Finding Green's Function Coefficients from the ARMA model form

Explicit method of finding Green's Function Coefficients from the ARMA model form

Green's function of the ARMA(2,1) system -- explicit method

$$(1 - \phi_1 B - \phi_2 B^2) = (1 - \lambda_1 B)(1 - \lambda_2 B)$$

$$\lambda_1 + \lambda_2 = \phi_1$$

$$\lambda_1 \lambda_2 = -\phi_2$$

$$\lambda^2 - \phi_1 \lambda - \phi_2 = 0 \qquad \lambda_1,\ \lambda_2 = \frac{1}{2}(\phi_1 \pm \sqrt{\phi_1^2 + 4\phi_2})$$

Real distinct roots:

$$X_t = \frac{(1-\theta_1 B)\, a_t}{(1-\phi_1 B - \phi_2 B^2)} = \frac{(1-\theta_1 B)\, a_t}{(1-\lambda_1 B)(1-\lambda_2 B)} = \left[\frac{(\lambda_1 - \theta_1)}{(\lambda_1 - \lambda_2)} \frac{1}{(1-\lambda_1 B)} + \frac{(\lambda_2 - \theta_1)}{(\lambda_2 - \lambda_1)} \frac{1}{(1-\lambda_2 B)} \right] a_t$$

$$= \sum_{j=0}^{\infty} \left[\left(\frac{\lambda_1 - \theta_1}{\lambda_1 - \lambda_2}\right) \lambda_1^j + \left(\frac{\lambda_2 - \theta_1}{\lambda_2 - \lambda_1}\right) \lambda_2^j \right] a_{t-j}$$

$$G_j = \left(\frac{\lambda_1 - \theta_1}{\lambda_1 - \lambda_2}\right) \lambda_1^j + \left(\frac{\lambda_2 - \theta_1}{\lambda_2 - \lambda_1}\right) \lambda_2^j$$

Above explicit form of Green's function can also be derived as the solution of an nth order homogeneous difference equation:

$$(1 - \phi_1 B - \phi_2 B^2)\, G_j = 0$$

with initial conditions of

$$G_0 = 1 \qquad G_1 = \phi_1 - \theta_1$$

The solution of the difference equation is a linear combination of terms, λ^j,

$$G_j = g_1 \lambda_1^j + g_2 \lambda_2^j$$

$$G_0 = g_1 + g_2 = 1$$

$$G_1 = g_1 \lambda_1 + g_2 \lambda_2 = \phi_1 - \theta_1$$

Thus,

$$g_1 = \frac{\lambda_1 - \theta_1}{\lambda_1 - \lambda_2} \qquad g_2 = \frac{\lambda_2 - \theta_1}{\lambda_2 - \lambda_1}$$

<u>Complex Roots:</u>

$$\phi_1^2 + 4\phi_2 < 0$$

$$\lambda_1, \lambda_2 = \frac{1}{2}(\phi_1 \pm \sqrt{\phi_1^2 + 4\phi_2}\,) = r\,e^{\pm i\,\omega}$$

$$r = |\lambda_1| = |\lambda_2| = \sqrt{-\phi_2}$$

$$\omega = \cos^{-1}\frac{\phi_1}{2\sqrt{-\phi_2}} = \cos^{-1}\frac{(\lambda_1+\lambda_2)}{2\sqrt{\lambda_1\lambda_2}}$$

$$g_1, g_2 = g\,e^{\pm i\,\beta}$$

$$G_j = g_1\lambda_1^j + g_2\lambda_2^j = g\,e^{i\,\beta}(r\,e^{i\,\omega})^j + g\,e^{-i\,\beta}(r\,e^{-i\,\omega})^j$$

* Green's function for special models

AR(2): $\theta_1=0$

$$G_j = \frac{1}{\lambda_1 - \lambda_2}[\,\lambda_1^{j+1} - \lambda_2^{j+1}\,]$$

General Case of ARMA(n,n-1)

$$G_l = g_1\lambda_1^l + g_2\lambda_2^l + \ldots + g_n\lambda_n^l$$

where $\lambda_i, i = 1,2,\ldots,n$ are roots of the characteristic AR polynomial, while coefficients $g_i, i = 1,2,\ldots,n$ are calculated from the ARMA(n,n-1) model parameters (from $\phi_i, i = 1,2,\ldots,n$ and $\theta_j, j = 1,2,\ldots,n-1$ coefficients) using formulae (3.1.26) in the textbook.