Homework 4 – Solutions
ME384Q.3 / ORI 390R.3: Time-Series Analysis

Problem 5.1

ARMA(2,1) model: $X_t - X_{t-1} + 0.3\,X_{t-2} = a_t + 0.4_{t-1}$, $a_t \sim NID(0,256)$.

(i) $a_{t-3} = X_{t-3} - X_{t-4} + 0.3\,X_{t-5} - 0.4\,a_{t-4} = 1 - 36 + 0.3(34) = -24.8$

$$a_{t-2} = X_{t-2} - X_{t-3} + 0.3\,X_{t-4} - 0.4\,a_{t-3} = 20.72$$
$$a_{t-1} = X_{t-1} - X_{t-2} + 0.3\,X_{t-3} - 0.4\,a_{t-2} = -24.988$$
$$a_t = X_t - X_{t-1} + 0.3\,X_{t-2} - 0.4\,a_{t-1} = -8.7048$$

$$\hat{X}_t(1) = X_t - 0.3\,X_{t-1} + 0.4\,a_t = -33.68$$
$$\hat{X}_t(2) = \phi_1\,\hat{X}_t(1) + \phi_2\,X_t = -23.18$$
$$\hat{X}_t(3) = \phi_1\,\hat{X}_t(2) + \phi_2\,\hat{X}_t(1) = -13.08$$
$$\hat{X}_t(4) = \phi_1\,\hat{X}_t(3) + \phi_2\,\hat{X}_t(2) = -6.122$$

$$G_1 = \phi_1 - \theta_1 = 1.4 \;;\; G_2 = \phi_1\,G_1 + \phi_2 = 1.1 \;;\; G_3 = \phi_1\,G_2 + \phi_2\,G_1 = 0.68$$

90% Probability limits: $\hat{X}_t(1) \pm 1.65\,\sigma_a = -33.68 \pm 26.4$

$$\hat{X}_t(2) \pm 1.65\,\sigma_a\,\sqrt{1+(\phi_1-\theta_1)^2} = -23.18 \pm 45.42$$

$$\hat{X}_t(3) \pm 1.65\,\sigma_a\,\sqrt{(1 + G_1^2 + G_2^2)} = -13.08 \pm 53.91$$

$$\hat{X}_t(4) \pm 1.65\,\sigma_a\,\sqrt{(1 + G_1^2 + G_2^2 + G_3^2)} = -6.122 \pm 56.826$$

(ii) Updating $X_t(\ell)$, $\ell = 2,3,4$ given $X_{t+1} = -37$

$$\hat{X}_{t+1}(\ell) = X_t(\ell+1) + G_\ell\,a_{t+1} \;;\; a_{t+1} = X_{t+1} - \hat{X}_t(1) = -37 + 33.68 = -3.32$$

$$\hat{X}_{t+1}(2) = \hat{X}_t(3) + G_2\,a_{t+1} = -16.73$$

$$\hat{X}_{t+1}(3) = \hat{X}_t(4) + G_3\,a_{t+1} = -8.38$$

$$\hat{X}_{t+1}(4) = \hat{X}_t(5) + G_4\,a_{t+1} \;;\; \hat{X}_t(5) = \phi_1\,\hat{X}_t(4) + \phi_2\,\hat{X}_t(3) = -2.20$$

$$G_4 = \phi_1\,G_3 + \phi_2\,G_2 = 0.35$$

$$\hat{X}_{t+1}(4) = (-2.20) + 0.35(-3.32) = -3.36.$$

Problem 5.3 $X_t = 0.5X_{t-1} - 0.5X_{t-2} + a_t$

$X_{100} = 0.8$, $X_{99} = 1.8$ $\qquad \hat{X}_t(1) \pm 2\sigma_a^2$ is the interval $[-1.5, 0.5]$

$$\hat{X}_t(1) = E[X_{t+1} | t] = 0.5X_t - 0.5X_{t-1} \Rightarrow$$

$$\hat{X}_{100}(1) = 0.5 \cdot 0.8 - 0.5 \cdot 1.8 = -0.5$$

$$\Rightarrow 2\sigma_a^2 = 1 \Rightarrow \sigma_a^2 = 0.5$$

(i) $\hat{X}_{100}(2) = E[X_{102} | 100] = 0.5\hat{X}_{100}(1) - 0.5X_{100} =$
$= 0.5 \cdot (-0.5) - 0.5 \cdot 0.8 = -0.65$

$\hat{X}_{100}(3) = E[_{103} | 100] = 0.5\hat{X}_{100}(2) - 0.5\hat{X}_{100}(1) = -0.075$

$Var[\hat{e}_{100}(2)] = \sigma_a^2(G_0^2 + G_1^2)$; $Var[\hat{e}_{100}(3)] = \sigma_a^2(G_0^2 + G_1^2 + G_2^2)$

Using implicit method to find G_0, G_1 and G_2

– $(1 - 0.5B + 0.5B^2)X_t = a_t \Rightarrow$

$(1 - 0.5B + 0.5B^2)(G_0 + G_1B + G_2B^2 + \dots)a_t = a_t$

B^0: $1 = G_0$

B^1: $0 = G_1 - 0.5G_0 \Rightarrow G_1 = 0.5G_0 = 0.5$

B^2: $0 = G_2 - 0.5G_1 + 0.5G_0 \Rightarrow G_2 = -0.25$

$\Rightarrow Var[\hat{e}_{100}(2)] = 0.5(1^2 + 0.5^2) = 0.625$

$Var[\hat{e}_{100}(3)] = 0.5(1^2 + 0.5^2 + 0.25^2) = 0.6562$

95% Confidence intervals are:

- For $\hat{X}_{100}(2)$, it's $\hat{X}_{100}(2) \pm 2\,\mathrm{Var}[\hat{e}_{100}(2)] =$

$$= -0.65 \pm 1.25$$

- For $\hat{X}_{100}(3)$, it's $\hat{X}_{100}(3) \pm 2\,\mathrm{Var}[\hat{e}_{100}(3)] =$

$$= -0.075 \pm 1.3125$$

(c) $X_{101} = 0 \Rightarrow a_{101} = X_{101} - \hat{X}_{100}(1) = 0.5$

$\hat{X}_{101}(1) = \hat{X}_{100}(2) + G_1 a_{101} = -0.65 + 0.5 \cdot 0.5 = 0.4 \leftarrow$ update for $\hat{X}_{100}(2)$

$\mathrm{Var}[\hat{e}_{101}(1)] = G_0^2 \cdot \sigma_a^2 = 0.5 \Rightarrow$ 95% confidence interval becomes $0.4 \pm 2 \cdot 0.5 = 0.4 \pm 1$

$\hat{X}_{101}(2) = \hat{X}_{100}(3) + G_2 a_{101} = -0.075 - 0.25 \cdot 0.5 = -0.2$

$\mathrm{Var}[\hat{e}_{101}(2)] = \sigma_a^2 (G_0^2 + G_1^2) = 0.625 \leftarrow$ Update for $\hat{X}_{100}(3)$

$\Rightarrow$ 95% confidence interval becomes $-0.2 \pm 2 \cdot 0.625 =$
$= -0.2 \pm 1.25$

Problem 5.6

(i) $X_t = 0.5\, X_{t-1} + a_t$ (ii) $X_t - X_{t-1} = a_t$ (iii) $X_t - 2\, X_{t-1} + X_{t-2} = a_t$

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6.3 (i) $\Delta = 1$, $\sigma_a^2 = \frac{RSS}{N} = \frac{19355.9}{369} = 52.46$

$$\alpha_o = \frac{-\ell n\ \phi}{\Delta} = -\ \ell n\ 0.998 = 0.002$$

$$\sigma_z^2 = \frac{2\alpha_o\ \sigma_a^2}{1-\phi^2} = \frac{2(0.002)(52.46)}{(1-0.998^2)} = 52.51$$

(ii) $\alpha_o = \frac{-\ell n\ \phi}{\Delta} = -\ \ell n\ 0.98 = 0.0202$, $\sigma_a^2 = \frac{15.58}{160} = 9.738 \times 10^{-2}$

$$\sigma_z^2 = \frac{2\ \alpha_o\ \sigma_a^2}{(1-\phi^2)} = \frac{2(0.0202)(9.738\mathrm{x}10^{-2})}{(1-0.98^2)} = 0.09935$$

(iii) $\alpha_o = \frac{-\ell n\ \phi}{\Delta} = -\ \ell n\ 0.63 = 0.46$, $\sigma_a^2 = \frac{1618.88}{250} = 6.48$

$$\sigma_z^2 = \frac{2\ \alpha_o\ \sigma_a^2}{(1-\phi^2)} = \frac{2(0.46)(6.48)}{(1-0.63^2)} = 9.885$$

At 95% Confidence interval (i) and (ii) does include one and hence these may be taken as random walk. (iii) doesn't include one, so it can't be taken as random walk.

6.4 $\sigma_z^2 = 10$; $\alpha_o = -\frac{\ell n\ \phi}{\Delta}$; $\phi = e^{-\alpha_o \Delta}$; $\alpha_o = 5.5$

(a)

Δ	ϕ	AR(1)	σ_a^2
1	0.004	$X_t - 0.004\ X_{t-1} = a_t$	0.909
0.1	0.577	$X_t - 0.577\ X_{t-1} = a_t$	0.606
0.01	0.946	$X_t - 0.946\ X_{t-1} = a_t$	0.095
0.001	0.994	$X_t - 0.994\ X_{t-1} = a_t$	0.011

(b) Only when $\Delta = 0.001$, ϕ value $= 0.994 \pm 0.01$ includes $\phi_1 = 1$, which is the only sampling interval for which the system is a random walk.

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6.5

α_0	0.001	0.01	0.5	2	4	8	10
ϕ	0.999	0.99	0.607	0.135	0.018	0.00033	0.000045
σ_a^2	9.995	9.95	6.316	2.454	1.25	0.625	0.5

As $\alpha_0 \to \infty$, $\sigma_a^2 \to \dfrac{\sigma_z^2}{2\alpha_0}$ by (6.5.14) and in the present case $\alpha_0 = 10$ gives

$$\sigma_a^2 = 0.5 = \frac{\sigma_z^2}{2\alpha_0} = \frac{10}{2\text{x}10}$$

6.8 $\sigma_a^2 = \dfrac{\sigma_z^2}{2\alpha_0}(1 - e^{-2\alpha_0\Delta})$; $\alpha_0 = 5.1$, $\Delta \to 0.1$ to 0.2

For $\Delta = 0.1$: $\sigma_a^2 = \dfrac{\sigma_z^2}{2(5.1)}\;[1 - e^{-2(5.1)0.1}] = 0.063\;\sigma_z^2$

For $\Delta = 0.2$: $\sigma_a^2 = \dfrac{\sigma_z^2}{2(5.1)}\;[1 - e^{-2(5.1)0.2}] = 0.085\;\sigma_z^2$

% change in residual sum of squares is 34.9% since residual sum of squares $\simeq$ $(N-1)\,\sigma_a^2$.