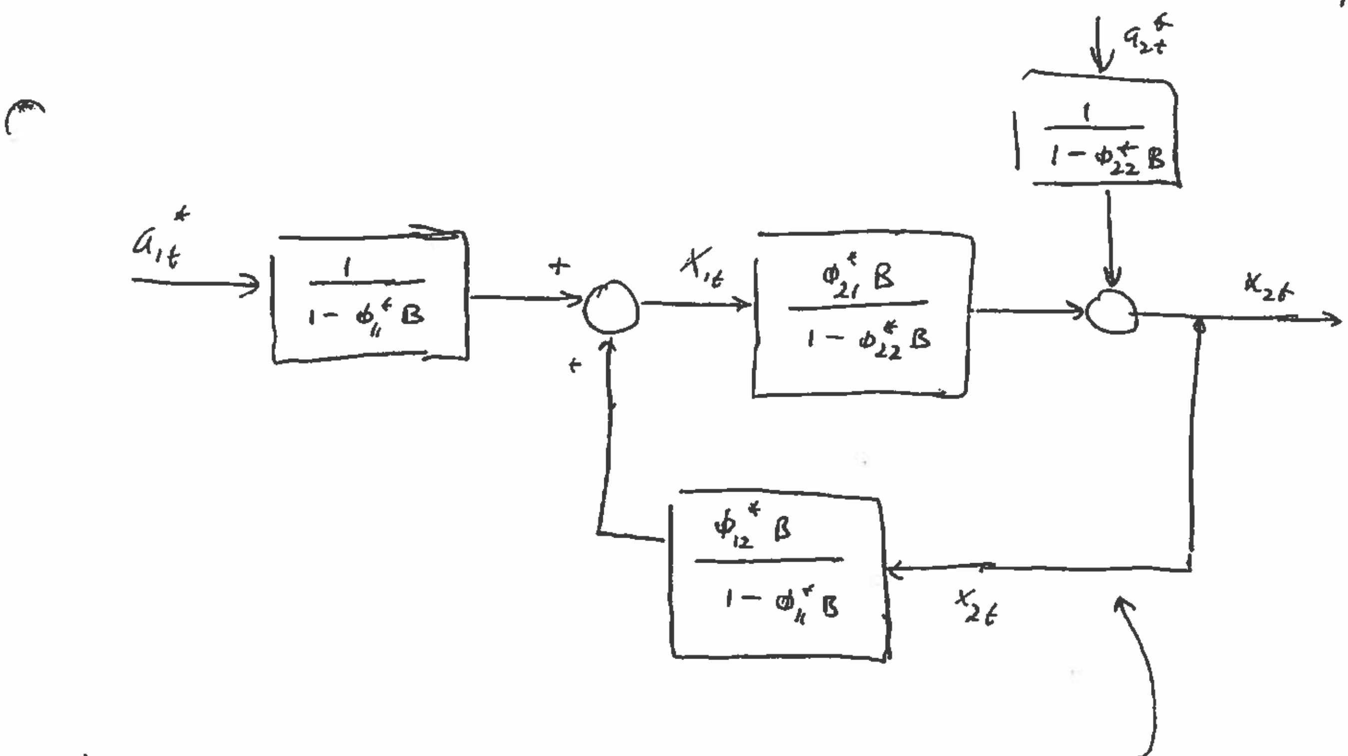
$$(1-\phi_{11}^{*}B)X_{1t} = \phi_{12}^{*} B X_{2t} + a_{1t}^{*}$$

$$(1-\phi_{22}^{*}B)X_{2t} = \phi_{21}^{*} B X_{1t} + a_{2t}^{*}$$

$$X_{1t} = \frac{\phi_{12}^{*}B}{1-\phi_{11}^{*}B} X_{2t} + \frac{1}{1-\phi_{11}^{*}B} a_{1t}^{*}$$

$$X_{2t} = \frac{\phi_{21}^{*}B}{1-\phi_{22}^{*}B} X_{1t} + \frac{1}{1-\phi_{22}B} a_{2t}^{*}$$

we can graphically represent this!



Also, I can see this as a system with a feedback

Vectorially, I can write this as

$$\bar{X}_t = \Phi_1^* \bar{X}_{t-1} + \vec{a}_t^*$$

$$\bar{X}_t = \begin{bmatrix} x_{1t} \\ x_{2t} \end{bmatrix} \qquad a_t^* = \begin{bmatrix} a_{1t} \\ a_{2t} \end{bmatrix} \qquad \Phi_1^* = \begin{bmatrix} \phi_{11}^* & \phi_{12}^* \\ \phi_{21}^* & \phi_{22}^* \end{bmatrix}$$

This is a vectorial AR(1) model, or in short ARV(1) model.
With vectorial MA parts, this becomes a vectorial ARMA model (ARMAV)

Vectorial (Multivariate) ARMA Models (ARMAV Models)

$\vec{X}_t$, $t \in \mathbb{Z}$ is a wss vectorial random process if:

* $E[\vec{X}_t] = \text{const.}$ (without loss of generality, let's assume it's zero)

* $$E[\vec{X}_t \, \vec{X}_{t-\ell}^T] = \begin{bmatrix} E[X_{1_t} X_{1_{t-\ell}}] & E[X_{1_t} X_{2_{t-\ell}}] & \dots & E[X_{1_t} X_{d_{t-\ell}}] \\ E[X_{2_t} X_{1_{t-\ell}}] & E[X_{2_t} X_{2_{t-\ell}}] & \dots & E[X_{2_t} X_{d_{t-\ell}}] \\ - & - & - & - \\ E[X_{d_t} X_{1_{t-\ell}}] & E[X_{d_t} X_{2_{t-\ell}}] & \dots & E[X_{d_t} X_{d_{t-\ell}}] \end{bmatrix}$$

$$= \Gamma_\ell$$

(matrices of 2nd order moments are origin independent)

Note: $\vec{X}_t = \begin{bmatrix} X_{1_t} \\ X_{2_t} \\ \vdots \\ X_{d_t} \end{bmatrix}$

Vectorial Wold's decomposition theorem

Any wss vectorial random process X_t, $t \in \mathbb{Z}$ can be represented as

$$\vec{X}_t = \vec{\varepsilon}_t + \Psi_1 \vec{\varepsilon}_{t-1} + \Psi_2 \vec{\varepsilon}_{t-2} + \dots + \Psi_\ell \varepsilon_{t-\ell} + \dots \qquad (*)$$

where

* $\sum_{\ell=0}^{\infty} \| \Psi_\ell \|^2 < \infty \qquad (\Psi_0 = I)$

* $E[\vec{\varepsilon}_t \vec{\varepsilon}_{t-\ell}^{\,T}] = \begin{cases} \Sigma, & \ell = 0 \\ 0, & \text{otherwise} \end{cases}$

If we introduce

$$\vec{a}_t = \mathbb{T} \cdot \vec{\varepsilon}_t$$

so that

$$E[\vec{a}_t \vec{a}_t^{\,T}] = D \qquad (\text{diagonalizing } \Sigma)$$

then $(*)$ can be expressed as

$$\vec{X}_t = \Theta_0 \vec{a}_t + \Theta_1 \vec{a}_{t-1} + \Theta_2 \vec{a}_{t-2} + \dots + \Theta_\ell \vec{a}_{t-\ell} + \dots$$

where $E[\vec{a}_t] = \vec{0} \qquad E[\vec{a}_t \vec{a}_{t-\ell}^{\,T}] = D\delta_\ell$