ME384Q / ORI 390R.3

Time-Series Analysis

Midterm Examination

Spring, 2019

Instructor: Dr. Dragan Djurdjanovic

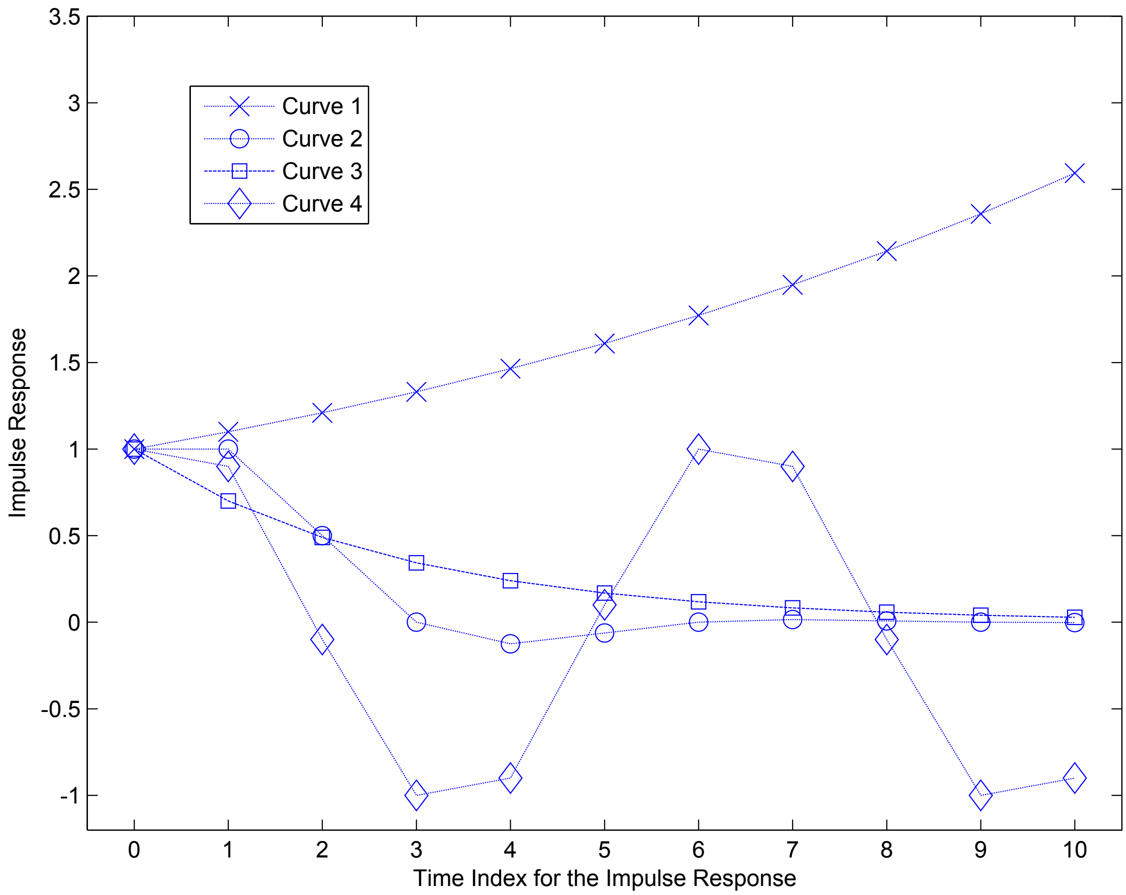
University of Texas at Austin

SOLUTIONS

Problem 1. (30 points)

Part a (16 points)

Match the impulse responses plotted in the figure below with the models listed below (one of the models is extra). Please explain your answers



Model No.	Model Formulation
1	$X_t - 0.7X_{t-1} = a_t$
2	$X_t - 1.1X_{t-1} = a_t$
3	$X_t - 0.9X_{t-1} + 0.2X_{t-2} = a_t - 0.1a_{t-1}$
4	$X_t - X_{t-1} + X_{t-2} = a_t - 0.1a_{t-1}$
5	$X_t - 0.5X_{t-1} + 0.25X_{t-2} = a_t$

Space for answers is on the next page!

Curve 1 -> Model No. ___2___; EXPLAIN

This curve has the shape of an exploding exponential which is the shape of the Green's Functions of AR(1) models with $|\phi_1| > 1$.

Curve 2 -> Model No. ___5___; EXPLAIN

Curve 2 looks like an exponentially decaying oscillation. We need to find a model whose AR characteristic roots are complex conjugate INSIDE the unit circle. Model 5 is the only plausible model, which has

$$\lambda^2 - 0.5\lambda + 0.25 = 0, \quad \text{and} \quad \lambda_{1/2} = 0.25 \pm 0.433j.$$

Curve 3 -> Model No. ___1___; EXPLAIN

This curve has the shape of a decaying exponential, which corresponds to the Green's function of an AR(1) with $|\phi_1| < 1$.

Curve 4 -> Model No. ___4___; EXPLAIN

Curve 4 is a pure sinusoidal curve, which implies that the AR characteristic polynomial roots of the underlying model are complex conjugates ON the unit circle. Only model 4 has such AR char. roots. That is

$$\lambda^2 - \lambda + 1 = 0 \Rightarrow \lambda_{1/2} = \frac{1}{2} \pm \frac{\sqrt{3}}{2} j$$

$$|\lambda_1| = |\lambda_2| = 1 \text{ and } \lambda_1 \neq \lambda_2.$$

Part b (11 points.)

Choose the most appropriate answer and explain your choice.

---

i) The system corresponding to the ARMA model below

$$X_t + X_{t-1} + X_{t-2} = a_t - a_{t-1} \qquad a_t \sim NIID(0, \sigma_a^2)$$

is stable | marginally stable | unstable (circle the most appropriate answer and get 1 point; Explain your answer and get another 3 points)

Characteristic AR equ is $\lambda^2 + \lambda + 1 = 0 \Rightarrow \lambda_{1/2} = -\frac{1}{2} \pm j\frac{\sqrt{3}}{2}$

$|\lambda_1| = |\lambda_2| = 1$ and $\lambda_1 \neq \lambda_2 \Rightarrow$ it's a marginally stable system

ii) The system

$$X_t - 3X_{t-1} + 3X_{t-2} - X_{t-3} = a_t + 2a_{t-1} + a_{t-2} \qquad a_t \sim NIID(0, \sigma_a^2)$$

is invertible | marginally invertible | non-invertible (circle the most appropriate answer and get 1 point; Explain your answer and get another 3 points)

Characteristic MA polynomial is $\lambda^2 + 2\lambda + 1 = 0 \Rightarrow$ $\lambda_1 = \lambda_2 = -1$. Hence, we have a repeated MA char. poly root ON the unit circle $\Rightarrow$ it's a non-invertible time-series model (system)

iii) Give an example of an **unstable**, but **invertible** system (1 points). Explain why it is unstable and invertible (2 points).

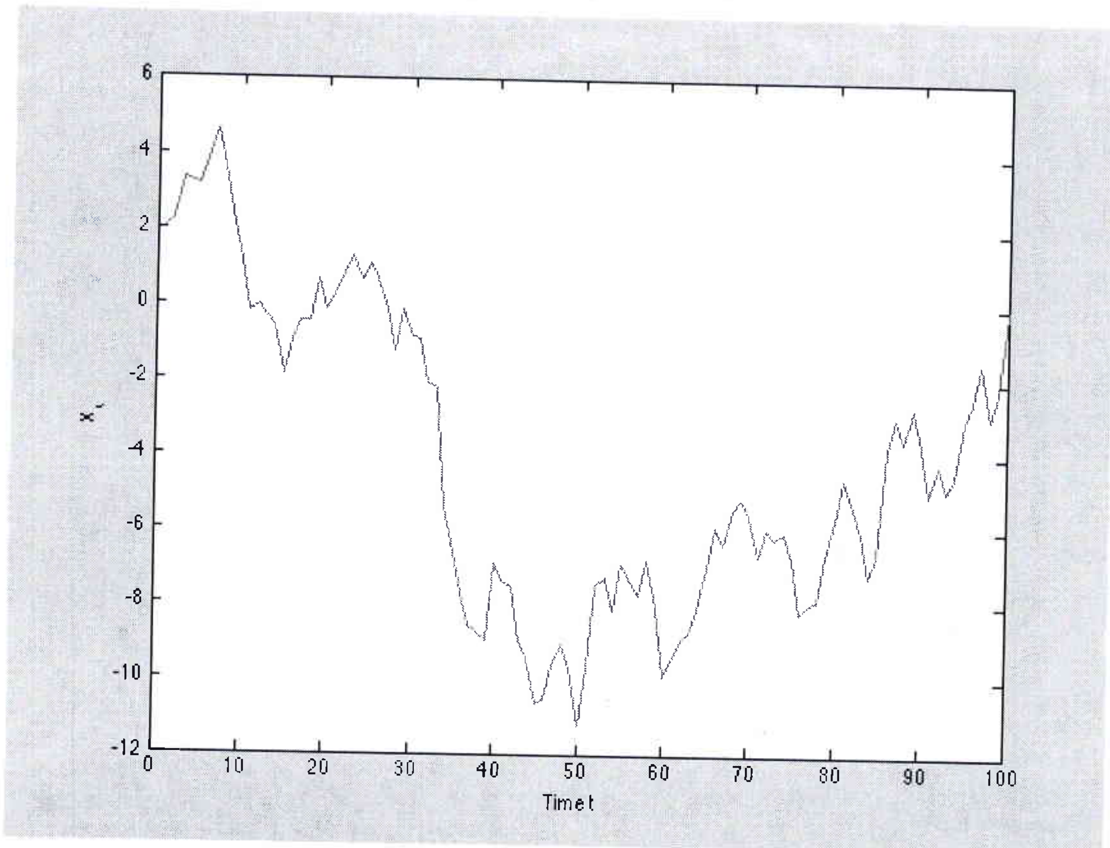
$$X_t - 2.5X_{t-1} + X_{t-2} = a_t - 0.1a_{t-1}$$

* This model is unstable because roots of the characteristic AR polynomial $\lambda^2 - 2.5\lambda + 1 = 0$ are $\lambda_1 = 2$, $\lambda_2 = 0.5$, and since $|\lambda_1| > 1$ (it's outside of the unit circle), it must be unstable.
* It's invertible because the one and only root of the MA char. poly. $s - 0.1 = 0$, which is $\nu = 0.1$, lays inside the unit circle.

Part c (3 points)

The figure below shows a realization of a time-series governed by an AR(1) model

$$X_t - X_{t-1} = a_t, \quad a_t \sim NIID(0,1)$$



Show that the variance of this time-series is infinite (use the Green's function decomposition). Now, please explain to me how come the values of this time-series are finite and not exploding? Hint-what would happen if I re-run my simulation 1000 times and obtain 1000 more realizations of this time-series?

$G_j = \phi_1^j \equiv 1$. $Var(X_t) = Var(\sum_{i=0}^{\infty} a_{t-i} G_i) = (\sum_{i=0}^{\infty} G_i) \cdot \sigma_a^2 = \infty$

The time series are finite and not exploding, because this is just one realization of the time series. If the simulation were run 1000 times, the realizations would be over the place.

Problem 2. (25 points)

The Green's function of the system is given as

$$G_k = \begin{cases} 0.7 \cdot (0.6)^k + 0.3 \cdot (-0.9)^k, k \geq 0 \\ 0, \quad \text{otherwise} \end{cases}$$

Part a (15 points)

Find the autoregressive and moving average parameters of the corresponding ARMA model.

$$X_t = \left(\sum_{k=0}^{\infty} G_k B^k\right) a_t = \left[\left(\sum_{k=0}^{\infty} 0.7 \cdot (0.6B)^k\right) + \left(\sum_{k=0}^{\infty} 0.3 \cdot (-0.9B)^k\right)\right] a_t =$$

$$\left(\frac{0.7}{1-0.6B} + \frac{0.3}{1+0.9B}\right) a_t = \frac{1+0.45B}{1+0.3B-0.54B^2} a_t$$

$$\Rightarrow (1+0.3B-0.54B^2) X_t = (1+0.45B) a_t \Rightarrow$$

$$X_t + 0.3X_{t-1} - 0.54X_{t-2} = a_t + 0.45a_{t-1}$$

Part (b) is on the next page

Part b (10 points)

Use the *explicit method* to find the inverse function coefficients of the system identified in part a. If you could not solve part a, use the *explicit method* to find inverse function coefficients for any ARMA(2,1) system in order to get partial credit.

$$a_t = \frac{1 + 0.3B - 0.54B^2}{1 + 0.45B} X_t$$

$$= \left(1 - 0.15B - 0.4725 \frac{B^2}{1 + 0.45B}\right) X_t$$

$$= \left(1 - 0.15B - 0.4725 B^2 \sum_{l=0}^{\infty} (-0.45)^l B^l\right) X_t$$

$$= \left(1 - 0.15B - 0.4725 \sum_{l=0}^{\infty} (-0.45)^l B^{l+2}\right) X_t$$

$$= -(I_0 + I_1 B + I_2 B^2 + I_3 B^3 + \dots) X_t$$

$$\Rightarrow I_0 = -1; \quad I_1 = 0.15; \quad I_l = 0.4725(-0.45)^{l-2}, \; l \geq 2$$

Problem 3. (25 points)

Part a (12 points)

A student fit an ARMA model to 250 samples of a stationary time-series obtained from an accelerometer mounted on a machine tool (in this case, accelerometers output time series centered around zero and hence the mean value of the time series was just taken for granted to be zero). Table below encloses the ARMA model parameters and residual sums of squares for a series of models output by this student's program. Please determine the adequate model according to the ARMA(2n, 2n-1) modeling procedure, using 95% confidence limits. Please carefully state the necessary statistical tests.

Parameters	AR(1)	AR(2)	ARMA(2,1)	ARMA(4,3)
ϕ_1 (95% C.I.)	0.72 ± 0.09	1.25 ± 0.21	1.38 ± 0.27	0.67 ± 1.00
ϕ_2 (95% C.I.)		-0.5 ± 0.28	-0.63 ± 0.39	0.45 ± 0.66
ϕ_3 (95% C.I.)				-0.12 ± 0.27
ϕ_4 (95% C.I.)				-0.43 ± 0.50
θ_1 (95% C.I.)			0.23 ± 0.31	-0.88 ± 1.00
θ_2 (95% C.I.)				-0.56 ± 0.60
θ_3 (95% C.I.)				0.29 ± 0.28
Residual Sum of Squares (RSS)	1705	1564	1563	1550

$F_{0.95}(1,\infty) = 3.84$, $F_{0.95}(2,\infty) = 3.00$, $F_{0.95}(3,\infty) = 2.60$, $F_{0.95}(4,\infty) = 2.37$,

$F_{0.95}(5,\infty) = 2.21$, $F_{0.95}(6,\infty) = 2.10$ etc. (this is enough for you to solve this problem).

Test 1: ARMA(4,3) vs ARMA(2,1)

$A_0 = 1550$; $A_1 = 1563$; $s = 4$; $N = 250$; $r = \underbrace{7}_{}+1 \leftarrow$ mean

unrestricted RSS — restricted RSS — # of restricted params — # of samples — # of estimated parameters in the unrestricted model (it's ok if you use 7 here - I accept that)

$$F = \frac{(A_1 - A_0)/s}{A_0/(N-r)} = 0.52 \qquad F^{0.95}_{4,\infty} = 2.37$$

Since $F < F^{0.95}_{4,\infty} \Rightarrow$ keep ARMA(2,1)

Test 2: ARMA(2,1) vs ARMA(1,0)

$A_0 = 1563$; $A_1 = 1705$; $s = 2$; $N = 250$; $r = 3+1 = 4$ (3 is ok here too)

$$F = \frac{(A_1 - A_0)/s}{A_0/(N-r)} = 11.17 \qquad F^{0.95}_{2,\infty} = 3$$

Since $F > F^{0.95}_{2,\infty}$ => keep ARMA(2,1)

Test 3: Since the confidence interval of θ_1 in the ARMA(2,1) model encompasses zero, let's check the AR(2) model, i.e. let's run

ARMA(2,1) vs AR(2)

$A_0 = 1563$; $A_1 = 1564$; $s = 1$; $N = 250$; $r = 3+1 = 4$ (3 is ok here too)

$$F = \frac{(A_1 - A_0)/s}{A_0/(N-r)} = 0.157 \qquad F^{0.95}_{1,\infty} = 3.84$$

Since $F < F^{0.95}_{1,\infty}$ => we can take AR(2) model as adequate!

Part b (5 points)

Let us assume that $X_{250} = 2$; $X_{249} = -1$. Estimate best least-squares prediction of x_{252}?

$X_{250}(2) = E[X_{252} \mid t=250]$; Our adequate model found in part (a) says that $X_{252} - 1.25X_{251} + 0.5X_{250} = a_{252}$

$\Rightarrow \hat{X}_{250}(2) = 1.25\hat{X}_{250}(1) - 0.5X_{250}$; Similarly, our model yields

$\hat{X}_{250}(1) = 1.25X_{250} - 0.5X_{249} = 3 \Rightarrow \hat{X}_{250}(2) = 2.75$

Part c (8 points)

What is the variance of the corresponding prediction error (hint – use results from the table to find the variance of the noise terms)

$$Var[\hat{e}_{250}(2)] = \sigma_a^2 [G_0^2 + G_1^2]$$

$$\sigma_a^2 \approx \frac{RSS}{N-r} = \frac{1564}{250-3} = 6.3$$

2 model params and the mean

Implicit method for finding G.F. coefficients yields

$$(1 - 1.25B + 0.5B^2)(G_0 + G_1B + G_2B^2 + \ldots)a_t = a_t$$

$\Rightarrow$

$$\Rightarrow G_0 = 1;\ G_1 = 1.25 \ldots$$

$$Var[\hat{e}_{250}(2)] = 6.3[1^2 + 1.25^2] = 16.22$$

$$\Rightarrow P(X_{2021 t=200} > 1) = P\left(\frac{X_{2021 t=200} - 0.54}{\sqrt{2.98}} > \frac{1-0.54}{\sqrt{2.98}} \right)$$

Standard normal distr.

$$= P(Z > 0.267) = 1 - \Phi(0.267) =$$

from the table in Appendix A

$$= 1 - 0.605 = 0.395$$

Problem 4. (20 points)

Gate-opening in the papermaking process is equidistantly sampled using the sampling interval of 0.3 seconds. The resulting AR(1) model was found to be

$$X_t - 0.7X_{t-1} = a_t\,, \quad a_t \sim NID(0,\sigma_a^2) \qquad (1)$$

where $\sigma_a^2 = 2.0$.

Part a (6 points)

Analytically describe the covariance and correlation functions for the time-series described by the AR(1) model (1).

$$\gamma_\ell = \frac{\sigma_a^2}{1-\phi_1^2}\phi_1^{|\ell|}, \quad \ell \in \mathbb{Z} \Rightarrow \gamma_\ell = \frac{2}{1-0.7^2}\cdot 0.7^{|\ell|} = 3.9\cdot 0.7^{|\ell|}$$

$$\rho_\ell = \frac{\gamma_\ell}{\gamma_0} = 0.7^{|\ell|}$$

Part b (6 points)

If $X_{200} = 1.1$, please evaluate the probability that $X_{202} > 1$.

$$\hat{X}_{200}(1) = 0.7X_{200} = 0.77; \quad \hat{X}_{200}(2) = 0.7\cdot\hat{X}_{200}(1) = 0.54$$

$$Var[\hat{e}_{200}(2)] = \sigma_a^2[G_0^2 + G_1^2] = 2[1^2 + 0.7^2] = 2.98$$

G.F. for an AR(1) model

$\Rightarrow$ At time $t=200$, X_{202} conditionally behaves as

$$X_{202|t=200} \sim \mathcal{N}(\underbrace{0.54}_{\hat{X}_{200}(2)}, \sqrt{2.98})$$

variance of 2 step prediction errors

$\Rightarrow$

Part c (8 points)

Assuming covariance-function equivalent sampling, please describe the stochastic linear ordinary differential equation characterizing the random process that describes the gate opening. Also, please comment on what kind of a result you'd obtain if you used the impulse response equivalent sampling approach.

AR(1) model is obtained via sampling of a 1st order linear stochastic ODE of the form

$$\dot{X}(t) + \alpha_0 X(t) = Z(t)$$

where $E[Z(t)] = 0$; $E[Z(t) Z(s)] = \sigma_z^2 \delta(t-s)$

We need to find α_0 and σ_z^2.

Regardless whether we use impulse response equivalent or covariance fcn equivalent sampling, we get

$$\alpha_0 = \left(-\frac{1}{\Delta}\right) \ln \phi_1 = \frac{\ln(0.7)}{(-0.3)} = 1.19$$

$$\sigma_z^2 = \frac{2 \alpha_0 \sigma_a^2}{1 - \phi_1^2} = \frac{2 \cdot 1.19 \cdot 2}{1 - 0.7^2} = 9.33$$