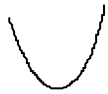
25. 3.a) $f(x)=x^2-6x+5 \quad (x\in\mathbb{R}) \quad R_f\,?$

$R_f=\{y\in\mathbb{R}\mid \exists x\in D_f : y=f(x)\} = \{y\in\mathbb{R}\mid \exists x\in\mathbb{R} : y=x^2-6x+5\}$

$x^2-6x+5-y=0$

$x_{1,2}=\frac{6\pm\sqrt{16+4y}}{2} \Rightarrow x_{1,2}=3\pm\sqrt{4+y} \Leftrightarrow 4+y\ge 0 \Rightarrow y\ge -4$



$R_f=[-4;+\infty)$

Megj.: $y=x^2-6x+5=(x-3)^2-4$

b) $R_f=\{y\in\mathbb{R}\mid \exists x\in[-1;6] : y=x^2-6x+5\}$

$x_1,x_2\in\mathbb{R} \Leftrightarrow y\ge -4$

$-1\le 3+\sqrt{4+y}\le 6 \quad \vee \quad -1\le 3-\sqrt{4+y}\le 6 \Rightarrow -4\le -\sqrt{4+y}\le 3 \;/\cdot(-1)$

$-4\le\sqrt{y+4}\le 3$

$y+4\le 9$

$y\le 5$

$y\in[-4;5]$

$4\ge\sqrt{y+4}\ge -3$

$-3\le\sqrt{y+4}\le 4 \Leftarrow$

$y+4\le 16$

$y\le 12$

$y\in[-4;12]$

$R_f=[-4;12]$

c) $R_f=\{y\in\mathbb{R}\mid \exists x\in[-2;3] : y=1-x^2\}$

$y=1-x^2$

$x^2=1-y$

$1-y\ge 0 \Rightarrow y\le 1$

$x_{1,2}=\pm\sqrt{1-y}\in[-2;3] \Rightarrow -2\le-\sqrt{1-y}\le 3 \;/\cdot(-1) \quad\vee\quad \underbrace{-2\le\sqrt{1-y}}_{\text{teljesül}}\le 3 \quad /()^2$

$2\ge\sqrt{1-y}\ge -3$

$\underbrace{-3\le\sqrt{1-y}}_{\text{teljesül}}\le 2 \quad /()^2$

$1-y\le 4$

$-3\le y$

$y\in[-3;1]$

$1-y\le 9$

$-8\le y$

$y\in[-8;1]$

$R_f = [-8; 1]$

8. d) $f(x) = \dfrac{3x+2}{x-1}$

invertálható $\Leftrightarrow$ $x \neq t \Rightarrow f(x) \neq f(t)$ $\quad x, t \in D_f = (+1; \infty)$, $x \neq t$

$$f(x) - f(t) = \frac{3x+2}{x-1} - \frac{3t+2}{t-1} = \frac{(3x+2)(t-1) - (3t+2)(x-1)}{(x-1)(t-1)} = \frac{5 \cdot (t-x)}{(x-1)(t-1)} =$$

$$= 0 \Leftrightarrow t - x = 0$$

$t = x$, de $t \neq x \Rightarrow f(x) - f(t) \neq 0$

$f(x) \neq f(t)$

f invertálható

$D_{f^{-1}} = R_f$

$f: \overset{x}{A} \to \overset{f(x)}{B}$

$A \leftarrow B : f^{-1}$

$$R_f = \left\{ y \in \mathbb{R} \mid \exists x \in (1; +\infty) : y = \frac{3x+2}{x-1} \right\}$$

$y = \dfrac{3x+2}{x-1}$

$y(x-1) = 3x+2$

$yx - y = 3x+2$

$yx - 3x = y+2$

$x(y-3) = y+2$ $\quad$ ha $y \neq 3$

$x = \dfrac{y+2}{y-3}$

Mivel $x \in (1; +\infty) \Rightarrow \dfrac{y+2}{y-3} > 1$

$\dfrac{y+2}{y-3} - 1 > 0$

$\dfrac{y+2}{y-3} - \dfrac{y-3}{y-3} > 0$

$\dfrac{5}{y-3} > 0 \Rightarrow y - 3 > 0 \Rightarrow y > 3$

$D_{f^{-1}} = R_f = (3; +\infty)$

$f^{-1}(y) = x = \dfrac{y+2}{y-3}$

8. b) $f(x) = x^2 - 2x + 2 \quad , x \in (-\infty, 1]$

$x, t \in (-\infty, 1] \quad , x \neq t$

$f(x) - f(t) = x^2 - 2x + 2 - (t^2 - 2t + 2) = x^2 - t^2 - 2(x - t) = \underbrace{(x-t)}_{\neq 0} \cdot \underbrace{(x+t-2)}_{\substack{x+t \neq 2 \\ \underbrace{x<1 \text{ v. } t<1}_{\neq 0}}}$

$\Rightarrow$ f invertálható

$D_{f^{-1}} = R_f$

$R_f = \{ y \in \mathbb{R} \mid x \in (-\infty ; 1] : y = x^2 - 2x + 2 \}$

$x^2 - 2x + 2 - y = 0 \Rightarrow x_{1,2} = 1 \pm \sqrt{y-1} \in \mathbb{R} \Rightarrow y - 1 \geq 0$

$y \geq 1$

$1 + \sqrt{y-1} \leq 1 \quad \vee \quad 1 - \sqrt{y-1} \leq 1$

$\sqrt{y-1} \leq 0 \qquad 0 \leq \sqrt{y-1}$

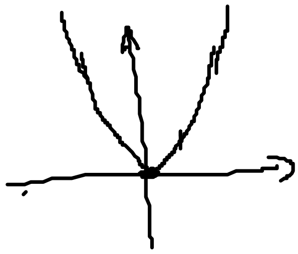
$y - 1 \leq 0 \qquad 0 \leq y - 1$

$y \leq 1 \qquad 1 \leq y$

$R_f = [1 ; +\infty)$

$f^{-1}(y) = 1 - \sqrt{y-1} \quad (y \geq 1)$

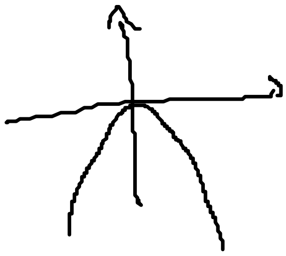
26. $f(x) = x^2$



$f(0) = 0$

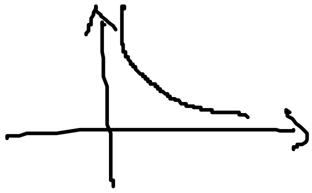
$0 \leq f(x) \; : x \in \mathbb{R}$

$f(x) = -x^2$



$f(x) \leq 0 \quad \forall x \in \mathbb{R}$

$\frac{1}{x}$, $x > 0$



0

26./ 12 a) kell. $\forall P>0\ \exists K>0:\ x>K:\ f(x)>P$

Legyen $P>0$ rögz.

$$f(x)=\frac{x^4-2x^3+x^2+7}{\underbrace{x^3+x+1}_{x^3+x^3+x^3}} \geq \frac{x^4-2x^3}{3x^3} \geq \frac{\frac{1}{2}x^4+x^3\overbrace{\left(\frac{1}{2}x-2\right)}^{\geq 0 \Rightarrow x>4}}{3x^3} \geq \frac{\frac{1}{2}x^4}{3x^3}=\frac{x}{6}\quad \text{, ha } x>4$$

$$\frac{x}{6}>P \iff x>6P$$

$$K:=\max\{4, 6P\}$$

$$\text{ha } x>K \Rightarrow f(x)\geq\frac{x}{6}>P \quad \checkmark$$

b) ha $x>K:\ |\underbrace{f(x)-2}_{\text{N.R.T.}}|<\varepsilon$