Lesson 04 Multiple Linear Regression Models

Lusine Zilfimian

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- How can we estimate the coefficient of linear regression?

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- How can we estimate the coefficient of linear regression?
- What are TSS, ESS, RSS?
- What is the distribution of $\varepsilon, y, and\ \hat{\beta}_1^{OLS}$?
- How can we estimate the model accuracy?

R-squared

R^2 - the proportion of variability in Y that can be explained using X.

- The total variance of the response variable:

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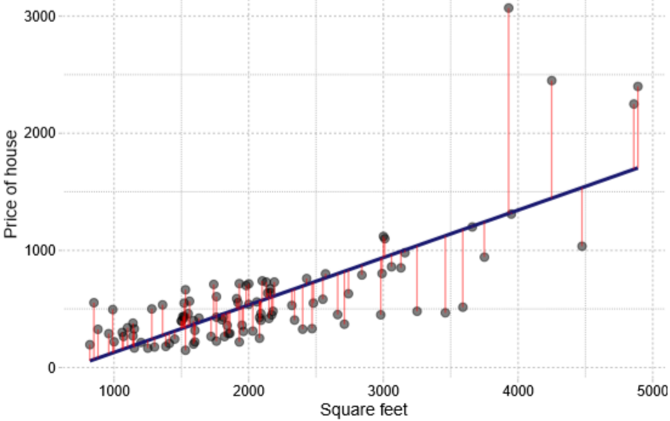
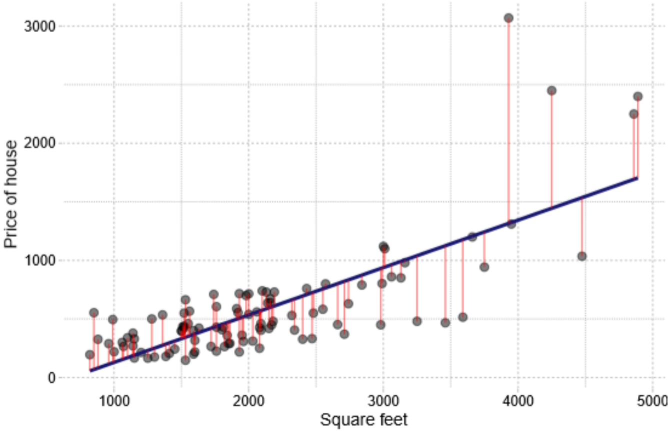
- By definition $0 \leq R^2 \leq 1$
- What is a good R^2 value?

TSS, RSS, ESS



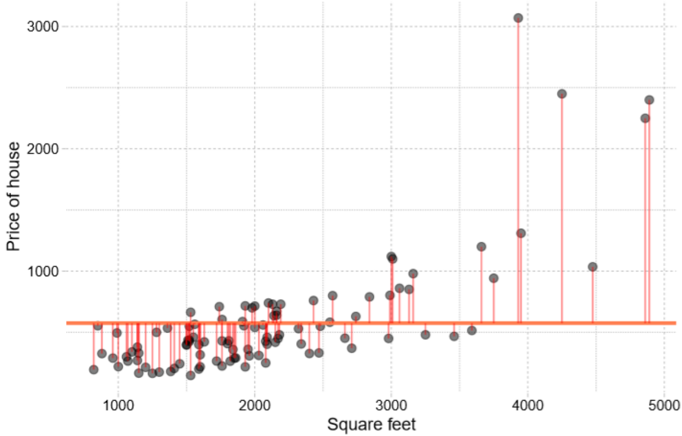
- This is . . .

TSS, RSS, ESS



- This is ...
- Residual Sum of Squares

TSS, RSS, ESS



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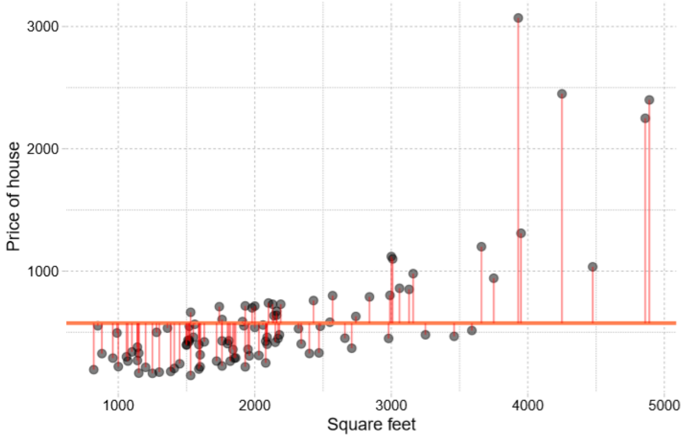
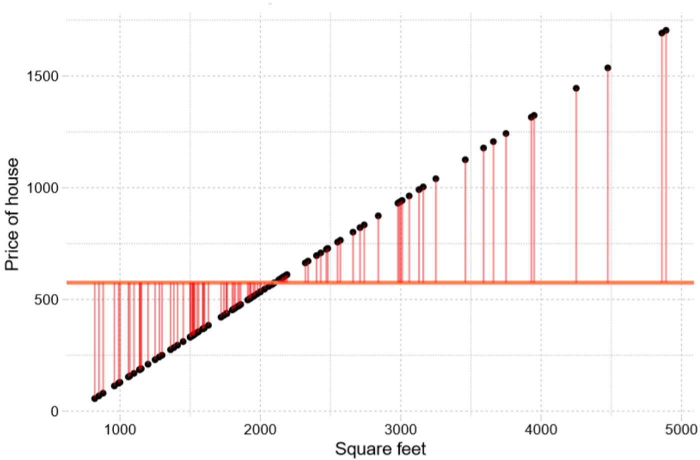


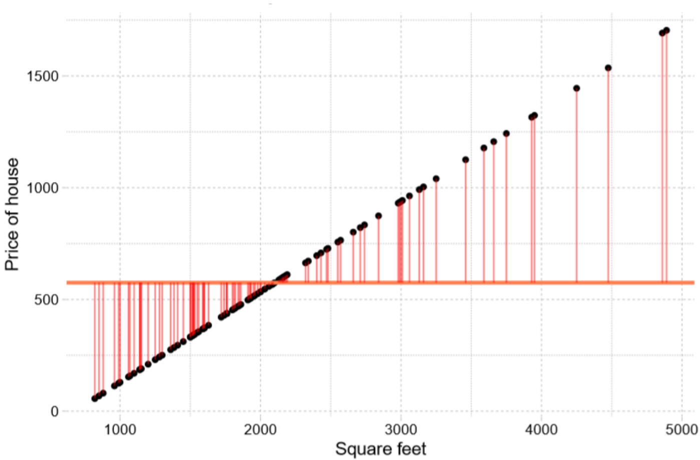
- This is ...
- Total Sum of Squares

TSS, RSS, ESS



- This is . . .

TSS, RSS, ESS



- This is . . .
- Estimated Sum of Squares

Multiple linear regression

Model specification:

- In practice we often have more than one predictor.

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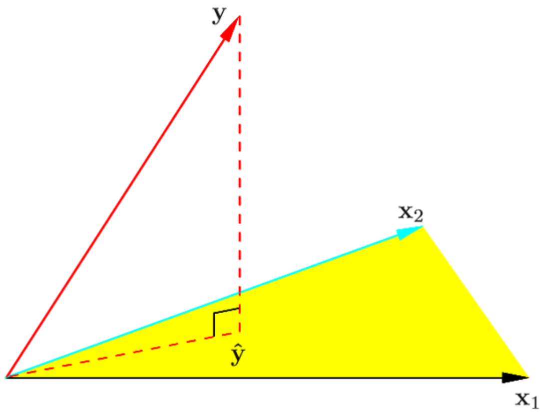
- $$\mathbb{Y} = \mathbb{X}\beta + \varepsilon$$

Geometry of least squares regression with two predictors

Source: T. Hastie, R. Tibshirani, J. Friedman, *The Elements of Statistical Learning*



Minimizing the RSS

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- **Task**: Derive all formulas by yourself.

Comparison of the models: Adjusted R^2

- The number of predictors $\uparrow \Rightarrow R^2 \uparrow$, even if those variables are only weakly associated with the response.

```
hd <- read.csv("housing.csv")[1:100,]
model1 <- lm(price ~ sqft_living + condition, data = hd)
summary(model1)$r.squared
```

```
## [1] 0.4847802
```

```
set.seed(2708)
model2 <- lm(price ~ sqft_living + condition +
    runif(dim(hd)[1]), hd)
summary(model2)$r.squared
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- $R^2 \uparrow \not\Rightarrow$ the quality of model $\uparrow$

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- **Task:** Show that $R^2_{adj} = 1 - (1 - R^2)\dfrac{n-1}{n-k}$
- **Task:** Prove that $R^2 \geq R^2_{adj}, k > 1$

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- **Task:** Prove that for the single linear regression $F = t$.

The Theorem of Gauss-Markov: OLS is BLUE

If the linear regression model satisfies the following six classical assumptions:

1. True model is linear in parameters

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- then ordinary least squares regression produces unbiased estimates that have the smallest variance of all possible linear estimations.

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- $$\varepsilon \sim \mathbb{N}(0, \sigma^2) \;\Rightarrow\; \hat{\beta}_{OLS} \sim \mathbb{N}(\beta, \sigma^2(X^TX)^{-1})$$
- We can apply statistical tests to them

Interpretations

Continuous predictors

Regression coefficients represent the mean change in the response variable for one unit of change in the predictor variable while holding other predictors in the model constant.

- $$\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x_1 + \hat{\beta}_2 x_2 + ... + \hat{\beta}_k x_k$$

Be attentive. Gayane's question is appropriate.

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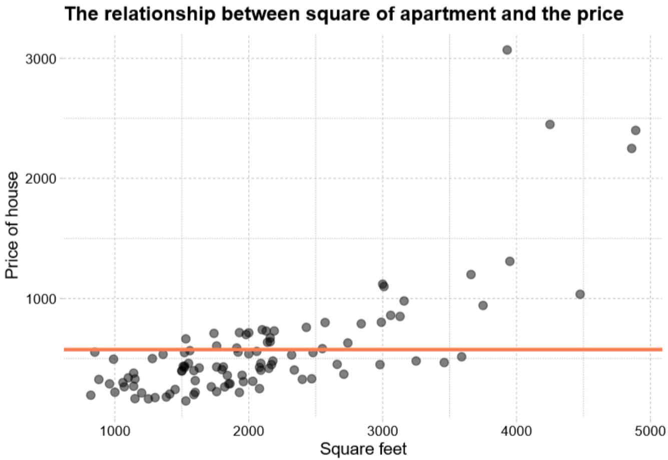
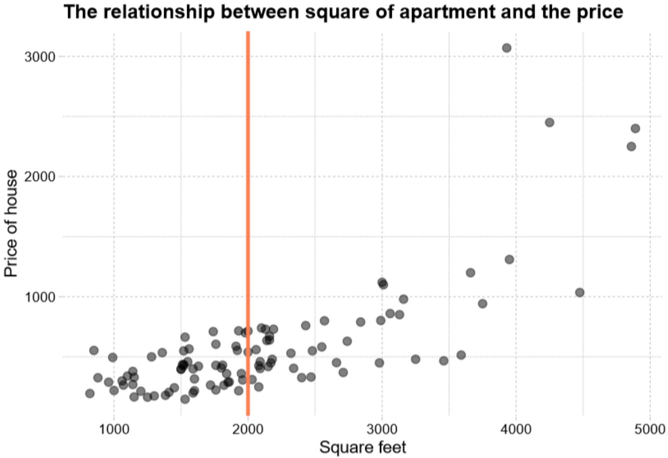
- Holding $x_2, ..., x_k$ constant, 1 unit $\uparrow x_1 \Rightarrow \ \hat{\beta}_1 \ unit \ \uparrow y$
- **Intercept** - $\hat{\beta}_0$ is the average value of y when $x = 0$.

Intercept-only model



Mean change in the response variable



Predictors with only two levels

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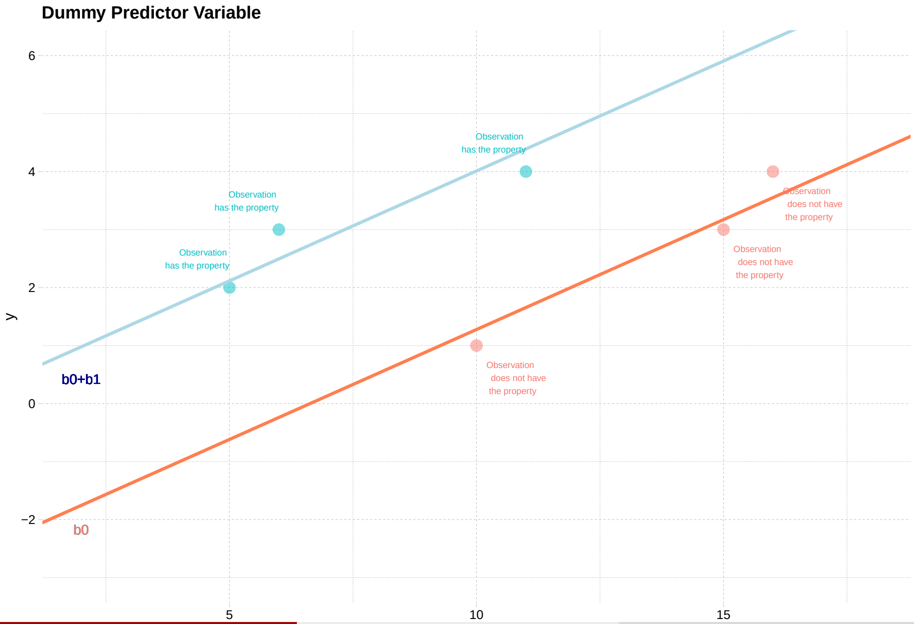
- $\beta_0 + \beta_1$ - the average of y among observations which have the property
- What about another way of coding?

Predictors with only two levels



Predictors with more that two levels

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- β_0 - the average of y among students from universities apart from YSU and AUA

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- $\beta_0 + \beta_1$ - the average of y among students from AUA
- $\beta_0 + \beta_2$ - the average of y among students from YSU

Some aspects of Linear Regression

Additivity

- The additive assumption means that the effect of changes in any predictor on the response is independent of the values of the other predictors.

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- Interaction term: $y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \beta_3 x_{i1} x_{i2} + \varepsilon_i$
- An interaction between a qualitative variable and a quantitative variable:

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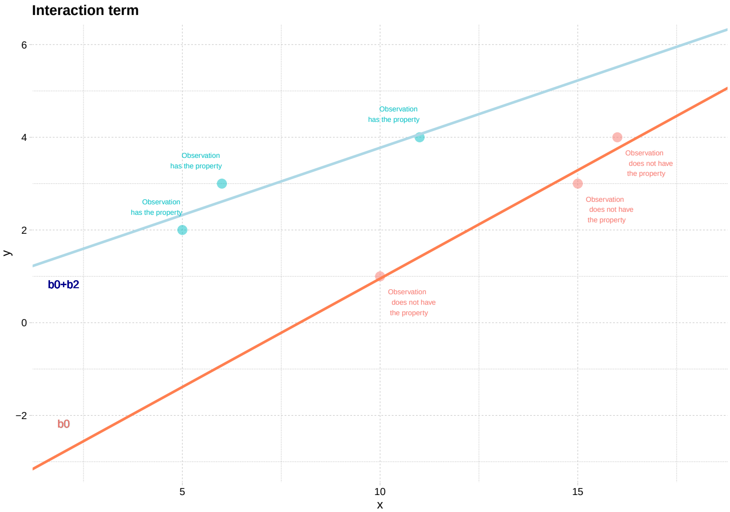
Additivity

- The additive assumption means that the effect of changes in any predictor on the response is independent of the values of the other predictors.
- Interaction term: $y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \beta_3 x_{i1} x_{i2} + \varepsilon_i$
- An interaction between a qualitative variable and a quantitative variable:
- $y_i = \beta_0 + \beta_1 x_{i1} + \begin{cases} \beta_2 + \beta_3 x_{i1} + \varepsilon_i \\ 0 + \varepsilon_i \end{cases}$

- Slope for blue line is $b_1 + b_3$



- Slope for blue line is $b_1 + b_3$
- Slope for red line is b_1