Autocovariance function

Autocovariance function gives a statistical characterization of the dependence between the sequence of random variables $X_t, X_{t-1}, X_{t-2}, \ldots$.

Distribution Properties of a_t

Fixed t: a_t is a random variable

all t: a_t is a stochastic process

$a_t \sim NID(0, \sigma_a^2)$ $E(a_t) = 0$; $Var(a_t) = \sigma_a^2$; $Cov(a_j, a_i) = 0$ if $i \neq j$

Covariance of a stochastic process with itself at different values of t, called **autocovariance**.

In general,

$$Cov(a_t, a_{t-k}) = E(a_t - \mu)(a_{t-k} - \mu)$$

since $\mu = 0$

$$Cov(a_t, a_{t-k}) = E(a_t a_{t-k}) \begin{array}{ll} = \sigma_a^2 & k=0 \\ = 0 & k \neq 0 \end{array}$$

$$E(a_t a_{t-k}) = \delta_k \sigma_a^2 \qquad \delta_k \begin{array}{ll} = 0 & k \neq 0 \\ = 1 & k = 0 \end{array}$$

Autocovariance and autocorrelation function

$$E(a_t a_{t-k}) = \delta_k \sigma_a^2 \qquad \delta_k \begin{array}{ll} = 0 & k \neq 0 \\ = 1 & k = 0 \end{array}$$

$$\gamma_k = E(X_t X_{t-k}) \qquad \rho_k = \frac{\gamma_k}{\gamma_0} \quad \text{with } \rho_0 = 1$$

<u>covariance between a and X</u>:

$$X_t = \sum_{j=0}^{\infty} G_j a_{t-j}$$

$$E(a_t X_t) = \sigma_a^2 \quad \text{since } G_0 = 1$$

$$E(a_t X_{t-k}) = 0, \quad \text{for } k > 0 \qquad X_{t-k} = \sum_{j=0}^{\infty} G_j a_{t-k-j}$$

$$E(a_t X_{t-k}) = \delta_k \sigma_a^2$$

$$\gamma_k = E(X_t X_{t-k}) = E(X_{t-k} X_t) = E(X_t X_{t+k}) = \gamma_{-k}$$

1). For AR(1) model:

$$X_t = \phi_1 X_{t-1} + a_t$$

$$\gamma_k = E(X_t X_{t-k}) = \phi_1 E(X_{t-1} X_{t-k}) + E(a_t X_{t-k})$$

$$\gamma_0 = \phi_1 \gamma_1 + \sigma_a^2$$

$$\gamma_k = \phi_1 \gamma_{k-1} \qquad k>0$$

$$\gamma_0 = \frac{\sigma_a^2}{1-\phi_1^2} = Var(X_t)$$

$$\rho_0 = 1 \quad \rho_k = \phi_1 \rho_{k-1}$$

implicit method

$$G_j = \phi_1^j$$

$$\gamma_k = E(X_t X_{t-k}) = E\left[\left(\sum_{i=0}^{\infty} G_i a_{t-i}\right)\left(\sum_{j=0}^{\infty} G_j a_{t-(j+k)}\right) \right] = \left(\sum_{j=0}^{\infty} G_{j+k} G_j\right) \sigma_a^2 = \left(\phi_1^{\kappa} \sum_{j=0}^{\infty} \phi_1^{2j}\right) \sigma_a^2$$

$$= \frac{\sigma_a^2}{1-\phi_1^2} \phi_1^k$$

$$\rho_0 = 1 \quad \rho_k = \phi_1^k$$

explicit method

2). For MA(1) model:

$$X_t = a_t - \theta_1 a_{t-1}$$

$$E(X_t X_{t-k}) = E(a_t X_{t-k}) - \theta_1 E(a_{t-1} X_{t-k})$$ since $G_0=1$, $G_1 = -\theta_1$, $G_j=0$ for $j \geq 2$

$$\gamma_0 = \sigma_a^2 - \theta_1 G_1 \sigma_a^2 = \sigma_a^2 + \theta_1^2 \sigma_a^2$$

$$\gamma_1 = -\theta_1 \sigma_a^2$$

$$\gamma_k = 0 \qquad \text{for } k \geq 2$$

$$\rho_0 = 1 \qquad \rho_1 = -\frac{\theta_1}{1+\theta_1^2} \qquad \rho_k = 0 \quad k \geq 2$$

$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} - \theta_1 a_{t-1} + a_t$

$E(X_t X_{t-k}) = \phi_1 E(X_{t-1}X_{t-k}) + \phi_2 E(X_{t-2}X_{t-k}) - \theta_1 E(a_{t-1}X_{t-k}) + E(a_t X_{t-k})$

k=0: $\gamma_0 = \phi_1\gamma_1 + \phi_2\gamma_2 - \theta_1 G_1 \sigma_a^2 + \sigma_a^2$

k=1: $\gamma_1 = \phi_1\gamma_0 + \phi_2\gamma_1 - \theta_1\sigma_a^2$

k: $\gamma_k = \phi_1\gamma_{k-1} + \phi_2\gamma_{k-2}, \quad k \geq 2$

implicit method , recursive

To find γ_k , γ_0 and γ_1 must be found first by simultaneously solving for γ_0 and γ_1 . (p.120)

$$G_j = g_1\lambda_1^j + g_2\lambda_2^j$$

$$\gamma_k = E(X_t X_{t-k}) = E\left[\left(\sum_{i=0}^{\infty} G_i a_{t-i}\right)\left(\sum_{j=0}^{\infty} G_j a_{t-(j+k)}\right)\right] = \left(\sum_{j=0}^{\infty} G_{j+k}G_j\right)\sigma_a^2$$

$$= \sigma_k^2 \sum_{j=0}^{\infty}\left(g_1\lambda_1^{j+k} + g_2\lambda_2^{j+k}\right)\left(g_1\lambda_1^j + g_2\lambda_2^j\right)$$

$$= \sigma_k^2\left[\frac{g_1^2}{1-\lambda_1^2}\lambda_1^k + \frac{g_2^2}{1-\lambda_2^2}\lambda_2^k + \frac{g_1 g_2}{1-\lambda_1\lambda_2}(\lambda_1^k + \lambda_2^k)\right]$$

$$\gamma_0 = \sigma_a^2\left[\frac{g_1^2}{1-\lambda_1^2} + \frac{g_2^2}{1-\lambda_2^2} + \frac{2g_1 g_2}{1-\lambda_1\lambda_2}\right]$$

$$\gamma_k = \sigma_a^2\left[\frac{g_1^2}{1-\lambda_1^2} + \frac{g_1 g_2}{1-\lambda_1\lambda_2}\right]\lambda_1^k + \sigma_a^2\left[\frac{g_2^2}{1-\lambda_2^2} + \frac{g_1 g_2}{1-\lambda_1\lambda_2}\right]\lambda_2^k$$

$\gamma_0 = d_1 + d_2$

$\gamma_k = d_1\lambda_1^k + d_2\lambda_2^k$

Remarks:

- The total variance γ_0 in an ARMA(2,1) system is decomposed into two components d_1 and d_2. d_1 and d_2 are respectively associated with the characteristic roots λ_1 and λ_2.
- There are similarities between Green's function and the autocovariance function.

Given: a_t's $\quad X_t = \sum_{j=0}^{\infty} G_j a_{t-j}$ convolution

Given: $E(a_t a_{t-k})$ $\quad \gamma_k = \sigma_a^2 \sum_{j=0}^{\infty} G_j G_{j+k}$ double convolution

4). General Results

ARMA(n,m) model:

$$X_t=\phi_1 X_{t-1}+\phi_2 X_{t-2}+\ldots+\phi_n X_{t-n} + a_t-\theta_1 a_{t-1}-\ldots-\theta_m a_{t-m}$$

$$\gamma_0=\phi_1\gamma_1 + \phi_2\gamma_2+\ldots+ \phi_n\gamma_n + (1-\theta_1 G_1-\theta_2 G_2-\ldots-\theta_m G_m)\sigma_a^2$$

$$\gamma_1=\phi_1\gamma_0 + \phi_2\gamma_1+\ldots+ \phi_n\gamma_{n-1} + (-\theta_1 G_0-\theta_2 G_1-\ldots-\theta_m G_{m-1})\sigma_a^2$$

.....

$$\gamma_m=\phi_1\gamma_{m-1} + \phi_2\gamma_{m-2} +\ldots+ \phi_n\gamma_{|m-n|} -\theta_m\sigma_a^2$$

$$\gamma_\kappa =\phi_1\gamma_{\kappa-1} + \phi_2\gamma_{\kappa-2} +\ldots+ \phi_n\gamma_{k-n} \qquad \kappa\geq m+1$$

$$\Phi(B)\gamma_k = 0, \qquad k\geq m+1$$

For an ARMA(n,n-1) model,

$$\gamma_k = d_1\lambda_1^\kappa+d_2\lambda_2^\kappa+ \ldots + d_n\lambda_n^\kappa$$

d can be found in Eqs.(3.3.26) and (3.1.26)

$$\gamma_0 = d_1+d_2+ \ldots + d_n$$

Applications in machine condition monitoring and diagnostics.

Theoretical/Sample Autocovariance/Autocorrelation function

$$\gamma_k = E(X_t X_{t-k}) \qquad \rho_k=\frac{\gamma_k}{\gamma_0} \quad \text{with } \rho_0=1$$

$$\hat{\gamma}_k = \frac{1}{N}\sum_{t=k+1}^{N} X_t X_{t-k} = \frac{1}{N}\sum_{t=k+1}^{N} (\dot{X}_t-\overline{X})(\dot{X}_{t-k}-\overline{X})$$

$$\hat{\rho}_k = \frac{\hat{\gamma}_k}{\hat{\gamma}_0}$$

Remarks:

- Sample autocovariance/autocorrelation functions can be obtained directly from the data before fitting a model.
- The use of autocorrelation functions for modeling and estimation of time series would be appropriate if a good estimate can be obtained.
- Sample autocorrelation functions often are very poor estimates and have large variances and present a distorted version of the true autocorrelations.