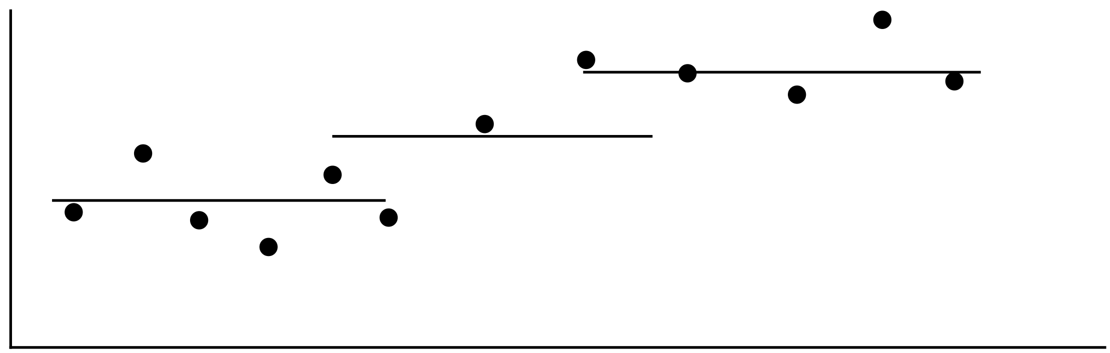
Exponentially weighted moving average (EWMA)

Idea simple and intuitively appealing
Assuming no knowledge of ARMA models / forecasts



(1) Intuitively, it will be safe and simple to take the average of the past data as the forecast.
(2) Random fluctuations in the data would be smoothed out.
(3) Taking the average of the entire past data seems unreasonable; the data in the distant past may be outdated and should be discarded after certain period of time.

$$\hat{X}_t(1) = \frac{1}{N}\sum_{j=0}^{N-1} X_{t-j} = \sum_{j=0}^{N-1} \frac{1}{N} X_{t-j}$$

(4) Equal weight --> unreasonable

Exponential weight $|\theta| < 1$ θ^j

$$\sum_{j=0}^{\infty} \theta^j = \frac{1}{1-\theta}$$

To get a weighted average, we should use

$(1-\theta)\,\theta^j$ Summation of all weights is equal to one.

$$\hat{X}_t(1) = \sum_{j=0}^{\infty} (1-\theta)\theta^j X_{t-j} \overset{\lambda=1-\theta}{=\!=\!=\!=\!=\!=} \sum_{j=0}^{\infty} \lambda(1-\lambda)^j X_{t-j}$$

EWMA or exponential smoothing.

Remarks

- The exponential smoothing constant λ determines the weight given to the past data in the forecast. When λ is large, the present observation is given more weight and the past observations have less influence on the forecast.
- Exponential smoothing can be obtained by a recursive formula that involves very little computations.

$$\hat{X}_{t-1}(1) = \sum_{j=0}^{\infty} \lambda(1-\lambda)^j X_{t-j-1}$$

$$\hat{X}_t(1) = \sum_{j=0}^{\infty} \lambda(1-\lambda)^j X_{t-j} = \lambda X_t + \sum_{j=1}^{\infty} \lambda(1-\lambda)^j X_{t-j}$$

$$= \lambda X_t + (1-\lambda) \sum_{j=0}^{\infty} \lambda(1-\lambda)^j X_{t-j-1} = \lambda X_t + (1-\lambda)\hat{X}_{t-1}(1) = \hat{X}_{t-1}(1) + \lambda[X_t - \hat{X}_{t-1}(1)]$$

- It can be proven that exponential smoothing is a special case of ARMA models: ARMA(1,1)

$$\hat{X}_t(1) = \phi_1 X_t - \theta a_t$$

$$= \phi_1 X_t - \theta[X_t - \hat{X}_{t-1}(1)] = (\phi_1 - \theta)X_t + \theta \hat{X}_{t-1}(1)$$

$$= (\phi_1 - \theta)X_t + \theta[(\phi_1 - \theta)X_{t-1} + \theta \hat{X}_{t-2}(1)]$$

$$= \sum_{j=0}^{\infty} (\phi_1 - \theta)\theta^j X_{t-j}$$

When $\phi_1 = 1$, it reduces to exponential smoothing.

That is,

$(1-B)X_t = (1-\theta B)\, a_t$ or $\nabla X_t = (1-\theta B)a_t$

corresponds to an exponential smoothing.

Uniformly sampled continuous-time stochastic linear systems

Linear stochastic systems are described in continuous time via linear ordinary differential equations with constant coefficients, where the forcing term is a continuous time white noise stochastic process.

General case of the continuous time covariance function

$$\gamma(s) = E[X(t)\,X(t-s)] = E[\int_0^\infty G(v')Z(t-v')dv' \int_0^\infty G(v)Z(t-s-v)dv\ ]$$

$$= \int_0^\infty\int_0^\infty G(v')G(v)\,E[Z(t-v')Z(t-s-v)]dv\,dv'$$

$$= \sigma_z^2 \int_0^\infty [\int_0^\infty G(v')G(v)\,\delta(v+s-v')dv']dv \ = \sigma_z^2 \int_0^\infty G(v)G(v+s)\,dv$$

$$\int_{-\infty}^{+\infty} f(t-u)\,\delta(u)\,du = \int_{-\infty}^{+\infty} f(u)\,\delta(t-u)\,du = f(t)$$

$$\int_{-\infty}^{+\infty} f(t-u)\,\delta^{(k)}(u)\,du = (-1)^k f^{(k)}(t)$$

(the last equation can be shown using integration by parts

For an A(1) system (first order cont. time stochastic system)

$$G(t) = s(t)\,e^{-\alpha_0 t}$$

$$\gamma(s) = \sigma_z^2 \int_0^\infty e^{-\alpha_0 v} e^{-\alpha_0(v+s)}dv \ = \sigma_z^2 e^{-\alpha_0 s} \int_0^\infty e^{-2\alpha_0 v}dv$$

$$= \sigma_z^2 e^{-\alpha_0 s}\left[\frac{-e^{-2\alpha_0 v}}{2\alpha_0}\right]_0^\infty = \frac{\sigma_z^2}{2\alpha_0} e^{-\alpha_0 s} \qquad s \geq 0$$

$$\gamma(0) = \frac{\sigma_z^2}{2\alpha_0} \qquad\qquad \rho(s) = \frac{\gamma(s)}{\gamma(0)} = e^{-\alpha_0 s}$$

Note: It can be easily shown that

$\rho(-s) = \rho(s)$ and $\gamma(-s) = \gamma(s)$

Thus,

$$\gamma(s) = \frac{\sigma_z^2}{2\,\alpha_0} e^{-\alpha_0 |s|} \qquad\qquad \rho(s) = e^{-\alpha_0\,|s|}$$

The autocovariance function

$$\gamma(s) = E[X(t)\,X(t-s)] = E[\int_0^\infty G(v')Z(t-v')dv' \int_0^\infty G(v)Z(t-s-v)dv\ ]$$

$$= \int_0^\infty\int_0^\infty G(v')G(v)\,E[Z(t-v')Z(t-s-v)]dv\,dv'$$

$$= \sigma_z^2 \int_0^\infty [\int_0^\infty G(v')G(v)\,\delta(v+s-v')dv']dv \ = \sigma_z^2 \int_0^\infty G(v)G(v+s)\,dv$$

$$\int_{-\infty}^{+\infty} f(t-u)\,\delta(u)\,du = \int_{-\infty}^{+\infty} f(u)\,\delta(t-u)\,du = f(t)$$

$$\int_{-\infty}^{+\infty} f(t-u)\,\delta^{(k)}(u)\,du = (-1)^k\, f^{(k)}(t)$$