Implementation and Simulation of Quantum Teleportation

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The basic principle of quantum teleportation (QT) is reviewed first in the introduction, and then some simple quantum gates which are necessary for QT are introduced in section 1. Next in section 2, we clarify how one could construct and measure a Bell state and relevant quantum circuits are illustrated. Finally, with all conceptions clear, a program for simulation of QT is shown to testify the quantum circuit really works.

Introduction

Quantum Teleportation (QT) appeared as a very fundamental but interesting topic in quantum information theory (QIT). The theory of QT was first published in 1993 by six scientists including Bennett[2]. According to their report, two qubits which form an EPR pair and a classic channel can be used to teleport another qubit from Alice to Bob (named for a comventional reason).

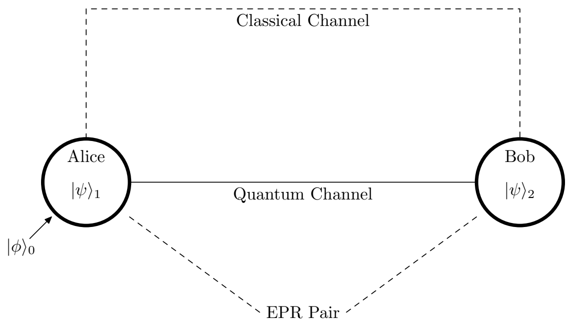

Figure 1: Quantum Teleportation

The whole system for this teleportation is shown in figure (1). Suppose that two qubits with label 1, 2 controlled by Alice and Bob respectively, form an EPR pair (or Bell state):

$$|\psi\rangle_{12} = \frac{1}{\sqrt{2}}(|0\rangle_1|1\rangle_2 - |1\rangle_1|0\rangle_2), \tag{1}$$

which is used as a quantum channel and the distance between Alice and Bob can be any value. The task of this teleportation is to send another qubit

$$|\phi\rangle_0 = a|0\rangle_0 + b|1\rangle_0 \tag{2}$$

from Alice to Bob (Alice could know nothing about $|\phi\rangle_0$). To complete this task, Alice put $|\phi\rangle_0$ in her box together with qubit 1, yielding a tri-qubit state in the whole system:

$$\begin{aligned} |\psi\rangle_{012} =& \frac{a}{\sqrt{2}}(|011\rangle - |100\rangle) \\ &+ \frac{b}{\sqrt{2}}(|011\rangle - |101\rangle). \end{aligned} \tag{3}$$

Next, using some apparatus, Alice measures and identifies which Bell state the pair 0 & 1 are at. To clarify this, we should rewrite formula (3) by projecting them to the four Bell states of qubits $0, 1$

$$\begin{aligned} |\psi\rangle_{012} =& \frac{1}{2}\Big[|\psi^-\rangle_{01}(-a|0\rangle_2 - b|1\rangle_2) \\ &+ |\psi^+\rangle_{01}(-a|0\rangle_2 + b|1\rangle_2) \\ &+ |\phi^-\rangle_{01}(a|1\rangle_2 + b|0\rangle_2) \\ &+ |\phi^+\rangle_{01}(a|1\rangle_2 - b|0\rangle_2)\Big]. \end{aligned} \tag{4}$$

So when Alice performs the measurement, $|\psi\rangle_{123}$ will collapse to one of the four parts of right hand of equation (4) with identical probabilities $1/4$. Meanwhile, when Bob detects qubit 2, he will get the corresponding state which can be restored to $|\phi\rangle_0$ by a unitary transformation. To know this unitary transformation, Alice need to tell Bob what is the Bell state of qubit 0&1 she obtained just now, through the classical channel. For example, if Alice obtains $|\psi^-\rangle_{01}$, then the state that Bob gets must be $-a|0\rangle_2 - b|1\rangle_2$ or $[-a, -b]$. After receiving Alice's information through classical channel, Bob can then restore qubit 0 by

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -a \\ -b \end{pmatrix}. \tag{5}$$

Note that because the classical channel is necessary, information can not be transported at speed beyond light, even though the collapsing occurs at both sides instantaneously (a common theory).

Above is just a basic and rough description of QT. There are 2 problems remaining: a) how to generate an EPR pair; b) how to measure (identify) a Bell state. To clarify these issues, knowledge about Quantum Gate, which will be introduced in next section. Because matrix operations can be perfomed conveniently in a computer, a numerical simulation will be shown in the last section.

1 Quantum Gates

1.1 Quantum Gates for Single Qubit

A Quantum gate is a unitary operator for qubit(s), which can be represented, for single qubit, as a 2×2 matrix. Such operators are all composed of Pauli matrices and are all reversible. Based on Pauli matrices, some simple quantum gates can be formed

$$\begin{aligned} I &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ Y &= \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \ Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ H &= \frac{1}{\sqrt{2}}(X + Z) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \end{aligned} \tag{6}$$

Given a qubit $|phi\rangle = a|0\rangle + b|1\rangle$, it is easy to check

$$I|\phi\rangle = a|0\rangle + b|1\rangle \tag{7}$$
$$X|\phi\rangle = b|0\rangle + a|1\rangle \tag{8}$$
$$Y|\phi\rangle = -\mathrm{i}b|0\rangle + \mathrm{i}a|1\rangle \tag{9}$$
$$Z|\phi\rangle = a|0\rangle - b|1\rangle \tag{10}$$
$$\begin{aligned} H|\phi\rangle &= \frac{1}{\sqrt{2}}\Big[a(|0\rangle + |1\rangle) + b(|0\rangle - |1\rangle)\Big] \\ &= a|+\rangle + b|-\rangle. \end{aligned} \tag{11}$$

where X, Y, Z are just Pauli matrices $\sigma_x, \sigma_y, \sigma_z$ respectively, and H is Hadamard[1] matrix gate which can transform any qubit into a superposition state $a|+\rangle + b|-\rangle$. We will use H to construct EPR pair later.

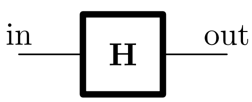

Figure 2: Quantum gate for a single qubit.

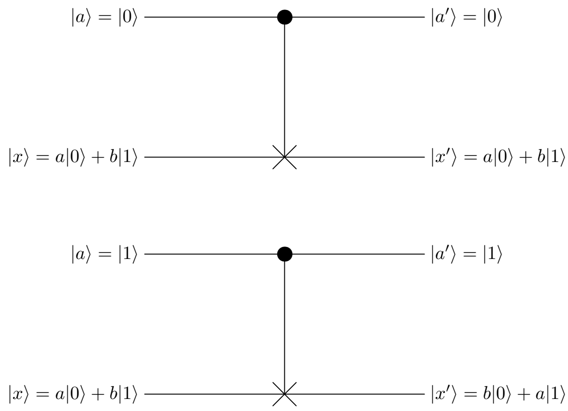

Figure 3: CNOT gate with control qubit $|a\rangle$ and target qubit $|x\rangle$.

A gate for a single qubit has one input and one output. An example is illustrated in figure (2).

1.2 CNOT Gate

CNOT stands for Controlled NOT gate with a control terminal a and a target terminal x. In classical circuits, a and x is certainly 0 or 1. x holds the value when $a = 0$ and becomes $\bar{x}$ (NOT x) when $a = 1$. In a word, x is always left xXORa after a CNOT gate with control bit a.

In the quantum situation, the difference is that both $|a\rangle$ and $|x\rangle$ (Dirac notation for quantum) can be some superposition states, but the basic idea of CNOT is the same as a classical one. Two typical kinds of inputs and outputs are shown in figure (3). Note that the two lines in figure (3) represented for qubits $|a\rangle$ and $|x\rangle$ can also be entangled, so it is better to find the 2×2 matrix representation of CNOT for convenience.

The direct product of $|a\rangle$ and $|x\rangle$ can be written as a 4D vector

$$\begin{aligned} |a\rangle \otimes |x\rangle &= u_1|00\rangle + u_2|01\rangle \\ &\quad + u_3|10\rangle + u_4|11\rangle \\ &= \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}, \end{aligned} \tag{12}$$

where the direct product symbol "$\otimes$" is omitted at right hand. And the "XOR idea" is also available here, which means $|a, x\rangle \mapsto |a, a \text{ XOR } x\rangle$ under a CNOT operation. Let A_{CNOT} be the matrix of

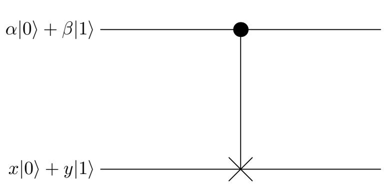

Figure 4: CNOT with two superposition states as inputs.

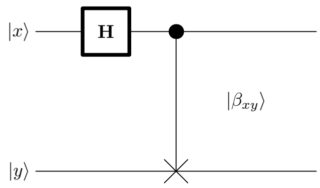

Figure 5: Quantum circuit for constructing entangled pair.

CNOT, then

$$\begin{aligned} A_{\text{CNOT}}|a\rangle \otimes |x\rangle &= u_1|00\rangle + u_2|01\rangle \\ &\quad + u_3|11\rangle + u_4|10\rangle \\ &= \begin{pmatrix} u_1 \\ u_2 \\ u_4 \\ u_3 \end{pmatrix}. \end{aligned} \tag{13}$$

Now it's easy to find the matrix transforming the vector in equation (12) to that in equation (13), which just keeps $u_1,\ u_2$ unchanged and makes $u_3,\ u_4$ swaped:

$$A_{\text{CNOT}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{14}$$

2 Realization of Quantum Teleportation

With quantum gates, we can build, like the classical logic circuit, quantum circuit. In this section, kinds of quantum circuits are introduced, which can make QT come true step by step.

2.1 Constructing Entangled Pairs

Consider the general situation of a CNOT gate, i.e. inputs with two qubits which are both at superposition states, as shown in figure (4). Because the outcome may be a entangled pair, we can not treat the two lines separately, which requires us calculate the outcome in a direct product form:

$$\begin{aligned} &A_{\text{CNOT}}(\alpha|0\rangle + \beta|1\rangle) \otimes (x|0\rangle + y|1\rangle) \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha x \\ \alpha y \\ \beta x \\ \beta y \end{pmatrix} \\ &= \begin{pmatrix} \alpha x \\ \alpha y \\ \beta y \\ \beta x \end{pmatrix} \end{aligned} \tag{15}$$

Note that the two qubits of inputs are separable. Now we shall ask "in what situation should the outcome be entangled?". Rewrite the outcome in formula (15) as

$$\begin{bmatrix} \alpha \begin{pmatrix} x \\ y \end{pmatrix} \\ \beta \begin{pmatrix} y \\ x \end{pmatrix} \end{bmatrix}. \tag{16}$$

It is seen that if $\alpha \neq 0$, $\beta \neq 0$ (namely the control qubit is a superposition state) and $x \neq y$, than the outcome can not be decomposited into direct product, which means an entangled outcome.

So the conclusion is that we can use a CNOT with a control qubit at superposition state to construct entangled pairs. We already know Hadamard gate generate superposition state, so a quantum circuit[3] for constructing entangled pairs can be built, as shown in figure (5).

The notation $|\beta_{ax}\rangle$ is for Bell states

$$|\beta_{00}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \tag{17}$$

$$|\beta_{01}\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \tag{18}$$

$$|\beta_{10}\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) \tag{19}$$

$$|\beta_{11}\rangle = \frac{1}{\sqrt{2}}(|10\rangle - |01\rangle) \tag{20}$$

We will see these binary encoding for Bell states exactly show the corresponding relation between the input and outcome of the circuit. For example, inputs $|a\rangle = |0\rangle$, $|x\rangle = |0\rangle$ lead to the outcome $|\beta_{00}\rangle$, and it's convenient to check the others.

2.2 Measuring Bell States

As mentioned in Instruction, in a QT system, Alice need some apparatus to identify which Bell state qubit 0 &1 are at. We have seen, in subsection 2.1, inputs with two separable qubits lead to an entangled outcome. In turn, we will soon check, an input with an entangled pair for the inverse circuit leads to a separable pair.

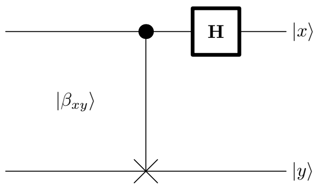

Figure 6: Quantum circuit for separating an entangled pair.

Figure (6) shows the inverse version of figure (5). This can be checked as follows. Suppose an entangled input, say $|\beta_{00}\rangle$. CNOT acts on it first:

$$\begin{aligned}A_{\text{CNOT}}|\beta_{00}\rangle &= A_{\text{CNOT}}\frac{1}{\sqrt{2}}(|00\rangle+|11\rangle)\\ &= \frac{1}{\sqrt{2}}(|00\rangle+|10\rangle),\end{aligned} \tag{21}$$

then the Hadamard follows. Note that because the Hadamard lives in a subspace of the whole direct space, we need to make a direct product with an identity matrix on it before acting it on the entangled pair:

$$\begin{aligned}&(H\otimes I)\frac{1}{\sqrt{2}}(|00\rangle+|10\rangle)\\ =&\frac{1}{\sqrt{2}}(|{+}0\rangle+|{-}0\rangle)\\ =&\frac{1}{2}(|00\rangle+|01\rangle+|00\rangle-|01\rangle)\\ =&|00\rangle=|0\rangle\otimes|0\rangle.\end{aligned} \tag{22}$$

As you can see, the outcome is a separable pair and each qubit of the pair is at an eigenstate, which means if we measure each line of the outcome, we will get eigen values 0 and 0 respectively. See the binary encoding always matches with the Bell states. Outcomes of other entangled inputs can be checked in the same way.

2.3 The Entire Circuit

Now we are in the position to build the entire circuit for QT.

Assemble the quantum gates introduced before following figure (7). As shown in figure, Bob controls $|\psi\rangle_2$ and Alice controls the other two qubits. Let $|\phi\rangle_0 = a|0\rangle + b|b\rangle$ be the qubit waiting for teleportation.

The first part is the entangled-pair-constructor at the left bottom. Let the inputs $|\psi\rangle_1 = |0\rangle$ and $|\psi\rangle_2 = |0\rangle$, then $|\beta_{00}\rangle_{12}$ is yielded[1]. Next, Alice add

[1] All of the four Bell states can be used in QT, while different choice leads to different unitary matrix that Bob uses to restore $|\phi\rangle_0$.

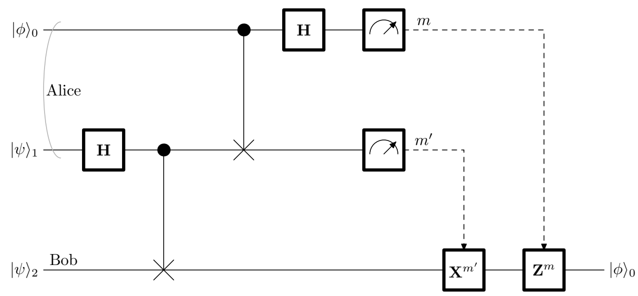

Figure 7: Entire circuit for QT.

$|\phi\rangle_0$ to the system, making the system a tri-qubit

$$\begin{aligned}&|\phi\rangle_0\otimes|\beta_{00}\rangle_{12}\\ =&(a|0\rangle+b|1\rangle)\otimes\frac{1}{\sqrt{2}}(|00\rangle+|11\rangle)\\ =&\frac{1}{\sqrt{2}}(a|000\rangle+a|011\rangle+b|100\rangle+b|111\rangle),\end{aligned} \tag{23}$$

which is a 8D vector.

Next part at right top is for Alice to identify Bell states for $|\psi\rangle_{01}$. The icons like gauge meters stand for measurement apparatus. After the operation of CNOT, the tri-qubit becomes (use "XOR idea")

$$\frac{1}{2}(a|000\rangle+a|011\rangle+b|110\rangle+b|101\rangle). \tag{24}$$

The Hadamard in this part acts on qubit 0, and the result is easy to obtained:

$$\begin{aligned}&\frac{1}{2}[a|000\rangle+a|100\rangle+a|011\rangle+a|111\rangle\\ &\quad+b|010\rangle-b|110\rangle+b|001\rangle-b|101\rangle],\end{aligned} \tag{25}$$

or (because direct product satisfies distributive law):

$$\begin{aligned}&|00\rangle_{01}\frac{a|0\rangle_2+b|1\rangle_2}{2}+|01\rangle_{01}\frac{b|0\rangle_2+1|1\rangle_2}{2}\\ +&|10\rangle_{01}\frac{a|0\rangle_2-b|1\rangle_2}{2}+|11\rangle_{01}\frac{-b|0\rangle_2+a|1\rangle_2}{2}.\end{aligned} \tag{26}$$

Notice that the $|00\rangle_{01}$, $|01\rangle_{01}$, $|10\rangle_{01}$, $|11\rangle_{01}$ in equation 26 generated from Alice's apparatus are not, but corresponding to Bell states $|\beta_{00}\rangle_{01}$, $|\beta_{01}\rangle_{01}$, $|\beta_{10}\rangle_{01}$, $|\beta_{11}\rangle_{01}$ respectively. After Alice's measurement the tri-qubit will collapse into one of the four items in equation (26) with equivalent probabilities.

Finally, Bob uses $X^{m'}$ gate and Z^m gate to restore $|\phi\rangle_0$. m and m' stand for Alice's measurement result, corresponding to Bell state $|\beta_{mm'}\rangle_{01}$, and Bob is told this result through classical channel. Next Bob acts matrix $Z^m X^{m'}$ on qubit 2, then he will restore the $|\phi\rangle_0$. All situations are listed in table (1).

m	m'	Alice	Bob	$Z^m X^{m'}$
0	0	$\|00\rangle$	$a\|0\rangle + b\|1\rangle$	I
0	1	$\|01\rangle$	$b\|0\rangle + a\|1\rangle$	X
1	0	$\|10\rangle$	$a\|0\rangle - b\|1\rangle$	Z
1	1	$\|11\rangle$	$-b\|0\rangle + a\|1\rangle$	ZX

Table 1: Relations between Alice and Bob's measurement. After Alice's measurement, Bob can restore $|\phi\rangle_0$ in his qubit 2 with corresponding matrix.

3 Simulation for QT

At last a program will be introduced here for testifying the theory of QT. I coded with Python and the program just follows the process shown in figure (7).

In my simulation, normalized $|\phi\rangle_0$s are generated randomly. And pseudo-random numbers are used to simulate collapsing. One simulation result are shown as follow.

```
Initial qubit 1 and 2:
[[1 0]],[[1 0]]
Qubit 0 to teleportation
[[ 0.17067118+0.83265808j]
 [ 0.46270451-0.25190554j]]


Constructed EPR pair:
[[ 0.70710678+0.j]
 [ 0.00000000+0.j]
 [ 0.00000000+0.j]
 [ 0.70710678+0.j]]
the tri-qubit:
[[ 0.12068275+0.58877818j]
 [ 0.00000000+0.j         ]
 [ 0.00000000+0.j         ]
 [ 0.12068275+0.58877818j]
 [ 0.32718150-0.17812412j]
 [ 0.00000000+0.j         ]
 [ 0.00000000+0.j         ]
 [ 0.32718150-0.17812412j]]
State before Alice's measurement:
[[ 0.08533559+0.41632904j]
 [ 0.23135226-0.12595277j]
 [ 0.23135226-0.12595277j]
 [ 0.08533559+0.41632904j]
 [ 0.08533559+0.41632904j]
 [-0.23135226+0.12595277j]
 [-0.23135226+0.12595277j]
 [ 0.08533559+0.41632904j]]
Restored qubit 0:
[[ 0.17067118+0.83265808j]
 [ 0.46270451-0.25190554j]]
```

As shown above, the restored qubit 0 is exactly the same as the original one, which means the quantum circuit does work well. Because there are no quantum things really happens, this is just a toy program for fun[2].

[2] The code is uploaded to https://github.com/Varato/QT_sims

Summary

As the program testified, if neglecting decoherence, QT works very well. This provides a way to send a quantum state from one to one. It is worth mentioning that classical and quantum channel are both necessary for QT. Only quantum channel can not transmit any information. It can be explained as follows:

1. The amount of information Bob can obtain when measuring his qubit depends on the partial entropy of entanglement (equals 1 for Bell state), and Alice's local measurement can not cause any change of this entropy. Further to say, qubit 2 is in a mixed state for Bob who can only access qubit 2 until he measures it.

2. It is believed collapsing is instantaneous and nonlocal, which means Bob's qubit can responce Alice's measurement without any delay, while transmission of information faster than light is not physical. This also ensures that along the quantum channel, no one other than Alice and Bob could fetch the quantum state of qubit 0.

And Through the classical channel, only part of information can Alice transmit to Bob, which means even though the classical information is eavesdropped, without Bob's qubit, the eavesdropper can not fetch the qubit 0 ether. This feature provide a scheme for Quantum private key distribution (QKD)[4].

Because noise and decoherence are all neglected in the theory and simulation above, so we are still far away from the real physical situation.

References

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