Uniformly sampled continuous-time stochastic linear systems

Linear stochastic systems are described in continuous time via linear ordinary differential equations with constant coefficients, where the forcing term is a continuous time white noise stochastic process.

General case of the continuous time covariance function

$$\gamma(s) = E[X(t)\,X(t-s)] = E[\int_0^\infty G(v')Z(t-v')dv' \int_0^\infty G(v)Z(t-s-v)dv\ ]$$

$$= \int_0^\infty\int_0^\infty G(v')G(v)\ E[Z(t-v')Z(t-s-v)]dv\,dv'$$

$$= \sigma_z^2 \int_0^\infty [\int_0^\infty G(v')G(v)\ \delta(v+s-v')dv']dv \ = \sigma_z^2 \int_0^\infty G(v)G(v+s)\ dv$$

$$\int_{-\infty}^{+\infty} f(t-u)\ \delta(u)\ du = \int_{-\infty}^{+\infty} f(u)\ \delta(t-u)\ du = f(t)$$

$$\int_{-\infty}^{+\infty} f(t-u)\ \delta^{(k)}(u)\ du = (-1)^k\ f^{(k)}(t)$$

(the last equation can be shown using integration by parts

For an A(1) system (first order cont. time stochastic system)

$$G(t) = s(t)\ e^{-\alpha_0 t}$$

$$\gamma(s) = \sigma_z^2 \int_0^\infty e^{-\alpha_0 v}\ e^{-\alpha_0 (v+s)}\ dv \ = \sigma_z^2\ e^{-\alpha_0 s} \int_0^\infty e^{-2\alpha_0 v} dv$$

$$= \sigma_z^2\ e^{-\alpha_0 s}\ [\frac{-e^{-2\alpha_0 v}}{2\alpha_0}]_0^\infty \ = \frac{\sigma_z^2}{2\alpha_0} e^{-\alpha_0 s} \qquad s \geq 0$$

$$\gamma(0) = \frac{\sigma_z^2}{2\alpha_0} \qquad \rho(s) = \frac{\gamma(s)}{\gamma(0)} = e^{-\alpha_0 s}$$

Note: It can be easily shown that

$\rho(-s) = \rho(s)$ and $\gamma(-s) = \gamma(s)$

Thus,

$$\gamma(s) = \frac{\sigma_z^2}{2\,\alpha_0} e^{-\alpha_0 |s|} \qquad \rho(s) = e^{-\alpha_0\,|s|}$$

1. The autocovariance function

$$\gamma(s) = E[X(t)\,X(t-s)] = E[\int_0^\infty G(v')Z(t-v')dv' \int_0^\infty G(v)Z(t-s-v)dv\ ]$$

$$= \int_0^\infty\int_0^\infty G(v')G(v)\ E[Z(t-v')Z(t-s-v)]dv\,dv'$$

$$= \sigma_z^2 \int_0^\infty [\int_0^\infty G(v')G(v)\ \delta(v+s-v')dv']dv\ = \sigma_z^2 \int_0^\infty G(v)G(v+s)\ dv$$

$$\int_{-\infty}^{+\infty} f(t-u)\ \delta(u)\ du = \int_{-\infty}^{+\infty} f(u)\ \delta(t-u)\ du = f(t)$$

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$$\gamma(0) = \frac{\sigma_z^2}{2\alpha_0}$$

$$\rho(s) = \frac{\gamma(s)}{\gamma(0)} = e^{-\alpha_0 s}$$

Note: It can be easily shown that

$$\rho(-s) = \rho(s) \quad \text{and } \gamma(-s) = \gamma(s)$$

Thus,

$$\gamma(s) = \frac{\sigma_z^2}{2\,\alpha_0} e^{-\alpha_0 |s|} \qquad\qquad \rho(s) = e^{-\alpha_0\,|s|}$$

2. Uniformly sampled first order autoregressive system

Criterion:

Covariance equivalence --- the covariance of the discrete model coincide with that of the continuous model at sampling points.

$$\gamma_k = \gamma(k\Delta) = E[X(t)\,X(t - k\Delta)] = E[X_t\,X_{t-k}]$$

$$\gamma_k = \gamma(s{=}k\Delta) = \frac{\sigma_z^2}{2\alpha_0} e^{-\alpha_0 k\Delta} = \gamma(0)\, e^{-\alpha_0 k\Delta} = \gamma(0)\,\phi^k$$

Thus,

$$\phi = e^{-\alpha_0 \Delta}$$

For every A(1) system, there will be a uniformly sampled AR(1) model. But, every AR(1) is not necessarily a uniformly sampled A(1) model.

$$X_t - \phi X_{t-1} = a_t$$

For positive parameter, then the AR(1) is a uniformly sampled A(1) model.

Expression for σ_a^2

For an AR(1) model,

$$\gamma_0 = \phi_1 \gamma_1 + \sigma_a^2 \quad \text{and} \quad \gamma_\kappa = \phi\gamma_{\kappa-1} \quad k > 0$$

$$\gamma_0 = \phi^2\gamma_0 + \sigma_a^2 \qquad \sigma_a^2 = (1-\phi^2)\gamma_0$$

$$\sigma_a^2 = (1-\phi^2)\gamma_0 = \frac{\sigma_z^2}{2\alpha_0}(1-\phi^2)$$

$$\alpha_0 = -\frac{\ln\phi}{\Delta} \qquad \sigma_z^2 = \frac{2\alpha_0\sigma_a^2}{(1-\phi^2)}$$

3. Uniformly sampled first order autoregressive system by impulse response invariant

Criterion: the impulse response function of the discrete model should be equivalent to that of the continuous model at sampling points.

For AR(1) system,

$$G_j = \phi^j \qquad j=0,\ 1,\ 2,\ \ldots$$

For A(1) system,

$$G(t) = s(t)\, e^{-\alpha_0 t}$$

At sampling points, $t=j\Delta$

$$G(t=j\Delta) = e^{-\alpha_0 (j\Delta)} = \phi^j$$

Thus,

$$\phi = e^{-\alpha_0 \Delta}$$