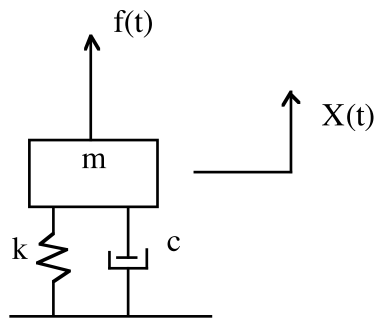
Uniformly Sampled Continuous-Time Second Order System Driven by White Noise

(Some notes repeated from Lecture 17)

1. Differential equation for a damped spring mass system



$$\frac{d^2X(t)}{dt^2} + \frac{c}{m}\frac{dX(t)}{dt} + \frac{k}{m}X(t) = \frac{1}{m}f(t)$$

$$\omega_n^2 = \frac{k}{m} \qquad \zeta = \frac{c}{2\sqrt{km}} = \frac{\text{actual damping}}{\text{critical damping}}$$

$$(D^2 + 2\zeta\omega_n D + \omega_n^2)\,X(t) = \frac{1}{m}f(t)$$

Solution:

$$X(t) = C_1 e^{\mu_1 t} + C_2 e^{\mu_2 t}$$

$$(D^2 + 2\zeta\omega_n D + \omega_n^2) = (D - \mu_1)(D - \mu_2) = 0$$

$$\mu_1, \mu_2 = \frac{1}{2}(-\alpha_1 \pm \sqrt{\alpha_1^2 - 4\alpha_0}\ ) = \omega_n(-\zeta \pm \sqrt{\zeta^2 - 1}\ )$$

$$\mu_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

$\zeta > 1$ X(t) is a sum of exponentials

$\zeta < 1$ X(t) is a damped sine wave

$\zeta = 0$ X(t) is undamped sine wave

$\zeta = 1$ the system has two real roots, then

$$X(t) = C_1 e^{-\omega_n t} + C_2\, t\, e^{-\omega_n t}$$

2. Autocovariance of the A(2) system

$$G(t) = \frac{e^{\mu_1 t} - e^{\mu_2 t}}{\mu_1 - \mu_2}\ , \quad \text{for } t \geq 0$$

$$\gamma(s) = \sigma_z^2 \int_0^\infty G(v)\, G(v+s)\, dv = \frac{\sigma_z^2}{2\mu_1\mu_2(\mu_1^2 - \mu_2^2)}(\mu_2 e^{\mu_1 s} - \mu_1 e^{\mu_2 s})$$

$$\gamma(0) = -\frac{\sigma_z^2}{2\mu_1\mu_2(\mu_1+\mu_2)} = \frac{\sigma_z^2}{4\zeta\omega_n^3} \qquad \left(\mu_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}\right)$$

$$\rho(s) = \frac{\mu_2 e^{\mu_1 s} - \mu_1 e^{\mu_2 s}}{\mu_2 - \mu_1}$$

3. Covariance function equivalent sampling:

$$\gamma_k = d_1\lambda_1^\kappa + d_2\lambda_2^\kappa = \frac{\sigma_z^2}{2\mu_1\mu_2(\mu_1^2-\mu_2^2)}(\mu_2 e^{\mu_1 k\Delta} - \mu_1 e^{\mu_2 k\Delta}) = \gamma(k\Delta)$$

$$d_1 = \frac{\sigma_z^2}{2\mu_1(\mu_1^2-\mu_2^2)} \qquad d_2 = \frac{-\sigma_z^2}{2\mu_2(\mu_1^2-\mu_2^2)}$$

$$\lambda_1 = e^{\mu_1\Delta} \qquad \lambda_2 = e^{\mu_2\Delta}$$

$$d_1 = \frac{\sigma_a^2(\lambda_1-\theta_1)}{(\lambda_1-\lambda_2)^2}\left[\frac{\lambda_1-\theta_1}{1-\lambda_1^2} - \frac{\lambda_2-\theta_1}{1-\lambda_1\lambda_2}\right]$$

in discrete model

$$d_2 = \frac{\sigma_a^2(\lambda_2-\theta_1)}{(\lambda_1-\lambda_2)^2}\left[\frac{\lambda_2-\theta_1}{1-\lambda_2^2} - \frac{\lambda_1-\theta_1}{1-\lambda_1\lambda_2}\right]$$

4. Example

For a given ARMA(2,1)

$$X_t - 1.52\, X_{t-1} + 0.55\, X_{t-2} = a_t + 0.26\, a_{t-1}$$

with a sampling interval $\Delta = 0.02$, find the corresponding differential equation.

Solution:

(1) For discrete system:

$$\lambda_1, \lambda_2 = \frac{1}{2}(\phi_1 \pm \sqrt{\phi_1^2 + 4\phi_2}) = \frac{1}{2}(1.52 \pm \sqrt{1.52^2 - 4 \times 0.55})$$

$$= 0.76 \pm 0.166 = 0.926 \text{ or } 0.594$$

$$\nu_1 = \theta_1 = -0.26$$

$$G_j = g_1\lambda_1^j + g_2\lambda_2^j$$

$$g_1 = \frac{\lambda_1 - \nu_1}{\lambda_1 - \lambda_2} = \frac{0.926 + 0.26}{0.926 - 0.594} = 3.572$$

$$g_2 = \frac{\lambda_2 - \nu_1}{\lambda_2 - \lambda_1} = \frac{0.594 + 0.26}{0.594 - 0.926} = -2.572$$

(2) For continuous system:

$$\mu_1 = \frac{1}{\Delta}\ln\lambda_1 = \frac{\ln 0.926}{0.02} = -3.844, \quad \tau_1 = \frac{1}{3.844} = 0.26$$

$$\mu_2 = \frac{1}{\Delta}\ln\lambda_2 = \frac{\ln 0.594}{0.02} = -26.04, \quad \tau_2 = \frac{1}{26.04} = 0.038$$

$$C_1 = g_1 = 3.572$$

$$C_2 = g_2 = -2.572$$

$$G(S) = \frac{C_1}{S - \mu_1} + \frac{C_2}{S - \mu_2} = \frac{3.572}{S + 3.844} + \frac{-2.572}{S + 26.04} = \frac{S + 83.13}{S^2 + 29.88S + 100.1} = \frac{X(S)}{Z(S)}$$

$$(S^2 + 29.88S + 100.1)X(S) = (S + 83.13)Z(S)$$

Take inverse Laplace transform with zero initial conditions:

$$\frac{d^2X(t)}{dt^2} + 29.88\frac{dX(t)}{dt} + 100.1X(t) = \frac{dZ(t)}{dt} + 83.13Z(t)$$

5. Stability Region of a discrete system

$$(D^2 + 2\zeta\omega_n D + \omega_n^2) = (D - \mu_1)(D - \mu_2) = 0$$

$$\mu_1, \mu_2 = \frac{1}{2}(-\alpha_1 \pm \sqrt{\alpha_1^2 - 4\alpha_0}\ ) = \omega_n(-\zeta \pm \sqrt{\zeta^2 - 1}\ )$$

$$\mu_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

$$(D^2 + \alpha_1 D + \alpha_0)\, X(t) = Z(t)$$

Static and dynamic stability:

<u>Static stability</u>: A random vibration system is statically stable if a displacement X(t) from the equilibrium position sets up a force or torque that tends to bring the system back to its equilibrium position.

i.e., a restoring force due to $\alpha_0 X(t)$

static stability:

$\alpha_0 > 0$

<u>Dynamic stability</u>: A random vibration system is dynamically stable if its velocity sets up a force or torque that tends to bring the system back to its equilibrium position.

i.e., a restoring force due to $\alpha_1 \dfrac{dX(t)}{dt}$

dynamic stability:

$\alpha_1 > 0$

$$\lambda_1 = e^{\mu_1\Delta} \qquad \lambda_2 = e^{\mu_2\Delta}$$

$$\phi_1 = \lambda_1 + \lambda_2 = e^{\mu_1\Delta} + e^{\mu_2\Delta}$$

$$\phi_2 = -\lambda_1\lambda_2 = -e^{(\mu_1+\mu_2)\Delta}$$

Stability region of ordinary ARMA(2,1) system:

$\phi_1 + \phi_2 < 1$

$\phi_2 - \phi_1 < 1$

$|\phi_2| < 1$

Additional restrictions on uniformly sampled ARMA(2,1) system:

(1) because

$$\mu_1 + \mu_2 = -2\zeta\omega_n = -\alpha_1 \quad \text{is real}$$

therefore,

$$\phi_2 = -\lambda_1\lambda_2 = -e^{(\mu_1+\mu_2)\Delta} \qquad \phi_2 = -e^{-\alpha_1\Delta}$$

implies that:

$$\phi_2 < 0$$

when $\alpha_1 < 0$ $\phi_2 < -1$ dynamically unstable

(2) if

μ_1, μ_2 are real $\alpha_0 < 0$ static instability

then,

λ_1, λ_2 are real positives

it implies that

$\phi_1 \geq 0$

(3) if

μ_1, μ_2 are complex conjugate

then

ϕ_1 could be positive or negative.

6. Nonuniqueness of A(2) model parameters

For $\phi_1^2 + 4\phi_2 \geq 0$

$$\zeta = \sqrt{\frac{[\ln(-\phi_2)]^2}{[\ln(-\phi_2)]^2 - 4[\cosh^{-1}(\frac{\phi_1}{2\sqrt{-\phi_2}})]^2}} \geq 1$$

$$\omega_n = \frac{1}{\Delta}\sqrt{\frac{[\ln(-\phi_2)]^2}{4} - [\cosh^{-1}(\frac{\phi_1}{2\sqrt{-\phi_2}})]^2}$$

For $\phi_1^2 + 4\phi_2 < 0$

$$\zeta = \sqrt{\frac{[\ln(-\phi_2)]^2}{[\ln(-\phi_2)]^2 + 4[\cos^{-1}(\frac{\phi_1}{2\sqrt{-\phi_2}})]^2}} < 1$$

$$\omega_n = \frac{1}{\Delta}\sqrt{\frac{[\ln(-\phi_2)]^2}{4} + [\cos^{-1}(\frac{\phi_1}{2\sqrt{-\phi_2}})]^2}$$

$$\mu_1, \mu_2 = -a \pm ib = -\zeta\omega_n \pm i\,\omega_n\sqrt{1-\zeta^2}$$

$$\phi_1 = \lambda_1 + \lambda_2 = e^{\mu_1\Delta} + e^{\mu_2\Delta} = 2e^{-a\Delta}\cos(b\Delta)$$

$$\phi_2 = -\lambda_1\lambda_2 = -\,e^{(\mu_1+\mu_2)\,\Delta} = -\,e^{-2a\Delta}$$

$$a = -\frac{\ln(-\phi_2)}{2\Delta}$$

a can be uniquely determined because ϕ is a real number.

$$\cos(b\Delta) = \frac{\phi_1}{2\sqrt{-\phi_2}}$$

$$b = \frac{\left| \pm\cos^{-1}\left(\dfrac{\phi_1}{2\sqrt{-\phi_2}}\right) + 2n\pi \right|}{\Delta}$$

This shows the multiplicity in b, or ω.

When sampling interval is sufficiently small, so that the dampled (natural) frequency is smaller than the highest frequency, which in the usual spectral analysis is known as Nyquist frequency, of $\frac{1}{2\Delta}$

$$\frac{\omega_n\sqrt{1-\zeta^2}}{2\pi} < \frac{1}{2\Delta}$$

$$\omega_n\sqrt{1-\zeta^2}\ \Delta = b\Delta < \pi$$

in this case, n is zero.