Lecture Notes: Sum of Independent ARMA Processes

Let $\mathrm{X(t)}$ and $\mathrm{Y(t)}$ be two ARMA time series that are independent at all leads and lags, then we have the following results.

1. **MA(1) + White noise = MA(1)**

$\mathrm{X(t) = Z(t) + \beta Z(t-1)}$

$\mathrm{Z(t)}$ is iid white noise with mean 0 and variance ${\sigma_Z}^2$

$\mathrm{Y(t)}$ is iid white noise with mean 0 and variance ${\sigma_Y}^2$

$\mathrm{W(t) = X(t) + Y(t)}$

$$\mathrm{E\big(W(t)\big) = E\big(X(t) + Y(t)\big) = 0}$$

$$\mathrm{Var\big(W(t)\big) = Var\big(X(t) + Y(t)\big)}$$

$$= (1 + \beta^2){\sigma_Z}^2 + {\sigma_Y}^2$$

$$\mathrm{Cov\big(W(t), W(t+h)\big)}$$

$$\mathrm{= Cov\big(X(t) + Y(t), X(t+h) + Y(t+h)\big)}$$

$$\mathrm{= Cov\big(X(t), X(t+h)\big) + Cov\big(Y(t), Y(t+h)\big) + Cov\big(X(t), Y(t+h)\big) + Cov\big(Y(t), X(t+h)\big)}$$

$$\mathrm{= Cov\big(X(t), X(t+h)\big) + Cov\big(Y(t), Y(t+h)\big) + 0 + 0}$$

$$\mathrm{= Cov\big(X(t), X(t+h)\big) + Cov\big(Y(t), Y(t+h)\big)}$$

For $h \geq 1$, we have:

$$\mathrm{Cov}\big(X(t), X(t+h)\big)$$

$$= \mathrm{Cov}(Z(t) + \beta Z(t-1), Z(t+h) + \beta Z(t-1+h))$$

$$\mathrm{Cov}\big(Y(t), Y(t+h)\big) = 0$$

$$\gamma_W(h) = \mathrm{Cov}\big(W(t), W(t+h)\big) = \begin{cases} h = 1 & \beta{\sigma_Z}^2 \\ h \geq 2 & 0 \end{cases}$$

According to the $\gamma_W(h)$, we know that $W(t)$ is an $MA(1)$ series. Now assuming:

$W(t) = V(t) + \theta V(t-1)$, $V(t)$ is a i.i.d. white noise with mean 0 and variance ${\sigma_V}^2$

$$\mathrm{Var}\big(W(t)\big) = (1+\theta^2){\sigma_V}^2 = (1+\beta^2){\sigma_Z}^2 + {\sigma_Y}^2$$

$$\mathrm{Cov}\big(W(t), W(t+1)\big) = \theta{\sigma_V}^2 = \beta{\sigma_Z}^2$$

Solving the above two equations we can obtain the values of θ and ${\sigma_V}^2$

Following the same steps, we can easily prove that:

MA(q) + White noise = MA(q), for any $q \geq 1$

2. MA(q) + MA(p) = MA(max(p,q))

$$X(t) = Z(t) + \beta_1 Z(t-1) + \cdots \beta_q Z(t-q)$$

$$Y(t) = V(t) + \alpha_1 V(t-1) + \cdots \alpha_p V(t-p)$$

$$W(t) = X(t) + Y(t)$$

$$E(W(t)) = 0$$

$$\mathrm{Var}\big(\mathrm{W(t)}\big) = \big(1 + \beta_1{}^2 + \cdots + \beta_q{}^2\big)\sigma_Z{}^2 + (1 + \alpha_1{}^2 + \cdots + \alpha_p{}^2)\sigma_V{}^2$$

Same as in the derivation of (1), we have:

$$\mathrm{Cov}(\mathrm{W(t)}, W(t+h)) = \mathrm{Cov}(\mathrm{X(t) + Y(t), X(t+h) + Y(t+h)})$$

$$= \mathrm{Cov}(\mathrm{X(t), X(t+h)}) + \mathrm{Cov}(\mathrm{Y(t), Y(t+h)})$$

$$\gamma_W(\mathrm{h}) = \gamma_X(\mathrm{h}) + \gamma_Y(\mathrm{h})$$

$$\gamma_X(\mathrm{h}) = \mathrm{Cov}\Big(\mathrm{Z(t)} + \cdots + \beta_q \mathrm{Z(t-q)}, \mathrm{Z(t+h)} + \cdots + \beta_q \mathrm{Z(t-q+h)}\Big)$$

Let $\beta_0 = 1$, $\alpha_0 = 1$, we have

$$\gamma_X(\mathrm{h}) = \begin{cases} \sum_{i=0}^{q-h} \beta_i \beta_{i+h} & (h = 1 \cdots q) \\ 0 & (h > q) \end{cases}$$

$$\gamma_Y(\mathrm{h}) = \begin{cases} \sum_{i=0}^{p-h} \alpha_i \alpha_{i+h} & (h = 1 \cdots p) \\ 0 & (h > p) \end{cases}$$

Assuming p > q

$$\gamma_W(\mathrm{h}) = \begin{cases} \sum_{i=0}^{q-h} \beta_i \beta_{i+h} + \sum_{i=0}^{p-h} \alpha_i \alpha_{i+h} & (h = 1 \cdots q) \\ \sum_{i=0}^{p-h} \alpha_i \alpha_{i+h} & (q < h \le p) \\ 0 & (h > \max(p,q)) \end{cases}$$

Given that $\mathrm{W(t)}$ is a weakly stationary times series, and according to the $\gamma_W(\mathrm{h})$, we assume $\mathrm{W(t)}$ is a MA(max(p,q)) process, Assuming p > q and $\theta_0 = 1$,

$$\mathrm{W(t)} = \varepsilon(t) + \theta_1\varepsilon(t-1) + \cdots \theta_p\varepsilon(t-p)$$

$$\mathrm{Var}\big(\mathrm{W(t)}\big) = \left(1 + {\theta_1}^2 + \cdots + {\theta_p}^2\right){\sigma_\varepsilon}^2$$

$$= \left(1 + \cdots {\beta_q}^2\right){\sigma_Z}^2 + (1 + \cdots {\alpha_p}^2){\sigma_V}^2$$

$$\gamma_W(\mathrm{h}) = \begin{cases} \sum_{i=0}^{p-h} \theta_i\theta_{i+h} & (0 < h \le p) \\ 0 & (h > p) \end{cases}$$

Then we solve for the coefficients $\theta_i,\ i = 1, \cdots, p$.

3. <u>AR(1) + AR(1) = ARMA(2,1)</u>

$$\mathrm{X(t)} - \alpha X(t-1) = Z(t)$$

$$\mathrm{Y(t)} - \beta Y(t-1) = V(t)$$

$$\mathrm{W(t)} = \mathrm{X(t)} + \mathrm{Y(t)}$$

$$\mathrm{Var}\big(\mathrm{W(t)}\big) = \mathrm{Var}\big(\mathrm{X(t)} + \mathrm{Y(t)}\big)$$

$$= \frac{{\sigma_Z}^2}{1-\alpha^2} + \frac{{\sigma_V}^2}{1-\beta^2}$$

$$\gamma_W(\mathrm{h}) = \gamma_X(\mathrm{h}) + \gamma_Y(\mathrm{h})$$

$$\gamma_X(\mathrm{h}) = \alpha\gamma_X(\mathrm{h}-1)$$

$$\gamma_Y(\mathrm{h}) = \beta\gamma_Y(\mathrm{h}-1)$$

Assuming:

$$\mu_1 = \alpha + \beta,\ \ \mu_2 = -\alpha\beta$$

$$\gamma_W(\mathrm{h}) = \mu_1\gamma_W(\mathrm{h}-1) + \mu_2\gamma_W(\mathrm{h}-2), \qquad \mathrm{h} \ge 2$$

Assuming:

$$\mathrm{M(t)} = \mathrm{W(t)} - \mu_1 W(t-1) - \mu_2 W(t-2)$$

$$= \mathrm{W(t)} - (\alpha + \beta) W(t-1)$$

$$+ \alpha\beta W(t-2)$$

$$= \mathrm{X(t)} - (\alpha + \beta) X(t-1) + \alpha\beta X(t-2) + \mathrm{Y(t)}$$

$$- (\alpha + \beta) Y(t-1) + \alpha\beta Y(t-2)$$

$$\mathrm{X(t)} - \alpha \mathrm{X(t-1)} = \mathrm{Z(t)}$$

$$\mathrm{Y(t)} - \beta \mathrm{Y(t-1)} = \mathrm{V(t)}$$

$$\mathrm{X(t)} - (\alpha + \beta) X(t-1) + \alpha\beta X(t-2)$$

$$= \mathrm{Z(t)} - \beta \mathrm{Z(t-1)}$$

$$\mathrm{Y(t)} - (\alpha + \beta) Y(t-1) + \alpha\beta Y(t-2)$$

$$= \mathrm{V(t)} - \alpha \mathrm{V(t-1)}$$

$$\mathrm{M(t)} = \mathrm{Z(t)} - \beta \mathrm{Z(t-1)} + \mathrm{V(t)} - \alpha \mathrm{V(t-1)}$$

According to (2), MA(1)+MA(1)=MA(1)

$\mathrm{M(t)}$ is also a MA(1) process.

Therefore,

$$\mathrm{W(t)} - \mu_1 W(t-1) - \mu_2 W(t-2) = \mathrm{Z(t)} - \beta \mathrm{Z(t-1)} + \mathrm{V(t)} - \alpha \mathrm{V(t-1)}$$

W(t) is an ARMA(2,1).

4. **<u>Sum of Independent ARMA Processes</u>**

<u>Theorem:</u> Suppose that X_{1t} and X_{2t} are two independent ARMA series of orders $(\mathrm{p}_1, \mathrm{q}_1)$ and $(\mathrm{p}_2, \mathrm{q}_2)$ respectively. Let $\mathrm{W(t)} = X_{1t} + X_{2t}$ Then, $\mathrm{W(t)}$ is an ARMA(p,q) process with $\mathrm{p} \leq \mathrm{p}_1 + \mathrm{p}_2$ and $\mathrm{q} \leq \max\{\mathrm{p}_1 + \mathrm{q}_2, \mathrm{p}_2 + \mathrm{q}_1\}$

The reason that $\leq$ is used is because of the possibility of common factors in the polynomials involved – when there is no common factor, the equal sign = will hold.

<u>Proof:</u> Write the model for X_{it} as

$$\emptyset_i(\mathrm{B})X_{it} = \theta_i(B)Z_{it}, i = 1,2$$

Apply $\emptyset_1(B)\emptyset_2(B)$ to $\mathrm{W(t)}$, we have

$$\begin{aligned}\emptyset_1(\mathrm{B})\emptyset_2(\mathrm{B})\mathrm{W(t)} &= \emptyset_1(\mathrm{B})\emptyset_2(\mathrm{B})(X_{1t} + X_{2t}) \\ &= \emptyset_2(\mathrm{B})\theta_1(B)Z_{1t} + \emptyset_1(\mathrm{B})\theta_2(B)Z_{2t}\end{aligned}$$

Example 1: Sum of independent AR(1) and MA(1)

Solution:

AR(1):

$$\mathrm{X(t)} - \alpha X(t-1) = Z(t)$$

$$(1 - \alpha\mathrm{B})\mathrm{X(t)} = Z(t)$$

MA(1):

$$\mathrm{Y(t)} = V(t) + \beta V(t-1)$$

$$\mathrm{Y(t)} = (1 + \beta\mathrm{B})V(t)$$

Let:

$$\mathrm{W(t)} = \mathrm{X(t)} + \mathrm{Y(t)}$$

$$(1 - \alpha\mathrm{B})\mathrm{W(t)} = \mathrm{Z(t)} + (1 - \alpha\mathrm{B})(1 + \beta\mathrm{B})V(t)$$

Notice that the right-hand side is the sum of independent white noise + MA(2), and thus following the general results from (1), we know that this sum follows MA(2). Therefore, we have proven that $\mathrm{W(t)}$ is ARMA(1,2).

Example 2: Sum of independent AR(1) and AR(1), Revisited

Solution:

AR(1):

$\mathrm{X(t)} - \alpha X(t-1) = Z(t)$

$(1 - \alpha\mathrm{B})\mathrm{X(t)} = Z(t)$

AR(1):

$\mathrm{Y(t)} - \beta Y(t-1) = V(t)$

$(1 - \beta\mathrm{B})\mathrm{Y(t)} = V(t)$

Let:

$\mathrm{W(t)} = \mathrm{X(t)} + \mathrm{Y(t)}$

$$(1 - \alpha\mathrm{B})(1 - \beta\mathrm{B})\mathrm{W(t)}$$

$$= (1 - \beta\mathrm{B})\mathrm{Z(t)} + (1 - \alpha\mathrm{B})V(t)$$

Notice that the right-hand side is the sum of independent MA(1) + MA(1), and thus following results from (2), we know that this sum follows MA(1). Therefore, we have proven that $\mathrm{W(t)}$ is ARMA(2,1).