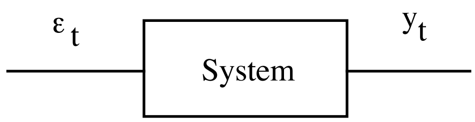
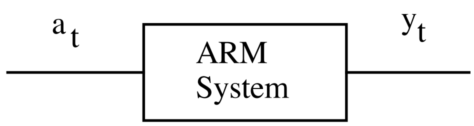
Chapter 3 --- Analysis of ARMA models

Green's Function

Static vs. dynamic dependence:



A disturbance ε_t entering a regression system at time t affects only y_t but not y_{t+1}.



A disturbance at affecting the system is "remembered" and continues to affect the system at subsequent times.

Green's function of an AR(1) model:

$$\frac{dy(t)}{dt} + k\,y(t) = u(t) \qquad \text{continuous differential equation}$$

Solution: $$y(t) = \int_0^\infty h(\tau)\, u(t-\tau)\, d\tau \qquad \text{convolution integral}$$

$D = \frac{d}{dt}(\quad)$: D is differential operator $\qquad (D + k)\,y(t) = u(t)$

Remarks:

- h(t) is the impulse response function, which describes the characteristics of a dynamic system, i.e., from h(t), we can find system transfer function, determine system stability, response speed, and other physical characteristics.
- left-hand side of equation represents the homogeneous part and u(t) is a forcing function. The system characteristic equation can be found from the homogeneous part.
- the convolution integral can be interpreted as:
  the current system response, y(t), is affected by all previous forcing input, u(t).

$$X_t - \phi_1 X_{t-1} = a_t \qquad \text{discrete difference equation}$$

Solution: $$X_t = \phi_1(\phi_1 X_{t-2} + a_{t-1}) + a_t = a_t + \phi_1 a_{t-1} + \phi_1^2 a_{t-2} + \ldots = \sum_{j=0}^{\infty} \phi_1^j a_{t-j}$$

$G_j = \phi_1^j$ Green's function

Remarks:

- G_j characterizes the dynamics or the memory of a system. It describes the influence of past "forcing input", a_t's, on X_t.
- G_j indicates how well the system remembers the shocks a_{t-j}.
  The larger the value of ϕ_1 in the AR(1) model, the more clearly is the shock a_{t-j} remembered. G_j is like a "weighting" function.
- G_j determines how slow or fast the dynamic response of the system to any particular a_t decays.
- G_j is the impulse response function.

B (Backshift) Operator:

$BX_t = X_{t-1}$ and $B^j X_t = X_{t-j}$

AR(1): $(1 - B\phi_1)X_t = a_t$

$$X_t = \frac{a_t}{(1 - B\phi_1)} = (1 + \phi_1 B + \phi_1^2 B^2 + \phi_1^3 B^3 + \ldots)\, a_t = a_t + \phi_1 a_{t-1} + \phi_1^2 a_{t-2} + \ldots$$

$$= \sum_{j=0}^{\infty} \phi_1^j a_{t-j} = \sum_{j=0}^{\infty} G_j a_{t-j}$$

$$X_t = \sum_{j=0}^{\infty} G_j a_{t-j} = \sum_{j=-\infty}^{t} G_{t-j} a_j$$

(Example of response generation)

Another interpretation of Green's function--Orthogonal or Wold's Decomposition

AR(1): $(1 - B\phi_1)X_t = a_t$

$$X_t = \frac{a_t}{(1 - B\phi_1)} = (1 + \phi_1 B + \phi_1^2 B^2 + \phi_1^3 B^3 + \ldots)\, a_t = a_t + \phi_1 a_{t-1} + \phi_1^2 a_{t-2} + \ldots$$

$$= \sum_{j=0}^{\infty} \phi_1^j a_{t-j} = \sum_{j=0}^{\infty} G_j a_{t-j}$$

For random variables, independence <--> orthogonality

If a_t's are considered as perpendicular "axes" and G_j as "coordinate" of X_t corresponding to the axis a_{t-j}:

$$X_t = \sum_{j=0}^{\infty} G_j a_{t-j} = \sum_{j=-\infty}^{t} G_{t-j} a_j$$

- It expresses X_t as the sum of perpendicular vectors $G_j a_{t-j}$ in an infinite dimensional space. In other words, a stationary stochastic process or time series X_t is decomposed into the sum of uncorrelated random variables.
- The Wold's decomposition is very important and can be used to derive all the statistical properties of a time series. For example, the variance of Xt is

$$\gamma_0 = \mathrm{Var}(X_t) = \mathrm{Var}\left(\sum_{j=0}^{\infty} \phi_1^j a_{t-j}\right) = \sum_{j=0}^{\infty} \mathrm{Var}(\phi_1^j a_{t-j}) = \left(\sum_{j=0}^{\infty} \phi_1^{2j}\right) \sigma_a^2 = \frac{\sigma_a^2}{1-\phi_1^2}$$