$$\begin{bmatrix} a_{11} & 0 & \dots & & 0 \\ 0 & a_{22} & & & \\ 0 & & a_{33} & 0 & \\ \vdots & 0 & & \ddots & \\ & & & & 0 \\ 0 & \dots & & 0 & a_{nn} \end{bmatrix}$$

Diagonális mátrix

Mátrix diagonalizálás

Cél: ~~Adott~~ A mátrix $\to D$ diagonális mátrix úgy, hogy A és D sajátértékei megegyeznek.

Keressünk C mátrixot:
$\exists C^{-1}$

és

$C^{-1}AC$ diagonális

Tétel: A adott és C-nel $\exists C^{-1}$ A, $C^{-1}AC = B$
A és B sajátértékei és determinánsa megegyezik.

Def. A mátrix diagonalizálható, ha
$\exists C$: $C^{-1}AC$ diagonális

Tétel. A diagonalizálható $\Leftrightarrow$ van sajátvektorokból álló bázis

$$C = \begin{bmatrix} c_1 & c_2 & \dots & c_n \end{bmatrix}$$

$c_1, c_2, \dots, c_n$: sajátbázis

$D = C^{-1}AC$

$$\begin{bmatrix} d_{11} & & & 0 \\ & d_{22} & & \\ & & \ddots & \\ 0 & & & d_{nn} \end{bmatrix} \leftarrow$$ főátlóbeli elemek ∴ A sajátértékei

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1.a, 1. példa $\begin{bmatrix} 2 & -1 & -1 \\ 3 & -2 & -3 \\ -1 & 1 & 2 \end{bmatrix}$ $\lambda_1 = 0$ $a(0) = 1$

$\lambda_2 = 1$ $a(1) = 2$

$$W_{\lambda_1} = W_0 = \left\{ x_1 \cdot \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix} \middle| x_1 \in \mathbb{R} \right\} \qquad g(0) = 1$$

$$W_{\lambda_2} = W_1 = \left\{ x_2 \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \middle| x_2, x_3 \in \mathbb{R} \right\} \qquad g(1) = 2$$

$$g(0) + g(1) = a(0) + a(1) = 1 + 2 = 3 = n$$

S. BÁZIS: $\left\{ \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$

$\Downarrow$

A diag. ható

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ A diagonális alakja

$$C = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \qquad C^{-1} A C = D$$

1. b)

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 0 & -1 & 2 \end{bmatrix} \qquad \begin{matrix} \lambda_1 = 1 & a(1) = 2 \\ \lambda_2 = 2 & a(2) = 1 \end{matrix}$$

$$W_{\lambda_1} = W_1 = \left\{ x_2 \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \middle| x_2 \in \mathbb{R} \right\} \quad \dim W_1 = 1 \quad g(1) = 1$$

$$W_{\lambda_2} = W_2 = \left\{ x_1 \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \middle| x_1 \in \mathbb{R} \right\} \quad \dim W_2 = 1 \quad g(2) = 1$$

$$g(1) + g(2) = 2$$
$$\#$$
$$a(1) + a(2) = 3$$

$\Rightarrow$ nincs sajátbázis $\Leftrightarrow$ A nem diag.ható

1/c) $A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 2 & -1 & 0 \end{bmatrix}$

- $\det(A - \lambda \underline{I}) = 0$
  $\Rightarrow \lambda$
  $\Rightarrow a(\lambda)$
- $(A - \lambda \underline{I})x = 0$
  $\Rightarrow W_\lambda$, $\dim W_\lambda$
  $\Rightarrow g(\lambda)$, bázis
- $\sum_i g(\lambda)$
  $= n \to$ S.B. C D
  $\leq n$ nincs

$\det(A - \lambda \underline{I}) = \begin{vmatrix} 1-\lambda & -1 & 1 \\ 1 & 1-\lambda & -1 \\ 2 & -1 & -\lambda \end{vmatrix} =$ 1. oszlop szerint kifejtve

$= (1-\lambda)\left[(1-\lambda)(-\lambda) - (-1)(-1)\right] - 1\left[(-1)(-\lambda) - 1(-1)\right] + 2\cdot\left[(-1)(-1) - 1(1-\lambda)\right] =$

$= (-1)(\lambda-1)(\lambda+1)(\lambda-2) = 0$

$\lambda_1 = 1$ $\quad a(1) = 1$
$\lambda_2 = -1$ $\quad a(-1) = 1$
$\lambda_3 = 2$ $\quad a(2) = 1$

(1.) $\lambda_1 = 1$ $\quad A - \lambda \underline{I}$

$\begin{bmatrix} 0 & -1 & 1 \\ \boxed{1} & 0 & -1 \\ 2 & -1 & -1 \end{bmatrix} x = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\left[\begin{array}{ccc|c} 0 & -1 & 1 & 0 \\ \underline{1} & 0 & -1 & 0 \\ 0 & -1 & \boxed{1} & 0 \end{array}\right]$

$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ \underline{1}x_1 + & -1x_2 + & 0x_3 = & 0 \\ 0x_1 & -1x_2 + & \underline{1}x_3 = & 0 \end{array}\right]$

$x_1 - x_2 = 0$
$x_1 = x_2$

$-x_2 + x_3 = 0$
$x_3 = x_2$

x_2 szabad

$x = \begin{pmatrix} x_2 \\ x_2 \\ x_2 \end{pmatrix} = x_2 \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$W_{\lambda_1} = W_1 = \left\{ x_2 \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \,\middle|\, x_2 \in \mathbb{R} \right\} \Rightarrow \dim W_1 = 1$

$g(1) = 1$

(2.) $\lambda_2 = -1$ $\quad (A + \underline{I})$

$(A - \lambda_2 \underline{I})$

$\begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & -1 \\ 2 & -1 & 1 \end{bmatrix} x = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

x_1 szabad
$x_1 \in \mathbb{R}$

$x = x_1 \cdot \begin{pmatrix} 1 \\ -3 \\ \end{pmatrix}$

x_1 szabad

$x_1 \in \mathbb{R}$

$x_2 = -3x_1$

$x_3 = -5x_1$

$$x = x_1 \cdot \begin{pmatrix} 1 \\ -3 \\ -5 \end{pmatrix}$$

$$W_{\lambda_2} = W_{-1} = \left\{ x_1 \cdot \begin{pmatrix} 1 \\ -3 \\ -5 \end{pmatrix} \middle| \, x_1 \in \mathbb{R} \right\}$$

$\dim W_{-1} = 1$

$g(-1) = 1$

3. $\lambda_3 = 2 \quad (A - 2I)$

$$\begin{bmatrix} -1 & -1 & 1 \\ 1 & -1 & -1 \\ 2 & -1 & -2 \end{bmatrix} x = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_2 = 0$

$x_3 = x_1$

$x_1 \in \mathbb{R}$

$$x = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \cdot x_1, \quad x_1 \in \mathbb{R}$$

$$W_{\lambda_3} = \quad = \left\{ x_1 \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \middle| \, x_1 \in \mathbb{R} \right\}$$

$\dim(W_2) = 1$

$g(2) = 1$

$\sum_i g(\lambda) = \sum_i a(\lambda) = 1+1+1 = 3 \Rightarrow$ van s.B.

s.B: $\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -3 \\ -5 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\} \Longleftrightarrow A$ diag.ható

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$C = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -3 & 0 \\ 1 & -5 & 1 \end{bmatrix} \qquad C^{-1} A C = D$$