ME384Q.3 / ORI 390R.3: Time-Series Analysis

Prof. Dragan Djurdjanovic

Homework 2 - Solutions

Problem 1:

$$X_t - 0.2X_{t-1} + 0.25X_{t-2} - 0.05X_{t-3} = a_t - a_{t-1} + a_{t-2}$$

Green's Function Coefficients:

$$X_t = (G_0 + G_1 B + G_2 B^2 + G_3 B^3 + \dots) a_t \Rightarrow$$

$$(1 - 0.2B + 0.25B^2 - 0.05B^3)\underbrace{(G_0 + G_1 B + G_2 B^2 + G_3 B^3 + \dots) a_t}_{X_t} =$$

$$= (1 - B + B^2) a_t$$

B^0: $G_0 = 1$

B^1: $G_1 - 0.2G_0 = -1 \Rightarrow G_1 = -1 + 0.2 = -0.8$

B^2: $G_2 - 0.2G_1 + 0.25G_0 = 1 \Rightarrow G_2 = 1 - 0.25G_0 + 0.2G_1 = 0.59$

B^3: $G_3 - 0.2G_2 + 0.25G_1 - 0.05G_0 = 0 \Rightarrow G_3 = 0.05G_0 - 0.25G_1 + 0.2G_2 = 0.368$

B^4: $G_4 - 0.2G_3 + 0.25G_2 - 0.05G_1 = 0 \Rightarrow G_4 = 0.05G_1 - 0.25G_2 + 0.2G_3 = -0.1139$

B^5: $G_5 - 0.2G_4 + 0.25G_3 - 0.05G_2 = 0 \Rightarrow G_5 = 0.05G_2 - 0.25G_3 + 0.2G_4 = -0.08528$

Problem 2:

$$(1 - B + B^2) X_t = (1 - 3B) a_t$$

Char. AR poly is $s^2 - s + 1$

Roots of this poly are

$$\lambda_{1/2} = \frac{1}{2} \pm \frac{\sqrt{3}}{2} j$$

(a) Easiest way to find 3 Green's Fcn coefficients is to use the the implicit method:

$$X_t = (G_0 + G_1 B + G_2 B^2 + G_3 B^3 + \dots) a_t$$

$$\Rightarrow (1 - B + B^2)(G_0 + G_1 B + G_2 B^2 + G_3 B^3 + \dots) a_t = (1 - 3B) a_t$$

Equating coefficients yields

B^0: $G_0 = 1$

B^1: $-G_0 + G_1 = -3 \Rightarrow G_1 = -3 + G_0 = -2$

B^2: $G_2 - G_1 + G_0 = 0 \Rightarrow G_2 = G_1 - G_0 = -2 - 1 = -3$

B^3: $G_3 - G_2 + G_1 = 0 \Rightarrow G_3 = G_2 - G_1 = -3 + 2 = -1$

(b) $G_\ell = g_1 \lambda_1^\ell + g_2 \lambda_2^\ell$

$$\lambda_1 = \frac{1}{2} - \frac{\sqrt{3}}{2} j = e^{-\frac{\pi}{3} j}$$

$$\lambda_2 = \frac{1}{2} + \frac{\sqrt{3}}{2} j = e^{\frac{\pi}{3} j} = \lambda_1^*$$

$$g_1 = \frac{\lambda_1 - \theta_1}{\lambda_1 - \lambda_2} = \frac{\frac{1}{2} - \frac{\sqrt{3}}{2} j - 3}{-\sqrt{3} j} = \frac{5 + \sqrt{3} j}{\sqrt{3} j} = \frac{1}{2} - \frac{5\sqrt{3}}{6} j$$

$$g_2 = \frac{\lambda_2 - \theta_1}{\lambda_2 - \lambda_1} = g_1^* = \frac{5 - \sqrt{3} j}{\sqrt{3} j} = \frac{1}{2} + \frac{5\sqrt{3}}{6} j$$

Using Euler's notation yields

$$g_1 = 1.5275\, e^{-1.2373 j} \qquad g_2 = 1.5275\, e^{1.2373 j}$$

$$\Rightarrow G_\ell = 1.5275 \left(e^{-\frac{\pi}{3} \ell j} e^{-1.2373 j} + e^{\frac{\pi}{3} \ell j} e^{1.2373 j} \right) =$$

$$= 1.5275 \cdot 2 \cdot \cos\left(\frac{\pi}{3} \ell + 1.2373\right) =$$

$$= 3.0550 \cos\left(\frac{\pi}{3} \ell + 1.2373\right), \quad \ell = 0, 1, 2, \ldots$$

Problem 3.2 from the textbook

(i) $\phi_1 = 1.5 \qquad \phi_2 = -0.6 \qquad \theta_1 = 0.5$

$G_0 = 1$

$G_1 = \phi_1 - \theta_1 = 1.5 - 0.5 = 1.0$

$G_2 = \phi_1 G_1 + \phi_2 G_0 = 1.5(1) - 0.6(1) = 0.9$

$G_3 = \phi_1 G_2 + \phi_2 G_1 = 1.5(0.9) - 0.6(1) = 0.75$

$G_4 = 0.5850, \; G_5 = 0.4275$

(iii) $\phi_1 = 2 \qquad \phi_2 = -0.6 \qquad \phi_3 = 0.2 \qquad \theta_1 = 0.6 \qquad \theta_2 = 0.5$

$G_0 = 1$

$G_1 = \phi_1 G_0 - \theta_1 = 2(1) - 0.6 = 1.4$

$G_2 = \phi_1 G_1 + \phi_2 G_0 - \theta_2 = 1.7, \; G_3 = \phi_1 G_2 + \phi_2 G_1 + \phi_3 G_0 = 2.76$

$G_4 = 4.78 \qquad G_5 = 8.244$

Problem 3.10 from the textbook

(ii) $G_j = A(0.8)^j \sin(0.6+0.8j)$ $\phi_2 = -r^2 = -0.64$

$G_o = 1 = A\sin(0.6)$; $A = 1.771$

$G_1 = \phi_1 - \theta_1 = 1.771(0.8)\sin(1.4) = 1.396$

$G_2 = \phi_1 G_1 + \phi_2 G_o = \phi_1(1.396) - 0.64 = 1.771(0.8)^2 \sin(2.2) = 0.916$,

i.e. $\phi_1 = 1.115$. Now $\theta_1 = \phi_1 - 1.396 = -0.281$

ARMA(2,1) model: $X_t - 1.115X_{t-1} + 0.64X_{t-2} = a_t + 0.281a_{t-1}$

(iii) $G_j = A(0.4)^j \cos(0.6+0.8j) + 0.4(0.6)^j$

$G_o = 1 = A\cos(0.6) + 0.4$; $A = 0.727$

$r = \sqrt{\lambda_1\lambda_2} = 0.4$; $\lambda_1\lambda_2 = 0.16$; $\lambda_3 = 0.6$

Hence $\phi_3 = \lambda_1\lambda_2\lambda_3 = (0.16)(0.6) = 0.096$

$\cos(0.8) = \dfrac{\lambda_1+\lambda_2}{2\sqrt{\lambda_1\lambda_2}} = \dfrac{\lambda_1+\lambda_2}{0.8}$, so $\lambda_1+\lambda_2 = 0.557$,

and $\phi_1 = \lambda_1+\lambda_2+\lambda_3 = 0.557 + 0.6 = 1.157$

$G_1 = \phi_1 - \theta_1 = 0.289$; $\theta_1 = 0.868$

$G_3 = \phi_1 G_2 + \phi_2 G_1 + \phi_3 G_0 = 0.040$; $\phi_2 = -0.496$

$G_2 = \phi_1 G_1 + \phi_2 G_o - \theta_2 = 0.0755$; $\theta_2 = -0.0237$

ARMA(3,2) model: $X_t - 1.157X_{t-1} + 0.496X_{t-2} - 0.096X_{t-3} = a_t - 0.868a_{t-1} + 0.0237a_{t-2}$

Problem 3.11 from the textbook

(a) Equation (3.1.4): $X_t = \sum_{j=0}^{\infty} G_j a_{t-j}$

$X_5 = G_o a_5 + G_1 a_4 + G_2 a_3 + G_3 a_2 + G_4 a_1 + G_5 a_o$

From $G_j = 0.4(0.9)^{j-1}$ we have $G_o = 1$; $G_1 = 0.4$; $G_2 = 0.36$; $G_3 = 0.324$; $G_4 = 0.2916$

$G_5 = 0.26244$. By substitution we get $X_5 = 1.382$

(b) Since there is only one root, AR orders n=1 with $\lambda_1 = \phi_1 = 0.9$.

Since $G_j - \phi_1 G_{j-1} = 0$, $j \geq 2$, by Eq. (3.1.12), MA order m=1.

Hence $G_1 = 0.4 = \phi_1 - \theta_1 = 0.9 - \theta_1$ gives $\theta_1 = 0.5$ and the model is ARMA(1,1):

$X_t = 0.9X_{t-1} + a_t - 0.5a_{t-1}$

(c) Difference form: $X_t = 0.9X_{t-1} + a_t - 0.5a_{t-1}$

$X_o = 0$; $a_o = 0$

$X_1 = a_1 = 0.5$

$X_2 = 0.9X_1 + a_2 - 0.5a_1 = -0.8$

$X_3 = 0.9X_2 + a_3 - 0.5a_2 = 0.78$

$X_4 = 0.9X_3 + a_4 - 0.5a_3 = -1.798$

$X_5 = 0.9X_4 + a_5 - 0.5a_4 = 1.3818$ checks with (a)