Stability of time-series with ARMA(n,m) models

Definition of Bounded Input Bounded Output (BIBO) Stability: A system is BIBO stable iff any bounded input into that system generates a bounded output.

Stability of a time-series will be assessed based on the stability of the dynamic system that generated that time series when a Gaussian white noise input was fed into it. In other words, we will assess stability of a time-series based on the stability of the system whose impulse response is the Green's Function of that time-series.

Let time-series X_t have a model of the form

$$X_t - \phi_1 X_{t-1} - \phi_2 X_{t-2} - \ldots - \phi_n X_{t-n} = a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2} - \ldots - \theta_m a_{t-m}$$

and let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the roots of the autoregressive characteristic polynomial

$$s^n - \phi_1 s^{n-1} - \phi_2 s^{n-2} - \ldots - \phi_{n-1} s - \phi_n = 0$$

Then, the following theorem holds.

Theorem 1: Time-series X_t is:

- BIBO stable iff for all the roots of the autoregressive characteristic polynomial $|\lambda_i| < 1$.
- BIBO unstable otherwise.

A very special case occurs when one or more simple autoregressive roots (autoregressive roots of multiplicity 1) appear on the unit circle. In this case, most bounded inputs will still produce a bounded output. However, hitting this system with an input corresponding exactly to the roots on the unit circle, will produce an unbounded output (which makes them technically BIBO unstable). This special class of systems is referred to as the "marginally stable systems". Hence, in summary, we can classify time series in the following way.

Theorem: Time-series X_t is:

- BIBO stable iff for all the roots of the autoregressive characteristic polynomial $|\lambda_i| < 1$.
- Marginally stable if for all the roots of the autoregressive characteristic polynomial $|\lambda_i| \leq 1$ and if for some j, $|\lambda_j| = 1$, then λ_j must be a root of multiplicity 1 (a simple root of the autoregressive characteristic polynomial).
- Unstable otherwise

Inverse function

$$X_t = \sum_{j=1}^{\infty} I_j X_{t-j} + a_t$$

or $a_t = (1-I_1B - I_2B^2-\ldots)\, X_t$

Inverse function, I_j, expresses X_t as a linear combination of the past X_t's.

Inverse function for AR(1) model:

$X_t = \phi_1 X_{t-1} + a_t$

$I_1 = \phi_1$ and $I_j = 0$ for $j > 1$

$\frac{1}{1-\phi_1 B}$ operator leads to Green's function $(1-\phi_1 B)$ operator leads to Inverse function

Inverse function for MA(1) model:

MA(1): $X_t = (1-\theta_1 B)\, a_t$ $a_t = \frac{1}{1-\theta_1 B} X_t = (1+\theta_1 B+\theta_1^2 B^2+\ldots)X_t$

$$X_t = \sum_{j=1}^{\infty} -\theta_1^j X_{t-j} + a_t \quad \Rightarrow \quad I_j = -\theta_1^j$$

Invertibility condition: $|\theta_1| < 1$

Reason for invertibility: If roots are greater than one in absolute value, I_j increases w/o bound. It means the more distant we go in the past, the greater the influence of the past X_t's on the present one.

Inverse function of ARMA(1,2) models -- Implicit method

$(1-\phi_1 B)\, X_t = (1-\theta_1 B -\theta_2 B^2)a_t$

$(1-\phi_1 B) = (1-\theta_1 B -\theta_2 B^2)\,(1-I_1B - I_2B^2-\ldots)$

$-I_1 - \theta_1 = -\phi_1 \Rightarrow I_1 = \phi_1 - \theta_1$

$I_j = \theta_1 I_{j-1}+\theta_2 I_{j-2}$ for $j\geq 2$ and $I_0 = -1$

$(1-\theta_1 B -\theta_2 B^2)I_j = 0$, $j\geq 2$

Inverse function of ARMA(1,2) models -- Explicit method

$(1-\theta_1 B -\theta_2 B^2) = (1-v_1B)(1-v_2B) \Rightarrow v_1+v_2=\theta_1 \quad v_1v_2= -\theta_2$

$v^2 - \theta_1 v -\theta_2 = 0$

$$I_j = -\left(\frac{v_1-\phi_1}{v_1-v_2}\right) v_1^j - \left(\frac{v_2-\phi_1}{v_2-v_1}\right) v_2^j$$