A Good Mythical Morning Darts Problem

Varaun Ramgoolie

A Late Tuesday in May, 2024

1 Introduction

1.1 Setup

There exists a game where two players throw darts randomly at a dartboard. After two darts are thrown, a point on the dart board is selected at random, and then the respective distances from the point and the players' darts are measured. These distances are then added to the corresponding player's score. The player with the lowest score wins.

1.2 Problem

Say at the final round of the game, player B has x more points than player A (they are losing) and player A is throwing their dart last. What can player A do to ensure that they don't lose the game?

2 Attempt at Solution

2.1 Abstracting the Problem

We can translate this problem into randomly selecting two points in the Cartesian plane (the darts), selecting another random point on the plane (the reference point for the game), and measuring their distances. We will adopt this notion moving forward.

2.2 Motivation

At first, the problem seems relatively trivial. I'm actually just doing this to refresh myself with LaTeX before resuming work on some Applied Mathematics solutions (go check that out lol!). The problem is probably trivial, but thanks for making it so far.

The actual motivation comes about from this observation. If two right angles share an entire edge, two vertices (including the vertex where the right angle lies), and (at least part) of a base, the difference of the hypotenuses of these right triangles is less than or equal to the difference of the shared component of the base (that is a lot of words, everything will make sense soon).

We use the term base relatively loosely here to refer to the edge of the triangles which is perpendicular to the wholly shared edge.

2.3 Obscure Triangle Proof

We will employ the science that is high-school geometry for this. Let us define Triangle 1 to have vertices A, B. and C, and Triangle 2 will have vertices A, B, and C'. Convince yourself that we can express these points on the Cartesian plane as follows:

$$\begin{aligned} A &= (0, y), \\ B &= (0, 0), \\ C &= (x, 0), \\ C' &= ((x+k), 0), \end{aligned}$$

for positive real numbers x, y, and k (practically, the mathematics holds even if these are not strictly positive, due to the effects of translation, symmetry, exchanging of variables, and most importantly, hand-waving).

We want to show that:

$$AC' - AC \leq k. \tag{1}$$

First, let us notice that that, by the power of the mighty Pythagorean Theorem, we get that:

$$\begin{aligned} AC' &= \sqrt{(x+k)^2 + y^2}, \\ AC &= \sqrt{x^2 + y^2}, \\ \implies AC' - AC &= \sqrt{(x+k)^2 + y^2} - \sqrt{x^2 + y^2}. \end{aligned}$$

Now, hold my hand while we go along a treacherous path of very scary inequalities (the proofs of these are left as an exercise for the reader, but they are true). First, let us rewrite $AC' - AC$ as,

$$\begin{aligned} \sqrt{(x+k)^2 + y^2} - \sqrt{x^2 + y^2} &= \frac{(x+k)^2 + y^2 - x^2 - y^2}{\sqrt{(x+k)^2 + y^2} + \sqrt{x^2 + y^2}} && (2) \\ &= \frac{2kx + k^2}{\sqrt{(x+k)^2 + y^2} + \sqrt{x^2 + y^2}} && (3) \end{aligned}$$

Using a string of inequalities, we find that,

$$
\begin{aligned}
x^2 &\leq x^2 + y^2, \\
x &\leq \sqrt{x^2 + y^2}, \\
2kx &\leq 2k\sqrt{x^2 + y^2}, \\
x^2 + k^2 + 2kx &\leq x^2 + k^2 + 2k\sqrt{x^2 + y^2}, \\
(x+k)^2 &\leq x^2 + k^2 + 2k\sqrt{x^2 + y^2}, \\
(x+k)^2 + y^2 &\leq x^2 + k^2 + 2k\sqrt{x^2 + y^2} + y^2, \\
(x+k)^2 + y^2 &\leq (\sqrt{x^2 + y^2} + k)^2, \\
\sqrt{(x+k)^2 + y^2} &\leq \sqrt{x^2 + y^2} + k,
\end{aligned}
$$

and so,

$$\sqrt{(x+k)^2 + y^2} - \sqrt{x^2 + y^2} \leq k. \tag{4}$$

If you are actually attempting these inequalities, be careful with your square root inequalities.

As we have gotten that out of the way, we have shown that $AC' - AC \leq k$. We can now actually go on to explain why this was necessary in the first place.

2.4 Road to Success (Probably)

From our game, we know that the lowest score wins and we also are given that the person who has the lower score is the last to act. We will claim that this person can guarantee they win if the throw their dart at most x units away from the first thrower's dart (assuming that the score difference between them is x). Here is why.

Convince yourself that, regardless of where the darts are thrown and the reference point is chosen, the scenario can be abstracted to our Cartesian plane idea from above. We can then shift and rotate our plane suitably so that we form a triangle similar to the one mentioned in the proof above (all while preserving the plane's mathematical properties).

Let us now introduce our mathematical notation. Say we have fitted our Cartesian plane in such a way that the first dart lands on $(b, 0)$, the second lands on $(a, 0)$, and the reference point is $(0, c)$. We can absolutely do this because mathematical relationships are preserved regardless of if the plane is upside down, rotated, moved away, or even under water. Our claim is that, if $|b - a| \leq x$, then the second thrower is guaranteed to win. Let's see.

Our very obscure triangle is realized if we add our shared right-angled vertex at $(0, 0)$. What we get is equivalent to our proof above, where our hypotenuses are the distances our darts are from the reference point and the distance between both darts is realized to be k (from our proof). Thus, we get that the difference of the hypotenuses is less than (or equal to) the distance between our darts. Since the distance between our darts is less than x (by our expert strategy that we claim), and the second thrower has x less points that the first thrower, there is then no way for the first thrower's dart to be in a position where their dart is closer to the reference point compared to the second dart by more than x units (in order to win the game).

3 Conclusion

Congrats for making it so far. I left out a few things in this 'proof' because I was kinda lazy just typing. You can perhaps break this problem up into 3 cases, show that 2 of those cases are very obvious, and then throw this mess at the last case to finish your proof. My proofs prof would definitely have some nightmares about how messy this was, but I'm pretty sure it works. Throw some numbers at it and get back to me (or not, your choice). I found this from one of Good Mythical Morning's games on their show, and I thought it was kinda cool to show (with a decent amount of rigour) that there might be an optimal game theory strategy for their darts game, albeit a very specific situation.

Thanks for reading if you did, and take care!