ME 384Q.3 / ORI 390R.3

Time Series Analysis

Final Examination SOLUTIONS

Spring 2018

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(2pm – 5pm)

Problem 1 (25 points)

There are parts (a), (b) and (c) in this problem.

Part a) True/False Questions (9 points):

Please fill in "T" for True and "F" for false in the brackets before every statement. For full credit, explain your answer verbally.

(F) Let Δ be the sampling interval and τ be the time constant of a continuous system that is being equidistantly sampled. For covariance equivalent AR(1) model, we have that as

$$\frac{\Delta}{\tau} \to 0$$

the corresponding discrete model takes the form $X_t = a_t$. (2 points)

$$\lambda_1 = \phi_1 = e^{-\alpha_0 \Delta} = e^{-\frac{\Delta}{\tau}} \xrightarrow[\frac{\Delta}{\tau} \to 0]{} 1$$ => equivalent model becomes $X_t - X_{t-1} = a_t$

=> False statement

(T) For a covariance equivalent ARMA(2,1) model, as $\omega_n \Delta \to \infty$, the discrete model takes the form $X_t = a_t$. (3 points)

$$\mu_{1/2} = \omega_n (-\zeta \pm j\sqrt{1-\zeta^2}) \qquad \lambda_{1/2} = e^{\mu_{1/2} \cdot \Delta} =$$

$$= e^{-\omega_n \Delta (\zeta \pm j\sqrt{\zeta^2 - 1})} \longrightarrow 0 \text{ as } \omega_n \Delta \to \infty$$ => white noise dynamics as Δ becomes large

(F) Optimal stochastic regulation in systems described by vectorial ARMA models is achieved by pushing the impulse response of the controlled system to zero. If not – what is pushed to zero? (4 points)

One should be driving L-step ahead prediction of the system output, where L is the lag between input and output in the open-loop model (model without control)

Part b) equidistant sampling of continuous systems (8 points):
Time constant of a 1st order ordinary differential equation modeling how the torque of a DC motor depends on the input voltage is evaluated to be 5 seconds.

(1) Please describe the differential equation governing this system. Please assume the scaling factor with which input comes into the system as being 1 (i.e. assume a canonical 1st order system) (3 points)

$$\dot{X}(t) + \frac{X(t)}{5} = u(t)$$

(2) If this system is driven by a continuous time white noise with covariance function

$$\gamma(\tau) = 10\delta(\tau)$$

where $\delta(\tau)$ denotes a continuous-time Dirac's delta function, please describe the model of the discrete time-series obtained by equidistantly sampling its response, with sampling interval of 0.2 seconds. (5 points)

$$X_t - \phi_1 X_{t-1} = a_t$$

$$\phi_1 = e^{-\alpha_0 \Delta} = e^{-\frac{\Delta}{\tau}} = e^{-\frac{0.2}{5}} = 0.96$$

$$\sigma_a^2 = \frac{\sigma_z^2(1-\phi_1^2)}{2\alpha_0} = \frac{10(1-0.96^2)}{2 \cdot \frac{1}{5}} = 1.96$$

Part c) equidistant sampling of continuous systems (8 points):
Given an ARMA model:

$$X_t = 1.50X_{t-1} - 0.60X_{t-2} + a_t + 0.20a_{t-1},$$
$$\Delta = 0.05, \gamma_0 = 4$$

Find the natural frequency and damping ratio of the equivalent A(2) system.

AR char. polynomial $s^2 - 1.5s + 0.6 = 0$

$\lambda_{1,2} = 0.75 \pm \frac{1}{2}\sqrt{1.5 - 4 \cdot 0.6} = 0.75 \pm 0.194j$

$\mu_{1,2} = -a \pm b \cdot j \qquad a = \xi\omega_n = -\dfrac{\ln(-\phi_2)}{2\Delta}$

$= -\dfrac{\ln 0.6}{2 \cdot 0.05} = 5.11$

$b = \dfrac{\left|\arccos \dfrac{\phi_1}{2\sqrt{-\phi_2}}\right|}{\Delta} = 5.0536$

$\omega_n^2 = a^2 + b^2 = 51.6337 \left(\dfrac{rad}{s}\right)^2 \Rightarrow \omega_n = 7.2 \dfrac{rad}{s}$

$\xi = \dfrac{a}{\omega_n} = 0.71$

$\sigma_z^2 = 4 \cdot \gamma_0 \cdot \xi \cdot \omega_n^3 = 4 \cdot 4 \cdot 0.71 \cdot (7.2)^3 = 4240.1$

Diff. equation: $\ddot{x}(t) + 2\xi\omega_n \dot{x}(t) + \omega_n^2 x(t) = z(t)$

$\ddot{x}(t) + 10.22\dot{x}(t) + 51.634x(t) = z(t)$

You can also use eqns (7.5.4)-(7.5.5) to get the same result.

Problem 2 (25 points)
There are parts (a), (b), (c) and (d) in this problem!

A young researcher needs to predict the behavior of the time-series shown in Figure 2.1.

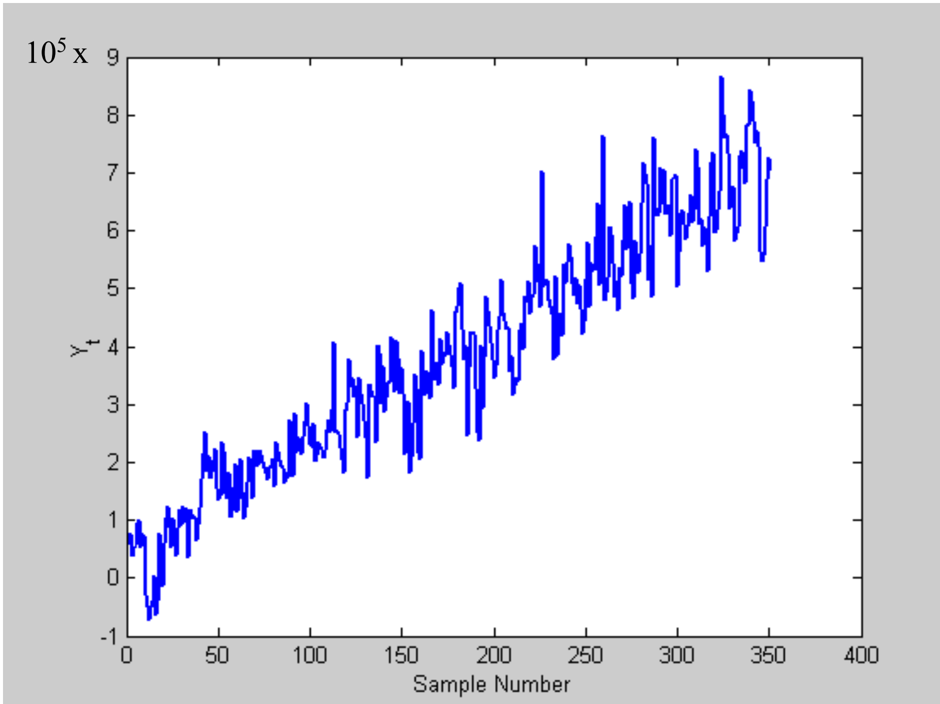


Figure 2.1: Time-series whose samples need to be predicted.

Part a) 4 points

After running the experiment, the researcher obtained N=350 samples of the time-series and it became clear to the researcher that the series displays non-stationary behavior. What is the reason why the researcher concluded that the experimental time-series Y_t displays non-stationary behavior?

The mean of this time-series seems to be continuously growing and does not appear to be constant.

Part b) 4 points

The researcher decides to fit a polynomial trend to the experimental data, and use ARMA modeling to model the residuals left after fitting the polynomial trend. Why does the researcher even bother to remove the polynomial trend from the data, before fitting the ARMA model?

The data seems to have a linearly increasing trend. Removing that trend will bring us back into the realm of stationary processes.

Part c) 12 points

A model of the form

$$Y_t = m + \sum_{j=1}^{l} R_j t^j + X_t$$

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \ldots + \phi_n X_{t-n} + a_t - \theta_1 a_{t-1} - \ldots - \theta_m a_{t-m}$$

was pursued to model the time-series Y_t observed in the experiment. Since the order of the polynomial is not known. the researcher successively increased by 1 the order of the polynomial that was fit to the time-series Y_t and the order of the ARMA(n,n-1) model describing the time-series X_t until the residual sum of square (RSS) did not reduce significantly. In other words, the researcher first fit a model with a first order polynomial (a line), one Autoregressive (AR) term and 0 Moving Average (MA) terms, then he/she fit a model with a 2nd order polynomial, 2 AR terms and 1 MA term, followed by a model with a 3rd order polynomial, 3 AR and 2 MA terms, etc. Table enclosed below describes the modeling procedure and order of the model (r_1, r_2, r_3) denotes a model with a polynomial of order r_1, r_2 AR terms and r_3 MA terms.

Parameters \ Model Order	(1,1,0)	(2,2,1)	(3,3,2)	(4,4,3)	(3,3,1)
M	12.2 ± 0.6	9.1 ± 0.7	9.0 ± 0.7	9.1 ± 0.4	9.1 ± 0.4
R_1	3.3 ± 3.1	2.2 ± 1.2	1.6 ± 1.0	1.8 ± 1.0	1.5 ± 1.0
R_2		-1.2 ± 0.2	-1.6 ± 0.1	-1.5 ± 0.1	-1.5 ± 0.1
R_3			0.02 ± 0.03	0.03 ± 0.01	0.02 ± 0.01
R_4				0.002 ± 0.01	
ϕ_1	0.6 ± 0.1	0.9 ± 0.2	0.85 ± 0.3	0.85 ± 0.2	0.84 ± 0.2
ϕ_2		-0.14 ± 0.1	-0.15 ± 0.1	-0.14 ± 0.1	-0.15 ± 0.1
ϕ_3			0.1 ± 0.1	0.15 ± 0.1	0.13 ± 0.1
ϕ_4				0.2 ± 0.1	
θ_1		0.1 ± 0.2	0.1 ± 0.2	0.12 ± 0.1	0.12 ± 0.1
θ_2			0.1 ± 0.2	0.25 ± 0.1	
θ_3				0.1 ± 0.1	
RSS	218	200	189	188	190

What model did the engineer choose to be adequate for the time-series Y_t (conduct the necessary F-tests).

$F_{0.95}(1,\infty) = 3.8601$; $F_{0.95}(2,\infty) = 3.0138$; $F_{0.95}(3,\infty) = 2.6347$ $F_{0.95}(4,\infty) = 2.3898$;

Test 1 : (2,2,1) vs (1,1,0)

$s = 3$; $RSS_1 = 218$; $RSS_0 = 200$; $r = 6$; $N = 350$

$$F = \frac{(RSS_1 - RSS_0)/s}{RSS_0/(N-r)} = 10.32 > F^{0.95}_{s,\infty} = F^{0.95}_{3,\infty} = 2.64 \Rightarrow$$

Drop in RSS is significant and we should try a higher order model.

Test 2: (3, 3, 2) vs (2, 2, 1) $RSS_1 = 200$ $RSS_0 = 189$

$r = 9,\ s = 3,\ N = 350$

$$F = \frac{(A_1 - A_0)/s}{A_0/(N-r)} = 6.62 > F^{0.95}_{s,\infty} = F^{0.95}_{3,\infty} = 2.64 \Rightarrow$$

$\Rightarrow$ Drop in RSS is significant and we should try a higher order model

Test 3: (4, 4, 3) vs (3, 3, 2) $RSS_1 = 189$ $RSS_0 = 188$

$r = 12,\ s = 3,\ N = 350$

$$F = \frac{(A_1 - A_0)/s}{A_0/(N-r)} = 0.6 < F^{0.95}_{s,\infty} = F^{0.95}_{3,\infty} = 2.64 \Rightarrow$$

$\Rightarrow$ It's an insignificant drop in RSS $\Rightarrow$ no need to go for higher order models. Keep (3,3,2)!

Note $\rightarrow$ confidence interval for θ_2 in the (3,3,2) model contains a zero. Hence, let us check the adequacy of (3, 3, 1) (obtained when θ_2 is forced to be zero).

Test 4: (3,3, 2) vs (3, 3, 1) $RSS_1 = 190$ $RSS_0 = 189$

$s = 1,\ r = 9,\ N = 350$

$$F = \frac{(RSS_1 - RSS_0)/s}{RSS_0/(N-r)} = 1.8 < F^{0.95}_{s,\infty} = F^{0.95}_{1,\infty} = 3.86 \Rightarrow$$

$\Rightarrow$ RSS drop is insignificant and we can take (3,3,1) as the adequate model.

Part d) 5 points

If $Y_{350} = 674291$, $Y_{349} = 907069$, $Y_{348} = 907072$ and $a_{350} = 4$, please determine the best prediction of Y_{352}.

Problem 3 (25 points)

There are parts (a), (b) and (c) in this problem.

For a single input (X_{1t}), single output (X_{2t}) system sampled at N=300 points, the following model is found to be adequate.

$$X_{2_t} = 0.3X_{1_{t-2}} + 0.9X_{2_{t-1}} - 0.14X_{2_{t-2}} + a_{2_t}$$
$$\text{RSS} = 1600$$

Part a) 10 points

Derive the mean-square error optimal control equation.

$$\hat{X}_{2_t}(2) = 0 \Rightarrow \hat{X}_{2_t}(2) = 0.3X_{1_t} + 0.9\hat{X}_{2_t}(1) - 0.14X_{2_t} = 0$$

$$\hat{X}_{2_t}(1) = 0.3X_{1_{t-1}} + 0.9X_{2_t} - 0.14X_{2_{t-1}}$$

$$\Rightarrow 0 = 0.3X_{1_t} + 0.9(0.3X_{1_{t-1}} + 0.9X_{2_t} - 0.14X_{2_{t-1}}) - 0.14X_{2_t}$$

$$\Rightarrow X_{1_t} = -0.9X_{1_{t-1}} - 2.23X_{2_t} + 0.42X_{2_{t-1}}$$

Part b) (5 points)

Write the model of the output after optimal control is applied to it.

$$X_{2_t} = \frac{0.3}{1-0.9B+0.14B^2}X_{1_{t-2}} + \frac{1}{1-0.9B+0.14B^2}a_{2_t}$$

$$X_{2_{t+2}} = \frac{0.3}{1-0.9B+0.14B^2}X_{1_t} + \overbrace{[G_0 + G_1B]a_{t+2}}^{\hat{e}_{2_t}(2)} + [G_2B^2 + \dots]a_{t+2}$$

$$\hat{X}_{2_t}(2) = 0$$

$$X_{2_{t+2}} = \hat{e}_{2_t}(2) = G_0 a_{t+2} + G_1 a_{t+1}$$

$G_\ell = g_1 \lambda_1^\ell + g_2 \lambda_2^\ell$ where

$\lambda_{1/2}$ are roots of $s^2 - 0.9s + 0.14 = 0$

$\Rightarrow \lambda_1 = 0.2 \quad \lambda_2 = 0.7$

$g_1 = -0.4 \quad g_2 = 1.4$ (Partial Fraction Expansion to get the Green's function)

$G_0 = 1$ $\quad G_1 = -0.4 \cdot 0.2 + 0.7 \cdot 1.4 = 0.9$

Model after control

$$X_{2_{t+2}} = a_{2_{t+2}} + 0.9 a_{2_{t+1}}$$

or equivalenty

$$\boxed{X_{2_t} = a_{2_t} + 0.9 a_{2_{t-1}}}$$

Part c) (10 points)
What is the control efficiency, i.e. what is the variance of the controlled output?
Note: RSS denotes the residual sum of squares for the output and you need to estimate $\gamma_{a_{2,2}}$ from it.

Problem 4 (25 points)

There are parts (a) and (b) and in this problem.

For two time series X_{1_t} and X_{2_t}, the researcher obtained N=250 points and the following model is found to be adequate.

$$X_{1_t} = 0.8X_{1_{t-1}} - 0.4X_{2_{t-1}} + a_{1_t}\text{ ; RSS}_1 = 1200$$

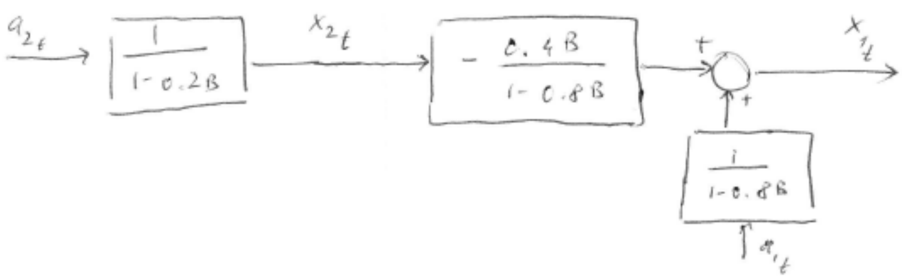
$$X_{2_t} = 0.2X_{2_{t-1}} + a_{2_t}\text{ ; RSS}_2 = 1300$$

Part a) (5 points)

Draw a block diagram of this system.

$$(1-0.8B)X_{1_t} = -0.4BX_{2_t} + a_{1_t} \Rightarrow X_{1_t} = \frac{a_{1_t}}{1-0.8B} + \frac{-0.4B}{1-0.8B}X_{2_t}$$

$$(1-0.2B)X_{2_t} = a_{2_t} \Rightarrow X_{2_t} = \frac{1}{1-0.2B}a_{2_t}$$



Part b) (7 points)

If $X_{1_{300}} = 5$ and $X_{2_{300}} = -2$, estimate $X_{1_{302}}$ and $X_{2_{302}}$. Please make sure to show all intermediate steps of your work

$$\hat{X}_{2_{300}}(2) = 0.2\hat{X}_{2_{300}}(1) \;;\; \hat{X}_{2_{300}}(1) = 0.2X_{2_{300}} = -0.4$$

$$\Rightarrow \hat{X}_{2_{300}}(2) = -0.08$$

$$\hat{X}_{1_{300}}(2) = 0.8\hat{X}_{1_{300}}(1) - 0.4\hat{X}_{2_{300}}(1)$$

$$\hat{X}_{1_{300}}(1) = 0.8X_{1_{300}} - 0.4X_{2_{300}} = 4.8 \Rightarrow \hat{X}_{1_{300}}(2) = 4$$

Part c) (13 points)

Express the variances of the errors of 2-step ahead predictions for time-series X_{1_t} and X_{2_t}. Please make sure to show all intermediate steps of your work.