Random Walk as a limit of AR(1)

In general, an AR(1) model is a good approximation for many systems characterized by inertia.

e.g.

for IBM stock prices data,

$\hat{\phi}_1 = 0.999 \qquad \hat{\sigma}_a^2 = 52.61$

$X_t = 0.999 X_{t-1} + a_t \qquad a_t \sim NID(0,\ 52.61)$

From figs. 2-13 & 2-14, it can be seen that above AR(1) model is adequate.

$X_t = X_{t-1} + a_t$ or $X_t - X_{t-1} = a_t \qquad \nabla X_t = a_t$

Remarks:

- The system is characterized by high inertia, or strong dependence / memory.
- Its response or value remains unchanged from t-1 to t, except for an random independent increment a_t. But, $E(a_t)=0$, the system would stay in the same position indefinitely.
- $\hat{X}_{t-1}(1) = X_{t-1}$

AR(2) model

$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + a_t$

conditional multiple linear regression

$$Y = \begin{vmatrix} X_3 \\ X_4 \\ X_N \end{vmatrix} \qquad X = \begin{vmatrix} X_2 & X_1 \\ X_3 & X_2 \\ X_{N-1} & X_{N-2} \end{vmatrix}$$

$$\begin{bmatrix} \hat{\phi}_1 \\ \hat{\phi}_2 \end{bmatrix} = (\mathbf{X'X})^{-1}\mathbf{X'Y}$$

ARMA(2,1) model

Consider Wolfer's sunspot numbers data.

AR(1) : $\hat{\phi}_1 = 0.81$ $\hat{\sigma}_a^2 = 409.08$

$\hat{\rho}(a_t, a_{t-1}) = 0.53$ $\hat{\rho}(a_t, X_{t-2}) = -0.38$

dependence of a_t on X_{t-2} and a_{t-1}

$$X_t = \phi_1 X_{t-1} + a_t'$$

$$a_t' = \phi_2 X_{t-2} - \theta_1 a_{t-1} + a_t$$

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} - \theta_1 a_{t-1} + a_t$$

when X_t is known,

$$a_t = X_t - \phi_1 X_{t-1} - \phi_2 X_{t-2} + \theta_1 a_{t-1}$$

Remarks:

- The model expresses the dependence of X_t on its two preceeding values, i.e., has an "autoregressive dependence" of order two.
- It also includes the dependence on preceeding a_t values of order one. Thus, called ARMA(2,1).
- Assumptions:
  a_t is independent of a_{t-2}, a_{t-3}, ...
  a_t is independent of X_{t-3}, X_{t-4}, ...
- The estimation of the ARMA(2,1) model is much more complicated compared to the AR(1) model.

Special Case: AR(2) model

$\theta_1 = 0$

$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + a_t$

conditional multiple linear regression

$$Y = \begin{vmatrix} X_3 \\ X_4 \\ X_N \end{vmatrix} \qquad X = \begin{vmatrix} X_2 & X_1 \\ X_3 & X_2 \\ X_{N-1} & X_{N-2} \end{vmatrix}$$

$$\begin{bmatrix} \hat{\phi}_1 \\ \hat{\phi}_2 \end{bmatrix} = (\mathbf{X'X})^{-1}\mathbf{X'Y}$$

Consider Wolfer's sunspot numbers data again:

AR(2) : $\hat{\phi}_1 = 1.34$ $\hat{\phi}_2 = -0.65$ $\sigma_a^2 = 236.85$

Adequacy checking (comparing AR(2) with AR(1) for sunspot series)

- The value ϕ_2 is fairly large, which shows the importance of including the X_{t-2}. Thus, AR(1) model, limiting the dependence to only X_{t-1} is inadequate.
- The drastic reduction in RSS indicates the AR(2) model accounts for a much larger portion of the total variance of the X_t series through the dependence of X_t on X_{t-1} and X_{t-2} than does the AR(1) model.

Q: How large should the value of the estimated additional parameters (ϕ_2, θ_1) or the reduction in the RSS be, to justify going from AR(1) model to a higher order model? (Ch. 4)

ARMA(1,1), MA(1) , and AR(1) are also special cases of ARMA(2,1) models

Non-linear regression of the ARMA(2,1) model

Conditional regressions for both AR(2) and ARMA(2,1) models are linear.

$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + a_t$

$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} - \theta_1 a_{t-1} + a_t$

Unconditional regression for AR(2) model is still linear, but for ARMA(2,1) model is non-linear.

$$a_t = X_t - \phi_1 X_{t-1} - \phi_2 X_{t-2} + \theta_1 a_{t-1}$$

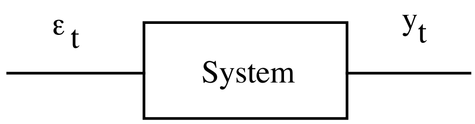
$$a_{t-1} = X_{t-1} - \phi_1 X_{t-2} - \phi_2 X_{t-3} + \theta_1 a_{t-2}$$

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} - \theta_1 a_{t-1} + a_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} - \theta_1 [X_{t-1} - \phi_1 X_{t-2} - \phi_2 X_{t-3} + \theta_1 a_{t-2}] + a_t$$

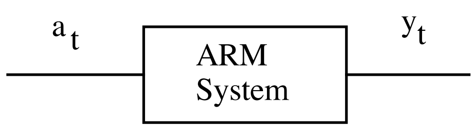
$$= (\phi_1 - \theta_1) X_{t-1} + (\phi_2 + \theta_1 \phi_1) X_{t-2} + \theta_1 \phi_2 X_{t-3} - \theta_1^2 a_{t-2} + a_t$$

Green's Function

Static vs. dynamic dependence:



A disturbance ε_t entering a regression system at time t affects only y_t but not y_{t+1}.



A disturbance at affecting the system is "remembered" and continues to affect the system at subsequent times.

Green's function of an AR(1) model:

$$\frac{dy(t)}{dt} + k\, y(t) = u(t) \qquad \text{continuous differential equation}$$

Solution: $$y(t) = \int_0^\infty h(\tau)\, u(t-\tau)\, d\tau \qquad \text{convolution integral}$$

$D = \frac{d}{dt}(\quad)$: D is differential operator $\qquad (D + k)\, y(t) = u(t)$

Remarks:

- h(t) is the impulse response function, which describes the characteristics of a dynamic system, i.e., from h(t), we can find system transfer function, determine system stability, response speed, and other physical characteristics.
- left-hand side of equation represents the homogeneous part and u(t) is a forcing function. The system characteristic equation can be found from the homogeneous part.
- the convolution integral can be interpreted as:

the current system response, y(t), is affected by all previous forcing input, u(t).

$X_t - \phi_1 X_{t-1} = a_t$ discrete difference equation

Solution: $X_t = \phi_1(\phi_1 X_{t-2} + a_{t-1}) + a_t = a_t + \phi_1 a_{t-1} + \phi_1^2 a_{t-2} + \ldots = \sum_{j=0}^{\infty} \phi_1^j a_{t-j}$

$G_j = \phi_1^j$ Green' s function

Remarks:

- G_j characterizes the dynamics or the memory of a system. It describes the influence of past "forcing input", a_t's, on X_t.
- G_j indicates how well the system remembers the shocks a_{t-j}.
  The larger the value of ϕ_1 in the AR(1) model, the more clearly is the shock a_{t-j} remembered. G_j is like a "weighting" function.
- G_j determines how slow or fast the dynamic response of the system to any particular a_t decays.
- G_j is the impulse response function.