Finding Inverse Function Coefficients from ARMA models

(i) AR(n) models

$$X_t - \phi_1 X_{t-1} - \phi_2 X_{t-2} - \dots - \phi_n X_{t-n} = a_t$$

This is already in the inverse function shape!

$$I_1 = \phi_1 \, ; \; I_2 = \phi_2 \, ; \dots ; \; I_n = \phi_n \, ; \; I_{n+1} = 0 \, ; \; I_{n+2} = 0 \dots$$

(ii) MA(1) models

$$X_t = a_t - \theta_1 a_{t-1} = (1 - \theta_1 B) a_t$$

$$(1 - \theta_1 B)^{-1} X_t = a_t$$

$$(1 + \theta_1 B + \theta_1^2 B^2 + \dots) X_t = a_t$$

$$\Rightarrow \; I_1 = -\theta_1 \, ; \; I_2 = -\theta_1^2 \, ; \; I_3 = -\theta_1^3 \, ; \dots$$

Remember ~~MA~~ AR(1) model $X_t - \phi_1 X_{t-1} = a_t$

$\rightarrow$ G.F. for this model was $G_\ell = \phi_1^{\ell}$

iii) Let us observe an ARMA(1,2) model

$$X_t - \phi_1 X_{t-1} = a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2}$$

$$(1 - \phi_1 B) X_t = (1 - \theta_1 B - \theta_2 B^2) a_t$$

This is the only time we'll look into such a "non-physical" model – I do it to draw a parallel between I.F. and G.F.

$$a_t = (1 - I_1 B - I_2 B^2 - I_3 B^3 - \dots) X_t \qquad (I.F.)$$

$$(1 - \phi_1 B) X_t = (1 - \theta_1 B - \theta_2 B^2)(1 - I_1 B - I_2 B^2 - \dots) X_t$$

B^0 : $1 = 1 = -I_0 \Rightarrow I_0 = -1$

B^1 : $-\phi_1 = -\theta_1 - I_1 \Rightarrow I_1 = \phi_1 - \theta_1$

B^2 : $0 = -I_2 + \theta_1 I_1 - \theta_2 = I_2 - \theta_1 I_1 - \theta_2 I_0$

B^3 : $0 = I_3 - \theta_1 I_2 - \theta_2 I_1$

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$$I_n - \theta_1 I_{n-1} - \theta_2 I_{n-2} = 0 \qquad n \geq 2$$

$$I_0 = -1; \quad I_1 = \phi_1 - \theta_1$$

Explicit way of finding I_e-s?

Just like in the case of G.F.-s

ARMA(2,1) $\to$ Implicit method for I.F. coefficients

$$(1-\phi_1 B-\phi_2 B^2)X_t = (1-\theta_1 B)a_t$$

$$a_t = -(I_0 + I_1 B + I_2 B^2 + \dots)X_t$$

$$(1-\phi_1 B-\phi_2 B^2)X_t = -(1-\theta_1 B)(I_0 + I_1 B + I_2 B^2 + \dots)X_t$$

B^0: $1 = -I_0$

B^1: $-\phi_1 = \theta_1 I_0 - I_1 \Rightarrow I_1 = \phi_1 - \theta_1$

B^2: $-\phi_2 = \theta_1 I_1 - I_2 \Rightarrow I_2 = \phi_2 + \theta_1(\phi_1 - \theta_1)$

B^3: $0 = \theta_1 I_2 - I_3 \Rightarrow I_3 = \theta_1 I_2$

etc.

Explicit method $\to$ using long division of polynomials

$$a_t = \frac{1-\phi_1 B - \phi_2 B^2}{1-\theta_1 B}X_t = \Big(\underbrace{1}_{-I_0} + \underbrace{(\theta_1-\phi_1)}_{-I_1}B + \overbrace{\frac{\theta_1(\theta_1-\phi_1)-\phi_2}{1-\theta_1 B}}^{-I_2}B^2\Big)X_t$$

$\Rightarrow I_0 = -1, \quad I_1 = \phi_1 - \theta_1, \quad I_2 = \phi_2 + \theta_1(\phi_1 - \theta_1)$

$$I_n = \theta_1^{n-2}(\theta_1(\phi_1-\theta_1)-\phi_2) \qquad n \geq 2$$

Auto-Covariance Function

$$\text{cov}(V_1, V_2) = E[(V_1 - E[V_1])(V_2 - E[V_2])] =$$
$$= E[V_1 V_2] - E[V_1]E[V_2]$$

High covariance between 2 variables implies that knowing one variables says a lot about the other variable.

Auto Covariance of X_t-s obeying an ARMA model can be assessed from

$$\text{cov}(X_t, X_{t-k}) = E[X_t X_{t-k}] = E[X_t X_{t+k}] = \gamma_k$$
$$\gamma_k = \gamma_{-k} \text{ - autocovariance is a symetric fcn}$$

Autocorrelation: $\rho_k = \dfrac{\gamma_k}{\gamma_0}$ (Cauchy-Schwartz inequality says $|\gamma_k| \leq \gamma_0$

Note: our basic assumption was that $\text{cov}[a_t, a_{t-k}] = \sigma_a^2 \delta_k$

$$\text{cov}[a_t\ a_{t-k}] = \begin{cases} \sigma_a^2, & k=0 \\ 0, & k \neq 0 \end{cases}$$

In order to characterize $\text{cov}[X_t, X_{t-k}]$ we'll use the orthogonal decomposition of X_t

$$X_t = \left(\sum_{\ell=0}^{\infty} G_\ell B^\ell\right) a_t$$

i) $E[X_t a_{t-k}] = \sum_{\ell=0}^{\infty} G_\ell \, a_{t-\ell} \, a_{t-k} = G_k \sigma_a^2 \qquad k \geq 0$

ii) $E[X_t a_{t+k}] = 0$ for $k > 0$

iii) $E[X_t X_{t-k}] = E[(\sum_{j_1=0}^{\infty} G_{j_1} a_{t-j_1}) \sum_{j_2=0}^{\infty} G_{j_2} a_{t-k-j_2})]$

$$= \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} G_{j_1} G_{j_2} E[a_{t-j_1} a_{t-k-j_2}]$$

$$= \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} G_{j_1} G_{j_2} \underbrace{\delta_{k+j_2-j_1}}_{\substack{\text{only exists} \\ \text{when } j_2+k=j_1}} \sigma_a^2$$

$$= \sum_{j_2=0}^{\infty} G_{j_2+k} G_{j_2} \sigma_a^2$$

Case of AR(1) models $\qquad X_t - \phi_1 X_{t-1} = a_t$

$$G_\ell = \phi_1^\ell$$

$$\gamma_k = \sigma_a^2 \sum_{j_2=0}^{\infty} \phi_1^{j_2} \phi_1^{j_2+k} = \sigma_a^2 \phi_1^k \sum_{j_2=0}^{\infty} \phi_1^{2j_2} = \frac{\sigma_a^2 \phi_1^k}{1-\phi_1^2}$$

(for $|\phi_1| < 1$)

ARMA(2,1) $\qquad X_t - \phi_1 X_{t-1} - \phi_2 X_{t-2} = a_t - \theta_1 a_{t-1}$

$$G_\ell = g_1 \lambda_1^\ell + g_2 \lambda_2^\ell$$

λ_1, λ_2 - roots of the AR characteristic poly

$$\gamma_k = \sigma_a^2 \sum_{j=0}^{\infty} G_j G_{j+k} = \sigma_a^2 \sum_{j=0}^{\infty} (g_1 \lambda_1^j + g_2 \lambda_2^j)(g_1 \lambda_1^{j+k} + g_2 \lambda_2^{k+j})$$

$$= \sigma_a^2 \sum_{j=0}^{\infty} (g_1^2 \lambda_1^{2j+k} + g_1 g_2 \lambda_1^j \lambda_2^{j+k} + g_1 g_2 \lambda_1^{j+k} \lambda_2^j + g_2^2 \lambda_2^{2j+k})$$

$$= \sigma_a^2 \Big[g_1^2 \lambda_1^k \sum_{j=0}^{\infty} \lambda_1^{2j} + g_1 g_2 \lambda_2^k \sum_{j=0}^{\infty} \lambda_1^j \lambda_2^j + g_1 g_2 \lambda_1^k \sum_{j=0}^{\infty} \lambda_2^j \lambda_1^j + g_2^2 \lambda_2^k \sum \lambda_2^{2j} \Big]$$

$$= \sigma_a^2 \left[\frac{g_1^2 \lambda_1^k}{1-\lambda_1^2} + \frac{g_1 g_2 \lambda_1^k}{1-\lambda_2\lambda_1} + \frac{g_1 g_2 \lambda_2^k}{1-\lambda_1\lambda_2} + \frac{g_2^2 \lambda_2^k}{1-\lambda_2^2} \right]$$

$$= \sigma_a^2 \left[d_1 \lambda_1^k + d_2 \lambda_2^k \right]$$

$$d_1 = \frac{g_1^2}{1-\lambda_1^2} + \frac{g_1 g_2}{1-\lambda_2\lambda_1} \qquad d_2 = \frac{g_2^2}{1-\lambda_2^2} + \frac{g_1 g_2}{1-\lambda_1\lambda_2}$$

Generally, for an ARMA(n, n-1) model

$$X_t - \phi_1 X_{t-1} - \dots - \phi_n X_{t-n} = a_t - \theta_1 a_{t-1} - \dots - \theta_{n-1} a_{t-n+1}$$

$$G_\ell = g_1 \lambda_1^\ell + g_2 \lambda_2^\ell + \dots + g_n \lambda$$

$$\gamma_k = cov(X_t, X_{t+k}) = d_1 \lambda_1^k + d_2 \lambda_2^k + \dots + d_n \lambda_n^k$$

$$d_i = \left(\frac{g_i g_1}{1-\lambda_i \lambda_1} + \frac{g_i g_2}{1-\lambda_i \lambda_2} + \dots + \frac{g_i g_n}{1-\lambda_i \lambda_n} \right) \sigma_a^2$$

Variance Decomposition

$$Var[X_t] = \gamma_0 = \underbrace{d_1 + d_2 + \dots + d_n}$$

contributions of different dynamic modes to the total variance

One can use this in tool condition monitoring to see ("track") contributions of different dynamic modes.

Example: $X_t - 0.7X_{t-1} + 0.12X_{t-2} = a_t - 0.5a_{t-1}$

$\sigma_a^2 = 1$

Char. AR poly: $s^2 - 0.7s + 0.12 = 0 \Rightarrow$

Its roots are $\lambda_1 = 0.3$, $\lambda_2 = 0.4$

$G_i = g_1\lambda_1^i + g_2\lambda_2^i$ where

$$g_1 = \frac{\lambda_1 - \theta_1}{\lambda_1 - \lambda_2} = \frac{0.3-0.5}{0.3-0.4} = 2$$

$$g_2 = \frac{\lambda_2 - \theta_1}{\lambda_2 - \lambda_1} = \frac{0.4-0.5}{0.4-0.3} = -1$$

$$\gamma_\ell = (d_1\lambda_1^\ell + d_2\lambda_2^\ell)$$

$$\hookrightarrow d_1 = \sigma_a^2\left(\frac{g_1^2}{1-\lambda_1^2} + \frac{g_1 g_2}{1-\lambda_2\lambda_1}\right) = \frac{2^2}{1-0.3^2} - \frac{2}{1-0.4.0.3} = 2.0751$$

$$d_2 = \sigma_a^2\left(\frac{g_2^2}{1-\lambda_2^2} + \frac{g_1 g_2}{1-\lambda_1\lambda_2}\right) = \frac{1}{1-0.4^2} - \frac{2}{1-0.4\cdot 0.3} = -1.0823$$