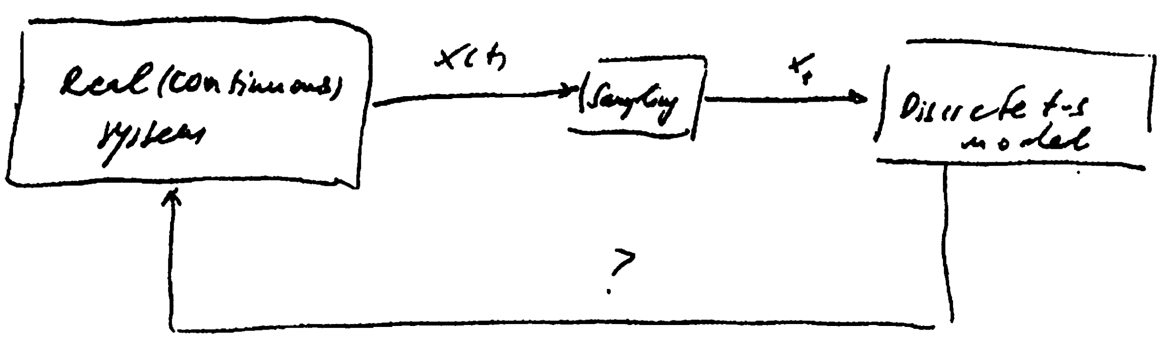
Uniform Sampling of Continuous Time-Series
(Chapter 6)

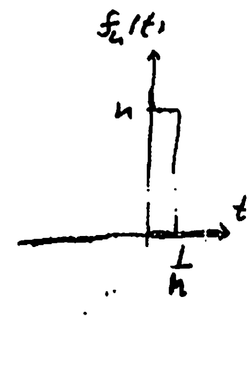
Actual physical phenomena are continuous in nature & discrete time-series are sampled versions of what actually happens in continuous time

Can we observe the discrete-time time-series & make judgement about the continuous-time (actual, physical) system?



To find this connection, let's review continuous-time Dirac's delta function

Def. $f_n(t) = \begin{cases} n, & |t| < 1/n \\ 0, & \text{otherwise} \end{cases}$ $\quad \delta(t) = \lim_{n \to \infty} f_n(t)$



Properties of $\delta(t)$

$$\delta(t) = \begin{cases} \infty, & t=0 \\ 0, & t \neq 0 \end{cases} \qquad \int_{-\tau}^{\tau} \delta(t)\,dt = 1$$

$$\int_{-\infty}^{t} \delta(u)\,du = \begin{cases} 1 & t>0 \\ 0 & t \leq 0 \end{cases} \qquad \int_{-\infty}^{\infty} f(u)\,\delta(t-u)\,du = f(t)$$

$\Rightarrow \delta(t) = \frac{d}{dt} S(t) \leftarrow$ step function

Integration by parts can give $\int_{-\infty}^{\infty} f(t-u)\,\delta^{(k)}(u)\,du = (-1)^k f^{(k)}(t)$

Review of Linear Ordinary Differential Equations with Constant Coefficients

i) Linear Homogeneous Differential Eqns with Constant Coefficients

$$X^{(n)}(t) + \alpha_{n-1} X^{(n-1)}(t) + \dots + \alpha_0 X(t) = 0$$

$$(D^n + \alpha_{n-1} D^{n-1} + \dots + \alpha_0)\, X(t) = 0$$

Solution must be of the shape

$$X(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} + \dots + C_n e^{\lambda_n t}$$

where $\lambda_i,\ i=1,2,\dots,n$ are roots of the characteristic polynomial

$$s^n + \alpha_{n-1} s^{n-1} + \dots + \alpha_0 = 0$$

and constants $c_1, c_2, \dots, c_n$ are obtained from the initial values of the function $x(0), x'(0), \dots, x^{(n-1)}(0)$

First Order H.O.D.E.

$$x'(t) + \alpha_0 x(t) = 0$$

Physically: changes of signals are proportional to signal values!

$$x(t) = c_1 e^{-\alpha_0 t} = c_1 e^{-\frac{t}{\tau}}$$ where $\tau = \frac{1}{\alpha_0}$ - time constant

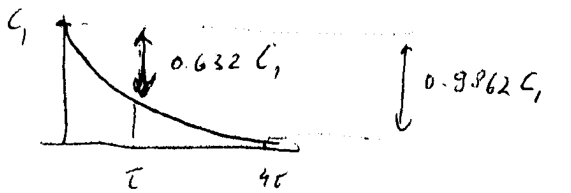
$\alpha_0 > 0 \Rightarrow x(t) \to 0$ (stable system)

$\alpha_0 = 0 \Rightarrow x(t) = c_1$ (marginally stable system)

$\alpha_0 < 0 \Rightarrow x(t) \to \infty$ (unstable system)

Large α_0 (small τ) $\Rightarrow$ ~~to~~ rapid decay (quick system)

Small α_0 (large τ) $\Rightarrow$ slow decay (slow system)



General Linear ODE-s with Constant Coefficients

$$X^{(n)}(t) + \alpha_{n-1} X^{(n-1)}(t) + \dots + \alpha_0 X(t) = \beta_m U^{(m)}(t) + $$
$$+ \beta_{m-1} U^{(m-1)}(t) + \dots + \beta_0 U(t)$$

where $m < n$ for causal systems

Solutions of these eqns can be found as

$$X(t) = \int_{-\infty}^{\infty} G(\tau) U(t-\tau)\, d\tau \underset{\uparrow}{=} \int_{0}^{\infty} G(\tau) U(t-\tau)\, d\tau$$

for a causal system

Where G is the impulse response of this system (this formula holds only when all initial conditions are 0 - if not true, one needs some modifications here, but that becomes control theory already).

Hence

$$G^{(n)}(t) + \alpha_{n-1} G^{(n-1)}(t) + \dots + \alpha_0 G(t) = \beta_m \delta^{(m)}(t) + \beta_{m-1} \delta^{(m-1)}(t) + \dots$$
$$+ \dots + \beta_0 \delta(t)$$

How to find $G(t)$?

$\boxed{\text{Thm}}$ For a Lin. Differential Eqn with constant coeff.

$$X^{(n)}(t) + \alpha_{n-1} X^{(n-1)}(t) + \dots + \alpha_0 X(t) = V(t)$$

$G(t)$ can be found by solving homogeneous ODE

$$G^{(n)}(t) + \alpha_{n-1} G^{(n-1)}(t) + \dots + \alpha_0 G(t) = 0$$

with initial conditions

$$G(0) = 0;\quad G'(0) = 0;\ \dots;\quad G^{(n-1)}(0) = 1$$

(this is for $t \geq 0$; for $t<0$, $G(t)=0$, since it's a causal system).

Proof: Easiest using Laplace transforms - please feel free to discuss with me this proof, so we can avoid turning this into a controls class

Ex. For a first order system;

$$X'(t) + \alpha_0 X(t) = u(t) \qquad G'(t) + \alpha_0 G(t) = 0$$

$$G(0) = 1 \Rightarrow$$

$$\Rightarrow G(t) = e^{-\alpha_0 t}$$

Let's now swith gears & observe discrete version of differential eqns $\rightarrow$ difference equations.

AR(1) model can be seen as a 1st order ~~[illegible]~~

Homogeneous Difference Eqn.

$$X_t - \phi_1 X_{t-1} = 0$$

Driven by white noise a_t as

$$X_t - \phi_1 X_{t-1} = a_t$$

Now, before introducing "stochastic continuous ODE and systems, I need to define continuous-time white noise.

$\boxed{\text{Def}}$ $Z(t)$, $t \in R$ is a white noise stochastic process iff.

i) $E[Z(t)] = 0$ ii) $E[Z(t) Z(t-s)] = \text{cov}(Z(t), Z(t-s))$
$= \sigma_z^2 \, \delta(s)$

Hence, it's a stochastic process where each sample is independent of samples infinitely close to it - does NOT really exist in nature, but very convenient for analysis because of the orthogonality property, i.e. because of property ii).

$\boxed{\text{Def}}$ Stochastic linear ODE with constant coefficients is an equation of the form

$$X^{(n)}(t)+\alpha_{n-1}X^{(n-1)}(t)+\ldots+\alpha_0 X(t)=$$
$$=\beta_m Z^{(m)}(t)+\beta_{m-1}Z^{(m-1)}(t)+\ldots+\beta_0 Z(t)$$

where $Z(t)$ is a Gaussian white noise process.

Solution of this equation for any trace (realization) of the random process $Z(t)$ is

$$X(t)=\int_0^\infty G(\tau)Z(t-\tau)\,d\tau \qquad (**)$$

where $G(\tau)$ is the impulse response of the system above, i.e. $G(\tau)$ is solution of the equation

$$G^{(n)}(t)+\alpha_{n-1}G^{(n-1)}(t)+\ldots+\alpha_0 G(t)=$$
$$=\beta_m\delta^{(m)}(t)+\beta_{m-1}\delta^{(m-1)}(t)+\ldots+\beta_0\delta(t)$$

where $\delta(t)$ is the impulse function.

Obviously, $(**)$ denotes a stochastic process

* $E[X(t)] = 0$ since $E[Z(t)] = 0$

* $\gamma(s) = E[X(t)X(t-s)] = E[\int_0^\infty\int_0^\infty G(s_1) Z(t-s_1) G(s_2) Z(t-s-s_2) ds_1 ds_2$

$$= \int_0^\infty\int_0^\infty G(s_1) G(s_2) E[Z(t-s_1) Z(t-s-s_2)] ds_1 ds_2 =$$

$$= \int_0^\infty\int_0^\infty G(s_1) G(s_2) \delta(t-s_1-t+s+s_2) \sigma_z^2 ds_1 ds_2 =$$

$$= \int_0^\infty\int_0^\infty G(s_1) G(s_2) \delta(s+s_2-s_1) \sigma_z^2 ds_1 ds_2 =$$

$$= \sigma_z^2 \int_0^\infty G(s_2) ds_2 \int_0^\infty ds_1 \, G(\underset{u}{\underset{\uparrow}{s_1}}) \, \delta(\underbrace{s+s_2}_{t} - \underset{u}{\underset{\uparrow}{s_1}}) =$$

$$= \sigma_z^2 \int_0^\infty G(s_2) G(s+s_2) ds_2$$

For a first order system, $X'(t) + \alpha_0 X(t) = u(t)$, impulse response is $G(s) = e^{-\alpha_0 s}$ and hence:

$$\gamma(s) = \sigma_z^2 \int_0^\infty e^{-\alpha_0 s_2} e^{-\alpha_0 (s+s_2)} ds_2 =$$

$$= \sigma_z^2 \int_0^\infty e^{-\alpha_0 s} e^{-2\alpha_0 s_2} ds_2 = \sigma_z^2 \frac{e^{-\alpha_0 s}}{2\alpha_0}$$

Can we connect this to some discrete system?

Note: $X'(t) + \alpha_0 X(t) = Z(t)$ is often referred to as a stochastic autoregressive model of order 1 (label as $A(1)$)

Two strategies will be explored in our course in order to accomplish equidistant sampling of a continuous-time time-series (or rather to establish a connection between a continuous-time system and a discrete-time system).

(a) Impulse response equivalent sampling

$$G_j = G(\Delta j) \qquad \Delta\text{ - sampling interval}$$

- it ensures that the continuous-time and discrete-time systems have <u>impulse responses</u> that match at appropriate samples
- it ensures that stability properties of the continuous-time and discrete-time systems are the same.

(b) Covariance function equivalent sampling

$$\gamma_j = \gamma(\Delta j)$$

- it ensures that the continuous-time and discrete-time systems have <u>covariance functions of their responses</u> match at appropriate samples
- once again, stability properties of the continuous and discrete-time systems are the same.

Impulse Response Equivalent Sampling

For a 1st order system $X'(t) + \alpha_0 X(t) = Z(t)$, $G(t) = e^{-\alpha_0 t}$

$$\Rightarrow G(j\Delta) = G_j \Leftrightarrow$$

$$\Leftrightarrow G(j\Delta) = e^{-\alpha_0 \Delta j} = G_j = \phi^j \Rightarrow \phi = e^{-\alpha_0 \Delta}$$

σ_a^2 can be found the same way as in the covariance equivalence sampling.

Covariance Function Equivalent Sampling

$$\gamma_\ell = E[X_t X_{t-\ell}] = \gamma^e(\ell \cdot \Delta) \qquad \Delta - \text{sam}$$

$$\Rightarrow \gamma_\ell = \frac{\sigma_z^2}{2\alpha_0} e^{-\alpha_0 \ell \Delta} = \frac{\sigma_z^2}{2\alpha_0}\left(e^{-\alpha_0 \Delta}\right)^\ell =$$

$$= \frac{\sigma_z^2}{2\alpha_0} \phi^\ell$$

where $\phi = e^{-\alpha_0 \Delta}$

This corresponds to an AR(1) model

$$X_t - \phi X_{t-1} = a_t$$

where a_t is white noise process whose variance can be found as

$$\gamma_0 = \frac{\sigma_a^2}{1-\phi^2} = \gamma(0) = \frac{\sigma_z^2}{2\alpha_0} \Rightarrow \sigma_a^2 = \frac{\sigma_z^2(1-\phi^2)}{2\alpha_0}$$

1^{st} Note: This time, two approaches gave the same result. For the 2^{nd} order systems, it won't be true!

2^{nd} Note: For a discrete system to originate from a real-life continuous 1^{st} order system, ϕ must be larger than 0!

$$\phi = e^{-\alpha_0 \Delta} > 0$$

Cont → Discr

$$\phi_1 = e^{-\alpha_0 \Delta}$$

$$\sigma_a^2 \doteq \frac{\sigma_Z^2 (1-\phi_1^2)}{2\alpha_0}$$

Discr → Cont

$$\alpha_0 = \frac{-\ln \phi_1}{\Delta}$$

$$\sigma_Z^2 = \frac{2\alpha_0 \sigma_a^2}{1-\phi^2}$$

Discussion about Equidistant Sampling of First Order Continuous Stochastic Systems

Cont → Discr

$$\phi_1 = e^{-\alpha_0 \Delta}$$

$$\sigma_a^2 = \frac{\sigma_z^2 (1-\phi_1^2)}{2\alpha_0}$$

Discr → Cont

$$\alpha_0 = \frac{-\ln \phi_1}{\Delta}$$

$$\sigma_z^2 = \frac{2\alpha_0 \sigma_a^2}{1-\phi^2}$$

Influence of the Sampling Rate (Interval)

i) $\Delta \uparrow \Rightarrow \phi_1 = e^{-\alpha_0 \Delta} \downarrow \Rightarrow$ dependence of X_t-s on each other drops. If we sample very, very far, $X(t)$ starts looking like a white noise process!

$$\sigma_a^2 \to \frac{\sigma_z^2}{2\alpha_0}$$

(ii) $\Delta \downarrow \Rightarrow \phi_1 = e^{-\alpha_0 \Delta} \to 1 \Rightarrow$ Resulting discrete time series model becomes

$$\sigma_a^2 = \frac{\sigma_z^2(1-\phi_1^2)}{2\alpha_0} \to 0$$

$$X_t - X_{t-1} = a_t$$

As we sample faster & faster, samples are so close that we almost know everything about the next sample from the previous one!

Influence of the continuous system dynamics (time-constant)

(iii) $\alpha_0 \uparrow \Rightarrow \tau \downarrow \Rightarrow$ less memory in the system & $G(t)$ as well as $\delta(t)$ drop off rapidly

$$\lim_{\alpha_0 \to \infty} \phi_1 = 0 \qquad \lim_{\alpha_0 \to \infty} \sigma_a^2 = \lim_{\alpha_0 \to \infty} \frac{\sigma_z^2(1-\phi_1^2)}{2\alpha_0} = 0$$

drops off slower than ϕ_1!

As I make large α_0, starts looking like a white noise process

$$\boxed{X_t = a_t}$$

(iv) $\alpha_0 \downarrow \Rightarrow \tau \uparrow \Rightarrow$ Greater memory, $G(t)$, $\delta(s)$ drop off slower!

$$\lim_{\alpha_0 \to 0} \phi_1 = 1 \qquad \lim_{\alpha_0 \to 0} \sigma_a^2 = \sigma_z^2 \Delta$$

Random walk is the resulting t-s (since memory is so strong, the best guess for the future is where I am now)