Stochastic Trends & Seasonalities

<u>Def.</u> Stochastic trends and/or seasonalities exist if appropriate trends/seasonalities exist in the Green's Function of a time-series

Ex. If Green's Function of a time-series shows polynomial trend of 3^{rd} order => we say that that time-series displays a <u>3^{rd} order polynomial stochastic trend!</u>

In the case of deterministic trends $E[X_t] = f_\psi(t) \neq 0$

In the case of stochastic trends or seasonalities $E[X_t] = 0$

i) Stochastic trends

Polynomial stochastic trend of order ℓ exists if one AR root of multiplicity $\ell+1$ exists exactly at 1, while all other AR roots are inside the unit circle, or if they are ON the unit circle, they are of multiplicity 1.

Note: root of multiplicity $\ell+1$ carries the term

$$c_0 \lambda_1^{2} + c_1 2 \lambda_1^{2} + \dots + c_\ell 2^\ell \lambda_1^{2}$$

into the G.F.

Hence, if λ_1 is at 1, we have a polynomial trend in the G.F.

ii) Stochastic seasonalities

Stochastic seasonality of period $\frac{2\pi}{\omega}$ exists if AR characteristic polynomial has a pair of roots

$$\lambda_{1,2} = e^{\pm j\omega}$$

which are of multiplicity one, while all other AR characteristic roots are either inside the unit circle or if they are ON the unit circle, they are of multiplicity 1.

Note: A pair of complex conjugate AR roots $\lambda_{1,2} = e^{\pm \omega j}$ corresponds to a 2nd order polynomial factor

$$(1-\lambda_1 B)(1-\lambda_2 B) = 1 - 2\cos\omega\, B + B^2$$

inside the AR characteristic polynomial

"Interesting" seasonalities:

a) Period of 12 $\rightarrow$ $\omega = \frac{2\pi}{12}$ $\Rightarrow$ Block corresponding to this seasonality is

$$1 - 2\cos\frac{2\pi}{12} B + B^2 =$$

$$= 1 - \sqrt{3} B + B^2$$

b) Period of 3 (quarterly) $\rightarrow$ $\omega = \frac{2\pi}{3}$ $\Rightarrow$ Block corresponding to this seasonality is

$$1 - 2\cos\frac{2\pi}{3} B + B^2 =$$

$$= 1 - B + B^2$$

How to confirm or disconfirm existance of some stochastic seasonality? AR roots in reality NEVER fall onto "nice" seasonalities...

If $\lambda_{1,2}$ fall "near" some seasonalities that make sense (quarterly, yearly, weekly...), we should fix 2 roots of the AR polynomial to exactly those "nice" values and see if the RSS increases significantly or not. If not $\rightarrow$ then the corresponding seasonality <u>exists</u>.

Eg. Let's suspect that there is a seasonality with period p in a model

$$(1-\phi_1 B - \dots - \phi_n B^n) X_t = (1-\theta_1 B - \dots - \theta_m B^m) a_t$$

AR block corresponding to periodicity p is

$$(1-\lambda_1 B)(1-\lambda_2 B) = 1 - 2\cos\frac{2\pi}{p} B + B^2$$

since $\lambda_{1/2} = e^{j\frac{2\pi}{p}}$

i) Create a new time series $\overset{\vee}{Y}_t = (1-2\cos\frac{2\pi}{p}B + B^2) X_t$

ii) Fit an ARMA$(n-2, m)$ model to $\overset{\vee}{Y}_t$ and note the new RSS. This new model of $\overset{\vee}{Y}_t$ is referred to as the "parsimonious" model!

iii) If the RSS corresponding to the parsimonious model does not increase significantly, then seasonality of period p indeed exists!

RSS of the parsimonious model

only 2 parameters are restricted

$$F = \frac{(RSS_{pars} - RSS_{original})/2}{RSS_{original}/(N-r)} \sim F_{2,N-r}$$

Number of samples

Number of parameters of the original model ARMA(n,m)