Lesson 06 Logistic Regression

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March 23 (Monday), 2020

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- Binominal Logistic Regression

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- Why Not Linear Regression and OLS?

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- What is the difference between binomial and multinomial LogReg?

Last Lecture ReCap

- Why Not Linear Regression and OLS?
- What is the difference between binomial and multinomial LogReg?
- Which type of dependent/independent variable is used in LogReg?

Binary simple case

- Binary response using one predictor

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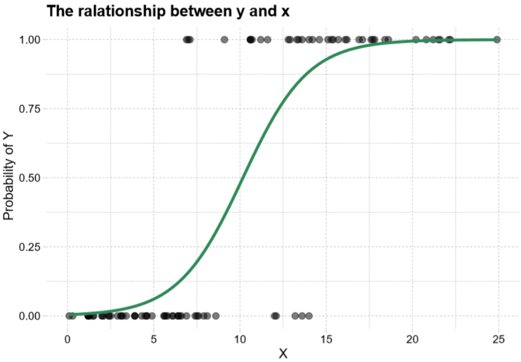
- When $p = 2$, there is only a single linear function to estimate.

Binary simple case

- Binary response using one predictor
- When p $= 2$, there is only a single linear function to estimate.
- The probability: $\mathbb{P}(y_i = 1) = \frac{e^{\beta_0 + \beta_1 x_i}}{1 + e^{\beta_0 + \beta_1 x_i}}$

S-shaped curve



Terminology

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- Odds ratio: $\frac{\frac{\mathbb{P}(y=1|x=1)}{1-\mathbb{P}(y=1|x=1)}}{\frac{\mathbb{P}(y=1|x=0)}{1-\mathbb{P}(y=1|x=0)}}$

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- Likelihood function:
- $$L(\beta_0, \beta_1) = \prod_{y_i=1} F(\beta_0 + \beta_1 x_i) \prod_{y_i=0} (1 - F(\beta_0 + \beta_1 x_i)) \to max$$

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- Likelihood is not a Probability - it can be larger than 1. It is not a PDF either, it is a function of the parameter.

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- $\underset{\beta}{argmax} L(\beta_0, \beta_1) = \underset{\beta}{argmax} lnL(\beta_0, \beta_1)$
- The estimates for coefficients are calculated using iterative procedure.

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- $ln\frac{\mathbb{P}(y=1)}{1-\mathbb{P}(y=1)} = \beta_0 + \beta_1 x_1 + ... + \beta_k x_k$
- $\mathbb{P}(y=1) = \frac{e^{\beta_0+\beta_1 x_1+...+\beta_k x_k}}{1+e^{\beta_0+\beta_1 x_1+...+\beta_k x_k}}$

Interpretations (Simple LogReg case)

Numeric predictors (Simple LogReg case)

- $ln\frac{\mathbb{P}(y=1)}{1-\mathbb{P}(y=1)} = \beta_0 + \beta_1 x$, where X is numeric

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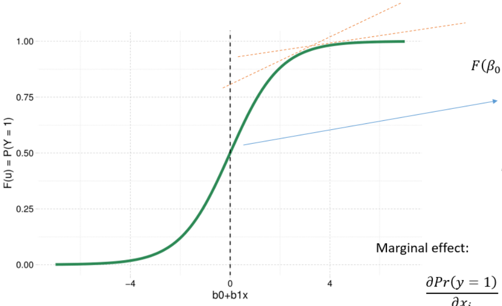
Numeric predictors (Simple LogReg case)



$$F(\beta_0+\beta_1 x) = \frac{1}{1+e^{-(\beta_0+\beta_1 x)}} = $$
$$= \frac{1}{1+e^0} = 0.5$$

$$\beta_0+\beta_1 x = 0 \Rightarrow \mathrm{x} = -\beta_0/\beta_1$$

Marginal effect:

$$\frac{\partial Pr(y=1)}{\partial x_j} = F(\beta_0+\beta_1 x_1+\cdots+\beta_k x_k) * \beta_j$$

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- $e^{\beta_1} = \frac{\frac{\mathbb{P}(y=1|x=1)}{1-\mathbb{P}(y=1|x=1)}}{\frac{\mathbb{P}(y=1|x=0)}{1-\mathbb{P}(y=1|x=0)}} = Odds\ ratio$

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- By changing x from 0 to 1, the odds ratio of $y = 1$ will be changed by $\frac{1}{e^{\beta_1}}$ times

Multinomial logistic regression

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- ...
- $ln \frac{\mathbb{P}(Y=P-1|X=x)}{1-\mathbb{P}(Y=P|X=x)} = \beta_{(P-1)0} + \beta_{(P-1)1}X_1 + ... + \beta_{(P-1)k}X_k$

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- $\mathbb{P}(Y = 1|X = x) = \frac{e^{\beta_{10}+\beta_{11}X_1+\ldots+\beta_{1k}X_k}}{1+\sum_{l=1}^{P-1} e^{\beta_{l0}+\beta_{l1}X_1+\ldots+\beta_{lk}X_k}}$

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- **Derive these formulas for p=3 case.**

Goodness of fit

- Confusion matrix

		Predicted	
		Negative (0)	Positive (1)
Actual	Negative (0)	TN	FP
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- $Sensitivity = \frac{TP}{TP+FN}$
- $Specificity = \frac{TN}{TN+FP}$

Goodness of fit

- How to choose the threshold?

Predicted	Actual1	Actual2	Actual3	Actual4	Actual5
0.1000000	0	1	0	0	0
0.1888889	0	1	0	0	1
0.2777778	0	1	0	0	0
0.3666667	0	1	0	0	1
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0.5444444	0	1	1	1	1
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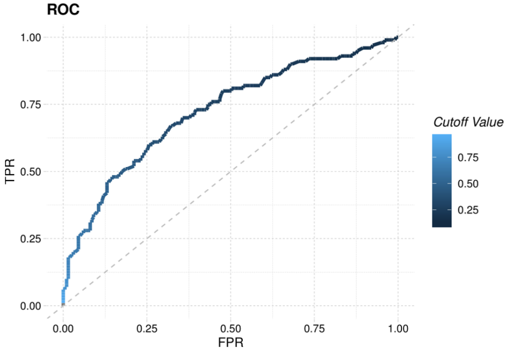
Goodness of fit: ROC and AUC



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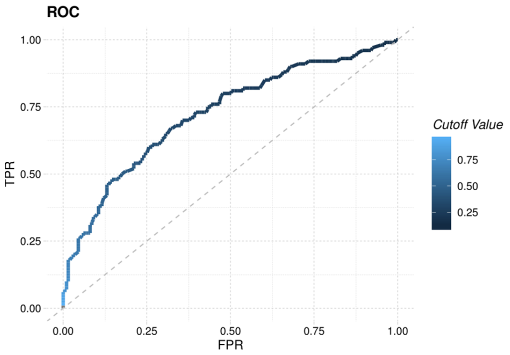
Goodness of fit: ROC and AUC



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- AUC – Area under ROC

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Ideas for final Project

- Play with Goodness of fit

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- LogReg for ordinal response dependent variable.

Coding examples in R

- See in Lab 06