Discussion on the behavior of Green fcn coefficients in ARMA(2,1) models

$$X_t - \phi_1 X_{t-1} - \phi_2 X_{t-2} = a_t - \theta_1 a_{t-1}$$

$a_t \sim$ 2nd order wht. noise with $E[a_t]=$
$\sigma_a^2 = Var[a_t]$
(e.g. NIID $N(0, \sigma_a^2)$)

$$X_t = \sum_{\ell=0}^{\infty} G_\ell a_{t-\ell} \quad \text{where}$$

$$G_\ell = g_1 \lambda_1^\ell + g_2 \lambda_2^\ell$$

λ_1, λ_2 are roots of $s^2 - \phi_1 s - \phi_2$ and

$$g_1 = \frac{\lambda_1 - \theta_1}{\lambda_1 - \lambda_2}, \quad g_2 = \frac{\lambda_2 - \theta_1}{\lambda_2 - \lambda_1}$$

Obviously $\lambda_{1/2} = \frac{1}{2}\left(\phi_1 \pm \sqrt{\phi_1^2 + 4\phi_2}\right)$

- If $\phi_1^2 + 4\phi_2 > 0 \Rightarrow \lambda_{1/2} \in R$
- If $\phi_1^2 + 4\phi_2 < 0 \Rightarrow \lambda_{1/2}$ are complex conjugate numbers

If $\lambda_1 = \lambda_2^* \in \mathbb{C}$, it's interesting to write out G_ℓ using Euler's notation of complex numbers

$$\lambda_{1/2} = r e^{\pm j \cdot w}$$

where $r = |\lambda_1| = |\lambda_2| = |\phi_2|$

$$w = \cos^{-1}\left(\frac{Re\,\lambda_1}{|\lambda_1|}\right) = \cos^{-1}\left(\frac{\lambda_1 + \lambda_2}{2\sqrt{\lambda_1 \lambda_2}}\right) = \cos^{-1}\frac{\phi_1}{2\sqrt{|\phi_2|}}$$

This gives

$$G_\ell = r^\ell \underbrace{(g_1 e^{jw\ell} + g_2 e^{-jw\ell})}_{\text{oscillatory term}}$$

Coefficients g_1 and g_2 (which is equal to g_1^*) can be written as

$$g_{1,2} = g \cdot e^{\pm j\beta}$$

$$g = \frac{1}{2}\sqrt{1 + \left[\frac{\phi_1 - 2\theta_1}{\sqrt{-(\phi_1^2 + 4\phi_2)}}\right]^2}$$

$$\beta = \tan^{-1}\left[\frac{-\phi_1 + 2\theta_1}{\sqrt{-(\phi_1^2 + 4\phi_2)}}\right]$$

Implicit method of finding GF coefficients for ARMA(n, n-1) models

Let's observe an ARMA(2,1) model

$$X_t - \phi_1 X_{t-1} - \phi_2 X_{t-2} = a_t - \theta_1 a_{t-1} \qquad (1)$$

a_t is wss white noise process, $E[a_t]=0$; $Var[a_t]=\sigma_a^2$

If $X_t = \sum_{l=0}^{\infty} G_l a_{t-l}$, coefficients can be recursively determined as follows.

From (1), we have that

$$(1-\phi_1 B - \phi_2 B^2) X_t = (1-\theta_1 B) a_t \quad \Rightarrow$$

$$(1-\phi_1 B - \phi_2 B^2)(G_0 + G_1 B + G_2 B^2 + \dots + G_l B^l + \dots) a_t =$$
$$= (1-\theta_1 B) a_t$$

Equating coefficients next to each power of B, we get

B^0: $G_0 = 1$

B^1: $G_1 - \phi_1 G_0 = -\theta_1$
$\hookrightarrow G_1$

B^2: $G_2 - \phi_1 G_1 - \phi_2 G_0 = 0 \rightarrow G_2$

$\vdots$

B^l: $G_l - \phi_1 G_{l-1} - \phi_2 G_{l-2} = 0 \rightarrow G_l$

$\dots$

This is a difference eqn

$$G_\ell - \phi_1 G_{\ell-1} - \phi_2 G_{\ell-2} = 0 \qquad \ell \geq 2 \qquad (2)$$

with initial conditions $G_0 = 1$, $G_1 = \phi_1 - \theta_1$

This difference equation can be explicitly solved as

$$G_\ell = g_1 \lambda_1^\ell + g_2 \lambda_2^\ell \qquad (3)$$

where $\lambda_{1/2} = \frac{1}{2}(\phi_1 \pm \sqrt{\phi_1^2 + 4\phi_2})$ are roots of the AR char. polynomial $s^2 - \phi_1 s - \phi_2 = 0$ and

$$g_1 = \frac{\lambda_1 - \theta_1}{\lambda_1 - \lambda_2} \qquad g_2 = \frac{\lambda_2 - \theta_1}{\lambda_2 - \lambda_1}$$

(this is the explicit solution we've already seen - recursive procedure of finding G_ℓ-s successively from (2), constitutes <u>the implicit method.</u>

NOTE: Eqn. (3) is ALWAYS the explicit solution of (2), but the resulting Wold's decomposition implies X_t is NOT wide sense stationary (no finite variance)

Let's observe a general ARMA(n, n-1) model

$$X_t - \phi_1 X_{t-1} - \dots - \phi_n X_{t-n} = a_t - \theta_1 a_{t-1} - \dots - \theta_{n-1} a_{t-n+1} \quad \dots \quad (4)$$

where a_t is a wss white noise with $E[a_t]=0$, $Var[a_t]=\sigma_a^2$

In order to find GF coefficients G_ℓ in the Wold's decomposition

$$X_t = \sum_{\ell=0}^{\infty} G_\ell a_{t-\ell} = \left[\sum_{\ell=0}^{\infty} G_\ell B^\ell\right] a_t \quad \dots \quad (5)$$

from (4) we get that

$$(1-\phi_1 B - \dots - \phi_n B^n) X_t = (1-\theta_1 B - \dots - \theta_{n-1} B^{n-1}) a_t \implies$$

$$(1-\phi_1 B - \phi_2 B^2 - \dots - \phi_n B^n)(G_0 + G_1 B + G_2 B^2 + \dots + G_n B^n + \dots) a_t =$$
$$(1 - \theta_1 B - \theta_2 B^2 - \dots - \theta_{n-1} B^{n-1}) a_t$$

Equating coefficients next to powers of B, we get

B^0: $G_0 = 1$ (i-1)

B^1: $G_1 - \phi_1 G_0 = -\theta_1 \rightarrow G_1$ (i-2)

B^2: $G_2 - \phi_1 G_1 - \phi_2 G_0 = -\theta_2 \rightarrow G_2$ (i-3)

$\vdots$

B^{n-1}: $G_{n-1} - \phi_1 G_{n-2} - \dots - \phi_{n-1} G_0 = -\theta_{n-1} \rightarrow G_{n-1}$ (i-n)

B^{n}: $G_n - \phi_1 G_{n-1} - \dots - \phi_n G_0 = 0 \rightarrow G_n$

$\vdots \qquad \vdots$

B^{ℓ}: $G_\ell - \phi_1 G_{\ell-1} - \dots - \phi_n G_{n-\ell} = 0$ .... (6)

Eqn. (6) is a general n^{th} order difference equ. with initial conditions (i-1) ÷ (i-n), whose explicit solution is

$$G_\ell = g_1 \lambda_1^\ell + g_2 \lambda_2^\ell + \dots + g_n \lambda_n^\ell \quad \text{-------} \quad (7)$$

where λ_i, $i = 1, 2, \dots, n$ are roots of the AR char. poly

$$s^n - \phi_1 s^{n-1} - \phi_2 s^{n-2} - \dots - \phi_n = 0$$

and coefficients g_i, $i = 1, 2, \dots, n$ are given by (3.1.26)

Form (7) is solution to (6) and initial cond. (i-1) - (i-n) regardless of λ-s. However, if not all λ-s are inside the unit circle, then X_t won't be a WSS process (its variance will be infinite).

$$\sum_{l=0}^{\infty} G_l^2 < \infty$$

Since complex exponentials form a basis function set, we can find $n \in \mathbb{N}$ and $\lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{C}$, $g_1, g_2, \dots, g_n \in \mathbb{C}$ such that

$$\sum_{l=0}^{\infty} \left(G_l - \sum_{i=1}^{n} g_i \lambda_i^l \right)^2 < \frac{\varepsilon}{\sigma_a^2}$$

Then, let's observe

$$\tilde{X}_t = \sum_{l=0}^{\infty} \tilde{G}_l a_{t-l}$$

where $\tilde{G}_l = \sum_{i=1}^{n} g_i \lambda_i^l$. We know that $\tilde{X}_t$ must follow an ARMA$(n, n-1)$ model. In addition,

$\forall t \in \mathbb{Z}$, $E[(X_t - \tilde{X}_t)^2] = E[(\sum_{l=0}^{\infty} (G_l - \tilde{G}_l) a_t)^2] =$

$$= \sum_{l=0}^{\infty} (G_l - \tilde{G}_l)^2 \sigma_a^2 < \cancel{\sigma_a^2} \cdot \frac{\varepsilon}{\cancel{\sigma_a^2}} = \varepsilon \qquad \text{QED}$$