Lesson 03 Intro to Linear Regression Models

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February 24 (Monday), 2020

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- Did you see the HW?

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- Are you shocked?

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- Well, we will have OH this Wednesday (instead of Shiny?)?

Last Lecture ReCap

- Bring an example of the power of the Visualization.

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- Give some structural differences between the Barplot and Histogram.

Intro to Simple Linear Regression

Suggested materials to read (master) regression

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- Magnus J., et al., Introduction to Econometrics, Chapter 2, 3, 4

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- Original data from Galton's notebook http://www.medicine.mcgill.ca/epidemiology/hanley/galton/notebook/index.html lists 963 children in 205 families ranging from 1-15 adult children.

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- Original data from Galton's notebook http://www.medicine.mcgill.ca/epidemiology/hanley/galton/notebook/index.html lists 963 children in 205 families ranging from 1-15 adult children.
- You can find Galton dataset `GaltonFamilies` from the package `HistData`.

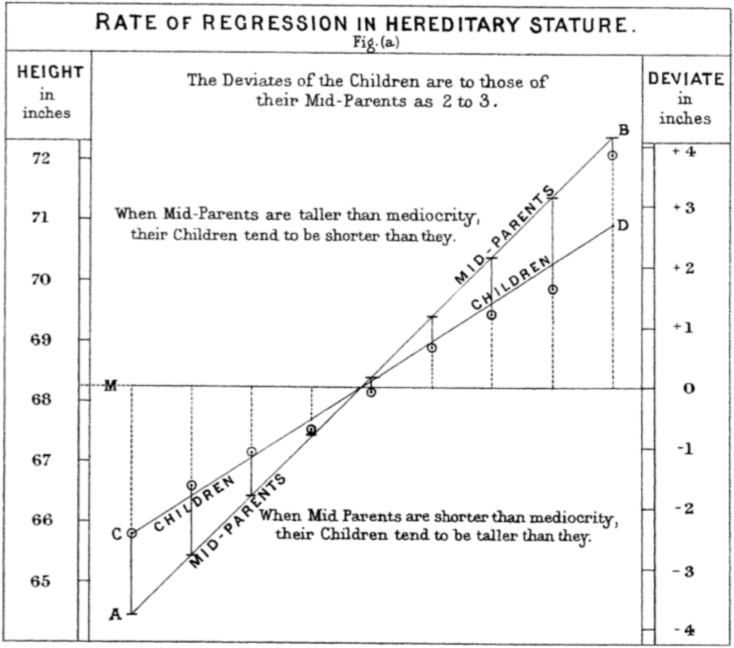
Source: Galton's classic paper *"Regression Towards Mediocrity in Hereditary Stature"*:
http://www.stat.ucla.edu/~nchristo/statistics100C/history_regression.pdf

TABLE I.

Number of Adult Children of various statures born of 205 Mid-parents of various statures.
(All Female heights have been multiplied by 1·08).

Heights of the Mid-parents in inches.	Heights of the Adult Children.														Total Number of		Medians.
	Below	62·2	63·2	64·2	65·2	66·2	67·2	68·2	69·2	70·2	71·2	72·2	73·2	Above	Adult Children.	Mid-parents.	
Above ..	..	..	..	..	..	..	..	..	..	..	..	1	3	..	4	5	..
72·5	..	..	..	..	..	..	..	1	2	1	2	7	2	4	19	6	72·2
71·5	..	..	..	..	1	3	4	3	5	10	4	9	2	2	43	11	69·9
70·5	1	..	1	..	1	1	3	12	18	14	7	4	3	3	68	22	69·5
69·5	..	..	1	16	4	17	27	20	33	25	20	11	4	5	183	41	68·9
68·5	1	..	7	11	16	25	31	34	48	21	18	4	3	..	219	49	68·2
67·5	..	3	5	14	15	36	38	28	38	19	11	4	..	..	211	33	67·6
66·5	..	3	3	5	2	17	17	14	13	4	..	..	..	..	78	20	67·2
65·5	1	..	9	5	7	11	11	7	7	5	2	1	..	..	66	12	66·7
64·5	1	1	4	4	1	5	5	..	2	..	..	..	..	..	23	5	65·8
Below ..	1	..	2	4	1	2	2	1	1	..	..	..	..	..	14	1	..
Totals ..	5	7	32	59	48	117	138	120	167	99	64	41	17	14	928	205	..
Medians ..	..	..	66·3	67·8	67·9	67·7	67·9	68·3	68·5	69·0	69·0	70·0	..	..	..	..	..

Note.—In calculating the Medians, the entries have been taken as referring to the middle of the squares in which they stand. The reason why the headings run 62·2, 63·2, &c., instead of 62·5, 63·5, &c., is that the observations are unequally distributed between 62 and 63, 63 and 64, &c., there being a strong bias in favour of integral inches. After careful consideration, I concluded that the headings, as adopted, best satisfied the conditions. This inequality was not apparent in the case of the Mid-parents.


RATE OF REGRESSION IN HEREDITARY STATURE.
Fig. (a)
HEIGHT in inches
The Deviates of the Children are to those of their Mid-Parents as 2 to 3.
DEVIATE in inches
72
71
70
69
68
67
66
65
+4
+3
+2
+1
0
-1
-2
-3
-4
B
D
M
C
A
When Mid-Parents are taller than mediocrity, their Children tend to be shorter than they.
When Mid Parents are shorter than mediocrity, their Children tend to be taller than they.
MID-PARENTS
CHILDREN
CHILDREN
MID-PARENTS

Simple (Modern) Linear Regression

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- Relationship between **one continuous dependent** variable and **one or more (any) explanatory variables**
- Simple LR - one explanatory variable
- Multiple LR - two or more explanatory variables
- Used for prediction and estimation

Simple linear regression

True relationship between X and Y:

- $$Y = f(X) + \varepsilon,$$

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- $$Y = f(X) + \varepsilon,$$
- ε-mean-zero random error term
- Approximately linear relationship between X and Y.
- $$Y \approx \beta_0 + \beta_1 X + \varepsilon$$
- **Unknown** constants: β_0-intercept, β_1-slope.

Reasons for having ε:

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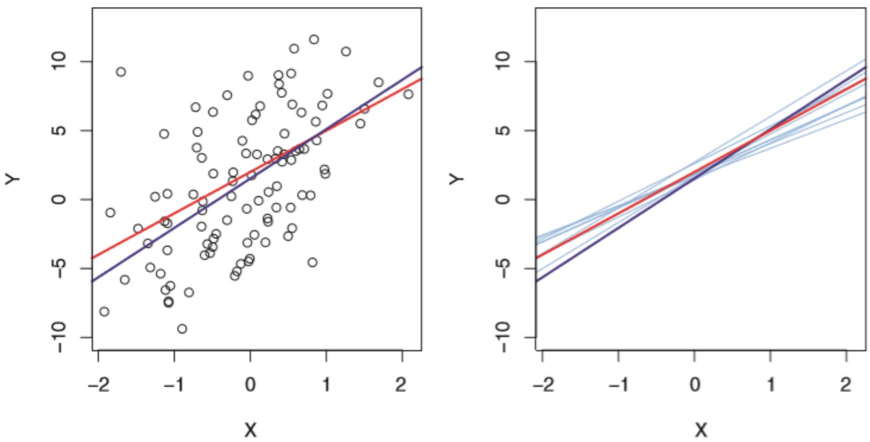
- $$\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x + \hat{\varepsilon}$$

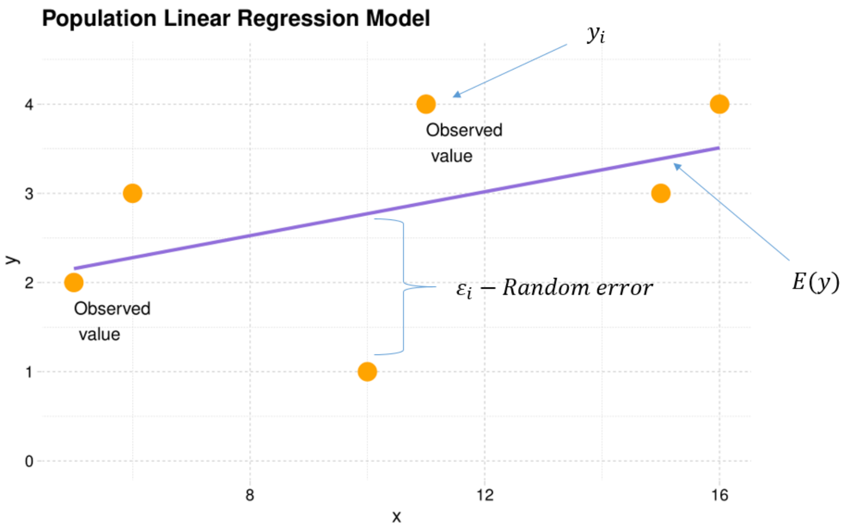
Population regression line vs least squares line

Source: G. James, D. Witten, et al., *An Introduction to Statistical Learning*

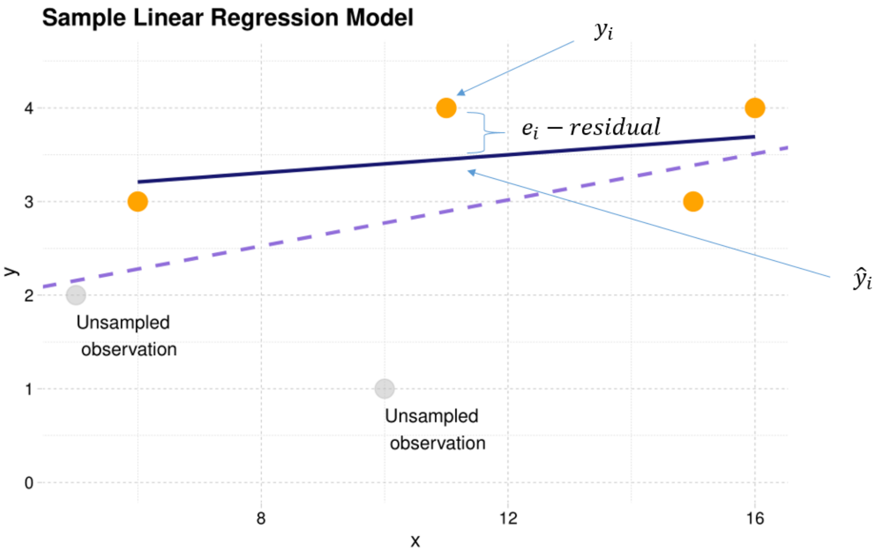


- Least squares line is computed using set of observations, however, the population regression line is unobserved.

Population Linear Regression Model



Sample Linear Regression Model



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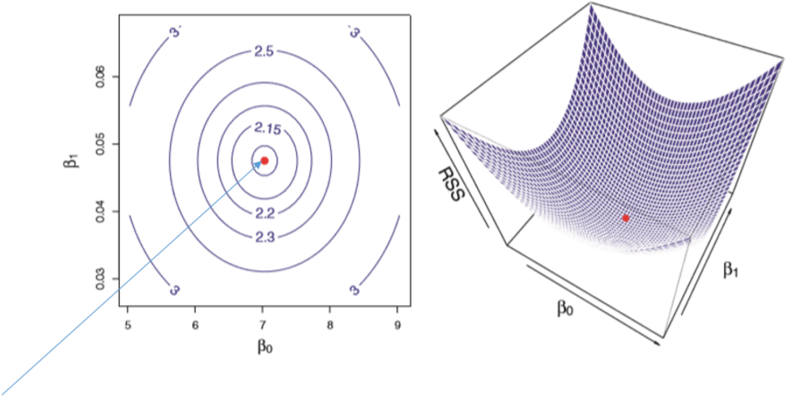
- $\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}$
- **Prove that:** $corr(x, y) = \hat{\beta}_1 \sqrt{\frac{var(x)}{vary}}$

Graphical representation

Source: G. James, D. Witten, et al., *An Introduction to Statistical Learning*



- The pair of least squares estimates $(\hat{\beta}_0, \hat{\beta}_1)$

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- **Task**: Derive all formulas by yourself.

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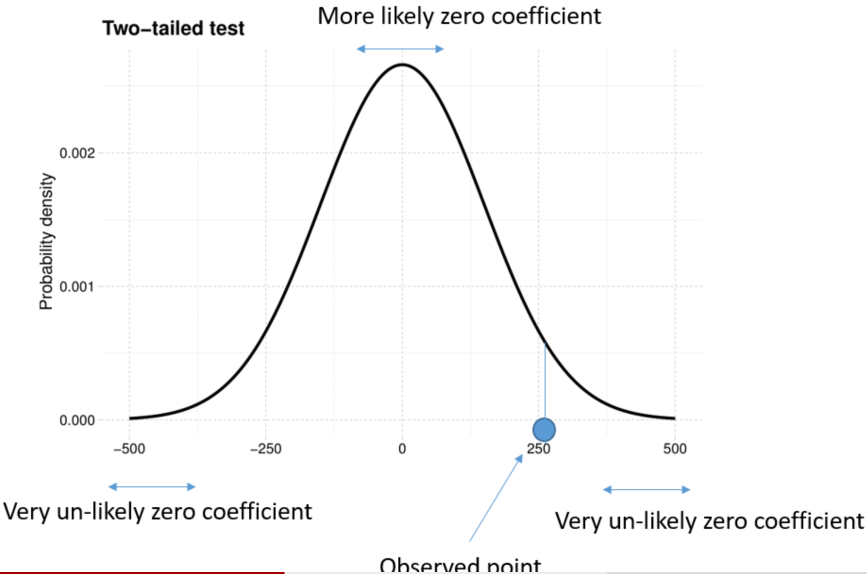
- $$|t| = \frac{|\hat{\beta}_1|}{SE(\hat{\beta}_1)} > t_{n-(k+1), \frac{\alpha}{2}}$$

- p-value - the probability of observing any value equal to $|t|$ or larger $(\mathbb{P} > |t|)$



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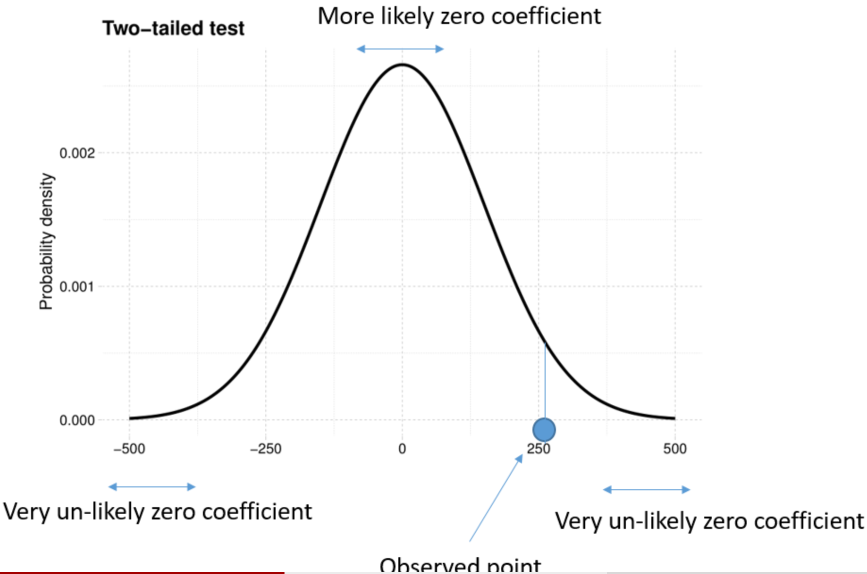

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3. F test

R-squared

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- By definition $0 \leq R^2 \leq 1$

Ideas for project

- Piecewise Polynomials, Regression Splines

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- Relaxing the assumtions of G-M theorem