Lecture 7

Stability

<u>Def</u>: Time series with an ARMA$(n, n-1)$ model is referred to as stable if the system that generated that time series when white noise was fed into it is stable

From the systems theory, this is true iff

$|\lambda_i| < 1$ λ_i-s are ~~the~~ roots of the characteristic polynomial obtained from the AR part of the model

In other words, checking for stability is easy!

Let's say X_t has a model

$$\underbrace{X_t - \phi_1 X_{t-1} - \dots - \phi_n X_{t-n}} = a_t - \theta_1 a_{t-1} - \dots - \theta_{n-1} a_{t-n+1}$$

$$S^n - \phi_1 S^{n-1} - \dots - \phi_n = 0 \rightarrow \lambda_1, \lambda_2, \dots, \lambda_n$$

$|\lambda_i| < 1$ for all i?

YES — Stable system

NO — Unstable system

Heuristic

$$G_\ell = g_1 \lambda_1^\ell + g_2 \lambda_2^\ell + \dots + g_n \lambda_n^\ell$$

* Does impulse response die out?

* $X_t = G_t * a_t$ If I take a_t to have a finite variance, then

$$\mathrm{Var}[X_t] = \sigma_a^2 \sum_{t=0}^{\infty} G_t^2$$

→ infinite if $\exists \lambda_i, |\lambda_i| \geq 1$

→ finite if $\forall \lambda_i, |\lambda_i| < 1$

Levels of Stability

- Stable system (time-series) $\longleftrightarrow$ all roots of the characteristic AR polynomial are inside the unit circle ($|\lambda_i| < 1$ for all $i = 1, 2, \ldots, n$)
- Marginally stable system (time-series) $\longleftrightarrow$ all roots of the characteristic AR polynomial are inside or ON the unit circle, with those roots ON the unit circle having multiplicity 1.
- Unstable system (time-series) $\longleftrightarrow$ one root of the AR characteristic poly is OUTSIDE the unit circle, or some root of multiplicity higher than 1 is ON the unit circle

a) $(1 - 0.6B)(1 + 0.2B) X_t = (1 - 3B) a_t$ $\rightarrow$ stable

b) $(1 - 1.3B)(1 + 0.7B) X_t = (1 - 0.1B) a_t$ $\rightarrow$ unstable

c) $\underbrace{(1 + 2B + B^2)}_{(s+1)(s+1)} (1 - 0.2B) X_t = (1 - 0.2B) a_t$ $\rightarrow$ unstable

d) $(1 - B)(1 + B) X_t = a_t$ $\rightarrow$ marginally stable!

Inverse Function & Invertibility

In the case of the Green's function, we transformed ARMA model

$$X_t - \phi_1 X_{t-1} - \dots - \phi_n X_{t-n} = a_t - \theta_1 a_{t-1} - \dots - \theta_{n-1} a_{t-n+1}$$

into

$$X_t = G_0 a_t + G_1 a_{t-1} + G_2 a_{t-2} + \dots =$$
$$= \sum_{\ell=0}^{\infty} G_\ell a_{t-\ell}$$

In other words, we decomposed X_t onto orthogonal components a_t.

Another way of exploring dynamics behind an ARMA model is to decompose X_t into past X_t's:

$$X_t = I_1 X_{t-1} + I_2 X_{t-2} + \dots + I_\ell X_{t-\ell} + \dots + a_t$$

or equivalently

$$a_t = X_t - \left[\sum_{\ell=1}^{\infty} I_\ell B^\ell\right] X_t$$

Coefficients I_ℓ of this decomposition are referred to as the "Inverse Function" coefficients

Why "Inverse"? It's "inverse" from the Green's fcn!

G.F. does $\quad X_t = \text{Operator}_{GF}[a_t]$

I.F. does $\quad a_t = \text{Operator}_{IF}[X_t]$

$$\overbrace{\text{Operator}_{GF} * \text{Operator}_{IF}}^{\text{Identity Operator}}[X_t] = X_t$$

$$\underbrace{\text{Operator}_{IF} * \text{Operator}_{GF}}_{\text{Identity Operator}}[a_t] = a_t$$

Why is I.F. important?

If I can truncate the I.F. decomposition after some "n", what I get is AR(n) model, which I can easily identify (it's a simple Least Squares estimation problem). It would be great if I could estimate I_ℓ-s (easy) and from that get ϕ_i-s and θ_i-s!

Finding Inverse Function Coefficients from ARMA models

(i) AR(n) models

$$X_t - \phi_1 X_{t-1} - \phi_2 X_{t-2} - \dots - \phi_n X_{t-n} = a_t$$

This is already in the inverse function shape!

$$I_1 = \phi_1 \; ; \; I_2 = \phi_2 \; ; \dots ; \; I_n = \phi_n \; ; \; I_{n+1} = 0 \; ; \; I_{n+2} = 0 \dots$$

(ii) MA(1) models

$$X_t = a_t - \theta_1 a_{t-1} = (1 - \theta_1 B) a_t$$

$$(1 - \theta_1 B)^{-1} X_t = a_t$$

$$(1 + \theta_1 B + \theta_1^2 B^2 + \dots) X_t = a_t$$

$$\Rightarrow \quad I_1 = -\theta_1 \; ; \; I_2 = -\theta_1^2 \; ; \; I_3 = -\theta_1^3 \; ; \dots$$

Remember ~~MA~~ AR(1) model $X_t - \phi_1 X_{t-1} = a_t$

$\rightarrow$ G.F. for this model was $G_\ell = \phi_1^{\,\ell}$

iii) Let us observe an ARMA(1,2) model

$$X_t - \phi_1 X_{t-1} = a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2}$$

$$(1-\phi_1 B) X_t = (1-\theta_1 B - \theta_2 B^2) a_t$$

This is the only time we'll look into such a "non-physical" model – I do it to draw a parallel between I.F. and G.F.

$$a_t = (1 - I_1 B - I_2 B^2 - I_3 B^3 - \dots) X_t \qquad (\text{I.F.})$$

$$(1-\phi_1 B) X_t = (1-\theta_1 B - \theta_2 B^2)(1 - I_1 B - I_2 B^2 - \dots) X_t$$

B^0 : $1 = 1 = -I_0 \Rightarrow I_0 = -1$

B^1 : $-\phi_1 = -\theta_1 - I_1 \Rightarrow I_1 = \phi_1 - \theta_1$

B^2 : $0 = -I_2 + \theta_1 I_1 - \theta_2 = I_2 - \theta_1 I_1 - \theta_2 I_0$

B^3 : $0 = I_3 - \theta_1 I_2 - \theta_2 I_1$

— — — —

$$I_n - \theta_1 I_{n-1} - \theta_2 I_{n-2} = 0 \qquad n \geq 2$$

$$I_0 = -1; \quad I_1 = \phi_1 - \theta_1$$

Explicit way of finding I_e-s?

Just like in the case of G.F.-s