Summary of Deterministic Trends & Seasonalities

If we assume $E[X_t] = f_\Psi(t)$, where Ψ is a set of parameters of the deterministic trend/seasonality, then time series modeling can be pursued in the form

$$\min_{\phi, \theta, \Psi} \sum_{t=0}^{N} a_t^2$$

where

$$a_t = (X_t - f_\Psi(t)) - \phi_1 (X_{t-1} - f_\Psi(t-1)) - \dots - \phi_n (X_{t-n} - f_\Psi(t-n))$$
$$+ \theta_1 a_{t-1} + \dots + \theta_{n-1} a_{t-n+1} \qquad (*)$$

Deterministic trends/seasonalities frequently encountered:

i) $f_\Psi(t) = \alpha_0 + \alpha_1 t + \dots + \alpha_\ell t^\ell$ (polynomial trend with parameters $\Psi = \{\alpha_0, \alpha_1, \dots, \alpha_\ell\}$

ii) $f_\Psi(t) = C_1 e^{-\frac{t}{\tau_1}} + \dots + C_\ell e^{-\frac{t}{\tau_\ell}}$ (exponential trends with parameters $\Psi = \{C_1, \tau_1, \dots, C_\ell, \tau_\ell\}$)

iii) $f_\Psi(t) = B_1 e^{b_1 t} \sin(\omega_1 t + \varphi_1) + B_2 e^{b_2 t} \sin(\omega_2 t + \varphi_2) + \dots + B_\ell e^{b_\ell t} \sin(\omega_\ell t + \varphi_\ell)$

(sinusoidal trends or seasonalities with parameters $\Psi = \{B_1, b_1, \varphi_1, B_2, b_2, \varphi_2, \dots, B_\ell, b_\ell, \varphi_\ell\}$)

Easy way of finding parameters

i) Fit increasing orders of deterministic models (increasing orders of polynomials, number of exponentials, number of exponentially modulated sinusoids) until criterion says "enough" → parameters Ψ

ii) Use F-testing or AIC based procedure on the residuals of step i) to obtain parameters ϕ and θ.

iii) Use parameters Ψ from step i) and ϕ & θ from step ii) as initial guesses for the optimization (*)

Many (most) people do not go to step 3. Doing it in your project would be a plus ☺.

Problem 1.

An engineer decides to do system identification by exposing the system to a Sinusoidal input and fitting a model of the form

$$Y_t = m + \sum_{j=1}^{l}\left[A_j \sin(j\omega_0 t) + B_j \cos(j\omega_0 t)\right] + X_t$$

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \ldots + \phi_n X_{t-n} + a_t - \theta_1 a_{t-1} - \ldots - \theta_m a_{t-m}$$

to the output time-series Y_t. The output time-series has $N=200$ samples and the engineer successively increased the number of harmonics and the order of the ARMA(n,n-1) model describing the time-series X_t by 1 until the residual sum of square (RSS) did not reduce significantly. In other words, he first fits a model with one harmonic (one sinusoid and cosine term), one Autoregressive (AR) term and 0 Moving Average (MA) terms, then he fits a model with 2 harmonics (two sinusoidal and cosine terms), 2 AR terms and 1 MA term, followed by a model with 3 harmonics (three sinusoidal and cosine terms), 3 AR and 2 MA terms, etc. Table 1 describes the modeling procedure and order of the model (r_1, r_2, r_3) denotes a model with r_1 harmonics, r_2 AR terms and r_3 MA terms.

Table 1

Parameters \ Model Order	(1,1,0)	(2,2,1)	(3,3,2)	(2,2,0)
m	12.2 ± 0.6	9.1 ± 0.7	9.0 ± 0.7	9.2 ± 0.6
A_1	130.3 ± 3.1	120.2 ± 4.2	119 ± 4.0	120.1 ± 4.0
B_1	0.2 ± 0.1	2.1 ± 0.5	1.9 ± 0.5	2.0 ± 0.5
A_2		5.2 ± 0.2	5.1 ± 0.1	5.0 ± 0.1
B_2		0.3 ± 0.1	0.4 ± 0.2	0.3 ± 0.2
A_3			0.2 ± 0.3	
B_3			9.0 ± 4.2	
ϕ_1	0.6 ± 0.1	0.9 ± 0.2	0.85 ± 0.3	0.8 ± 0.1
ϕ_2		-0.15 ± 0.1	-0.15 ± 0.1	-0.12 ± 0.07
ϕ_3			0.1 ± 0.2	
θ_1		0.1 ± 0.2	0.1 ± 0.2	
θ_2			0.05 ± 0.1	
RSS	400	300	290	305

YOUR QUESTIONS START ON THE NEXT PAGE. PLEASE. TURN THE PAGE!

Part a)

What model did the engineer choose to be adequate for the time-series Y_t (conduct the necessary F-tests).

$F_{0.95}(1,\infty) = 3.8601$; $F_{0.95}(2,\infty) = 3.0138$; $F_{0.95}(4,\infty) = 2.3898$; $F_{0.95}(6,\infty) = 2.1167$

Test 1: $(2,2,1)$ vs $(1,1,0)$

$RSS_0 = 300 \quad RSS_1 = 400$

$s = 4 \quad r = 8$

$$F = \frac{(RSS_1 - RSS_0)/s}{RSS_0/(N-r)}$$

$$F = \frac{(400-300)/4}{300/(200-8)} = 16$$

$F_{0.95}(4,\infty) = 2.3858 \qquad F > F_{0.95}(4,\infty) \Rightarrow$ continue with more complex models

Test 2: $(3,3,2)$ vs $(2,2,1)$

$RSS_0 = 250 \quad RSS_1 = 300$

$s = 4 \quad r = 12$

$$F = \frac{(RSS_1 - RSS_0)/s}{RSS_0/(N-r)} = \frac{(300-290)/4}{250/(200-12)} = 1.621$$

$F_{0.95}(4,\infty) = 2.3858 \Rightarrow F < F_{0.95}(4,\infty) \Rightarrow$ no need for higher order models!

Note that $\hat{\theta}_2$ estimate is such that its confidence interval encompasses 0 $\Rightarrow$ we can try to test $(2,2,1)$ vs $(2,2,0)$

Test 3: $(2,2,1)$ vs. $(2,2,0)$

$RSS_0 = 300$ $RSS_1 = 305$

$S = 1$ $r = 8$

$$F = \frac{(RSS_1 - RSS_0)/S}{RSS_0/(N-r)} = \frac{(305-300)/1}{300/(200-8)} = 3.2$$

$F_{0.95}(1,\infty) = 3.86 \Rightarrow F < F_{0.95}(1,\infty) \Rightarrow$ we can use the model $(2,2,0)$

$$Y_t = 9.2 + 12.1 \sin\omega_0 t + 2.0\cos\omega_0 t + 5.0 \sin 2\omega_0 t + 0.3 \sin 2\omega_0 t + X_t$$

$$X_t - 0.8X_{t-1} + 0.12X_{t-2} = a_t$$

Deterministic seasonalities with frequencies ω_0 & $2\omega_0$

Do we have stochastic trends/seasonalities in X_t?

$\lambda_{1,2} = 0.2$ & 0.6 (not even on the unit circle)

Stochastic Trends & Seasonalities

<u>Def.</u> Stochastic trends and/or seasonalities exist if appropriate trends/seasonalities exist in the Green's Function of a time-series

Ex. If Green's Function of a time-series shows polynomial trend of 3rd order => we say that that time-series displays a <u>3rd order polynomial stochastic trend!</u>

In the case of deterministic trends $E[X_t] = f_\psi(t) \neq 0$

In the case of stochastic trends or seasonalities $E[X_t] = 0$

i) Stochastic trends

Polynomial stochastic trend of order ℓ exists if one AR root of multiplicity $\ell+1$ exists exactly at 1, while all other AR roots are inside the unit circle, or if they are ON the unit circle, they are of multiplicity 1.

Note: root of multiplicity $\ell+1$ carries the term

$$C_0 \lambda_1^{2} + C_1 2 \lambda_1^{2} + \dots + C_\ell 2^{\ell} \lambda_1^{2}$$

into the G.F.

Hence, if λ_1 is at 1, we have a polynomial trend in the G.F.

ii) Stochastic seasonalities

Stochastic seasonality of period $\frac{2\pi}{\omega}$ exists if AR characteristic polynomial has a pair of roots

$$\lambda_{1/2} = e^{\pm j\omega}$$

which are of multiplicity one, while all other AR characteristic roots are either inside the unit circle or if they are ON the unit circle, they are of multiplicity 1.

Note: A pair of complex conjugate AR roots $\lambda_{1/2} = e^{\pm \omega j}$ corresponds to a 2nd order polynomial factor

$$(1-\lambda_1 B)(1-\lambda_2 B) = 1 - 2\cos\omega \, B + B^2$$

inside the AR characteristic polynomial

"Interesting" seasonalities:

a) Period of 12 $\rightarrow$ $\omega = \frac{2\pi}{12}$ => Block corresponding to this seasonality is

$$1 - 2\cos\frac{2\pi}{12}B + B^2 =$$

$$= 1 - \sqrt{3}B + B^2$$

b) Period of 3 (quarterly) $\rightarrow$ $\omega = \frac{2\pi}{3}$ => Block corresponding to this seasonality is

$$1 - 2\cos\frac{2\pi}{3}B + B^2 =$$

$$= 1 - B + B^2$$

How to confirm or disconfirm existance of some stochastic seasonality? AR roots in reality NEVER fall onto "nice" seasonalities...

If $\lambda_{1/2}$ fall "near" some seasonalities that make sense (quarterly, yearly, weekly...), we should fix 2 roots of the AR polynomial to exactly those "nice" values and see if the RSS increases significantly or not. If not $\rightarrow$ then the corresponding seasonality exists.

Eg. Let's suspect that there is a seasonality with period p in a model

$$(1-\phi_1 B - \dots - \phi_n B^n) X_t = (1-\theta_1 B - \dots - \theta_m B^m) a_t$$

AR block corresponding to periodicity p is

$$(1-\lambda_1 B)(1-\lambda_2 B) = 1 - 2\cos\frac{2\pi}{p} B + B^2$$

since $\lambda_{1/2} = e^{j\frac{2\pi}{p}}$

i) Create a new time series $Y_t = (1 - 2\cos\frac{2\pi}{p} B + B^2) X_t$

ii) Fit an ARMA$(n-2, m)$ model to Y_t and note the new RSS. This new model of Y_t is referred to as the "parsimonious" model!

iii) If the RSS corresponding to the parsimonious model does not increase significantly, then seasonality of period p indeed exists!

$$F = \frac{(\overset{\text{RSS of the parsimonious model}}{RSS_{pars}} - RSS_{original})/\overset{\text{only 2 parameters are restricted}}{2}}{RSS_{original}/(\underset{\text{Number of samples}}{N} - \underset{\text{Number of parameters of the original model ARMA}(n,m)}{r})} \sim F_{2, N-r}$$