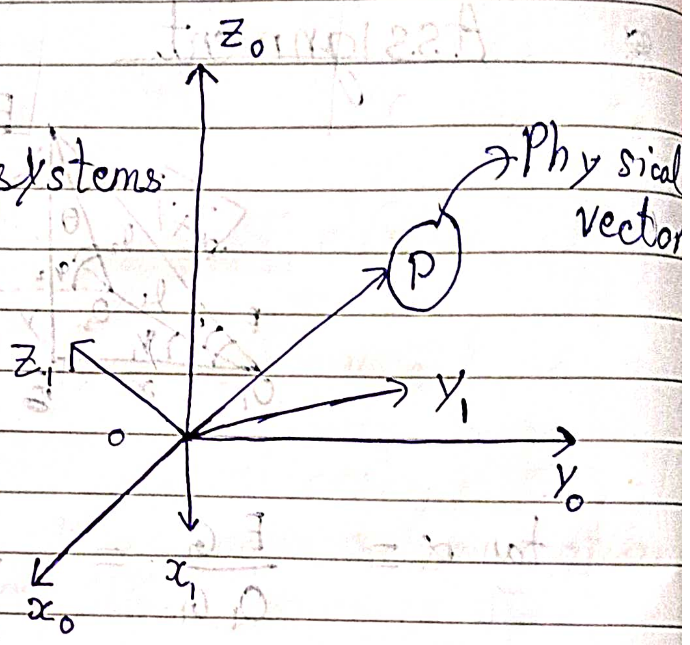
chapter-2-

Rigid motions & homogeneous transformat

Rotations-



* Both frames are Right handed systems with same origin.

* we have two relationships

(a) Rep. of $\vec{P}$ in both frames

(b) Rotating vector P

(a)#

$$\boxed{P = P_0 = P_1}$$

$$P_0 = P_{0x}\hat{i}_0 + P_{0y}\hat{j}_0 + P_{0z}\hat{k}_0$$

$$P_1 = P_{1x}\hat{i}_1 + P_{1y}\hat{j}_1 + P_{1z}\hat{k}_1$$

$$P_{0x} = P_0 \cdot \hat{i}_0 = P_1 \cdot \hat{i}_0 +$$

$$= P_{1x}\hat{i}_1 \cdot \hat{i}_0 + P_{1y}\hat{j}_1 \cdot \hat{i}_0 + P_{1z}\hat{k}_1 \cdot \hat{i}_0$$

$$P_{0y} = P_{1x}\hat{i}_1 \cdot \hat{j}_0 + P_{1y}\hat{j}_1 \cdot \hat{j}_0 + P_{1z}\hat{k}_1 \cdot \hat{j}_0$$

$$P_{0z} = P_{1x}\hat{i}_1 \cdot \hat{k}_0 + P_{1y}\hat{j}_1 \cdot \hat{k}_0 + P_{1z}\hat{k}_1 \cdot \hat{k}_0$$

$$\begin{bmatrix} P_{0x} \\ P_{0y} \\ P_{0z} \end{bmatrix} = \begin{bmatrix} \hat{i}_1 \cdot \hat{i}_0 & \hat{j}_1 \cdot \hat{k}_0 & \hat{k}_1 \cdot \hat{i}_0 \\ \hat{i}_1 \cdot \hat{j}_0 & \hat{j}_1 \cdot \hat{j}_0 & \hat{k}_1 \cdot \hat{j}_0 \\ \hat{k}_1 \cdot \hat{k}_0 & \hat{j}_1 \cdot \hat{k}_0 & \hat{k}_1 \cdot \hat{k}_0 \end{bmatrix} \begin{bmatrix} P_{1x} \\ P_{1y} \\ P_{1z} \end{bmatrix}$$

$$\boxed{P_0 = R_0^1 \, P_1}$$

Rotation matrix - converts coordinates of 1 frame to O.

b) # if we do reverse.

$$P^0_{x} = P_{0x} \, \hat{i}_0 \cdot \hat{i}_1 + P_{0y} \, \hat{j}_0 \cdot \hat{i}_1 + P_{0z} \, \hat{k}_0 \cdot \hat{i}_1$$

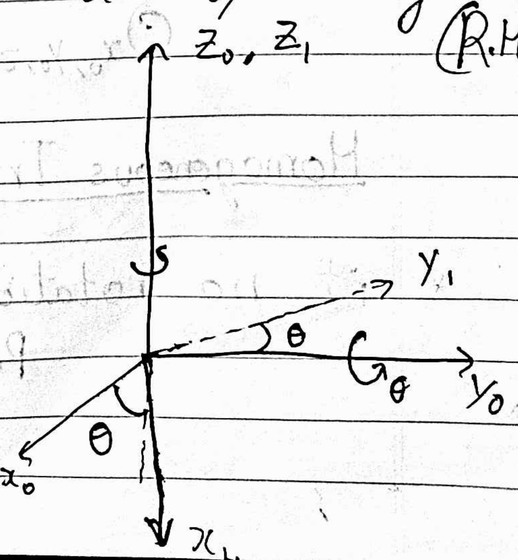
$$\boxed{P^0_{x_1} = R_1^0 \, P_0}$$ $R \in$ Special orthogonal matrix of order 3

$$R_1^0 = \boxed{R_0^1}^T$$

* $R_1^0 = (R_0^1)^{-1} = (R_0^1)^T$
* $R_1^0 \, R_0^1 = I$
* $|R| = 1 \Rightarrow$ pure rotation.

* Example - Rotation about z axis by an angle θ & (R.H.R)

$$R_0^1 = \begin{bmatrix} \cos\theta & -S_\theta & 0 \\ S_\theta & C_\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Basic Rotation matrix
$R_{z,\theta}$.

Similarly

$$R_{y,\theta} = \begin{bmatrix} C_\theta & 0 & S_\theta \\ 0 & 1 & 0 \\ -S_\theta & 0 & C_\theta \end{bmatrix} \qquad R_{x,\theta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C_\theta & -S_\theta \\ 0 & S_\theta & C_\theta \end{bmatrix}$$

Composition of rotations -

$P_0 = R_0^1 P_1$, $P_1 = R_1^2 P_2$, ...

$\boxed{P_0 = R_0^1 R_1^2 P_2}$ → Any relationships of rotation b/w two frames is broken into two sequence of rotation

$\boxed{R_0^n = R_0^1 \; R_1^2 \; R_2^3 \ldots \; R_{n-1}^n}$

works only when we talk about successive rotations about current axis

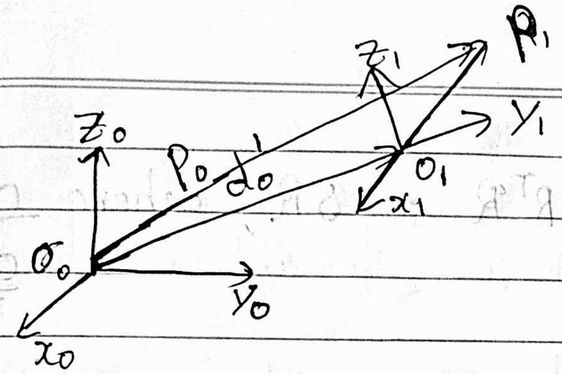
if successive rotation are about original fixed axis, the order of mul is different (textbook)

$$O_{x_0,y_0,z_0} \Rightarrow O_{x_1,y_1,z_1} \Rightarrow O_{x_2,y_2,z_2}$$

Homogeneous Transformation -

* if no rotation then

$P_0 = P_1 + d_0^1$ (→ 3d vector)



translation first then rotation

in general;-

$$P_0 = R_0^1 p_1 + d_0^1$$

similarly

$$P_1 = R_1^2 P_2 + d_1^2$$

on substitution

$$\boxed{P_0 = \underbrace{R_0^1 R_1^2}_{R_0^2} P_2 + \underbrace{R_0^1 d_1^2 + d_0^1}_{d_0^2}}$$

can be expressed as matrix,:

$$H = \begin{bmatrix} R & d \\ 0 & 1 \end{bmatrix}, \quad R \in SO(3) \text{ mul. with } \begin{bmatrix} P \\ 1 \end{bmatrix}$$

$\Rightarrow$ new transformed coords.

$$\begin{bmatrix} P_0 \\ 1 \end{bmatrix} = H_0^1 \begin{bmatrix} P_1 \\ 1 \end{bmatrix} \quad \& \quad \begin{bmatrix} P_0 \\ 1 \end{bmatrix} = H_0^1 H_1^2 \begin{bmatrix} P_2 \\ 1 \end{bmatrix}$$

skew symmetric matrix -

$$S = -S^T \text{ or } S + S^T \quad \& \quad S \in ss(3)$$

$$S = \begin{bmatrix} 0 & -S_1 & S_2 \\ S_1 & 0 & -S_3 \\ -S_2 & S_3 & 0 \end{bmatrix}$$

if $a = [a_x \;\; a_y \;\; a_z]^T$ vector

$$S(a) = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}$$

$$\boxed{a \times p = S(a)\, p}$$

Note -

(a) $$\frac{dR}{d\theta} = \frac{dR}{d\theta} R^T R = SR, \text{ where } \frac{dR}{d\theta} R^T = S$$

Similarly

$$\dot{R}(t) = S(t) R(t),$$ where $S(t)$ is $S(\omega)$ where ω is angular velocity vector.

$$\frac{dr}{dt} = \omega \times r$$ } → in pure rotation

(b) $$\dot{R}_0^n = S(\omega_0^n) R_0^n, \text{ where } \omega_0^n = \omega_0^1 + R_0^1 \omega_1^2 + R_0^2 \omega_2^3 + \dots + R_0^{n-1} \omega_{n-1}^n$$

Q Example - for a basic $R_{y,\theta}$, $\theta(t)$ find $\dot{R}_{y,t}$

$$R_{y,\theta} = \begin{bmatrix} C_\theta & 0 & S_\theta \\ 0 & 1 & 0 \\ -S_\theta & 0 & C_\theta \end{bmatrix} \Rightarrow \dot{R}_{y,\theta} = \begin{bmatrix} -S_\theta \cdot \dot{\theta} & 0 & C_\theta \cdot \dot{\theta} \\ 0 & 0 & 0 \\ -C_\theta \dot{\theta} & 0 & -S_\theta \cdot \dot{\theta} \end{bmatrix}$$

Now,

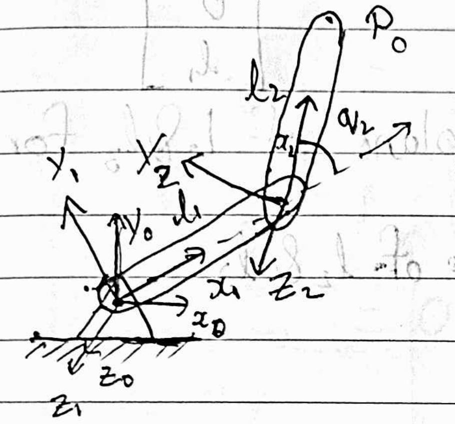
$$\omega = \dot{\theta} = [0 \ \dot{\theta} \ 0]^T \quad \text{(in y- direction)}$$

$$\therefore S(\omega) = \begin{bmatrix} 0 & 0 & \dot{\theta} \\ 0 & 0 & 0 \\ -\dot{\theta} & 0 & 0 \end{bmatrix}$$

$$\dot{R} = S(\omega) R = \begin{bmatrix} 0 & 0 & \dot{\theta} \\ 0 & 0 & 0 \\ -\dot{\theta} & 0 & 0 \end{bmatrix} \begin{bmatrix} C_\theta & 0 & S_\theta \\ 0 & 1 & 0 \\ -S_\theta & 0 & C_\theta \end{bmatrix}$$

$$= \begin{bmatrix} -S_\theta \dot{\theta} & 0 & C_\theta \cdot \theta \\ 0 & 0 & 0 \\ -C_\theta \dot{\theta} & 0 & -S_\theta \cdot \theta \end{bmatrix}$$

Example: Elbow manipulator, find P_0



z_0, z_1, z_2 are all in the same direction

$$H_0^1 \to R_0^1,\ d_0^1 \qquad H_1^2 \to R_1^2,\ d_1^2$$

$$H = \begin{bmatrix} R & d \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} P_0 \\ 1 \end{bmatrix} = H_0^1 H_1^2 \begin{bmatrix} P_2 \\ 1 \end{bmatrix} \qquad P_2 = \begin{bmatrix} l_2 \\ 0 \\ 0 \end{bmatrix}$$

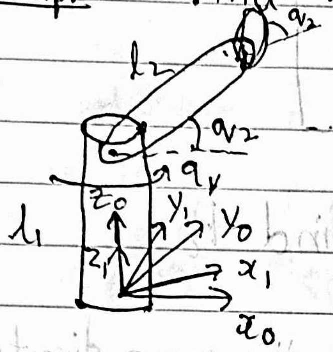
$$R_1^2 = R_{z_2, q_2} \quad d_1^2 = \begin{bmatrix} l_1 \\ 0 \\ 0 \end{bmatrix} \; ; \; R_0^1 = R_{z_0, q_1} \quad d_0^1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} P_{0x} \\ P_{0y} \\ P_{0z} \\ 1 \end{bmatrix} = \begin{bmatrix} C_{q_1} & -S_{q_1} & 0 & 0 \\ S_{q_1} & C_{q_1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C_{q_2} & -S_{q_2} & 0 & l_1 \\ S_{q_2} & C_{q_2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} l_2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} C_{q_1} C_{q_2} l_2 + C_{q_1} l_1 - S_{q_1} S_{q_2} l_2 \\ S_{q_1} C_{q_2} l_2 + S_{q_1} l_1 + C_{q_1} S_{q_2} l_2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} l_2\, C_{(q_1+q_2)} + l_1 C_{q_1} \\ l_2\, S_{(q_1+q_2)} + l_1 S_{q_1} \\ 0 \\ 1 \end{bmatrix}$$

Same as ① in 2R manipulator but q_1 & q_2 are defined differently.

Example - find P_0 from PUMA560



$R_0^1 = R_{z,q_1}$; $d_0^1 = 0$

$R_1^2 = R_{x,q_2}$; $d_1^2 = \begin{bmatrix} 0 \\ 0 \\ l_1 \end{bmatrix}$

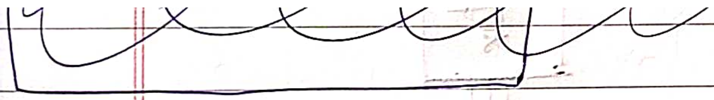
if we set x_0 being $\perp$ to the plane of l_2 & l_3 for the baselink,

x_1 will always be $\perp$ to the plane of l_2 & l_3

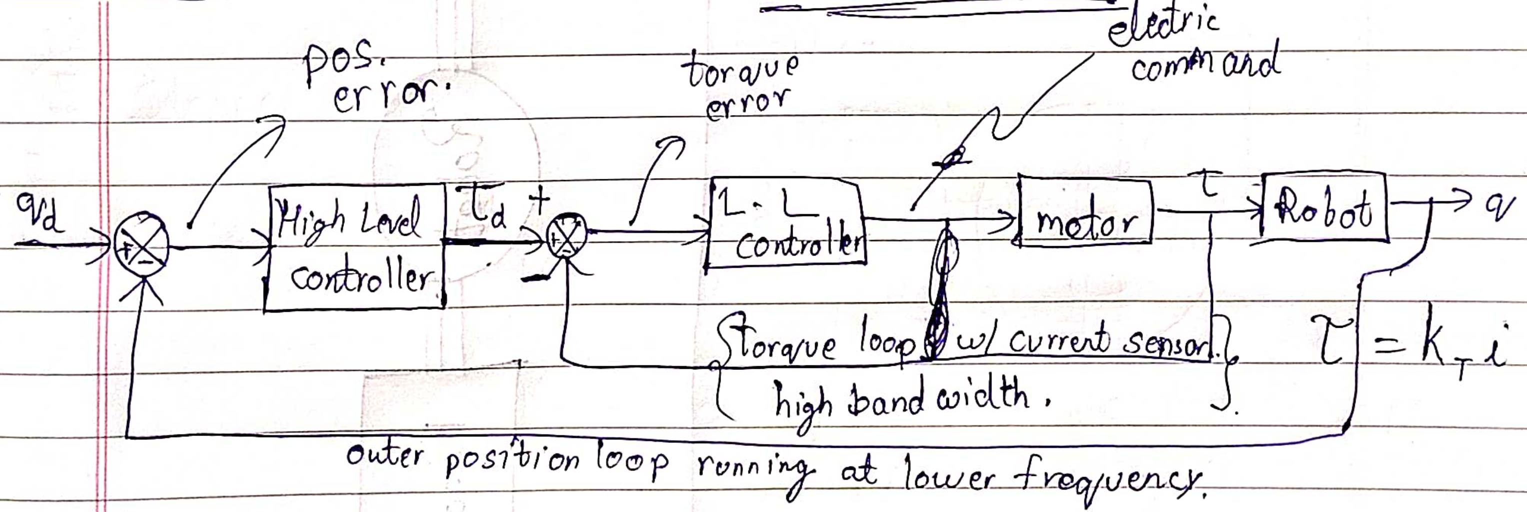
other 2 frames same as elbow manipulator

$R_2^3 = R_{x,q_3}$; $d_2^3 = \begin{bmatrix} 0 \\ l_2 \\ 0 \end{bmatrix}$

$P_3 = \begin{bmatrix} 0 \\ l_3 \\ 0 \end{bmatrix}$



2 level control



Translation to hardware -